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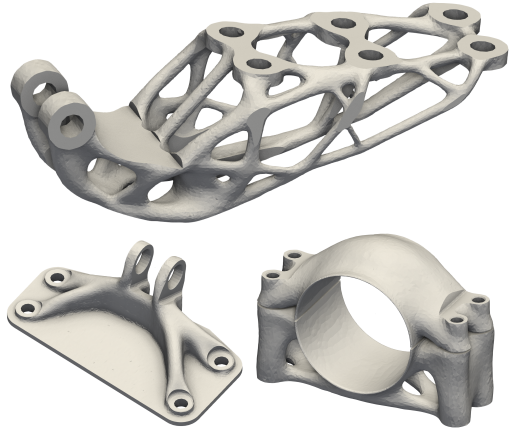
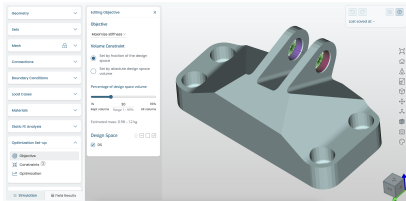
Mesh-independent topology optimisation in FEniCSx

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About Rafinex SARL

- based in Luxembourg, Berlin and London,
- development of cloud-based topology optimization software called Möbius,
- features: stochastic optimization for robust designs, **adaptive anisotropic mesh refinement**, manufacturing constraints, stress constraint, web based GUI.



Topology Optimization for minimum compliance

Find $u \in U(\Omega)$, $\varphi \in V(\Omega)$, s.t.

$$\begin{aligned}\min C(u) &= \min \int_{\Omega} f \cdot u \, ds, \\ \varphi &\in [\delta, 1], \\ \int_{\Omega} \varphi \, dx - \text{Vol} &= 0, \\ \int_{\Omega} \varepsilon(u) : E(\varphi) \varepsilon(v) \, dx &= \int_{\Gamma} f \cdot v \, ds, \quad \forall v \in U(\Omega)\end{aligned}$$

i.e., find critical points of the Lagrange functional $\mathcal{L}$,

$$\mathcal{L}(\varphi, u, v, \lambda) = C(u) - \left(\int_{\Omega} \varepsilon(u) : E(\varphi) \varepsilon(v) \, dx - \int_{\Gamma} f \cdot v \, ds \right) + \lambda \left(\int_{\Omega} \varphi \, dx - \text{Vol} \right),$$

+ bounds constraint. SIMP penalization $E(\varphi) = \varphi^p E_0$.

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- local or global slope (gradient) constraints, which add a bound on the spatial gradient of the density,
- phasefield or diffused interface formulations, where Landau-Ginzburg functional is considered in order to introduce a finite transition layer,
- $W^{1,p}$ and Tikhonov regularization methods where an L^p norm of spatial gradient is added as additional objective term,
- density filtering which formulates the optimization problem in terms of filtered density $\hat{\varphi} = \mathcal{F}(\varphi)$, where $\mathcal{F}$ is a filter operator,

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- density filtering which formulates the optimization problem in terms of filtered density $\hat{\varphi} = \mathcal{F}(\varphi)$, where $\mathcal{F}$ is a filter operator,
- sensitivity filters, where functional gradient is heuristically filtered using a filter operator, $\widehat{\nabla \mathcal{L}} = \mathcal{F}(\nabla \mathcal{L})$.

Sensitivity of the reduced functional (wrt. L^2 metric)

$$\frac{\partial \mathcal{L}_C}{\partial \varphi} = -\varepsilon(u) : \frac{\partial \mathbb{E}}{\partial \varphi} \varepsilon(v)$$

$$\varphi_1 = \varphi_0 - \alpha \frac{\partial \mathcal{L}}{\partial \varphi}$$

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Riesz representation theorem. Let H be a Hilbert space equipped with an inner product denoted $(\cdot, \cdot)_H$ and the induced norm $\|u\|_H := \sqrt{(u, u)_H}$. For every continuous linear functional $f \in H^*$ there exists a unique vector $u_f \in H$, called the *Riesz representation* of f , such that

$$f(x) = \langle x, u_f \rangle_H, \quad \forall x \in H. \quad (1)$$

The map $\mathcal{R}_H : H' \longrightarrow H$ is called the *Riesz map*.

