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From the Lie Operad to the Grothendieck–Teichmüller Group

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We study the deformation complex of the natural morphism from the degree d shifted Lie operad to its polydifferential version, and prove that it is quasi-isomorphic to the Kontsevich graph complex $\mathbf{GC}_d$. In particular, we show that in the case $d = 2$ the Grothendieck–Teichmüller group $\mathbf{GRT}_1$ is a symmetry group (up to homotopy) of the aforementioned morphism. We also prove that in the case $d = 1$, corresponding to the usual Lie algebras, the natural morphism admits a unique homotopy non-trivial deformation, which is described explicitly with the help of the universal enveloping construction. Finally, we prove the rigidity of the strongly homotopy version of the universal enveloping functor in the Lie theory.

1 Introduction

The theory of operads, props, and graph complexes underwent a rapid development in recent years; its applications can be seen nowadays almost everywhere [11, 12, 23]: in algebraic topology, in homological algebra, in (algebraic) geometry, in string topology, in deformation theory, in quantization theory, etc. This theory unifies several ideas from various areas in mathematics.

There is a remarkable polydifferential functor $\mathcal{O}$ from the category of props to the category of operads,

$$\begin{array}{rccc} \mathcal{O}: & \mathbf{Props} & \longrightarrow & \mathbf{Operads} \\ & P & \longmapsto & \mathcal{O}(P), \end{array}$$

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whose main defining property is the following one: given any representation of the prop P in a differential graded vector space V, there is an associated representation of the operad $\mathcal{O}(P)$ on the symmetric tensor algebra $\odot V$ given in terms of polydifferential operators with respect to the standard graded commutative product in $\odot V$. It was introduced in [15] to study ribbon graph complexes in the theory of moduli spaces of algebraic curves, and it was used in [1] to classify all homotopy classes of M. Kontsevich formality maps and in [16] to study universal quantizations of Lie bialgebras for example.

This paper is devoted to the study of the operad $\mathcal{O}(\mathbf{Lie}_d)$ obtained by applying that functor $\mathcal{O}$ to the prop closure of the classical operad $\mathbf{Lie}_d$ controlling degree d shifted Lie algebras (the case $d = 1$ corresponding to usual Lie algebras, often abbreviated $\mathbf{Lie} := \mathbf{Lie}_1$). This operad can be described in terms of graphs with two types of vertices and, as we discuss below, occurs naturally in the study of the well-known universal enveloping functor $\mathfrak{U}$ from the category of Lie algebras to the category of associative algebras. This functor can be understood as a morphism of operads

$$\mathcal{U} : \mathbf{Ass} \to \mathcal{O}(\mathbf{Lie}), \tag{1}$$

satisfying some natural non-triviality condition (see §7 below).

This operad comes equipped with a natural morphism of operads

$$i : \mathbf{Lie}_d \to \mathcal{O}_c(\mathbf{Lie}_d), \tag{2}$$

where $\mathcal{O}_c(\mathbf{Lie}_d)$ is the suboperad of $\mathcal{O}(\mathbf{Lie}_d)$ spanned by connected graphs.

The first main (and perhaps, very surprising) result of this paper says that the deformation complex $\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}_c(\mathbf{Lie}_d))$ of the above natural morphism can be identified (via an explicit quasi-isomorphism shown in §6) with M. Kontsevich's famous graph complex $\mathbf{GC}_d$! In particular, this result implies that the mysterious Grothendieck–Teichmüller group acts non-trivially and almost faithfully on homotopy classes of the map i.

This result gives us one of the simplest incarnations of the Grothendieck–Teichmüller group, and explains, perhaps, why it occurs in two seemingly different deformation quantization problems, the universal quantization of Poisson structures (solved by M. Kontsevich in [9]) and the universal quantization of Lie bialgebras (formulated by V. Drinfeld and solved by Etingov–Kazhdan in [5]) as both these theories involve $\mathcal{O}(\mathbf{Lie})$ as a sub-structure.

The universal enveloping construction associates to a Lie algebra V the corresponding associative algebra $\mathfrak{U}(V)$, which is isomorphic by the Poincaré–Birkhoff–Witt (PBW) theorem to the symmetric tensor algebra $\odot V$. Hence, it is not that surprising that this construction can be described in terms of the operad $\mathcal{O}(\mathbf{Lie})$. Indeed, following S. Gutt [6] and K. Vinay [7], one first interprets the universal enveloping algebra construction $\mathfrak{U}(V)$ as a star product construction on $\odot V$ given in terms of polydifferential operators. Second, one notices that this interpretation can be encoded as a morphism of operads $\mathbf{Ass} \to \mathbf{End}_{\odot V}$ satisfying some non-triviality condition. The point is that this map factors through some morphism $\mathbf{Ass} \to \mathcal{O}(\mathbf{Lie})$ and the canonical map $\mathcal{O}(\mathbf{Lie}) \to \mathbf{End}_{\odot V}$ induced by the functor $\mathcal{O}$ applied to the given representation of $\mathbf{Lie}$ in V. Hence, all subtleties of the universal enveloping functor $\mathfrak{U}$ get encoded into the morphism of operads $\mathbf{Ass} \to \mathcal{O}(\mathbf{Lie})$ satisfying some non-triviality condition (see §7 below).

We study in this paper the deformation complex $\mathbf{Def}(\mathbf{Ass} \xrightarrow{\mathcal{U}} \mathcal{O}(\mathbf{Lie}))$ and prove that its cohomology is one-dimensional, the unique cohomology class corresponding to the rescaling freedom of the Lie bracket. The conclusion is that the morphism $\mathcal{U}$ is unique (up to gauge equivalence). This result is not surprising, of course. One can infer it from the classification theorem of Kontsevich formality maps given in terms of the graph complex $\mathbf{GC}_2$ [4]. So we just give a new very short proof of this uniqueness. This proof has a small advantage that it carries over to $\mathbf{Lie}_\infty$ straightforwardly.

There are several constructions generalizing the universal enveloping functor from Lie algebras (controlled by the operad $\mathbf{Lie}$) to strongly homotopy Lie algebras (which are controlled by $\mathbf{Lie}_\infty$, the minimal resolution of $\mathbf{Lie}$). All constructions involve the notion of strongly homotopy associative algebras, which was introduced by J. Stasheff in [20] and which are controlled by the dg operad $\mathbf{Ass}_\infty$, the minimal resolution of $\mathbf{Ass}$. One such generalization of the functor $\mathfrak{U}$ is offered by M. Kontsevich formality map applied to linear polyvector fields; that generalization uses, in general, graphs with wheels, but, as has been proven by B. Shoikhet [19], the graphs with wheels can be removed so that one gets a strongly homotopy extension of the functor $\mathfrak{U}$, which works well for infinite dimensional $\mathbf{Lie}_\infty$ algebras. Another construction was given by V. Baranovsky [2] as the cobar construction of the Cartan–Chevalley–Eilenberg coalgebra associated to an $\mathbf{Lie}_\infty$ algebra, by J. M. Moreno-Fernández [17] as an $\mathbf{Ass}_\infty$ algebra isomorphic as graded vector spaces to the free symmetric algebra associated to an $\mathbf{Lie}_\infty$ algebra and by T. Lada and M. Markl in [10].

