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LIMIT THEOREMS FOR HIGH-FREQUENCY DATA: FROM SEMIMARTINGALES TO AMBIT FIELDS

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Limit theorems for high-frequency data: from semimartingales to ambit fields

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Abstract

Understanding the asymptotic behavior of statistics based on high frequency observations has gained considerable attention in the recent years, notably due to the ever-growing availability of datas. We think of the numerous applications in economics and econometrics, among others. This thesis explores the limit theory of realized quadratic variation and related functionals for two important classes of processes, namely semimartingales and ambit fields.

Chapter I is an introductory chapter to the main mathematical concepts encountered in the thesis. We preface the classes of semimartingales and of ambit fields and introduce the commom problematic to all the following chapters, that is the establishment of the asymptotic theory for functionals of increments of the processes of interest. We finally present the different methodologies to answer this problematic in the various chapters of the sequel.

Chapter II contains the paper [44]: "Limit theorems for general functionals of Brownian local times", in collaboration with Simon Campese and Mark Podolskij. *Electronic Journal of Probability,* 29:1–18, 2024. We prove a stable central limit theorem for a class of integrated functionals of increments of the local time of a Brownian motion. This result generalizes a number of prior works in the unified framework of semimartingales' limit theory.

Chapter III contains the preprint: "Limit theorems for asynchronously observed bivariate pure jump semimartingales", in collaboration with Mark Podolskij, 2024. In this chapter we prove a non-central limit theorem for the Hayashi-Yoshida estimator of the quadratic covariation process of an asynchronously observed stable process. This result is one of the first to establish the asymptotic theory for non-synchronous high-frequency statistics of pure jump processes.

Chapter IV contains the preprint: "Limit theorems for two dimensional ambit fields observed along curves", in collaboration with Mikko S. Pakkanen, Mark Podolskij and Bezirgen Veliyev, work in progress. The main result is a stable central limit theorem for the power variation of increments of a two-dimensional ambit field, observed with high frequency along some curve embedded in the field. This result is a direct extension of the univariate case.

Finally, the appendix contains technical results that complement the various concepts covered in the introduction.

Contents

Notations

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denotes respectively the set of positive integers, integers, rational numbers, real numbers and complex numbers.
- $\mathbb{N}_{\geq q}$ denotes the set of positive integers greater or equal than q.
- $\mathbb{R}_+$, $\mathbb{R}^d$ denotes the set of real positive real numbers, resp. the d-fold cartesian product of the set of real numbers.
- $\mathcal{B}(\mathbb{K})$, $\mathcal{B}_b(\mathbb{K})$ denotes the Borel σ-algebra containing all borel sets of $\mathbb{K}$, resp. all bounded borel sets of $\mathbb{K}$.
- $x^\intercal$ denotes the transpose of x.
- If $x, y \in \mathbb{R}^d$, then $x \odot y = xy^\intercal + yx^\intercal$ denotes the symmetric tensor product of x and y.
- $x \vee y$ and $x \wedge y$ denote the maximum of x and y, resp. the minimum of x and y.
- $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$.
- *a.s.* stands for *almost surely.*
- $\overset{d}{=}$ denotes equality in distribution.
- $\xrightarrow{u.c.}$ denotes the uniform convergence on compact sets.
- $\xrightarrow{d}$, $\xrightarrow{d_{st}}$, $\xrightarrow{\mathbb{P}}$, $\xrightarrow{u.c.p.}$, $\xrightarrow{L^p}$ and $\xrightarrow{a.s.}$ denote respectively convergence in distribution (or in law), stable convergence, convergence in probability, uniform convergence in probability on compact sets, convergence in L^p and convergence almost sure.

Chapter I

Introduction

For nearly three decades the field of high-frequency statistics for stochastic processes has grown in popularity and expanded quickly, partially due to the numerous applications. We can notably think of the countless applications in economics and financial econometrics.

In this thesis we focus on deriving estimators for the quadratic and power variations for different classes of processes, and then establish in a loose sense generalized central and non central limit theorem for the proposed estimator. The quadratic variation is a stochastic quantity that measures broadly speaking the noisiness of a typical trajectory of a stochastic process.

In the high-frequency framework, we investigate first the asymptotic theory for quadratic variation and other functionals of increments of Itô semimartingales, in the Gaussian and pure jump case. Second, we establish the limit theory for power variations for a class of spatiotemporal ambit process driven by a Gaussian white noise.

In this chapter we introduce the high-frequency framework, shared in all chapters in the sequel, and the main mathematical objects and tools of the dissertation. We provide a brief summary of past findings and techniques linked to the results presented in this thesis. More specifically, in Section I.1 we define the high-frequency framework, the cornerstone of this dissertation. In Section I.2 we introduce the main mathematical objects of the thesis. In Section I.3 we provide the methods used in the different chapter of this dissertation: for Chapter II in Section I.3.1, for Chapter III in Section I.3.2 and for Chapter IV in Section I.3.3. We present our results in Section I.4.

I.1 High-frequency framework

In this section we introduce the high-frequency framework for statistics of stochastic processes. This asymptotic regime will be commom to Chapter III and Chapter IV and "hidden" in the statistic of interest of Chapter II.

Let $X = (X_t)_{t \geq 0}$ be a stochastic process and assume that we observe one path $X_t(\omega)$

discretely over a finite and fixed time interval $[0,T]$. More precisely, we assume that we are given discrete observations $X_{t_i^n}$, $1 \leq i \leq n$ over a time grid $(t_i^n)_{1\leq i\leq n}$ with $0 \leq t_i^n \leq T$ for all $1 \leq i \leq n$. We consider then a statistic of the form

$$S^n((X_{t_i^n})_{1\leq i\leq n}) := S^n(X),$$

using our discrete observations. The goal is to establish the asymptotic behavior of the statistic $S^n(X)$ when n goes to infinity, or equivalenty when the mesh of the partition $(t_i^n)_{1\leq i\leq n}$ tends to zero. In the sequel we call this partition the **observation scheme** or **sampling scheme**. From this limiting behavior of the statistic, we can deduce features of the continuous-time process $(X_t)_{t\geq 0}$.

This asymptotic framework is very useful in pratice, especially for financial applications where quantities of interest, e.g. transactions, price movements, are observed with a high frequency (e.g. every hours, minutes or seconds) or even ultra-high frequency (when all possible transactions on some market are being recorded).

Within this framework we distinguish different observation schemes:

- If for all i, $t_i^n := i\Delta_n$ with Δ_n a deterministic sequence of real numbers going to zero, we say that the sampling scheme is **regularly spaced** or that the datas are **regularly spaced**.
- Suppose that we observe discretely d processes X^j, $1 \leq j \leq d$, over a sampling scheme $(t_i^n(j))_{1\leq i\leq n, 1\leq j\leq d}$.
  - If $t_i^n(j) = t_i^n$ for all j, then the sampling scheme is called **synchronous**.
  - If there exists j,k such that $t_i^n(j) \neq t_i^n(k)$, then the sampling scheme is called **asynchronous**.

Chapter IV uses synchronous and regularly spaced datas for the statistic whereas Chapter III uses asynchronous and irregularly spaced datas.

I.2 Mathematical objects

In this section we introduce the two main objects of this thesis. In Section I.2.1 we introduce the class of semimartingales, with a particular attention to the subclass of Lévy processes. Section I.2.2 presents a class of spatiotemporal ambit field, driven by a Gaussian white noise. Before delving into the properties of specific classes of processes, we give two important definitions for the sequel.

Throughout this section, all processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e. a family of sub σ-fields of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$ (the family is said to be **increasing**) and we assume that $\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$ (the family is said to be **right-continuous**). We call the quadruplet $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ a **filtered probability space** or a **stochastic basis.**

Let $(X_t)_{t\geq 0}$ be a stochastic process. The quadratic variation process can be defined as the limit in probability of the sum of squared increments of X over a sequence of partitions with mesh going to zero, if it exists. Formally:

Definition I.2.1. Let $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ be two real-valued stochastic processes. We define the **quadratic variation** of X over $[0,t]$, resp. the **quadratic covariation** of X and Y over $[0,t]$, as the limit under covergence in probability, if it exists

$$[X]_t = \lim_{|\pi_n|\to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2, \tag{I.2.1}$$

$$[X,Y]_t = \lim_{|\pi_n|\to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}), \tag{I.2.2}$$

where the limit is taken over all partitions π_n of $[0,t]$ with $0 = t_0 < t_1 < ... < t_n = t$ and $|\pi_n| = \max_{1\leq i\leq n} |t_i - t_{i-1}|$.

Informally, the quadratic variation of some process X "accumulates" in some sense the randomness of the process X and measures the variability of its path.

These definitions can be extended directly to the multivariate case. We finish the introduction of this section with a last definition.

Definition I.2.2. Let $(X_t)_{t\geq 0}$ be a càdlàg stochastic process. The process X is called a **finite variation process** if the paths $X(\omega)$ are of finite variation on each compact set of $\mathbb{R}_+$ almost surely.

Remark I.2.3. One can prove that the quadratic variation exists for any càdlàg finite variation process and that the quadratic variation of a continuous finite variation process is identically 0.

Assume that X is a càdlàg pure jump finite variation process. Denote by X_{s-} the left limit of X with respect to s. This limit exists due to the càdlàg assumption on X. Denote by $\Delta_s X := X_s - X_{s-}$ the jump of X at time s. The quadratic variation of X over $[0,t]$ is given by the sum of the squared jumps of X up to time t, i.e.

$$[X]_t = \sum_{0\leq s\leq t} (\Delta_s X)^2 .$$

I.2.1 Semimartingales and Lévy processes

We start with the definition of a semimartingale.

Definition I.2.4. A real-valued stochastic process $Y = (Y_t)_{t\geq 0}$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, is called a **semimartingale** if

(i) Y is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e. for every $t \geq 0$, Y_t is $\mathcal{F}_t$-measurable.

(ii) For all $\omega \in \Omega$, $Y_t(\omega)$ is càdlàg.

(iii) Y can be decomposed as:

$$Y_t = Y_0 + M_t + A_t$$

where Y_0 is finite-valued and $\mathcal{F}_0$-measurable, M is a local martingale and A is a finite variation process.

Definition I.2.5. A $\mathbb{R}^d$-valued stochastic process $Y = (Y^1, ..., Y^d)$ is a semimartingale if Y^i is a $\mathbb{R}$-valued semimartingale, for all $1 \leq i \leq d$.

Remark I.2.6. The decomposition in (iii) is not unique for any semimartingale. In particular, if Y is a continuous semimartingale, then we can decompose uniquely

$$Y_t = M_t + A_t$$

where M is a continuous local martingale and A is a continuous finite variation process (see [138]).

Most of the usual stochastic processes are semimartingales. Such examples include (the list is not exhaustive):

- Càdlàg martingales and local martingales;
- Lévy processes, including Brownian motion and Poisson processes;
- Itô processes;
- Hawkes processes;
- Local time of a Brownian motion.

Apart from including many useful examples of stochastic processes, the class of semimartingales is the largest (in the sense of inclusion) class of stochastic integrators: for any left continuous, locally bounded and adapted process H and any semimartingale Y the stochastic Itô integral of H with respect to Y exists [38, 39]. Such integral is defined as a stochastic extension of the Riemann-Stieljes integral

$$X_t = \int_0^t H_s dY_s = \lim_{n\to\infty} \sum_{t_i, t_{i+1} \in \pi_n} H_{t_i}(Y_{t_{i+1}} - Y_{t_i})$$

where π_n is a sequence of finite random partition of $[0, t]$ with mesh going to zero and the sum converges in probability. This property can be used to define the class of semimartingales. The **Bichteler-Dellacherie** theorem (see [133, Theorem 47]) ensures the equivalence of the two definitions.

For any semimartingale X one can show that the **quadratic variation** of X exists. Similarly, for any two semimartingales X and Y one can show that the **quadratic covariation** of the processes X and Y exists. In the semimartingale framework, we have another definition for the quadratic (co)variation, equivalent to the one given in Definition I.2.1.

Definition I.2.7. Let X, Y be semimartingales. Then the **quadratic variation process** $([X]_t)_{t\geq 0}$ of X is defined by

$$[X] := X^2 - 2\int X_- dX \tag{I.2.3}$$

where $X_{t-}(\omega) := \lim_{s\to t, s<t} X_s(\omega)$ (see I.2.1). Similarly, the **quadratic covariation process** $([X,Y])_{t\geq 0}$ of X and Y is defined by

$$[X,Y] := XY - \int X_- dY - \int Y_- dX. \tag{I.2.4}$$

Note that the map $(X,Y) \mapsto [X,Y]$ is bilinear and symmetric. From the definition we deduce an integration by part formula:

$$XY = \int X_- dY + \int Y_- dX + [X,Y]. \tag{I.2.5}$$

This process is of the utmost interest as it is a key element of the Itô isometry and Itô's lemma (see e.g. [133, Theorem 32]), a lemma useful notably for statistical inference: when we need to evaluate statistics on the process $f(X)$ for X a semimartingale, Itô's Lemma ensures that we can compute the dynamic $df(X_t)$ under mild conditions on the regularity of f.

The definition (I.2.1) is more convenient for statistical purpose. Indeed, assume that we observe a semimartingale Y over a synchronous sampling scheme $(t_i^n)_{1\leq i\leq n}$ on some time interval $[0,t]$ (see Section I.1). Then Definition I.2.1 provides us a way to define a statistic for the quadratic variation (resp. covariation). Let X, Y be real-valued semimartingale. Then by Definition I.2.1

$$V^n(X,2)_t := \sum_i (X_{t_i^n} - X_{t_{i-1}^n})^2 \xrightarrow{\mathbb{P}} [X]_t, \tag{I.2.6}$$

$$V^n(2,X,Y)_t := \sum_i (X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_i^n} - Y_{t_{i-1}^n}) \xrightarrow{\mathbb{P}} [X,Y]_t. \tag{I.2.7}$$

In other words, $V^n(X,2)_t$, resp. $V^n(2,X,Y)_t$, is a **consistent** estimator of the quadratic variation $[X]_t$, resp. the quadratric covariation $[X,Y]_t$. We call this estimator the **realized variance**, resp. the **realized covariance**. We can define the same estimators for two d-dimensional semimartingales $X = (X^1, ..., X^d)$ and $Y = (Y^1, ..., Y^d)$ observed on the sampling scheme defined above:

$$V^n(X,2)_t := \sum_i (X_{t_i^n} - X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n})^\intercal \xrightarrow{\mathbb{P}} [X]_t, \tag{I.2.8}$$

$$V^n(2,X,Y)_t := \sum_i (X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_i^n} - Y_{t_{i-1}^n})^\intercal \xrightarrow{\mathbb{P}} [X,Y]_t, \tag{I.2.9}$$

where $x^\intercal$ is the transpose of the vector x. Estimation of the quadratic covariation and the second order limit associated is the main subject of Chapter III.

Remark I.2.8. Let Y be a real-valued semimartingale. Let $(\pi_n)_n$ be a sequence of randoms partitions over the time interval $[0,t]$ with mesh going to zero. Let $p \geq 0$. We can define another important quantity when studying the behavior of semimartingales, the so called **power variation of order** p or p**-variation**, denoted $[Y]_t^{[p]}$ with the special case $[Y]_t^{[2]} = [Y]_t$. We define it as the limit, if it exists:

$$[Y]_t^{[p]} := \lim_{n\to\infty} \sum_{T_i^n \in \pi_n} |Y_{T_i^n} - Y_{T_{i-1}^n}|^p$$

where the convergence holds in probability. We can correspondingly define a consistent estimator of the power variation, called the **realized power variation** as

$$V^n(Y,p)_t := \sum_i |Y_{t_i^n} - Y_{t_{i-1}^n}|^p.$$

Althought we consider ambit processes instead of semimartingales, Chapter IV is devoted to the estimation of the p-variation. Note that contrary to the quadratic variation, depending on the value of p, this quantity may not be defined for any semimartingale [100].

Coming back to the main properties of a semimartingale, one can characterize entirely the distribution of a semimartingale by a triplet, called **characteristics** of a semimartingale. We observe first the following theorem [92, Theorem 23.14]:

Theorem I.2.9. *Let Y be a semimartingale. Then almost surely X has an unique decomposition*

$$Y = Y_0 + Y^c + Y^d,$$

where Y^c is a continuous local martingale satisfying $Y_0^c = 0$ and Y^d is a purely discontinuous semimartingale. Denote by $[Y]^c$, resp. $[Y]^d$ the continous part, resp. the purely discontinuous part of the quadratic variation $[Y]$. Then, almost surely,

$$[Y^c] = [Y]^c \qquad \text{and} \qquad [Y^d] = [Y]^d.$$

In the decomposition of Theorem I.2.9, Y^d is a purely discontinuous semimartingale, therefore we can decompose Y^d as a sum of a local martingale and a finite variation process.

Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a compactly supported function, bounded and such that $h(x) = x$ in a neighborhood of 0. We can for example consider the function $h(x;\epsilon) = \mathbb{1}_{\{\|x\|\le\epsilon\}}$ for some $\epsilon > 0$ fixed. We use this cutoff function h to further decompose the semimartingale Y, depending on the size of the jumps. We use the notation $\Delta Y_t = Y_t - Y_{t-}$. Define

$$\begin{cases} Y'(h)_t & := \sum_{s\le t} \left(\Delta Y_s - h(\Delta Y_s)\right), \\ Y(h) & := Y - Y'(h). \end{cases} \tag{I.2.10}$$

We observe that $Y'(h)$ is a finite variation process and $Y(h)$ is a semimartingale with canonical decomposition

$$Y(h) := Y_0 + M(h) + B(h),$$

with $M(h)$ a semimartingale and $B(h)$ a predictable finite variation process. To sum up, we have

$$Y = Y_0 + B(h) + M(h) + Y'(h) = Y_0 + B(h) + M(h)^c + M(h)^d + Y'(h)$$

where $M(h)^c$ and $M(h)^d$ denote respectively the continuous and purely discontinuous part of the local martingale $M(h)$. Finally, denote by $\mu(\omega, dt, dx)$ the random jump measure of Y, defined as

$$\mu(\omega,(0,t],A) := \sum_{0<s\le t} \mathbb{1}_A(\Delta Y_s), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Observe that $Y(h)$ has bounded jump, hence it is locally integrable and its compensator $A(h)$ exists, i.e. an increasing process such that $Y(h) - A(h)$ is a local martingale. Denote by ν the compensator of the random jump measure μ. We have the following definition:

Definition I.2.10. Let h be some fixed cutoff function, let Y be a d-dimensional semimartingale. We call **characteristics** of Y (associated with h) the triplet (B, C, ν) where

(i) $B = B(h)$ is a d-dimensional predictable process of locally bounded variation.

(ii) $C = C^{i,j},\ 1 \leq i, j \leq d$ is a $d \times d$-dimensional predictable increasing continuous process defined as

$$C^{i,j} = \langle Y^{i,c}, Y^{j,c} \rangle$$

where Y^c is the continuous martingale part of Y.

(iii) μ is the compensator of the random measure ν associated to the jumps of Y.

Remark I.2.11. If the characteristics of the semimartingale Y are absolutely continuous with respect to the Lebesgue measure, we call Y an **Itô semimartingale**. This assumption on the characteristics is crucial for statistical purpose [7].

The characteristics of a semimartingale are uniquely determined, due to Theorem I.2.9. This unicity is crucial to define a method to prove the weak convergence:

$$Y^n \xrightarrow{d} Y \tag{I.2.11}$$

where $(Y^n)_{n\geq 0}$ is a sequence of semimartingales with characteristics (B^n, C^n, ν^n) and Y is a semimartingale with characteristics (B, C, ν). Indeed, to prove (I.2.11) one only need to show the tightness of the sequence $(Y^n)_{n\geq 0}$ and the convergence of the sequence of characteristics $((B^n, C^n, \nu^n))_{n\geq 0}$ towards (B, C, ν). This method is the main tool used to prove Theorem III.2.3 of Chapter III.

We turn our attention to another class of stochastic processes called **Lévy processes**. We will see in the sequel that Lévy processes is a subclass of semimartingales with deterministic characteristics, ensuring for statistical purpose the identifiability of the characteristics. We begin with the definition of such processes.

Definition I.2.12. Let $Y = (Y_t)_{t\geq 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Y is called a **Lévy process** if it satisfies each of the following properties:

(i) $Y_0 = 0$ almost surely.

(ii) Y has independent increments, i.e. for any $0 \leq t_1 < t_2 < \ldots < t_n < \infty$ the random variables $(Y_{t_{i+1}} - Y_{t_i})_{1\leq i\leq n-1}$ are mutually independent.

(iii) Y has stationary increments, i.e. for any $s < t$, $Y_t - Y_s \stackrel{d}{=} Y_{t-s}$.

(iv) Y is stochastically continuous, i.e. for any $\varepsilon > 0$ and any $t \geq 0$,

$$\lim_{h\to 0} \mathbb{P}\left(|Y_{t+h} - Y_t| > \varepsilon\right) \longrightarrow 0.$$

If Y is a Lévy process, there exists a version $\widetilde{Y}$ of Y with càdlàg paths, meaning that $\mathbb{P}(Y_t = \widetilde{Y}_t) = 1$ for all t and for any $\omega \in \Omega$, $\widetilde{Y}_\cdot(\omega)$ is càdlàg.

Some example of Lévy processes include Brownian motion, (compound) Poisson processes, β-stable processes.

An important property is that the distribution of any Lévy process is **infinitely divisible**: let Y be a Lévy process. Then for any $t \geq 0$ and any $n \in \mathbb{N}$, there exists n i.i.d. random vectors $X_1, ..., X_n$ such that

$$Y_t \overset{d}{=} X_1 + ... + X_n.$$

Due to (ii) and (iii), we can chose $X_i = Y_{it/n} - Y_{(i-1)t/n}$ for any $1 \leq i \leq n$.

The characteristic function of any infinite divisible distribution is given by the **Lévy-Khintchine formula** (see *inter alia* [143, Theorem 8.1] for a proof):

Theorem I.2.13. *Let X be a random vector on $\mathbb{R}^d$ with distribution μ. Assume that μ is infinitely divisible. Then the characteristic function of X is given by the* ***Lévy-Khintchine formula****:*

$$\begin{aligned}\mathbb{E}\left[e^{i\langle u,X\rangle}\right] &= \exp(\psi_X(u)) \qquad \textit{with} \\ \psi_X(u) &= -\frac{1}{2}\langle u, Au\rangle + i\langle \gamma, u\rangle \\ &\quad + \int_{\mathbb{R}^d} \left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{\{\|x\|\leq 1\}}\right) \nu(dx),\end{aligned} \tag{I.2.12}$$

where A is a symmetric nonnegative definite $d \times d$ matrix called the ***Gaussian covariance matrix****, $\gamma \in \mathbb{R}^d$ and ν is a measure on $\mathbb{R}^d$, called the* ***Lévy measure****, satisfying*

$$\nu(\{0\}) = 0 \qquad \textit{and} \qquad \int_{\mathbb{R}^d} (\|x\|^2 \wedge 1)\nu(dx) < \infty.$$

The vector γ appearing in the drift term depends on the choice of the cut-off function. The triplet (γ, A, ν) is called the ***generating triplet*** *of the distribution μ and is unique. Conversely we can associate to any generating triplet (γ, A, ν) a corresponding infinitely divisible distribution.*

Let $X = (X_t)_{t\leq 0}$ be a $\mathbb{R}^d$-valued Lévy process. As we have seen in the prequel, the distribution of X_t is infinitely divisible for any $t \geq 0$. We then have a Lévy-Khintchine formula for Lévy processes:

Theorem I.2.14. *Let X be a $\mathbb{R}^d$-valued Lévy process. Then, for all $t \geq 0$, for all $u \in \mathbb{R}^d$, there exists a triplet (γ, A, ν), called* ***Lévy triplet****, such that*

$$\mathbb{E}\left[e^{i\langle u,X_t\rangle}\right] = \exp(t\psi_X(u))$$

with $\psi_X(u)$ defined in (I.2.12).

Definition I.2.15. Let X be a $\mathbb{R}^d$-valued Lévy process with Lévy triplet (γ, A, ν). If $A = 0$, then X is called a **pure jump** Lévy process.

We recall that the characteristic function of the sum of independent random vectors is the product of the characteristic function of each random vectors. Using this property and the formula (I.2.12) we can prove the **Lévy-Itô decomposition**: every Lévy process can be decomposed as the independent sum of a deterministic drift, a Brownian motion and a jump process. Formally (see [143, Theorems 19.2, 19.3]):

Theorem I.2.16. *Let X be a d-dimensional Lévy process with characteristic exponent define in* (I.2.12). *Then there exists three independent Lévy processes $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ such that*

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)} \tag{I.2.13}$$

where $X^{(1)}$ is a Brownian motion with drift, $X^{(2)}$ is a compound Poisson process with intensity $\nu(\mathbb{R}^d \setminus \{\|x\| \leq 1\})$ and child distribution $\nu \mathbb{1}_{\{\mathbb{R}^d \setminus \{\|x\| \leq 1\}\}} / \nu(\mathbb{R}^d \setminus \{\|x\| \leq 1\})$. Its characteristic exponent is given by

$$\psi_{X^{(2)}}(u) = \int_{\mathbb{R}^d \setminus \{\|x\| \leq 1\}} (e^{i\langle u,x\rangle} - 1)\nu(dx).$$

$X^{(3)}$ is a compensated generalized Poisson process with characteristic exponent given by

$$\psi_{X^{(3)}}(u) = \int_{\{\|x\| \leq 1\} \setminus \{0\}} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\nu(dx).$$

We can show that $X^{(3)}$ is a martingale and $X^{(2)}$ is a finite variation process. Adding the fact that $X^{(1)}$ minus its drift is a martingale, we get that X is a semimartingale by definition. In particular, X is a semimartingale with deterministic characteristics (γ, A, ν). Therefore, for any sequence of Lévy processes X^n with characteristics (γ^n, A^n, ν^n), we can prove the weak convergence

$$X^n \xrightarrow{d} X \tag{I.2.14}$$

by using the methodology presented in (I.2.11): provided that the sequence $(X^n)_{n \geq 0}$ is tight, one can show (I.2.14) by proving the convergence of the characteristics (γ^n, A^n, ν^n) towards the characteristics of X.

In the following we focus on two subclasses of Lévy processes. The sequel is divided into two subsections: in Section I.2.1.1 we introduce the notion of **local time** of a Brownian motion, the main mathematical object of study of Chapter II. In Section I.2.1.2, we define the class of β-stable processes, which is the core subject of Chapter III.

I.2.1.1 Local time of a Brownian motion

The local time of a semimartingale X at a level x is a continuous increasing process that measures the amount of time that the semimartingale spends at the given level. We denote it L_t^x or L^x if $t = 1$.

In this section we focus on the special case of the local time of a Brownian motion, or **Brownian local time**. Following Section I.2,we recall that a real-valued Brownian motion $W = (W_t)_{t\geq 0}$ is a Lévy process (and therefore a semimartingale) with characteristic exponent

$$\psi_W(u) = -\frac{\sigma^2}{2}.$$

It is the only Lévy process with almost surely continuous paths.

We define the Brownian local time via the following theorem, called the **Tanaka formula** (see a proof in e.g. [136, Theorem VI.1.2]):

Theorem I.2.17. *Let $(W_t)_{t\geq 0}$ be a real-valued Brownian motion. For any $x \in \mathbb{R}$ there exists an increasing continuous process (L^x_t), called the* ***local time*** *of W at level x such that*

$$|W_t - x| = |W_0 - x| + \int_0^t \operatorname{sgn}(W_s - x) dW_s + L^x_t \tag{I.2.15}$$

where $\operatorname{sgn}(x) := \begin{cases} x/|x|, & x \neq 0 \\ 0 & \textit{else} \end{cases}$ *is the sign function.*

The proof of this theorem relies essentially on a generalisation of the Itô formula for convex function.

Remark I.2.18. There exists another approach to define the Brownian local time. Let $A \in \mathcal{B}(\mathbb{R})$ be a borel set and define the occupation time of W in the set A up to time t by the measure $O_t(A)$ defined as

$$O_t(A) := \int_0^t \mathbb{1}_A(W_s) d[W]_s = \int_0^t \mathbb{1}_A(W_s) ds.$$

We can show that the occupation time is absolutely continuous with respect to the Lebesgue measure $\lambda(\cdot)$. Then we define the Brownian local time as the Radon-Nikodym derivative

$$L^x_t = \frac{O_t(dx)}{\lambda(dx)}.$$

From Theorem I.2.17 we deduce the following corollary which will be our definition of reference for the Brownian local time in the sequel.

Corollary I.2.19. *Let $(W_t)_{t\geq 0}$ be a real-valued Brownian motion. Then L^x_t is defined as the almost sure limit:*

$$L^x_t := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^1 \mathbb{1}_{(x-\varepsilon, x+\varepsilon)}(W_s) ds. \tag{I.2.16}$$

We refer to [50, 61, 110, 126] for some initial applications of the Brownian local time. A key application of Brownian local time in probability theory is the possibility to extend the Itô formula to non-differentiable function. It can be used to study stochastic differential equations and their solutions, we refer to the book of Ikeda and Watanabe [74] for a general presentation of such application and e.g. the articles

[59,127,150] or more recently [63,99] for more complex applications of the local time related to the study of SDE. We can also think of financial application, notably for options pricing, see e.g. [71, 97, 111, 125]. There are also applications in physics, when modeling self-interacting particles, see e.g. [1] or [112].

We finish this section by mentioning the following result of Perkins [124] (see also [88]), crucial to prove the main result of Chapter II.

Theorem I.2.20. *Let $(W_t)_{t\geq 0}$ be a real-valued Brownian motion and $(L^x)_{x\in\mathbb{R}}$ the associated local time up to time* 1*. Then there exists a real-valued Brownian motion $(B_t)_{t\in\mathbb{R}}$ such that the following representation holds:*

$$L^x = L^z + \int_z^x 2\sqrt{L^y}dB_y + \int_z^x a_y dy, \qquad x \geq z, \tag{I.2.17}$$

where a is a predictable, locally bounded process. It follows that L^x is a semimartingale.

Observe that the diffusion coefficient $\sigma_y := 2\sqrt{L^y}$ is not differentiable around 0 and therefore is not a semimartingale in itself. This will prove to add another layer of difficulty when dealing with statistics involving the Brownian local time, as the classical techniques in the study of semimartingales cannot be applied (see Section II.3 of Chapter II). We think in particular of the Itô formula that does not hold for the volality.

I.2.1.2 β-stable processes

In this section we define the object of study of Chapter III: the class of **stable processes**. Many books have studied extensively these processes and their properties, in particular let us acknowledge the amazing books of Sato [143], Applebaum [4] and Samorodnitsky and Taqqu [142]. We also invite the reader to refer to the books of Zolotarev [153] and Nolan [113] for a comprehensive study of (univariate) stable distributions.

Informally, a (β-)stable process is a Lévy process that satisfies the **self-similarity** property:

$$X_t \overset{d}{=} t^{1/\beta} X_1 \qquad \text{for all } t \geq 0. \tag{I.2.18}$$

The coefficient β is called the **exponent** of the stable process X. One can show (see [143, Chapter 3]) that $\beta \in (0,2]$, $\beta = 2$ implies that the distribution of X_1 is Gaussian and $\beta = 1$ implies that the distribution of X_1 is Cauchy. We exclude the Gaussian case in our study and focus on β-stable process with $\beta \in (0,2)$. We will see in the sequel that such process is a pure jump process. We will denote by (γ, A, G) instead of the usual notation (γ, A, ν) its Lévy triplet to be closer to the notation used in Chapter III. We have the following result [143, Theorem 14.3] on the characteristic exponent of a β-stable process.

Theorem I.2.21. *Let $X = (X_t)_{t\geq 0}$ be a $\mathbb{R}^d$-valued Lévy process satisfying the property* (I.2.18) *for some $\beta \in (0,2)$. Let (γ, A, G) be its Lévy triplet. Then*

(i) $A = 0$, implying that X is a pure jump process.

(ii) *There exists a finite measure H on the d-dimensional unit sphere $\mathbb{S}_d$, called the* ***spherical part*** *of the Lévy measure G, such that*

$$\psi_X(u) = i\langle \gamma, u\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{\{\|x\|\leq 1\}}\right) G(dx) \qquad \text{(I.2.19)}$$

with

$$G(A) = \int_{\mathbb{S}_d} H(d\theta) \int_0^\infty \mathbb{1}_A(\rho\theta)\rho^{-1-\beta} d\rho, \qquad A \in \mathcal{B}(\mathbb{R}^d). \qquad \text{(I.2.20)}$$

Equivalently,

$$G(dx) = \frac{1}{\rho^{1+\beta}} d\rho H(d\theta) \qquad \textit{with} \quad x = (\rho, \theta) \in \mathbb{S}_d \times \mathbb{R}_+.$$

Remark I.2.22. One can show that β-stable processes with $\beta \in (0, 1]$ have no first moment. When $\beta \in (1, 2)$ they have no second moment.

Stable processes with exponent $\beta \in (0, 2)$ have heavy-tailed distribution. This explains partially their gain in popularity in the last two decades due to their ability to model with more accuracy financial phenomenons that empirically exhibit greater deviations from the mean or produce many outliers. We refer to the books [52, 91, 134] and the survey [109] for a large exposition of the different financial applications.

Althought pure jump stable processes are better at modeling some financial phenomenon than Gaussian processes, contrary to the latter they are difficultly tractable, partly due to the fact that there is no closed form for their marginal probability distributions, except for a handful of value of the exponent β. Numerous papers try to adress this issue, e.g. [5, 51, 132, 137] to name a few.

I.2.2 Ambit fields

This section is devoted to the presentation of ambit field, the mathematical object studied in Chapter IV. Ambit fields is a class of random fields used to model the dynamics of a given system along curves embedded in that field.

It was introduced by Ole E. Barndorff-Nielsen and Jürgen Schmiegel in 2005 to model turbulence (see [19, 20]), but since then, ambit fields have be used to model many different stochastic phenomenons, with notably applications in finance (e.g. [8, 9]), in biology to model the growth of a tumor e.g. [17], in physics [18].

Informally, ambit fields are defined by a stochastic integral that incorporate additional stochastic inputs and model the sphere of influence around a given point in space-time.

We present first the definition of an independently scattered random measure:

Definition I.2.23. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a set and $\mathcal{E}$ a σ-field on $\mathcal{E}$. A **random measure** L on $(E, \mathcal{E})$ is a collection of random variables $(L(B))_{B\in\mathcal{E}}$ such that

(i) $L(\emptyset) = 0$,

(ii) Let $A, B \in \mathcal{E}$. If $A \cap B = \emptyset$ then

$$L(A \cup B) = L(A) + L(B).$$

L is said to be **independently scattered** if the following property holds:

(iii) Let $A_1, ..., A_n \in \mathcal{E}$ be mutually disjoint sets. Then the random variables $L(A_1), ..., L(A_n)$ are mutually independent.

(iv) Assume that W is a random measure satisfying (i), (ii) and (iii). If for all $A \in \mathcal{E}$, $W(A) < \infty$ and $W(A)$ is normally distributed, then W is called a **white noise**.

Remark I.2.24. Let us mention that an independently scattered random measure such that for all $A \in \mathcal{E}$ the random variable $L(A)$ is infinitely divisible (and therefore can be characterize by a Lévy-Khintchine formula) is called a **Lévy basis**.

We define in a very general way the class of **spatio-temporal ambit fields**:

Definition I.2.25. Consider a stochastic field $Y = \{Y_t(x)\}_{t\in\mathbb{R},x\in\mathcal{X}}$ over a d-dimensional space-time $\mathbb{R} \times \mathcal{X}$ taking values in $\mathbb{R}$. Define

$$Y_t(x) = \mu + \int_{A_t(x)} g(t,s,x,\xi)\sigma_s(\xi)L(ds,d\xi) + \int_{D_t(x)} q(t,s,x,\xi)a_s(\xi)dsd\xi, \quad \text{(I.2.21)}$$

where μ is a constant, $A_t(x)$ and $D_t(x)$ are **ambit sets** in $\mathbb{R} \times \mathcal{X}$ (defined later on), g and q are deterministic weight (or kernel) functions, a is a stochastic drift field, σ is a volatility (or intermittency in the context of turbulence modeling) field and L is a independently scattered random measure. Then Y is called an spatio-temporal **ambit field**.

Remark I.2.26.

1. The volatility field σ is in general of stochastic nature and can be define notably as an ambit field itself.

2. We can define ambit field over some d-dimensional space without making the distinction between time and space, depending on the definition chosen for the stochastic integral in the formula (I.2.21).

3. The stochastic integral in (I.2.21) is defined in the Walsh sense (we refer to [149] or the survey [129]). Section A.1 of the appendix contains a precise definition and basic properties of such integral. Under some midly restrictive moment assumptions on the stochastic integrator $L(\cdot)$ the Walsh approach remarkably allows us to integrate stochastic integrands. On the downside, the Walsh integral is not defined for β-stable random measures with $\beta \in (0, 2)$.

4. It is possible to consider different definition of the stochastic integral, such as the Rajput and Rosinski approach (we refer to [135] or the survey [129]). We don't need anymore conditions on the moments of the stochastic integrator $L(\cdot)$, which allows us to integrate with respect to β-stable random measures for any $\beta \in (0, 2]$. On the other hand, we can only define the stochastic integral in (I.2.21) for deterministic integrands.

5. The approach of Chong and Klüpperlberg [47] defines the stochastic integral for any finite L^0-valued random measures, following the idea from [41] and using the correspondance between the class of semimartingales and the class of finite L^0-valued random measures. Their definition of the stochastic integral (I.2.21) allows to integrate stochastic integrands without moment conditions on the L^0-valued random measure. Provided that we have a Lévy-Khintchine formula (see Remark I.2.24), we can characterize L^0-random measures by a characteristic triplet. Under tractable integrability conditions on this triplet, we can ensure the integrability (or non-integrability) of the stochastic integrand.

For the sequel, we only need to keep in mind that the stochastic integral is well defined in the Walsh sense if and only if the process $(L_t(A))_{t\in\mathbb{R},A\in\mathcal{B}(\mathcal{X})}$, defined as

$$L_t(A) := L((0,t]\times A), \quad A \in \mathcal{B}(\mathcal{X}),$$

is a square integrable martingale with respect to some filtration to be defined in Section A.1.

Under this assumption, the stochastic integral behaves roughly speaking like an Itô integral. In particular we have an L^2 isometry similar to the Itô isometry. We also have Burkholder-Davis-Gundy (BDG) type inequality.

We state the definition of ambit set. Note that the name ambit comes from the latin *ambitus* which can mean sphere of influence, boundary or neighborhood.

Definition I.2.27. For any given point $(t,x)\in\mathbb{R}\times\mathcal{X}$ we define two sets, $A_t(x)$ and $D_t(x)$, called **ambit sets**, which represent the region of influence around (t,x). In other words, $A_t(x)$ and $D_t(x)$ are the only regions in space-time which affect the value of Y at (t,x). In the sequel, we consider ambit sets of the form $A+(t,x)$, $D+(t,x)$ where $A,D\subset\mathbb{R}\times\mathcal{X}$ are fixed.

Ambit fields are not *per se* the object of study: we are more interested in understanding the behavior of an ambit field along a curve embedded in the space-time $\mathbb{R}\times\mathcal{X}$. This lead to the following definition:

Definition I.2.28. Let $\tau(\theta)=(t(\theta),x(\theta))$ be a curve in $\mathbb{R}\times\mathcal{X}$. Define

$$X_\theta = Y_{t(\theta)}(x(\theta)).$$

Then X is called an **ambit process**.

In Section I.4 and Chapter IV we restrict the scope of our study to the subclass of spatio-temporal 2-dimensional ambit fields driven by a white noise. Set the notation $\boldsymbol{t} := (t_1,t_2)\in\mathbb{R}^2$. We consider 2-dimensional ambit fields defined as

$$\begin{aligned} X_{\boldsymbol{t}} &= \int_{-\infty}^{\boldsymbol{t}} g(\boldsymbol{t}-\boldsymbol{s})\sigma_{\boldsymbol{s}} W(d\boldsymbol{s}) \qquad \text{(I.2.22)}\\ &= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} g(t_1-s_1,t_2-s_2)\sigma_{s_1,s_2}W(ds_1,ds_2) \end{aligned}$$

where W is a white noise process on $\mathbb{R}^2$, $g : \mathbb{R}^2_+ \to \mathbb{R}$ is a deterministic weight function and σ is a continuous volatility process, ensuring that the stochastic integral (I.2.22) is well-defined in the Walsh sense (see Section A.1).

I.3 Methodology for high-frequency statistics

This section is devoted to the presentation of the shared problematic of Chapter II, Chapter III and Chapter IV. We then present the methods and tools used to answer this problematic.

In the sequel we denote by $\xrightarrow{d_{st}}$ or $\xrightarrow{\mathcal{L}-s}$ the **stable convergence**. We refer to Section A.2 of the appendix for a presentation of this mode of convergence and some of its properties.

Let X be a d-dimensional process defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Assume for simplicity of exposition that the process X is observed discretely over the interval over time $[0,t]$, following a synchronous and regularly spaced high-frequency sampling scheme. More precisely, for Δ_n a sequence of positive real numbers going to 0, we observe the process X at time $i\Delta_n$ for $1 \leq i \leq \lfloor t/\Delta_n \rfloor$. Define the increment

$$\Delta_i^n X := X_{i\Delta_n} - X_{(i-1)\Delta_n}. \tag{I.3.1}$$

Let f be a $\mathbb{R}^d$-valued function with some assumptions on its regularity to be defined. Define the functional (or statistic, depending on the context):

$$V^n(X,f)_t := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f(\Delta_i^n X), \tag{I.3.2}$$

with the convention that if $f(x) = |x|^p$, we write $V^n(X,p)_t$ and $V^n(X,2)_t := [X]_t^n$.

Remark I.3.1. To get a limit for the functional (I.3.2), understanding the behavior of f around the origin is crucial. In certain situations it could be necessary to introduce a normalizing sequence a_n to rescale either the increment $\Delta_i^n X$ or the function of the increment $f(\Delta_i^n X)$.

When X is a Lévy process with Lévy triplet $(\gamma, A, 0)$, i.e. a Brownian motion with drift γ, it could be easier to work with the normalized functional

$$V^n(X,f)_t = \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f\left(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right).$$

We would like to prove a law of large numbers for the functional (I.3.2), i.e. find a stochastic process $V(X,f)$ such that the following functional convergence holds:

$$V^n(X,f)_t \xrightarrow{u.c.p.} V(X,f)_t \quad \text{as } n \to \infty. \tag{I.3.3}$$

We also call this result first order limit (of the functional/statistic). In a second step, we would like to find a normalizing sequence $\delta_n \to$ and establish a central or non central limit theorem for the **error process** U_t^n defined as

$$U_t^n := \delta_n \left(V^n(X,f)_t - V(X,f)_t\right). \tag{I.3.4}$$

More precisely, under some assumptions on the process X, we want to find a stochastic process U such that the following functional stable convergence holds:

$$U^n \xrightarrow{d_{st}} U, \tag{I.3.5}$$

where U is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{\mathbb{P}})$ of the original filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ (we refer again to Section A.2 of the appendix). We also denote this result as second order limit.

Finding first and second order limits for statistics like (I.3.2) is a key part of today's research in mathematical statistics, especially since high-frequency datas are more and more available. We provide examples of such result in Section I.4. An thorough study of the theory supporting these asymptotic results can be found in the books of Jacod and Shiryaev [84] and Jacod and Protter [83], among others. We also recommend [65] for a compendium on high-frequency based applications.

I.3.1 Limit theory for Lévy processes: the Gaussian case

In this subsection we introduce the main mathematical tool to prove the convergence (I.3.5) when X is a d-dimensional semimartingale with a non-vanishing Gaussian part: Jacod's stable central limit theorem. Although there exists different versions of this theorem (see e.g. [84, Chapter IX]), we choose to only provide the version ensuring the convergence towards a semimartingale [84, Chapter IX.7.28].

We start with some prerequisite hypothesis. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and let Z be a d-dimensional continuous local $(\mathcal{F}_t)_{t\geq 0}$-martingale, called the **reference martingale.** Define the set $\mathcal{M}_b(Z^\perp)$ by

$$\mathcal{M}_b(Z^\perp) := \{N \text{ bounded martingales such that } \langle N, Z\rangle = 0\},$$

where $\langle N, Z\rangle$ is the continuous part of the quadratic covariation $[N, Z]$. If $\langle N, Z\rangle = 0$ we say that N is **orthogonal** to Z. Let

$$X_t^n := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \chi_i^n$$

be a q-dimensional semimartingale with respect to the discretized filtration $(\mathcal{F}_{i\Delta_n})_{1\leq i\leq \lfloor t/\Delta_n\rfloor}$. Then we observe the following theorem [84, Chapter IX.7.28]:

Theorem I.3.2. *Assume that Z is square integrable and assume that each χ_i^n is square integrable. Let B be a $\mathbb{R}^q$-valued continuous finite variation process, F a $\mathbb{R}^{q\times q}$-valued continuous process and G a $\mathbb{R}^{q\times d}$-valued continuous process. Assume that B, F and G are adapted to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. If for all $t > 0$, $\varepsilon > 0$ and all $N \in \mathcal{M}_b(Z^\perp)$, we have*

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\chi_i^n | \mathcal{F}_{(i-1)\Delta_n}\right] \xrightarrow{u.c.p.} B_t, \tag{I.3.6}$$

$$\sum_{i=1}^{\lfloor t/\Delta_n \lfloor} \left(\mathbb{E}\left[\chi_i^n (\chi_i^n)^\intercal | \mathcal{F}_{(i-1)\Delta_n}\right] - \mathbb{E}\left[\chi_i^n | \mathcal{F}_{(i-1)\Delta_n}\right] \mathbb{E}\left[\chi_i^n | \mathcal{F}_{(i-1)\Delta_n}\right]^\intercal\right)$$

$$\xrightarrow{\mathbb{P}} F_t = \int_0^t (v_s u_s u_s^\intercal v_s^\intercal + w_s w_s^\intercal)\, ds, \tag{I.3.7}$$

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\chi_i^n (\Delta_i^n Z)^\intercal | \mathcal{F}_{(i-1)\Delta_n}\right] \xrightarrow{\mathbb{P}} G_t = \int_0^t v_s u_s u_s^\intercal ds, \tag{I.3.8}$$

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\|\chi_i^n\|^2 \mathbb{1}_{\{\|\chi_i^n\| > \varepsilon\}} | \mathcal{F}_{(i-1)\Delta_n}\right] \xrightarrow{\mathbb{P}} 0, \tag{I.3.9}$$

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\chi_i^n (\Delta_i^n N)^\intercal | \mathcal{F}_{(i-1)\Delta_n}\right] \xrightarrow{\mathbb{P}} 0, \tag{I.3.10}$$

then we obtain the following functional stable convergence

$$X^n \xrightarrow{d_{st}} X = B + \int_0^\cdot v_s dZ_s + \int_0^\cdot w_s dW'_s \tag{I.3.11}$$

where W' is a q-dimensional Brownian motion defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{\mathbb{P}})$ of the original filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and is independent of $\mathcal{F}$.

The proof of this theorem relies partly on the method defined to study (I.2.11), i.e. by showing the convergence of the characteristics of the semimartingale X^n towards the characteristics of X. However in practice these five conditions are rather easy to verify, whereas the method using characteristics triplet is not very tractable, as computing the characteristics of X_t^n define above can be challenging or even impossible.

Jacod proved a version of his stable central limit theorem in the context of continuous conditional Gaussian martingale already in 1997 in the paper [77], building upon numerous articles paving the way towards this powerful theorem. We refer to [76, 79, 81, 82] to cite a few.

We observe that in condition (I.3.6) we compute the limit of the conditional mean of X_t^n, which converges towards a drift process B of finite variation. Condition (I.3.7) asks us to study the limit of the conditional variance of the empirical process X_t^n. Condition (I.3.9) is a Lindeberg type condition to ensure that there is no χ_i^n having a too big contribution to the limit. Condition (I.3.10) is easy to verify in pratice by applying a martingale representation theorem.

Coming back to the result (I.3.11) of Theorem I.3.2, we observe that X is define as the sum of three independent components: a finite variation process, a local martingale and a Brownian motion. Since the sum of two independent local martingale is a local martingale, by definition, X is a semimartingle.

For statistical purpose, as it is presented in the theorem, (I.3.11) is useless due the $\mathcal{F}$-conditional biais appearing in the limit, as we cannot jointly estimate the distribution of

$$\int_0^\cdot v_s dZ_s + \int_0^\cdot w_s dW'_s.$$

However, when conditions (I.3.8) is verified with $G = 0$, then the conclusion (I.3.11) reads as

$$X_t^n \xrightarrow{d_{st}} B_t + \int_0^t w_s dW'_s. \tag{I.3.12}$$

Since W' is independent of the original σ-algebra $\mathcal{F}$, we can construct estimators for its distribution. In pratice, it is common to verify (I.3.8) with $G = 0$.