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Gradient Descent in Hilbert spaces

$$\varphi_1 = \varphi_0 - \alpha \mathcal{R}_V(D_\varphi \mathcal{L}) .$$

For the compliance functional there is

$$D_\varphi \mathcal{L}_C[\delta\varphi] = - \int_{\Omega} \varepsilon(u) : \frac{\partial \mathbb{E}}{\partial \varphi} \varepsilon(v) \delta\varphi \, dx$$

and we denote

$$\nabla_\varphi \mathcal{L} = \mathcal{R}_V(D_\varphi \mathcal{L}) .$$

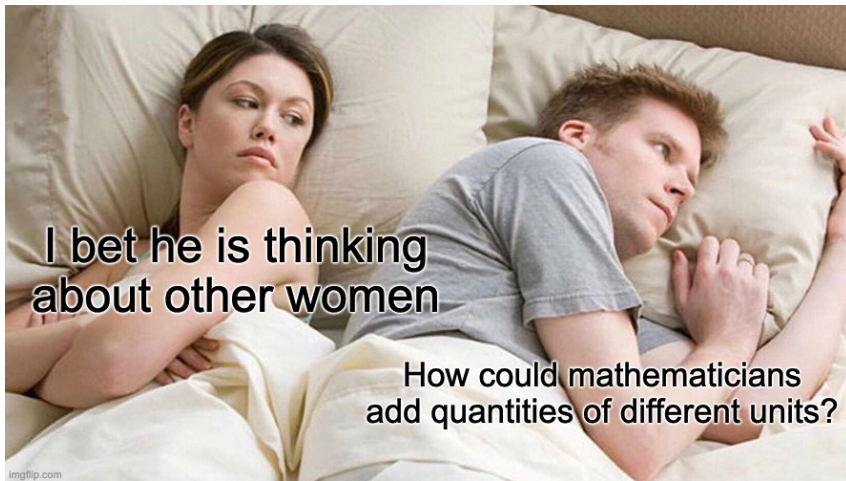
Let us consider $V(\Omega) := H_{\kappa}^1(\Omega)$ with the scalar product

$$\langle u, v \rangle_{H_{\kappa}^1} := \int_{\Omega} uv \, dx + \kappa^2 \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

Then $g \in H_{\kappa}^1(\Omega)$ s.t.

$$\int_{\Omega} g \tau \, dx + \kappa^2 \int_{\Omega} \nabla g \cdot \nabla \tau \, dx = - \int_{\Omega} \varepsilon(u) : \frac{\partial \mathbb{E}}{\partial \varphi} \varepsilon(v) \tau \, dx$$

could be seen as **sensitivity filter** based on positive definite Helmholtz PDE.



Optimality Criteria 1

- Gradient Descent could be slow,
- Sequential Quadratic Programming methods costly,
- special case of SAO (Sequential Approximate Optimization), along with the MMA (Method of Moving Asymptotes),
- for the case of the compliance functional we know that it behaves locally as φ^{-1} ,

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$$\begin{aligned}\mathcal{L} \approx \mathcal{L}_{\text{rec}}(\varphi) &= \mathcal{L}(\varphi_0) + \langle \varphi - \varphi_0, \nabla_{\varphi} \mathcal{L} \rangle_{L^2} + \frac{1}{2} \left\langle \varphi - \varphi_0, \frac{-2\nabla_{\varphi} \mathcal{L}_C}{\varphi} (\varphi - \varphi_0) \right\rangle_{L^2} = \\ &\quad \dots + \frac{1}{2} \langle \varphi - \varphi_0, H(\varphi - \varphi_0) \rangle_{L^2}\end{aligned}$$

where the Hessian approximation $H = \frac{-2\nabla_{\varphi} \mathcal{L}_C}{\varphi}$.