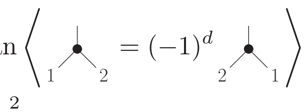
These constructions can be understood as a morphism of operads $\mathbf{Ass}_\infty \to \mathcal{O}(\mathbf{Lie}_\infty)$. We study the deformation complex of any of these two maps and prove that it is quasi-isomorphic to the deformation complex of the map considered in (**??**), implying that all constructions are gauge equivalent. Moreover, any other attempt to construct such a generalization satisfying some natural conditions must be gauge equivalent to these ones as the cohomology of the complex $\mathbf{Def}(\mathbf{Ass}_\infty \to \mathcal{O}(\mathbf{Lie}_\infty))$ is one-dimensional (see Corollary 7.8). This cohomology result is in a full agreement with the derived PBW theorem established by A. Khoroshkin and P. Tamaroff in [8].

2 Reminder on Classical Operads

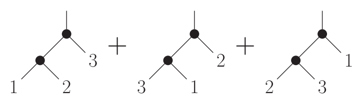
Let us remind the reader the construction of two classical operads [11, 12].

Let $E = \{E(n)\}$ be the $\mathbb{S}$-module defined by $E(n) = 0$ except that

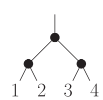
$$E(2) = \mathrm{sgn}_2^d[d-1] = \mathrm{span}$$



The operad of degree d shifted Lie algebras is defined to be the quotient $\mathbf{Lie}_d := \mathrm{Free}(E)/I$, where I is generated by the element

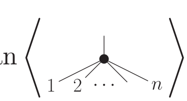


When studying linear combinations of graphs built from this generating corolla, it is useful, not to make sign mistakes, to view each such a corolla as a degree d vertex with two degree $-d$ incoming half-edges and one degree 1 outgoing half-edge. Hence for d odd, such a graph has vertices of odd degree (and thus an ordering of this set, up to a permutation σ and multiplication by $\mathrm{sgn}(\sigma)$, has to be chosen), while internal edges have degree $1-d$. In the case d even, the vertices are even, but the internal edges are odd so that an ordering of this set is chosen (up to permutation). This rule helps us understand what kind of implicit ordering is hidden in a graph like this:



The minimal resolution of $\mathbf{Lie}_d$ is given by the dg quasi-free operad $\mathbf{HoLie}_d = (\mathbf{Free}\langle E\rangle, \delta)$ generated by the $\mathbb{S}$-module

$$E = \{E(n) = (\mathrm{sgn}_n)^{\otimes |d|}[nd-d-1] = \mathrm{span}$$

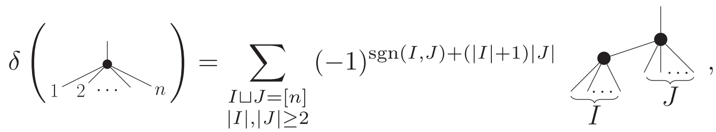


$$\}_{n\geq 2},$$

for $n \geq 2$, where

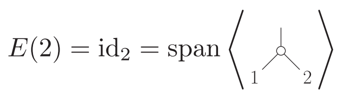
$$(\mathrm{sgn}_n)^{\otimes |d|} = \begin{cases} \mathrm{sgn}_n & \text{if } d \text{ is odd} \\ 1_n & \text{if } d \text{ is even,} \end{cases}$$

where sgn_n denotes the 1-dimensional sign representation and 1_n denotes the 1-dimensional trivial representation of $\mathbb{S}_n$. The differential is given on generators by

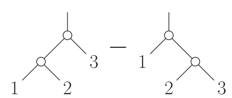


where $\mathrm{sgn}(I, J)$ denotes the sig of the shuffle permutation induced by the ordered partition $I \sqcup J$ of $[n]$. The case $d = 1$ is also denoted by $\mathbf{Lie}_\infty := \mathbf{HoLie}_1$.

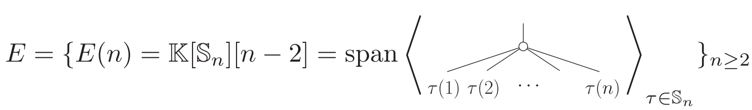
Let $E = \{E(n)\}$ be the $\mathbb{S}$-module defined by $E(n) = 0$ except that



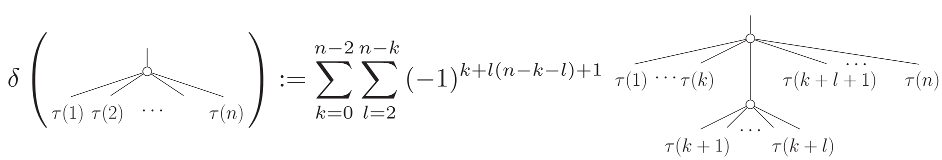
The operad of associative algebras is defined by the quotient $\mathbf{Ass} := \mathrm{Free}(E)/I$ where I is generated by



Its minimal resolution is given by the dg free operad $\mathbf{Ass}_\infty := (\mathrm{Free}(E), \delta)$, where E is the $\mathbb{S}$-module generated by

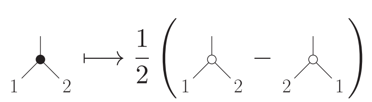


and the differential is given on generators by



Representations of $\mathbf{Ass}_\infty$ in a dg vector space V are precisely the strongly homotopy associative algebras introduced by J. Stasheff in [20].

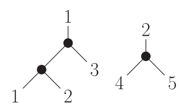
As a final remark, we recall that there is a morphism of operads $\mathbf{Lie} \to \mathbf{Ass}$ given by



This map will be used in Section 8.

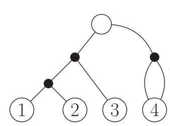
3 Polydifferential Functor

For our purposes, we need to recall the construction of an endofunctor in the category of dg operads, see [15] for additional details. Given an augmented prop $\mathcal{P}$ there is an operad $\mathcal{O}(\mathcal{P})$ such that for any representation $\mathcal{P} \to \mathbf{End}_V$ in a dg vector space V there is a representation $\mathcal{O}(\mathcal{P}) \to \mathbf{End}_{\odot^\bullet(V)}$ in the graded algebra $\odot^\bullet(V)$ where the elements of $\mathcal{O}(\mathcal{P})$ act as polydifferential operators. When $\mathcal{P}$ is an operad, we consider its prop closure, which we also denote $\mathcal{P}$. The elements of the prop closure are given by disjoint unions of elements of $\mathcal{P}$ with inputs and outputs labelled differently. For example,



$\in \mathbf{Lie}_d(2,5).$

We are mainly interested in the operad $\mathcal{O}(\mathbf{Lie}_d)$ obtained by applying the functor $\mathcal{O}$ to the prop closure of the operad of degree d shifted Lie algebras. Elements of $\mathcal{O}(\mathbf{Lie}_d)$ are obtained joining the outputs of an element of the prop closure of $\mathbf{Lie}_d$ to a new white vertex and also by partitioning all of the input legs into groups and joining each group them to a new labelled white vertex. For example,



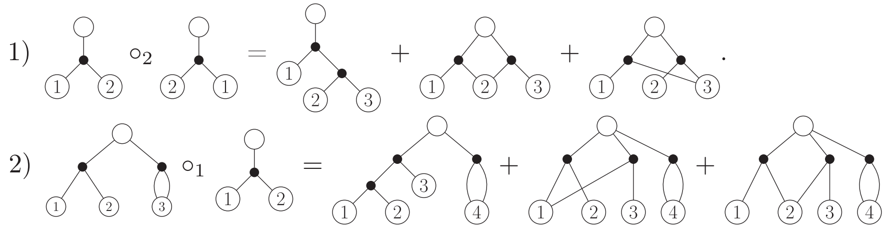
$. \in \mathcal{O}(\mathbf{Lie}_d)(4).$

The black vertices are called *internal vertices* and edges between internal vertices are called *internal edges*. By erasing all white vertices, we obtain a collection of disjoint trees, which will be called *internal irreducible components* (i.i.c.).