In Chapter II, we want to derive a law of large numbers and a stable central limit theorem for an integrated functional of scaled increments of a Brownian local time. After a discretization of the integral and using the semimartingale represention of the Brownian local time, we can make profit of Jacod's theorem. In that case we will see that condition (I.3.8) is satisfied with a non-trivial limit. However, and surprisingly, when we let the time $t \to \infty$, the drift part compensates the second term in (I.3.11), hence the usability of the associated central limit theorem for statistical purpose.

I.3.2 Limit theory for Lévy processes: the β-stable case

In the sequel all processes are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. This section is devoted to the introduction of the methodology used to derived, for X a symmetric $\mathbb{R}^d$-valued β-stable process with $\beta \in (0,2)$ and δ_n a suitable normalizing sequence, the functional stable convergence:

$$\delta_n \left([X]_t^n - [X]_{\Delta_n \lfloor t/\Delta_n \rfloor}\right) := U_t^n \xrightarrow{d_{st}} U_t \tag{I.3.13}$$

when U is a symmetric $\mathbb{R}^d$-valued β-stable process with the same exponent $\beta \in (0,2)$, defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{\mathbb{P}})$ of the original space and is independent of the σ-algebra $\mathcal{F}$. Note that we introduced a discretization of the true quadratic variation $[X]_t$.

Indeed, following the results of Section A.2 of the appendix, to prove the stable convergence above it is sufficient to show the joint convergence in distribution

$$([X]_{\Delta_n \lfloor t/\Delta_n \rfloor}, U_t^n) \xrightarrow{d} ([X]_t, U_t).$$

Let X be a $\mathbb{R}^d$-valued symmetric β-stable process without drift. Using the results presented in Section I.2.1, X is describe by the Lévy triplet $(0, 0, G)$ with G a symmetric Lévy measure satisfying

$$G(A) = \int_{\mathbb{S}_d} H(d\theta) \int_0^\infty \mathbb{1}_A(\rho\theta)\rho^{-1-\beta} d\rho, \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{I.3.14}$$

$$\int_{\mathbb{R}^d} (\|x\|^2 \wedge 1) G(dx) < \infty \tag{I.3.15}$$

and H the spherical part of G is a symmetric measure on $\mathbb{S}_d$. The characteristic exponent of X can be written as

$$\psi_X(u) = \int_{\mathbb{R}^d} \left(\exp(i\langle x, u\rangle) - 1 - i\langle x, u\rangle \mathbb{1}_{\{\|x\|\leq 1\}}\right) G(dx).$$

In this setting, we cannot use Theorem I.3.2 to prove the functional stable convergence as β-stable processes have infinite second moment.

Assume that we are given high-frequency datas $(X_{i\Delta_n})_{1\leq i \leq \lfloor t/\Delta_n \rfloor}$ of the process X define above over some time interval $[0, t]$ and let f be some $\mathbb{R}^d$-valued function.

Denote by Γ^n the functional

$$\Gamma_t^n = \Gamma^n(f)_t := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f(\Delta_i^n X). \tag{I.3.16}$$

Observe that the random vectors $(f(\Delta_i^n X))_{1\le i\le \lfloor t/\Delta_n \lfloor}$ are i.i.d.

We use the following criterion to show the functional stable convergence of Γ^n towards a symmetric stable process Γ with Lévy triplet $(0,0,\nu)$ and exponent $\beta \in (0,2)$ (see [58, Lemma 6.8]): let $u \in \mathbb{R}^d$. Then the convergence

$$\prod_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[e^{i\langle u, f(\Delta_i^n X)\rangle}\right] \xrightarrow{u.c.} \begin{cases} \exp\left\{t \int \left\{e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\right\} \nu(dx)\right\} & \text{if} \quad \beta \in (1,2) \\ \exp\left\{t \int \left\{e^{i\langle u,x\rangle} - 1\right\} \nu(dx)\right\} & \text{if} \quad \beta \in (0,1) \\ \exp\left\{t \int \left\{e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle \mathbb{1}_{\{0<\|x\|\le 1\}}\right\} \nu(dx)\right\} & \text{if} \quad \beta = 1 \end{cases} \tag{I.3.17}$$

implies the functional stable convergence

$$\Gamma^n \xrightarrow{d_{st}} \Gamma.$$

This result is an application of the method for semimartingale sketched to solve the problem (I.2.11). This criterion is the core of the proof of the main result in Chapter III. It was used to prove second order limit, notably in [58] for the realized quadratic variation of a 1-dimensional stochastic integral with respect to a non-homogeneous pure jump Lévy process that behaves locally like a β-stable process. We can also think of the result of [70], where the authors considered a d-dimensional extension of the previous setting. We present in detail this two results in Section I.4.

I.3.3 Limit theory for Ambit fields driven by Gaussian white noise

In this last subsection, we present the main tools used in Chapter IV. As we no longer deal with semimartingales but with ambit processes, we use a more classical approach: to get limit theorem on some statistic, we prove the tightness and the finite dimensional weak convergence of this statistic.

Let $(X_{\boldsymbol{t}})_{t\in\mathbb{R}^2}$ be a 2-dimensional ambit field driven by a Gaussian white noise, defined as in (I.2.22). Assume that we observe X along a curve $z \ : \ t \mapsto (z_1(t), z_2(t))$ discretely in a high frequency regime over $[0,t]$. We denote these observations

$$Y_{i\Delta_n} := X_{z_1(i\Delta_n), z_2(i\Delta_n)}, \quad 1 \le i \le \lfloor t/\Delta_n \rfloor$$

and the associated increment

$$\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}.$$

We want to study the asymptotic behavior of the statistic

$$V^n(Y,p)_t := \Delta_n \tau_n^{-p} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n Y|^p$$

with $p > 0$ and τ_n a normalizing sequence. We will see in Chapter IV that we can prove a law of large numbers for this quantity, using classical techniques. The difficulty comes from the non-stationarity of the increments. Denote by $V(Y,p)$ the functional limit of the statistic $V^n(Y,p)$. Define the error process:

$$U_t^n = \Delta_n^{-1/2} \left(V^n(Y,p)_t - V(Y,p)_t \right).$$

We would like to prove a second order theorem, using Malliavin calculus techniques. Let $0 \leq t_1 < t_2 < ... < t_d$ with $d \in \mathbb{N}$. the goal is to show the weak convergence:

$$\left(U_{t_1}^n, ..., U_{t_d}^n \right) \xrightarrow{d_{st}} \left(U_{t_1}, ..., U_{t_d} \right). \tag{I.3.18}$$

We proceed in four steps.

(i) We prove first that the increments $(\Delta_i^n Y)_{1 \leq i \leq \lfloor t/\Delta_n \rfloor}$ can be approximated by increments of some Gaussian process $\widetilde{Y}$. We denote these new increments $\Delta_i^n \widetilde{Y}$, $1 \leq i \leq \lfloor t/\Delta_n \rfloor$.

(ii) We expand f in a basis of polynomial functions, orthogonals with respect to the Gaussian distribution. These polynomials are the Hermite polynomials, denoted $(H_k)_{k \geq 0}$. We get that

$$f(x) = \sum_{k=0}^{\infty} a_k H_k(x).$$

(iii) Using its chaotic decomposition, we rewrite the error process in terms of iterated Wiener-Itô integrals.

(iv) From the asymptotic behavior of the iterated Wiener-Itô integrals we deduce the weak convergence presented in (I.3.18).

We refer to Section A.3 of the appendix for an introduction to Malliavin calculus and more specifically to all the mathematical concepts and tools appearing in the four steps above.

I.4 Contributions of the thesis

In this final section we present the three primary results of this thesis, contained in Chapter II, Chapter III and Chapter IV, and contrast them with the state of the art in each related field.

I.4.1 Chapter II: Limit theorems for general functionals of Brownian local times

In this subsection we present the main result of Chapter II, based on the paper [44]:

- "Limit theorems for general functionals of Brownian local times", in collaboration with Simon Campese and Mark Podolskij. *Electronic Journal of Probability,* 29:1–18, 2024.

Understanding the probabilistic and statistical characteristics of local times have attracted a lot of attention in the mathematical literature. The asymptotic behavior of functionals of Brownian local time has been recently studied, notably when considering integrated moments in space of increments of Brownian local time (see [43, 45, 72, 73, 139, 140]).

The framework is the following: let $(W_t)_{t\in[0,1]}$ be a standard Brownian motion and denote by $(L^x)_{x\in\mathbb{R}}$ the local time of W over $[0,1]$ (see Section I.2.1.1). Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with $f(0) = 0$. We give later assumptions on the regularity of f. We consider statistics of the form

$$V(f)^h_{\mathbb{R}} := \int_{\mathbb{R}} f(h^{-1/2}(L^{x+h} - L^x))dx, \qquad h > 0. \tag{I.4.1}$$

We want to establish a law of large numbers for the statistic $V(f)^h_{\mathbb{R}}$ and derive a central limit theorem for the error process $U(f)^h_{\mathbb{R}} := h^{-1/2}\left(V(f)^h_{\mathbb{R}} - V(f)_{\mathbb{R}}\right)$.

The case where f is a power function has be established, in [45, 72, 140] when $f(x) = x^2$, in [73, 139] when $f(x) = x^3$ and more generally in [43] when $f(x) = x^q$ with $q \in \mathbb{N}_{\geq 2}$. Let us mention this last result:

Theorem I.4.1. *Let Z be a standard Gaussian random variable independent of the local time $(L^x_t)_{x\in\mathbb{R}}$. Let $f(x) = x^q$ with $q \in \mathbb{N}_{\geq 2}$. Then the following weak convergence holds:*

$$\frac{1}{h^{\frac{q+1}{2}}}\left(\int_{\mathbb{R}} (L^{x+h}_t - L^x_t)^q dx + R_{q,h}\right) \xrightarrow{d} c_q\sqrt{\int_{\mathbb{R}} (L^x_t)^q dx} \times Z$$

where the random variable $R_{q,h}$ is given by

$$R_{q,h} := \sum_{k=1}^{\lfloor q/2 \rfloor} a_{q,k} \int_{\mathbb{R}} (L^{x+h} - L^x)^{q-2k}\left(4\int_x^{x+h} L^u du\right)^k dx,$$

and the constants $a_{q,k}$ and c_k are defined as

$$a_{q,k} = \frac{(-1)^k q!}{2^k k!(q-2k)!} \quad \text{and} \quad c_q = \sqrt{\frac{2^{2q+1}q!}{q+1}}.$$

There is no clear methodology to prove the weak convergence presented above. The quadratic case, when $f(x) = x^2$ has been proven by Rosen in [140] using the method of moments and by Hu and Nualart in [72] where the authors used Malliavin Calculus techniques and asymptotic Ray-Knight type theorem to get their result.

Rosen extended his result to the cubic case with $f(x) = x^3$ in [139], using again the method of moments. Hu and Nualart in [73] also managed to use Malliavin calculus techniques to obtain the asymptotic theory in the cubic case. However, the method of moments cannot be extended to integer power higher than 3.

Campese in [43] achieved a spectacular leap in the theory by establishing the limit theory for any positive integer powers, see Theorem I.4.1 above. Denote by

$$V(L,q)^h_{\mathbb{R}} := \int_{\mathbb{R}} (L^{x+h} - L^x)^q dx$$

the statistic of interest. His methodology consists of using the semimartingale decomposition presented in Theorem I.2.20. From this, he deduces a semimartingale decomposition of the statistic of interest into a local martingale M and a finite variation process A and proceeds to show that the asymptotic behavior of the statistic on the local time is driven by the asymptotic behavior of the statistic on its local martingale part, that is to say

$$V(L,q)^h_{\mathbb{R}} \approx V(M,q)^h_{\mathbb{R}}.$$

He then expresses, using the Kailath-Segall formula [144], the statistic $V(M,q)^h_{\mathbb{R}}$ as an iterated integral with respect to M. In a last step he obtain the limit of this iterated integral using an asymptotic Ray-Knight theorem [128]. Unfortunately, we cannot extend his result with the same methodology to a wider class of function f.

Our contribution is to extend the result of Theorem I.4.1 to C^1 function f where f and f' have polynomial growth. Our methodology follows the first idea of Campese: to make use of the semimartingale representation of the Brownian local time, allowing us to handle the statistic $V(f)^h_{\mathbb{R}}$ using semimartingale techniques. More precisely, we would like to use Jacod's stable central limit theorem I.3.2. To do so, we consider in a first step a functional version of the statistic $V(f)^h_{\mathbb{R}}$: let $T > 0$ fixed and $t \in [0,T]$. We define

$$V(f)^h_t := \int_0^t f\left(h^{-1/2}(L^{x+h} - L^x)\right) dx.$$

Since the increments $L^{x+h} - L^x$ and $L^{y+h} - L^y$ are asymptotically correlated whenever $|x-y| < h$, we use Bernstein blocking technique [35] to break this correlation and discretize the functional statistic $V(f)^h_t$. We split the interval $[0,t]$ into big and small blocks, where the small blocks ensure the asymptotic independence between the big blocks and are sufficiently small to not contribute to the limit. Roughly speaking, we obtain a functional of the form

$$V(f)^h_t \approx \sum_i \int_{A_i} f(h^{-1/2}(L^{x+h} - L^x))dx$$

where A_i denote big blocks. Then, using Jacod's stable central limit theorem, we establish the limit theory for the functional statistic $V(f)^h_t$. In a final step, we let $t \to \infty$ and deduce the limit theory for the statistic $V(f)^h_{\mathbb{R}}$. Our result is the following:

Theorem I.4.2. *Let $f \in C(\mathbb{R})$ be a function with polynomial growth satisfying $f(0) = 0$. Define the quantity*

$$\rho_u(f) := \mathbb{E}[f(\mathcal{N}(0, u^2))] \qquad \text{for } u \in \mathbb{R}. \tag{I.4.2}$$

Then it holds that

$$V(f)^h_{\mathbb{R}} \xrightarrow{\mathbb{P}} V(f)_{\mathbb{R}} := \int_{\mathbb{R}} \rho_{\sigma_u}(f) du \qquad \text{where} \qquad \sigma_u := 2\sqrt{L^u} \tag{I.4.3}$$

as $h \to 0$. If moreover $f \in C^1(\mathbb{R})$ and f, f' have polynomial growth we deduce the stable convergence

$$U(f)^h_{\mathbb{R}} := h^{-1/2} \left(V(f)^h_{\mathbb{R}} - V(f)_{\mathbb{R}}\right) \xrightarrow{d_{st}} U(f)_{\mathbb{R}} := \int_{\mathbb{R}} \sqrt{v^2_{\sigma_u} - \sigma^2_u \rho^2_{\sigma_u}(f')}\, dW'_u, \tag{I.4.4}$$

where W' is a Brownian motion defined on an extended probability space and independent of $\mathcal{F}$. The quantity v_x is defined as

$$v^2_x := 2 \int_0^1 \operatorname{cov}\left(f(xB_1), f(x(B_{s+1} - B_s))\right) ds \tag{I.4.5}$$

with B being a standard Brownian motion.

Remark I.4.3. In the stable central limit theorem for the functional statistic $V(f)^h_t$, the limit obtained exhibits an $\mathcal{F}$-conditional biais, rendering the result useless for statistical application. Indeed, condition (I.3.8) in Jacod's stable central limit theorem I.3.2 is verified with a non-zero limit. Moreover, a non-vanising drift term is also added to this limit, and is due to an approximation of the increments of L made during the proof. However, when letting $t \to \infty$, these two terms cancel each others and we obtain the statistically usable limit (I.4.4).

I.4.2 Chapter III: Limit theorems for asynchronously observed bivariate pure jump semimartingales

In this subsection we present the main result of Chapter III, based on the preprint:

- "Limit theorems for asynchronously observed bivariate pure jump semimartingales", in collaboration with Mark Podolskij, 2024.

In this paper we delve into the asymptotic theory for the realized quadratic covariation of a bivariate β-stable process. We are given high-frequency observations of both components of the process, with the specificity that each component has a different and non-regularly spaced sampling scheme. We start by introducing the framework and prior works.

Let $X = (X_t)_{t \geq 0}$ be an d-dimensional Itô semimartingale and assume that we have synchronous and regularly spaced observations of X over some time interval $[0, t]$, i.e. we have observations $(X_{i\Delta_n})_{1 \leq i \leq \lfloor t/\Delta_n \rfloor}$ with Δ_n a sequence of real numbers going to 0. A big part of the literature on asymptotic theory for semimartingales was focused on obtaining first and second order limit theorems for the realized quadratic variation of the semimartingale X, based on the high-frequency observations that we presented above. We recall that the realized quadratic variation was defined in (I.2.6) as:

$$V^n(X,2)_t := [X]^n_t = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)(\Delta_i^n X)^\intercal \quad \text{with} \quad \Delta_i^n := X_{i\Delta_n} - X_{(i-1)\Delta_n}.$$

By definition, $[X]^n_t$ is a consistent estimator for the quadratic variation $[X]_t$. We recall to the reader the definition of the error process, also sometimes called **realized volatility error**:

$$U_t^n := \delta_n \left([X]^n_t - [X]_t \right)$$

with δ_n a normalizing sequence.

Some of the first asymptotic results on the realized volatility error were obtained by Barndorff-Nielsen and Shephard in 2002 in [22,23] in the one dimensional continuous case: assume that X is a semimartingale of the form

$$X_t = a_t + \int_0^t \sigma_s dW_s,$$

where $\sigma > 0$ and a are processes pathwise of local bounded variation and independent of the Brownian motion W. Assume that the process X is observed $M+1$ times over some fixed time interval $[t_1, t_2]$ with $0 \le t_1 < t_2$ fixed and that the observations are regularly space, i.e. we have datas $(X_{t_1+(t_2-t_1)j/M})_{0\le j \le M}$. Denote by $\Delta_j^M X = X_{t_1+(t_2-t_1)j/M} - X_{t_1+(t_2-t_1)(j-1)/M}$. Then they obtained the following weak convergence:

Theorem I.4.4. *Under some mild regularity assumptions on the drift process, as $M \to \infty$:*

$$\frac{\sum_{j=1}^M (\Delta_j^M X)^2 - \int_{t_1}^{t_2} \sigma_s^2 ds}{\sqrt{\frac{2}{3}\sum_{j=1}^M (\Delta_j^M X)^4}} \xrightarrow{d} \mathcal{N}(0,1).$$

In particular, the asymptotic distribution of $\sum_{j=1}^M (\Delta_j^M X)^2 - \int_{t_1}^{t_2} \sigma_s^2 ds$ *is a mixed normal distribution.*

An extension of this result to the d-dimensional has been proven in 2006 by Barndorff-Nielsen et al. [15, 16], where the authors not only consider the realized quadratic variation but also the realized power and bipower variations, again in the continuous case. Let X be a continuous d-dimensional Itô semimartingales, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s,$$

where a is a predictable locally bounded d-dimensional drift, σ is a $\mathbb{R}^{d\times d}$-valued càdlàg volatility process and W a d-dimensional Brownian motion. Assume that we are given synchronous and regularly spaced observations $(X_{i\Delta_n})_{1\le i\le \lfloor t/\Delta_n \rfloor}$ of the semimartingale X. Assume furthermore than σ is an Itô semimartingale. They obtain the following stable central limit theorem:

Theorem I.4.5. *Under previous assumptions on a and σ we obtain the following stable convergence*

$$\Delta_n^{1/2}\left([X]_t^n - [X]_t\right) \xrightarrow{d_{st}} M_t,$$

where, conditionally on $\mathcal{F}$, $(M_t)_{t\geq 0}$ is a Gaussian martingale with mean zero and conditional covariance function

$$\mathbb{E}\left[M_t^{jk} M_t^{j'k'} | \mathcal{F}\right] = c_t^{jj'} c_t^{kk'} + c_t^{jk'} c_t^{kj'} \quad \text{with } c_t = \sigma_t \sigma_t^T.$$

The asymptotic theory when considering semimartingales with non-vanishing Gaussian part has been widely studied. We refer *inter alia* to the work of Barndorff-Nielsen et al. [25] where they adjoin to the Gaussian semimartingale X a jump process with finite and/or infinite activities. Other authors considered similar semimartingales but assumed that the datas were perturbed by a (Gaussian) noise, for example Podolskij and Vetter [130] or Jacod et al. [80].

Finally, some authors started to work with non-synchronous datas: each coordinates of the multivariate process is observed with high-frequency and non-synchronously, see e.g. [49] among others for multivariate continuous Itô semimartingales or the papers [6,37,48,66] where the authors considered at the same time noisy and non-synchronous datas. In all these results, the second order limit is always conditionally Gaussian, due to the non-vanishing Gaussian part of the semimartingale of interest.

When we consider a pure-jump semimartingale, the asymptotic theory has been much less developed. The univariate case has been establish Jacod et al. [58] when the authors consider general assumptions on the semimartingale of interest. The model is the following

$$X_t = \int_0^t \sigma_{s-} dZ_s + Y_t$$

with Z a non-homogeneous Lévy process that behaves like a locally β-stable process, σ an Itô semimartingale and X an Itô semimartingale without Gaussian part. Provided that the process X is observed on a synchronous and regularly spaced high-frequency sampling scheme, the authors managed to prove a stable central limit theorem for the realized quadratic variation. Their proof relies on the semimartingale representation of the process Z, noticing that the local martingale part of Z, that is the compensated sum of the "small" jumps (smaller than some threshold to be determined), is driving the limit theory when $\beta \in (1,2)$ whereas the finite variation part, that is the sum of the "big" jumps is driving the limit theory when $\beta \in (0,1]$.

This result was extend to the multivariate case by Heiny and Podolskij [70], in particular for a d-dimensional symmetric β-stable process with $\beta \in (0,2)$, observed synchronously and regularly in a high-frequency setting. They proved a stable convergence result for the error process related to the realized quadratic variation and showed that the limit is a $d \times d$ matrix-valued β-stable process. Denote by $x \odot y$ the symmetric tensor product between two vectors. We have the following theorem:

Theorem I.4.6. *Let $\delta_n = (\Delta_n \log(1/\Delta_n))^{-1/\beta}$, $n \geq 1$. For any $\beta \in (0,2)$ we obtain the functional stable convergence*

$$U_t^n := \delta_n \left([L]_t^n - [L]_{\Delta_n \lfloor t/\Delta_n \rfloor}\right) \xrightarrow{d_{st}} U_t,$$

where $(U_t)_{t\geq 0}$ is an $\mathbb{R}^{d\times d}$-valued Lévy process with characteristic triplet $(0,0,\nu_U)$ with ν_U the Lévy measure given by

$$\nu_U(B) = \frac{1}{2\beta}\int_{\mathbb{S}_{d\times d}} \mu(dz)\int_{\mathbb{R}_+} \mathbb{1}_B(\rho z)\rho^{-1-\beta}d\rho, \quad B\in\mathcal{B}(\mathbb{R}^d\odot\mathbb{R}^d),$$

and

$$\mu(z) = \int_{\mathbb{S}_d^2} \mathbb{1}_z\left(\frac{\theta_1\odot\theta_2}{||\theta_1\odot\theta_2||}\right)||\theta_1\odot\theta_2||^\beta H(d\theta_1)H(d\theta_2), \quad z\in\mathcal{B}\left(\mathbb{S}_{d\times d}\right)$$

with B bounded away from 0. The process U is defined on an extension $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t)_{t\geq 0},\overline{\mathbb{P}})$ of the original space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ and is independent of the σ-algebra $\mathcal{F}$.

Futher work on high-frequency observations of a pure jump process includes [146–148], investigating the estimation of the jump activity and the non-parametric estimation of the spectral density. We also refer to [89,90] where the authors developed a statistical test to discrimate whereas we should model financial datas by pure jump processes or not. We finally refer to the recent preprint [33] where the authors worked on joint estimation of the drift, the scaling and the jump activity index for a pure-jump stable Cox-Ingersoll-Ross process.

To the best of our knowledge, the limit theory for the approximate quadratic variation of a pure jump semimartingale when considering high-frequency non-synchronous and irregularly spaced datas has not been established. This is the main contribution of Chapter III.

Consider a bivariate β-stable Lévy process $(L_t)_{t\geq 0} = (L_t^1, L_t^2)_{t\geq 0}$ with Lévy triplet $(0,0,G)$ and denote by H the spherical part of the Lévy measure G. We would like to establish a weak limit theorem for the error process associated to an estimator of the quadratic covariation process $[L^1,L^2]$, over the time interval $[0,1]$. Assume the following: we have two sets of datas, $(L^1_{t_i^1})_{1\leq i\leq n_1}$ and $(L^2_{t_i^2})_{1\leq i\leq n_2}$ and the sampling schemes $(t_i^1)_{1\leq i\leq n_1}$ and $(t_i^2)_{1\leq i\leq n_2}$ satisfies the conditions :

1. There exist strictly monotonic (deterministic) C^2 functions $f_k : [0,1] \longrightarrow [0,1]$ with non-zero right and left derivative in 0 and 1 respectively and with $f_k(0)=0,\ f_k(1)=1$ such that

$$t_i^k = f_k\left(\frac{i}{n_k}\right), \quad 0\leq i\leq n_k, \quad k=1,2.$$

2. There exists a natural number $M>0$ such that

$$\frac{1}{M} < \inf_{x\in[0,1]} f_k'(x) < \sup_{x\in[0,1]} f_k'(x) < M, \quad k=1,2.$$

3. Set $n=n_1+n_2$. It holds that

$$\frac{n_k}{n} \longrightarrow m_k \in (0,1], \quad k=1,2.$$

As we deal with non-synchronous data, we cannot approximate the quadratic covariation process $[L^1, L^2]$ by the usual realized quadratic covariation. Instead, we use the **Hayashi-Yoshida** estimator [67], defined as:

$$\widehat{[L^1, L^2]}_1^{HY} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \Delta_{t_i^1} L^1 \Delta_{t_j^2} L^2 \mathbb{1}_{\{(t_{i-1}^1, t_i^1]\cap(t_{j-1}^2, t_j^2]\neq\emptyset\}},$$

where $\Delta_{t_i^1} L^1 = L^1_{t_i^1} - L^1_{t_{i-1}^1}$ and $\Delta_{t_j^2} L^2 = L^2_{t_j^2} - L^2_{t_{j-1}^2}$. We obtain the following stable weak convergence theorem:

Theorem I.4.7. *For any $\beta \in (0,2)$, we obtain the functional stable convergence*

$$U_1^n = \delta_n \left(\widehat{[L^1, L^2]}_1^{HY} - [L^1, L^2]_{\lfloor 1/t^1_{n_1} \rfloor} \right) \xrightarrow{d_{st}} U_1$$

where $(U_t)_{t\geq 0}$ is an $\mathbb{R}$-valued symmetric β-stable process with $U_1 \overset{d}{=} S_\beta(c, 0, 0)$ a symmetric β-stable distribution with characteristic exponent

$$\varphi_{S_\beta(c,0,0)}(t; \beta, c, 0, 0) = \exp(-c|t|^\beta).$$

The scaling parameter c is defined as

$$c := \frac{\sigma_\beta^0}{m_1} \int_0^1 (f_1'(t))^2 dt + \frac{2\sigma_\beta^1}{m_2} \int_0^1 f_1'(t) f_2'(t) dt$$

with

$$\sigma_\beta^0 := \begin{cases} \dfrac{-\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)}{2\beta} \displaystyle\int_{\mathbb{S}_2^2} \left|\theta_1^1\theta_2^2 + \theta_1^2\theta_2^1\right|^\beta H(d\theta_1)H(d\theta_2) & \text{if } \beta \in (0,1)\cup(1,2), \\ \dfrac{\pi}{4} \displaystyle\int_{\mathbb{S}_2^2} \left|\theta_1^1\theta_2^2 + \theta_1^2\theta_2^1\right| H(d\theta_1)H(d\theta_2) & \text{if } \beta = 1, \end{cases}$$

$$\sigma_\beta^1 := \begin{cases} \dfrac{-\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)}{2\beta} \displaystyle\int_{\mathbb{S}_2^2} \left|\theta_1^2\theta_2^1\right|^\beta H(d\theta_1)H(d\theta_2) & \text{if } \beta \in (0,1)\cup(1,2), \\ \dfrac{\pi}{4} \displaystyle\int_{\mathbb{S}_2^2} \left|\theta_1^2\theta_2^1\right| H(d\theta_1)H(d\theta_2) & \text{if } \beta = 1. \end{cases}$$

$\mathbb{S}_2$ denotes the unit sphere on $\mathbb{R}^2$ with respect to the Euclidean norm and $\theta_i = (\theta_i^1, \theta_i^2)$, $i = 1, 2$. Moreover, the process U is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{\mathbb{P}})$ of the original space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 1}, \mathbb{P})$ and is independent of the σ-algebra $\mathcal{F}$.

We would like to apply the criterion (I.3.17), presented in Section I.3.2. However we cannot apply it directly due to the dependency appearing in the Hayashi-Yoshida estimator: increments of L^1 overlap with increments of L^2. To deal with this structural dependency, we use an approximation of the characteristic function of the error process (see [69]) and then apply the criterion (I.3.17).

Remark I.4.8. Using the same techniques of proof that we used in Chapter III, we cannot extend our result to a d-dimensional β-stable process. Indeed, contrary to the Gaussian case where we can obtain the limit by computing component-wise the covariance matrix of the limiting process, in the pure jump setting we cannot recover the multidimensional quadratic variation of the process from the quadratic covariation between two coordinates of the process.

I.4.3 Chapter IV: Limit theorems for two dimensional ambit fields observed along curves

This subsection is devoted to the presentation of the main result of Chapter IV, based on the paper:

- "Limit theorems for two dimensional ambit fields observed along curves", in collaboration with Mikko S. Pakkanen, Mark Podolskij and Bezirgen Veliyev, work in progress.

This work is an extension of the univariate case. We start by presenting this case and then delve into our result.

Define on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ a subclass of purely temporal ambit fields, called **Brownian semi-stationary process**, by

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(ds)$$

where W is an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted white noise on $\mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is a deterministic weight function such that $g(t) = 0$ for $t \leq 0$ and $g \in L^2(\mathbb{R})$. The volatility σ is an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted càdlàg process.

Define the **power variation** of Y as

$$V(Y,p)^n_t := \Delta_n \tau_n^{-p} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n Y|^p, \quad t \in [0,T],\ p > 0,\ \Delta_n \to 0,$$

where $\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$ and $\tau_n^2 = \mathbb{E}[(\Delta_i^n G)^2]$. The process G is a zero-mean stationary Gaussian process defined by

$$G_t := \int_{-\infty}^{t} g(t-s) W(ds), \quad t \in \mathbb{R}$$

called the **Gaussian core**. The limit theory for the power variation hinges particularly on the behavior of the function g around the origin. Note that if $g' \in L^2(\mathbb{R}_+)$, one can prove that the process Y is a semimartingale. In that case, one can use all the techniques for high-frequency statistics of semimartingales to obtain central limit theorem for the error process associated to the realized quadratic variation. Therefore in the following, we exclude from our assumptions this case.

We would like to have a kernel g that explodes around 0, verifying $g' \notin L^2(\mathbb{R}_+)$ and that such that the behavior of g outside the origin is controlled: $g(x)$ converges fast enough to 0 when $x \to \infty$ and is as smooth as we need outside of a neighborhood of the origin. A prototype example of a kernel satisfying these conditions is the **Gamma kernel**:

$$g(x) = |x|^\alpha e^{-\lambda x}, \quad \alpha \in (-1/2, 0).$$

The following law of large numbers and stable central limit theorem for the realized power variation has been proven in [12, 54]:

Theorem I.4.9. *Assume that the process* Y *is defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Further assume that* $g(x) \sim cx^\alpha$ *as* $x \to 0$ *with* $\alpha \in (-1/2, 1/2)$. *Then, under conditions of [54, Theorem 3.1], we deduce the uniform convergence in probability*

$$\sup_{t\in[0,T]} |V(X,p)^n_t - V(X,p)_t| \xrightarrow{\mathbb{P}} 0 \qquad \text{with} \qquad V(X,p)_t := m_p \int_0^t |\sigma_s|^p ds,$$

where $m_p := \mathbb{E}[|\mathcal{N}(0,1)|^p]$. *When further* $\alpha \in (-1/2, 0)$ *and conditions of [54, Theorem 3.2] are satisfied, we obtain the stable convergence in law:*

$$\Delta_n^{-1/2} \left(V(X,p)^n_t - V(X,p)_t\right) \xrightarrow{st} \lambda_p \int_0^t |\sigma_s|^p B(ds),$$

where B *is a new Brownian motion independent of the initial* σ*-algebra* $\mathcal{F}$, *and the constant* λ_p *is defined in [54, Eq. (3.3)].*

We want to extend this result to the two-dimensional case. Few results exists on the asymptotic behavior of spatio-temporal ambit fields. We note the paper from Pakkanen [120] where he established a law of large numbers and the associated stable central limit theorem for the normalized power variations over rectangle on the space-time called rectangular increment. We mention also the two papers [34, 121] where the authors consider trawl processes: a class of ambit fields where $g \equiv 1$ and $\sigma \equiv 1$ and the ambit sets $A_t(x)$ are translation of a fixed borel set A. We refer to [10, Chapter 8] for a detailled presentation of trawl processes.

We recall the setting presented in (I.2.22): We consider a 2-dimensional ambit field defined as

$$\begin{aligned} X_{\boldsymbol{t}} &= \int_{-\infty}^{\boldsymbol{t}} g(\boldsymbol{t}-\boldsymbol{s})\sigma_{\boldsymbol{s}} W(d\boldsymbol{s}) \\ &= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} g(t_1-s_1, t_2-s_2)\sigma_{s_1,s_2} W(ds_1, ds_2) \end{aligned}$$

where W is a white noise process on $\mathbb{R}^2$, $g : \mathbb{R}^2_+ \to \mathbb{R}$ is a deterministic weight function such that $g \in L^2(\mathbb{R}^2_+)$ with $g(s_1, s_2) = 0$ if $s_1 < 0$ or $s_2 < 0$ and σ is a continuous volatility field. The stochastic integral is to be understood in the Walsh sense (see Section A.1 of the appendix). To ensure the necessary integrability condition:

$$\int_{-\infty}^{\boldsymbol{t}} g^2(\boldsymbol{t}-\boldsymbol{s})\sigma^2_{\boldsymbol{s}} d\boldsymbol{s} < \infty,$$

we assume that $\sup_{\boldsymbol{t}\in\mathbb{R}^2} \mathbb{E}[\sigma^2_{\boldsymbol{t}}] < \infty$.

We are not interested in the probabilistic and statistic properties of the ambit field X but rather on its probabilistic characteristics when observed along a curve. From this, the field X is observed discretly along the curve

$$\boldsymbol{z} \,:\, [0,t] \to \mathbb{R}^2, \quad \boldsymbol{z}(s) = (z_1(s), z_2(s)).$$

Define Y by

$$Y_u = X_{\boldsymbol{z}(u)}$$

and G by

$$G_u = \int_{-\infty}^{\boldsymbol{z}(u)} g(\boldsymbol{z}(u) - \boldsymbol{s}) W(d\boldsymbol{s}), \quad u \in [0, T].$$

Assume that we are given high-frequency observations $Y_{i\Delta_n} = X_{\boldsymbol{z}(i\Delta_n)}$, $i \geq 0$ with $\Delta_n \to 0$. For $p > 0$, we define the power variation of Y as

$$V(Y,p)_t^n := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n Y|^p \quad \text{where } \Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}.$$

Before delving into the asymptotic results, we present the necessary assumptions on the curve $\boldsymbol{z}$, the kernel g and the volatility field σ:

(A1) The curve $t \mapsto \boldsymbol{z}(t)$ is C^2 and the derivatives $z_1'(t), z_2'(t)$ are positive and bounded away from 0.

(A2) For $\alpha \in (-1, 0)$, the kernel g admits the representation

$$g(\boldsymbol{x}) = \|\boldsymbol{x}\|^\alpha f(\boldsymbol{x})$$

where $\|\cdot\|$ denotes the Euclidean norm. The function $f : \mathbb{R}_+^2 \to \mathbb{R}$ is bounded, $f(\boldsymbol{0}) \neq 0$ and $f \in C^1(\mathbb{R}_+^2) \cap L^2(\mathbb{R}_+^2)$ with

$$\|\nabla f(\boldsymbol{x})\| \leq C(\|\boldsymbol{x}\|^{-1} \wedge 1), \quad \boldsymbol{x} \in \mathbb{R}_+^2, \quad C > 0 \text{ some constant.}$$

(A3) There exist $\gamma > 1/2$ such that for any $q > 0$,

$$\mathbb{E}[|\sigma_{\boldsymbol{t}} - \sigma_{\boldsymbol{s}}|^q]^{1/q} \leq C_q \|\boldsymbol{t} - \boldsymbol{s}\|^\gamma$$

for some constant $C_q > 0$ and $\boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}_+^2$.

(A4) There exists $\boldsymbol{a} \in \mathbb{R}_+^2$ such that the partial derivatives satisfy $|\partial_j g(\boldsymbol{x})| \leq |\partial_j g(\boldsymbol{y})|$ for any $\boldsymbol{x} \geq \boldsymbol{y} \geq \boldsymbol{a}$, $j = 1, 2$. Furthermore it holds that

$$F_{\boldsymbol{t}} := \int_{\mathbb{R}_+^2 \setminus [\boldsymbol{0}, \boldsymbol{1}]} (\partial_1 g(\boldsymbol{s})^2 + \partial_2 g(\boldsymbol{s})^2) \sigma_{\boldsymbol{t}-\boldsymbol{s}}^2 d\boldsymbol{s} < \infty \quad \mathbb{P}-\text{a.s.} \quad \text{for all } \boldsymbol{t} \in \mathbb{R}_+^2.$$

We finally define the asymptotic variance of the non-stationary Gaussian core G:

$$\phi_t^2 = z_1'(t) z_2'(t) f(\boldsymbol{0})^2 \Big(\int_{\mathbb{R}_+^2 \setminus (1,\infty)^2} \|\boldsymbol{z}'(t) \circ \boldsymbol{x}\|^{2\alpha} d\boldsymbol{x} + \int_{\mathbb{R}_+^2} (\|\boldsymbol{z}'(t) \circ (\boldsymbol{x} + \boldsymbol{1})\|^\alpha - \|\boldsymbol{z}'(t) \circ \boldsymbol{x}\|^\alpha)^2 d\boldsymbol{x} \Big),$$

defined as the scaled limit of $\Delta^{-(2\alpha+2}\mathrm{var}(G_{t+\Delta} - G(t)$.

The law of large numbers for the realized power variations and the associated stable central limit theorem are the main results of Chapter IV. The stable central limit theorem is proven following the methodology presented in Section I.3.3. We have the following result:

Theorem I.4.10. *Assume that conditions (A1), (A2) and (A4) hold.*

(i) If $\alpha \in (-1, -1/2)$*, we obtain that*

$$\Delta_n^{1-p(1+\alpha)} V(Y,p)_t^n \xrightarrow{u.c.p.} m_p \int_0^t |\phi_s \sigma_{\boldsymbol{z}(s)}|^p ds. \qquad \text{(I.4.6)}$$

(ii) If $\alpha \in (-1/2, 0)$*, we deduce the convergence*

$$\Delta_n^{1-p/2} V(Y,p)_t^n \xrightarrow{u.c.p.} m_p \int_0^t |w_s|^p ds. \qquad \text{(I.4.7)}$$

Futhermore, assume that conditions (A1)-(A4) are satisfied and $\alpha \in (-1, -3/4)$*. We also assume that* $\gamma(p \wedge 1) > 1/2$*. Let* $(B_t)_{t\geq 0}$ *be a Brownian motion defined on an extension* $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ *of the original probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, and being independent of the* σ*-field* $\mathcal{F}$*. We deduce the functional stable convergence*

$$\Delta_n^{-1/2} \left(\Delta_n^{1-p(1+\alpha)} V(Y,p)_t^n - m_p \int_0^t |\phi_s \sigma_{\boldsymbol{z}_s}|^p ds \right) \xrightarrow{st} \int_0^t \kappa_s |\sigma_{\boldsymbol{z}_s}|^p dB_s. \qquad \text{(I.4.8)}$$

where κ_s^2 *is defined as*

$$\kappa_s^2 = \phi_s^{2p} \sum_{k=2}^{\infty} \lambda_k^2 k! \left(1 + 2 \sum_{l=1}^{\infty} \rho(l)^k \right).$$

Observe that we cannot prove a central limit theorem when $\alpha \in (-1/2, 0)$, due to a non-negligible intrinsic biais. We also highlight the fact that contrary to the usual limit results, the realized power variation does not estimate the integrated volatility along the curve $\boldsymbol{z}$ but an integrated product of the quantity ϕ_t and the volatility $\sigma_{\boldsymbol{z}(t)}$.

Remark I.4.11. We mention at the beginning of the current section that Chapter IV is based on a work in progress. Instead of assuming (A2) for the kernel function g, we may assume the following:

$$\text{(A2')} \quad g(\boldsymbol{x}) = |x_1|^\alpha |x_2|^\alpha f(\boldsymbol{x}), \quad \alpha \in (-1/2, 0) \cup (0, 1/2)$$

with $f(\mathbf{0}) \neq 0$ and f decaying fast enough at infinity. The asymptotic regime changes drastically compare to the asymptotic regime under assumption (A2). So far we only managed to establish partial asymptotic results under (A2') but the limit theory is still misunderstood in that case.

Chapter II

Limit theorems for general functionals of Brownian local times

Abstract: In this paper, we present the asymptotic theory for integrated functions of increments of Brownian local times in space. Specifically, we determine their first-order limit, along with the asymptotic distribution of the fluctuations. Our key result establishes that a standardized version of our statistic converges stably in law towards a mixed normal distribution. Our contribution builds upon a series of prior works by S. Campese, X. Chen, Y. Hu, W.V. Li, M.B. Markus, D. Nualart and J. Rosen [43, 45, 72, 73, 107, 139, 140], which delved into special cases of the considered problem. Notably, [45, 72, 73, 139, 140] explored quadratic and cubic cases, predominantly utilizing the method of moments technique, Malliavin calculus and Ray-Knight theorems to demonstrate asymptotic mixed normality. Meanwhile, [43] extended the theory to general polynomials under a non-standard centering by exploiting Perkins' semimartingale representation of local time and the Kailath-Segall formula. In contrast to the methodologies employed in [45, 72, 73, 139], our approach relies on infill limit theory for semimartingales, as formulated in [77, 84]. Notably, we establish the limit theorem for general functions that satisfy mild smoothness and growth conditions. This extends the scope beyond the polynomial cases studied in previous works, providing a more comprehensive understanding of the asymptotic properties of the considered functionals.

II.1 Introduction

Over the past five decades, the mathematical literature has witnessed a surge in interest regarding the probabilistic and statistical properties of local times. Originating from the structure of a Hamiltonian in a specific polymer model, numerous investigations have been dedicated to the asymptotic theory concerning functionals derived from the local time of a Brownian motion. A notable body of work in this domain includes [43, 45, 72, 73, 107, 139]. Recall that the local time $(L^x)_{x\in\mathbb{R}}$ of a Brownian motion $(W_t)_{t\in[0,1]}$ over a time interval $[0,1]$ is defined as the almost sure limit

$$L^x := \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^1 \mathbb{1}_{(x-\varepsilon,x+\varepsilon)}(W_s)ds. \tag{II.1.1}$$

The primary focus of our paper centers around statistics of the form:

$$V(f)^h_{\mathbb{R}} := \int_{\mathbb{R}} f\left(h^{-1/2}(L^{x+h} - L^x)\right) dx, \qquad h > 0, \tag{II.1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth enough function with $f(0) = 0$. Our objective is to ascertain the asymptotic behavior of the statistic $V(f)^h_{\mathbb{R}}$ as $h \to 0$. The theorem below summarizes several special cases, extensively explored in the existing literature, that fall within the scope of our investigation.

Theorem II.1.1. *Let Z be a standard Gaussian random variable independent of the local time $(L^x)_{x\in\mathbb{R}}$.*

(i) Chen et al. [45], Hu and Nualart [72], Rosen [140], case $f(x) = x^2$: As $h \to 0$

$$\frac{1}{h^{3/2}}\left(\int_{\mathbb{R}} \left(L^{x+h} - L^x\right)^2 dx - 4h\right) \xrightarrow{d} \sqrt{\frac{64}{3}\int_{\mathbb{R}} (L^x)^2 dx} \times Z.$$

(ii) Hu and Nualart [73], Rosen [139], case $f(x) = x^3$: As $h \to 0$

$$\frac{1}{h^2}\int_{\mathbb{R}} \left(L^{x+h} - L^x\right)^3 dx \xrightarrow{d} \sqrt{192\int_{\mathbb{R}} (L^x)^3 dx} \times Z.$$

(iii) Campese [43], case $f(x) = x^q$ with $q \in \mathbb{N}_{\geq 2}$: As $h \to 0$

$$\frac{1}{h^{\frac{q+1}{2}}}\left(\int_{\mathbb{R}} \left(L^{x+h} - L^x\right)^q dx + R_{q,h}\right) \xrightarrow{d} c_q\sqrt{\int_{\mathbb{R}} (L^x)^q dx} \times Z,$$

where the random variable $R_{q,h}$ is given by

$$R_{q,h} := \sum_{k=1}^{\lfloor q/2\rfloor} a_{q,k} \int_{\mathbb{R}} \left(L^{x+h} - L^x\right)^{q-2k} \left(4\int_x^{x+h} L^u du\right)^k dx,$$

and the constants $a_{q,k}$ and c_k are defined as

$$a_{q,k} = \frac{(-1)^k q!}{2^k k!(q-2k)!} \qquad \text{and} \qquad c_q = \sqrt{\frac{2^{2q+1}q!}{q+1}}.$$

The weak convergence established in Theorem II.1.1(i) for the quadratic case has been demonstrated via the method of moments in [45]. On the other hand, in [72], techniques from Malliavin calculus, along with a version of the Ray-Knight theorem, were employed to derive the same result. Similar methodologies were applied in [73, 139] to establish the cubic case outlined in Theorem II.1.1(ii). The more general outcome of [43], as presented in Theorem II.1.1(iii), employs a distinct technique to establish asymptotic mixed normality. The starting point in [43] is the semimartingale representation of the local time $(L^x)_{x\in\mathbb{R}}$, initially proven by Perkins in [124]. This representation, in turn, implies a semimartingale decomposition of the statistic $\int_{\mathbb{R}} \left(L^{x+h} - L^x\right)^q dx$. The somewhat intricate standardization $R_{q,h}$ is derived from the Kailath-Segall formula [144], ensuring that the normalized object is a martingale. In the final step, the asymptotic Ray-Knight theorem is applied to deduce weak convergence.

It is worth noting that Theorem II.1.1(iii) extends the results of Theorem II.1.1(i) and (ii) due to $R_{2,h} = -4h$ and $R_{3,h} = 0$ (cf. [43]). However, in other cases, the standardization $R_{q,h}$ is somewhat unnatural as it depends on the parameter h. Our main result, presented below, not only extends Theorem II.1.1 to general functions but also employs a much more natural standardization. Unless stated otherwise, all random variables are defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem II.1.2. *Let $f \in C(\mathbb{R})$ be a function with polynomial growth satisfying $f(0) = 0$. Define the quantity*

$$\rho_u(f) := \mathbb{E}[f(\mathcal{N}(0, u^2))] \qquad \text{for } u \in \mathbb{R}. \tag{II.1.3}$$

Then it holds that

$$V(f)^h_{\mathbb{R}} \xrightarrow{\mathbb{P}} V(f)_{\mathbb{R}} := \int_{\mathbb{R}} \rho_{\sigma_u}(f) du \qquad \text{where} \qquad \sigma_u := 2\sqrt{L^u} \tag{II.1.4}$$

as $h \to 0$. If moreover $f \in C^1(\mathbb{R})$ and f, f' have polynomial growth we deduce the stable convergence

$$U(f)^h_{\mathbb{R}} := h^{-1/2} \left(V(f)^h_{\mathbb{R}} - V(f)_{\mathbb{R}}\right) \xrightarrow{d_{st}} U(f)_{\mathbb{R}} := \int_{\mathbb{R}} \sqrt{v^2_{\sigma_u} - \sigma^2_u \rho^2_{\sigma_u}(f')}\, dW'_u, \tag{II.1.5}$$

where W' is a Brownian motion defined on an extended probability space and independent of $\mathcal{F}$. The quantity v_x is defined as

$$v^2_x := 2 \int_0^1 \operatorname{cov}\left(f(xB_1), f(x(B_{s+1} - B_s))\right) ds \tag{II.1.6}$$

with B being a standard Brownian motion.