Optimality Criteria 2

Alternatively, we take constant Hessian approximation based on previous iterate

$$H = \frac{-\gamma \nabla_{\varphi} \mathcal{L}_C}{\varphi_0}.$$

which would be satisfied for

$$\varphi_1 = \left(\frac{(1+\gamma)}{\gamma} + \frac{\lambda}{\gamma \nabla_{\varphi} \mathcal{L}_C} \right) \varphi_0.$$

- quadratic approximation

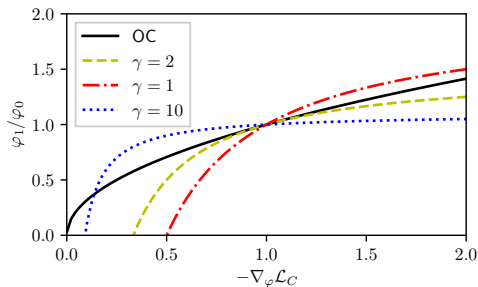


Figure 1: Comparison of the traditional OC method with the OC method based on quadratic approximation.

Optimality Criteria 3

Traditional results on the existence of minimizers for the SIMP scheme work for the following assumptions (not exclusive)

- set of admissible densities is bounded in H^1 ,
- SIMP exponent $p < d/(d - 2)$,
- set of displacement solutions for all admissible densities is bounded in $[H_0^1]^d$.

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Set of admissible densities is bounded in H^1 ?

- add Tikhonov penalty term to the reciprocal approximation

$$\mathcal{L}_{\text{tikh}} = \mathcal{L}_{\text{rec}} + \frac{\alpha}{42} \langle \nabla \varphi, \nabla \varphi \rangle_{L^2}$$

- explicit expression for $\varphi_1 = (\dots) \varphi_0$ does not exist, since Optimality Criteria step is now a PDE.

Algorithm summary

Start $\varphi_0 = 0.5$;

while $\|\varphi_{k+1} - \varphi_k\|_{H^1_\kappa} > 10^{-2}$ **do**

Find u_k , solve state equation

$$\mathbf{K}(\varphi_k)u_k = f;$$

$$v_k \leftarrow u_k;$$

Find g_k , solve Riesz representation

$$\mathbf{H}g_k = D_\varphi \mathcal{L}_C;$$

Estimate $\lambda_{\min}, \lambda_{\max}$;

while volume constraint $> 10^{-8}$ **do**

Find φ_{k+1}^* , solve Optimality Criteria;

Project into (box) bounds constraint,

$$\varphi_{k+1} \leftarrow \mathcal{P}(\varphi_{k+1}^*);$$

end

$$k \leftarrow k + 1, \varphi_k \leftarrow \varphi_{k+1}$$

end

Toolchain:

- **gmsh**, triangular/tetrahedral mesh generation,
- **FEniCSx**, FEM assembly and MPI mesh parallelism,
- **PETSc**, matrix solvers,
 - **GMRES+GAMG**,
 - **CG+Jacobi**,
- **ParaView**, visualization.