Note that for d even we implicitly assume an ordering of the internal edges as well as an ordering of the in-edges of the white out-vertex (up to a sign). For d odd, we assume implicitly an ordering of the internal vertices as well as an ordering (up to a sign) of the out-edges attached to each white in-vertex.

The compositions $\Gamma_1 \circ_i \Gamma_2$ in $\mathcal{O}(\mathbf{Lie}_d)$ work as follows: first, we erase the white out-vertex of Γ_2 and the ith white in-vertex of Γ_1. This step creates many “hanging” out-edges in Γ_2 and in-edges in Γ_1. Second, we sum over all possible attachments of the hanging out-edges of Γ_2 to the hanging in-edges of Γ_1 and the white out-vertex of Γ_1. Finally, we sum over all attachments of remaining in-edges of Γ_1 to the white in-vertices of Γ_2.

Example 3.1.

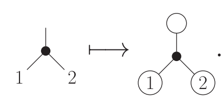


We define $\mathcal{O}_c(\mathbf{Lie}_d)$ to be the suboperad spanned by connected graphs, that is, we assume that the graphs remain connected when we erase the white output vertex.

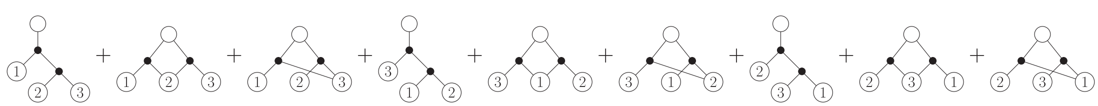
Lemma 3.2. There is a morphism of operads

$$i : \mathbf{Lie}_d \longrightarrow \mathcal{O}_c(\mathbf{Lie}_d) \tag{3}$$

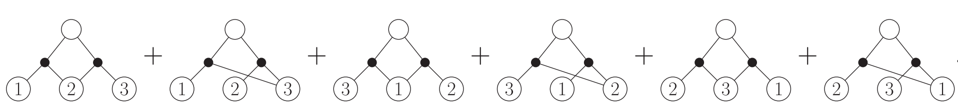
given by



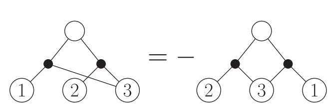
Proof. It suffices to check that the Jacobi identity is mapped to 0. Indeed, we need that



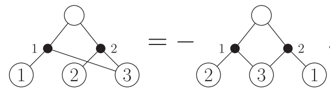
vanishes. This expression reduces to



If d is odd, we observe that



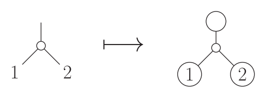
and thus for any permutation of $\{1, 2, 3\}$. For d even, we see that



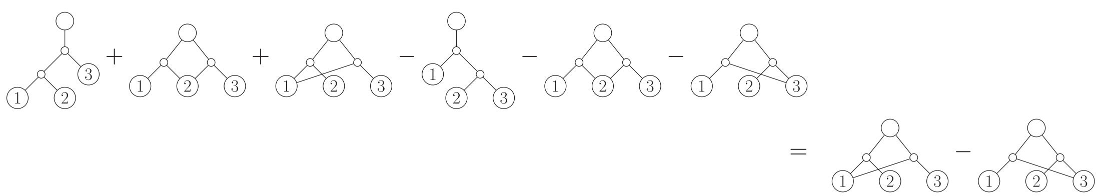
In any case, we see that the Jacobi identity is mapped to zero. ∎

Remark 3.3. The map

$$\mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Ass})$$



is not a morphism of operads as the associativity condition is not mapped to zero. Indeed, the associativity condition would require that the following relation vanishes,



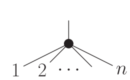
which is not the case.

4 Deformation Complexes

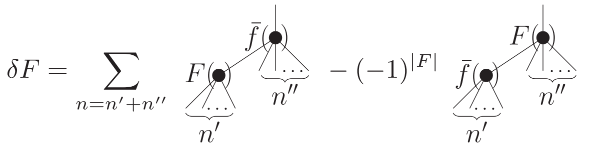
We follow [14] to introduce deformation complexes. We consider a morphism of operads $f : \mathbf{Lie}_d \to \mathcal{P}$ and define the associated deformation complex by

$$\begin{aligned}\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{f} \mathcal{P}) \quad = \quad \mathbf{Def}(\mathbf{HoLie}_d \xrightarrow{\bar{f}} \mathcal{P}) \quad &= \quad \prod_{n\geq 1} [E(n)^* \otimes \mathcal{P}(n)]^{\mathbb{S}_n}[-1] \\ &= \quad \prod_{n\geq 1} [\mathrm{sgn}_n^{\otimes |d|} \otimes \mathcal{P}(n)]^{\mathbb{S}_n}[d - dn].\end{aligned}$$

The map $\bar{f} : \mathbf{HoLie}_d \to \mathbf{Lie}_d \xrightarrow{f} \mathcal{P}$ induces a differential as follows: an element $F \in [\mathrm{sgn}_n^{\otimes|d|} \otimes \mathcal{P}(n)]^{\mathbb{S}_n}$ can be interpreted as the image of

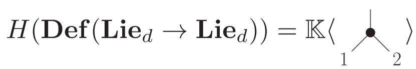


under a derivation $F : \mathbf{HoLie}_d \to \mathcal{P}$. The differential is given by



We recall the following facts:

Fact. [21]



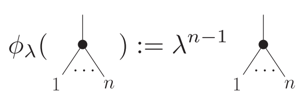
Fact. [14]

Let $f : \mathcal{P} \to \mathcal{Q}_1$, $s : \mathcal{Q}_1 \to \mathcal{Q}_2$ be morphism of dg operads such that s is a quasi-isomorphism. Then there is a quasi-isomorphism

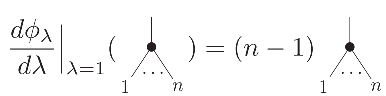
$$\mathbf{Def}(\mathcal{P} \xrightarrow{f} \mathcal{Q}_1) \cong \mathbf{Def}(\mathcal{P} \xrightarrow{sf} \mathcal{Q}_2).$$

Combining these two facts we see that $H(\mathbf{Def}(\mathbf{HoLie}_d \to \mathbf{HoLie}_d))$ is one-dimensional.

Now consider for $\lambda \in \mathbb{K}^*$ the map ϕ_λ, defined by



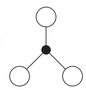
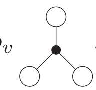
This is an automorphism of $\mathbf{HoLie}_d$. We compute that,



which implies that $\Lambda := \sum_{n\geq 2} (n-1)$  $\in \mathbf{Def}(\mathbf{HoLie}_d \to \mathbf{HoLie}_d)$ is a cohomology class and that

$$H^0(\mathbf{Def}(\mathbf{HoLie}_d \to \mathbf{HoLie}_d)) = \mathbb{K}\langle \Lambda \rangle.$$

We are interested in the deformation complex $\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}(\mathbf{Lie}_d))$ of the morphism of operads described in (3). The elements of this complex can be described by formal series of elements in $\mathcal{O}(\mathbf{Lie}_d)$ with either unlabelled white input vertices if d is even or with an ordering of the input vertices up to even permutations. The differential can be described more explicitly for $\Gamma \in \mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}(\mathbf{Lie}_d))$ by

$\delta\Gamma =$  $\circ\, \Gamma - (-1)^{|\Gamma|} \sum_{v \in V_{\mathrm{in}}(\Gamma)} \Gamma \circ_v$  ,

where $V_{\mathrm{in}}(\Gamma)$ denotes the set of white input vertices of Γ.