Building on the insights presented in [43], our approach begins by leveraging the semimartingale representation of the local time. This transformation allows us to recast the original problem into an asymptotic statistic of a semimartingale. Employing a series of approximation techniques from stochastic analysis, we then apply the limit theory for high-frequency observations of semimartingales, as established in [77]. This application yields the stable convergence result expressed in (II.1.5). However, it's important to highlight that our framework diverges from classical results established in works such as [15,83,95,131] in several aspects. Firstly, we need to introduce a blocking technique as a necessity to break the correlation in the statistic $V(f)^h_{\mathbb{R}}$. Another notable departure from standard high-frequency theory lies in the assumption regarding the semimartingale property of the diffusion coefficient. This property, crucial for obtaining the necessary smoothness for a stable limit theorem, is absent in our model. Instead, our diffusion coefficient is represented by the process σ_u defined in (II.1.4), which is not a semimartingale. Consequently, we employ more nuanced techniques to derive the asymptotic theory.

A surprising distinction, in comparison to [95], is observed in the form of the limit at (II.1.5). Generally, when the function f is not even, the limit typically comprises three terms, revealing an $\mathcal{F}$-conditional bias (cf. [77, 95] and Theorem II.2.1 below). However, in our scenario, we obtain a simpler limit denoted as $U(f)_{\mathbb{R}}$, devoid of an $\mathcal{F}$-conditional bias. This holds true regardless of whether the function f is even or not.

The paper is structured as follows. Section II.2 provides crucial technical results, including the semimartingale decomposition of the local time $(L^x)_{x\in\mathbb{R}}$ and a functional stable central limit theorem. In Section 3, we delve into the proof of the main result.

Notation

Unless explicitly stated otherwise, all random variables and stochastic processes are defined on a filtered probability space denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. All positive constants are denoted by C (or by C_p if we want to emphasise the dependence on an external parameter p) although they may change from line to line. We use the notation

$$I_t := [\min(0,t), \max(0,t)].$$

We say that a function $f:\mathbb{R}\to\mathbb{R}$ has polynomial growth if it holds that $|f(x)| \leq C(1+|x|^p)$ for some $p \geq 0$. We denote by $\langle X, Y\rangle$ the covariation process of two semimartingales X and Y. For real-valued stochastic processes Y^n and Y, we employ the notation $Y^n \xrightarrow{u.c.p.} Y$ to signify uniform convergence in probability, specifically:

$$\sup_{t\in A} |Y^n_t - Y_t| \xrightarrow{\mathbb{P}} 0$$

for any compact set $A \subset \mathbb{R}$. For a sequence of random variables $(Y^n)_{n\in\mathbb{N}}$ defined on a Polish space $(E, \mathcal{E})$, we say that Y^n converges stably in law towards Y ($Y^n \xrightarrow{d_{st}} Y$), which lives on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if and only if

$$\lim_{n\to\infty} \mathbb{E}[Fg(Y^n)] = \mathbb{E}'[Fg(Y)]$$

for all bounded $\mathcal{F}$-measurable random variables F and all bounded continuous functions $g: E\to\mathbb{R}$.

II.2 Definitions and preliminary results

To begin, we utilize the semimartingale representation of the local time process $(L^x)_{x\in\mathbb{R}}$, as derived in [124]. This representation posits the existence of a Brownian motion $(B_t)_{t\in\mathbb{R}}$ such that the local time is expressed as follows:

$$L^x = L^z + \int_z^x \sigma_y dB_y + \int_z^x a_y dy, \qquad x \geq z, \tag{II.2.1}$$

where the diffusion coefficient σ is defined at (II.1.4), and the drift coefficient a is a predictable, locally bounded process. This representation, as emphasized in the introduction, serves as a fundamental tool for establishing the stable limit theorem in (II.1.5). Additionally, we introduce two random times

$$\underline{S} := \inf\{a \leq 0 : \; L^a > 0\}, \qquad \overline{S} := \sup\{a \geq 0 : \; L^a > 0\}, \tag{II.2.2}$$

and remark that $\underline{S}$ and $\overline{S}$ are stopping times with respect to the filtration generated by L. Note $L^x = 0$ for any $x \notin [\underline{S}, \overline{S}]$.
To establish the results outlined in Theorem II.1.2, it is essential to introduce a functional version of the statistic $V(f)^h_{\mathbb{R}}$. For a fixed $T > 0$, we define this functional as follows:

$$V(f)^h_t := \int_{I_t} f\left(h^{-1/2}(L^{x+h} - L^x)\right) dx, \qquad t \in [-T, T]. \tag{II.2.3}$$

We obtain the following theorem.

Theorem II.2.1. *Let $f \in C(\mathbb{R})$ be a function with polynomial growth satisfying $f(0) = 0$. Then it holds that*

$$V(f)^h \xrightarrow{u.c.p.} V(f) \text{ as } h \to 0 \qquad \text{where} \qquad V(f)_t := \int_{I_t} \rho_{\sigma_u}(f) du. \tag{II.2.4}$$

Assume moreover that $f \in C^1(\mathbb{R})$ and f, f' have polynomial growth, and define the process

$$U(f)^h_t := h^{-1/2}\left(V(f)^h_t - V(f)_t\right).$$

Then, as $h \to 0$, we obtain the functional stable convergence $U(f)^h \xrightarrow{d_{st}} U(f)$ on $(C([-T,T]), \|\cdot\|_\infty)$, where

$$U(f)_t := \int_{I_t} r_{a_x,\sigma_x} dx + \int_{I_t} w_{\sigma_x} dB_x + \int_{I_t} \sqrt{v^2_{\sigma_x} - w^2_{\sigma_x}} dW'_x. \tag{II.2.5}$$

The processes v_u and W' have been introduced in Theorem II.1.2, while the quantities w_u and r_{u_1,u_2} are defined as

$$w_u := u\rho_u(f'), \tag{II.2.6}$$

$$r_{u_1,u_2} := u_1\rho_{u_2}(f') + \int_0^1 \mathbb{E}\left[f'(u_2(B_{x+1} - B_x))(B_2^2 - 2)\right] dx.$$

We now demonstrate that the consistency statement in (II.1.4), as mentioned in Theorem II.1.2, follows from the more general results provided in Theorem II.2.1. Initially, we observe the identities

$$V(f)^h_{\mathbb{R}} = V(f)^h_{\underline{S}} + V(f)^h_{\overline{S}} + O_{\mathbb{P}}(h), \qquad V(f)_{\mathbb{R}} = V(f)_{\underline{S}} + V(f)_{\overline{S}} \tag{II.2.7}$$

hold. For any $\varepsilon > 0$, we conclude that

$$\mathbb{P}\left(|V(f)^h_{\underline{S}} - V(f)_{\underline{S}}| > \varepsilon\right) \le \mathbb{P}\left(\sup_{t\in[-T,T]} |V(f)^h_t - V(f)_t| > \varepsilon, |\underline{S}| \le T\right) + \mathbb{P}\left(|\underline{S}| > T\right), \tag{II.2.8}$$

and a similar estimate holds for the probability $\mathbb{P}(|V(f)^h_{\overline{S}} - V(f)_{\overline{S}}| > \varepsilon)$. Consequently, the uniform convergence in (II.2.4) implies the statement in (II.1.4) when we choose T to be sufficiently large and then h to be sufficiently small. This establishes the consistency result in the context of Theorem II.1.2.
Subject to an additional smoothness condition on the function f, the expression for the limit $U(f)_t$ simplifies as demonstrated in the following proposition.

Proposition II.2.2. *Assume that $f(0) = 0$, $f \in C^3(\mathbb{R})$, and f and its first three derivatives have polynomial growth. Define the function*

$$G(u) := \int_0^u \rho_{2\sqrt{x}}(f')dx, \qquad u \ge 0. \tag{II.2.9}$$

Then we obtain the identity

$$U(f)_t = G(L^t) - G(L^0) + \int_{I_t} \sqrt{v^2_{\sigma_x} - w^2_{\sigma_x}} dW'_x. \tag{II.2.10}$$

Proof. First of all, we note that $\rho_u(g) < \infty$ when the function g has polynomial growth. Observing the semimartingale decomposition at (II.2.1), an application of Itô formula gives

$$F(L^b) = F(L^a) + \int_a^b F'(L^u)dL^u + 2\int_a^b F''(L^u)L^u du,$$

for any $F \in C^2(\mathbb{R})$ and any $b > a$. A twofold application of an integration by parts formula implies that

$$\int_0^1 \mathbb{E}\left[f'(u_2(B_{x+1} - B_x))(B_2^2 - 2)\right] dx = u_2^2 \rho_{u_2}(f''').$$

According to definition (II.2.9) it holds that

$$G'(u) = \rho_{2\sqrt{u}}(f') \qquad \text{and} \qquad G''(u) = 2\rho_{2\sqrt{u}}(f''').$$

Consequently, we deduce the identity

$$\int_{I_t} r_{a_x,\sigma_x} dx + \int_{I_t} w_{\sigma_x} dB_x = \int_{I_t} \rho_{\sigma_u}(f')dL^u + \int_{I_t} \sigma_u^2 \rho_{\sigma_u}(f''')du$$

$$= \int_{I_t} G'(\sigma_u^2/4) dL^u + \frac{1}{2} \int_{I_t} \sigma_u^2 G''(\sigma_u^2/4) du$$

$$= \int_{I_t} G'(L^u) dL^u + 2 \int_{I_t} L_u^2 G''(L^u) du = G(L^t) - G(L^0).$$

This completes the proof of the proposition. □

As a consequence of Proposition II.2.2 and the identity $L^x = 0$ for $x \notin [\underline{S}, \overline{S}]$, we infer that

$$U(f)_{\mathbb{R}} = U(f)_{\underline{S}} + U(f)_{\overline{S}} = \int_{\mathbb{R}} \sqrt{v_{\sigma_x}^2 - w_{\sigma_x}^2} dW'_x, \tag{II.2.11}$$

provided the function f satisfies the conditions outlined in Proposition II.2.2. Therefore, the stable convergence asserted in Theorem II.1.2 follows from Proposition II.2.2 when accompanied by a suitable approximation argument. The details of this argument will be explained in Section II.3.4.

Example II.2.3. Here, we illustrate that Theorem II.1.2 includes the results of Theorem II.1.1(i) and (ii) as specific cases.

(i) Consider the quadratic case $f(x) = x^2$ and note the identities $\rho_u(f) = u^2$, $\rho_u(f') = 0$. We immediately conclude that

$$V(f)_{\mathbb{R}} = \int_{\mathbb{R}} \sigma_u^2 du = 4 \int_{\mathbb{R}} L^u du = 4,$$

where the last equality follows from the occupation time formula. We also deduce that

$$v_x^2 = 2x^4 \int_0^1 \operatorname{cov}\left(B_1^2, (B_{s+1} - B_s)^2\right) ds = 4x^4 \int_0^1 \operatorname{cov}\left(B_1, B_{s+1} - B_s\right)^2 ds = \frac{4}{3} x^4,$$

and consequently we get $v_{\sigma_u}^2 = \frac{64}{3}(L^u)^2$. Thus we recover the statement of Theorem II.1.1(i).

(ii) Now, consider the cubic setting $f(x) = x^3$. In this scenario we deduce the identities $\rho_u(f) = 0$ and $\rho_u(f') = 3u^2$. Consequently, $V(f)_{\mathbb{R}} = 0$ and $w_u^2 = 9u^6$. A straightforward computation shows that

$$v_x^2 = 2 \int_0^1 \operatorname{cov}\left(f(xB_1), f(x(B_{s+1} - B_s))\right) ds = 12x^6.$$

Thus we deduce the identity $v_{\sigma_u}^2 - w_{\sigma_u}^2 = 192(L^u)^3$ and we recover the statement of Theorem II.1.1(ii). □

II.3 Proofs

II.3.1 Preliminary results

To simplify our analysis, we begin by establishing stronger assumptions on the involved stochastic processes. Analogous to the reasoning provided in (II.2.8), we can perform all proofs on the set $\{\underline{S}, \overline{S} \in [-T,T]\}$ for some $T>0$. Provided $L^z > 0$ for all $z \in [y,x]$, we will also use the identity

$$\sigma_z = \sigma_y + 2(B_z - B_y) + \int_y^z \tilde{a}_u \cdot (L^u)^{-1/2} dy, \tag{II.3.1}$$

where $\tilde{a}$ is a locally bounded process. This identity follows from $\sigma_x = 2\sqrt{L^x}$ and an application of the Itô formula to (II.2.1).
Additionally, given that $\tilde{a}, a$ and σ are locally bounded processes, we may, without loss of generality, assume that

$$\sup_{\omega\in\Omega,\ x\in[-T,T]} \left(|\tilde{a}_x(\omega)| + |a_x(\omega)| + |\sigma_x(\omega)|\right) \leq C$$

by employing a standard localization argument (cf. [15]).
Due to the boundedness of coefficients a and σ we deduce from Burkholder inequality for any $a<b$ and $p>0$:

$$\mathbb{E}\left[\sup_{x\in[a,b]} |L^x - L^a|^p\right] \leq C_p |b-a|^{p/2}. \tag{II.3.2}$$

Consequently, due to definition in (II.1.4), we also deduce the inequality

$$\mathbb{E}\left[\sup_{x\in[a,b]} |\sigma_x - \sigma_a|^p\right] \leq C_p |b-a|^{p/4}. \tag{II.3.3}$$

Furthermore, for any function $g : \mathbb{R} \to \mathbb{R}$ with polynomial growth we have that

$$\mathbb{E}\left[g\left(h^{-1/2}(L^{x+h} - L^x)\right)\right] \leq C, \tag{II.3.4}$$

which follows directly from (II.3.2).
We will often use the following lemmata, which are well known results.

Lemma II.3.1. *Consider the process $Y_t^n = \sum_{i=1}^{\lfloor nt \rfloor} \chi_i^n$, $t\in[0,T]$, where the random variables are χ_i^n are $\mathcal{F}_{i/n}$-measurable and square integrable. Assume that*

$$\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}\left[\chi_i^n | \mathcal{F}_{(i-1)/n}\right] \xrightarrow{u.c.p.} Y_t \qquad \text{and} \qquad \sum_{i=1}^{\lfloor nT \rfloor} \mathbb{E}\left[(\chi_i^n)^2 | \mathcal{F}_{(i-1)/n}\right] \xrightarrow{\mathbb{P}} 0.$$

Then $Y^n \xrightarrow{u.c.p.} Y$ as $n\to\infty$.

Lemma II.3.2. *Consider a sequence of stochastic processes Y^n and $Y^{n,m}$. Assume that*

$$Y^{n,m} \xrightarrow{d_{st}} Z^m \quad \text{as } n\to\infty, \qquad Z^m \xrightarrow{d_{st}} Y \quad \text{as } m\to\infty, \qquad \text{and}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |Y_t^{n,m} - Y_t^n| > \varepsilon\right) = 0 \qquad \text{for any } \varepsilon > 0.$$

Then it holds $Y^n \xrightarrow{d_{st}} Y$ as $n\to\infty$.

The following estimate is important for the mathematical arguments below.

Proposition II.3.3. *It holds that*

$$\sup_{x\in[-T,T]} \mathbb{P}\left(x \in (\underline{S}, \overline{S}),\ L^x \in [0,\varepsilon)\right) \le C_T\varepsilon.$$

Proof. Let τ_x be the first time the Brownian motion W hits the level $x \in \mathbb{R}$. For $x \in (\underline{S}, \overline{S})$, it must satisfy $\tau_x < 1$. Now, applying the strong Markov property of Brownian motion, we introduce a new process $\widetilde{W}_t := W_{t+\tau_x} - W_{\tau_x}$, where $t \ge 0$. Consequently, $\widetilde{W}$ is a new Brownian motion independent of τ_x. Let $L_t^x(W)$ denote the local time of W at point x up to time t. This leads to the relation $L_1^x(W) = L^0_{1-\tau_x}(\widetilde{W})$. A well-known result asserts that $L^0_u(\widetilde{W}) \stackrel{d}{=} |\widetilde{W}_u|$ for any fixed u. Thus, by conditioning on τ_x, we infer that

$$\begin{aligned}\mathbb{P}\left(x \in (\underline{S}, \overline{S}),\ L^x \in [0,\varepsilon)\right) &\le \mathbb{P}\left(\tau_x < 1,\ L^x \in [0,\varepsilon)\right)\\ &= \mathbb{P}\left(\tau_x < 1,\ \sqrt{1-\tau_x}\cdot|\mathcal{N}(0,1)| \in [0,\varepsilon)\right)\\ &\le C\varepsilon\mathbb{E}\left[\mathbb{1}_{\{\tau_x<1\}}(1-\tau_x)^{-1/2}\right].\end{aligned}$$

The density of τ_x is given by $p(u) = (2\pi)^{-1/2}|x|u^{-3/2}\exp(-x^2/2u)\mathbb{1}_{\{u>0\}}$. Hence, we conclude that

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_x<1\}}(1-\tau_x)^{-1/2}\right] < \infty,$$

which completes the proof. □

II.3.2 Law of large numbers

In this section we show the uniform convergence in probability as stated in (II.2.4). An application of (II.2.8) implies the statement (II.1.4).
The basic idea of all proofs is to consider the approximation

$$h^{-1/2}(L^{x+h} - L^x) \approx h^{-1/2}\sigma_x(B_{x+h} - B_x).$$

Observing this approximation we see that the two increments $h^{-1/2}(L^{x+h} - L^x)$ and $h^{-1/2}(L^{y+h} - L^y)$ are asymptotically correlated when $|x-y| < h$. To break this dependence we use a classical blocking technique. For $i \ge 0$ we introduce the sets

$$\begin{aligned}A_i(m) &= [i(m+1)h, i(m+1)h + mh],\\ B_i(m) &= [i(m+1)h + mh, i(m+1)h + (m+1)h].\end{aligned}$$

Note that the length of $A_i(m)$ is mh (big block) while $B_i(m)$ has the length h (small block). In the first step we obtain the following decomposition:

$$V(f)_t^h = Z_t^{h,m}(f) + R_t^{h,m}(f) + D_t^{h,m}(f),$$

where

$$Z_t^{h,m}(f) := \sum_{i\in\mathbb{N}:i(m+1)h+mh\in I_t} \int_{A_i(m)} f\left(h^{-1/2}(L^{x+h} - L^x)\right)dx,$$

$$R_t^{h,m}(f) := \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \int_{B_i(m)} f\left(h^{-1/2}(L^{x+h}-L^x)\right) dx,$$

and $D_t^{h,m}(f)$ comprises the edge terms and satisfies

$$D^{h,m}(f) \xrightarrow{u.c.p.} 0 \qquad \text{as } h\to 0 \tag{II.3.5}$$

due to (II.3.4) and the polynomial growth of f. Next, we will analyse the asymptotic behaviour of the processes $Z^{h,m}(f)$ and $R^{h,m}(f)$.

(a) *Negligibility of $R_t^{h,m}(f)$:* First, we observe the inequality

$$\sup_{t\in[-T,T]} |R_t^{h,m}(f)| \le R_T^{h,m}(|f|) + R_{-T}^{h,m}(|f|)$$

Since f has polynomial growth we deduce that $|f(x)|\le C(1+|x|^p)$ for some $p>0$. Due to inequality (II.3.2) we get

$$\mathbb{E}\left[R_T^{h,m}(|f|) + R_{-T}^{h,m}(|f|)\right] \le Cm^{-1}.$$

Thus, we conclude that

$$\lim_{m\to\infty} \limsup_{h\to 0} \mathbb{P}\left(\sup_{t\in[-T,T]} |R_t^{h,m}(f)| > \varepsilon\right) = 0, \tag{II.3.6}$$

for any $\varepsilon>0$. This proves the negligibility of the term $R^{h,m}(f)$. □

(b) *Law of large numbers for the approximation:* We introduce the following approximation of the statistic $Z_t^{h,m}(f)$:

$$\overline{Z}_t^{h,m}(f) := \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \alpha_i^h(m), \tag{II.3.7}$$

$$\alpha_i^h(m) := \int_{A_i(m)} f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right) dx,$$

where $t_i^h(m) = i(m+1)h$ is the left boundary of the interval $A_i(m)$. Due to Riemann integrability we deduce that

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[\alpha_i^h(m)|\mathcal{F}_{t_i^h(m)}] = mh \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \rho_{\sigma_{t_i^h(m)}}(f) \xrightarrow{u.c.p.} \frac{m}{m+1}V(f)_t$$

as $h\to 0$ and $m/(m+1)V(f) \xrightarrow{u.c.p.} V(f)$ as $m\to\infty$. By Lemma II.3.1 it suffices to prove that

$$\sum_{i\in\mathbb{N}:\le i(m+1)h+mh\in[-T,T]} \mathbb{E}\left[\left|\alpha_i^h(m)\right|^2|\mathcal{F}_{t_i^h(m)}\right] \xrightarrow{\mathbb{P}} 0 \quad \text{as } h\to 0.$$

By (II.3.4) we readily deduce that

$$\mathbb{E}[|\alpha_i^h(m)|^2|\mathcal{F}_{t_i^h(m)}] \le C(mh)^2.$$

Hence, we conclude

$$\overline{Z}^{h,m}(f) \stackrel{u.c.p.}{\longrightarrow} \frac{m}{m+1}V(f) \text{ as } h \to 0, \qquad \text{and} \qquad \frac{m}{m+1}V(f) \stackrel{u.c.p.}{\longrightarrow} V(f) \text{ as } m \to \infty. \tag{II.3.8}$$

□

(c) In view of steps (a) and (b) we are left to proving the statement

$$\overline{Z}^{h,m}(f) - Z^{h,m}(f) \stackrel{u.c.p.}{\longrightarrow} 0. \tag{II.3.9}$$

Since f has polynomial growth we have the following inequality for $\varepsilon, A > 0$:

$$|f(x) - f(y)| \le C \left(w_f(A,\varepsilon) + (1 + |x|^p + |y|^p)(\mathbb{1}_{\{|x|>A\}} + \mathbb{1}_{\{|y|>A\}} + \mathbb{1}_{\{|x-y|>\varepsilon\}}) \right),$$

where $w_f(A,\varepsilon) := \sup\{|f(x) - f(y)| : \; |x|, |y| \le A, \; |x-y| \le \varepsilon\}$ is the modulus of continuity of f. Using this inequality and (II.3.2), and also $\mathbb{1}_{\{|x|>A\}} \le A^{-1}|x|$, $\mathbb{1}_{\{|x-y|>\varepsilon\}} \le \varepsilon^{-1}|x-y|$, we conclude that

$$\mathbb{E}\left[\sup_{t\in[-T,T]} \left| \overline{Z}_t^{h,m}(f) - Z_t^{h,m}(f) \right| \right] \le C\Big(w_f(A,\varepsilon) + A^{-1}$$

$$+ \varepsilon^{-1} \sum_{i\in\mathbb{N}:\; i(m+1)h+mh\in[-T,T]} \int_{A_i(m)} \left(h^{1/2} + h^{-1/2}\mathbb{E}\left[\int_x^{x+h} |\sigma_u - \sigma_{t_i^h(m)}|^2 du \right]^{1/2} \right) dx.$$

Since σ is continuous and bounded we see that the third term converges to 0 as $h \to 0$. On the other hand, we have that $\lim_{\varepsilon\to 0} w_f(A,\varepsilon) = 0$ for a any fixed A. Hence, we deduce that

$$\overline{Z}^{h,m}(f) - Z^{h,m}(f) \stackrel{u.c.p.}{\longrightarrow} 0$$

by letting first $h \to 0$, then $\varepsilon \to 0$ and $A \to \infty$. Due to statements (II.3.6) and (II.3.8), we obtain the convergence in (II.2.4). □

II.3.3 Stable central limit theorem

Demonstrating the stable central limit theorem as stated in Theorem II.2.1 poses a more intricate challenge. Our approach is primarily based upon limit theorems for semimartingales, notably in works such as [15]. It is crucial to highlight that the diffusion coefficient $\sigma_x = 2\sqrt{L^x}$ is not a semimartingale, introducing a heightened level of complexity to the proofs. We will continue to employ the blocking technique introduced in the preceding section.

First of all, we decompose our statistic into several terms:

$$U(f)^h = \sum_{k=1}^{3} Z^{h,m,k}(f) + \sum_{k=1}^{3} R^{h,m,k}(f) + \overline{D}^{h,m}. \tag{II.3.10}$$

Here the processes $Z^{h,m,k}(f)$, $k = 1, 2, 3$, are big blocks approximations, which are defined by

$$Z_t^{h,m,1}(f) := h^{-1/2} \sum_{i\in\mathbb{N}:\; i(m+1)h+mh\in I_t} \Big(\alpha_i^h(m) - \mathbb{E}[\alpha_i^h(m)|\mathcal{F}_{t_i^h(m)}] \Big),$$

$$Z_t^{h,m,2}(f) := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \left(\int_{A_i(m)} f\left(h^{-1/2}(L^{x+h}-L^x)\right) dx - \alpha_i^h(m) \right),$$

$$Z_t^{h,m,3}(f) := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \int_{A_i(m)} \left(\rho_{\sigma_{t_i^h(m)}}(f) - \rho_{\sigma_x}(f) \right) dx.$$

The small block processes $R^{h,m,k}(f)$, $k=1,2,3$, are introduced in exactly the same way with the set $A_i(m)$ being replaced by $B_i(m)$ in all relevant definitions. Finally, the process $\overline{D}^{h,m}$ comprises all the edge terms. Similarly to the treatment of the term $D^{h,m}$ in (II.3.5), we immediately conclude that

$$\overline{D}^{h,m} \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to 0. \tag{II.3.11}$$

In the following subsections we will show that all small blocks terms are negligible in the sense

$$\lim_{m\to\infty} \limsup_{h\to 0} \mathbb{P}\left(\sup_{t\in[-T,T]} |R_t^{h,m,k}(f)| > \varepsilon \right) = 0 \qquad \text{for any } \varepsilon > 0, \tag{II.3.12}$$

for all $k=1,2,3$. Finally, we will show that

$$Z^{h,m,1}(f) \xrightarrow{d_{st}} U'^m(f), \qquad Z^{h,m,2}(f) \xrightarrow{u.c.p.} U''^m(f), \qquad Z^{h,m,3}(f) \xrightarrow{u.c.p.} 0$$

as $h \to 0$, and moreover

$$U'^m(f) \xrightarrow{d_{st}} U'(f) = \int_0^\cdot w_{\sigma_x} dB_x + \int_0^\cdot \sqrt{v_{\sigma_x}^2 - w_{\sigma_x}^2}\, dW'_x, \tag{II.3.13}$$

$$U''^m(f) \xrightarrow{u.c.p.} U''(f) = \int_0^\cdot r_{a_x,\sigma_x} dx$$

as $m \to \infty$. Consequently, due to (II.3.11)-(II.3.13), an application of Lemma II.3.2 and properties of stable convergence imply the statement of Theorem II.2.1.

II.3.3.1 Central limit theorem for the approximation

Recalling the notation from the previous subsection, we set

$$Z_t^{h,m,1}(f) =: \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} X_i^h(m).$$

We now prove the stable central limit theorem for $Z^{h,m,1}(f)$ as $h \to 0$. According to Theorem [84, Theorem IX.7.28] we need to show that

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[|X_i^h(m)|^2 | \mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} \int_{I_t} v_{\sigma_x}^2(m) dx \tag{II.3.14}$$

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[X_i^h(m)(B_{t_i^h(m)+(m+1)h} - B_{t_i^h(m)}) | \mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} c_m \int_{I_t} w_{\sigma_x} dx \tag{II.3.15}$$

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[|X_i^h(m)|^2 \mathbb{1}_{\{|X_i^h(m)|>\varepsilon\}}|\mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} 0 \quad \forall \varepsilon > 0 \tag{II.3.16}$$

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[X_i^h(m)(N_{t_i^h(m)+(m+1)h} - N_{t_i^h(m)})|\mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} 0 \tag{II.3.17}$$

where the last statement should hold for all bounded continuous martingales N with $\langle B, N\rangle = 0$, $c_m = m/(m+1)$, and the function $v_u(m)$ will be introduced below.

We start by showing the condition (II.3.14). A straightforward computation using the substitution $x = hz_1, y = hz_2$ shows that

$$\mathbb{E}[|X_i^h(m)|^2|\mathcal{F}_{t_i^h(m)}] = h^{-1} \int_{A_i^2(m)} \Big(\mathbb{E}\Big[f\Big(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\Big)$$

$$\times f\Big(h^{-1/2}\sigma_{t_i^h(m)}(B_{y+h} - B_y)\Big)\Big|\,\mathcal{F}_{t_i^h(m)}\Big] - \rho^2_{\sigma_{t_i^h(m)}}(f)\Big)\,\mathbb{1}_{\{|x-y|<h\}}dxdy$$

$$= h\int_{[i(m+1),i(m+1)+m]^2} \Big(\mathbb{E}\Big[f\Big(\sigma_{t_i^h(m)}(B_{z_1+1} - B_{z_1})\Big)$$

$$\times f\Big(\sigma_{t_i^h(m)}(B_{z_2+1} - B_{z_2})\Big)\Big|\,\mathcal{F}_{t_i^h(m)}\Big] - \rho^2_{\sigma_{t_i^h(m)}}(f)\Big)\,\mathbb{1}_{\{|z_1-z_2|<1\}}dz_1dz_2.$$

Hence, by Riemann integrability we deduce that

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[|X_i^h(m)|^2|\mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} \int_{I_t} v^2_{\sigma_x}(m)dx,$$

where

$$v_u^2(m) := \frac{1}{m}\int_{[0,m]^2} \mathrm{cov}\left(f\left(u(B_{z_1+1} - B_{z_1})\right), f\left(u(B_{z_2+1} - B_{z_2})\right)\right)\mathbb{1}_{\{|z_1-z_2|<1\}}dz_1dz_2.$$

We note that

$$\lim_{m\to\infty} v_u^2(m) = v_u^2, \tag{II.3.18}$$

where v_u^2 has been introduced in (II.1.6).

In the next step we show condition (II.3.15). By the integration by parts formula we deduce the identity

$$\mathbb{E}[X_i^h(m)(B_{t_i^h(m)+(m+1)h} - B_{t_i^h(m)})|\mathcal{F}_{t_i^h(m)}]$$

$$= \int_{A_i(m)} \mathbb{E}\Big[f\Big(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\Big)\, h^{-1/2}(B_{x+h} - B_x)|\mathcal{F}_{t_i^h(m)}\Big]\,dx$$

$$= (mh)^{-1} w_{\sigma_{t_i^h(m)}}$$

where the function w_u has been defined in (II.2.6). This implies condition (II.3.15) by Riemann integrability.

To show condition (II.3.16), we observe the inequality

$$\mathbb{E}\left[|X_i^h(m)|^2 1_{\{|X_i^h(m)|>\varepsilon\}}|\mathcal{F}_{t_i^h(m)}\right] \leq \varepsilon^{-2}\mathbb{E}\left[|X_i^h(m)|^4|\mathcal{F}_{t_i^h(m)}\right] \leq C\varepsilon^{-2}m^4h^2.$$

Hence, we deduce the statement of (II.3.16).

To prove condition (II.3.17), we apply a martingale representation theorem to deduce the representation

$$X_i^h(m) = \int_{A_i(m)} \eta_{i,x}^{h,m} dB_x,$$

where $\eta_i^{h,m}$ is a predictable square integrable process. Now, applying Itô isometry, we obtain that

$$\mathbb{E}\left[X_i^h(m)(N_{t_i^h(m)+(m+1)h} - N_{t_i^h(m)})|\mathcal{F}_{t_i^h(m)}\right] = \mathbb{E}\left[\int_{A_i(m)} \eta_{i,x}^{h,m} d\langle B, N\rangle_x|\mathcal{F}_{t_i^h(m)}\right] = 0$$

Consequently, we showed condition (II.3.17).

Now, due to (II.3.14)-(II.3.17), we conclude the stable convergence $Z^{h,m,1}(f) \xrightarrow{d_{st}} U'^m(f)$ as $h \to 0$ with

$$U'^m(f)_t := c_m \int_{I_t} w_{\sigma_x} dB_x + \int_{I_t} \sqrt{v_{\sigma_x}^2(m) - c_m^2 w_{\sigma_x}^2} dW'_x.$$

On the other hand, since $c_m \to 1$ and $v_u^2(m) \to v_u^2$ as $m \to \infty$, we obtain that

$$U'^m(f) \xrightarrow{d_{st}} U'(f) = \int_{I.} w_{\sigma_x} dB_x + \int_{I.} \sqrt{v_{\sigma_x}^2 - w_{\sigma_x}^2} dW'_x \tag{II.3.19}$$

as $m \to \infty$.

II.3.3.2 Negligibility of the small blocks: the martingale term

Here we show that the small block term $R^{h,m,1}(f)$ is negligible. We recall

$$R_t^{h,m,1}(f) = h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \left(\beta_i^h(m) - \mathbb{E}[\beta_i^h(m)|\mathcal{F}_{t_i^h(m)+mh}]\right),$$

where

$$\beta_i^h(m) := \int_{B_i(m)} f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\right) dx.$$

Since $R^{h,m,1}(f)$ is a martingale, f has polynomial growth and σ is bounded, we conclude that

$$\mathbb{E}\left[|R_T^{h,m,1}(f)|^2 + |R_{-T}^{h,m,1}(f)|^2\right] \leq Cm^{-1}.$$

Hence, by Lemma II.3.1 we obtain that

$$\lim_{m\to\infty} \limsup_{h\to 0} \mathbb{P}\left(\sup_{t\in[-T,T]} |R_t^{h,m,1}(f)| > \varepsilon\right) = 0 \qquad \text{for any } \varepsilon > 0.$$

II.3.3.3 Riemann sum approximation error

We now consider the Riemann sum approximation error associated with big blocks. We need to show that

$$Z_t^{h,m,3}(f) = h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \int_{A_i(m)} \Big(\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f)\Big)\, dx \overset{u.c.p.}{\longrightarrow} 0.$$

(The corresponding statement for the small block term $R^{h,m,3}(f)$ is shown in exactly the same way). For this purpose we introduce the threshold

$$\varepsilon_h = h^r \qquad \text{for some } r \in (1/4, 1/2). \tag{II.3.20}$$

On each big block $A_i(m)$, we will distinguish two cases according to whether $L^{t_i^h(m)} < \varepsilon_h$ or $L^{t_i^h(m)} \geq \varepsilon_h$.
We start with the first case. Since $f \in C^1(\mathbb{R})$ the map $u \mapsto \rho_u(f)$ is C^1. Also note that $\sup_{u\in A} |\rho_u'(f)|$ is bounded if A is a compact set. Due to mean value theorem and boundedness of σ we have

$$\begin{aligned} &\mathbb{1}_{\left\{t_i^h(m)\in(\underline{S},\overline{S}),\ L^{t_i^h(m)}<\varepsilon_h\right\}} \int_{A_i(m)} \Big|\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f)\Big|\, dx \\ &\leq C\mathbb{1}_{\left\{t_i^h(m)\in(\underline{S},\overline{S}),\ L^{t_i^h(m)}<\varepsilon_h\right\}} \int_{A_i(m)} \Big|\sigma_x - \sigma_{t_i^h(m)}\Big|\, dx \end{aligned}$$

Now, we use Proposition II.3.3, inequality (II.3.3) as well as Hölder inequality with conjugates $p, q > 1$, $1/p + 1/q = 1$, to deduce that

$$\mathbb{E}\left[\mathbb{1}_{\left\{t_i^h(m)\in(\underline{S},\overline{S}),\ L^{t_i^h(m)}<\varepsilon_h\right\}} \Big|\sigma_x - \sigma_{t_i^h(m)}\Big|\right] \leq Ch^{1/4}\varepsilon_h^{1/q}.$$

Thus, we obtain that

$$\begin{aligned} &h^{-1/2}\mathbb{E}\left[\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in[-T,T]} \mathbb{1}_{\left\{t_i^h(m)\in(\underline{S},\overline{S}),\ L^{t_i^h(m)}<\varepsilon_h\right\}} \int_{A_i(m)} \Big|\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f)\Big|\, dx\right] \\ &\leq Ch^{-1/4}\varepsilon_h^{1/q} \to 0, \end{aligned} \tag{II.3.21}$$

where we use the definition at (II.3.20) and choose q close enough to 1.
Now, we treat the case $L^{t_i^h(m)} \geq \varepsilon_h$. For a fixed $m \in \mathbb{N}$, we conclude by Borel–Cantelli lemma and $\varepsilon_h = h^r$ for $r < 1/2$ that there exists a $h_0 > 0$ such that $\mathbb{P}$-almost surely

$$\sup_{x\in A_i(m)} |L^x - L^{t_i^h(m)}| \leq \varepsilon_h/2$$

for any $h < h_0$. In the scenario $L^{t_i^h(m)} \geq \varepsilon_h$ the latter implies that

$$\inf_{x\in A_i(m)} |L^x| \geq \varepsilon_h/2 \qquad \mathbb{P}\text{-almost surely,} \tag{II.3.22}$$

for $h < h_0$. We introduce the following process:

$$Z_t^{h,m,3.1} := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} 1_{\left\{L^{t_i^h(m)}\geq\varepsilon_h\right\}} \int_{A_i(m)} \left(\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f)\right) dx.$$

To handle the process $Z^{h,m,3.1}$ we will apply the decomposition (II.3.1). First of all, we use the mean value theorem to deduce that

$$\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f) = \rho'_{\sigma_{t_i^h(m)}}(f)(\sigma_x - \sigma_{t_i^h(m)}) + (\rho'_{\sigma_{x_i^h}}(f) - \rho'_{\sigma_{t_i^h(m)}}(f))(\sigma_x - \sigma_{t_i^h(m)}),$$

where x_i^h is a certain point in the interval $(t_i^h(m), x)$. Now, applying (II.3.1), we decompose $Z^{h,m,3.1} = Z^{h,m,3.2} + Z^{h,m,3.3} + Z^{h,m,3.4}$ as

$$Z_t^{h,m,3.2} := 2h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{1}_{\left\{L^{t_i^h(m)}\geq\varepsilon_h\right\}} \int_{A_i(m)} \rho'_{\sigma_{t_i^h(m)}}(f) \left(B_x - B_{\sigma_{t_i^h(m)}}\right) dx,$$

$$Z_t^{h,m,3.3} := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{1}_{\left\{L^{t_i^h(m)}\geq\varepsilon_h\right\}} \int_{A_i(m)} \rho'_{\sigma_{t_i^h(m)}}(f) \left(\int_{t_i^h(m)}^x \tilde{a}_y \cdot (L^y)^{-1/2} dy\right) dx,$$

$$Z_t^{h,m,3.4} := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{1}_{\left\{L^{t_i^h(m)}\geq\varepsilon_h\right\}} \int_{A_i(m)} (\rho'_{\sigma_{x_i^h}}(f) - \rho'_{\sigma_{t_i^h(m)}}(f))(\sigma_x - \sigma_{t_i^h(m)}) dx.$$

Since $Z^{h,m,3.2}$ is a martingale, we obtain that

$$\mathbb{E}\left[|Z_T^{h,m,3.2}|^2 + |Z_{-T}^{h,m,3.2}|^2\right] \leq C_T h.$$

By Lemma II.3.1 we deduce that

$$Z^{h,m,3.2} \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to 0. \tag{II.3.23}$$

Due to (II.3.22) we know that $(L^x)^{-1/2} < 2\varepsilon_h^{-1/2}$ for all $x \in A_i(m)$ and $h < h_0$. Since $\sigma, \tilde{a}$ are bounded, we conclude that

$$\mathbb{E}\left[\sup_{t\in[-T,T]} |Z_t^{h,m,3.3}|\right] \leq C_T h^{1/2} \varepsilon_h^{-1/2} \to 0 \qquad \text{as } h \to 0. \tag{II.3.24}$$

To handle the last term $Z^{h,m,3.4}$ we apply a similar technique as in step (c) of Section II.3.2. Notice that the quantity $\rho'_{\sigma_x}(f)$ is bounded, because σ is bounded. We obtain that

$$\left|\rho'_{\sigma_{x_i^h}}(f) - \rho'_{\sigma_{t_i^h(m)}}(f)\right| \leq C\left(w_{\rho'(f)}(A,\varepsilon) + \mathbb{1}_{\{|\sigma_{x_i^h} - \sigma_{t_i^h(m)}| > \varepsilon\}}\right).$$

Since $L^x \geq \varepsilon_h/2$ for all $x \in A_i(m)$ and $h < h_0$, we deduce from representation (II.3.1) that

$$\mathbb{E}\left[|\sigma_x - \sigma_{t_i^h(m)}|^p\right] \leq C\left(h^{p/2} + h^p \varepsilon_h^{-p/2}\right)$$

for any $p > 0$, for all $x \in A_i(m)$ and $h < h_0$. Hence, we now obtain from (II.3.3) that

$$\mathbb{E}\left[\sup_{t\in[-T,T]} |Z_t^{h,m,3.4}|\right] \leq C\left(w_{\rho'(f)}(A,\varepsilon)\left(1 + h^{1/2}\varepsilon_h^{-1/2}\right) + \varepsilon^{-3} h^{1/2}\right)$$

Thus, we conclude that

$$Z^{h,m,3.4} \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to 0. \tag{II.3.25}$$

A combination of (II.3.21) and (II.3.23)-(II.3.25) implies the statement

$$Z^{h,m,3}(f) \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to \infty.$$

Similarly, $R^{h,m,3}(f) \xrightarrow{u.c.p.} 0$ as $h \to 0$.

II.3.3.4 The terms $Z^{h,m,2}(f)$ and $R^{h,m,2}(f)$

In view of the previous steps, we are left with handling the terms $Z^{h,m,2}(f)$ and $R^{h,m,2}(f)$. We start with the term $Z^{h,m,2}(f)$. First, we consider an approximation of $Z^{h,m,2}(f)$ given as

$$\overline{Z}_t^{h,m,2}(f) := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \int_{A_i(m)} \mathbb{E}\left[f\left(h^{-1/2}(L^{x+h}-L^x)\right)\right.$$
$$\left. -f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)|\mathcal{F}_{t_i^h(m)}\right] dx.$$

Applying Lemma II.3.1 and following the same arguments as presented in part (c) of Section II.3.2 (see the proof of (II.3.9)), we deduce that

$$\overline{Z}^{h,m,2}(f) - Z^{h,m,2}(f) \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to 0. \tag{II.3.26}$$

We use again the mean value theorem to obtain the decomposition

$$\mathbb{E}\left[f\left(h^{-1/2}(L^{x+h}-L^x)\right) - f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)|\mathcal{F}_{t_i^h(m)}\right]$$
$$= \mathbb{E}\left[h^{-1/2} f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)\left((L^{x+h}-L^x) - \sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)|\mathcal{F}_{t_i^h(m)}\right]$$
$$+\mathbb{E}\left[h^{-1/2}\left(f'(z_i^h) - f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)\right)\right.$$
$$\left.\times\left((L^{x+h}-L^x) - \sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)|\mathcal{F}_{t_i^h(m)}\right],$$

where z_i^h is a point between $h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)$ and $h^{-1/2}(L^{x+h}-L^x)$. As in the previous subsection we need to discuss the cases $L^{t_i^h(m)} \geq \varepsilon_h$ and $L^{t_i^h(m)} < \varepsilon_h$ separately. The easier case $L^{t_i^h(m)} < \varepsilon_h$ is handled in exactly the same way as presented in (II.3.21), so we focus on the scenario $L^{t_i^h(m)} \geq \varepsilon_h$. Similarly to the treatment of $Z^{h,m,3.4}$ we conclude that

$$h^{-1/2}\sum_{i\in\mathbb{N}:\ i(m+1)h+mh<t} \mathbb{1}_{\{L^{t_i^h(m)}\geq\varepsilon_h\}} \int_{A_i(m)} \mathbb{E}\left[h^{-1/2}\left(f'(z_i^h) - f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)\right)\right.$$
$$\left.\times\left((L^{x+h}-L^x) - \sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)|\mathcal{F}_{t_i^h(m)}\right] dx \xrightarrow{u.c.p.} 0$$

as $h \to \infty$. Thus, we need to show that

$$\overline{Z}_t^{h,m,2.1} := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{1}_{\{L^{t_i^h(m)}\geq\varepsilon_h\}} \int_{A_i(m)} \mathbb{E}\Big[h^{-1/2} f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)$$
$$\times \left(\int_x^{x+h}(a_u - a_{t_i^h(m)})du + \int_x^{x+h}\left(\int_{t_i^h(m)}^u \tilde{a}_s\cdot(L^s)^{-1/2}ds\right)dB_u\right)|\mathcal{F}_{t_i^h(m)}\Big]\,dx \xrightarrow{u.c.p.} 0$$

and

$$\overline{Z}_t^{h,m,2.2} := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} 1_{\{L^{t_i^h(m)}\geq\varepsilon_h\}} \int_{A_i(m)} \mathbb{E}\Big[h^{-1/2} f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)$$
$$\times \left(h a_{t_i^h(m)} + 2\int_x^{x+h}(B_u - B_{t_i^h(m)})dB_u\right)|\mathcal{F}_{t_i^h(m)}\Big]\,dx \xrightarrow{u.c.p.} \frac{m}{m+1}\int_{I_t} r_{a_x,\sigma_x}dx \tag{II.3.27}$$

as $h\to 0$. The statement $\overline{Z}^{h,m,2.1} \xrightarrow{u.c.p.} 0$ is obtained along the lines of the arguments presented in the previous subsection. Finally, observe the identities

$$\mathbb{E}\left[f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)a_{t_i^h(m)}|\mathcal{F}_{t_i^h(m)}\right] = a_{t_i^h(m)}\rho_{\sigma_{t_i^h(m)}}(f')$$

and

$$2h^{-1}\int_{A_i(m)}\mathbb{E}\left[f'\left(h^{-1/2}u(B_{x+h}-B_x)\right)\int_x^{x+h}(B_u - B_{t_i^h(m)})dB_u\right]dx$$
$$= 2h^{-1}\int_{A_i(m)}\mathbb{E}\left[f'\left(h^{-1/2}u(B_{x+h}-B_x)\right)\int_x^{x+h}(B_u - B_x)dB_u\right]dx$$
$$= 2mh\int_0^1\mathbb{E}\left[f'\left(u(B_{y+1}-B_y)\right)\int_y^{y+1}(B_u - B_y)dB_u\right]dy$$
$$= 2mh\int_0^1\mathbb{E}\left[f'\left(u(B_{y+1}-B_y)\right)\int_0^2 B_u dB_u\right]dy$$
$$= mh\int_0^1\mathbb{E}\left[f'\left(u(B_{y+1}-B_y)\right)(B_2^2-2)\right]dy,$$

where we used the substitution $x = hy$ and the self-similarity of the Brownian motion. Hence, the convergence in (II.3.27) follows from Riemann integrability. Following exactly the same arguments we conclude that

$$\lim_{m\to\infty}\limsup_{h\to 0}\mathbb{P}\left(\sup_{t\in[-T,T]}|R_t^{h,m,2}(f)|>\varepsilon\right) = 0 \qquad \text{for any } \varepsilon>0.$$

This completes the proof of stable convergence $U(f)^h \xrightarrow{d_{st}} U(f)$.

II.3.4 Proof of Theorem II.1.2

Here we prove the statements of Theorem II.1.2 via an application of Theorem II.2.1 and Proposition II.2.2. Recall that we have already shown the convergence at (II.1.4); see (II.2.8). Thus we are left to proving the stable central limit theorem presented in Theorem II.1.2.

We recall that it suffices to show all convergence results under the restriction $\underline{S}, \overline{S} \in [-T,T]$ for some $T>0$. Theorem II.2.1 states that $U(f)^h \xrightarrow{d_{st}} U(f)$ on $(C([-T,T]), \|\cdot\|_\infty)$. Since the mapping $F : [-T,T]^2 \times C([-T,T]) \to \mathbb{R}$ defined as $F((t_1,t_2),H) := H(t_1)+H(t_2)$ is continuous, we deduce by the properties of stable convergence and (II.2.7):

$$\begin{aligned} U(f)^h_{\mathbb{R}} &\xrightarrow{d_{st}} U(f)_{\underline{S}} + U(f)_{\overline{S}} \\ &= \int_{\mathbb{R}} r_{a_x,\sigma_x} dx + \int_{\mathbb{R}} w_{\sigma_x} dB_x + \int_{\mathbb{R}} \sqrt{v^2_{\sigma_x} - w^2_{\sigma_x}} dW'_x, \end{aligned}$$

under conditions of Theorem II.2.1. Our task now boils down to demonstrating that the sum of the first two terms in the limit are equal to zero. This assertion has already been established in (II.2.11) under the condition $f(0)=0$, with $f \in C^3(\mathbb{R})$, and f and its first three derivatives exhibiting polynomial growth. Therefore, our focus shifts to confirming that this statement carries over under the weaker assumptions of Theorem II.1.2.