Examples: Cantilever

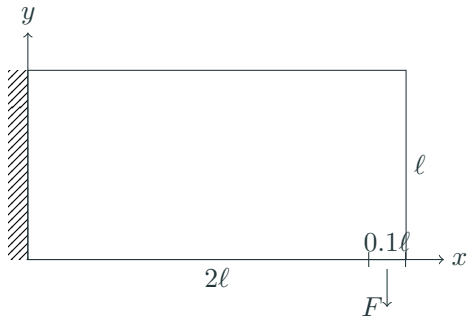
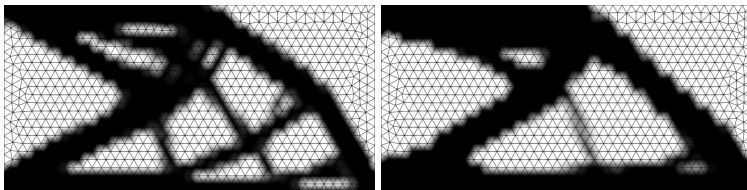


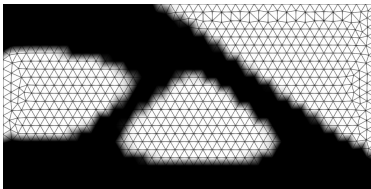
Figure 2: Cantilever beam setup.

Example: Cantilever



(a) $\kappa = 0$.

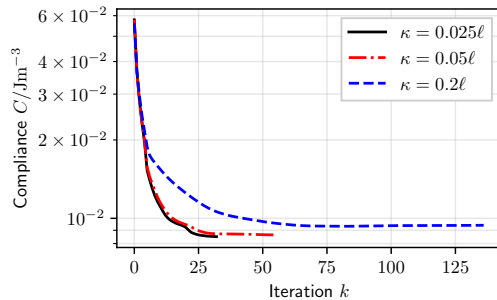
(b) $\kappa = 0.05\ell$.



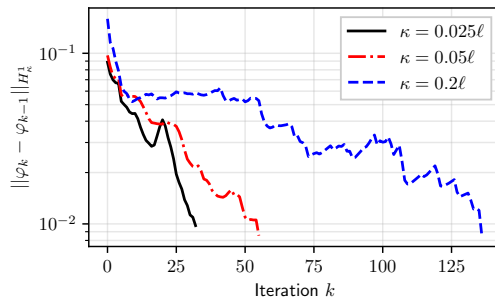
(c) $\kappa = 0.2\ell$.

Figure 3: The effect of Riesz scale parameter κ on the optimal design in the cantilever beam example, $\alpha = 0$.

Example: Cantilever



(a) Compliance.



(b) H_κ^1 norm of the difference between consecutive iterations.

Figure 4: Convergence of the OC method for the cantilever beam problem.

Example: Cantilever

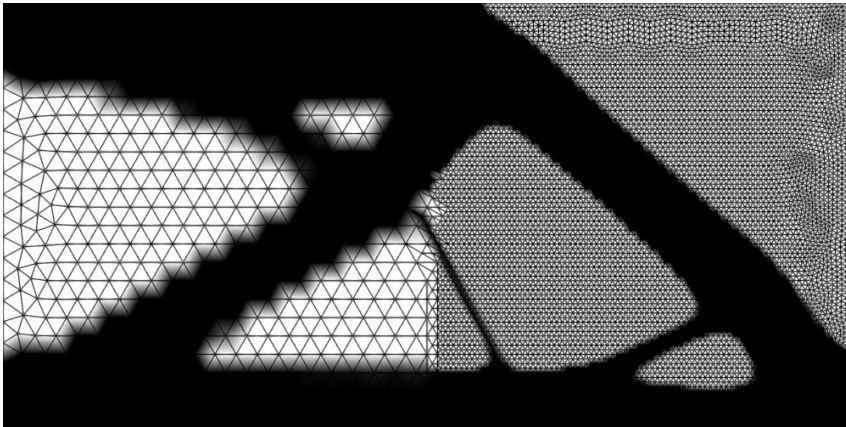
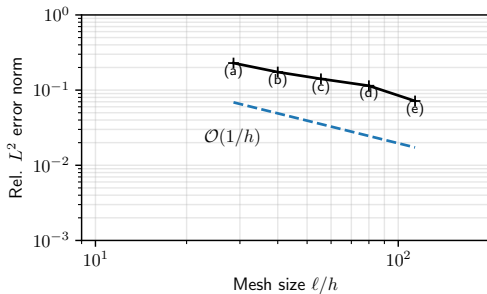


Figure 5: Mesh-independent optimal cantilever beam topology for locally refined mesh ($x > 0.5\ell$), $\kappa = 0.05\ell$ and $\alpha = 0$.