5 Kontsevich Graph Complexes

We recall some facts about the operad of graphs and graph complexes. For completeness, we recall the definition of a directed graph:

Definition 5.1. A graph with hairs is a triple $\Gamma = (H(\Gamma), \sqcup, \tau)$ where

1) $H(\Gamma)$ is a finite set of *half-edges*,
2) $\sqcup$ is a partition of $H(\Gamma)$

$$H(\Gamma) = \bigsqcup_{v \in V(\Gamma)} H(v),$$

   parametrized by a set $V(\Gamma)$ called the *set of vertices* of Γ. For a vertex v, the set $H(v)$ is called the *set of half-edges attached to* v. The valency of a vertex v is defined to be the cardinality of $H(v)$.
3) $\tau : H(\Gamma) \to H(\Gamma)$ is an involution. The orbits of cardinality two are called the *(internal) edges* and the set of edges is denoted by $E(\Gamma)$. The orbits of cardinality one are called *hairs* and the set of hairs is denoted by $L(\Gamma)$

If $L(\Gamma) = \emptyset$, then Γ is simply called a *graph*. A graph Γ is called *directed* if each edge $e = (h, \tau(h))$ comes with a choice of an ordering of half-edges.

We denote by $G_{v,e}$ the set of connected directed graphs with no hairs, with v vertices labelled by $\{1, \cdots, v\}$, and e edges labelled by $\{1, \cdots, e\}$. We define an $\mathbb{S}_v$ module

$$\mathbf{Gra}_d(v) := \begin{cases} \bigoplus_{e \geq 0} \mathbb{K}\langle G_{v,e} \rangle \otimes_{\mathbb{S}_e \ltimes (\mathbb{S}_2)^e} \mathrm{sgn}_e[e(d-1)] & \text{if } d \text{ is even,} \\ \bigoplus_{e \geq 0} \mathbb{K}\langle G_{v,e} \rangle \otimes_{\mathbb{S}_e \ltimes (\mathbb{S}_2)^e} \mathrm{sgn}_2^{\otimes e}[e(d-1)] & \text{if } d \text{ is odd} \end{cases}$$

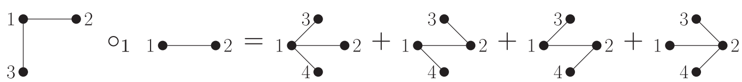
where d is an integer, and thus an operad $\mathbf{Gra}_d = \{\mathbf{Gra}_d(v)\}_{v\geq 1}$ called the operad of graphs.

If d is even, elements of $\mathbf{Gra}_d$ can be seen as undirected graphs with edges having degree $1-d$ together with an ordering of the edges up to an even permutation, while an odd permutation acts by multiplication by -1.

If d is odd, $\mathbf{Gra}_d$ consists of directed graphs where changing the direction of an edge yields a multiplication by -1.

The operadic composition $\Gamma_1 \circ_v \Gamma_2$ works by substituting the graph Γ_2 in the vertex v of Γ_1 and by summing over all possibilities of attaching the edges of v to the vertices of Γ_2.

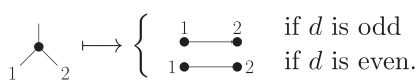
Example 5.2.



There is a morphism of operads [21]

$$\mathbf{Lie}_d \longrightarrow \mathbf{Gra}_d$$

by mapping

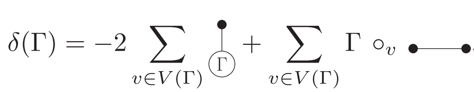


Thus, we can define the *full graph complex* as the deformation complex

$$\mathbf{fcGC}_d := \mathbf{Def}(\mathbf{Lie}_d \longrightarrow \mathbf{Gra}_d).$$

Elements of $\mathbf{fcGC}_d$ can be interpreted as connected graphs together with an ordering of the edges, up to an even permutation when d is even. For d odd, we have an ordering of the vertices, up to even permutations, together with a choice on the orientation of each edge, up to a flip yielding a multiplication by -1.

Its differential can be represented by



where the first term is given by summing over all attachments of a univalent vertex to a vertex of Γ.

We consider the subcomplex $\mathbf{GC}_d \subset \mathbf{fcGC}_d$ spanned by connected graphs with vertices having valency at least 3. Let $(\mathbf{GC}_d^2 := \bigoplus_{\substack{p \geq 1 \\ p \equiv 2d+1 \mod 4}} \mathbb{K}[d-p], 0)$. It has been shown [21] that the natural projection $\mathbf{fcGC}_d \to \mathbf{GC}_d^{\geq 2} := \mathbf{GC}_d \oplus \mathbf{GC}_d^2$ is a quasi-isomorphism, that is, $H(\mathbf{GC}_d^{\geq 2}) = H(\mathbf{fcGC}_d)$.

We recall the definition of the Grothendieck–Teichmüller group [21].

Consider $\mathbb{F}_2 = \mathbb{K}\langle\langle X, Y\rangle\rangle$ the completed free associative algebra generated by X, Y. Define a coproduct Δ by setting $\Delta X = X \otimes 1 + 1 \otimes X$ and $\Delta Y = Y \otimes 1 + 1 \otimes Y$. An element $\alpha \in \mathbb{F}_2$ is called *group-like* if $\Delta\alpha = \alpha \otimes \alpha$.

We also need the Drinfeld–Kohno Lie algebra t_n defined by generators $t_{ij} = t_{ji}$ for $1 \leq i, j \leq n, i \neq j$, which satisfy $[t_{ij}, t_{ik} + t_{kj}] = 0$ and $[t_{ij}, t_{kl}] = 0$ for any distinct i, j, k, l.

The Grothendieck–Teichmüller group $\mathbf{GRT}_1$ is defined to be set of group-like elements $\alpha \in \mathbb{F}_2$, which satisfy the following relations:

$$\begin{aligned} \alpha(t_{12}, t_{23} + t_{24})\alpha(t_{13} + t_{23}, t_{34}) &= \alpha(t_{23}, t_{34})\alpha(t_{13} + t_{13}, t_{24} + t_{34})\alpha(t_{12}, t_{23}) \\ 1 &= \alpha(t_{13}, t_{12})\alpha(t_{13}, t_{23})^{-1}\alpha(t_{12}, t_{23}) \\ \alpha(x, y) &= \alpha(y, x)^{-1}. \end{aligned}$$

The group structure is given by

$$\alpha_1(X, Y) \cdot \alpha_2(Y, X) = \alpha_1(X, Y)\alpha_2(X, \alpha_1(X, Y)^{-1} Y \alpha_1(X, Y)).$$

The Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$ is given by the elements $\alpha \in \hat{\mathbb{F}}_2(X, Y)$, the completed free Lie algebra generated by X, Y, satisfying

$$\begin{aligned} \alpha(t_{12}, t_{23} + t_{24}) + \alpha(t_{13} + t_{23}, t_{34}) &= \alpha(t_{23}, t_{24}) + \alpha(t_{12} + t_{13}, t_{24} + t_{34}) + \alpha(t_{12}, t_{23}) \\ \alpha(X, Y) + \alpha(Y, -X - Y) + \alpha(-X - Y, X) &= 0 \\ \alpha(X, Y) &= -\alpha(Y, X). \end{aligned}$$

We can now state the following theorem concerning the cohomology of $\mathbf{GC}_d$.