Let $f \in C^1(\mathbb{R})$ be an arbitrary function satisfying the conditions of Theorem II.1.2. Then there exists a sequence of functions $(f_n)_{n\geq 1} \in C^3(\mathbb{R})$ that fulfils the conditions $f_n(0)=0$,

$$|f_n(x)| + |f'_n(x)| + |f''_n(x)| + |f'''_n(x)| \leq C(1+|x|^p) \qquad \text{for some } p>0,$$

and

$$\sup_{x\in A} \left(|f_n(x) - f(x)| + |f'_n(x) - f'(x)|\right) \to 0 \qquad \text{as } n \to 0, \tag{II.3.28}$$

for any compact set $A \subset \mathbb{R}$. In view of Lemma II.3.2 it suffices to show that

$$U(f_n)_{\mathbb{R}} \xrightarrow{\mathbb{P}} U(f)_{\mathbb{R}} \qquad \text{as } n\to\infty, \qquad \text{and} \tag{II.3.29}$$

$$\lim_{n\to\infty} \limsup_{h\to 0} \mathbb{P}\left(\left|U(f_n)^h_{\mathbb{R}} - U(f)^h_{\mathbb{R}}\right| > \varepsilon\right) = 0 \qquad \text{for any } \varepsilon > 0. \tag{II.3.30}$$

We start by proving the statement (II.3.29). For this purpose we introduce the notation $r_{a_x,\sigma_x}(f)$, $w_{\sigma_x}(f)$ and $v_{\sigma_x}(f)$ to explicitly denote the dependence of these quantities on the function f. Since $\underline{S}, \overline{S} \in [-T,T]$ it suffices to prove the convergence

$$\mathbb{E}\left[\int_{-T}^{T} |r_{a_x,\sigma_x}(f_n) - r_{a_x,\sigma_x}(f)| + |w_{\sigma_x}(f_n) - w_{\sigma_x}(f)| + |v_{\sigma_x}(f_n) - v_{\sigma_x}(f)| dx\right] \to 0$$

as $n\to\infty$, to conclude (II.3.29). But the latter follows directly from (II.3.28) since the processes a and σ are bounded.

Now, we show condition (II.3.30). Applying Theorem II.2.1 we deduce that

$$\begin{aligned} \mathbb{P}\left(\left|U(f_n)^h_{\mathbb{R}} - U(f)^h_{\mathbb{R}}\right| > \varepsilon\right) &\leq \mathbb{P}\left(\sup_{t\in[-T,T]} \left|U(f_n)^h_t - U(f)^h_t\right| > \varepsilon\right) \\ &\to \mathbb{P}\left(\sup_{t\in[-T,T]} |U(f_n)_t - U(f)_t| > \varepsilon\right) \qquad \text{as } h \to 0. \end{aligned}$$

Using Markov and Burkholder inequalities, and the same arguments as in the proof of (II.3.29), we obtain that

$$\mathbb{P}\left(\sup_{t\in[-T,T]} |U(f_n)_t - U(f)_t| > \varepsilon\right) \to 0 \qquad \text{as } n \to \infty.$$

Thus we deduce (II.3.30), which completes the proof of Theorem II.1.2.

Chapter III

Limit theorems for asynchronously observed bivariate pure jump semimartingales

Abstract: In this article we establish the asymptotic theory of the Hayashi-Yoshida estimator for the quadratic covariation process of a bivariate pure jump process. Specifically, we consider the problem of estimating the quadratic covariation process for a bivariate symmetric β-stable processes $(L_t^1, L_t^2)_{t\geq 0}$, with $\beta \in (0, 2)$, from high frequency non-synchronous observations of $(L_t^1)_{t\geq 0}$ and $(L_t^2)_{t\geq 0}$. The main focus is on the derivation of a stable non-central limit theorem for the integrated covariance process. Building on the insight of [36] we use a method called *pseudo-aggregation* to handle the asynchronicity of the sampling scheme. Following the methodology of [58, 70], we will show that the limiting process in the stable non-central limit theorem is a β-stable process.

III.1 Introduction

Since Jacod's preliminary work more than 20 years ago [77], the asymptotic theory of statistics for Itô semimartingales has been studied in many different directions, with a predominant part of the literature focusing on the estimation of the power variation of Lévy processes in a high frequency regime, a quantity of crucial interest notably in financial applications.

The *realized quadratic variation process* of a $\mathbb{R}^d$-valued of an Itô semimartingale $(Y_t)_{t\geq 0}$, observed at time points $i\Delta_n$, $i = 0, ..., \lfloor t/\Delta_n \rfloor$, is defined as

$$[Y]_t^n := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n Y)(\Delta_i^n Y)^\intercal \quad \text{with } \Delta_i^n := Y_{i\Delta_n} - Y_{(i-1)\Delta_n}.$$

A well-known fact from the general theory of semimartingales is the convergence in probability

$$[Y]_t^n \xrightarrow{\mathbb{P}} [Y]_t$$

towards the quadratic variation process of Y. As a consequence of this convergence, a natural question arises: the establishment of an associated Central Limit Theorem, namely the convergence in distribution of the error process

$$U_t^n := \delta_n([Y]_t^n - [Y]_t) \xrightarrow{d} U_t$$

towards a non-trivial process $(U_t)_{t\geq 0}$, with $(\delta_n)_{n\geq 0}$ a suitable normalizing sequence.

In the Gaussian framework, namely when the Gaussian part of an Itô semimartingale Y is non-vanishing, the asymptotic theory is well-known and we can establish the convergence in law of the error process towards a non-trivial conditionally Gaussian limiting process (see e.g. [15, 24, 49, 130]). In the simplest case, consider a d-dimensional Itô semimartingale Y defined as

$$Y_t = Y_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s$$

with a a predictable locally bounded d-dimensional drift, σ a $\mathbb{R}^{d\times d}$-valued càdlàg volatility process and W a d-dimensional Brownian process.

Theorem III.1.1. *[15] Under previous assumptions on a and σ we obtain the following stable convergence*

$$\Delta_n^{1/2}\left([X]_t^n - [X]_t\right) \xrightarrow{d_{st}} M_t,$$

where, conditionally on $\mathcal{F}$, $(M_t)_{t\geq 0}$ is a Gaussian martingale with mean zero and conditional covariance function

$$\mathbb{E}\left[M_t^{jk} M_t^{j'k'} |\mathcal{F}\right] = c_t^{jj'} c_t^{kk'} + c_t^{jk'} c_t^{kj'} \quad \text{with } c_t = \sigma_t \sigma_t^\intercal.$$

Numerous extension has been studied, in particular in the presence of jumps (e.g. [24]), with noisy data (see e.g. [130]) or with asynchronous data (see e.g. [36, 67]).

Now, when considering pure jump Itô semimartingales, the probabilistic tools and assumptions on the model differ drastically from the Gaussian case. In the context of synchronous high-frequency data, the one-dimensional case has been demonstrated in a very general framework in [58] with

$$Y_t = \int_0^t \sigma_{s-} dZ_s + X_t,$$

where Z is a non-homogeneous Lévy process, σ an Itô semimartingale and Y an Itô semimartingale with vanishing Gaussian part. In the multivariate case, the following theorem has been proven in [70] for β-stable pure jump processes.

Theorem III.1.2. *[70] Let $\delta_n = (\Delta_n \log(1/\Delta_n))^{-1/\beta}$, $n \geq 1$. For any $\beta \in (0, 2)$ we obtain the functional stable convergence*

$$U_t^n := \delta_n \left([L]_t^n - [L]_{\Delta_n \lfloor t/\Delta_n \rfloor}\right) \xrightarrow{d_{st}} U_t,$$

where $(U_t)_{t\geq 0}$ is an $\mathbb{R}^{d\times d}$-valued Lévy process with characteristic triplet $(0, 0, \nu_U)$ with ν_U the Lévy measure given by

$$\nu_U(B) = \frac{1}{2\beta} \int_{\mathbb{S}_{d\times d}} \mu(dz) \int_{\mathbb{R}_+} \mathbb{1}_B(\rho z)\rho^{-1-\beta} d\rho, \quad B \in \mathcal{B}(\mathbb{R}^d \odot \mathbb{R}^d),$$

and

$$\mu(z) = \int_{\mathbb{S}_d^2} \mathbb{1}_z \left(\frac{\theta_1 \odot \theta_2}{||\theta_1 \odot \theta_2||}\right) ||\theta_1 \odot \theta_2||^\beta H(d\theta_1)H(d\theta_2), \quad z \in \mathcal{B}\left(\mathbb{S}_{d\times d}\right)$$

with B bounded away from 0. The process U is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{\mathbb{P}})$ of the original space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and is independant of the σ-algebra $\mathcal{F}$.

The aim of this paper is to provide similar result when considering a bivariate β-stable pure jump process L, with the particularity that we observe each component of L asynchronously. We will show a stable weak limit theorem for the Hayashi-Yoshida estimator of the quadratic covariation $[L^1, L^2]_t$. We refer to the sequel for a definition of stable convergence and of the Hayashi-Yoshida estimator. The limiting process will be a $\mathbb{R}$-valued symmetric β-stable process with a scaling parameter dependent of the asynchronous observation scheme.

In Section 2 we introduce the model and the main result. Section 3 collects some preliminary results for the proof of the main result. Section 4 is devoted to the proof of the main result in the case $\beta \in (1, 2)$ and Section 5 for the proof in the case $\beta \in (0, 1]$.

III.2 The model, notation and main results

III.2.1 Notation

In this subsection we introduce the main notations used throughout the paper. For $a \in \mathbb{C}$ we write $|a|$ to denote the norm of a. For a vector $x \in \mathbb{R}^2$ its transpose is denoted $x^\intercal$. The notation $\|x\|$ stands for the Euclidean norm of x. The set $\mathbb{S}_d$ denotes the Euclidean unit sphere in $\mathbb{R}^d$. Let $x = (x_1, x_2)^\intercal$, $y = (y_1, y_2)^\intercal \in \mathbb{R}^2$. We denote by $\langle x, y\rangle = x_1y_1 + x_2y_2$ the standard scalar product in $\mathbb{R}^2$.

The sequence $\Delta_n = O(1/n)$, $n \geq 1$, will be used in the sequel.

For two sequences of real numbers $(u_n)_n$ and $(v_n)_n$, we say that $(u_n)_n$ is asymptotically equivalent to $(v_n)_n$, and we denote this relation $u_n \sim v_n$ if $u_n/v_n \overset{n\to\infty}{\longrightarrow} 1$, provided $v_n \neq 0 \ \forall n$. In particular, if $u_n \sim v_n$, then they have the same limit.

Let $(Y_t)_{t\geq 0}$ be a càdlàg stochastic process. We denote by Y_{t-} the left limit of Y at time t and by $\Delta Y_t = Y_t - Y_{t-}$ the jump at t. Thoughout this paper $(t_i^k)_{i\in\mathbb{N}}$, $k = 1, 2$, is a sequence of positive numbers satisfying $0 \leq t_i^k < t_j^k \leq 1$ for all $i < j$ and $\sup_i |t_{i+1}^k - t_i^k| \to 0$.

We write $\Delta_{t_i^k} Y := Y_{t_i^k} - Y_{t_{i-1}^k}$ and $\Delta_i^n Y := Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$ with $(\Delta_n)_{n\geq 1}$ defined above.

For stochastic processes Y^n and Y we denote by $Y^n \overset{u.c.p.}{\longrightarrow} Y$ the uniform convergence in probability, that is

$$\sup_{t\in[0,T]} |Y_t^n - Y_t| \overset{\mathbb{P}}{\longrightarrow} 0 \quad \text{for any } T > 0.$$

We recall that a sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to converge stably with limit Y defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, denoted $Y_n \overset{d_{st}}{\longrightarrow} Y$, iff for any bounded continuous function g and any bounded $\mathcal{F}$-measurable random variable Z it holds that

$$\mathbb{E}\left[g(Y_n)Z\right] \longrightarrow \overline{\mathbb{E}}\left[g(Y)Z\right], \quad \text{as } n \to \infty.$$

Finally, we will consider $\mathbb{R}^2$-valued pure jump Lévy processes $(Y_t)_{t\geq 0} = (Y_t^1, Y_t^2)_{t\geq 0}$, without Gaussian part and drift. They are characterised by the Lévy triplet $(0, 0, \nu)$, namely

$$\mathbb{E}\left[\exp(i\langle u, Y_t\rangle)\right] = \exp\left(t\int_{\mathbb{R}^2} \{\exp(i\langle x, u\rangle) - 1 - i\langle x, u\rangle \mathbb{1}_{\{\|x\|\leq 1\}}\}\nu(\mathrm{d}x)\right)$$

for $u \in \mathbb{R}^2$ and ν a measure, called the Lévy measure, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^2}(1 \wedge \|x\|^2)\nu(\mathrm{d}x) < \infty$.

III.2.2 The setting

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. We define on this filtered probability space a 2-dimensional symmetric β-stable Lévy process $(L_t)_{t\geq 0}$ with Lévy triplet $(0, 0, G)$, where G denotes the Lévy measure of L and admits the representation

$$G(dx) = \frac{1}{\rho^{1+\beta}} d\rho H(d\theta)$$

where $x = (\rho, \theta) \in \mathbb{R}_+ \times \mathbb{S}_d$ and H denotes a symmetric finite measure on $\mathbb{S}_d$, called the *directional measure*.

Without loss of generality, we suppose that $(L_t)_{t\geq 0}$ is observed at high frequency over the time interval $[0, 1]$. More specifically, we suppose that L^k is observed at times $(t_i^k)_{0\leq i\leq n_k}$, $k = 1, 2$. We assume the following conditions on the observation times :

1. There exist strictly monotonic (deterministic) C^2 functions $f_k : [0, 1] \longrightarrow [0, 1]$ with non-zero right and left derivative in 0 and 1 respectively and with $f_k(0) = 0,\ f_k(1) = 1$ such that
$$t_i^k = f_k\left(\frac{i}{n_k}\right), \quad 0 \leq i \leq n_k, \quad k = 1, 2.$$

2. There exists a natural number $M > 0$ such that
$$\frac{1}{M} < \inf_{x\in[0,1]} f_k'(x) < \sup_{x\in[0,1]} f_k'(x) < M, \quad k = 1, 2.$$

3. Set $n = n_1 + n_2$. It holds that
$$\frac{n_k}{n} \longrightarrow m_k \in (0, 1], \quad k = 1, 2.$$

Note that with our notation, $n_1 = O(\Delta_n)$, $n_2 = O(\Delta_n)$. The main focus of this paper is the *Hayashi-Yoshida* estimator of the quadratic covariation, which is defined as

$$\widehat{[Y^1, Y^2]}_1^{HY} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \Delta_{t_i^1} Y^1 \Delta_{t_j^2} Y^2 \mathbb{1}_{\{(t_{i-1}^1, t_i^1]\cap(t_{j-1}^2, t_j^2]\neq\emptyset\}}$$

for any bivariate semimartingale $(Y_t)_{t\geq 0} = (Y_t^1, Y_t^2)_{t\geq 0}$. The main goal of this paper is to develop a fluctuation theorem for the Hayashi-Yoshida estimator applied to our process of interest. In the following we denote by U_1^n the error process defined as

$$U_1^n = \delta_n \left(\widehat{[L^1, L^2]}_1^{HY} - [L^1, L^2]_{\lfloor 1/t_{n_1}^1 \rfloor}\right)$$

where δ_n is a sequence that will be defined later on. Observe that we stop the true quadratic covariation at time $\lfloor 1/t_{n_1}^1 \rfloor$ (see Section 2.3.1).

III.2.3 Asynchronicity of the sampling scheme

III.2.3.1 Pseudo-aggregation

Following the methodology of [36, Section 3], to handle the asynchronicity of the data, we use a method called *pseudo-aggregation*. We can rewrite the estimator without indicator functions as it is sufficient to aggregate addends for which partial sums are telescoping. Suppose without loss of generality that $n_1 \leq n_2$ and consider L^1 as the reference process. Rewrite the Hayashi-Yoshida estimator by taking the sum of the products of all increments of L^1 with the telescoping sums of aggregated observed increments of L^2 where the observation time instants overlap with the according observation time instant of L^1. This aggregation procedure is possible due to the independence of non-overlapping increments of L^2.

For each $0 \leq i \leq n_1$ we define the next-tick interpolation $t_{i,+} := \min_{0\leq j\leq n_2} \{t_j^2 \mid t_j^2 \geq t_i^1\}$ and the previous-tick interpolation $t_{i,-} := \max_{0\leq j\leq n_2} \{t_j^2 \mid t_j^2 \leq t_i^1\}$. We can rewrite the Hayashi-Yoshida estimator as

$$\begin{aligned}
\widehat{[L^1, L^2]}_1^{HY} &= \sum_{i=1}^{n_1} \Delta_{t_i^1} L^1 \left(L^2_{t_{i,+}} - L^2_{t_{i-1,-}}\right) \\
&= \sum_{i=1}^{n_1} \Delta_{t_i^1} L^1 \left((L^2_{t_{i,+}} - L^2_{t_i^1}) + \Delta_{t_i^1} L^2 + (L^2_{t_{i-1}^1} - L^2_{t_{i-1,-}})\right) \\
&:= \sum_{i=1}^{n_1} \Delta_{t_i^1} L^1 \left(\Delta_{t_{i,+}} L^2 + \Delta_{t_i^1} L^2 + \Delta_{t_{i-1,-}} L^2\right)
\end{aligned}$$

where $\Delta_{t_{i,+}} L^2 = L^2_{t_{i,+}} - L^2_{t_i^1}$ and $\Delta_{t_{i-1,-}} L^2 = L^2_{t_{i-1}^1} - L^2_{t_{i-1,-}}$. In the sequel, for all $0 \leq i \leq n_1$, we denote the observation time instants by $\Delta_{t_i^1} = t_i^1 - t_{i-1}^1$, $\Delta_{t_{i,+}} = t_{i,+} - t_i^1$ and
$\Delta_{t_{i-1,-}} = t_{i-1}^1 - t_{i-1,-}$. Note that for each i these observation time instants are non-overlapping.

With this new expression for the Hayashi-Yoshida estimator we can rewrite the error process as

$$U_1^n = \delta_n \sum_{i=1}^{n_1} \Delta_{t_i^1} L^1 \Delta_{t_i^1} L^2 - \Delta_{t_i^1}[L^1, L^2] + \Delta_{t_i^1} L^1 \left(\Delta_{t_{i,+}} L^2 + \Delta_{t_{i-1,-}} L^2\right).$$

Remark III.2.1. In the sequel, the difference $\Delta_{t_i^1} L^1 \Delta_{t_i^1} L^2 - \Delta_{t_i^1}[L^1, L^2]$ will be called the *synchronous part* of the error process and the term $\Delta_{t_i^1} L^1 \left(\Delta_{t_{i,+}} L^2 + \Delta_{t_{i-1,-}} L^2\right)$ will be called *asynchronous part* of the error process. Indeed, if there is no asynchronicity in the sampling scheme, i.e. if $t_{i,-} = t_{i,+}$, then

$$U_1^n = \delta_n \sum_{i=1}^{n_1} \Delta_{t_i^1} L^1 \Delta_{t_i^1} L^2 - \Delta_{t_i^1}[L^1, L^2].$$

III.2.3.2 Asymptotic of the sampling scheme

For any $n \in \mathbb{N}$ we define the following sequences of functions, for $t \in [0, 1]$

$$F^n(t) = n \sum_{t_i^1 \leq t} (\Delta_{t_i^1})^2 \ ,$$

$$G^n(t) = n \sum_{t_{i,+} \le t} \Delta_{t_i^1} \left(\Delta_{t_{i,+}} + \Delta_{t_{i-1,-}} \right).$$

Taking the terminology from [36, Definition 2], we call $F^n(t)$ **sequence of quadratic variation of times** and $G^n(t)$ **sequence of quadratic covariation of times**. The sequence of functions G^n describes the interaction between the sampling scheme of the first process and the interpolation steps of the second.

To ensure the convergence of U^n we may assume the following :

Assumption A. For the sequences of quadratic (co)-variation of times, the following holds:

$$F^n(t) \to F(t) \quad \text{and} \quad G^n(t) \to G(t)$$

uniformly as $n \to \infty$, where F and G are continuous functions on $[0, 1]$.

Assumption (A) is general and can be verified for many sampling schemes, including e.g. Poisson random sampling scheme. However there exists sampling where the limit of F^n and/or G^n is not tractable, hence the need to assume (A) in these settings. In our setting, we can compute directly the limit of F^n and G^n.

Lemma III.2.2. *Under conditions* (1), (2) *and* (3) *on the sampling scheme, we have:*

$$F^n(1) \overset{n\to\infty}{\longrightarrow} \frac{1}{m_1} \int_0^1 (f_1'(t))^2 dt \quad \text{and} \quad G^n(1) \overset{n\to\infty}{\longrightarrow} \frac{1}{m_2} \int_0^1 f_1'(t) f_2'(t) dt.$$

Proof. Using a first order Taylor approximation, we obtain that

$$(\Delta_{t_i^1})^2 = \left(f_1\left(\frac{i}{n_1}\right) - f_1\left(\frac{i-1}{n_1}\right) \right)^2 = \frac{1}{n_1^2} \left(f_1'\left(\frac{i-1}{n_1}\right) \right)^2 + o\left(\frac{1}{n_1^2}\right)$$

and therefore by a Riemann sum argument

$$F^n(1) = n \sum_{i=1}^{n_1} (\Delta_{t_i^1})^2 = \frac{n}{n_1} \sum_{i=1}^{n_1} \frac{1}{n_1} \left(f_1'\left(\frac{i-1}{n_1}\right) \right)^2 + o(1) \overset{n\to\infty}{\longrightarrow} \frac{1}{m_1} \int_0^1 (f_1'(t))^2 dt.$$

We use a similar approach to compute the limit of G^n. Denote by $j_{i,-}$ the integer such that $t^2_{j_{i,-}} = t_{i,-}$ and by $j_{i,+}$ the integer such that $t^2_{j_{i,+}} = t_{i,+}$. Observe that by construction $t^2_{j_{i,+}} = t^2_{j_{i,-}+1}$. We have

$$\begin{aligned} n \sum_{i=1}^{n_1} \Delta_{t_i^1} \left(\Delta_{t_{i,+}} + \Delta_{t_{i-1,-}} \right) =& n \sum_{i=1}^{n_1} \Delta_{t_i^1} \left(\Delta_{t_{i,+}} + \Delta_{t_{i,-}} \right) + n \sum_{i=1}^{n_1-1} \left(\Delta_{t_{i+1}^1} - \Delta_{t_i^1} \right) \Delta_{t_{i,-}} \\ &+ n \left(\Delta_{t_1^1} \Delta_{t_{0,-}} + \Delta_{t_{n_1}^1} \Delta_{t^1_{n_1,-}} \right) \end{aligned}$$

and since $\Delta_{t_{0,-}} = 0$, $\Delta_{t_{n_1}^1} = O(\Delta_n)$ and $\Delta_{t^1_{n_1,-}} = O(\Delta_n)$ we get that

$$n \left(\Delta_{t_1^1} \Delta_{t_{0,-}} + \Delta_{t_{n_1}^1} \Delta_{t^1_{n_1,-}} \right) = o(1).$$

Using Taylor approximations, we observe that

$$\Delta_{t^1_{i+1}} - \Delta_{t^1_i} = \frac{1}{n_1} f_1'\left(\frac{i}{n_1}\right) - \frac{1}{n_1} f'\left(\frac{i-1}{n_1}\right) + o\left(\frac{1}{n_1}\right)$$
$$= \frac{1}{n_1^2} f_1''\left(\frac{i-1}{n_1}\right) + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_1^2}\right).$$

Since $\Delta_{t_{i,-}} = O(\Delta_n) = O(1/n_1)$ we obtain

$$n\sum_{i=1}^{n_1-1}\left(\Delta_{t^1_{i+1}} - \Delta_{t^1_i}\right)\Delta_{t_{i,-}} = n\sum_{i=1}^{n_1-1}\left(\frac{1}{n_1^2} f_1''\left(\frac{i-1}{n_1}\right) + o\left(\frac{1}{n_1}\right)\right)\Delta_{t_{i,-}}$$
$$= O(1/n_1) + o(1) = o(1)$$

and therefore

$$n\sum_{i=1}^{n_1}\Delta_{t^1_i}\left(\Delta_{t_{i,+}} + \Delta_{t_{i-1,-}}\right) = n\sum_{i=1}^{n_1}\Delta_{t^1_i}\left(\Delta_{t_{i,+}} + \Delta_{t_{i,-}}\right) + o(1).$$

Now, using again a first order Taylor approximation:

$$\Delta_{t_{i,+}} + \Delta_{t_{i,-}} = f_2\left(\frac{j_{i,-}+1}{n_2}\right) - f_1\left(\frac{i}{n_1}\right) + f_1\left(\frac{i}{n_1}\right) - f_2\left(\frac{j_{i,-}}{n_2}\right)$$
$$= \frac{1}{n_2} f_2'\left(\frac{j_{i,-}}{n_2}\right) + o\left(\frac{1}{n_2}\right),$$

hence, using a Riemann sum argument:

$$n\sum_{i=1}^{n_1}\Delta_{t^1_i}\left(\Delta_{t_{i,+}} + \Delta_{t_{i-1,-}}\right) = n\sum_{i=1}^{n_1}\Delta_{t^1_i}\left(\Delta_{t_{i,+}} + \Delta_{t_{i,-}}\right) + o(1)$$
$$= n\sum_{i=1}^{n_1}\left(\frac{1}{n_1} f_1'\left(\frac{i-1}{n_1}\right) + o\left(\frac{1}{n_1}\right)\right)\left(\frac{1}{n_2} f_2'\left(\frac{j_{i,-}}{n_2}\right) + o\left(\frac{1}{n_2}\right)\right)$$
$$= \frac{n}{n_2}\sum_{i=1}^{n_1}\frac{1}{n_1} f_1'\left(\frac{i-1}{n_1}\right) f_2'\left(\frac{j_{i,-}}{n_2}\right) + o(1)$$
$$\overset{n\to\infty}{\longrightarrow} \frac{1}{m_2}\int_0^1 f_1'(t) f_2'(t)dt.$$

□

Examples :

- Synchronous case : $n_1 = n_2 = n$ and $t^1_i = t^2_i = f\left(\frac{i}{n}\right)$. Then

$$F^n(1) \to \int_0^1 f'(t)^2 dt \quad \text{and} \quad G^n(1) = 0.$$

- Perfect asynchronicity scheme : $n_1 = n_2$, $t^1_i = \frac{i}{n}$ and $t^2_i = \frac{i}{n} + \frac{1}{2n}$. In that case,

$$F^n(1) \to 1 \quad \text{and} \qquad G^n(1) \to 1.$$

III.2.4 Limit theorem

In this section we introduce the asymptotic theory for the Hayashi-Yoshida estimator applied to a symmetric β-stable bivariate Lévy process. We denote

$$\delta_n = (\Delta_n \log(1/\Delta_n))^{-1/\beta}, \quad n \geq 1,$$

which will be proven to be the rate of convergence of the error process. The following theorem holds.

Theorem III.2.3. *For any $\beta \in (0,2)$, we obtain the stable convergence*

$$U_1^n = \delta_n \left(\widehat{[L^1, L^2]}_1^{HY} - [L^1, L^2]_1 \right) \xrightarrow{d_{st}} U_1$$

where $(U_t)_{t\geq 0}$ is an $\mathbb{R}$-valued symmetric β-stable process with $U_1 \stackrel{d}{=} S_\beta(c,0,0)$ a symmetric β-stable distribution with characteristic exponent

$$\varphi_{S_\beta(c,0,0)}(t;\beta,c,0,0) = \exp(-c|t|^\beta).$$

The scaling parameter c is defined as

$$c := \frac{\sigma_\beta^0}{m_1} \int_0^1 (f_1'(t))^2 dt + \frac{2\sigma_\beta^1}{m_2} \int_0^1 f_1'(t) f_2'(t) dt$$

with

$$\sigma_\beta^0 := \begin{cases} \dfrac{-\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)}{2\beta} \displaystyle\int_{\mathbb{S}_2^2} \left|\theta_1^1\theta_2^2 + \theta_1^2\theta_2^1\right|^\beta H(d\theta_1)H(d\theta_2) & \text{if } \beta \in (0,1)\cup(1,2), \\ \dfrac{\pi}{4} \displaystyle\int_{\mathbb{S}_2^2} \left|\theta_1^1\theta_2^2 + \theta_1^2\theta_2^1\right| H(d\theta_1)H(d\theta_2) & \text{if } \beta = 1, \end{cases}$$

$$\sigma_\beta^1 := \begin{cases} \dfrac{-\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)}{2\beta} \displaystyle\int_{\mathbb{S}_2^2} \left|\theta_1^2\theta_2^1\right|^\beta H(d\theta_1)H(d\theta_2) & \text{if } \beta \in (0,1)\cup(1,2), \\ \dfrac{\pi}{4} \displaystyle\int_{\mathbb{S}_2^2} \left|\theta_1^2\theta_2^1\right| H(d\theta_1)H(d\theta_2) & \text{if } \beta = 1. \end{cases}$$

$\mathbb{S}_2$ denotes the unit sphere on $\mathbb{R}^2$ with respect to the Euclidean norm and $\theta_i = (\theta_i^1, \theta_i^2)$, $i = 1,2$. Moreover, the process U is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{\mathbb{P}})$ of the original space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 1}, \mathbb{P})$ and is independent of the σ-algebra $\mathcal{F}$.

III.3 Proof of Theorem III.2.3

In the sequel we denote by C all positive constants appearing in the proofs, although they may change from line to line. The idea behind the proof of Theorem III.2.3 is to show the functional weak convergence (with respect to the Skorokhod J_1-topology)

$$\left((L^1_{t^1_{n_1}\lfloor t/t^1_{n_1}\rfloor}, L^2_{t^2_{n_2}\lfloor t/t^2_{n_2}\rfloor}), U_t^n \right) \xrightarrow{d} \left((L_t^1, L_t^2), U_t \right),$$

which implies the functional stable convergence $U^n \xrightarrow{d_{st}} U$, according to e.g. [58, Lemma 6.9]. The laws of $(L^1_{t^1_{n_1}\lfloor\cdot/t^1_{n_1}\rfloor}, L^2_{t^2_{n_2}\lfloor\cdot/t^2_{n_2}\rfloor})$ and U^n factorize asymptotically, cf [58, Proof of Proposition 7.3], which guarantees the independence of the limits L and U. Hence we need to prove the functional weak convergence

$$U^n \xrightarrow{d} U,$$

which we will show in the following.

III.3.1 Main decompositions

We are following the decompositions proposed in [70], adapted to the bivariate setting and the observation scheme. Following the argument in [58, Assumption S2], we can truncate the original Lévy measure specified in (2.1) and work with G restricted to the unit ball $\{\|x\| \leq 1\}$ instead of the original Lévy measure. We assume now that

$$G(dx) = \frac{\mathbb{1}_{(0,1]}(\rho)}{\rho^{1+\beta}} d\rho H(d\theta), \quad x = (\rho, \theta) \in \mathbb{R}_+ \times \mathbb{S}_2.$$

The main decomposition is given by

$$L = M(v) + A(v), \quad v \in (0,1),$$

where $A(v)_t = \sum_{s\leq t} \Delta L_s \mathbb{1}_{\{\|\Delta L_s\|>v\}}$, which corresponds to the classical Lévy-Itô decomposition. When $\beta > 1$, $M(v)$ is a martingale. Denote by $\Delta_n := \sup_{k=1,2} |t^k_{i,n_k} - t^k_{i-1,n_k}|$. We set

$$v_n = \begin{cases} \Delta_n^{1/(2\beta)} \log(1/\Delta_n), & \text{if } \beta > 1 \\ (\Delta_n \log(1/\Delta_n))^{1/\beta}, & \text{if } \beta \leq 1 \end{cases}$$

and define $M^n = M(v_n)$, $A^n = A(v_n)$. The process M^n has the Lévy triplet $(0, 0, G(dx)\mathbb{1}_{\{\|x\|\leq v_n\}})$ and A^n is a compound Poisson process with intensity $\overline{G}(v_n) := G(\{x \in \mathbb{R}^2 \ : \ v_n < \|x\| \leq 1\})$ and jump distribution $G(dx)\mathbb{1}_{\{v_n<\|x\|\leq 1\}}/\overline{G}(v_n)$. Note that A^n and M^n are independent.

We recall that

$$U^n_t = \delta_n \sum_{i=1}^{n_1} \underbrace{\Delta_{t^1_i} L^1 \Delta_{t^1_i} L^2 - \Delta_{t^1_i}[L^1, L^2]}_{\text{synchronous part}} + \overbrace{\Delta_{t^1_i} L^1 \left(\Delta_{t_{i,+}} L^2 + \Delta_{t_{i-1,-}} L^2\right)}^{\text{asynchronous part}}.$$

We further decompose U^n in terms of M^n and A^n and according to the number of jumps within the observation time instants $\Delta_{t^1_i}$, $\Delta_{t_{i,+}}$ and $\Delta_{t_{i-1,-}}$. We write $M^n = (M^{n,1}, M^{n,2})$ and $A^n = (A^{n,1}, A^{n,2})$. We denote by τ_i (resp. $\tau_{i,+}$, $\tau_{i-1,-}$) the number of jumps within $\Delta_{t^1_i}$ (resp. $\Delta_{t_{i,+}}$ and $\Delta_{t_{i-1,-}}$). We also denote by $T(n,i)_j$ (resp. $T(n,i,+)_j$, $T(n,i-1,-)_j$) the time of the j-th jump within $\Delta_{t^1_i}$ (resp. $\Delta_{t_{i,+}}$ and $\Delta_{t_{i-1,-}}$). Due to Itô formula, we have that

$$U_t^n = \sum_{i=1}^{n_1} \xi_i^n$$

with

$$\xi_i^n = \delta_n \Bigg(\underbrace{\int_{t_{i-1}^1}^{t_i^1} \left(L_{s-}^1 - L_{t_{i-1}^1}^1 \right) dL_s^2 + \int_{t_{i-1}^1}^{t_i^1} \left(L_{s-}^2 - L_{t_{i-1}^1}^2 \right) dL_s^1}_{\text{synchronous part}} + \overbrace{\Delta_{t_i^1} L^1 \left(\Delta_{t_{i,+}} L^2 + \Delta_{t_{i-1,-}} L^2 \right)}^{\text{asynchronous part}} \Bigg).$$

More specifically, we have that $\xi_i^n = \sum_{j=1}^{13} \xi_i^n(j)$ with

$$\begin{aligned}
\xi_i^n(1) &= \delta_n \left(\Delta_{t_i^1} M^{n,1} \Delta L^2_{T(n,i)_1} + \Delta L^1_{T(n,i)_1} \Delta_{t_i^1} M^{n,2} \right) \mathbb{1}_{\{\tau_i = 1\}} \\
\xi_i^n(2) &= \delta_n \left(\Delta L^1_{T(n,i)_1} \Delta L^2_{T(n,i)_2} + \Delta L^1_{T(n,i)_2} \Delta L^2_{T(n,i)_1} \right) \mathbb{1}_{\{\tau_i = 2\}} \\
\xi_i^n(3) &= \delta_n \left(\Delta_{t_i^1} M^{n,1} \Delta_{t_i^1} A^{n,2} + \Delta_{t_i^1} A^{n,1} \Delta_{t_i^1} M^{n,2} \right) \mathbb{1}_{\{\tau_i \geq 2\}} \\
\xi_i^n(4) &= \delta_n \left(\int_{t_{i-1}^1}^{t_i^1} (M^{n,1}_{s-} - M^{n,1}_{t_{i-1}^1}) dM^{n,2}_s + \int_{t_{i-1}^1}^{t_i^1} (M^{n,2}_{s-} - M^{n,2}_{t_{i-1}^1}) dM^{n,1}_s \right) \\
\xi_i^n(5) &= \delta_n \left(\int_{t_{i-1}^1}^{t_i^1} (A^{n,1}_{s-} - A^{n,1}_{t_{i-1}^1}) dA^{n,2}_s + \int_{t_{i-1}^1}^{t_i^1} (A^{n,2}_{s-} - A^{n,2}_{t_{i-1}^1}) dA^{n,1}_s \right) \mathbb{1}_{\{\tau_i \geq 3\}} \\
\xi_i^n(6) &= \delta_n \Delta_{t_i^1} M^{n,1} \left(\Delta_{t_{i,+}} M^{n,2} + \Delta_{t_{i-1,-}} M^{n,2} \right) \\
\xi_i^n(7) &= \delta_n \Delta_{t_i^1} M^{n,1} \left(\Delta L^2_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+} = 1\}} + \Delta L^2_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-} = 1\}} \right) \\
\xi_i^n(8) &= \delta_n \Delta_{t_i^1} M^{n,1} \left(\Delta_{t_{i,+}} A^{n,2} \mathbb{1}_{\{\tau_{i,+} \geq 2\}} + \Delta_{t_{i-1,-}} A^{n,2} \mathbb{1}_{\{\tau_{i-1,-} \geq 2\}} \right) \\
\xi_i^n(9) &= \delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i = 1\}} \left(\Delta_{t_{i,+}} M^{n,2} + \Delta_{t_{i-1,-}} M^{n,2} \right) \\
\xi_i^n(10) &= \delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i = 1\}} \left(\Delta L^2_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+} = 1\}} + \Delta L^2_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-} = 1\}} \right) \\
\xi_i^n(11) &= \delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i = 1\}} \left(\Delta_{t_{i,+}} A^{n,2} \mathbb{1}_{\{\tau_{i,+} \geq 2\}} + \Delta_{t_{i-1,-}} A^{n,2} \mathbb{1}_{\{\tau_{i-1,-} \geq 2\}} \right) \\
\xi_i^n(12) &= \delta_n \Delta_{t_i^1} A^{n,1} \mathbb{1}_{\{\tau_i \geq 2\}} \left(\Delta_{t_{i,+}} M^{n,2} + \Delta_{t_{i-1,-}} M^{n,2} \right) \\
\xi_i^n(13) &= \delta_n \Delta_{t_i^1} A^{n,1} \mathbb{1}_{\{\tau_i \geq 2\}} \left(\Delta_{t_{i,+}} A^{n,2} \mathbb{1}_{\{\tau_{i,+} \geq 1\}} + \Delta_{t_{i-1,-}} A^{n,2} \mathbb{1}_{\{\tau_{i-1,-} \geq 1\}} \right).
\end{aligned}$$

We will see that $\xi_i^n(1) + \xi_i^n(7) + \xi_i^n(9)$ represents the dominating part when $\beta > 1$, while $\xi_i^n(2) + \xi_i^n(10)$ is dominating when $\beta \leq 1$.

Remark III.3.1. Observe that the terms $\xi_i^n(j)$, $j = 1, 2, 3, 4, 5$ come from the synchronous part of the error process. They are very similar in nature to the terms of the decomposition that appears in [70, Section 4.1], the difference being that they consider a sampling scheme with synchronous data (i.e. all components of the process are observed at the same time points) and regularly spaced time points $t_i^n = i\Delta_n$. The assumption of regularly spaced time points is not crucial in the proof and can be omitted, as long as all the observation time instants (i.e. the difference between two consecutives times points) have the same order Δ_n, which is the case in our setting. It follows that these five terms in our setting will have the same order as the five terms that appear in the decomposition of [70, Section 4.1].

III.3.2 Preliminary results

In this section we list a few technical results required to demonstrate Theorem III.2.3. The first result gives some conditions to ensure negligibility of certain partial sums [58, Lemma 6.6].

Lemma III.3.2. *Let ξ_i^n be real-valued $\mathcal{F}_{i\Delta_n}$-measurable random variables. Then each of the following conditions implies the uniform convergence $\sum_{i=1}^{n_k}\xi_i^n \overset{u.c.p.}{\longrightarrow} 0$:*

$$\sum_{i=1}^{n_k}\mathbb{E}[|\xi_i^n|\wedge 1]\longrightarrow 0, \tag{III.3.1}$$

$$\sum_{i=1}^{n_k}\mathbb{E}[\xi_i^n|\mathcal{F}_{(i-1)\Delta_n}]\overset{u.c.p.}{\longrightarrow} 0 \quad \textit{and} \quad \sum_{i=1}^{n_k}\mathbb{E}[(\xi_i^n)^2]\longrightarrow 0, \tag{III.3.2}$$

$$\left.\begin{array}{l}\sum_{i=1}^{n_k}\mathbb{E}\left[\xi_i^n\mathbb{1}_{\{|\xi_i^n|\le 1\}}|\mathcal{F}_{(i-1)\Delta_n}\right]\overset{u.c.p.}{\longrightarrow} 0\\ \sum_{i=1}^{n_k}\mathbb{E}\left[(\xi_i^n)^2\mathbb{1}_{\{|\xi_n^i|\le 1\}}\right]\longrightarrow 0,\ \sum_{i=1}^{n_k}\mathbb{P}\left(|\xi_i^n|>1\right)\longrightarrow 0.\end{array}\right\} \tag{III.3.3}$$

The next lemma provides us some inequalities for the moments of M^n. Both of them comes from [83, Proposition 2.1.10].

Lemma III.3.3. *Let W be a predictable $\mathbb{R}$-valued process and $u>0$ fixed. Then it holds that*

$$\mathbb{E}\left[\sup_{s\le t}\left|\int_u^{u+s}W_s dM_s(v_n)\right|^p\right]\le Cv_n^{p-\beta}\mathbb{E}\left[\int_u^{u+t}|W_s|^p\,ds\right], \tag{III.3.4}$$

$$\mathbb{E}\left[\sup_{s\le t}|M_{u+s}(v_n)-M_u(v_n)|^p\right]\le Ctv_n^{p-\beta}, \tag{III.3.5}$$

for $1\le\beta<p\le 2$ when $\beta\ge 1$ and $\beta<p\le 1$ if $\beta<1$.

Below we collect some estimates on the jumps of the process $A(v_n)$. They follow from [58, Lemma 6.2 and 6.3].

Lemma III.3.4. *Let $w>0$, $b=H(\mathbb{S}_2)/\beta$ and recall the definition of τ_i and $T(n,i)_j$.*

- *For any $1\le j\le m$, it holds on the set $\{\tau_i\ge j-1\}$ that*

$$\mathbb{P}(\tau_i\ge m)\le C\left(\Delta_n/v_n^\beta\right)^{m-j+1}. \tag{III.3.6}$$

- *For any $1\le j\le m$, it holds on the set $\{\tau_i\ge j-1\}$ that*

$$\mathbb{E}\left[\left(|\Delta L_{T(n,i)_k}|\wedge w\right)^p\mathbb{1}_{\{\tau_i\ge m\}}\right]\le C\begin{cases}\Delta_n\left(\frac{b\Delta_n}{v_n^\beta}\right)^{m-j}w^{p-\beta} & \textit{for } p>\beta\\ \Delta_n\left(\frac{b\Delta_n}{v_n^\beta}\right)^{m-j}\log(1/\Delta_n) & \textit{for } p=\beta\\ \Delta_n\left(\frac{b\Delta_n}{v_n^\beta}\right)^{m-j}v_n^{p-\beta} & \textit{for } p<\beta.\end{cases} \tag{III.3.7}$$

- *For any $1 \leq j \leq k < r \leq m$, it holds on the set $\{\tau_i \geq j-1\}$ that*

$$\mathbb{E}\left[\left(|\Delta L_{T(n,i)_k}||\Delta L_{T(n,i)_r}| \wedge w\right)^p \mathbb{1}_{\{\tau_i \geq m\}}\right] \leq C \begin{cases} \Delta_n^2 \log(1/\Delta_n) \left(\frac{b\Delta_n}{v_n^\beta}\right)^{m-j-1} w^{p-\beta}, & p > \beta \\ \Delta_n^2 \left(\log(1/\Delta_n)\right)^2 \left(\frac{b\Delta_n}{v_n^\beta}\right)^{m-j-1}, & p = \beta. \end{cases} \tag{III.3.8}$$

Next lemma is a result from [69, Lemma 1 and 2] that provides an approximation for the characteristic function of a sequence of 1-dependent random variables.

Lemma III.3.5. *Let X_1, X_2, ... be a sequence of 1-dependent random variables. If $\max_{1\leq k\leq n} \mathbb{E}\left[|e^{itX_k}-1|\right] \leq 1/36$, then*

$$\Big| \log\left(\mathbb{E}\left[e^{it\sum_{k=1}^n X_k} - 1\right]\right) - \sum_{k=1}^n \mathbb{E}\left[e^{itX_k}-1\right] - \sum_{k=2}^n \mathbb{E}\left[\left(e^{itX_{k-1}}-1\right)\left(e^{itX_k}-1\right)\right]$$
$$- \sum_{k=3}^n \mathbb{E}\left[\left(e^{itX_{k-2}}-1\right)\left(e^{itX_{k-1}}-1\right)\left(e^{itX_k}-1\right)\right] \Big| \leq C \max_{1\leq k\leq n} \left|\mathbb{E}\left[e^{itX_k}-1\right]\right| \sum_{k=1}^n \left|\mathbb{E}\left[e^{itX_k}-1\right]\right|.$$

Moreover we have the following estimates

$$\left|\mathbb{E}\left[\left(e^{itX_1}-1\right)\left(e^{itX_2}-1\right)\left(e^{itX_3}-1\right)\right]\right| \leq \left(\mathbb{E}\left[|e^{itX_1}-1|^2\right]\right)^{3/2} \tag{III.3.9}$$
$$\mathbb{E}\left[|e^{itX_1}-1|^2\right] \leq 2\left|\mathbb{E}\left[e^{itX_1}-1\right]\right| \tag{III.3.10}$$

The following lemma is a statement of complex analysis that comes from [58, Lemma 6.7].