Example: Cantilever



(a) $h = 0.035\ell$, 48 its.



(b) $h = 0.025\ell$, 80 its.



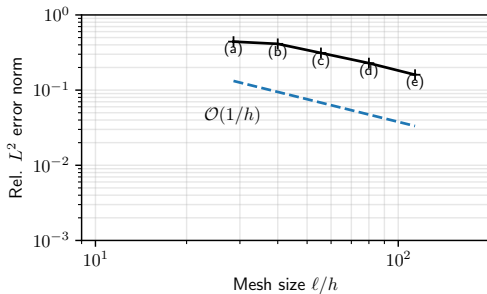
(c) $h = 0.018\ell$, 88 its.



(d) $h = 0.0125\ell$, 96 its.

Figure 6: Convergence of the cantilever optimal solution under uniform mesh refinement for a fixed $\kappa = 0.05\ell$, $\alpha = 0$.

Example: Cantilever



(a) $h = 0.035\ell$, 37 its.



(b) $h = 0.025\ell$, 49 its.



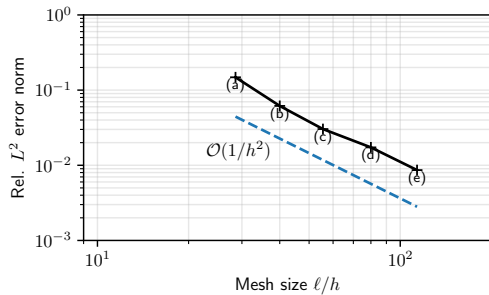
(c) $h = 0.018\ell$, 49 its.



(d) $h = 0.0125\ell$, 56 its.

Figure 7: Convergence of the cantilever optimal solution under uniform mesh refinement for $\kappa = h$, $\alpha = 0$.

Example: Cantilever



(a) $h = 0.035\ell$, 30 its.



(b) $h = 0.025\ell$, 31 its.



(c) $h = 0.018\ell$, 32 its.



(d) $h = 0.0125\ell$, 32 its.

Figure 8: Convergence of the cantilever optimal solution under uniform mesh refinement for $\kappa = 0.02\ell$, $\alpha = 0.042\ell$.

Example: L-shape

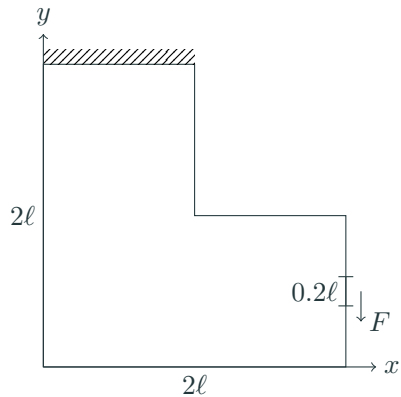


Figure 9: L-shape beam setup.