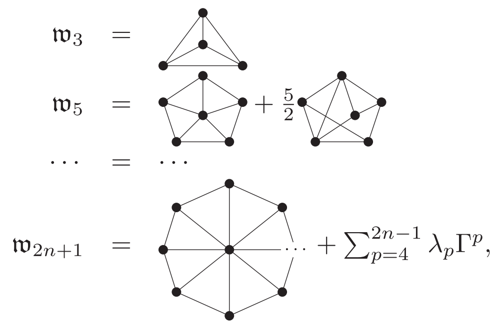
Theorem 5.3. T. Willwacher,[21]

For $d = 2$, one has

$$H^0(\mathbf{GC}_2) = \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ denotes the Grothendieck–Teichmüller Lie algebra (see below).

It has been shown [18] that $H(\mathbf{GC}_2)$ contains the so-called *wheel classes* $\mathfrak{w}_{2n+1}$ given by



where Γ^p is a linear combination of graphs with exactly one vertex of valency p and all other vertices of valency strictly less than p and λ_p is some coefficient.

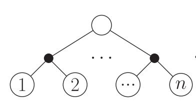
6 Kontsevich Graph Complex From the Operad of Lie Algebras

There is a relation between the operad of graphs and the operad $\mathcal{O}(\mathbf{Lie}_d)$.

Lemma 6.1. Let I be the ideal in the operad $\mathcal{O}_c(\mathbf{Lie}_d)$ generated by graphs having at least one internal edge. Then the quotient operad can be identified with the operad $\mathbf{Gra}_d$, that is,

$$\mathcal{O}_c(\mathbf{Lie}_d)/I \cong \mathbf{Gra}_d.$$

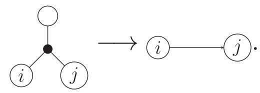
Proof. The operad $\mathcal{O}(\mathbf{Lie}_d)/I$ is generated by elements Γ of the form



Consider the linear map

$$\begin{array}{rccc} \alpha: & \mathcal{O}_c(\mathbf{Lie}_d)/I & \longrightarrow & \mathbf{Gra}_d \\ & \Gamma & \longmapsto & \alpha(\Gamma), \end{array}$$

where $\alpha(\Gamma)$ is a graph in $\mathbf{Gra}_d$, which by definition has labelled white vertices identical to the white vertices of Γ, while the edges between vertices in $\alpha(\Gamma)$ correspond to irreducible components in Γ. If d is even, the ordering of edges attached to the white output vertex of Γ induces an ordering of the edges in $\alpha(\Gamma)$. For d odd, an orientation of the edges in $\alpha(\Gamma)$ is given by the ordering of the output edges attached to the white input vertices,



Conversely, given a generator $g \in \mathbf{Gra}_d$, we can construct a generator $\Gamma \in \mathcal{O}_c(\mathbf{Lie}_d)$ with white input vertices corresponding the vertices of g and each edge corresponds to one irreducible component between the two output vertices. It follows that α defines a bijection on generators and it remains to show that it respects operadic compositions. By its very definition as qoutient operad, for generators $\Gamma_1, \Gamma_2 \in \mathcal{O}_c(\mathbf{Lie}_d)$, the operadic composition $\Gamma_1 \circ_i \Gamma_2$ is given by erasing input vertex i of Γ_1 and summing over all attachments of the hanging edges to the input vertices of Γ_2. This corresponds by definition of $\mathbf{Gra}_d$ to the operadic composition $\alpha(\Gamma_1) \circ_i \alpha(\Gamma_2)$ and the result follows. ■

This map α induces a map $\mathcal{O}_c(\mathbf{Lie}_d) \to \mathcal{O}_c(\mathbf{Lie}_d)/I \to \mathbf{Gra}_d$ and thus a map of the deformation complexes

$$F : \mathbf{Def}(\mathbf{Lie}_d \to \mathcal{O}_c(\mathbf{Lie}_d)) \to \mathbf{fcGC}_d.$$

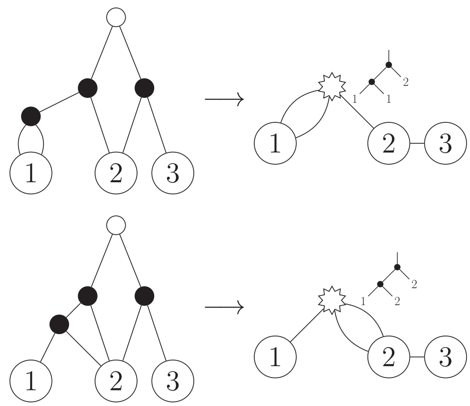
We can now state the main theorem:

Theorem 6.2. The map F is a quasi-isomorphism.

Proof. Recall that the natural projection $\mathbf{fcGC}_d \to \mathbf{GC}_d^{\geq 2} = \mathbf{GC}_d^2 \oplus \mathbf{GC}_d$ is a quasi-isomorphism and thus it is enough to show that the map $C := \mathbf{Def}(\mathbf{Lie}_d \to \mathcal{O}_c(\mathbf{Lie}_d)) \to \mathbf{GC}_d^{\geq 2}$ is a quasi-isomorphism.

We notice that the generators of $\mathcal{O}_c(\mathbf{Lie}_d)$ can be suitably represented as connected graphs having two types of vertices, the white vertices corresponding to the input vertices and the star vertices corresponding to the irreducible connected components. This graph is obtained by first erasing the output vertex and all attached edges. Then we contract the internal vertices of each irreducible component to a single vertex, denoted , which is decorated by the i.i.c. it originates. Last, the star vertices corresponding to  is replaced by an edge between the associated white vertices. ■

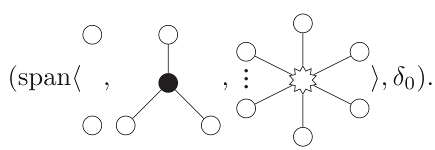
Example 6.3.

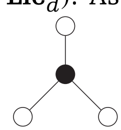
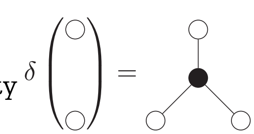


An edge between a white vertex and a star vertex is called *star edge*. Note that by construction, there are no edges between star vertices.

The complex C splits as a direct sum $C = (C^{\leq 1}, \delta) \oplus (C^{\geq 2}, \delta)$ where $C^{\leq 1}$ is generated by graphs with all white vertices having valency less or equal to 1 and $C^{\geq 2}$ is generated by graphs with at least one white vertex of valency at least 2.

The complex $C^{\leq 1}$ is acyclic. Indeed, this complex is equal to



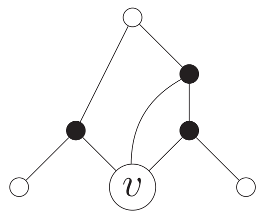
As the differential δ cannot create univalent white vertices we see that the complex where we omit the first graph is isomorphic to $\mathbf{Def}(\mathbf{Lie}_d \to \mathbf{Lie}_d)$. As this complex is one-dimensional with unique cohomology class given by , the equality  and the above isomorphism imply that $C^{\leq 1}$ is acyclic.