Lemma III.3.6. *Let a_i^n be complex numbers such that*

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} a_i^n \overset{u.c.p.}{\longrightarrow} g(t)$$

where g is a complex-valued continuous function. Then

$$\prod_{i=1}^{\lfloor t/\Delta_n \rfloor} (1+a_i^n) \overset{u.c.p.}{\longrightarrow} \exp(g(t)).$$

Remark III.3.7. Let $(\xi_i^n)_{1\leq i\leq n}$ be a sequence of i.i.d. random variables and let $\Gamma^n = \sum_{i=1}^{\lfloor 1/\Delta_n \rfloor} \xi_i^n$. Throughout the paper we will use the following criterion to show the convergence of Γ^n towards a symmetric stable process Γ with Lévy triplet $(0,0,\nu_\Gamma)$ and stability index $\beta \in (0,2)$ (see [58, Lemma 6.8]). We need to show that

$$\prod_{i=1}^{\lfloor 1/\Delta_n \rfloor} \mathbb{E}\left[e^{it\xi_i^n}\right] \overset{u.c.p.}{\longrightarrow} \begin{cases} \exp\left\{\int \left\{e^{itx}-1-itx\right\}\nu_\Gamma(dx)\right\} & \text{if} \quad \beta \in (1,2) \\ \exp\left\{\int \left\{e^{itx}-1\right\}\nu_\Gamma(dx)\right\} & \text{if} \quad \beta \in (0,1) \\ \exp\left\{\int \left\{e^{itx}-1-itx\mathbb{1}_{\{0<|x|\leq 1\}}\right\}\nu_\Gamma(dx)\right\} & \text{if} \quad \beta = 1 \end{cases}$$

or equivalently, using Lemma III.3.6

$$\sum_{i=1}^{\lfloor 1/\Delta_n \rfloor} \mathbb{E}\left[e^{it\xi_i^n} - 1\right] \xrightarrow{u.c.p.} \begin{cases} \int \left\{e^{itx} - 1 - itx\right\} \nu_\Gamma(dx) & \text{if} \quad \beta \in (1,2) \\ \int \left\{e^{itx} - 1\right\} \nu_\Gamma(dx) & \text{if} \quad \beta \in (0,1) \\ \int \left\{e^{itx} - 1 - itx\mathbb{1}_{\{0<|x|\leq 1\}}\right\} \nu_\Gamma(dx) & \text{if} \quad \beta = 1. \end{cases}$$

III.4 Proof of Theorem III.2.3 in the case $\beta \in (1,2)$

III.4.1 Negligible terms

We start by the terms $\xi_i^n(j)$ for $j = 2,3,4,5$. Observe that these terms come from the synchronous part of the decomposition of the error process U_1^n. Combined with Remark III.3.1 we can use the result of [70, Section 4.3.1] to show their negligibility, since asynchronicity does not play any role for these terms and since the observation time instants don't need to be identical but only comparable in size, which is the case here where $\Delta_{t_i^1} = O(\Delta_n)$ for all $1 \leq i \leq n_1$. From this, we have that $\xi_i^n(j)$, $j = 2,3,4,5$ is negligible. For $j = 12,13$ we have

$$\mathbb{E}\left[|\xi_i^n(j)| \wedge 1\right] \leq \mathbb{P}(\tau_i \geq 2) \leq C\Delta_n(\log(1/\Delta_n))^{-2\beta}$$

by (III.3.6) applied to $m = 2$. Hence we conclude that $\sum_{i=1}^{n_1} \mathbb{E}\left[|\xi_i^n(j)| \wedge 1\right] \to 0$ and we deduce by (III.3.1) that

$$\sum_{i=1}^{n_1} \xi_i^n(j) \xrightarrow{u.c.p.} 0, \quad j = 12, 13.$$

For $j = 8,11$ by (III.3.6) applied to $m = 2$ we have

$$\mathbb{E}\left[|\xi_i^n(j)| \wedge 1\right] \leq \mathbb{P}(\tau_{i,+} \geq 2) + \mathbb{P}(\tau_{i-1,-} \geq 2) \leq C\Delta_n(\log(1/\Delta_n))^{-2\beta}.$$

It follows that $\sum_{i=1}^{n_1} \mathbb{E}[|\xi_i^n| \wedge 1] \to 0$ and by (III.3.1)

$$\sum_{i=1}^{n_1} \xi_i^n(j) \xrightarrow{u.c.p.} 0.$$

We now consider the case $j = 6$. We first have $\xi_i^n(6) = \eta_{i,-}^n + \eta_{i,+}^n$ with

$$\eta_{i,-}^n := \delta_n \Delta_{t_i^1} M^{n,1} \Delta_{t_{i-1,-}} M^{n,2}, \quad \eta_{i,+}^n := \delta_n \Delta_{t_i^1} M^{n,1} \Delta_{t_{i,+}} M^{n,2}.$$

We have

$$\mathbb{E}[\eta_{i,-}^n | \mathcal{F}_{t_{i-1}^1}] = \delta_n \mathbb{E}[\Delta_{t_i^1} M^{n,1}] \mathbb{E}[\Delta_{t_{i-1,-}} M^{n,2} | \mathcal{F}_{t_{i-1}^1}] = 0.$$

On the other hand, using the independence of $\Delta_{t_i^1} M^{n,1}$ and $\Delta_{t_{i-1,-}} M^{n,2}$ we obtain that

$$\mathbb{E}[(\eta_{i,-}^n)^2] = \delta_n^2 \mathbb{E}[(\Delta_{t_i^1} M^{n,1})^2] \mathbb{E}[(\Delta_{t_{i-1,-}} M^{n,2})^2].$$

By (III.3.5) applied to $p = 2$ we get that

$$\mathbb{E}[(\eta_{i,-}^n)^2] \leq C\Delta_n^2\delta_n^2 v_n^{4-2\beta} = C\Delta_n(\log(1/\Delta_n))^{4-2(\beta+1/\beta)}.$$

Since $\beta + 1/\beta > 2$ for $\beta > 1$ we obtain that

$$\sum_{i=1}^{n_1} \mathbb{E}[(\eta_{i,-}^n)^2] \longrightarrow 0.$$

Therefore, by condition (III.3.2) we have

$$\sum_{i=1}^{n_1} \eta_{i,-}^n \xrightarrow{u.c.p.} 0.$$

We consider now the term $\eta_{i,+}^n$. As before we have

$$\mathbb{E}[\eta_{i,+}^n | \mathcal{F}_{t_i^1}] = 0$$

and

$$\mathbb{E}[(\eta_{i,+}^n)^2] \leq C\Delta_n^2\delta_n^2 v_n^{4-2\beta} = C\Delta_n(\log(1/\Delta_n))^{4-2(\beta+1/\beta)}$$

by (III.3.5) applied to $p = 2$. Therefore by condition (III.3.2) we have

$$\sum_{i=1}^{n_1} \eta_{i,+}^n \xrightarrow{u.c.p.} 0.$$

Finally we get that

$$\sum_{i=1}^{n_1} \xi_i^n(6) \xrightarrow{u.c.p.} 0.$$

It remains to prove that $\xi_i^n(10)$ is asymptotically negligible. We have

$$\begin{aligned}
&\mathbb{E}\left[\left|\delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}} \left(\Delta L^2_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} + \Delta L^2_{T(n,i-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}}\right)\right| \wedge 1\right] \\
&\leq \mathbb{E}\left[\left|\delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}} \Delta L^2_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}}\right| \wedge 1\right] \\
&\qquad + \mathbb{E}\left[\left|\delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}} \Delta L^2_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}}\right| \wedge 1\right] \\
&\leq \mathbb{E}\left[\left|\delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}} \Delta L^2_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}}\right|\right] \wedge 1 \\
&\qquad + \mathbb{E}\left[\left|\delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}} \Delta L^2_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}}\right|\right] \wedge 1 \\
&= \left(\mathbb{E}\left[\left|\delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}}\right|\right] \mathbb{E}\left[\left|\Delta L^2_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}}\right|\right]\right) \wedge 1 \\
&\qquad + \left(\mathbb{E}\left[\left|\delta_n \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}}\right|\right] \mathbb{E}\left[\left|\Delta L^2_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}}\right|\right]\right) \wedge 1 \\
&\leq \mathbb{P}(\tau_i = 1)\left(\mathbb{P}(\tau_{i,+} = 1) + \mathbb{P}(\tau_{i-1,-} = 1)\right) \\
&\leq \mathbb{P}(\tau_i \geq 1)\left(\mathbb{P}(\tau_{i,+} \geq 1) + \mathbb{P}(\tau_{i-1,-} \geq 1)\right).
\end{aligned}$$

By (III.3.6) applied to $m = 1$ we obtain that

$$\mathbb{P}(\tau_i \geq 1)\left(\mathbb{P}(\tau_{i,+} \geq 1) + \mathbb{P}(\tau_{i-1,-} \geq 1)\right) \leq C(\Delta_n/v_n^\beta)^2 = C\Delta_n(\log(1/\Delta_n))^{-2\beta}.$$

It follows that

$$\sum_{i=1}^{n_1} \mathbb{E}[|\xi_i^n(10)| \wedge 1] \longrightarrow 0$$

and therefore by condition (III.3.1) we have

$$\sum_{i=1}^{n_1} \xi_i^n(10) \overset{u.c.p.}{\longrightarrow} 0.$$

To sum up we have shown that $\sum_{i=1}^{n_1} \xi_i^n(j) \overset{u.c.p.}{\longrightarrow} 0$ for $j = 2, 3, 4, 5, 6, 8, 10, 11, 12, 13$.

III.4.2 The dominating term

III.4.2.1 Technical results

Lemma III.4.1. *Let $p, q \geq 2$ and let $(\Delta_i^{n,1})_{1\leq i\leq p}$ and $(\Delta_j^{n,2})_{1\leq j\leq q}$ be two families of observation time instants with $\sup_{i,j}(\Delta_i^{n,1} + \Delta_j^{n,2}) \leq C\Delta_n$. For each $1 \leq j \leq q$ define a set $S_j \subset \{1, ..., p\}$, $S_j \neq \emptyset$, such that $\bigcup_{1\leq j\leq q} S_j = \{1, ..., p\}$. For all $t \in \mathbb{R}$ we have*

$$\prod_{i=1}^{p} \alpha_i^n \Delta_i^{n,1} \int_{\|y_1\|,...,\|y_p\|\in(v_n,1]} \exp\left\{ \sum_{j=1}^{q} \Delta_j^{n,2} \right.$$
$$\left. \times \int_{\|x_j\|\in(0,v_n]} \left(e^{it\delta_n\langle x_j, \sum_{k\in S_j} y_k\rangle} - 1 - it\delta_n\langle x_j, \sum_{k\in S_j} y_k\rangle \right) G(dx_j) \right\} G(dy_1)...G(dy_p) = o(\Delta_n).$$

Proof. There exists $C > 0$ such that $\sup_i \Delta_i^{n,1} \leq C\Delta_n$. Since $\overline{G}(v_n) \leq C\Delta_n^{-1/2}\log(1/\Delta_n)^{-\beta}$, it follows that $\alpha_i^n = \exp(-\Delta_i^{n,1}\overline{G}(v_n)) \to 1$ and therefore

$$\prod_{i=1}^{p} \alpha_i^n \Delta_i^{n,1} \int_{\|y_1\|,...,\|y_p\|\in(v_n,1]} \exp\left\{ \sum_{j=1}^{q} \Delta_j^{n,2} \right.$$
$$\left. \times \int_{\|x_j\|\in(0,v_n]} \left(e^{it\delta_n\langle x_j, \sum_{k\in S_j} y_k\rangle} - 1 - it\delta_n\langle x_j, \sum_{k\in S_j} y_k\rangle \right) G(dx_j) \right\} G(dy_1)...G(dy_p)$$
$$\leq C\Delta_n^p \int_{\|y_1\|,...,\|y_p\|\in(v_n,1]} \exp\left\{ \sum_{j=1}^{q} \Delta_j^{n,2} \right.$$
$$\left. \times \int_{\|x_j\|\in(0,v_n]} \left(e^{it\delta_n\langle x_j, \sum_{k\in S_j} y_k\rangle} - 1 - it\delta_n\langle x_j, \sum_{k\in S_j} y_k\rangle \right) G(dx_j) \right\} G(dy_1)...G(dy_p)$$
$$= C\Delta_n^p \int_{\|y_1\|,...,\|y_p\|\in(v_n,1]} \exp\left\{z_n(t, y_1, ..., y_p)\right\} G(dy_1)...G(dy_p)$$
$$:= R^n(t)$$

with

$$z_n(t, y_1, ..., y_p) = \sum_{j=1}^{q} \Delta_j^{n,2} \int_{\|x_j\| \in (0, v_n]} \left(e^{it\delta_n \langle x_j, \sum_{k \in S_j} y_k \rangle} - 1 - it\delta_n \langle x_j, \sum_{k \in S_j} y_k \rangle \right) G(dx_j).$$

We further decompose $R^n(t) = \rho_n^1(t) + \rho_n^2(t) + \rho_n^3(t)$ with

$$\rho_n^1(t) = \Delta_n^p \int_{\|y_1\|, ..., \|y_p\| \in (v_n, 1]} \{\exp\{z_n(t, y_1, ..., y_p)\} - 1 - z_n(t, y_1, ..., y_p)\} G(dy_1)...G(dy_p)$$

$$\rho_n^2(t) = \Delta_n^p \int_{\|y_1\|, ..., \|y_p\| \in (v_n, 1]} z_n(t, y_1, ..., y_p) G(dy_1)...G(dy_p)$$

$$\rho_n^3(t) = \Delta_n^p \int_{\|y_1\|, ..., \|y_p\| \in (v_n, 1]} G(dy_1)...G(dy_p).$$

We start with the term $\rho_n^1(t)$. For any $w > 0$ it holds that

$$\int (w\|x\|) \wedge (w\|x\|)^2 G(dx) = C \left(w \int_{w^{-1}}^{1} r^{-\beta} dr + w^2 \int_0^{w^{-1}} r^{1-\beta} dr \right) \leq C w^\beta$$

and

$$|\exp(iw) - 1 - iw| \leq C(|w| \wedge w^2).$$

We deduce for a fixed $t \in \mathbb{R}$ that

$$\begin{aligned} |z_n(t, y_1, ..., y_p)| &\leq C \sum_{j=1}^{q} \Delta_j^{n,2} \int \left(\delta_n \left| \langle x_j, \sum_{k \in S_j} y_k \rangle \right| \right) \wedge \left(\delta_n \left| \langle x_j, \sum_{k \in S_j} y_k \rangle \right| \right)^2 G(dx_j) \\ &\leq C \Delta_n^q \sum_{j=1}^{q} \int \left(\delta_n \left| \langle x_j, \sum_{k \in S_j} y_k \rangle \right| \right) \wedge \left(\delta_n \left| \langle x_j, \sum_{k \in S_j} y_k \rangle \right| \right)^2 G(dx_j) \end{aligned}$$

where we used the fact that $\sup_j \Delta_j^{n,2} \leq C\Delta_n$. Observe that

$$\left| \langle x_j, \sum_{k \in S_j} y_k \rangle \right| \leq \|x_j\| \left\| \sum_{k \in S_j} y_k \right\| \leq \|x_j\| \sum_{k \in S_j} \|y_k\|$$

hence

$$\begin{aligned} |z_n(t, y_1, ..., y_p)| &\leq C \Delta_n^q \sum_{j=1}^{q} \int \left(\delta_n \|x_j\| \sum_{k \in S_j} \|y_k\| \right) \wedge \left(\delta_n \|x_j\| \sum_{k \in S_j} \|y_k\| \right)^2 G(dx_j) \\ &\leq C \Delta_n^q \delta_n^\beta \sum_{j=1}^{q} \left(\sum_{k \in S_j} \|y_k\| \right)^\beta = C \Delta_n^{q-1} \log(1/\Delta_n)^{-1} \sum_{j=1}^{q} \left(\sum_{k \in S_j} \|y_k\| \right)^\beta. \end{aligned}$$

Using the fact that $|e^x - 1 - x| \leq x^2$ for $|x| \leq 1$, and since $\Delta_n^{q-1}\log(1/\Delta_n)^{-1} \to 0$, there exists a rank n_0 such that for all $n \geq n_0$, $|z_n(t, y_1, ..., y_p)| \leq 1$. Then

$$|\exp\{z_n(t, y_1, ..., y_p)\} - 1 - z_n(t, y_1, ..., y_p)| \leq |z_n(t, y_1, ..., y_p)|^2$$
$$\leq C\Delta_n^{2q-2}\log(1/\Delta_n)^{-2}\left(\sum_{j=1}^{q}\left(\sum_{k\in S_j}\|y_k\|\right)^{\beta}\right)^2.$$

By Jensen inequality, we have

$$\left(\sum_{k\in S_j}\|y_k\|\right)^{\beta} \leq |S_j|^{\beta-1}\sum_{k\in S_j}\|y_k\|^{\beta} \leq C\sum_{k\in S_j}\|y_k\|^{\beta}$$

and, for some $a_j \in \mathbb{R}$,

$$\left(\sum_{j=1}^{q} a_j\right)^2 \leq q\sum_{j=1}^{q} a_j^2.$$

Therefore

$$\left(\sum_{j=1}^{q}\left(\sum_{k\in S_j}\|y_k\|\right)^{\beta}\right)^2 \leq C\sum_{j=1}^{q}\sum_{k\in S_j}\|y_k\|^{2\beta}.$$

Since $\int\|y\|^{2\beta}G(dx) \leq C$, this implies that

$$|\rho_n^1(t)| \leq C\Delta_n^p \int\limits_{\|y_1\|,...,\|y_p\|\in(v_n,1]} |z_n(t, y_1, ..., y_p)|^2 G(dy_1)...G(dy_p)$$
$$\leq C\Delta_n^{p+2q-2}\log(1/\Delta_n)^{-2}\sum_{j=1}^{q}\sum_{k\in S_j}\int\|y_k\|^{2\beta}G(dy_k)$$
$$\leq C\Delta_n^{p+2q-2}\log(1/\Delta_n)^{-2}.$$

We continue with the term $\rho_n^3(t)$. Recall that $\overline{G}(v_n) \leq C\Delta_n^{-1/2}\log(1/\Delta_n)^{-\beta}$. We have

$$|\rho_n^3(t)| = \Delta_n^p \int\limits_{\|y_1\|,...,\|y_p\|\in(v_n,1]} G(dy_1)...G(dy_p)$$
$$= \Delta_n^p\overline{G}(v_n)^p$$
$$\leq C\Delta_n^{p/2}\log(1/\Delta_n)^{-p\beta}.$$

Concerning $\rho_n^2(t)$, we have

$$\rho_n^2(t) = \sum_{j=1}^{q}\rho_{n,j}^2(t)$$

with

$$\rho^2_{n,j}(t) =$$

$$\Delta^p_n \Delta^{n,2}_j \int\limits_{\|y_1\|,\ldots,\|y_p\| \in (v_n,1]} \int\limits_{\|x\| \in (0,v_n]} \left\{ e^{it\delta_n \langle x, \sum_{k\in S_j} y_k \rangle} - 1 - it\delta_n \langle x, \sum_{k\in S_j} y_k \rangle \right\} G(dx)G(dy_1)\ldots G(dy_p).$$

Let $1 \le j \le q$. Denote $|S_j| = m_j$ and observe that $1 \le m_j \le p$. Since each variable y_k is interchangeable we obtain

$$\begin{aligned}
&\rho^2_{n,j}(t) \\
&= \Delta^p_n \Delta^{n,2}_j \int\limits_{\|y_1\|,\ldots,\|y_p\| \in (v_n,1]} \int\limits_{\|x\| \in (0,v_n]} \left(e^{it\delta_n \langle x, \sum_{k\in S_j} y_k \rangle} - 1 - it\delta_n \langle x, \sum_{k\in S_j} y_k \rangle \right) G(dx)G(dy_1)\ldots G(dy_p) \\
&= \Delta^p_n \Delta^{n,2}_j \int\limits_{\|y_{m_j+1}\|,\ldots,\|y_p\| \in (v_n,1]} G(dy_{m_j+1})\ldots G(dy_p) \\
&\times \int\limits_{\|y_1\|,\ldots,\|y_{m_j}\| \in (v_n,1]} \int\limits_{\|x\| \in (0,v_n]} \left(e^{it\delta_n \langle x, \sum_{k=1}^{m_j} y_k \rangle} - 1 - it\delta_n \langle x, \sum_{k=1}^{m_j} y_k \rangle \right) G(dx)G(dy_1)\ldots G(dy_{m_j}) \\
&= \int \left\{ e^{itz} - 1 - itz \right\} \nu_{n,j}(dz)
\end{aligned}$$

where $\nu_{n,j}$ is a Lévy measure defined on $\mathbb{R}$ by

$$\nu_{n,j}(A) = \Delta^p_n \Delta^{n,2}_j \overline{G}(v_n)^{p-m_j} \int_{(v_n,1]^{m_j}} \left(\int_{(0,v_n]} \mathbb{1}_A \left(\delta_n \sum_{k=1}^{m_j} \langle x, y_k \rangle \right) G(dx) \right) G(dy_1)\ldots G(dy_{m_j})$$

with $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$. We will show that $\nu_{n,j}$ converges strongly to the null measure. Set $x = (\rho, \theta) \in \mathbb{R}_+ \times \mathbb{S}_2$ and $y_k = (\rho_k, \theta_k) \in \mathbb{R}_+ \times \mathbb{S}_2$ for all $1 \le k \le m_j$. Note that

$$\sum_{k=1}^{m_j} \langle x, y_k \rangle = \rho \sum_{k=1}^{m_j} \rho_k \langle \theta, \theta_k \rangle.$$

We obtain, for $w > 0$ and $B \in \mathcal{B}(\mathbb{S}_1)$

$$\begin{aligned}
\nu_{n,j}(B \times (w,\infty)) = \Delta^p_n \Delta^{n,2}_j \overline{G}(v_n)^{p-m_j} \int_{\mathbb{S}_2^{m_j+1}} \int_{(0,v_n]\times(v_n,1]^{m_j}} \mathbb{1}_B \left(\frac{\sum_{k=1}^{m_j} \rho_k \langle \theta, \theta_k \rangle}{\left| \sum_{k=1}^{m_j} \rho_k \langle \theta, \theta_k \rangle \right|} \right) \\
\times \mathbb{1}_{(w,\infty)} \left(\delta_n \rho \left| \sum_{k=1}^{m_j} \rho_k \langle \theta, \theta_k \rangle \right| \right) \left(\rho \prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho d\rho_1\ldots d\rho_{m_j} H(d\theta)H(d\theta_1)\ldots H(d\theta_{m_j}).
\end{aligned}$$

We compute the integral with respect to $d\rho d\rho_1\ldots d\rho_{m_j}$. For any $r_1, \ldots, r_{m_j} \in \mathbb{R}$ we have

$$\Delta^p_n \Delta^{n,2}_j \overline{G}(v_n)^{p-m_j} \int\limits_{(0,v_n]\times(v_n,1]^{m_j}} \mathbb{1}_{(w,\infty)} \left(\delta_n \rho \left| \sum_{k=1}^{m_j} \rho_k r_k \right| \right) \left(\rho \prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho d\rho_1\ldots d\rho_{m_j}$$

$$
\begin{aligned}
&= \Delta_n^p \Delta_j^{n,2} \overline{G}(v_n)^{p-m_j} \int_{(v_n,1]^{m_j}} \left(\int_{w/\delta_n \left|\sum_{k=1}^{m_j} \rho_k r_k\right|}^{v_n} \rho^{-1-\beta} d\rho \right) \left(\prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho_1 ... d\rho_{m_j} \\
&= -\Delta_n^p \Delta_j^{n,2} \overline{G}(v_n)^{p-m_j} \frac{v_n^{-\beta}}{\beta} \int_{(v_n,1]^{m_j}} \left(\prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho_1 ... d\rho_{m_j} \\
&\qquad + \Delta_n^p \Delta_j^{n,2} \overline{G}(v_n)^{p-m_j} \frac{\delta_n^\beta}{\beta w^\beta} \int_{(v_n,1]^{m_j}} \left| \sum_{k=1}^{m_j} \rho_k r_k \right|^\beta \left(\prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho_1 ... d\rho_{m_j} \\
&= -\Delta_n^p \Delta_j^{n,2} \overline{G}(v_n)^{p-m_j} \frac{v_n^{-\beta}}{\beta} \left(\int_{v_n}^1 \rho_1^{-1-\beta} d\rho_1 \right)^{m_j} \\
&\qquad + \Delta_n^p \Delta_j^{n,2} \overline{G}(v_n)^{p-m_j} \frac{\delta_n^\beta}{\beta w^\beta} \int_{(v_n,1]^{m_j}} \left| \sum_{k=1}^{m_j} \rho_k r_k \right|^\beta \left(\prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho_1 ... d\rho_{m_j}.
\end{aligned}
$$

We consider the first integral. We have

$$
\int_{v_n}^1 \rho_1^{-1-\beta} d\rho_1 = \frac{1}{\beta} \left(v_n^{-\beta} - 1 \right)
$$

hence

$$
-\Delta_j^{n,2} \frac{v_n^{-\beta}}{\beta} \left(\int_{v_n}^1 \rho_1^{-1-\beta} d\rho_1 \right)^{m_j} = \Delta_j^{n,2} \frac{-v_n^{-\beta}}{\beta^{m_j+1}} \left(v_n^{-\beta} - 1 \right)^{m_j}.
$$

Finally,

$$
\begin{aligned}
&\left| \Delta_n^p \Delta_j^{n,2} \overline{G}(v_n)^{p-m_j} \frac{-v_n^{-\beta}}{\beta^{m_j+1}} \left(v_n^{-\beta} - 1 \right)^{m_j} \right| \le C \Delta_n^{p+1} \Delta_n^{(m_j-p)/2} \log(1/\Delta_n)^{\beta(m_j-p)} v_n^{-\beta} (v_n^{-\beta m_j} + 1) \\
&\quad \le C \Delta_n^{(p+m_j+2)/2} \log(1/\Delta_n)^{\beta(m_j-p)} \left(\Delta_n^{-(m_j+1)/2} \log(1/\Delta_n)^{-\beta(m_j+1)} + \Delta_n^{-1/2} \log(1/\Delta_n)^{-\beta} \right) \\
&\quad \le C \Delta_n^{(p+1)/2} \log(1/\Delta_n)^{-\beta(p+1)} + C \Delta_n^{(p+m_j+1)/2} \log(1/\Delta_n)^{-\beta(p+1-m_j)}.
\end{aligned}
$$

For the second integral, observe that

$$
\left| \sum_{k=1}^{m_j} \rho_k r_k \right|^\beta \le \left(\sum_{k=1}^{m_j} \rho_k |r_k| \right)^\beta \le m_j^{\beta-1} \sum_{k=1}^{m_j} \rho_k^\beta |r_k|^\beta \le m_j^{\beta-1} \sup_{1 \le k \le m_j} |r_k| \sum_{k=1}^{m_j} \rho_k^\beta \le C \sum_{k=1}^{m_j} \rho_k^\beta.
$$

We obtain that

$$
\begin{aligned}
&\int_{(v_n,1]^{m_j}} \left| \sum_{k=1}^{m_j} \rho_k r_k \right|^\beta \left(\prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho_1 ... d\rho_{m_j} \\
&\le C \int_{(v_n,1]^{m_j}} \sum_{k=1}^{m_j} \rho_k^\beta \left(\prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho_1 ... d\rho_{m_j} \\
&= C \sum_{k=1}^{m_j} \int_{(v_n,1]^{m_j}} \rho_k^{-1} \prod_{\substack{l=1 \\ l \neq k}}^{m_j} \rho_l^{-1-\beta} d\rho_1 ... d\rho_{m_j} \\
&= C m_j \int_{(v_n,1]^{m_j}} \rho_1^{-1} \prod_{k=2}^{m_j} \rho_k^{-1-\beta} d\rho_1 ... d\rho_{m_j}
\end{aligned}
$$

$$= Cm_j \int_{v_n}^1 \rho_1^{-1} d\rho_1 \left(\int_{v_n}^1 \rho_2^{-1-\beta} d\rho_2 \right)^{m_j-1}$$
$$= Cm_j \log(1/v_n) \left(\frac{1}{\beta}(v_n^{-\beta} - 1) \right)^{m_j-1}.$$

Using the fact that $2\beta \log(1/v_n)/\log(1/\Delta_n) \to 1$, we get

$$\frac{Cm_j}{\beta^{m_j-1}} \left| \log(1/v_n) \left(v_n^{-\beta} - 1 \right)^{m_j-1} \right| \leq C\Delta_n^{-(m_j-1)/2} \log(1/\Delta_n)^{1-\beta(m_j-1)} + C \log(1/\Delta_n)$$

It follows that

$$\left| \Delta_n^p \Delta_j^{n,2} \overline{G}(v_n)^{p-m_j} \frac{\delta_n^\beta}{\beta w^\beta} \int_{(v_n,1]^{m_j}} \left| \sum_{k=1}^{m_j} \rho_k r_k \right|^\beta \left(\prod_{k=1}^{m_j} \rho_k \right)^{-1-\beta} d\rho_1 ... d\rho_{m_j} \right|$$
$$\leq C\Delta_n^{p+1} \left(\Delta_n^{-1/2} \log(1/\Delta_n)^{-\beta} \right)^{p-m_j} \delta_n^\beta \left(\Delta_n^{-(m_j-1)/2} \log(1/\Delta_n)^{1-\beta(m_j-1)} + \log(1/\Delta_n) \right)$$
$$\leq C\Delta_n^{(p+1)/2} \log(1/\Delta_n)^{-\beta(p-1)} + C\Delta_n^{(p+m_j)/2} \log(1/\Delta_n)^{-\beta(p-m_j)}$$

Finally,

$$|\nu_{n,j}(B \times (w, \infty))| \leq C\Delta_n^{(p+1)/2} \left(\log(1/\Delta_n)^{-\beta(p+1)} + \log(1/\Delta_n)^{-\beta(p-1)} \right)$$
$$+ C\Delta_n^{(p+m_j)/2} \left(\log(1/\Delta_n)^{-\beta(p-m_j)} + \Delta_n^{1/2} \log(1/\Delta_n)^{-\beta(p+1-m_j)} \right)$$

hence the strong convergence of $\nu_{n,j}$ towards the null measure. In particular we have that

$$\rho_{n,i}^2(t) = o(\Delta_n)$$

and therefore

$$\rho_n^2(t) = o(\Delta_n).$$

Finally,

$$R^n(t) \leq C\Delta_n^{p+2q-2} \log(1/\Delta_n)^{-2} + C\Delta_n^{p/2} \log(1/\Delta_n)^{-p\beta} + o(\Delta_n) = o(\Delta_n)$$

hence the result. □

Lemma III.4.2. *Under the same conditions on the observation time instants as Lemma III.4.1, we have, for $t \in \mathbb{R}$:*

$$\mathbb{E}\left[e^{it\delta_n \sum_{j=1}^p \langle \Delta_j^n M^n, \Delta_0^n A^n \rangle \mathbb{1}_{\{\tau_0 = 1\}}} - 1 \right] = \sum_{j=1}^p \left[e^{it\delta_n \langle \Delta_j^n M^n, \Delta_0^n A^n \rangle \mathbb{1}_{\{\tau_0 = 1\}}} - 1 \right] + o(1).$$

Proof. We start by expanding the indicator function. We have

$$\mathbb{E}\left[e^{it\delta_n \sum_{j=1}^p \langle \Delta_j^n M^n, \Delta_0^n A^n \rangle \mathbb{1}_{\{\tau_0 = 1\}}} - 1 \right] = \alpha_0^n \Delta_0^n \overline{G}(v_n) \mathbb{E}\left[e^{it\delta_n \sum_{j=1}^p \langle \Delta_j^n M^n, \Delta_0^n A^n \rangle} - 1 \right].$$

Recall that the increments of M^n are independent since the observation time instants are non-overlapping.

By conditioning on A^n we obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{it\delta_n \sum_{j=1}^p \langle \Delta_j^n M^n, \Delta_0^n A^n\rangle}\right] \\
&= \mathbb{E}\left[\prod_{j=1}^p \exp\left\{\Delta_j^n \int_{(0,v_n]} \left\{e^{it\delta_n\langle x_j, \Delta_0^n A^n\rangle} - 1 - it\delta_n\langle x_j, \Delta_0^n A^n\rangle\right\} G(dx_j)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{\sum_{j=1}^p \Delta_j^n \int_{(0,v_n]} \left\{e^{it\delta_n\langle x_j, \Delta_0^n A^n\rangle} - 1 - it\delta_n\langle x_j, \Delta_0^n A^n\rangle\right\} G(dx_j)\right\}\right]
\end{aligned}$$

hence

$$\mathbb{E}\left[e^{it\delta_n \sum_{j=1}^p \langle \Delta_j^n M^n, \Delta_0^n A^n\rangle \mathbb{1}_{\{\tau_0=1\}}} - 1\right] = \alpha_0^n \Delta_0^n \int_{(v_n,1]} \{\exp\{z_n(t,y)\} - 1\}\, G(dy) := R^n(t)$$

with

$$z_n(t,y) = \sum_{j=1}^p \Delta_j^n \int_{(0,v_n]} \left\{e^{it\delta_n\langle x_j, y\rangle} - 1 - it\delta_n\langle x_j, y\rangle\right\} G(dx_j).$$

We decompose $R^n(t) = \rho_n^1(t) + \rho_n^2(t)$ with

$$\begin{aligned}
\rho_n^1(t) &= \alpha_0^n \Delta_0^n \int_{(v_n,1]} \{\exp\{z_n(t,y)\} - 1 - z_n(t,y)\}\, G(dy) \\
\rho_n^2(t) &= \alpha_0^n \Delta_0^n \sum_{j=1}^p \Delta_j^n \int_{(v_n,1]} \left(\int_{(0,v_n]} \left\{e^{it\delta_n\langle x_j, y\rangle} - 1 - it\delta_n\langle x_j, y\rangle\right\} G(dx_j)\right) G(dy).
\end{aligned}$$

Starting with the term $\rho_n^1(t)$, we use the same inequality as in the proof of Lemma III.4.1. We obtain for a fixed $t \in \mathbb{R}$ that

$$\begin{aligned}
|z_n(t,y)| &\le C \sum_{j=1}^p \Delta_j^n \int_{(0,v_n]} (\delta_n|\langle x_j, y\rangle|) \wedge (\delta_n|\langle x_j, y\rangle|)^2\, G(dx) \\
&\le C \sum_{j=1}^p \Delta_j^n \int_{(0,v_n]} (\delta_n\|x_j\|\|y\|) \wedge (\delta_n\|x_j\|\|y\|)^2\, G(dx) \\
&\le C\delta_n^\beta \|y\|^\beta \sum_{j=1}^p \Delta_j^n \le C\Delta_n^{p-1}\|y\|^\beta \log(1/\Delta_n)^{-1}.
\end{aligned}$$

Consequently, we obtain for a fixed $t \in \mathbb{R}$

$$|\rho_n^1(t)| \le C\alpha_0^n \Delta_0^n \int_{(v_n,1]} |z_n(t,y)|^2 G(dy) \le C\alpha_0^n \Delta_n^{2p-1} \log(1/\Delta_n)^{-2} = o(\Delta_n).$$

Therefore we get that

$$\begin{aligned}
R^n(t) &= \rho_n^2(t) + o(\Delta_n) \\
&= \alpha_0^n \Delta_0^n \sum_{j=1}^p \Delta_j^n \int_{(v_n,1]} \left(\int_{(0,v_n]} \left\{e^{it\delta_n\langle x_j, y\rangle} - 1 - it\delta_n\langle x_j, y\rangle\right\} G(dx_j)\right) G(dy) + o(\Delta_n)
\end{aligned}$$

We know from the synchronous case presented in [70, Section 4.3.2] that, for $1 \leq p \leq j$

$$\alpha_0^n \Delta_0^n \Delta_j^n \int_{(v_n,1]} \left(\int_{(0,v_n]} \left\{ e^{it\delta_n \langle x_j, y\rangle} - 1 - it\delta_n \langle x_j, y\rangle \right\} G(dx_j) \right) G(dy)$$
$$= \mathbb{E}\left[e^{it\delta_n \langle \Delta_j^n M^n, \Delta_0^n A^n \rangle \mathbb{1}_{\{\tau_0 = 1\}}} - 1 \right]$$

hence the result. $\square$

Proposition III.4.3. *Under the same conditions on the observation time instants as Lemma III.4.1, we have:*

$$\mathbb{E}\left[\exp\left\{ it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k \in S_j} \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k = 1\}} \rangle \right\} - 1 \right]$$
$$= \sum_{j=1}^{q} \sum_{k \in S_j} \mathbb{E}\left[\exp\{ it\delta_n \langle \Delta_j^{n,2} M^n, \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k=1\}} \rangle \} - 1 \right] + o(1).$$

Proof. We recall that M^n has the Lévy triplet $(0, 0, G(dx)\mathbb{1}_{\{\|x\| \leq v_n\}}$ and A^n is a compound Poisson process with intensity $\overline{G}(v_n) := G(\{x \in \mathbb{R}^2 \ : \ v_n < \|x\| \leq 1\})$ and jump distribution $G(dx)\mathbb{1}_{\{v_n < \|x\| \leq 1\}}$ and M^n and A^n are independent. We start by expanding the expectation on the left hand side with respect to the indicator functions. We obtain

$$\mathbb{E}\left[\exp\left\{ it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k \in S_j} \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k = 1\}} \rangle \right\} - 1 \right]$$
$$= \prod_{i=1}^{p} \mathbb{P}(\mathbb{1}_{\{\tau_i=1\}} = 1) \mathbb{E}\left[\exp\left\{ it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k \in S_j} \Delta_k^{n,1} A^n \rangle \right\} - 1 \right]$$
$$+ \sum_{l=1}^{p} \mathbb{P}(\mathbb{1}_{\{\tau_l=1\}} = 0) \prod_{\substack{i=1 \\ i \neq k}}^{p} \mathbb{P}(\mathbb{1}_{\{\tau_i=1\}} = 1) \mathbb{E}\left[\exp\left\{ it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k \in S_j \backslash \{l\}} \Delta_k^{n,1} A^n \rangle \right\} - 1 \right]$$
$$+ \sum_{l_1=1}^{p} \sum_{\substack{l_2=1 \\ l_2 \neq l_1}}^{p} \Bigg(\mathbb{P}(\mathbb{1}_{\{\tau_{l_1}=1\}} = 0) \mathbb{P}(\mathbb{1}_{\{\tau_{l_2}=1\}} = 0)$$
$$\times \prod_{\substack{i=1 \\ i \neq l_1, l_2}}^{p} \mathbb{P}(\mathbb{1}_{\{\tau_i=1\}} = 1) \mathbb{E}\left[\exp\left\{ it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k \in S_j \backslash \{l_1, l_2\}} \Delta_k^{n,1} A^n \rangle \right\} - 1 \right] \Bigg)$$
$$+ \sum_{l_1=1}^{p} \sum_{\substack{l_2=1 \\ l_2 \neq l_1}}^{p} \sum_{\substack{l_3=1 \\ l_3 \neq l_1, l_2}}^{p} \Bigg(\mathbb{P}(\mathbb{1}_{\{\tau_{l_1}=1\}} = 0) \mathbb{P}(\mathbb{1}_{\{\tau_{l_2}=1\}} = 0) \mathbb{P}(\mathbb{1}_{\{\tau_{l_3}=1\}} = 0)$$
$$\times \prod_{\substack{i=1 \\ i \neq l_1, l_2, l_3}}^{p} \mathbb{P}(\mathbb{1}_{\{\tau_i=1\}} = 1) \mathbb{E}\left[\exp\left\{ it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k \in S_j \backslash \{l_1, l_2, l_3\}} \Delta_k^{n,1} A^n \rangle \right\} - 1 \right] \Bigg)$$

$$
\begin{aligned}
&+\dots\\
&+\sum_{l_1=1}^{p}\sum_{\substack{l_2=1\\ l_2\neq l_1}}^{p}\sum_{\substack{l_3=1\\ l_3\neq l_1,l_2}}^{p}\Bigg(\mathbb{P}(\mathbb{1}_{\{\tau_{l_1}=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_{l_2}=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_{l_3}=1\}}=1)\\
&\qquad\times\prod_{\substack{i=1\\ i\neq l_1,l_2,l_3}}^{p}\mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0)\mathbb{E}\left[\exp\left\{it\delta_n\sum_{j=1}^{q}\langle\Delta_j^{n,2}M^n,\sum_{k\in S_j\cap\{l_1,l_2,l_3\}}\Delta_k^{n,1}A^n\rangle\right\}-1\right]\Bigg)\\
&+\sum_{l_1=1}^{p}\sum_{\substack{l_2=1\\ l_2\neq l_1}}^{p}\Bigg(\mathbb{P}(\mathbb{1}_{\{\tau_{l_1}=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_{l_2}=1\}}=1)\\
&\qquad\times\prod_{\substack{i=1\\ i\neq l_1,l_2}}^{p}\mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0)\mathbb{E}\left[\exp\left\{it\delta_n\sum_{j=1}^{q}\langle\Delta_j^{n,2}M^n,\sum_{k\in S_j\cap\{l_1,l_2\}}\Delta_k^{n,1}A^n\rangle\right\}-1\right]\Bigg)\\
&+\sum_{l=1}^{p}\mathbb{P}(\mathbb{1}_{\{\tau_l=1\}}=1)\prod_{\substack{i=1\\ i\neq l}}^{p}\mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0)\mathbb{E}\left[\exp\left\{it\delta_n\sum_{j=1}^{q}\langle\Delta_j^{n,2}M^n,\sum_{k\in S_j\cap\{l\}}\Delta_k^{n,1}A^n\rangle\right\}-1\right].
\end{aligned}
$$

We start by studying all the terms containing at least two increments of A^n. Such a term can be written, for $0\leq m\leq p-2$, as

$$
\prod_{l=1}^{m}\mathbb{P}(\mathbb{1}_{\{\tau_{k_l}=1\}}=0)\prod_{\substack{i=1\\ i\neq k_1,\dots,k_m}}^{p}\mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n\sum_{j=1}^{q}\langle\Delta_j^{n,1}M^n,\sum_{k\in S_j\setminus\{k_1,\dots,k_m\}}\Delta_k^{n,1}A^n\rangle\right\}-1\right]. \tag{III.4.1}
$$

We have that

$$
\begin{aligned}
\prod_{l=1}^{m}\mathbb{P}(\mathbb{1}_{\{\tau_{k_l}=1\}}=0)&=\prod_{l=1}^{m}(1-\alpha_{k_l}^{n,1}\Delta_{k_l}^{n,1}\overline{G}(v_n))\\
&=1-\sum_{l=1}^{m}\alpha_{k_l}^{n,1}\Delta_{k_l}^{n,1}\overline{G}(v_n)+o(\Delta_n)=1+o(\Delta_n^{1/2})
\end{aligned}
$$

where we used that $\Delta_{k_l}^{n,1}\leq C\Delta_n$, $\alpha_{k_l}^{n,1}=\exp(-\Delta_{k_l}^{n,1}\overline{G}(v_n))\to 1$ and $\overline{G}(v_n)\leq C\Delta_n^{-1/2}\log(1/\Delta_n)^{-\beta}$. From this, to get the order of (III.4.1) we only need to study the following expression, for some $k_1,...,k_m\in\{1,...,p\}$:

$$
\prod_{\substack{i=1\\ i\neq k_1,\dots,k_m}}^{p}\alpha_i^{n,1}\Delta_i^{n,1}\overline{G}(v_n)\mathbb{E}\left[\exp\left\{it\delta_n\sum_{j=1}^{q}\langle\Delta_j^{n,1}M^n,\sum_{k\in S_j\setminus\{k_1,\dots,k_m\}}\Delta_k^{n,1}A^n\rangle\right\}-1\right]. \tag{III.4.2}
$$

Since $\{1,...,p\}\setminus\{k_1,...,k_m\}=\{i_1,...,i_{p-m}\}$, after a change of index in the product we obtain

$$
\prod_{\substack{i=1\\ i\neq k_1,\dots,k_m}}^{p}\alpha_i^{n,1}\Delta_i^{n,1}\overline{G}(v_n)\mathbb{E}\left[\exp\left\{it\delta_n\sum_{j=1}^{q}\langle\Delta_j^{n,1}M^n,\sum_{k\in S_j\setminus\{k_1,\dots,k_m\}}\Delta_k^{n,1}A^n\rangle\right\}-1\right]
$$

$$= \prod_{l=1}^{p-m} \alpha_{i_l}^{n,1} \Delta_{i_l}^{n,1} \overline{G}(v_n) \mathbb{E}\left[\exp\left\{it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,1} M^n, \sum_{k\in S_j \cap \{i_1,\dots,i_{p-m}\}} \Delta_k^{n,1} A^n\rangle\right\}\right] + o(\Delta_n)^{1/2}$$

$$= o(\Delta_n^{1/2}) + \prod_{l=1}^{p-m} \alpha_{i_l}^{n,1} \Delta_{i_l}^{n,1} \int_{\|y_{i_1}\|,\dots,\|y_{i_{p-m}}\|\in(v_n,1]} \exp\left\{\sum_{j=1}^{q} \Delta_j^{n,2}\right.$$

$$\times \int_{\|x\|\in(0,v_n]} \left(e^{it\delta_n\langle x, \sum_{k\in S_j\cap\{i_1,\dots,i_{p-m}\}} y_k\rangle} - 1 - it\delta_n\langle x, \sum_{k\in S_j\cap\{i_1,\dots,i_{p-m}\}} y_k\rangle\right) G(dx)\left.\right\}$$

$$\times G(dy_{i_1})\dots G(dy_{i_{p-m}})$$

$$= o(\Delta_n^{1/2}) + o(\Delta_n) = o(\Delta_n^{1/2})$$

by Lemma III.4.1. Therefore we get that

$$\mathbb{E}\left[\exp\left\{it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k\in S_j} \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k=1\}}\rangle\right\} - 1\right]$$

$$= \sum_{l=1}^{p} \mathbb{P}(\mathbb{1}_{\{\tau_l=1\}}=1) \prod_{\substack{i=1\\ i\neq l}}^{p} \mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0) \mathbb{E}\left[\exp\left\{it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k\in S_j\cap\{l\}} \Delta_k^{n,1} A^n\rangle\right\} - 1\right]$$

$$+ o(\Delta_n^{1/2})$$

$$\sim \sum_{l=1}^{p} \alpha_l^{n,1} \Delta_l^{n,1} \overline{G}(v_n) \mathbb{E}\left[\exp\left\{it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k\in S_j\cap\{l\}} \Delta_k^{n,1} A^n\rangle\right\} - 1\right]$$

$$= \sum_{l=1}^{p} \mathbb{E}\left[\exp\left\{it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k\in S_j\cap\{l\}} \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k=1\}}\rangle\right\} - 1\right]$$

Since $\sum_{k\in S_j\cap\{l\}} \Delta_k^{n,1} A^n$ contains at most 1 increment of A^n we obtain, using Lemma III.4.2, that

$$\mathbb{E}\left[\exp\left\{it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k\in S_j\cap\{l\}} \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k=1\}}\rangle\right\} - 1\right]$$

$$= \sum_{j=1}^{q} \sum_{k\in S_j\cap\{l\}} \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_j^{n,2} M^n, \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k=1\}}\rangle\right\} - 1\right] + o(1)$$

and finally

$$\mathbb{E}\left[\exp\left\{it\delta_n \sum_{j=1}^{q} \langle \Delta_j^{n,2} M^n, \sum_{k\in S_j} \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k=1\}}\rangle\right\} - 1\right]$$

$$= \sum_{l=1}^{p} \sum_{j=1}^{q} \sum_{k\in S_j\cap\{l\}} \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_j^{n,2} M^n, \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k=1\}}\rangle\right\} - 1\right] + o(1)$$

$$= \sum_{j=1}^{q} \sum_{k\in S_j} \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_j^{n,2} M^n, \Delta_k^{n,1} A^n \mathbb{1}_{\{\tau_k=1\}}\rangle\right\} - 1\right] + o(1),$$

since $\sum_{l=1}^{p} \sum_{k\in S_j\cap\{l\}} a_k = \sum_{k\in S_j} a_k$. □

III.4.2.2 Approximation of the characteristic function

We rewrite

$$X_i := \xi_i^n(1) + \xi_i^n(7) + \xi_i^n(9)$$

in a vectorial form as follows :

$$\begin{aligned}
\xi_i^n(1) &= \delta_n \left(\Delta_{t_i^1} M^{n,1} \Delta L^2_{T(n,i)_1} + \Delta L^1_{T(n,i)_1} \Delta_{t_i^1} M^{n,2}\right) \mathbb{1}_{\{\tau_i=1\}} \\
&= \delta_n \langle \Delta_{t_i^1} M^n, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1}\rangle \mathbb{1}_{\{\tau_i=1\}} \\
\xi_i^n(7) &= \delta_n \Delta_{t_i^1} M^{n,1} \left(\Delta L^2_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} + \Delta L^2_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}}\right) \\
&= \delta_n \langle \Delta_{t_i^1} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)}\rangle \mathbb{1}_{\{\tau_{i,+}=1\}} \\
&+ \delta_n \langle \Delta_{t_i^1} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)}\rangle \mathbb{1}_{\{\tau_{i-1,-}=1\}} \\
&:= \xi_i^n(7,1) + \xi_i^n(7,2) \\
\xi_i^n(9) &= \Delta L^1_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}} \left(\Delta_{t_{i,+}} M^{n,2} + \Delta_{t_{i-1,-}} M^{n,2}\right) \\
&= \delta_n \langle \Delta_{t_{i,+}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1}\rangle \mathbb{1}_{\{\tau_i=1\}} \\
&+ \delta_n \langle \Delta_{t_{i-1,-}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1}\rangle \mathbb{1}_{\{\tau_i=1\}} \\
&:= \xi_i^n(9,1) + \xi_i^n(9,2)
\end{aligned}$$

and therefore, if we regroup the different terms with respect to each increment of M^n we obtain

$$\begin{aligned}
X_i = \delta_n \Bigg(&\langle \Delta_{t_{i-1,-}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1}\rangle \mathbb{1}_{\{\tau_i=1\}} \\
&+ \langle \Delta_{t_i^1} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)} \mathbb{1}_{\{\tau_{i-1,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)} \mathbb{1}_{\{\tau_{i,+}=1\}}\rangle \\
&+ \langle \Delta_{t_{i,+}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1}\rangle \mathbb{1}_{\{\tau_i=1\}} \Bigg)
\end{aligned} \tag{III.4.3}$$

The goal of this section is to show the approximation :

$$\log\left(\mathbb{E}\left[e^{it\sum_{i=1}^{n_1} X_i} - 1\right]\right) = \sum_{i=1}^{n_1} \mathbb{E}\left[e^{itX_i} - 1\right] + o(1). \tag{III.4.4}$$

Using Lemma III.3.5, it is sufficient to show that

$$\mathbb{E}\left[e^{itX_i} - 1\right] = O(\Delta_n). \tag{III.4.5}$$

$$\sum_{i=2}^{n_1} \mathbb{E}\left[(e^{itX_i}-1)(e^{itX_{i-1}}-1)\right] = o(\Delta_n), \tag{III.4.6}$$

$$\sum_{i=3}^{n_1} \mathbb{E}\left[(e^{itX_i}-1)(e^{itX_{i-1}}-1)(e^{itX_{i-2}}-1)\right] = o(\Delta_n). \tag{III.4.7}$$

Note that $(X_i)_i$ is a sequence of 1-dependent random variables by construction, since there is no overlapping observation time instants between X_i and X_{i+2}.