Example: L-shape

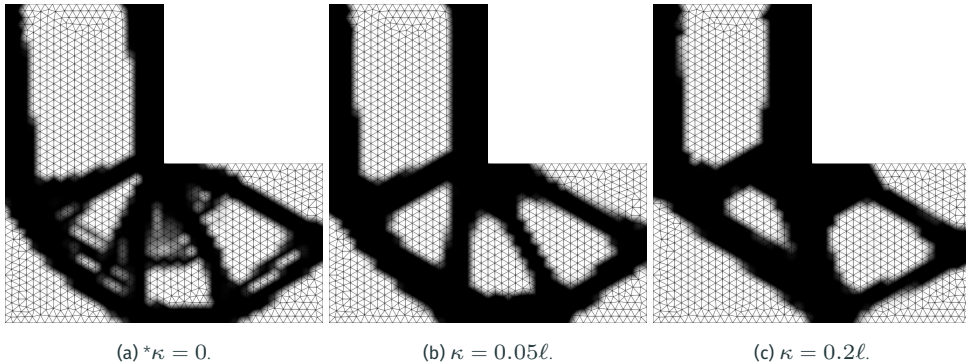
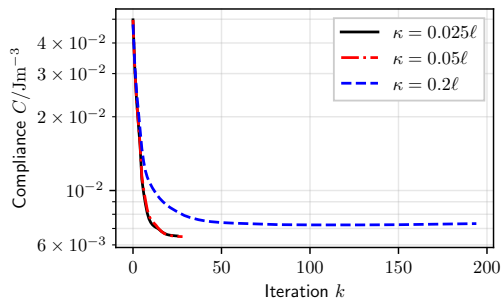
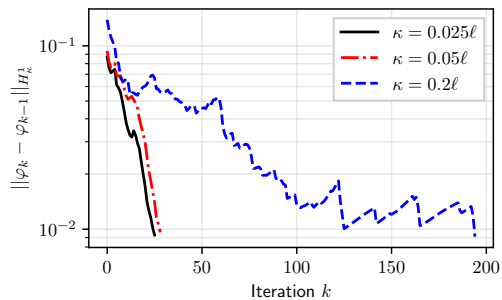


Figure 10: The effect of Riesz scale parameter κ on the optimal design in the L-shape beam example, $\alpha = 0$.

Example: L-shape



(a) Compliance.



(b) H_κ^1 norm of the difference between consecutive iterations.

Figure 11: Convergence of the OC method for the L-shape beam problem.

Example: L-shape

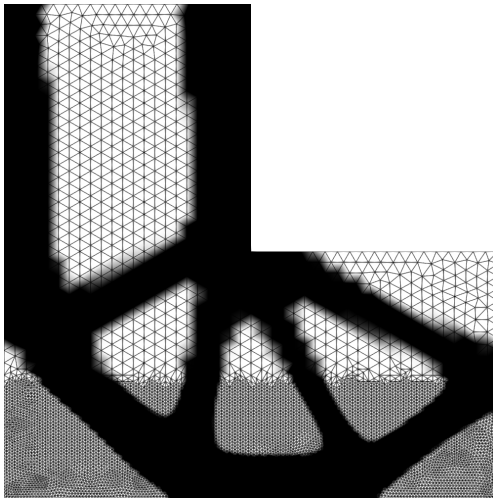
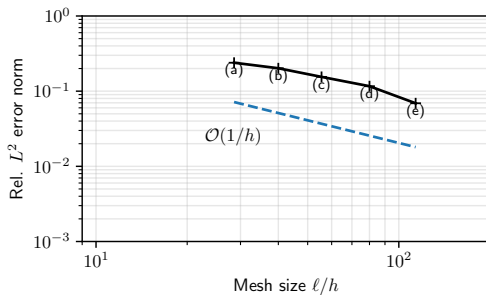


Figure 12: Mesh-independent optimal L-shaped beam topology for locally refined mesh ($y < 0.25\ell$), $\kappa = 0.05\ell$, $\alpha = 0$.

Example: L-shape



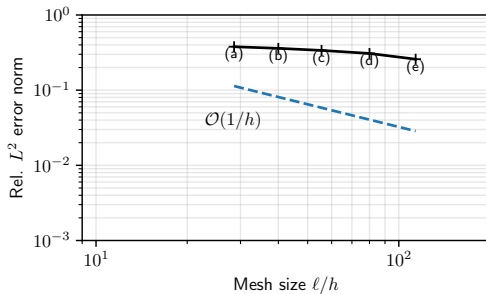
(a) $h = 0.035\ell$, 51 its. (b) $h = 0.025\ell$, 63 its.