Hence, it is enough to prove that the restriction $f : (C^{\geq 2}, \delta) \to (\mathbf{GC}^2_d, 0) \oplus (\mathbf{GC}_d, \delta)$ is a quasi-isomorphism. On the right-hand side, we consider a filtration on the number of vertices. Then on the first page the induced differential vanishes and the second page is equal to $(\mathbf{GC}^2_d, 0) \oplus (\mathbf{GC}_d, \delta)$.

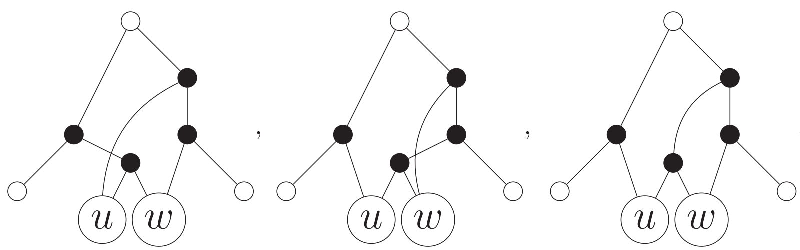
On the left-hand side, we consider a filtration defined by

$$F_{-p} = \text{span of graphs with } \#\{\text{white vertices of valency at least 3}\} + 3\#\{\text{internal edges}\} \geq p.$$

There is exactly one situation where the number of vertices of valency at least 3 is decreasing. Let v be a vertex of valency exactly 3, for example, consider



Then the differential can split v in to two bivalent vertices u and w. In the above example, the differential creates (up to changing u and w) three such terms



Hence, the number of white vertices of valency at least 3 can decrease by 1 but note that an internal edge needs to be created. In this case, the number F_{-p} increases by 1 and hence is respected by the differential.

As the differential does not create new univalent vertices, we see that the differential δ_0 on the first page can only create new white bivalent vertices. As the number of star vertices is preserved, we have a direct sum decomposition $\mathrm{gr}C^{\geq 2} = \bigoplus_{N\geq 0} C_N$ where C_N is the subcomplex spanned by graphs with N star vertices. It has been proven in [21] that $H(C_0, \delta_0) = \mathbf{GC}^2_d \oplus \mathbf{GC}_d$. It remains to show that $(\bigoplus_{N\geq 1} C_N, \delta_0)$ is acyclic.

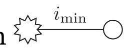
The differential δ_0 can be represented by

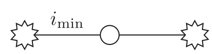
$$\delta_0(\Gamma) = -2 \sum_{v\in V_{\mathrm{in}}(\Gamma)} \overset{\circ}{\underset{\textcircled{\scriptsize Γ}}{|}} + \sum_{v\in V_{\mathrm{in}}(\Gamma)} \Gamma \circ_v \circ\!\!-\!\!\circ$$

where $V_{\mathrm{in}}(\Gamma)$ denotes the set of white vertices of Γ. The first part is given by summing over all attachments of a new univalent white vertex to a white vertex of Γ. The term $\Gamma \circ'_v \circ\!\!-\!\!\circ$ is given by splitting the white vertex v and summing over all attachments of the half-edges of v to the two newly created vertices. As the graphs Γ are connected this differential is equal to

$$\delta_0(\Gamma) = \sum_{\substack{v \in V_{\mathrm{in}}(\Gamma) \\ \text{valency of } v \geq 2}} \Gamma \circ_v \circ\!\!-\!\!\circ,$$

where we omit the summands creating a univalent vertex.

We show that the complex spanned by graphs having at least one star vertex and at least one white vertex with valency at least 2 is acyclic. It is enough to show that the complex spanned by such graphs with a labelling of star edges with integers is acyclic. In particular, the set of star edges is totally ordered. Consider the star edge with minimum label that is is attached to a white vertex of valency at least 2. We call *antenna* the sequence of white vertices starting from  and ending with a white vertex of valency different from 2 or a star vertex. The two-valent white vertices of this sequence are called *antenna vertices*. Consider a filtration by the number of non-antenna vertices. To finish the argument, it is sufficient to observe that the graphs with no antenna vertices and graphs with an antenna given by



are not cocycles and we conclude that $(\bigoplus_{N \geq 1} C_N, \delta_0)$ is acyclic.

On the next page of the spectral sequence, there is a map

$$(H^*(C_0, \delta_0), \delta_1) \longrightarrow (\mathbf{GC}_d^2, 0) \oplus (\mathbf{GC}_d, \delta),$$

where $H^*(C_0\delta_0) \cong \mathbf{GC}_d^2 \oplus \mathbf{GC}_d$ as vector spaces and δ_1 is the differential that increases the value of the parameter of the filtration by 1. Hence, δ_1 cannot create new star vertices (as this gives an increase by at least 2) and thus δ_1 can only split white vertices as the usual differential in $\mathbf{GC}_d$. Finally, this shows that we have an isomorphism of complexes on the second page and the result follows.

As a corollary, we obtain the following:

Corollary 6.4.

$$H^\bullet(\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}_c(\mathbf{Lie}_d))) = \mathbf{GC}_d^2 \oplus H^\bullet(\mathbf{GC}_d)$$

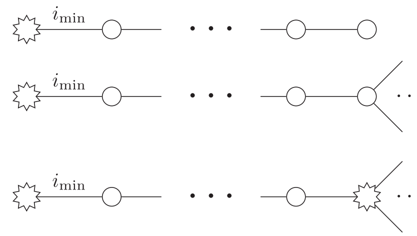


Fig. 1. The different types of antennas.

In particular,

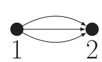
$$H^0(\mathbf{Lie}_2 \to \mathcal{O}(\mathbf{Lie}_2))) = \mathfrak{grt}_1.$$

Another application is the following: we know that $\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}_c(\mathbf{Lie}_d))$ controls the infinitesimal homotopy non-trivial deformations of i, which are given by $H^1(\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}_c(\mathbf{Lie}_d))) \cong H^1(\mathbf{GC}_d^2) \oplus H^1(\mathbf{GC}_d)$, while the obstructions to extending an infinitesimal deformation Δ of i to a genuine morphism of (completed) operads

$$i^{\Delta} : \mathbf{HoLie}_d \to \mathcal{O}_c(\mathbf{Lie}_d)$$

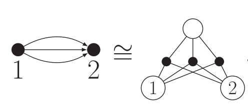
lie in $H^2(\mathbf{GC}_d^2) \oplus H^2(\mathbf{GC}_d)$.

We now consider the case $d = 1$ corresponding to the operad of usual Lie algebras. It is noticed in [22] that $H^1(\mathbf{GC}_1^2) = 0$ while $H^1(\mathbf{GC}_1)$ is one-dimensional and is generated by the theta graph

$\Delta :=$ 

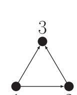
.

Therefore, the standard morphism $i : \mathbf{Lie} \to \mathcal{O}_c(\mathbf{Lie})$ admits precisely one homotopy non-trivial infinitesimal deformation corresponding to the above mentioned theta graph, which in our approach is incarnated as the following element in $\mathcal{O}_c(\mathbf{Lie})$:



.

Moreover, the second cohomology group is generated by



.