Using Proposition III.4.3 and the representation (III.4.3), we can decompose $\mathbb{E}\left[e^{itX_i}-1\right]$ as follows:

$$\begin{aligned}\mathbb{E}\left[e^{itX_i}-1\right] =& \mathbb{E}\left[e^{it\xi_i^n(1)}-1\right] + \mathbb{E}\left[e^{it\xi_i^n(7,1)}-1\right] + \mathbb{E}\left[e^{it\xi_i^n(7,2)}-1\right] + \mathbb{E}\left[e^{it\xi_i^n(9,1)}-1\right] \\ &+ \mathbb{E}\left[e^{it\xi_i^n(9,2)}-1\right]. \end{aligned} \tag{III.4.8}$$

We know from the synchronous case [70, Section 4.3.2] that each expectation in the decomposition (III.4.8) is of order Δ_n, hence

$$\mathbb{E}\left[e^{itX_i}-1\right] = O(\Delta_n)$$

and (III.4.5) is verified. Using (III.3.9) and (III.3.10) we obtain

$$\begin{aligned}\left|\mathbb{E}\left[\left(e^{itX_{i-2}}-1\right)\left(e^{itX_{i-1}}-1\right)\left(e^{itX_i}-1\right)\right]\right| &\le \left(\mathbb{E}\left[|e^{itX_i}-1|^2\right]\right)^{3/2} \\ &\le 2^{3/2}\left|\mathbb{E}\left[e^{itX_i}-1\right]\right|^{3/2} = O(\Delta_n^{3/2})\end{aligned}$$

therefore

$$\sum_{i=3}^{n_1} \mathbb{E}\left[\left(e^{itX_{i-2}}-1\right)\left(e^{itX_{i-1}}-1\right)\left(e^{itX_i}-1\right)\right] = O(\Delta_n^{1/2}).$$

and (III.4.7) is verified. It is left to verify the condition (III.4.6). We start by decomposing the term

$$\sum_{i=2}^{n_1} \mathbb{E}\left[\left(e^{itX_{i-1}}-1\right)\left(e^{itX_i}-1\right)\right] = \sum_{i=2}^{n_1}\left(\mathbb{E}\left[e^{it\delta_n(X_i+X_{i-1})}\right] - \mathbb{E}[e^{itX_i}+e^{itX_{i-1}}]+1\right).$$

Note that $[t_i^1, t_{i,+}] \subset [t_i^1, t_{i+1}^1]$ and $[t_{i,-}, t_i^1] \subset [t_{i-1}^1, t_i^1]$, hence we can decompose the observation time instants as

$$\begin{aligned}\Delta_{t_i^1} &= t_i^1 - t_{i-1}^1 = (t_i^1 - t_{i,-}) + (t_{i,-} - t_{i-1}^1) := \Delta_{t_{i,-}} + \overline{\Delta}_{t_i} \\ \Delta_{t_{i+1}^1} &= t_{i+1}^1 - t_i^1 = (t_{i+1}^1 - t_{i,+}) + (t_{i,+} - t_i^1) := \overline{\Delta}_{t_{i+1}} + \Delta_{t_{i,+}}.\end{aligned}$$

With these new notations, we obtain that

$$\begin{aligned}\Delta_{t_i^1} M^n &\overset{Law}{=} \overline{\Delta}_{t_i} M^n + \Delta_{t_{i,-}} M^n \\ \Delta_{t_{i+1}^1} M^n &\overset{Law}{=} \Delta_{t_{i,+}} M^n + \overline{\Delta}_{t_{i+1}} M^n.\end{aligned} \tag{III.4.9}$$

Furthermore we decompose each increment $\Delta L_{T(n,i)_1}$ (resp. $\Delta L_{T(n,i+1)_1}$) depending whether the jump occurs over the observation time instant $\Delta_{t_{i,-}}$ or $\overline{\Delta}_{t_i}$ (resp. $\Delta_{t_{i,+}}$

or $\overline{\Delta}_{t_{i+1}}$). Denote by $\overline{\tau}_i$ the number of jumps within $\overline{\Delta}_{t_i}$ and $\overline{T}(n,i)_1$ the time of the first jump of L within the observation time instant $\overline{\Delta}_{t_i}$, $\tau_{i,-}$ the number of jumps within $\Delta_{t_{i,-}}$ and $T(n,i,-)_1$ the time of the first jump of L within the observation time instant $\Delta_{t_{i,-}}$. We obtain

$$\Delta L_{T(n,i)_1}\mathbb{1}_{\{\tau_i=1\}} = \Delta L_{\overline{T}(n,i)_1}\mathbb{1}_{\{\overline{\tau}_i=1\}}\mathbb{1}_{\{\tau_{i,-}=0\}} + \Delta L_{T(n,i,-)_1}\mathbb{1}_{\{\overline{\tau}_i=0\}}\mathbb{1}_{\{\tau_{i,-}=1\}}.$$

Lemma III.4.4. *Let $\Delta_0^n M^n$ be any increment of M^n with $\Delta_0^n = O(\Delta_n)$. We observe the following approximation:*

$$\begin{aligned}
&\mathbb{E}\left[\exp\{it\delta_n\langle\Delta_0^n M^n, \Delta L_{T(n,i)_1}\rangle\mathbb{1}_{\{\tau_i=1\}}\} - 1\right]\\
&= \mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta_0^n M^n, \Delta L_{\overline{T}(n,i)_1}\mathbb{1}_{\{\overline{\tau}_i=1\}}\rangle\right\} - 1\right]\\
&+ \mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta_0^n M^n, \Delta L_{T(n,i,-)_1}\mathbb{1}_{\{\tau_{i,-}=1\}}\rangle\right\} - 1\right] + o(1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta_0^n M^n, \Delta L_{T(n,i+1)_1}\mathbb{1}_{\{\tau_i=1\}}\rangle\right\} - 1\right]\\
&= \mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta_0^n M^n, \Delta L_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}\rangle\right\} - 1\right]\\
&+ \mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta_0^n M^n, \Delta L_{\overline{T}(n,i+1)_1}\mathbb{1}_{\{\overline{\tau}_{i+1}=1\}}\rangle\right\} - 1\right] + o(1).
\end{aligned}$$

Proof. Let $t \in \mathbb{R}$. We denote by $\alpha_i^n := \exp(-\Delta_{t_i^1}\overline{G}(v_n))$, $\overline{\alpha}_i^n = \exp(-\overline{\Delta}_{t_i}\overline{G}(v_n))$ and $\alpha_{i,-}^n = \exp(-\Delta_{t_{i,-}}\overline{G}(v_n))$. Observe first that, for some increment $\Delta_0^n M$ (see [70, Section 4.3.2])

$$\begin{aligned}
&\mathbb{E}[\exp\{it\delta_n\langle\Delta_0^n M, \Delta L_{T(n,i)_1}\rangle\mathbb{1}_{\{\tau_i=1\}}\} - 1]\\
&= \alpha_i^n\Delta_0^n\Delta_{t_i^1}\int_{\|y\|\in(v_n,1]}\left(\int_{\|x\|\in(0,v_n]}(\exp\{it\delta_n\langle x,y\rangle\} - 1 - it\delta_n\langle x,y\rangle)\,G(dx)\right)G(dy) + o(\Delta_n)\\
&\sim \Delta_0^n\Delta_{t_i^1}\int_{\|y\|\in(v_n,1]}\left(\int_{\|x\|\in(0,v_n]}(\exp\{it\delta_n\langle x,y\rangle\} - 1 - it\delta_n\langle x,y\rangle)\,G(dx)\right)G(dy)
\end{aligned}$$

where we used the fact that $\alpha_i^n = \exp(-\Delta_{t_i^1}\overline{G}(v_n)) \to 1$ due to the upper bound $\overline{G}(v_n) \le C\Delta_n^{-1/2}\log(1/\Delta_n)^{-\beta}$. On the other hand,

$$\begin{aligned}
&\mathbb{E}[\exp\{it\delta_n(\langle\Delta_0^n M, \Delta L_{\overline{T}(n,i)_1}\rangle\mathbb{1}_{\{\overline{\tau}_i=1\}} + \langle\Delta_0^n M, \Delta L_{T(n,i,-)_1}\rangle\mathbb{1}_{\{\tau_{i,-}=1\}})\} - 1]\\
&= \mathbb{P}(\mathbb{1}_{\{\overline{\tau}_i=1\}} = 1)\mathbb{P}(\mathbb{1}_{\{\tau_{i,-}=1\}} = 1)\mathbb{E}[\exp\{it\delta_n(\langle\Delta_0^n M, \Delta L_{\overline{T}(n,i)_1}\rangle + \langle\Delta_0^n M, \Delta L_{T(n,i,-)_1}\rangle)\}]\\
&+ \mathbb{P}(\mathbb{1}_{\{\overline{\tau}_i=1\}} = 1)(1 - \mathbb{P}(\mathbb{1}_{\{\tau_{i,-}=1\}} = 1))\mathbb{E}[\exp\{it\delta_n\langle\Delta_0^n M, \Delta L_{\overline{T}(n,i)_1}\rangle\}]\\
&+ (1 - \mathbb{P}(\mathbb{1}_{\{\overline{\tau}_i=1\}} = 1))\mathbb{P}(\mathbb{1}_{\{\tau_{i,-}=1\}} = 1)\mathbb{E}[\exp\{it\delta_n\langle\Delta_0^n M, \Delta L_{T(n,i,-)_1}\rangle\}]\\
&+ (1 - \mathbb{P}(\mathbb{1}_{\{\overline{\tau}_i=1\}} = 1))(1 - \mathbb{P}(\mathbb{1}_{\{\tau_{i,-}=1\}} = 1)) - 1\\
&= \overline{\alpha}_i^n\alpha_{i,-}^n\overline{\Delta}_{t_i}\Delta_{t_{i,-}}\overline{G}(v_n)^2\mathbb{E}[\exp\{it\delta_n(\langle\Delta_0^n M, \Delta L_{\overline{T}(n,i)_1}\rangle + \langle\Delta_0^n M, \Delta L_{T(n,i,-)_1}\rangle)\}]\\
&+ \overline{\alpha}_i^n\overline{\Delta}_{t_i}\overline{G}(v_n)(1 - \alpha_{i,-}^n\Delta_{t_{i,-}}\overline{G}(v_n))\mathbb{E}[\exp\{it\delta_n\langle\Delta_0^n M, \Delta L_{\overline{T}(n,i)_1}\rangle\}] - \overline{\alpha}_i^n\overline{\Delta}_{t_i}\overline{G}(v_n)\\
&+ \alpha_{i,-}^n\Delta_{t_{i,-}}\overline{G}(v_n)(1 - \overline{\alpha}_i^n\overline{\Delta}_{t_i}\overline{G}(v_n))\mathbb{E}[\exp\{it\delta_n\langle\Delta_0^n M, \Delta L_{T(n,i,-)_1}\rangle\}] - \alpha_{i,-}^n\Delta_{t_{i,-}}\overline{G}(v_n)\\
&+ \overline{\alpha}_i^n\alpha_{i,-}^n\overline{\Delta}_{t_i}\Delta_{t_{i,-}}\overline{G}(v_n)^2.
\end{aligned}$$

In view of Lemma III.4.1, the first term in the decomposition is negligible. We also have the negligibility of the last term of the decomposition. It follows that

$$\begin{aligned}
&\mathbb{E}[\exp\{it\delta_n(\langle \Delta_0^n M, \Delta L_{\overline{T}(n,i)_1}\rangle \mathbb{1}_{\{\overline{\tau}_i=1\}} + \langle \Delta_0^n M, \Delta L_{T(n,i,-)_1}\rangle \mathbb{1}_{\{\tau_{i,-}=1\}})\} - 1] \\
&= (1-\alpha_{i,-}^n \Delta_{t_{i,-}} \overline{G}(v_n))\overline{\alpha}_i^n \overline{\Delta}_{t_i} \Delta_0^n \int_{\|y\|\in(v_n,1]} \left(\int_{\|x\|\in(0,v_n]} \left(e^{it\delta_n\langle x,y\rangle} - 1 - it\delta_n\langle x,y\rangle\right) G(dx)\right) G(dy) \\
&+ (1-\overline{\alpha}_i^n \overline{\Delta}_{t_i} \overline{G}(v_n))\alpha_{i,-}^n \Delta_{t_{i,-}} \Delta_0^n \int_{\|y\|\in(v_n,1]} \left(\int_{\|x\|\in(0,v_n]} \left(e^{it\delta_n\langle x,y\rangle} - 1 - it\delta_n\langle x,y\rangle\right) G(dx)\right) G(dy) \\
&+ o(\Delta_n) \\
&\sim (\overline{\Delta}_{t_i} + \Delta_{t_{i,-}})\Delta_0^n \int_{\|y\|\in(v_n,1]} \left(\int_{\|x\|\in(0,v_n]} \left(e^{it\delta_n\langle x,y\rangle} - 1 - it\delta_n\langle x,y\rangle\right) G(dx)\right) G(dy) \\
&= \Delta_{t_i^1} \Delta_0^n \int_{\|y\|\in(v_n,1]} \left(\int_{\|x\|\in(0,v_n]} \left(e^{it\delta_n\langle x,y\rangle} - 1 - it\delta_n\langle x,y\rangle\right) G(dx)\right) G(dy)
\end{aligned}$$

hence the result. □

We can rewrite X_i (and X_{i+1}) using Lemma III.4.4 and (III.4.9) :

$$\begin{aligned}
X_i = \delta_n \Bigg(&\langle \Delta_{t_{i-1,-}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i)_1} \mathbb{1}_{\{\overline{\tau}_i=1\}} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}} \rangle \\
&+ \langle \overline{\Delta}_{t_i} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i)_1} \mathbb{1}_{\{\overline{\tau}_i=1\}} \\
&\qquad + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} \rangle \\
&+ \langle \Delta_{t_{i,-}} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i)_1} \mathbb{1}_{\{\overline{\tau}_i=1\}} \\
&\qquad + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} \rangle \\
&+ \langle \Delta_{t_{i,+}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i)_1} \mathbb{1}_{\{\overline{\tau}_i=1\}} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}} \rangle \Bigg).
\end{aligned}$$

From this we get

$$\begin{aligned}
X_i + X_{i+1} = \delta_n \Bigg(&\langle \Delta_{t_{i-1,-}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i)_1} \mathbb{1}_{\{\overline{\tau}_i=1\}} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}} \rangle \\
&+ \langle \overline{\Delta}_{t_i} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i)_1} \mathbb{1}_{\{\overline{\tau}_i=1\}} \\
&\qquad + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} \rangle \\
&+ \langle \Delta_{t_{i,-}} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i)_1} \mathbb{1}_{\{\overline{\tau}_i=1\}} \\
&\qquad + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}} + \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} \\
&\qquad + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i+1)_1} \mathbb{1}_{\{\overline{\tau}_{i+1}=1\}} \rangle
\end{aligned}$$

$$+ \langle \Delta_{t_{i,+}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i)_1} \mathbb{1}_{\{\overline{\tau}_i=1\}} + \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}}$$

$$+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i+1)_1} \mathbb{1}_{\{\overline{\tau}_{i+1}=1\}}$$

$$+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i+1,+)_1} \mathbb{1}_{\{\tau_{i+1,+}=1\}} \rangle$$

$$+ \langle \overline{\Delta}_{t_{i+1}} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}}$$

$$+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i+1)_1} \mathbb{1}_{\{\overline{\tau}_{i+1}=1\}} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i+1,+)_1} \mathbb{1}_{\{\tau_{i+1,+}=1\}} \rangle$$

$$+ \langle \Delta_{t_{i+1,+}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i+1)_1} \mathbb{1}_{\{\overline{\tau}_{i+1}=1\}} \rangle \Big) .$$

Using Proposition III.4.3 the characteristic function of $X_i + X_{i+1}$ is the sum of the characteristic function of each term in $X_i + X_{i+1}$. Similarly, the characteristic function of X_i (resp. X_{i+1}) is the sum of the characteristic function of each term in X_i (resp. X_{i+1}). By observing that

$$\mathbb{E}\left[(e^{itX_i} - 1)(e^{itX_{i+1}} - 1)\right] = \mathbb{E}\left[e^{it(X_i + X_{i+1})} - 1\right] - \left(\mathbb{E}\left[e^{itX_i} - 1\right] + \mathbb{E}\left[e^{itX_{i+1}} - 1\right]\right)$$

we obtain

$$\begin{aligned}
&\mathbb{E}\left[(e^{itX_i} - 1)(e^{itX_{i+1}} - 1)\right] \\
&= \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_{t_{i,-}} M^n, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \rangle \mathbb{1}_{\{\tau_{i,+}=1\}} \right\} - 1\right] \\
&- \left(\mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_{t_{i,-}} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \rangle \mathbb{1}_{\{\tau_{i,+}=1\}} \right\} - 1\right]\right. \\
&\left. + \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_{t_{i,-}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \rangle \mathbb{1}_{\{\tau_{i,+}=1\}} \right\} - 1\right]\right) \qquad \text{(III.4.10)} \\
&+ \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_{t_{i,+}} M^n, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \rangle \mathbb{1}_{\{\tau_{i,-}=1\}} \right\} - 1\right] \\
&- \left(\mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_{t_{i,+}} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \rangle \mathbb{1}_{\{\tau_{i,-}=1\}} \right\} - 1\right]\right. \\
&\left. + \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta_{t_{i,+}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_1} \rangle \mathbb{1}_{\{\tau_{i,-}=1\}} \right\} - 1\right]\right) . \qquad \text{(III.4.11)}
\end{aligned}$$

Using again Proposition III.4.3 we can regroup the two terms in (III.4.10) and the two terms in (III.4.11) to obtain that

$$\mathbb{E}\left[(e^{itX_i} - 1)(e^{itX_{i+1}} - 1)\right] = 0$$

hence (III.4.6) is verified and the approximation (III.4.4) holds.

III.4.2.3 The limit distribution

We have that

$$\log\left(\mathbb{E}\left[\exp\left\{it \sum_{i=1}^{n_1} X_i\right\} - 1\right]\right)$$

$$= \sum_{i=1}^{n_1} \Bigg(\mathbb{E}\left[\exp\left\{ it\delta_n \langle \Delta_{t_i^1} M^n, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1} \rangle \mathbb{1}_{\{\tau_i=1\}} \right\} - 1\right] \tag{III.4.12}$$

$$+ \mathbb{E}\left[\exp\left\{ it\delta_n \langle \Delta_{t_i^1} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)_1} \rangle \mathbb{1}_{\{\tau_{i-1,-}=1\}} \right\} - 1\right] \tag{III.4.13}$$

$$+ \mathbb{E}\left[\exp\left\{ it\delta_n \langle \Delta_{t_i^1} M^n, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \rangle \mathbb{1}_{\{\tau_{i,+}=1\}} \right\} - 1\right] \tag{III.4.14}$$

$$+ \mathbb{E}\left[\exp\left\{ it\delta_n \langle \Delta_{t_{i-1,-}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1} \rangle \mathbb{1}_{\{\tau_i=1\}} \right\} - 1\right] \tag{III.4.15}$$

$$+ \mathbb{E}\left[\exp\left\{ it\delta_n \langle \Delta_{t_{i,+}} M^n, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_1} \rangle \mathbb{1}_{\{\tau_i=1\}} \right\} - 1\right] \Bigg) \tag{III.4.16}$$

$$+ o(1).$$

Using the results from [70, Section 4.3.2], we get that

$$\text{(III.4.12)} = n \sum_{i=1}^{n_1} \alpha_i^{n_1} (\Delta_{t_i^1})^2 \int \left\{ e^{itz} - 1 - itz \right\} \nu_n^0(dz) + o(1),$$

$$\text{(III.4.13)} + \text{(III.4.14)} = n \sum_{i=1}^{n_1} \Delta_{t_i^1} (\alpha_{i-1,-}^n \Delta_{t_{i-1,-}} + \alpha_{i,+}^n \Delta_{t_{i,+}}) \int \left\{ e^{itz} - 1 - itz \right\} \nu_n^1(dz) + o(1),$$

$$\text{(III.4.15)} + \text{(III.4.16)} = n \sum_{i=1}^{n_1} \alpha_i^{n_1} \Delta_{t_i^1} (\Delta_{t_{i-1,-}} + \Delta_{t_{i,+}}) \int \left\{ e^{itz} - 1 - itz \right\} \nu_n^2(dz) + o(1),$$

with

$$\nu_n^0(A) = \frac{1}{n} \int\limits_{\|x\| \le v_n} \left(\int_{v_n < \|y\| \le 1} \mathbb{1}_A \left(\delta_n \langle x, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y \rangle \right) G(dx) \right) G(dy),$$

$$\nu_n^1(A) = \frac{1}{n} \int\limits_{\|x\| \le v_n} \left(\int_{v_n < \|y\| \le 1} \mathbb{1}_A \left(\delta_n \langle x, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y \rangle \right) G(dx) \right) G(dy),$$

$$\nu_n^2(A) = \frac{1}{n} \int\limits_{\|x\| \le v_n} \left(\int_{v_n < \|y\| \le 1} \mathbb{1}_A \left(\delta_n \langle x, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y \rangle \right) G(dx) \right) G(dy).$$

Since G is a symmetric distribution, we have that $\nu_n^1(A) = \nu_n^2(A)$. We recall that $\alpha_i^{n_1} \to 1$, $\alpha_{i,-}^n \to 1$ and $\alpha_{i,+}^n \to 1$. Combining these two facts, we get that

$$\log \left(\mathbb{E}\left[\exp\left\{ it \sum_{i=1}^{n_1} X_i \right\} - 1 \right] \right)$$
$$\sim F^n(1) \int \left\{ e^{itz} - 1 - itz \right\} \nu_n^0(dz) + 2G^n(1) \int \left\{ e^{itz} - 1 - itz \right\} \nu_n^1(dz).$$

Denote $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $P_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. By [70, Section 4.3.2], we have $\nu_n^k(A) \to \nu_U^k(A)$, $k = 0, 1$ where

$$\nu_U^k(A) = \frac{1}{2\beta} \int_{\mathbb{S}_2^2} \mu^k(dz) \int_0^\infty \mathbb{1}_A(\rho z) \rho^{-1-\beta} d\rho, \quad k = 0, 1$$

with

$$\mu^k(z) = \int_{\mathbb{S}_2^2} \mathbb{1}_z\left(\frac{\langle \theta^x, P_k\theta^y\rangle}{|\langle \theta^x, P_k\theta^y\rangle|}\right) |\langle \theta^x, P_k\theta^y\rangle|^\beta \, H(d\theta^x)H(d\theta^y), \quad z \in \mathcal{B}(\mathbb{S}_1).$$

It follows that

$$\begin{aligned}&\log\left(\mathbb{E}\left[\exp\left\{it\sum_{i=1}^{n_1} X_i\right\} - 1\right]\right)\\ &\longrightarrow F(1)\int \left\{e^{itz} - 1 - itz\right\}\nu_U^0(dz) + 2G(1)\int \left\{e^{itz} - 1 - itz\right\}\nu_U^1(dz) \quad \text{(III.4.17)}\end{aligned}$$

We recall the following result from [143, Lemma 14.11]. For $\beta \in (1,2)$ and $x > 0$ we have

$$\int_0^\infty \left(e^{i\rho x} - 1 - i\rho x\right)\rho^{-1-\beta}d\rho = |x|^\beta\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)\left(1 - i\tan\left(\frac{\pi\beta}{2}\right)\operatorname{sgn}(x)\right).$$

From this lemma, we obtain

$$\begin{aligned}&\int \left(e^{itz} - 1 - itz\right)\nu_U^k(dz)\\ &= \frac{|t|^\beta}{2\beta}\int_{S_2^2} |\langle \theta^x, P_k\theta^y\rangle|^\beta\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)\left(1 - i\tan\left(\frac{\pi\beta}{2}\right)\operatorname{sgn}\left(\frac{t\langle \theta^x, P_k\theta^y\rangle}{|\langle \theta^x, P_k\theta^y\rangle|}\right)\right)H(d\theta^x)H(d\theta^y).\end{aligned}$$

Since H is a symmetric measure, it follows that

$$\int_{\mathbb{S}_2^2} |\langle \theta^x, P_k\theta^y\rangle|^\beta \operatorname{sgn}\left(\frac{t\langle \theta^x, P_k\theta^y\rangle}{|\langle \theta^x, P_k\theta^y\rangle|}\right)H(d\theta^x)H(d\theta^y) = 0$$

due to the integrand being an odd function. We conclude that

$$\int \left(e^{itz} - 1 - itz\right)\nu_U^k(dz) = -|t|^\beta\sigma_\beta^k$$

with

$$\sigma_\beta^k := \frac{-\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)}{2\beta}\int_{S_2^2} |\langle \theta^x, P_k\theta^y\rangle|^\beta H(d\theta^x)H(d\theta^y).$$

Finally,

$$\text{(III.4.17)} = -\left(F(1)\sigma_\beta^0 + 2G(1)\sigma_\beta^1\right)|t|^\beta$$

hence the result of Theorem III.2.3 in the case $\beta \in (1,2)$.

III.5 Proof of Theorem III.2.3 in the case $\beta \in (0,1]$

III.5.1 Negligible terms

As for the case $\beta \in (1,2)$, the terms $\xi_i^n(j),\ j = 1,3,4,5$ come from the synchronous part of the decomposition of the error process and therefore the asynchronicity of the sampling scheme does not appear in these terms. Combined with Remark III.3.1 we can use the result from [70, Section 4.4.1] and conclude that $\xi_i^n(j)$, $j = 1,3,4,5$ are negligible. We consider the case $j = 6$. Set $\xi_i^n(6) = \eta_{i,-}^n + \eta_{i,+}^n$ with

$$\eta_{i,-}^n = \delta_n \Delta_{t_i^1} M^{n,1} \Delta_{t_{i-1,-}} M^{n,2}, \quad \eta_{i,+}^n = \delta_n \Delta_{t_i^1} M^{n,1} \Delta_{t_{i,+}} M^{n,2}.$$

Suppose first that $\beta = 1$. Observe that

$$\mathbb{E}[\eta_{i,-}^n | \mathcal{F}_{t_{i-1}^1}] = \delta_n \mathbb{E}[\Delta_{t_i^1} M^{n,1}] \mathbb{E}[\Delta_{t_{i-1,-}} M^{n,2} | \mathcal{F}_{t_{i-1}^1}] = 0,$$
$$\mathbb{E}[(\eta_{i,-}^n)^2] = \delta_n^2 \mathbb{E}[(\Delta_{t_i^1} M^{n,1})^2] \mathbb{E}[(\Delta_{t_{i-1,-}} M^{n,2})^2].$$

By (III.3.5) applied to $p = 2$ we obtain that

$$\mathbb{E}[(\eta_{i,-}^n)^2] \le C \delta_n^2 \Delta_n^2 v_n^2 = C \Delta_n^2.$$

It follows that $\sum_{i=1}^{n_1} \mathbb{E}[(\eta_{i,-}^n)^2] \to 0$ and by (III.3.2) we have

$$\sum_{i=1}^{n_1} \eta_{i,-}^n \overset{u.c.p.}{\longrightarrow} 0.$$

Similarly,

$$\mathbb{E}[\eta_{i,+}^n | \mathcal{F}_{t_i^1}] = 0,$$
$$\mathbb{E}[(\eta_{i,+}^n)^2] \le C \Delta_n^2$$

by (III.3.5) applied to $p = 2$. Therefore $\sum_{i=1}^{n_1} \mathbb{E}[(\eta_{i,+}^n)^2] \to 0$. By (III.3.2), $\sum_{i=1}^{n_1} \eta_{i,+}^n \overset{u.c.p.}{\longrightarrow} 0$ and finally

$$\sum_{i=1}^{n_1} \xi_i^n(6) \overset{u.c.p.}{\longrightarrow} 0.$$

Suppose now that $\beta < 1$. By (III.3.5) applied to $p = 1$, we have

$$\mathbb{E}[|\eta_{i,-}^n| \wedge 1] + \mathbb{E}[|\eta_{i,+}^n| \wedge 1] \le C \delta_n \Delta_n^2 v_n^{1-2\beta} = C \Delta_n^{1/\beta} \log(1/\Delta_n)^{(1-2\beta)/\beta}$$

and by (III.3.1) it follows that

$$\sum_{i=1}^{n_1} \xi_i^n(6) \overset{u.c.p.}{\longrightarrow} 0.$$

For $j = 8$, observe that

$$|\delta_n \Delta_{t_i^1} M^{n,1} \Delta_{t_{i,+}} A^{n,2} \mathbb{1}_{\{\tau_{i,+} \ge 2\}}| \le \delta_n |\Delta_{t_i^1} M^{n,1}| \sum_{k \ge 1} |\Delta L^2_{T(n,i,+)_k}| \mathbb{1}_{\{\tau_{i,+} \ge 2 \vee k\}}.$$

By independence of A^n and M^n we also have

$$\mathbb{E}[|\delta_n \Delta_{t_i^1} M^{n,1} \Delta_{t_{i,+}} A^{n,2} \mathbb{1}_{\{\tau_{i,+} \ge 2\}}|] = \delta_n \mathbb{E}[|\Delta_{t_i^1} M^{n,1}|] \mathbb{E}[|\Delta_{t_{i,+}} A^{n,2}| \mathbb{1}_{\{\tau_{i,+} \ge 2\}}].$$

Suppose that $\beta = 1$. By (III.3.5) applied to $p = 2$ we have $\mathbb{E}[|\Delta_{t_i^1} M^{n,1}|^2] \le C \Delta_n v_n$ and therefore by Jensen inequality

$$\mathbb{E}[|\Delta_{t_i^1} M^{n,1}|] \le \sqrt{\mathbb{E}[|\Delta_{t_i^1} M^{n,1}|^2]} \le C \Delta_n^{1/2} v_n^{1/2}.$$

Applying (III.3.7) for $p=\beta=1$ and $w=1$, we get

$$\begin{aligned}\sum_{k\geq 1}\mathbb{E}[|\Delta L^2_{T(n,i,+)_k}|\mathbb{1}_{\{\tau_{i,+}\geq 2\vee k\}}] &\leq C\sum_{k\geq 1}\Delta_n\left(\frac{b\Delta_n}{v_n}\right)^{2\vee k-1}\log(1/\Delta_n)\\ &= C\Delta_n^2 v_n^{-1}\log(1/\Delta_n)\left(2b+\sum_{k\geq 1}\left(\frac{b\Delta_n}{v_n}\right)^k\right)\\ &\leq C\Delta_n^2 v_n^{-1}\log(1/\Delta_n)\end{aligned}$$

since there exists n_0 such that $b\Delta_n v_n^{-1}<1$ for all $n\geq n_0$. It follows that

$$\mathbb{E}[|\delta_n\Delta_{t_i^1}M^{n,1}\Delta_{t_{i,+}}A^{n,2}\mathbb{1}_{\{\tau_{i,+}\geq 2\}}|]\leq C\Delta_n^{5/2}v_n^{-1/2}\log(1/\Delta_n)=C\Delta_n^2\log(1/\Delta_n)^{1/2}.$$

We can similarly show that

$$\begin{aligned}&\mathbb{E}[|\delta_n\Delta_{t_i^1}M^{n,1}\Delta_{t_{i-1,-}}A^{n,2}\mathbb{1}_{\{\tau_{i-1,-}\geq 2\}}|]\leq C\Delta_n^2\log(1/\Delta_n)^{1/2}\\ &\mathbb{E}[|\delta_n\Delta_{t_i^1}A^{n,1}\mathbb{1}_{\{\tau_i\geq 2\}}(\Delta_{t_{i,+}}M^{n,2}+\Delta_{t_{i,-}}M^{n,2})]\leq C\Delta_n^2\log(1/\Delta_n)^{1/2}.\end{aligned}$$

Suppose now that $\beta<1$. By (III.3.5) applied to $p=1$ we have

$$\mathbb{E}[|\Delta_{t_i^1}M^{n,1}|]\leq C\Delta_n v_n^{1-\beta}.$$

Applying (III.3.7) with $w=1$ we obtain

$$\sum_{k\geq 1}\mathbb{E}[|\Delta L^2_{T(n,i,+)_k}|\mathbb{1}_{\{\tau_{i,+}\geq 2\vee k\}}]\leq C\sum_{k\geq 1}\Delta_n\left(\frac{b\Delta_n}{v_n^\beta}\right)^{2\vee k-1}\leq C\Delta_n^2 v_n^{-\beta}$$

and

$$\mathbb{E}[|\delta_n\Delta_{t_i^1}M^{n,1}\Delta_{t_{i,+}}A^{n,2}\mathbb{1}_{\{\tau_{i,+}\geq 2\}}|]\leq C\delta_n\Delta_n^3 v_n^{1-2\beta}=C\Delta_n^{1+1/\beta}\log(1/\Delta_n)^{(1-2\beta)/\beta}.$$

As before we can show similarly that

$$\begin{aligned}&\mathbb{E}[|\delta_n\Delta_{t_i^1}M^{n,1}\Delta_{t_{i-1,-}}A^{n,2}\mathbb{1}_{\{\tau_{i-1,-}\geq 2\}}|]\leq C\Delta_n^{1+1/\beta}\log(1/\Delta_n)^{(1-2\beta)/\beta}\\ &\mathbb{E}[|\delta_n\Delta_{t_i^1}A^{n,1}\mathbb{1}_{\{\tau_i\geq 2\}}(\Delta_{t_{i,+}}M^{n,2}+\Delta_{t_{i,-}}M^{n,2})]\leq C\Delta_n^{1+1/\beta}\log(1/\Delta_n)^{(1-2\beta)/\beta}.\end{aligned}$$

From this, by (III.3.1) and for $0<\beta\leq 1$ we have

$$\sum_{i=1}^{n_1}\xi_i^n(j)\xrightarrow{u.c.p.}0,\quad j=8,12.$$

We consider now the case $j=7,9$. Suppose first that $\beta=1$. Observe that

$$\begin{aligned}&\mathbb{E}[\delta_n\Delta_{t_i^1}M^{n,1}\Delta L^2_{T(n,i-1,-)_1}\mathbb{1}_{\{\tau_{i-1,-}=1\}}|\mathcal{F}_{t^1_{i-1}}]=\delta_n\Delta L^2_{T(n,i-1,-)_1}\mathbb{1}_{\{\tau_{i-1,-}=1\}}\mathbb{E}[\Delta_{t_i^1}M^{n,1}]=0,\\ &\mathbb{E}[\delta_n\Delta_{t_i^1}M^{n,1}\Delta L^2_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}|\mathcal{F}_{t_i^1}]=\delta_n\Delta_{t_i^1}M^{n,1}\mathbb{E}[\Delta L^2_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}]=0\end{aligned}$$

since G is symmetric and therefore $\Delta L^2_{T(n,i,+)_1}$ has a symmetric distribution. On the other hand, using (III.3.5) with $p=2$ and (III.3.7) with $p=2$, $w=1$ and $j=1$ we obtain

$$\mathbb{E}[(\delta_n\Delta_{t_i^1}M^{n,1}(\Delta L^2_{T(n,i-1,-)_1}\mathbb{1}_{\{\tau_{i-1,-}=1\}}+\Delta L^2_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}))^2]=C\delta_n^2\Delta_n^2 v_n=C\Delta_n\log(1/\Delta_n)^{-1}.$$

It follows by (III.3.2) that

$$\sum_{i=1}^{n_1} \xi_i^n(7) \xrightarrow{u.c.p.} 0.$$

If $\beta < 1$ we have by (III.3.5) for $p = 1$ and by (III.3.7) for $m = p = j = w = 1$ that

$$\mathbb{E}[\xi_i^n(7)| \wedge 1] \leq C\delta_n \Delta_n^2 v_n^{1-\beta} = C\Delta_n \log(1/\Delta_n)^{-1}.$$

By (III.3.1) we get that

$$\sum_{i=1}^{n_1} \xi_i^n(7) \xrightarrow{u.c.p.} 0.$$

We can prove similarly that

$$\sum_{i=1}^{n_1} \xi_i^n(9) \xrightarrow{u.c.p.} 0, \quad \beta \leq 1.$$

We finally consider the cases $j = 11, 13$. We will only prove the negligibility of $\xi_i^n(11)$, as the proof for $\xi_i^n(13)$ is essentially the same. Since $\Delta_{t_i^1}$ and $\Delta_{t_{i-1,-}}$ (resp. $\Delta_{t_{i,+}}$) are non overlapping, $\Delta L^1_{T(n,i)_1}$ and $\Delta_{t_{i-1,-}} A^{n,2}$ (resp. $\Delta_{t_{i,+}} A^{n,2}$) are independent. Therefore,

$$\mathbb{E}[|\xi_i^n(11)|] = \delta_n \mathbb{E}[|\Delta L^1_{T(n,i)_1}| \mathbb{1}_{\{\tau_i=1\}}] \left(\mathbb{E}[|\Delta_{t_{i-1,-}} A^{n,2}| \mathbb{1}_{\{\tau_{i-1,-}\geq 2\}}] + \mathbb{E}[|\Delta_{t_{i,+}} A^{n,2}| \mathbb{1}_{\{\tau_{i,+}\geq 2\}}]\right).$$

Suppose that $\beta = 1$. As before,

$$\sum_{k\geq 1} \mathbb{E}[|\Delta L^2_{t(n,i-1,-)_k}| \mathbb{1}_{\{\tau_{i-1,-}\geq 2\vee k\}}] + \sum_{k\geq 1} \mathbb{E}[|\Delta L^2_{t(n,i,+)_k}| \mathbb{1}_{\{\tau_{i,+}\geq 2\vee k\}}] \leq C\Delta_n^2 v_n^{-1} \log(1/\Delta_n)$$

and by (III.3.7) applied to $w = p = 1$,

$$\mathbb{E}[(|\Delta L^1_{T(n,i)_1}| \wedge 1)\mathbb{1}_{\{\tau_i=1\}}] \leq C\Delta_n \log(1/\Delta_n).$$

We obtain that

$$\mathbb{E}[|\xi_i^n(11)|] \leq C\Delta_n^3 v_n^{-1} \log(1/\Delta_n)^2 = C\Delta_n^2 \log(1/\Delta_n).$$

If $\beta < 1$,

$$\sum_{k\geq 1} \mathbb{E}[|\Delta L^2_{t(n,i-1,-)_k}| \mathbb{1}_{\{\tau_{i-1,-}\geq 2\vee k\}}] + \sum_{k\geq 1} \mathbb{E}[|\Delta L^2_{t(n,i,+)_k}| \mathbb{1}_{\{\tau_{i,+}\geq 2\vee k\}}] \leq C\Delta_n^2 v_n^{-\beta}$$

and by (III.3.7) applied with $w = p = 1$,

$$\mathbb{E}[(|\Delta L^1_{T(n,i)_1}| \wedge 1)\mathbb{1}_{\{\tau_i=1\}}] \leq C\Delta_n.$$

It follows that

$$\mathbb{E}[|\xi_i^n(11)|] \leq C\Delta_n^3 v_n^{-\beta} = C\Delta_n^2 \log(1/\Delta_n)^{-1}.$$

Finally, by (III.3.1), for $\beta \geq 1$,

$$\sum_{i=1}^{n_1} \xi_i^n(11) \xrightarrow{u.c.p.} 0.$$

We have shown that $\sum_{i=1}^{n_1} \xi_i^n(j) \xrightarrow{u.c.p.} 0$ for $j = 1, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13$.

III.5.2 The dominating term

III.5.2.1 The synchronous case

The weak convergence

$$\sum_{i=1}^{\lfloor 1/\Delta_n \rfloor} \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2} \rangle \mathbb{1}_{\{\tau_i=2\}} \xrightarrow{d_{st}} U_1$$

was proven in [70]. Their approach consists of a direct computation of the Lévy measure when $\beta \leq 1$, and characteristic function techniques when $\beta \in (1,2)$. Our result on asynchronicity relies on the approximation of the characteristic function from Lemma III.3.5 hence the need to derive the same result using the same techniques as for the case $\beta \in (1,2)$. Assume that for all i, $\Delta_i^n = \Delta_n$. Using Lemma III.3.6 and Remark III.3.7, we need to show that

$$\sum_{i=1}^{\lfloor 1/\Delta_n \rfloor} \mathbb{E}[e^{it\delta_n \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2} \rangle \mathbb{1}_{\{\tau_i=2\}}} - 1] \xrightarrow{u.c.p.} \begin{cases} \int \{e^{itx} - 1\} \nu_U(dx) & \text{if} \quad \beta \in (0,1) \\ \int \left\{e^{itx} - 1 - itx \mathbb{1}_{\{0<|x|\leq 1\}}\right\} \nu_U(dx) & \text{if} \quad \beta = 1, \end{cases}$$

where ν_U was defined in [70, Theorem 2.2]. We have

$$\begin{aligned}
&\sum_{i=1}^{\lfloor \Delta_n^{-1} \rfloor} \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2} \rangle \mathbb{1}_{\{\tau_i=2\}}\right\} - 1\right] \\
&= \sum_{i=1}^{\lfloor \Delta_n^{-1} \rfloor} \left(\mathbb{P}\left(\mathbb{1}_{\{\tau_i=2\}} = 1\right) \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2} \rangle\right\}\right] - 1 + \left(1 - \mathbb{P}\left(\mathbb{1}_{\{\tau_i=2\}} = 1\right)\right) \right) \\
&= \sum_{i=1}^{\lfloor \Delta_n^{-1} \rfloor} \left(\frac{(\Delta_i^n)^2}{2} \alpha_i^n \overline{G(v_n)}^2 \int_{||y|| \in (v_n,1)} \int_{||x|| \in (v_n,1)} e^{it\delta_n \langle x,y \rangle} \frac{G(dx)}{\overline{\overline{G(v_n)}}} \frac{G(dy)}{\overline{\overline{G(v_n)}}} - \frac{(\Delta_i^n)^2}{2} \alpha_i^n \overline{G(v_n)}^2 \right) \\
&= \begin{cases} \Delta_n^{-1} \dfrac{(\Delta_n)^2}{2} \alpha_n \displaystyle\int_{||y|| \in (v_n,1)} \int_{||x|| \in (v_n,1)} \left(e^{it\delta_n \langle x,y \rangle} - 1\right) G(dx) G(dy), \ \beta < 1 \\ \Delta_n^{-1} \dfrac{(\Delta_n)^2}{2} \alpha_n \displaystyle\int_{||y|| \in (v_n,1)} \int_{||x|| \in (v_n,1)} \left(e^{it\delta_n \langle x,y \rangle} - 1 - it\delta_n \langle x,y \rangle \mathbb{1}_{\{0<|\langle x,y \rangle| \leq 1\}}\right) G(dx) G(dy), \ \beta = 1 \end{cases} \qquad \text{(III.5.1)} \\
&= \begin{cases} \alpha_n \displaystyle\int \left(e^{itz} - 1\right) \nu_n(dz) & \text{if } \beta < 1 \\ \alpha_n \displaystyle\int \left(e^{itz} - 1 - itz \mathbb{1}_{\{0<|z|\leq 1\}}\right) \nu_n(dz) & \text{if } \beta = 1. \end{cases}
\end{aligned}$$

where

$$\nu_n(A) = \frac{\Delta_n}{2} \int_{||y|| \in (v_n,1)} \int_{||x|| \in (v_n,1)} \mathbb{1}_A(\delta_n \langle x,y \rangle) G(dx) G(dy).$$

The equality (III.5.1) is true due to the fact that G is symmetric and therefore

$$-it\delta_n \int_{\|y\| \in (v_n,1]} \int_{\|x\| \in (v_n,1]} \langle x,y \rangle G(dx) G(dy) = 0.$$

Observe that $\overline{G}(x) \le Cx^{-\beta}$, therefore $\overline{G}(v_n) \le C\Delta_n^{-1}\log(1/\Delta_n)^{-1}$ and $\alpha_n = \exp(-\Delta_n \overline{G}(v_n)) \to 1$ and we can omit this term in the limit. We will now show that the Levy measure ν_n converges. To do so, we need to show the conditions of [143, Theorem 8.7]. It is sufficient to show the following :

1. $\lim_{n\to\infty} \nu_n(B \times (w,\infty)) = \nu_U(B\times(w,\infty))$ for $B \in \mathcal{B}(\mathbb{S}_1)$ and $w > 0$;

2. $\lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \int_{|z|\le\varepsilon} |z|^2 \nu_n(dz) = 0.$

We start by proving the first condition. We recall that $G(dx) = \frac{\mathbb{1}_{(0,1]}}{\rho^{1+\beta}} d\rho H(d\theta)$, $x = (\rho,\theta) \in \mathbb{R}_+ \times \mathbb{S}_2$, with H a symmetric spherical measure. We therefore obtain the following expression :

$$\nu_n(B\times(w,\infty)) = \frac{\Delta_n}{2}\int_{\mathbb{S}_2^2}\int_{(v_n,1]^2} \mathbb{1}_B\left(\frac{\langle\theta_1,\theta_2\rangle}{|\langle\theta_1,\theta_2\rangle|}\right)\mathbb{1}_{(w,\infty)}\left(\delta_n\rho_1\rho_2|\langle\theta_1,\theta_2\rangle|\right) \times (\rho_1\rho_2)^{-1-\beta}\, d\rho_1 d\rho_2 H(d\theta_1)H(d\theta_2).$$

Let $r > 0$. We first assume that $\omega \in (0,1)$. We have that $\omega/(\delta_n\rho_1 r) < 1$ for any $y_1 > v_n$. We get that

$$\begin{aligned}
&\frac{\Delta_n}{2}\int_{(v_n,1]^2}\mathbb{1}_{(\omega,\infty)}\left(\delta_n\rho_1\rho_2 r\right)(\rho_1\rho_2)^{-1-\beta} d\rho_1 d\rho_2\\
&= \frac{\Delta_n}{2}\left(\int_{v_n}^{\omega}\rho_1^{-1-\beta}\left(\int_{\omega/(\delta_n\rho_1 r)}^{1}\rho_2^{-1-\beta}d\rho_2\right)d\rho_1 + \int_{\omega}^{1}\rho_1^{-1-\beta}d\rho_1\int_{v_n}^{1}\rho_2^{-1-\beta}d\rho_2\right)\\
&= \frac{\Delta_n\delta_n^\beta\log(\delta_n)r^\beta}{2\beta w^\beta} + \frac{\Delta_n\delta_n^\beta r^\beta\log(w)}{2\beta w^\beta} + \frac{\Delta_n}{2\beta^2}\left(1 + w^{-\beta} - w^{-\beta} - 2\delta_n^\beta + \delta_n^\beta/w^\beta\right)\\
&= \frac{\Delta_n\delta_n^\beta\log(\delta_n)r^\beta}{2\beta w^\beta} + o(1)\\
&= \frac{r^\beta}{2\beta^2 w^\beta} + o(1),
\end{aligned}$$

where we used that $2\beta\log(1/v_n)\log(1/\Delta_n)^{-1} \to 1$. Now we suppose that $w \ge 1$. For $w \ge 1$ we have $w/(\delta_n\rho_1 r) \le 1$ for any $\rho_1 \ge wv_n$. Consequently we have

$$\begin{aligned}
&\frac{\Delta_n}{2}\int_{(v_n,1]^2}\mathbb{1}_{(w,\infty)}(\delta_n\rho_1\rho_2 r)(\rho_1\rho_2)^{-1-\beta}d\rho_1 d\rho_2\\
&= \frac{\Delta_n}{2}\int_{wv_n}^{1}\rho_1^{-1-\beta}\left(\int_{w/(\delta_n\rho_1 r)}^{1}\rho_2^{-1-\beta}d\rho_2\right)d\rho_1\\
&= \frac{\Delta_n}{2\beta}\int_{wv_n}^{1}\rho_1^{-1-\beta}\left((w/(\delta_n\rho_1 r))^{-\beta} - 1\right)d\rho_1\\
&= \frac{\Delta_n\delta_n^\beta\log(\delta_n)r^\beta}{2\beta w^\beta} + o(1)\\
&= \frac{r^\beta}{2\beta^2 w^\beta} + o(1).
\end{aligned}$$

Hence we have proven the first condition and $\nu_n \to \nu_U$ with

$$\nu_U(A) = \frac{1}{2\beta}\int_{\mathbb{S}_2^2} \mu(dz) \int_0^\infty \mathbb{1}_A(\rho z)\rho^{-1-\beta} d\rho, \qquad A \in \mathcal{B}(\mathbb{R})$$

and

$$\mu(z) = \int_{\mathbb{S}_2^2} \mathbb{1}_z \left(\frac{\langle\theta_1, \theta_2\rangle}{|\langle\theta_1, \theta_2\rangle|}\right) |\langle\theta_1, \theta_2\rangle|^\beta H(d\theta_1)H(d\theta_2), \qquad z \in \mathcal{B}(\mathbb{S}_1).$$

We will now show the second condition. Let $\varepsilon > 0$. We have

$$\begin{aligned}\int_{|z|\le\varepsilon} |z|^2 \nu_n(dz) = &\frac{\Delta_n}{2}\int_{\mathbb{S}_2^2}\int_{(v_n,1]^2} \mathbb{1}_{(0,\varepsilon)}(\delta_n\rho_1\rho_2|\langle\theta_1,\theta_2\rangle|) \\ &\times (\rho_1\rho_2)^{-1-\beta}\left(\delta_n\rho_1\rho_2|\langle\theta_1,\theta_2\rangle|\right)^2 d\rho_1 d\rho_2 H(d\theta_1)H(d\theta_2).\end{aligned}$$

We will compute the integral with respect to $d\rho_1 d\rho_2$. Let $r > 0$. We have

$$\begin{aligned}&\frac{\Delta_n}{2}\delta_n^2 r^2 \int_{(v_n,1]^2} \mathbb{1}_{(0,\varepsilon)}(\delta_n\rho_1\rho_2 r)(\rho_1\rho_2)^{1-\beta} d\rho_1 d\rho_2 \\ &= \frac{\Delta_n}{2}\delta_n^2 r^2 \int_{v_n}^{\varepsilon/r}\left(\int_{v_n}^{\varepsilon/(\delta_n\rho_2 r)} \rho_1^{1-\beta} d\rho_1\right)\rho_2^{1-\beta} d\rho_2 \\ &= \frac{\Delta_n}{2(2-\beta)}\delta_n^2 r^2 \left((\varepsilon/r)^{2-\beta}\delta_n^{\beta-2}\int_{v_n}^{\varepsilon/r}\rho_2^{-1}d\rho_2 - \delta_n^{\beta-2}\int_{v_n}^{\varepsilon/r}\rho_2^{1-\beta}d\rho_2\right) \\ &= \frac{\Delta_n\delta_n^\beta r^\beta\varepsilon^{2-\beta}\log(\delta_n)}{2(2-\beta)} + \frac{\Delta_n\delta_n^\beta r^\beta\varepsilon^{2-\beta}\log(\varepsilon/r)}{2(2-\beta)} + \frac{\Delta_n\delta_n^{2\beta-2}r^2}{2(2-\beta)^2} - \frac{\Delta_n\delta_n^\beta\varepsilon^{2-\beta}r^\beta}{2(2-\beta)^2}.\end{aligned}$$

Since $\Delta_n\delta_n^\beta = \log(1/\Delta_n)^{-1} = o(1)$ and since $2\beta\log(1/v_n)\log(1/\Delta_n)^{-1} \to 1$, we have that

$$\begin{aligned}&\frac{\Delta_n\delta_n^\beta r^\beta\varepsilon^{2-\beta}\log(\delta_n)}{2(2-\beta)} + \frac{\Delta_n\delta_n^\beta r^\beta\varepsilon^{2-\beta}\log(\varepsilon/r)}{2(2-\beta)} + \frac{\Delta_n\delta_n^{2\beta-2}r^2}{2(2-\beta)^2} - \frac{\Delta_n\delta_n^\beta\varepsilon^{2-\beta}r^\beta}{2(2-\beta)^2} \\ &= \frac{\Delta_n\delta_n^\beta r^\beta\varepsilon^{2-\beta}\log(\delta_n)}{2(2-\beta)} + o(1) \\ &\to \frac{r^\beta\varepsilon^{2-\beta}}{4\beta(2-\beta)} \qquad \text{as } n\to\infty.\end{aligned}$$

Since $0 < \beta \le 1$ we can conclude that the second condition is verified, i.e.

$$\lim_{\varepsilon\to 0^+}\limsup_{n\to\infty}\int_{|z|\le\varepsilon}|z|^2\nu_n(dz) = 0.$$

Remark III.5.1. Consider two set of observation time instants $(\Delta_i^{n,1})_i$ and $(\Delta_j^{n,2})_j$ with $\Delta_i^{n,1} = \Delta_j^{n,2} = \Delta_n$ and $\Delta_i^{n,k} \cap \Delta_j^{n,l} = \emptyset$ for all i, j and $k, l = 1, 2$. With similar techniques, we can show that

$$\sum_{i=1}^{\lfloor\Delta_n^{-1}\rfloor}\mathbb{E}[\exp\{it\delta_n\langle\Delta L_{T(n,1,i)_1}, \Delta L_{T(n,2,i)_1}\rangle\mathbb{1}_{\{\tau_{1,i}=1\}}\mathbb{1}_{\{\tau_{2,i}=1\}}\} - 1]$$

$$
\begin{aligned}
&= \sum_{i=1}^{\lfloor \Delta_n^{-1} \rfloor} (\alpha_n \Delta_n)^2 \int_{\|y\| \in (v_n,1]} \int_{\|x\| \in (v_n,1]} \left(e^{it\delta_n \langle x,y\rangle} - 1\right) G(dx)G(dy) \\
&= 2\alpha_n^2 \int \left(e^{itz} - 1\right) \nu_n(dz)
\end{aligned}
$$

with ν_n is the Lévy measure defined above. Since $\alpha_n \to 1$, we get the convergence

$$
\sum_{i=1}^{\lfloor \Delta_n^{-1} \rfloor} \mathbb{E}[e^{it\delta_n \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle \mathbb{1}_{\{\tau_i = 2\}}} - 1] \xrightarrow{u.c.p.} \begin{cases} 2\int \left\{e^{itx} - 1\right\} \nu_U(dx) & \text{if} \quad \beta < 1, \\ 2\int \left\{e^{itx} - 1 - itx\mathbb{1}_{\{0<|x|\le 1|\}}\right\} \nu_U(dx) & \text{if} \quad \beta = 1. \end{cases}
$$

III.5.2.2 Technical results

For simplicity of exposure all the proofs in this section will be presented for the case $\beta < 1$. Similar result with $\beta = 1$ can be proven using the same arguments.