(c) $h = 0.018\ell$, 93 its. (d) $h = 0.0125\ell$, 114 its.

Figure 13: Convergence of the L-shape optimal solution under uniform mesh refinement for a fixed $\kappa = 0.05\ell$, $\alpha = 0$.

Example: L-shape



(a) $h = 0.035\ell$, 37 its. (b) $h = 0.025\ell$, 40 its.



(c) $h = 0.018\ell$, 35 its. (d) $h = 0.0125\ell$, 36 its.

Figure 14: Convergence of the L-shape optimal solution under uniform mesh refinement for $\kappa = h$, $\alpha = 0$.

Example: L-shape

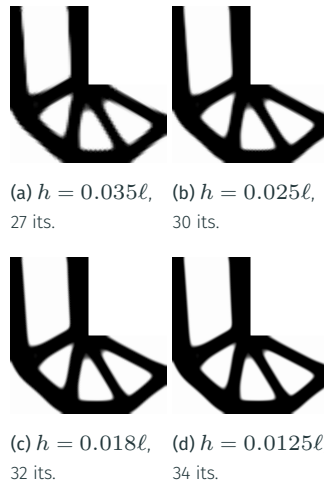
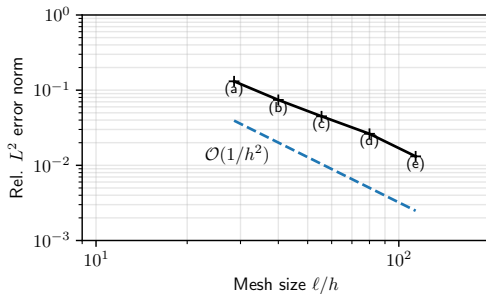
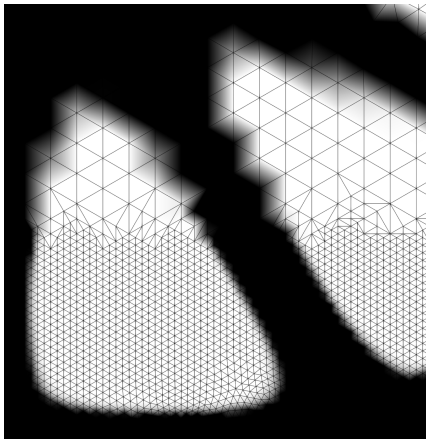
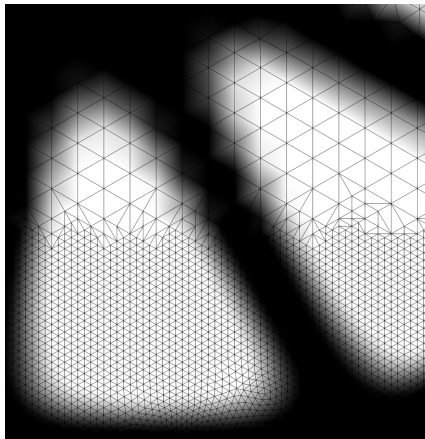


Figure 15: Convergence of the cantilever optimal solution under uniform mesh refinement for $\kappa = 0.02\ell$, $\alpha = 0.042\ell$.

Example: L-shape



(a) $\alpha = 0$.



(b) $\alpha = 0.063\ell$.

Figure 16: Independent diffused interface thickness and “feature size” control.

- constraints (castability, CNC machining, build-plate orientation and overhang angle, minimum/maximum member size, symmetry, maximum stress, ...),
- computational domain boundary effects,
- scalability and robustness of the elasticity state solver,
- re-CAD (implicit surface into CAD),
- parameter robustness (regularization terms, penalties, ...).



- density-based compliance TO formulated in H_{κ}^1 ,
- Helmholtz sensitivity filter interpreted as Riesz map,
- efficient OC-like reciprocal Hessian approximation,
- 2D mesh convergence study.