This obstruction cohomology class cannot be hit when we try to extend the infinitesimal deformation Δ to a genuine deformation (cf. [22] Section 5). Hence, we conclude that the standard morphism $i : \mathbf{Lie} \to \mathcal{O}_c(\mathbf{Lie})$ admits precisely one (up to homotopy equivalence) non-trivial deformation. Remarkably, this unique non-trivial deformation is directly related to the universal enveloping construction and the PBW theorem as we show in Section 8.

7 Gutt Quantizations

Let V be a vector space equipped with a Lie bracket $[,]$. This datum is equivalent to a morphism of operads

$$\rho : \mathbf{Lie} \longrightarrow \mathbf{End}_V.$$

Using the functor $\mathcal{O}$, we see that there is an associated morphism of operads

$$\hat{\rho} : \mathcal{O}(\mathbf{Lie}) \longrightarrow \mathbf{End}_{\odot V}$$

given by polydifferential operators.

The space $\odot V$ can be canonically given a structure of an associative algebra, with product denoted by $*$, as follows: denote by $\mathcal{U}V$ the universal enveloping algebra of V, that is,

$$\mathcal{U}V := \mathcal{T}V / \langle v_1 \otimes v_2 - v_2 \otimes v_1 - [v_1, v_2] | v_1, v_2 \in V \rangle,$$

where $\mathcal{T}V := \bigoplus_{n\geq 0} \bigotimes^n V$ is the tensor algebra. Then $\mathcal{U}V$ is an associative algebra with product $\odot$ given by $\otimes$ mod I.

The spaces $\odot V$ and $\mathcal{U}V$ are related as vector spaces by the PBW theorem:

Theorem 7.1. As a vector space $\odot V \cong \mathcal{U}V$.

More precisely, consider the linear map [7]

$$\begin{array}{cccc} \sigma : & \odot V & \longrightarrow & \mathcal{U}V \\ & v_1 \odot \cdots \odot v_n & \longmapsto & \sum_{\tau \in \mathbb{S}_n} \frac{1}{n!} v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}. \end{array}$$

By PBW, this is an isomorphism of vector spaces and this allows us to define a product of $p, q \in \odot V$ by

$$p * q := \sigma^{-1}(\sigma(p) \circ \sigma(q)).$$

It has been shown [6, 7] that

$$p * q = p \odot q + \sum_{m,n \geq 1} B_{m,n}(p, q), \tag{4}$$

where $B_{m,n}$ are bi-differential operators of order m on p and order n on q. The product $*$ is called $\mathcal{U}$-star product.

If p and q are polynomials, then the right-hand side is a finite sum. If we try to extend (**??**) to general smooth functions on v, then the sum can be infinite and thus we run into convergence issues. This can be solved by introducing a parameter $\hbar$ and by working with

$$\mathcal{U}_\hbar V := \mathcal{T}V[[\hbar]]\langle v_1 \otimes v_2 - v_2 \otimes v_1 - \hbar[v_1, v_2] | v_1, v_2 \in V\rangle.$$

Example 7.2.

1) If $v_1, v_2 \in V$, then

$$v_1 * v_2 = v_1 \odot v_2 + \frac{\hbar}{2}[v_1, v_2]. \tag{5}$$

2) Let $e^{v_1}, e^{v_2} \in C_V^\infty$. Then

$$e^{v_1} * e^{v_2} = e^{\mathrm{CBH}(v_1, v_2)},$$

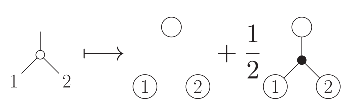
where $\mathrm{CBH}(v_1, v_2)$ denotes the Campbell–Baker–Hausdorff series

$$\mathrm{CBH}(v_1, v_2) = v_1 + v_2 + \frac{\hbar}{2}[v_1, v_2] + \frac{\hbar^2}{12}([v_1, [v_1, v_2]] + [[v_1, v_2], v_2]) + \cdots.$$

We conclude that the $\mathcal{U}$-star product gives us a map $\mathcal{U}_V : \mathbf{Ass} \to \mathbf{End}_{\odot V}$. As this map does not depend on the choice of V and is given by polydifferential operators, we infer that it factors through $\mathcal{O}(\mathbf{Lie})$, that is,

$$\begin{tikzcd} Ass \arrow[r, "\mathcal{U}"] \arrow[dr, dashed, "\mathcal{U}_V"'] & \mathcal{O}(Lie) \arrow[d, "\hat{\rho}"] \\ & End_{\odot V} \end{tikzcd},$$

for some morphism $\mathcal{U}$ of operads satisfying the non-triviality condition



$+$ terms with at least two internal vertices. (6)

This motivates the following definition:

Definition 7.3. Any morphism of operads $\mathcal{U} : \mathbf{Ass} \to \mathcal{O}(\mathbf{Lie})$ satisfying (**??**) is called an *S. Gutt quantization*.

That the standard universal enveloping construction gives us an example of such a quantization was first noticed by Simone Gutt in [6]. We classify below all such quantizations up to homotopy equivalence, and prove that they are all gauge equivalent to the universal enveloping one.

Theorem 7.4. Let U be an S. Gutt quantization. Then the cohomology of $\mathbf{Def}(\mathbf{Ass} \to \mathcal{O}(\mathbf{Lie}))$ is one-dimensional.

Proof. Recall that by its very definition the deformation complex $\mathbf{Def}(\mathbf{Ass} \xrightarrow{U} \mathcal{O}(\mathbf{Lie}))$ is given by

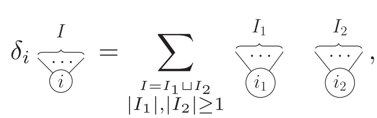
$$\prod_{n\geq 1} \mathbf{Ass}(n)^* \otimes (\mathcal{O}(\mathbf{Lie})(n))^{\mathbb{S}_n}[-1] = \prod_{n\geq 1} (\mathcal{O}(\mathbf{Lie})(n))^{\mathbb{S}_n}[1-n]$$

equipped with the differential d associated to U.

Consider a filtration by the number of internal vertices. As the differential d acts (up to signs) by the composition of an element $\Gamma \in \mathbf{Def}(\mathbf{Ass}\overset{U}{\to}\mathcal{O}\mathbf{Lie})$ with the right-hand side of (**??**) we see that d respects the filtration and we consider the associated spectral sequence $(\mathcal{E}_n, d_n)_{n\geq 0}$. The differential on the first page is only induced by the first term of U. We observe that if Γ is connected, then the surviving terms are also connected and the differential is given by

$$d_0\Gamma = \sum_i \delta_i(\Gamma),$$

where i ranges over the number of white vertices and δ_i acts by



This follows from the fact that the two terms with a disconnected white vertex only differ swapping i_1 and i_2 and thus by the skew-symmetrization of labels of white vertices these terms are opposites. In particular, $\delta_i(\Gamma) = 0$ and $d_0(\Gamma) = 0$ if all white vertices are univalent.