Lemma III.5.2. *Let $(\Delta_i^n)_{1\le i\le p}$, $p \ge 3$, be a collection of non-overlapping observation time instants with $\Delta_i^n = O(\Delta_n)$ for all i. Define, for a fixed $q < p$,*

$$
\begin{aligned}
&S := \{i_1, ..., i_q\} \subset \{1, ..., p\}, \\
&S_j := \{i_1, ..., i_{m_j}\} \subset \{1, ..., p\} \setminus S, \text{ for all } j \in S.
\end{aligned}
$$

We suppose that $S \cup \left(\bigcup_{j\in S} S_j\right) = \{1, ..., p\}$. We set, for all $1 \le i \le p$, $\alpha_i^n = \exp(-\Delta_i^n \overline{G}(v_n))$. Then

$$
\prod_{i=1}^{p} \alpha_i^n \Delta_i^n \int_{\|y_1\|,...,\|y_p\| \in (v_n,1]} \left(e^{it\delta_n \sum_{j\in S}\langle y_j, \sum_{k\in S_j} y_k\rangle} - 1\right) G(dy_1)...G(dy_p) = o(\Delta_n).
$$

Proof. Observe first that $\overline{G}(v_n) \le C\Delta_n^{-1}\log(1/\Delta_n)^{-1}$, hence $\alpha_i^n \to 1$ and therefore

$$
\begin{aligned}
&\prod_{i=1}^{p} \alpha_i^n \Delta_i^n \int_{\|y_1\|,...,\|y_p\| \in (v_n,1]} \left(e^{it\delta_n \sum_{j\in S}\langle y_j, \sum_{k\in S_j} y_k\rangle} - 1\right) G(dy_1)...G(dy_p) \\
&\sim \prod_{i=1}^{p} \Delta_i^n \int_{\|y_1\|,...,\|y_p\| \in (v_n,1]} \left(e^{it\delta_n \sum_{j\in S}\langle y_j, \sum_{k\in S_j} y_k\rangle} - 1\right) G(dy_1)...G(dy_p) \\
&\le C\Delta_n^p \int_{\|y_1\|,...,\|y_p\| \in (v_n,1]} \left(e^{it\delta_n \sum_{j\in S}\langle y_j, \sum_{k\in S_j} y_k\rangle} - 1\right) G(dy_1)...G(dy_p) \\
&= \int \left\{e^{itz} - 1\right\} \nu_n(dz)
\end{aligned}
$$

with, for $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$

$$
\nu_n(A) = C\Delta_n^p \int_{\|y_1\|,...,\|y_p\| \in (v_n,1]} \mathbb{1}_A \left(\delta_n \sum_{j\in S} \langle y_j, \sum_{k\in S_j} y_k\rangle\right) G(dy_1)...G(dy_p).
$$

To prove the lemma it is sufficient to show that $\nu_n(A)$ converges strongly to the null measure with $\nu_n(A) \le o(\Delta)$. Set $y_i = (\rho_i, \theta_i) \in \mathbb{R}_+ \times \mathbb{S}_2$ for all $1 \le i \le p$. We obtain, for $w > 0$ and $B \in \mathcal{B}(\mathbb{S}_1)$,

$$\nu_n(B \times (w, \infty)) = C\Delta_n^p \int_{\mathbb{S}_2^p} \int_{(v_n,1]^p} \mathbb{1}_B \left(\frac{\sum_{j\in S} \sum_{k\in S_j} \rho_j \rho_k \langle \theta_j, \theta_k \rangle}{\left| \sum_{j\in S} \sum_{k\in S_j} \rho_j \rho_k \langle \theta_j, \theta_k \rangle \right|} \right)$$
$$\times \mathbb{1}_{(w,\infty)} \left(\delta_n \left| \sum_{j\in S} \sum_{k\in S_j} \rho_j \rho_k \langle \theta_j, \theta_k \rangle \right| \right) \left(\prod_{i=1}^p \rho_i \right)^{-1-\beta} d\rho_1 ... d\rho_p H(d\theta_1) ... H(d\theta_p).$$

The above integral has the same order as

$$\Delta_n^p \int_{(v_n,1]^p} \mathbb{1}_{(w,\infty)} \left(\delta_n \sum_{j\in S} \sum_{k\in S_j} \rho_j \rho_k r_{j,k} \right) \left(\prod_{i=1}^p \rho_i \right)^{-1-\beta} d\rho_1 ... d\rho_p$$

for some $r_{j,k} > 0$. Denote by λ^p the Lebesgue measure on $\mathbb{R}^p$ and observe that

$$\lambda^p \left(\left\{ \rho_1, ..., \rho_p \in (v_n, 1]^p \ : \ \delta_n \sum_{j\in S} \sum_{k\in S_j} \rho_j \rho_k r_{j,k} > w \right\} \right)$$
$$\le \lambda^p \left(\{ \rho_1, ..., \rho_p \in (v_n, 1]^p \ : \ \delta_n \rho_1 \rho_2 r_{p,q} > w \} \right) \quad \text{for some } (p, q) \in S \times S_j.$$

It follows that

$$\Delta_n^p \int_{(v_n,1]^p} \mathbb{1}_{(w,\infty)} \left(\delta_n \sum_{j\in S} \sum_{k\in S_j} \rho_j \rho_k r_{j,k} \right) \left(\prod_{i=1}^p \rho_i \right) d\rho_1 ... d\rho_p$$
$$\le \Delta_n^p \left(\int_{(v_n,1]} \rho^{-1-\beta} d\rho \right)^{p-2} \int_{(v_n,1]^2} \mathbb{1}_{(w,\infty)} (\delta_n \rho_1 \rho_2 r_{p,q}) (\rho_1 \rho_2)^{-1-\beta} d\rho_1 d\rho_2.$$

In the proof of the synchronous case (see Section III.5.2.1), we have shown that

$$\Delta_n \int_{(v_n,1]^2} \mathbb{1}_{(w,\infty)} (\delta_n \rho_1 \rho_2) (\rho_1 \rho_2)^{-1-\beta} d\rho_1 d\rho_2 = O(1).$$

Combined with the fact that

$$\int_{(v_n,1]} \rho^{-1-\beta} d\rho = -\frac{1}{\beta} + \frac{\Delta_n^{-1} \log(1/\Delta_n)^{-1}}{\beta} = O(\Delta_n^{-1} \log(1/\Delta_n)^{-1})$$

we obtain

$$\Delta_n^p \left(\int_{(v_n,1]} \rho^{-1-\beta} d\rho \right)^{p-2} \int_{(v_n,1]^2} \mathbb{1}_{(w,\infty)} (\delta_n \rho_1 \rho_2 r_{p,q}) (\rho_1 \rho_2)^{-1-\beta} d\rho_1 d\rho_2 = O(\Delta_n \log(1/\Delta_n)^{2-p}),$$

hence the result. □

Corollary III.5.3. *Let $(\Delta_i^n)_{1\le i\le p}$, $p \ge 2$, be a collection of non-overlapping observation time instants with $\Delta_i^n = O(\Delta_n)$ for all i. With the usual notation, we have*

$$\mathbb{E}\left[\exp\left\{it\delta_n \sum_{i=1}^p \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle \mathbb{1}_{\{\tau_i=2\}}\right\} - 1\right]$$
$$= \sum_{i=1}^p \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle \mathbb{1}_{\{\tau_i=2\}}\right\} - 1\right] + o(1).$$

Proof. We start by expanding the expectation on the left hand side with respect to the indicator functions. Analogously to the proof of the Proposition III.4.3, we obtain

$$\mathbb{E}\left[\exp\left\{it\delta_n \sum_{i=1}^p \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle \mathbb{1}_{\{\tau_i=2\}}\right\} - 1\right]$$
$$= \sum_{l=1}^p \mathbb{P}(\mathbb{1}_{\{\tau_l=2\}}=1) \prod_{\substack{i=1\\ i\neq l}}^p \mathbb{P}(\mathbb{1}_{\{\tau_i=2\}}=0) \mathbb{E}\left[\exp\left\{it\delta_n \langle \Delta L_{T(n,l)_1}, \Delta L_{T(n,l)_2}\rangle\right\} - 1\right] + R_p^n$$

with R_p^n a finite sum of terms of the form, for $0 \le m \le p-2$,

$$\prod_{l=1}^m \mathbb{P}(\mathbb{1}_{\{\tau_{k_l}=2\}}=0) \prod_{\substack{i=1\\ i\neq k_1,\dots,k_m}}^p \mathbb{P}(\mathbb{1}_{\{\tau_i=2\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n \sum_{\substack{i=1\\ i\neq k_1,\dots,k_m}}^p \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle\right\} - 1\right].$$

Define as in the prequel $\alpha_i^n := \exp(-\Delta_i^n \overline{G}(v_n))$ and observe that $\alpha_i^n \to 1$ since $\overline{G}(v_n) \le C\Delta_n^{-1}\log(1/\Delta_n)^{-1}$. From this,

$$\prod_{l=1}^m \mathbb{P}(\mathbb{1}_{\{\tau_{k_l}=2\}}=0) = \prod_{l=1}^m \left(1 - \alpha_{k_l}^n \left(\Delta_{k_l}^n\right)^2 \overline{G}(v_n)^2/2\right) = 1 + O(\log(1/\Delta_n)^{-1})$$

and therefore

$$\prod_{l=1}^m \mathbb{P}(\mathbb{1}_{\{\tau_{k_l}=2\}}=0) \prod_{\substack{i=1\\ i\neq k_1,\dots,k_m}}^p \mathbb{P}(\mathbb{1}_{\{\tau_i=2\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n \sum_{\substack{i=1\\ i\neq k_1,\dots,k_m}}^p \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle\right\} - 1\right]$$
$$\sim \prod_{i\in\{1,\dots,p\}\setminus\{k_1,\dots,k_m\}} \mathbb{P}(\mathbb{1}_{\{\tau_i=2\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n \sum_{i\in\{1,\dots,p\}\setminus\{k_1,\dots,k_m\}} \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle\right\} - 1\right].$$

Denote $\{i_1,\dots,i_{p-m}\} = \{1,\dots,p\} \setminus \{k_1,\dots,k_m\}$. We have

$$\prod_{l=1}^{p-m} \mathbb{P}(\mathbb{1}_{\{\tau_{i_l}=2\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n \sum_{l=1}^{p-m} \langle \Delta L_{T(n,i_l)_1}, \Delta L_{T(n,i_l)_2}\rangle\right\} - 1\right]$$
$$= \prod_{l=1}^{p-m} \alpha_{i_l}^n \frac{\left(\Delta_{i_l}^n\right)^2}{2} \int_{\|y_1\|,\dots\|y_{2p-2m}\|\in(v_n,1]} \left(e^{it\delta_n \sum_{l=1}^{p-m}\langle y_l, y_{l+p-m}\rangle} - 1\right) G(dy_1)\dots G(dy_{2p-2m})$$

$$= o(\Delta_n)$$

by Lemma III.5.2. It follows that

$$\mathbb{E}\left[\exp\left\{it\delta_n\sum_{i=1}^{p}\langle\Delta L_{T(n,i)_1},\Delta L_{T(n,i)_2}\rangle\mathbb{1}_{\{\tau_i=2\}}\rangle\right\}-1\right]$$

$$= \sum_{l=1}^{p}\mathbb{P}(\mathbb{1}_{\{\tau_l=2\}}=1)\prod_{\substack{i=1\\i\neq l}}^{p}\mathbb{P}(\mathbb{1}_{\{\tau_i=2\}}=0)\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{T(n,l)_1},\Delta L_{T(n,l)_2}\rangle\right\}-1\right]+o(\Delta_n)$$

$$\sim \sum_{l=1}^{p}\mathbb{P}(\mathbb{1}_{\{\tau_l=2\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{T(n,l)_1},\Delta L_{T(n,l)_2}\rangle\right\}-1\right]$$

$$= \sum_{l=1}^{p}\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{T(n,l)_1},\Delta L_{T(n,l)_2}\rangle\mathbb{1}_{\{\tau_l=2\}}\right\}-1\right]$$

hence the result. □

Corollary III.5.4. *Let $(\Delta_i^n)_{1\leq i\leq p}$, $p\geq 3$, be a collection of non-overlapping observation time instants. Define, for a fixed $q<p$,*

$$S := \{i_1,...,i_q\}\subset\{1,...,p\},$$
$$S_j := \{i_1,...,i_{m_j}\}\subset\{1,...,p\}\setminus S,\ \textit{for all } j\in S.$$

We suppose that $S\cup\left(\bigcup_{j\in S}S_j\right)=\{1,...,p\}$. Then

$$\mathbb{E}\left[\exp\left\{it\delta_n\sum_{j\in S}\langle\Delta L_{T(n,j)_1}\mathbb{1}_{\{\tau_j=1\}},\sum_{k\in S_j}\Delta L_{T(n,k)_1}\mathbb{1}_{\{\tau_k=1\}}\rangle\right\}-1\right]$$
$$=\sum_{j\in S}\sum_{k\in S_j}\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{T(n,j)_1}\mathbb{1}_{\{\tau_j=1\}},\Delta L_{T(n,k)_1}\mathbb{1}_{\{\tau_k=1\}}\rangle\right\}-1\right]+o(1).$$

Proof. As for the previous corollary, we expand the expectation with respect to the indicator function to get

$$\mathbb{E}\left[\exp\left\{it\delta_n\sum_{j\in S}\langle\Delta L_{T(n,j)_1}\mathbb{1}_{\{\tau_j=1\}},\sum_{k\in S_j}\Delta L_{T(n,k)_1}\mathbb{1}_{\{\tau_k=1\}}\rangle\right\}-1\right]$$
$$=\sum_{j\in S}\sum_{k\in S_j}\mathbb{P}(\mathbb{1}_{\{\tau_j=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_k=1\}}=1)\prod_{i\in\{1,...p\}\setminus(S\cup S_j)}\mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0)$$
$$\times\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{T(n,j)_1},\Delta L_{T(n,k)_1}\rangle\right\}-1\right]+\overline{R}_p^n$$

with $\overline{R}_p^n$ a finite sum of terms of the form, for $\overline{S}\subset S$ and $\overline{S}_j\subset S_j$ for all $j\in\overline{S}$ with $3\leq\left|\overline{S}\cup\left(\bigcup_{j\in\overline{S}}\overline{S}_j\right)\right|\leq p$, (this condition is to ensure that there is at least two terms of the form $\langle\Delta L_{T(n,i)_1},\Delta L_{T(n,j)_1}\rangle$ inside the expectation)

$$\prod_{j\in\overline{S}}\prod_{k\in\bigcup_j\overline{S}_j}\mathbb{P}(\mathbb{1}_{\{\tau_j=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_k=1\}}=1)\prod_{i\in\{1,...,p\}\setminus\overline{S}\cup\left(\bigcup_{j\in\overline{S}}\overline{S}_j\right)}\mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0)$$

$$\times \mathbb{E}\left[\exp\left\{it\delta_n \sum_{j\in\overline{S}} \langle \Delta L_{T(n,j)_1}, \sum_{k\in\overline{S}_j} \Delta L_{T(n,k)_1}\rangle\right\} - 1\right].$$

Similarly to the previous corollary, we have

$$\prod_{i\in\{1,\dots,p\}\setminus\overline{S}\cup\left(\bigcup_{j\in\overline{S}}\overline{S}_j\right)} \mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0) = \prod_{i\in\{1,\dots,p\}\setminus\overline{S}\cup\left(\bigcup_{j\in\overline{S}}\overline{S}_j\right)} \left(1-\alpha_i^n\Delta_i^n\overline{G}(v_n)\right) = 1+O(\log(1/\Delta_n)^{-1})$$

and therefore

$$\prod_{j\in\overline{S}}\prod_{k\in\bigcup_j\overline{S}_j} \mathbb{P}(\mathbb{1}_{\{\tau_j=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_k=1\}}=1) \prod_{i\in\{1,\dots,p\}\setminus\overline{S}\cup\left(\bigcup_{j\in\overline{S}}\overline{S}_j\right)} \mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0)$$
$$\times \mathbb{E}\left[\exp\left\{it\delta_n \sum_{j\in\overline{S}} \langle \Delta L_{T(n,j)_1}, \sum_{k\in\overline{S}_j} \Delta L_{T(n,k)_1}\rangle\right\} - 1\right]$$
$$\sim \prod_{j\in\overline{S}}\prod_{k\in\bigcup_j\overline{S}_j} \mathbb{P}(\mathbb{1}_{\{\tau_j=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_k=1\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n \sum_{j\in\overline{S}} \langle \Delta L_{T(n,j)_1}, \sum_{k\in\overline{S}_j} \Delta L_{T(n,k)_1}\rangle\right\} - 1\right].$$

Denote $\overline{S}\cup\left(\bigcup_{j\in\overline{S}}\overline{S}_j\right) := \{i_1,\dots,i_m\}$ with $m := \left|\overline{S}\cup\left(\bigcup_{j\in\overline{S}}\overline{S}_j\right)\right|$. We have

$$\prod_{j\in\overline{S}}\prod_{k\in\bigcup_j\overline{S}_j} \mathbb{P}(\mathbb{1}_{\{\tau_j=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_k=1\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n \sum_{j\in\overline{S}} \langle \Delta L_{T(n,j)_1}, \sum_{k\in\overline{S}_j} \Delta L_{T(n,k)_1}\rangle\right\} - 1\right]$$
$$= \prod_{l=1}^m \alpha_{i_l}^n\Delta_{i_l}^n \int_{\|y_{i_1}\|,\dots,\|y_{i_m}\|\in(v_n,1]} \left(e^{it\delta_n\sum_{j\in\overline{S}}\langle y_j, \sum_{k\in\overline{S}_j} y_k\rangle}-1\right) G(dy_{i_1})\dots G(dy_{i_m}) = o(\Delta_n)$$

by Lemma III.5.2. It follows that

$$\mathbb{E}\left[\exp\left\{it\delta_n \sum_{j\in S} \langle \Delta L_{T(n,j)_1}\mathbb{1}_{\{\tau_j=1\}}, \sum_{k\in S_j} \Delta L_{T(n,k)_1}\mathbb{1}_{\{\tau_k=1\}}\rangle\right\} - 1\right]$$
$$= \sum_{j\in S}\sum_{k\in S_j} \mathbb{P}(\mathbb{1}_{\{\tau_j=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_k=1\}}=1) \prod_{i\in\{1,\dots p\}\setminus(S\cup S_j)} \mathbb{P}(\mathbb{1}_{\{\tau_i=1\}}=0)$$
$$\times \mathbb{E}\left[\exp\left\{it\delta_n\langle \Delta L_{T(n,j)_1}, \Delta L_{T(n,k)_1}\rangle\right\}-1\right] + o(\Delta_n)$$
$$\sim \sum_{j\in S}\sum_{k\in S_j} \mathbb{P}(\mathbb{1}_{\{\tau_j=1\}}=1)\mathbb{P}(\mathbb{1}_{\{\tau_k=1\}}=1)\mathbb{E}\left[\exp\left\{it\delta_n\langle \Delta L_{T(n,j)_1}, \Delta L_{T(n,k)_1}\rangle\right\}-1\right]$$
$$= \sum_{j\in S}\sum_{k\in S_j} \mathbb{E}\left[\exp\left\{it\delta_n\langle \Delta L_{T(n,j)_1}\mathbb{1}_{\{\tau_j=1\}}, \Delta L_{T(n,k)_1}\mathbb{1}_{\{\tau_k=1\}}\rangle\right\}-1\right],$$

hence the result. □

Proposition III.5.5. *Let $(\Delta_i^n)_{1\le i\le p}$, $p\ge 3$, be a collection of non-overlapping observation time instants. Define, for a fixed $q<p$ and a fixed $r\le p$*

$$S^{(q)} := \{i_1,\dots,i_q\} \subset \{1,\dots,p\},$$

$$S_j := \{i_1, ..., i_{m_j}\} \subset \{1, ..., p\} \setminus S^{(q)}, \text{ for all } j \in S^{(q)},$$
$$S^{(r)} := \{i_1, ..., i_r\} \subset \{1, ..., p\}.$$

We suppose that $S^{(q)} \cup \left(\bigcup_{j\in S} S_j\right) = \{1, ..., p\}$. *Then*

$$\mathbb{E}\left[\exp\left\{it\delta_n\left(\sum_{i\in S^{(r)}} \langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle \mathbb{1}_{\{\tau_i=2\}} + \sum_{j\in S^{(q)}} \langle \Delta L_{T(n,j)_1}\mathbb{1}_{\{\tau_j=1\}}, \sum_{k\in S_j} \Delta L_{T(n,k)_1}\mathbb{1}_{\{\tau_k=1\}}\rangle\right)\right\} - 1\right]$$
$$= \sum_{i\in S^{(r)}} \mathbb{E}\left[\exp\left\{it\delta_n\langle \Delta L_{T(n,i)_1}, \Delta L_{T(n,i)_2}\rangle \mathbb{1}_{\{\tau_i=2\}}\right\} - 1\right]$$
$$+ \sum_{j\in S^{(q)}} \sum_{k\in S^{(j)}} \mathbb{E}\left[\left\{\exp\{it\delta_n\langle \Delta L_{T(n,j)_1}\mathbb{1}_{\{\tau_j=1\}}, \Delta L_{T(n,k)_1}\mathbb{1}_{\{\tau_k=1\}}\right\} - 1\right] + o(1).$$

Proof. The proof, although tedious to write, is similar to the proof of the two previous corollary and relies on the expansion of the expectation with respect to the indicator functions and then we apply Lemma III.5.2. □

III.5.2.3 Approximation of the characteristic function

We recall that $\xi_i^n(2) + \xi_i^n(10)$ is the dominating part when $\beta \leq 1$. As in the previous section, we rewrite $X_i = \xi_i^n(2) + \xi_i^n(10)$ in a vectorial form as

$$\xi_i^n(2) = \delta_n(\Delta L^1_{T(n,i)_1}\Delta L^2_{T(n,i)_2} + \Delta L^1_{T(n,i)_2}\Delta L^2_{T(n,i)_1})\mathbb{1}_{\{\tau_i=2\}}$$
$$= \delta_n\langle \Delta L_{T(n,i)_1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_2}\rangle \mathbb{1}_{\{\tau_i=2\}},$$
$$\xi_i^n(10) = \delta_n \Delta L^1_{T(n,i)_1}\mathbb{1}_{\{\tau_i=1\}}\left(\Delta L^2_{T(n,i-1,-)_1}\mathbb{1}_{\{\tau_{i-1,-}=1\}} + \Delta L^2_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}\right)$$
$$= \delta_n\langle \Delta L_{T(n,i)_1}\mathbb{1}_{\{\tau_i=1\}}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)_1}\mathbb{1}_{\{\tau_{i-1,-}=1\}} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}\rangle$$
$$:= \xi^n i(10,1) + \xi_i^n(10,2).$$

We want to show the approximation :

$$\log\left(\mathbb{E}\left[e^{it\sum_{i=1}^{n_1} X_i} - 1\right]\right) = \sum_{i=1}^{n_1} \mathbb{E}\left[e^{itX_i} - 1\right] + o(1). \tag{III.5.2}$$

Using Lemma III.3.5, it is sufficient to show that

$$\mathbb{E}\left[e^{itX_i} - 1\right] = O(\Delta_n). \tag{III.5.3}$$
$$\sum_{i=2}^{n_1} \mathbb{E}\left[(e^{itX_i} - 1)(e^{itX_{i-1}} - 1)\right] = o(\Delta_n), \tag{III.5.4}$$
$$\sum_{i=3}^{n_1} \mathbb{E}\left[(e^{itX_i} - 1)(e^{itX_{i-1}} - 1)(e^{itX_{i-2}} - 1)\right] = o(\Delta_n). \tag{III.5.5}$$

We start with the condition (III.5.3). Using Proposition III.5.5, we obtain

$$\mathbb{E}\left[e^{itX_i}-1\right]=\mathbb{E}\left[e^{it\xi^n(2)}-1\right]+\mathbb{E}\left[e^{it\xi_i^n(10,1)}-1\right]+\mathbb{E}\left[e^{it\xi_i^n(10,2)}-1\right]+o(\Delta_n).$$

From the synchronous case, each term on the right hand side of the above decomposition is of order Δ_n, hence the condition (III.5.3) is verified. As for the case $\beta \in (1,2)$, due to (III.3.9) and (III.3.10), condition (III.5.5) is verified. We need to verify now (III.5.4). It is left to decompose the term

$$\sum_{i=2}^{n_1}\mathbb{E}\left[(e^{itX_{i-1}}-1)(e^{itX_i}-1)\right]=\sum_{i=2}^{n_1}\left\{\mathbb{E}\left[e^{it(X_{i-1}+X_i)}\right]-\mathbb{E}\left[e^{itX_{i-1}}+e^{itX_i}\right]+1\right\}.$$

We keep the same notation as the case $\beta \in (1,2)$ and we decompose the observation time instant as follows :

$$\begin{aligned}\Delta_{t_i^1} &:= \overline{\Delta}_{t_i}+\Delta_{t_{i,-}}\\ \Delta_{t_{i+1}^1} &:= \Delta_{t_{i,+}}+\overline{\Delta}_{t_{i+1}}.\end{aligned}$$

Lemma III.5.6. *We observe the following approximation:*

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{T(n,i)_1},\Delta L_{T(n,i)_2}\rangle\mathbb{1}_{\{\tau_i=2\}}\right\}-1\right]\\ &=\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{\overline{T}(n,i)_2}\rangle\mathbb{1}_{\{\overline{\tau}_i=2\}}\right\}-1\right]\\ &+\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{T(n,i,-)_1}\rangle\mathbb{1}_{\{\overline{\tau}_i=1\}}\mathbb{1}_{\{\tau_{i,-}=1\}}\right\}-1\right]\\ &+\mathbb{E}\left[\exp\left\{it\delta_n\langle\Delta L_{T(n,i,-)_1},\Delta L_{T(n,i,-)_2}\rangle\mathbb{1}_{\{\tau_{i,-}=2\}}\right\}-1\right]+o(1).\end{aligned}$$

Proof. We recall first that $v_n=(\Delta_n\log(1/\Delta_n))^{-1/\beta}$. Since $\overline{G}(x)\le Cx^{-\beta}$, it follows that $\overline{G}(v_n)\le C\Delta_n^{-1}\log(1/\Delta_n)^{-1}$. In particular, with the usual notation, we also have $\alpha_i^n=\exp(-\Delta_i^n\overline{G}(v_n))\to 1$. On one hand, we have, for $t\in\mathbb{R}$

$$\begin{aligned}&\mathbb{E}[\exp\{it\delta_n\langle\Delta L_{T(n,i)_1},\Delta L_{T(n,i)_2}\rangle\mathbb{1}_{\{\tau_i=2\}}\}-1]\\ &=\alpha_i^n\frac{(\Delta_i^n)^2}{2}\int_{(v_n,1]^2}\{\exp\{it\delta_n\langle x,y\rangle\}-1\}\,G(dx)G(dy)\\ &\sim\frac{(\Delta_i^n)^2}{2}\int_{(v_n,1]^2}\{\exp\{it\delta_n\langle x,y\rangle\}-1\}\,G(dx)G(dy).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mathbb{E}[\exp\{it\delta_n(\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{\overline{T}(n,i)_2}\rangle\mathbb{1}_{\{\overline{\tau}_i=2\}}+\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{T(n,i,-)_1}\rangle\mathbb{1}_{\{\overline{\tau}_i=1\}}\mathbb{1}_{\{\tau_{i,-}=1\}}\\ &\qquad+\langle\Delta L_{T(n,i,-)_1},\Delta L_{T(n,i,-)_2}\rangle\mathbb{1}_{\{\tau_{i,-}=2\}})\}-1]\\ &=\mathbb{P}(\overline{\tau}_i=2)\mathbb{P}(\tau_{i,-}=2)\mathbb{E}[\exp\{it\delta_n(\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{\overline{T}(n,i)_2}\rangle+\langle\Delta L_{T(n,i,-)_1},\Delta L_{T(n,i,-)_2}\rangle)\}-1]\\ &+\mathbb{P}(\overline{\tau}_i=2)(\mathbb{P}(\tau_{i,-}=1)+\mathbb{P}(\tau_{i,-}=0))\mathbb{E}[\exp\{it\delta_n\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{\overline{T}(n,i)_2}\rangle\}-1]\\ &+\mathbb{P}(\tau_{i,-}=2)(\mathbb{P}(\overline{\tau}_i=1)+\mathbb{P}(\overline{\tau}_i=0))\mathbb{E}[\exp\{it\delta_n\langle\Delta L_{T(n,i,-)_1},\Delta L_{T(n,i,-)_2}\rangle\}-1]\\ &+\mathbb{P}(\overline{\tau}_i=1)\mathbb{P}(\tau_{i,-}=1)\mathbb{E}[\exp\{it\delta_n\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{T(n,i,-)_1}\rangle\}-1]\\ &=\overline{\alpha}_i^n\alpha_{i,-}^n\frac{\left(\overline{\Delta}_i^n\right)^2\left(\Delta_{i,-}^n\right)^2}{4}\int_{(v_n,1]^4}(\exp\{it\delta_n(\langle x_1,x_2\rangle+\langle x_3,x_4\rangle)\}-1)\,G(dx_1)G(dx_2)G(dx_3)G(dx_4)\end{aligned}$$

$$+\,\overline{\alpha}_i^n\frac{(\overline{\Delta}_i^n)^2}{2}(\alpha_{i,-}^n\Delta_{i,-}^n\overline{G}(v_n)+\alpha_{i,-}^n)\int_{(v_n,1]^2}(\exp\{it\delta_n\langle x,y\rangle\}-1)\,G(dx)G(dy)$$
$$+\,\alpha_{i,-}^n\frac{(\Delta_{i,-}^n)^2}{2}(\overline{\alpha}_i^n\overline{\Delta}_i^n\overline{G}(v_n)+\overline{\alpha}_i^n)\int_{(v_n,1]^2}(\exp\{it\delta_n\langle x,y\rangle\}-1)\,G(dx)G(dy)$$
$$+\,\overline{\alpha}_i^n\alpha_{i,-}^n\overline{\Delta}_i^n\Delta_{i,-}^n\int_{(v_n,1]^2}(\exp\{it\delta_n\langle x,y\rangle\}-1)\,G(dx)G(dy).$$

Due to Lemma III.5.2 the first term in the decomposition is asymptotically negligible. It follows that

$$\begin{aligned}
&\mathbb{E}[\exp\{it\delta_n(\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{\overline{T}(n,i)_2}\rangle\mathbb{1}_{\{\overline{\tau}_i=2\}}+\langle\Delta L_{\overline{T}(n,i)_1},\Delta L_{T(n,i,-)_1}\rangle\mathbb{1}_{\{\overline{\tau}_i=1\}}\mathbb{1}_{\{\tau_{i,-}=1\}}\\
&\qquad\qquad+\langle\Delta L_{T(n,i,-)_1},\Delta L_{T(n,i,-)_2}\rangle\mathbb{1}_{\{\tau_{i,-}=2\}})\}-1]\\
&=\overline{\alpha}_i^n\frac{(\overline{\Delta}_i^n)^2}{2}(\alpha_{i,-}^n\Delta_{i,-}^n\overline{G}(v_n)+\alpha_{i,-}^n)\int_{(v_n,1]^2}(\exp\{it\delta_n\langle x,y\rangle\}-1)\,G(dx)G(dy)\\
&+\alpha_{i,-}^n\frac{(\Delta_{i,-}^n)^2}{2}(\overline{\alpha}_i^n\overline{\Delta}_i^n\overline{G}(v_n)+\overline{\alpha}_i^n)\int_{(v_n,1]^2}(\exp\{it\delta_n\langle x,y\rangle\}-1)\,G(dx)G(dy)\\
&+\overline{\alpha}_i^n\alpha_{i,-}^n\overline{\Delta}_i^n\Delta_{i,-}^n\int_{(v_n,1]^2}(\exp\{it\delta_n\langle x,y\rangle\}-1)\,G(dx)G(dy)+o(\Delta_n)\\
&\sim\frac{1}{2}\left((\overline{\Delta}_i^n)^2+2\overline{\Delta}_i^n\Delta_{i,-}^n+(\Delta_{i,-}^n)^2\right)\int_{(v_n,1]^2}(\exp\{it\delta_n\langle x,y\rangle\}-1)\,G(dx)G(dy)\\
&=\frac{(\Delta_i^n)^2}{2}\int_{(v_n,1]^2}(\exp\{it\delta_n\langle x,y\rangle\}-1)\,G(dx)G(dy),
\end{aligned}$$

hence the result. □

Using this lemma, we rewrite X_i (and similarly X_{i+1}) as

$$\begin{aligned}
X_i=\delta_n\Bigg(&\langle\Delta L_{\overline{T}(n,i)_1},\begin{pmatrix}0&1\\1&0\end{pmatrix}\Delta L_{\overline{T}(n,i)_1}\rangle\mathbb{1}_{\{\overline{\tau}_i=2\}}\\
&+\langle\Delta L_{\overline{T}(n,i)_1}\mathbb{1}_{\{\overline{\tau}_i=1\}},\begin{pmatrix}0&1\\1&0\end{pmatrix}\Delta L_{T(n,i,-)_1}\mathbb{1}_{\{\tau_{i,-}=1\}}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\Delta L_{T(n,i-1,-)_1}\mathbb{1}_{\{\tau_{i-1,-}=1\}}\\
&\qquad+\begin{pmatrix}0&1\\0&0\end{pmatrix}\Delta L_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}\rangle\\
&+\langle\Delta L_{T(n,i,-)_1},\begin{pmatrix}0&1\\1&0\end{pmatrix}\Delta L_{T(n,i,-)_2}\rangle\mathbb{1}_{\{\tau_{i,-}=2\}}\\
&+\langle\Delta L_{T(n,i,-)_1}\mathbb{1}_{\{\tau_{i,-}=1\}},\begin{pmatrix}0&1\\0&0\end{pmatrix}\Delta L_{T(n,i-1,-)_1}\mathbb{1}_{\{\tau_{i-1,-}=1\}}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\Delta L_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}\rangle\Bigg).
\end{aligned}$$

From this we get

$$\begin{aligned}
X_i+X_{i+1}=\delta_n\Bigg(&\langle\Delta L_{\overline{T}(n,i)_1},\begin{pmatrix}0&1\\1&0\end{pmatrix}\Delta L_{\overline{T}(n,i)_1}\rangle\mathbb{1}_{\{\overline{\tau}_i=2\}}\\
&+\langle\Delta L_{\overline{T}(n,i)_1}\mathbb{1}_{\{\overline{\tau}_i=1\}},\begin{pmatrix}0&1\\1&0\end{pmatrix}\Delta L_{T(n,i,-)_1}\mathbb{1}_{\{\tau_{i,-}=1\}}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\Delta L_{T(n,i-1,-)_1}\mathbb{1}_{\{\tau_{i-1,-}=1\}}\\
&\qquad+\begin{pmatrix}0&1\\0&0\end{pmatrix}\Delta L_{T(n,i,+)_1}\mathbb{1}_{\{\tau_{i,+}=1\}}\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \langle \Delta L_{T(n,i,-)_1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,-)_2} \rangle \mathbb{1}_{\{\tau_{i,-}=2\}} \\
&+ \langle \Delta L_{T(n,i,-)_1} \mathbb{1}_{\{\tau_{i,-}=1\}}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}} \\
&\qquad\qquad + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i+1)_1} \mathbb{1}_{\{\overline{\tau}_{i+1}=1\}} \\
&\qquad\qquad + \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} \rangle \\
&+ \langle \Delta L_{T(n,i,+)_1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_2} \rangle \mathbb{1}_{\{\tau_{i,+}=2\}} \\
&+ \langle \Delta L_{t(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i+1)_1} \mathbb{1}_{\{\overline{\tau}_{i+1}=1\}} \\
&\qquad\qquad + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i+1,+)_1} \mathbb{1}_{\{\tau_{i+1,+}=1\}} \rangle \\
&+ \langle \Delta L_{\overline{T}(n,i+1)_1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{\overline{T}(n,i+1)_2} \rangle \mathbb{1}_{\{\overline{\tau}_{i+1}=2\}} \\
&+ \langle \Delta L_{\overline{T}(n,i+1)_1} \mathbb{1}_{\{\overline{\tau}_{i+1}=1\}}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i+1,+)_1} \mathbb{1}_{\{\tau_{i+1,+}=1\}} \rangle \Bigg).
\end{aligned}
$$

Similarly to the previous case, using Proposition III.5.5 the characteristic function of $X_i + X_{i+1}$, X_i and X_{i+1} is the sum of the characteristic function of each term in the above decomposition. As in the case $\beta \in (1,2)$, up to rearranging the term using Proposition III.5.5, we obtain

$$
\mathbb{E}\left[(e^{itX_i} - 1)(e^{itX_{i+1}} - 1)\right] = 0.
$$

Condition (III.5.4) is therefore verified and the approximation (III.5.2) holds.

III.5.2.4 The limiting distribution

We have that

$$
\begin{aligned}
&\log\left(\mathbb{E}\left[\exp\left\{ it \sum_{i=1}^{n_1} X_i \right\} - 1 \right] \right) \\
&= \sum_{i=1}^{n_1} \Bigg(\mathbb{E}\left[\exp\left\{ it\delta_n \langle \Delta L_{T(n,i)_1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta L_{T(n,i)_2} \rangle \mathbb{1}_{\{\tau_i=2\}} \right\} - 1 \right] && \text{(III.5.6)} \\
&+ \mathbb{E}\left[\exp\left\{ it\delta_n \langle \Delta L_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i-1,-)_1} \mathbb{1}_{\{\tau_{i-1,-}=1\}} \rangle \right\} - 1 \right] && \text{(III.5.7)} \\
&+ \mathbb{E}\left[\exp\left\{ it\delta_n \langle \Delta L_{T(n,i)_1} \mathbb{1}_{\{\tau_i=1\}}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Delta L_{T(n,i,+)_1} \mathbb{1}_{\{\tau_{i,+}=1\}} \rangle \right\} - 1 \right] \Bigg) && \text{(III.5.8)} \\
&+ o(1).
\end{aligned}
$$

Now we apply the result from the synchronous case presented in the prequel. We have, if $\beta < 1$:

$$
\text{(III.5.6)} = n \sum_{i=1}^{n_1} \alpha_i^{n_1} (\Delta_{t_i^1})^2 \int \left(e^{itz} - 1\right) \nu_n^0(dz) + o(1),
$$

$$(\text{III.5.7}) + (\text{III.5.8}) = 2n \sum_{i=1}^{n_1} \alpha_i^{n_1} \Delta_{t_i^1} (\alpha_{i-1,-}^n \Delta_{t_{i-1,-}} + \alpha_{i,+}^n \Delta_{t_{i,+}}) \int \left(e^{itz} - 1\right) \nu_n^1(dz) + o(1),$$

and if $\beta = 1$:

$$(\text{III.5.6}) = n \sum_{i=1}^{n_1} \alpha_i^{n_1} (\Delta_{t_i^1})^2 \int \left(e^{itz} - 1 - itz\mathbb{1}_{\{0<|z|\leq 1\}}\right) \nu_n^0(dz) + o(1),$$

$$(\text{III.5.7}) + (\text{III.5.8}) = 2n \sum_{i=1}^{n_1} \alpha_i^{n_1} \Delta_{t_i^1} (\alpha_{i-1,-}^n \Delta_{t_{i-1,-}} + \alpha_{i,+}^n \Delta_{t_{i,+}}) \int \left(e^{itz} - 1 - itz\mathbb{1}_{\{0<|z|\leq 1\}}\right) \nu_n^1(dz) + o(1),$$

with

$$\nu_n^0(A) = \frac{n}{2} \int_{\|y\|\in(v_n,1]} \int_{\|x\|\in(v_n,1]} \mathbb{1}_A \left(\delta_n \langle x, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y \rangle\right) G(dx)G(dy),$$

$$\nu_n^1(A) = \frac{n}{2} \int_{\|y\|\in(v_n,1]} \int_{\|x\|\in(v_n,1]} \mathbb{1}_A \left(\delta_n \langle x, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y \rangle\right) G(dx)G(dy).$$

We recall that $\alpha_i^{n_1} \to 1$, $\alpha_{i,-}^n \to 1$ and $\alpha_{i,+}^n \to 1$. It follows that

$$\log\left(\mathbb{E}\left[\exp\left\{it\sum_{i=1}^{n_1} X_i\right\} - 1\right]\right)$$
$$\sim \begin{cases} F^n(1) \int \left\{e^{itz} - 1\right\} \nu_n^0(dz) + 2G^n(1) \int \left\{e^{itz} - 1\right\} \nu_n^1(dz), & \beta < 1 \\ F^n(1) \int \left\{e^{itz} - 1 - itz\mathbb{1}_{\{0<|z|\leq 1\}}\right\} \nu_n^0(dz) & \\ + 2G^n(1) \int \left\{e^{itz} - 1 - itz\mathbb{1}_{\{0<|z|\leq 1\}}\right\} \nu_n^1(dz) & , \beta = 1. \end{cases}$$

Denote $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $P_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. By [70, Section 4.4.2], we have $\nu_n^k(A) \to \nu_U^k(A)$, $k = 0, 1$ where

$$\nu_U^k(A) = \frac{1}{2\beta} \int_{\mathbb{S}_2^2} \mu^k(dz) \int_0^\infty \mathbb{1}_A(\rho z)\rho^{-1-\beta} d\rho, \quad k = 0, 1$$

with

$$\mu^k(z) = \int_{\mathbb{S}_2^2} \mathbb{1}_z \left(\frac{\langle P_k\theta^x, \theta^y\rangle}{|\langle P_k\theta^x, \theta^y\rangle|}\right) |\langle P_k\theta^x, \theta^y\rangle|^\beta \, H(d\theta^x)H(d\theta^y), \quad z \in \mathcal{B}(\mathbb{S}_1).$$

It follows that

$$\log\left(\mathbb{E}\left[\exp\left\{it\sum_{i=1}^{n_1} X_i\right\} - 1\right]\right)$$
$$\longrightarrow \begin{cases} F(1) \int \left\{e^{itz} - 1\right\} \nu_U^0(dz) + 2G(1) \int \left\{e^{itz} - 1\right\} \nu_U^1(dz), & \beta < 1 \\ F(1) \int \left\{e^{itz} - 1 - itz\mathbb{1}_{\{0<|z|\leq 1\}}\right\} \nu_U^0(dz) & \\ + 2G(1) \int \left\{e^{itz} - 1 - itz\mathbb{1}_{\{0<|z|\leq 1\}}\right\} \nu_U^1(dz) & , \beta = 1. \end{cases} \tag{III.5.9}$$

Using [143, lemma 14.11], we observe the equalities

$$\int_0^\infty \left(e^{i\rho x}-1\right)\rho^{-1-\beta}d\rho = |x|^\beta\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)\left(1-i\tan\left(\frac{\pi\beta}{2}\right)\operatorname{sgn}(x)\right),\ \text{if } \beta<1$$
$$\int_0^\infty \left(e^{i\rho x}-1-i\rho x\mathbb{1}_{\{0<|x|\le 1\}}\right)\rho^{-2}d\rho = -\frac{\pi x}{2} - ix\log(x)+icx,\ \beta=1$$

for $x>0$ with

$$c=\int_1^\infty \rho^{-2}\sin(\rho)d\rho+\int_0^1\rho^{-2}(\sin(\rho)-\rho)d\rho.$$

Suppose that $\beta<1$. Similarly to the case $\beta\in(1,2)$ we obtain, using the symmetry of H,

$$\int\left(e^{itz}-1\right)\nu_U^k(dz)$$
$$=\frac{|t|^\beta}{2\beta}\int_{\mathbb{S}_2^2}|\langle\theta^x,P_k\theta^y\rangle|^\beta\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)\left(1-i\tan\left(\frac{\pi\beta}{2}\right)\operatorname{sgn}\left(\frac{t\langle\theta^x,P_k\theta^y\rangle}{|\langle\theta^x,P_k\theta^y\rangle|}\right)\right)H(d\theta^x)H(d\theta^y)$$
$$=-|t|^\beta\sigma_\beta^k$$

with

$$\sigma_\beta^k=\frac{-\Gamma(-\beta)\cos\left(\frac{\pi\beta}{2}\right)}{2\beta}\int_{\mathbb{S}_2^2}|\langle\theta^x,P_k\theta^y\rangle|^\beta H(d\theta^x)H(d\theta^y).$$

Suppose now that $\beta=1$. We have

$$\int\left(e^{itz}-1-itz\mathbb{1}_{\{0<|x|\le1\}}\right)\nu_U^k(dz)$$
$$=\frac{1}{2}\int_{\mathbb{S}_2^2}|\langle\theta^x,P_k\theta^y\rangle|\left(-\frac{\pi}{2}|t|-it\frac{\langle\theta^x,P_k\theta^y\rangle}{|\langle\theta^x,P_k\theta^y\rangle|}\log|t|+ict\frac{\langle\theta^x,P_k\theta^y\rangle}{|\langle\theta^x,P_k\theta^y\rangle|}\right)H(d\theta^x)H(d\theta^y).$$

Since H is symmetric, we have

$$\int_{\mathbb{S}_2^2}\langle\theta^x,P_k\theta^y\rangle H(d\theta^x)H(d\theta^y)=0$$

and therefore

$$\int\left(e^{itz}-1-itz\mathbb{1}_{\{0<|x|\le1\}}\right)\nu_U^k(dz)=-\sigma_1^k|t|\quad\text{with}\quad\sigma_1^k=\frac{\pi}{4}\int_{\mathbb{S}_2^2}|\langle\theta^x,P_k\theta^y\rangle|H(d\theta^x)H(d\theta^y).$$

We conclude that

$$\text{(III.5.9)}=-\left(F(1)\sigma_\beta^0+2G(1)\sigma_\beta^1\right)|t|^\beta,\ \beta\le1$$

hence the result of Theorem III.2.3 in the case $\beta\le1$.