In [13], it has been shown that the cohomology on the first page $H(\mathcal{E}_0, d_0)$ is equal to

$$H(\mathcal{E}_0, d_0) = \mathrm{span}\langle \text{span of graphs with all white vertices univalent and skew-symmetrized, e.g.} \rangle.$$

This complex can be identified with the one in which white vertices are removed and where the outputs are symmetrized and the inputs are skew-symmetrized. This space can be identified with the deformation complex of the prop extension of **Lie**, $\mathbf{Def}(\mathbf{Lie} \to \mathbf{Lie})_{\mathrm{prop}}$, which in turn is isomorphic, as proven in [14], to $\odot(\mathbf{Def}(\mathbf{Lie} \to \mathbf{Lie})_{\mathrm{operad}}[-1])[1]$. Since we are working over a field, we see that the cohomology of this space is one-dimensional as the cohomology of $\mathbf{Def}(\mathbf{Lie} \to \mathbf{Lie})_{\mathrm{operad}}$ is concentrated in degree 1. ■

Corollary 7.5. Let U be an S. Gutt quantization. Then the map

$$\mathbf{Def}(\mathbf{Lie} \to \mathbf{Lie}) \longrightarrow \mathbf{Def}(\mathbf{Ass} \xrightarrow{U} \mathcal{O}(\mathbf{Lie}))$$

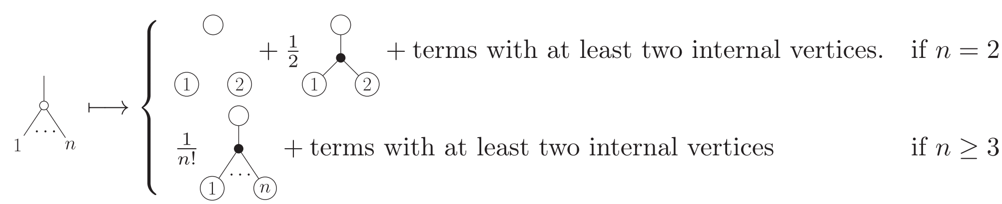
is a quasi-isomorphism.

We will now generalize the above results to $\mathbf{Lie}_\infty$ and $\mathbf{Ass}_\infty$ structures.
Let V be a $\mathbb{Z}$-graded vector space.

Definition 7.6. A homotopy S. Gutt quantization is a morphism of dg operads

$$\varphi : \mathbf{Ass}_\infty \longrightarrow \mathcal{O}(\mathbf{Lie}_\infty)$$

satisfying



Example 7.7.

1) The constructions given by Baranovsky [2], Moreno-Fernández [17], and Lada and Markl [10] can be interpreted as such a map.
2) The Kontsevich formality map [9] implies this formula, but as a map to a wheeled version $\mathcal{O}(\mathbf{Lie}_\infty^{\circlearrowright})$, which is ill-defined for infinite-dimensional Lie algebras. This was resolved by Shoikhet in [19] by showing that wheels can be set to zero and thus we get a map of the desired form.

Corollary 7.8. Let φ be a homotopy S. Gutt quantization. Then the complex $\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\varphi} \mathcal{O}(\mathbf{Lie}_\infty))$ is one-dimensional. In particular, the map

$$\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\varphi} \mathcal{O}(\mathbf{Lie}_\infty)) \cong \mathbf{Def}(\mathbf{Lie}_\infty \to \mathbf{Lie}_\infty)$$

is a quasi-isomorphism.

Proof. Since the canonical map $\pi : \mathbf{HoLie}_1 \to \mathbf{Lie}$ is a quasi-isomorphism and the functor $\mathcal{O}$ is exact, we see that the map $\mathcal{O}(\pi)$ is a quasi-isomorphism. Hence, the deformation complexes $\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\varphi} \mathcal{O}(\mathbf{HoLie}_1))$ and $\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\mathcal{O}(\pi)\varphi} \mathcal{O}(\mathbf{Lie}))$ are quasi-isomorphic. In addition, the map $\mathcal{O}(\pi)\varphi : \mathbf{Ass}_\infty \to \mathcal{O}(\mathbf{Lie})$ factors through $\mathbf{Ass}$, that is,

$$\begin{tikzcd} Ass_\infty \arrow[r, "\pi'"] \arrow[dr, dashed, "\mathcal{O}(\pi)\varphi"'] & Ass \arrow[d, "\mathcal{U}"] \\ & \mathcal{O}(Lie) \end{tikzcd},$$

where $\mathcal{U}$ is a S. Gutt quantization. Thus,

$$H(\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\mathcal{O}(\pi)\varphi} \mathcal{O}(\mathbf{Lie}))) = H(\mathbf{Def}(\mathbf{Ass} \xrightarrow{\mathcal{U}} \mathcal{O}(\mathbf{Lie}))).$$

By Theorem 7.4, the cohomology $H(\mathbf{Def}(\mathbf{Ass} \xrightarrow{\mathcal{U}} \mathcal{O}(\mathbf{Lie})))$ is one-dimensional and the result follows. ■

This result fully agrees with the PBW theorem obtained in [8] using the bar-cobar duality in the homotopy theory of operads.

8 On the Unique Non-Trivial Deformation of the Map i

Let $\mathfrak{g}$ be a Lie algebra over some field $\mathbb{K}$ with a countable basis $\{t_i\}_{i\in I}$. Then $\odot\mathfrak{g}$ can be identified with the polynomial ring $\mathbb{K}[t_I]$. The Gutt quantization formula or the PBW quantization formula applied to polynomials $P(t)$ and $Q(t)$ can be given in terms of differential operators as follows (see Theorem 5 in [3]):

$$P(t) * Q(t) = \exp(\sum_i t_i m^i(\frac{\partial}{\partial u}\frac{\partial}{\partial v}))\, P(u)Q(v)|_{u=v=t},$$

where m^i is defined as follows: let $X = \sum_i x^i t_i$, $Y = \sum_i y^i t_i$ be arbitrary elements of $\mathfrak{g}$ (we understand the numerical coefficients x^i and y^i as formal parameters). Then define

$$m(X, Y) := \log(e^X e^Y) - X - Y = \sum t_i m^i_{j_1,\cdots,j_p,k_1,\cdots,k_q} x^{j_1}\cdots x^{j_p} y^{k_1}\cdots y^{k_q}$$

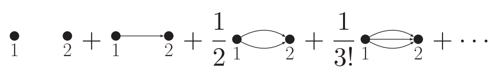
and set $m(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ to be the differential operator obtained from the above power series by replacing each $x^{j_1} \cdots x^{j_p}$ by $\frac{\partial}{\partial u_{j_1}} \cdots \frac{\partial}{\partial u_{j_p}}$ and similarly for $y^{k_1} \cdots y^{k_q}$. It is hard in general to rewrite this formula in terms of our graphs as an explicit morphism of operads

$$\mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Lie}),$$

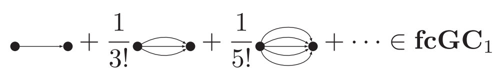
but we only need to see its quotient modulo the graphs $I \subset \mathcal{O}(\mathbf{Lie})$ containing at least one internal edge

$$\mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Lie})/I \cong \mathbf{Gra}_1$$

whose value on the generator of **Ass** is given by the following element in $\mathbf{Gra}_1$:



After skew-symmetrization of the indices 1 and 2, this series simplifies as follows:



As we see the first non-trivial term is precisely our theta class in $H^1(\mathbf{GC}_1)$, we complete the proof that the composition

$$\mathbf{Lie} \longrightarrow \mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Lie})$$

is precisely the unique homotopy non-trivial deformation of the naive map i. Thus, we proved the following theorem:

Theorem 8.1. In the case $d = 1$, the natural map $i : \mathbf{Lie} \to \mathcal{O}(\mathbf{Lie})$ has a unique homotopy non-trivial deformation $J : \mathbf{Lie} \to \mathcal{O}(\mathbf{Lie})$ corresponding to the composition

$$\mathbf{Lie} \longrightarrow \mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Lie}),$$

where the first arrow is the canonical map of operads, and the second arrow is the universal enveloping map.

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