Chapter IV

Limit theorems for two dimensional ambit fields observed along curves

Abstract: This article delves into the asymptotic behavior of power variations of continuous two-dimensional ambit fields observed along a curve in $\mathbb{R}^2$. Specifically, the ambit field under consideration is an integral involving a weight kernel $g : \mathbb{R}^2 \to \mathbb{R}$ and a stochastic intermittency process σ, driven by Gaussian white noise. Our investigation demonstrates that the limit theory for the power variation statistics critically hinges on the behavior of the weight kernel g around 0. We explore the case: $g(\mathbf{x}) \sim c\|\mathbf{x}\|^{\alpha}$ as $\mathbf{x} \to \mathbf{0} \in \mathbb{R}^2$. We establish the laws of large numbers and stable central limit theorems. It is noteworthy that the limit theory and proofs of these results significantly differ from those of one-dimensional Brownian semi-stationary processes, as discussed in, for example, [12, 13, 54, 62, 129].

IV.1 Introduction

In recent years, there has been an increasing interest surrounding ambit stochastics. Ambit fields, a category of spatio-temporal stochastic fields, were initially introduced by Barndorff-Nielsen and Schmiegel in a series of papers [20,21], primarily within the realm of turbulence modeling. However, their applications have since proliferated across various disciplines, including mathematical finance and biology, among others, as evidenced by works such as [9,17]. For a comprehensive exploration of the theory and applications of ambit fields we refer to the excellent monograph [10].

Ambit processes characterize the dynamics within a stochastically evolving field, such as turbulent wind field, along curves embedded within the field. What sets ambit fields apart from other models is their incorporation of additional inputs, often called volatility or intermittency, beyond the fundamental random input. Mathematically, a general ambit field is described by the formula:

$$X_t(\boldsymbol{x}) = \mu + \int_{A_t(\boldsymbol{x})} g(t,s,\boldsymbol{x},\boldsymbol{y})\sigma_s(\boldsymbol{y})L(ds,d\boldsymbol{y}) + \int_{D_t(\boldsymbol{x})} q(t,s,\boldsymbol{x},\boldsymbol{y})a_s(\boldsymbol{y})dsd\boldsymbol{y},$$

where t denotes time while $\boldsymbol{x} \in \mathbb{R}^d$ gives the position in space. Further, $A_t(\boldsymbol{x})$ and $D_t(\boldsymbol{x})$ are ambit sets, g and q are deterministic weight functions, σ represents the volatility or intermittency field, a is a drift field and L denotes a Lévy basis on $\mathbb{R}_+ \times \mathbb{R}^d$ (i.e. an independently scattered random measure on $\mathbb{R}_+ \times \mathbb{R}^d$).

While stochastic analysis and modeling can be investigated in general spatio-temporal scenarios, much of the literature predominantly focuses on purely temporal settings when exploring limit theory and statistical inference for ambit fields. In these cases, authors often investigate *Lévy semi-stationary processes*, which take the form:

$$X_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s L(ds) + \int_{-\infty}^{t} q(t-s)a_s ds. \tag{IV.1.1}$$

Here, L represents a two-sided one-dimensional Lévy motion. Numerous papers have delved into the statistical analysis of power variations for Brownian semi-stationary processes, as evidenced in works such as [12, 13, 54, 62, 129], which corresponds to $L = W$ being a Brownian motion. To provide a comparative overview, let us briefly examine the limit theory for high-frequency observations of Brownian semi-stationary processes. The focus lies on the power variations of X, defined as:

$$V(X,p)_t^n := \Delta_n \tau_n^{-p} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^p, \qquad t \in [0,T],\ p > 0,$$

where $\Delta_i^n X := X_{i\Delta_n} - X_{(i-1)\Delta_n}$ and $\tau_n^2 := \mathbb{E}[(\Delta_i^n G)^2]$ with

$$G_t := \int_{-\infty}^{t} g(t-s)W(ds) \tag{IV.1.2}$$

and W representing the Brownian motion. It is notable that the limit theory critically depends on the behavior of the kernel g at 0, which determines the local smoothness of the Brownian semi-stationary process. In particular, we obtain the following theorem which can be found in [12, 54].

Theorem IV.1.1. *Assume that the process X, introduced at* (IV.1.1) *with $L = W$ being a two-sided Brownian motion, is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Further assume that $g(x) \sim cx^\alpha$ as $x \to 0$ with $\alpha \in (-1/2, 1/2)$. Then, under conditions of [54, Theorem 3.1], we deduce the uniform convergence in probability*

$$\sup_{t\in[0,T]} |V(X,p)^n_t - V(X,p)_t| \xrightarrow{\mathbb{P}} 0 \qquad \textit{with} \qquad V(X,p)_t := m_p \int_0^t |\sigma_s|^p ds,$$

where $m_p := \mathbb{E}[|\mathcal{N}(0,1)|^p]$. When further $\alpha \in (-1/2, 0)$ and conditions of [54, Theorem 3.2] are satisfied, we obtain the stable convergence in law:

$$\Delta_n^{-1/2} \left(V(X,p)^n_t - V(X,p)_t\right) \xrightarrow{st} \lambda_p \int_0^t |\sigma_s|^p B(ds),$$

where B is a new Brownian motion independent of the initial σ-algebra $\mathcal{F}$, and the constant λ_p is defined in [54, Eq. (3.3)].

In addition to the exploration of the Brownian case, another branch of literature investigates the Lévy case [28–30,32,101–103,108], with most of the articles focusing on the simplified scenario of $\sigma = 1$ and $a = 0$. From a statistical perspective, researchers have examined the estimation of intermittency σ, the stability index of the driving motion L, and the Hurst parameter. Only a handful of papers [31, 34,120,121] delve into specific subclasses of spatio-temporal ambit fields. Typically, these studies concentrate on determining the limiting behavior of power variations of rectangular increments. A related analysis of high-frequency asymptotics of SPDEs can be found in [46].

In this paper, we investigate the limit theory for power variations of two-parameter ambit fields observed along a curve. Specifically, we examine an ambit field defined as

$$X_{\boldsymbol{t}} = \int_{-\infty}^{\boldsymbol{t}} g(\boldsymbol{t} - \boldsymbol{s})\sigma_{\boldsymbol{s}} W(d\boldsymbol{s}), \tag{IV.1.3}$$

where W represents the white noise process on $\mathbb{R}^2$, $g : \mathbb{R}^2_+ \to \mathbb{R}$ is a deterministic weight function satisfying $g \in L^2(\mathbb{R}^2_+)$, and σ is a continuous intermittency field, ensuring the integral at (IV.1.3) is well-defined in the Walsh sense. We set $g(s_1, s_2) = 0$ if $s_1 < 0$ or $s_2 < 0$. We assume that the process X can be observed discretely along the curve

$$\boldsymbol{z} : [0,t] \to \mathbb{R}^2, \qquad \boldsymbol{z}(s) = (z_1(s), z_2(s)).$$

and we introduce the new processes Y and G defined via

$$Y_u := X_{\boldsymbol{z}(u)}, \qquad G_u := \int_{-\infty}^{\boldsymbol{z}(u)} g(\boldsymbol{z}(u) - \boldsymbol{s}) W(d\boldsymbol{s}), \qquad u \in [0,T]. \tag{IV.1.4}$$

The importance of observing ambit fields along curves in time-space has been emphasized in [10, 14]. In related literature, statistical inference for Gaussian fields observed along curves has been investigated in [2, 104, 151].

Our primary focus is on the asymptotic theory for power variation statistics of Y. Similar to the one-dimensional theory illustrated in Theorem IV.1.1, the limit theory for the power variation of Y critically depends on the behavior of the weight kernel g near $\mathbf{0} \in \mathbb{R}^2$. However, the complexity increases significantly depending on the specific assumptions about the function g. Here, we explore the isotropic case where $g(\mathbf{x}) \sim c\|\mathbf{x}\|^{\alpha}$. In this, we apply techniques from Malliavin calculus to obtain the desired limit theory. We demonstrate convergence in probability for the statistics $V(X,p)^n$ and provide the corresponding fluctuation analysis.

The paper is organized as follows: Section IV.2 introduces the setting, main assumptions, and crucial decompositions. The main asymptotic statements are gathered in Section IV.3, while Section IV.4 is dedicated to proving the limit theorems. Proofs of some technical statements are collected in the Appendix.

Notations

In this section we introduce some notations that are used throughout the paper. All stochastic processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All positive constants are denoted by C, or C_p if they depend on an external parameter p, although they may change from line to line. Vectors in $\mathbb{R}^d$ or $\mathbb{Z}^d$ are denoted by bold letters; in particular, we write $\mathbf{1} := (1,\dots,1) \in \mathbb{R}^d$ and $\mathbf{0} := (0,\dots,0) \in \mathbb{R}^d$ (in most cases we will have $d = 2$). For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ the vector $\boldsymbol{x} \circ \boldsymbol{y} \in \mathbb{R}^d$ (resp. $\boldsymbol{x} + \boldsymbol{y}$) denotes the componentwise multiplication (resp. summation), i.e. $(\boldsymbol{x} \circ \boldsymbol{y})_i = x_i y_i$ (resp. $(\boldsymbol{x} + \boldsymbol{y})_i = x_i + y_i$) for all $i = 1,\dots,d$. Furthermore, we write $\boldsymbol{x} < \boldsymbol{y}$ when $x_i < y_i$ for all $i = 1,\dots,d$, and $[\boldsymbol{x}, \boldsymbol{y}] := [x_1, y_1] \times \dots \times [x_d, y_d]$. All stochastic fields are adapted to a filtration $(\mathcal{G}_{\boldsymbol{t}})_{\boldsymbol{t} \in \mathbb{R}^2}$. Furthermore, $W([\boldsymbol{s}, \boldsymbol{t}])$ is independent of the σ-algebra $\mathcal{G}_{\boldsymbol{s}}$ and $W([\boldsymbol{s}, \boldsymbol{t}]) \sim \mathcal{N}(0, \mathrm{Leb}[\boldsymbol{s}, \boldsymbol{t}])$.
For functions $h_1, h_2 : \mathbb{Z}^d \to \mathbb{R}$ we use the notation

$$h_1(\boldsymbol{z}) \lesssim h_2(\boldsymbol{z})$$

when there exists a constant $C > 0$ such that $h_1(\boldsymbol{z}) \le C h_2(\boldsymbol{z})$ for all $\boldsymbol{z} \in \mathbb{Z}^d$. We write $h_1(\boldsymbol{z}) \lesssim_{\theta} h_2(\boldsymbol{z})$ if we want to stress the dependence of the constant C on some external parameter θ. Furthermore, we write $h_1(\mathbf{x}) \sim h_2(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x_0} \in \mathbb{R}^2$ when

$$\lim_{\mathbf{x} \to \mathbf{x_0}} h_1(\mathbf{x})/h_2(\mathbf{x}) = 1.$$

In the following exposition we will use the notion of *stable convergence*. We recall that a sequence of stochastic processes $(Y_n)_{n \ge 1}$ in $D([0,T])$ equipped with the Skorohod topology and defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to converge stably with limit Y ($Y_n \xrightarrow{st} Y$) defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if and only if for any bounded, continuous function h and any bounded $\mathcal{F}$-measurable random variable Z it holds that

$$\mathbb{E}[h(Y_n)Z] \to \overline{\mathbb{E}}[h(Y)Z], \quad \text{as } n \to \infty. \tag{IV.1.5}$$

We refer to [3, 141] for a detailed exposition of stable convergence. We also write $Y_n \xrightarrow{\text{u.c.p.}} Y$ to denote convergence in probability uniformly on compact intervals, i.e.

$$\sup_{t \in [0,T]} |Y_n(t) - Y(t)| \xrightarrow{\mathbb{P}} 0 \qquad \text{for any } T > 0.$$

IV.2 The setting and main assumptions

IV.2.1 Assumptions

In this subsection, we outline the primary prerequisites for ensuring the existence of the integral introduced in (IV.1.3) and the necessary conditions for the limit theory. The stochastic field $(X_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^2}$ described in (IV.1.3) is defined in the Walsh sense, as detailed in [149]. Specifically, we require that $g \in L^2(\mathbb{R}^2_+)$ and $\sup_{\boldsymbol{t}\in\mathbb{R}^2} \mathbb{E}[\sigma^2_{\boldsymbol{t}}] < \infty$ to ensure the finiteness of the integral

$$\int_{-\infty}^{\boldsymbol{t}} g^2(\boldsymbol{t}-\boldsymbol{s})\sigma^2_{\boldsymbol{s}} d\boldsymbol{s} < \infty \quad \mathbb{P}-\text{a.s.} \tag{IV.2.1}$$

We assume that we are given high frequency observations $Y_{i\Delta_n}$, $i \geq 0$, where $\Delta_n \to 0$ and Y is defined at (IV.1.4). The power variation of Y (or equivalently, the power variation of X discretely observed along the curve $\boldsymbol{z}$) is defined as

$$V(Y,p)^n_t := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n Y|^p, \qquad \Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}, \tag{IV.2.2}$$

with $p > 0$. Similar to the definition (IV.1.2), the Gaussian process G introduced at (IV.1.4) governs the asymptotic theory for the statistic $V(Y,p)^n$ in certain scenarios. However, it is important to note that the stationarity of G—typically crucial in proving limit theorems—is lost due to time transformation by the curve $\boldsymbol{z}$ (unless $\boldsymbol{z}$ is a linear function). This loss of stationarity has significant implications for the presented asymptotic theory.

To determine the asymptotic theory for the power variation statistics we will require a set of assumptions on the kernel g, the intermittency field σ and the curve $\boldsymbol{z}$. We start with the following assumption on the curve $\boldsymbol{z}$:

(A1) The curve $t \mapsto \boldsymbol{z}(t) = \big(z_1(t), z_2(t)\big)$ is C^2 and the derivatives $z_1'(t), z_2'(t) > 0$ are bounded away from zero.

Assumption (A1) is very general, encompassing various practical curve forms. In [14], the authors specifically examine straight lines, a special case of this assumption. They establish corresponding laws of large numbers for quadratic variation (i.e., with $p = 2$). While (A1) doesn't address curves that remain constant across one component, the pertinent results are notably simpler, aligning more closely with the one-dimensional cases explored in [12, 13, 54]. On the other hand, the positivity of the derivatives is assumed for simplicity of expositions only.

The next set of assumptions concerns the kernel function g. In the univariate framework, as observed in [12, 13, 54, 62, 129], the behavior of g at $\mathbf{0}$ dictates the asymptotic theory for power variation statistics. In the multivariate context, various specifications for this behavior exist, yielding markedly different theoretical outcomes. We will delve into the following class:

(A2) For some $\alpha \in (-1, 0)$ the kernel g admits the representation

$$g(\boldsymbol{x}) = \|\boldsymbol{x}\|^{\alpha} f(\boldsymbol{x}). \tag{IV.2.3}$$

where $\|\cdot\|$ denotes the Euclidean norm and the function $f : \mathbb{R}_+^2 \to \mathbb{R}$ is bounded, $f(\mathbf{0}) \neq 0$ and satisfies $f \in C^1(\mathbb{R}_+^2) \cap L^2(\mathbb{R}_+^2)$ with

$$\|\nabla f(\boldsymbol{x})\| \leq C\left(\|\boldsymbol{x}\|^{-1} \wedge 1\right), \quad \boldsymbol{x} \in \mathbb{R}_+^2,$$

for some constant $C > 0$.

We note that under assumption (IV.2.3), the condition $g \in L^2(\mathbb{R}_+^2)$ necessitates the restriction $\alpha > -1$. Conversely, to prevent the process Y from being differentiable—where the limit of $V(Y,p)^n$ can be readily obtained via the mean value theorem and Riemann integrability—we exclude values $\alpha \geq 0$. Thus, we confine our considerations to the range $\alpha \in (-1, 0)$ in (A2).

In addition to the continuity assumption on the intermittency field σ, we will also stipulate its Hölder continuity in L^q, for any $q > 0$, to establish the central limit theorem:

(A3) There exists a $\gamma > 1/2$ such that for any $q > 0$

$$\mathbb{E}[|\sigma_{\boldsymbol{t}} - \sigma_{\boldsymbol{s}}|^q]^{1/q} \leq C_q||\boldsymbol{t} - \boldsymbol{s}||^\gamma \tag{IV.2.4}$$

for some constant $C_q > 0$ and $\boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}_+^2$.

Last but not least we will also require the following condition:

(A4) There exists $\mathbf{a} \in \mathbb{R}_+^2$ such that the partial derivatives satisfy $|\partial_j g(\mathbf{x})| \leq |\partial_j g(\mathbf{y})|$ for any $\mathbf{x} \geq \mathbf{y} \geq \mathbf{a}$, $j = 1, 2$. Furthermore it holds that

$$F_{\boldsymbol{t}} := \int_{\mathbb{R}_+^2 \setminus [\mathbf{0},\mathbf{1}]} \left(\partial_1 g(\boldsymbol{s})^2 + \partial_2 g(\boldsymbol{s})^2\right) \sigma_{\boldsymbol{t}-\boldsymbol{s}}^2 d\boldsymbol{s} < \infty \qquad \mathbb{P} - \text{a.s.}$$

for all $\boldsymbol{t} \in \mathbb{R}_+^2$.

IV.2.2 First remarks

Before delving into the investigation of the limit theory for power variation statistics $V(Y,p)^n$, let us provide some heuristic insights into the forthcoming results. The foundation of our statistics lies in the increments of Y, which we can express as:

$$\begin{aligned} Y_{t+\Delta} - Y_t &= \int_{\mathbb{R}^2} (g(\boldsymbol{z}(t+\Delta) - \boldsymbol{s}) - g(\boldsymbol{z}(t) - \boldsymbol{s})) \left(\mathbb{1}_{A_{t,\Delta}}(\boldsymbol{s}) + \mathbb{1}_{B_{t,\Delta}}(\boldsymbol{s})\right) \sigma_{\boldsymbol{s}} W(d\boldsymbol{s}) \\ &=: I_{t,\Delta}^1 + I_{t,\Delta}^2 \end{aligned}$$

with

$$A_{t,\Delta} := (-\infty, z_1(t)] \times (-\infty, z_2(t)] \cup (z_1(t), z_1(t+\Delta)] \times (z_2(t), z_2(t+\Delta)],$$

$$B_{t,\Delta} := (-\infty, z_1(t)] \times (z_2(t), z_2(t+\Delta)] \cup (z_1(t), z_1(t+\Delta)] \times (-\infty, z_2(t)].$$

It is evident that the terms $I_{t,\Delta}^1$ and $I_{t,\Delta}^2$ differ significantly in nature. Their stochastic order heavily relies on the form of the kernel g outlined in assumptions (A2) as well as on the value of α. Specifically, under assumptions (A2) and (A4), and for $\Delta \to 0$, we observe:

$$\begin{cases} I^1_{t,\Delta} = O_{\mathbb{P}}(\Delta^{1+\alpha}) = I^2_{t,\Delta} : & \alpha \in (-1,-1/2) \\ I^1_{t,\Delta} = O_{\mathbb{P}}(\Delta^{1+\alpha}),\ I^2_{t,\Delta} = O_{\mathbb{P}}(\Delta^{1/2}) : & \alpha \in (-1/2, 0) \end{cases} \tag{IV.2.5}$$

Consequently, when $\alpha \in (-1/2, 0)$, the term $I^2_{t,\Delta}$ dominates in the above decomposition.
We remark that the increments $Y_{t+\Delta} - Y_t$ are too intricate to handle independently, often necessitating a separation between the Gaussian component G of Y and the intermittency σ to establish the limit theory of $V(Y,p)^n$. Such a separation is feasible under assumptions (A1)-(A4) when $\alpha \in (-1,-1/2)$. Here, we can justify the approximation

$$Y_{t+\Delta} - Y_t \approx \sigma_{\boldsymbol{z}_t}(G_{t+\Delta} - G_t), \tag{IV.2.6}$$

where G has been introduced at (IV.1.4). Leveraging this approximation and employing further probabilistic techniques, including a blocking technique, enables us to infer the limit theory of $V(Y,p)^n$ from the corresponding asymptotic theory for $V(G,p)^n$, where the latter is a functional of a non-stationary Gaussian process.
In the remaining scenario—when (A2) holds with $\alpha \in (-1/2, 0)$—we instead infer that:

$$Y_{t+\Delta} - Y_t \approx H\left(W, (\sigma_{s,z_2(t)})_{s\le z_1(t)}, (\sigma_{z_1(t),s})_{s\le z_2(t)}\right).$$

Here, we necessitate vastly different techniques, relying on martingale approximations, to derive the asymptotic theory for $V(Y,p)^n$.

IV.3 Limit theorems

To demonstrate the main theoretical statements, we need to introduce additional notation. Recall that $m_p = \mathbb{E}[|\mathcal{N}(0,1)|^p]$ and define, for any $p > 0$, the function $f_p(x) := |x|^p - m_p$. We introduce the *Hermite expansion* of f_p as

$$f_p(x) = \sum_{k=2}^{\infty} \lambda_k H_k(x), \tag{IV.3.1}$$

where $(H_k)_{k\ge 0}$ are Hermite polynomials. Additionally, we define the function

$$\phi_t^2 = z_1'(t) z_2'(t) f(\mathbf{0})^2 \Big(\int_{\mathbb{R}^2_+ \setminus (1,\infty)^2} \|\boldsymbol{z}'(t) \circ \boldsymbol{x}\|^{2\alpha} d\boldsymbol{x} + \int_{\mathbb{R}^2_+} (\|\boldsymbol{z}'(t)\circ(\boldsymbol{x}+\mathbf{1})\|^{\alpha} - \|\boldsymbol{z}'(t)\circ\boldsymbol{x}\|^{\alpha})^2 d\boldsymbol{x} \Big), \tag{IV.3.2}$$

which pertains to the time variation of the curve $\boldsymbol{z}$ and the kernel g. Finally, we introduce the correlation function

$$\rho(l) := \operatorname{corr}\left(B_1^H, B_{l+1}^H - B_l^H\right),$$

where $(B_t^H)_{t\ge 0}$ denotes the standard fractional Brownian motion with Hurst parameter $H = \alpha + 1 \in (0,1)$.

IV.3.1 Law of large numbers

We introduce the stochastic processes

$$w_t^2 = z_2'(t)\int_0^\infty |x|^{2\alpha} f^2(x,0)\sigma^2_{z_1(t)-x,z_2(t)}dx + z_1'(t)\int_0^\infty |x|^{2\alpha} f^2(0,x)\sigma^2_{z_1(t),z_2(t)-x}dx. \tag{IV.3.3}$$

The following theorem demonstrates the law of large numbers under different specifications of the weight function g.

Theorem IV.3.1. *Assume that conditions (A1), (A2) and (A4) hold.*

(i) If $\alpha \in (-1,-1/2)$, we obtain that

$$\Delta_n^{1-p(1+\alpha)} V(Y,p)_t^n \xrightarrow{u.c.p.} m_p \int_0^t |\phi_s \sigma_{\boldsymbol{z}(s)}|^p ds. \tag{IV.3.4}$$

(ii) If $\alpha \in (-1/2,0)$, we deduce the convergence

$$\Delta_n^{1-p/2} V(Y,p)_t^n \xrightarrow{u.c.p.} m_p \int_0^t |w_s|^p ds. \tag{IV.3.5}$$

Some remarks are necessary to grasp the implications of Theorem IV.3.5. From a statistical perspective, the convergence described in (IV.3.4) signifies that the power variation doesn't directly estimate the integrated intermittency along the curve $\boldsymbol{z}$ (as in the univariate case), but rather an integrated product of $\sigma_{\boldsymbol{z}_t}$ and ϕ_t. Here, ϕ_t embodies the asymptotic scaled variance of the non-stationary Gaussian process G on small scales, i.e.

$$\lim_{\Delta\to 0} \Delta^{-(2\alpha+2)} \mathrm{var}(G_{t+\Delta} - G_t) = \phi_t^2.$$

A clear distinction arises between the convergence results (IV.3.4) and (IV.3.5). Firstly, the scaling of the statistics under consideration is inferred from the preceding subsection discussion (cf. (IV.2.5)). Secondly, while the limit at (IV.3.4) is solely dependent on the intermittency σ observed along the curve $\mathbf{z}$, the limit at (IV.3.5) relies on past observations of the intermittency field. This distinction also stems from the considerations outlined in (IV.2.5).

Remark IV.3.2. Similar to the one-dimensional case, the convergence in (IV.3.4) can be utilized for the estimation of the parameter α. One classical approach is to employ the change-of-frequency method for statistical estimation. Let $V(Y,p)_t^{n,2}$ denote the power variation statistic computed using observations $(Y_{2i\Delta_n})_{1\le i\le \lfloor t/(2\Delta_n)\rfloor}$. Then, under assumptions (A1), (A2), (A4), and $\alpha \in (-1,-1/2)$, we have:

$$\frac{V(Y,p)_t^{n,2}}{V(Y,p)_t^n} \xrightarrow{\mathbb{P}} 2^{1-p(1+\alpha)}.$$

This result can be used for the statistical estimation of the parameter α (cf. [12, 13, 54]). □

Remark IV.3.3. The critical case $\alpha = -1/2$ in Theorem IV.3.1(i)-(ii) is challenging to address. In this situation we expect a non-trivial contribution from both terms $I^1_{t,\Delta}$ and $I^2_{t,\Delta}$. However, these terms are handled using different probabilistic techniques in the proofs, making it difficult to manage the case $\alpha = -1/2$. Nevertheless, in the simple scenario where $p = 2$, our proofs suggest the convergence

$$\Delta_n^{1-p/2} V(Y,2)^n_t \xrightarrow{\text{u.c.p.}} \int_0^t \left(\phi_s^2 \sigma^2_{\boldsymbol{z}(s)} + w_s^2\right) ds.$$

when $\alpha = -1/2$. □

Remark IV.3.4. The findings of this and subsequent subsections can be extended to d-parameter stochastic fields relatively straightforwardly, albeit with much heavier notation. For instance, in the analogue of assumption (A2), the critical region for the parameter α becomes:

$$\alpha \in (-d/2, 1 - d/2).$$

However, explicit discussion of this scenario is beyond the scope of this paper. □

IV.3.2 Weak limit theorems

This subsection is dedicated to presenting fluctuation results associated with Theorem IV.3.1. Our main result is demonstrated in the following theorem.

Theorem IV.3.5. *Assume that conditions (A1)-(A4) are satisfied and* $\alpha \in (-1, -3/4)$. *Furthermore, we assume that* $\gamma(p \wedge 1) > 1/2$. *Let* $(B_t)_{t \geq 0}$ *be a Brownian motion defined on an extension* $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ *of the original probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *and being independent of the* σ*-field* $\mathcal{F}$. *We deduce the functional stable convergence*

$$\Delta_n^{-1/2} \left(\Delta_n^{1-p(1+\alpha)} V(Y,p)^n_t - m_p \int_0^t |\phi_s \sigma_{\boldsymbol{z}_s}|^p ds \right) \xrightarrow{st} \int_0^t \kappa_s |\sigma_{\boldsymbol{z}_s}|^p dB_s. \qquad \text{(IV.3.6)}$$

where κ_s^2 *is defined as*

$$\kappa_s^2 = \phi_s^{2p} \sum_{k=2}^{\infty} \lambda_k^2 k! \left(1 + 2 \sum_{l=1}^{\infty} \rho(l)^k \right).$$

Remark IV.3.6. We remark that $\sum_{l=1}^{\infty} \rho(l)^2 < \infty$ when $H = \alpha + 1 \in (0, 3/4)$. Hence, we conclude that

$$\kappa_s^2 \leq \phi_s^{2p} \left(1 + 2 \sum_{l=1}^{\infty} \rho(l)^2 \right) \sum_{k=2}^{\infty} \lambda_k^2 k! = \phi_s^{2p} \mathrm{var}(f_p(\mathcal{N}(0,1))) \left(1 + 2 \sum_{l=1}^{\infty} \rho(l)^2 \right),$$

which ensures finiteness of the function κ. □

Let us provide some insights into the results of Theorem IV.3.5. First, compared to the law of large numbers presented in (IV.3.4), the stable limit theorem in (IV.3.6) requires an additional restriction to $\alpha \in (-1, -3/4)$. This is due to an intrinsic bias that may explode when multiplied by the convergence rate $\Delta_n^{-1/2}$. Similar bias issues arise in the setting of (IV.3.5), hence explaining the absence of associated weak limit theorems in this case.

Proving the statement in (IV.3.6) hinges on demonstrating the validity of the approximation (IV.2.6) and then employing a blocking technique to derive the convergence at (IV.3.6) from the corresponding result for the statistic $V(G,p)^n$. However, handling the functional $V(G,p)^n$ poses several technical challenges since G is a non-stationary Gaussian process. Consequently, the Malliavin calculus techniques established [114, 123], utilized to show asymptotic normality and tightness of functionals of stationary Gaussian sequences, require modification to accommodate the non-stationary setting.

IV.4 Proofs

It is worth noting that σ is locally bounded due to its continuity. Additionally, because all our asymptotic results are stable under localization, we can assume without loss of generality that the intermittency field σ is bounded on compact sets; see e.g. [15] for a detailed argument. Via the same argument we can assume that the random field F introduced in Assumption (A4) is bounded on compact sets.
Furthermore, we assume that conditions (A1) and (A4) hold throughout the proof.
Recalling that $z_1(t), z_2(t) > 0$, we introduce the filtration

$$\mathcal{F}_t := \mathcal{G}_{\boldsymbol{z}(t)}, \qquad t \geq 0. \tag{IV.4.1}$$

It will be convenient to use the following notation (cf. Figure 1):

$$\begin{aligned}
\mathcal{A}_{i,n} &:= \big[z_1\big((i-1)\Delta_n\big), z_1(i\Delta_n)\big) \times \big[z_2\big((i-1)\Delta_n\big), z_2(i\Delta_n)\big), \\
\mathcal{B}_{i,n} &:= \big(-\infty, z_1\big((i-1)\Delta_n\big)\big) \times \big[z_2\big((i-1)\Delta_n\big), z_2(i\Delta_n)\big), \\
\mathcal{B}'_{i,n} &:= \big[z_1\big((i-1)\Delta_n\big), z_1(i\Delta_n)\big) \times \big(-\infty, z_2\big((i-1)\Delta_n\big)\big), \\
\mathcal{C}_{i,n} &:= \big(-\infty, z_1(i\Delta_n)\big) \times \big(-\infty, z_2(i\Delta_n)\big),
\end{aligned}$$

and

$$g_{i,n}(\boldsymbol{x}) := g\big(\boldsymbol{z}(i\Delta_n) - \boldsymbol{x}\big), \quad \boldsymbol{x} \in \mathcal{C}_{i,n}, \tag{IV.4.2}$$

for any $i = 0, 1, 2 \ldots$ and $n \in \mathbb{N}$.

IV.4.1 Auxiliary results

Recalling the notation $\boldsymbol{x} \circ \boldsymbol{y}$, we observe the inequality

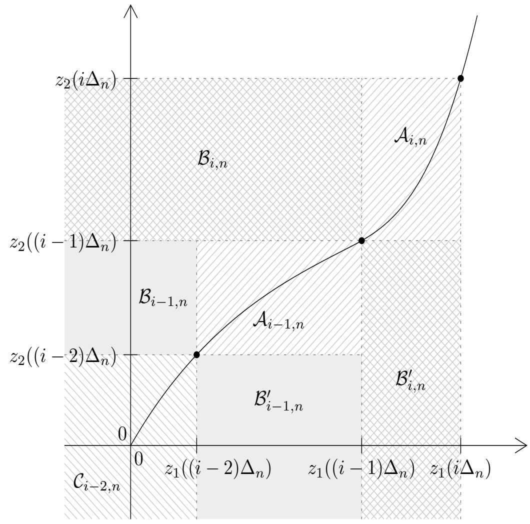
$$\|\boldsymbol{x} \circ \boldsymbol{y}\| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|. \tag{IV.4.3}$$

In this section, we shall derive various auxiliary results, including the ones related to the asymptotic behavior of the covariance kernel of the increments of the Gaussian core $(G_t)_{t\geq 0}$ that has been introduced in (IV.1.4). Note that Assumption (A1) implies that for any $T > 0$, there exist constants $0 < \underline{z}'_T \leq \overline{z}'_T < \infty$ such that

$$\underline{z}'_T(t-s) \leq z_i(t) - z_i(s) \leq \overline{z}'_T(t-s) \tag{IV.4.4}$$

for any $0 \leq s \leq t \leq T$ and $i = 1, 2$.
Let us write $\Delta_i^n G := G_{i\Delta_n} - G_{(i-1)\Delta_n}$ and $\gamma_n(i,j) := \operatorname{cov}(\Delta_i^n G, \Delta_j^n G)$ for any $i, j = 0, 1, 2 \ldots$ and $n \in \mathbb{N}$. We start with the following technical results.

Figure IV.1: An illustration of some sets defined in the proof of Lemma IV.4.1



Lemma IV.4.1. *Suppose that Assumptions (A1) and (A2) hold, and that $\alpha \in (-1, -1/4)$. Then for any $T > 0$, there exists a sequence $\left(\overline{\rho}(k)\right)_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} \overline{\rho}(k)^2 < \infty$ and*

$$|\gamma_n(i,j)| \lesssim_{\alpha,f,\mathbf{z}} \Delta_n^{2\alpha+2} \overline{\rho}(|i-j|)$$

for any $n \in \mathbb{N}$ and $i, j = 0, 1, \ldots, \lfloor T/\Delta_n \rfloor$.

Proof. See Appendix. □

Lemma IV.4.2. *Suppose that Assumptions (A1) and (A2) hold. Then we deduce the inequalities*

$$\int_{\mathcal{A}_{i,n}} g_{i,n}^2(\boldsymbol{s}) d\mathbf{s} \lesssim_{\alpha,f,\mathbf{z}} \Delta_n^{2+2\alpha}, \quad \int_{\mathcal{B}_{i,n}\cup\mathcal{B}'_{i,n}} g_{i,n}^2(\boldsymbol{s}) d\boldsymbol{s} \lesssim_{\alpha,f,\mathbf{z}} \Delta_n^{2+2\alpha},$$

$$\int_{\mathcal{C}_{i-1,n}} \left\{g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s})\right\}^2 d\boldsymbol{s} \lesssim_{\alpha,f,\mathbf{z}} \Delta_n^{2+2\alpha},$$

$$\int_{(-\boldsymbol{\infty},\boldsymbol{z}(i\Delta_n)-\boldsymbol{\varepsilon})} \left(g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s})\right)^2 d\mathbf{s} \lesssim_{\alpha,f,\mathbf{z}} \Delta_n^2 \varepsilon^{2\alpha},$$

for any $\boldsymbol{\varepsilon} = (\varepsilon, \varepsilon)$ with $\varepsilon \in (\Delta_n, 1)$.

Proof. The first three inequalities follow directly from the proof of Lemma IV.4.1 when handling the variance term $\gamma_n(i,i)$. To show the last inequality we use the substitution $\boldsymbol{y} = \boldsymbol{z}(i\Delta_n) - \boldsymbol{s}$ and Assumption (A2) to get

$$\int_{(-\boldsymbol{\infty},\boldsymbol{z}(i\Delta_n)-\boldsymbol{\varepsilon})} \left(g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s})\right)^2 d\mathbf{s}$$
$$= \int_{(\boldsymbol{\varepsilon},\boldsymbol{\infty})} \left(g(\boldsymbol{y}) - g(\boldsymbol{y} + \boldsymbol{z}((i-1)\Delta_n) - \boldsymbol{z}(i\Delta_n))\right)^2 d\mathbf{s}$$

$$= \int_{(\boldsymbol{\varepsilon},\boldsymbol{\infty})} (\|\boldsymbol{y}\|^\alpha f(\boldsymbol{y}) - \|\boldsymbol{y}+\boldsymbol{z}((i-1)\Delta_n) - \boldsymbol{z}(i\Delta_n)\|^\alpha f(\boldsymbol{y}+\boldsymbol{z}((i-1)\Delta_n) - \boldsymbol{z}(i\Delta_n)))^2\, d\mathbf{s}.$$

Using again Assumption (A2), we thus conclude that

$$\int_{(-\boldsymbol{\infty},\boldsymbol{z}(i\Delta_n)-\boldsymbol{\varepsilon})} (g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s}))^2\, d\mathbf{s}$$
$$\lesssim_{\alpha,f,\mathbf{z}} \int_{(\boldsymbol{\varepsilon},\boldsymbol{\infty})} (\|\boldsymbol{y}\|^\alpha - \|\boldsymbol{y}+\boldsymbol{z}((i-1)\Delta_n) - \boldsymbol{z}(i\Delta_n)\|^\alpha)^2\, d\mathbf{y} + \Delta_n^2.$$

For the latter integral we use the substitution $\boldsymbol{y} = \boldsymbol{\varepsilon}\circ\boldsymbol{x}$ to obtain:

$$\int_{(\boldsymbol{\varepsilon},\boldsymbol{\infty})} (\|\boldsymbol{y}\|^\alpha - \|\boldsymbol{y}+\boldsymbol{z}((i-1)\Delta_n) - \boldsymbol{z}(i\Delta_n)\|^\alpha)^2\, d\mathbf{y}$$
$$= \varepsilon^{2+2\alpha} \int_{(\mathbf{1},\boldsymbol{\infty})} \left(\|\boldsymbol{x}\|^\alpha - \|\boldsymbol{x}+\varepsilon^{-1}\{\boldsymbol{z}((i-1)\Delta_n) - \boldsymbol{z}(i\Delta_n)\}\|^\alpha\right)^2 d\mathbf{x}.$$

Finally, applying the mean value theorem and the differentiability of the curve $\boldsymbol{z}$, we obtain that

$$\int_{(\mathbf{1},\boldsymbol{\infty})} \left(\|\boldsymbol{x}\|^\alpha - \|\boldsymbol{x}+\varepsilon^{-1}\{\boldsymbol{z}((i-1)\Delta_n) - \boldsymbol{z}(i\Delta_n)\}\|^\alpha\right)^2 d\mathbf{x}$$
$$\lesssim_{\alpha,f,\mathbf{z}} \Delta_n^2\varepsilon^{-2} \int_{(\mathbf{1},\boldsymbol{\infty})} \|\boldsymbol{x}\|^{2\alpha-2} d\mathbf{x} = \mathcal{O}(\Delta_n^2\varepsilon^{-2}).$$

This completes the proof of Lemma IV.4.2. □

Lemma IV.4.3. *Suppose that Assumptions (A1) and (A2) hold, and that* $\alpha \in (-1,-1/2)$. *Then we have,*

(i) *For any* $T>0$,
$$\lim_{\Delta\to 0}\sup_{t\in[0,T]} |\Delta^{-(2\alpha+2)} v_{t,t+\Delta} - \phi_t^2| = 0,$$
where
$$v_{t,t+\Delta} := \mathrm{var}(G_{t+\Delta} - G_t) \qquad \text{(IV.4.5)}$$
and ϕ_t^2 *has been defined in* (IV.3.2).

(ii) *For any* $T>0$,
$$\sup_{t\in[0,T]} |\phi_{t+\Delta} - \phi_t| \lesssim_{f,T,\boldsymbol{z}} \Delta.$$

(iii) *For any* $T>0$,
$$\inf_{n\in\mathbb{N},t\in[0,T]} \Delta_n^{-2\alpha-2} v_{t,t+\Delta_n} > 0.$$

Proof. See Appendix. □

Lemma IV.4.4. *Let* $\alpha \in (-1,-1/2)$. *Under Assumptions (A1), (A2) and (A4), for any* $q>0$, *we obtain that*

$$\mathbb{E}[|\Delta_i^n Y|^q] + \mathbb{E}[|\Delta_i^n G|^q] \lesssim_q \Delta_n^{(1+\alpha)q}$$

Proof. Recalling (IV.4.5), we deduce that

$$\mathbb{E}[|\Delta_i^n G|^q] = m_q v_{(i-1)\Delta_n, i\Delta_n}^{q/2} \lesssim_q \Delta_n^{(1+\alpha)q}$$

where we used Lemma IV.4.3.
Let us proceed to the second part. We recall Assumption (A4) and assume without loss of generality that $\mathbf{a} > \mathbf{1}$. Applying Burkholder's inequality yields:

$$\mathbb{E}[|\Delta_i^n Y|^q] \lesssim_q \left(\int_{[\boldsymbol{z}((i-1)\Delta_n), \boldsymbol{z}(i\Delta_n)]} g_{i,n}^2(\boldsymbol{s}) \sigma_{\boldsymbol{s}}^2 d\boldsymbol{s} \right)^{q/2} + \left(\int_{(-\boldsymbol{\infty}, \boldsymbol{z}(i\Delta_n)] \setminus [\boldsymbol{z}((i-1)\Delta_n), \boldsymbol{z}(i\Delta_n)]} \{g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s})\}^2 \sigma_{\boldsymbol{s}}^2 d\boldsymbol{s} \right)^{q/2}$$

Recalling that σ is bounded on compact intervals and applying Assumption (A4), we deduce that

$$\mathbb{E}[|\Delta_i^n Y|^q] \lesssim_q \mathbb{E}\left[\left(v_{(i-1)\Delta_n, i\Delta_n} + \Delta_n^2 F_{\boldsymbol{z}((i-1)\Delta_n)} \right)^{q/2} \right].$$

This completes the proof of Lemma IV.4.4. □

IV.4.2 Proofs of laws of large numbers

Next, we prove a law of large numbers for the power variations of the Gaussian core G.

Lemma IV.4.5. *Under the assumptions of Theorem IV.3.1(i), we have for any fixed $t > 0$:*

$$\Delta_n^{1-p(1+\alpha)} V(G,p)_t^n \xrightarrow{\mathbb{P}} m_p \phi_t^{p+} \quad \textit{as } n \to \infty,$$

where

$$\phi_t^{p+} := \int_0^t |\phi_s|^p ds.$$

Proof. We obtain that

$$\begin{aligned}
\Delta_n^{1-p(1+\alpha)} \mathbb{E}[V(G,p)_t^n] &= \Delta_n^{1-p(1+\alpha)} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[|\Delta_i^n G|^p] \\
&= m_p \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \Delta_n^{-p(1+\alpha)} v_{(i-1)\Delta_n, i\Delta_n}^{p/2} \\
&= m_p \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\phi_{(i-1)\Delta_n}^2 + u_{\Delta_n}((i-1)\Delta_n) \right)^{p/2} \xrightarrow[n\to\infty]{} m_p \phi_t^{p+},
\end{aligned}$$

where we used Lemma IV.4.3 in the last part with the notation $u_\Delta(t) := \Delta^{-(2\alpha+2)} v_{t,t+\Delta} - \phi_t^2$.
For two jointly normal random variables X_1 and X_2 with mean zero and unit variance, we observe the inequality (recall that $f_p(x) = \sum_{k\geq 2} \lambda_k H_k(x)$):

$$|\mathrm{cov}(|X_1|^p, |X_2|^p)| = \left| \sum_{k\geq 2} \lambda_k^2 k! \mathbb{E}[X_1 X_2]^k \right| \leq \mathbb{E}[X_1 X_2]^2 \sum_{k\geq 2} \lambda_k^2 k!.$$

As a consequence, we conclude for the variance of $V(G,p)_t^n$ that

$$\begin{aligned}
\Delta_n^{2-2p(1+\alpha)}\mathrm{var}\big(V(G,p)_t^n\big) &= \Delta_n^2 \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \mathrm{var}\Big(\Big|\frac{\Delta_i^n G}{\Delta_n^{1+\alpha}}\Big|^p\Big) \\
&\quad + \Delta_n^2 2 \sum_{k=1}^{\lfloor t/\Delta_n\rfloor -1} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor -k} \mathrm{cov}\Big(\Big|\frac{\Delta_i^n G}{\Delta_n^{1+\alpha}}\Big|^p, \Big|\frac{\Delta_{i+k}^n G}{\Delta_n^{1+\alpha}}\Big|^p\Big) \\
&\lesssim_{\alpha,f,\boldsymbol{z},p} \Delta_n^2 \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \overline{\rho}(0)^2 + 2\Delta_n^2 \sum_{k=1}^{\lfloor t/\Delta_n\rfloor -1} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor -k} \overline{\rho}(k)^2 \\
&\lesssim_{\alpha,f,\boldsymbol{z},p} \Delta_n \sum_{k=0}^{\infty} \overline{\rho}(k)^2 \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

□

IV.4.2.1 Proof of Theorem IV.3.1(i)

By Pólya's theorem, it suffices to establish (IV.3.4) for a fixed $t \geq 0$. To this end, let us consider for any $\ell > 0$ the decomposition

$$\begin{aligned}
&\Delta_n^{1-p(1+\alpha)} V(Y,p)_t^n - m_p \int_0^t |\phi_s \sigma_{\mathbf{z}(s)}|^p ds \\
&= \Delta_n^{1-p(1+\alpha)} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \big(|\Delta_i^n Y|^p - |\sigma_{\mathbf{z}((i-1)\Delta_n)} \Delta_i^n G|^p\big) \\
&\quad + \Delta_n^{1-p(1+\alpha)} \sum_{j=1}^{\lfloor \ell t\rfloor} \sum_{i\in I_n(j,\ell)} \big(|\sigma_{\mathbf{z}((i-1)\Delta_n)}|^p - |\sigma_{\mathbf{z}((j-1)/\ell)}|^p\big)|\Delta_i^n G|^p \\
&\quad + \sum_{j=1}^{\lfloor \ell t\rfloor} |\sigma_{\mathbf{z}((j-1)/\ell)}|^p \Big(\Delta_n^{1-p(1+\alpha)} \sum_{i\in I_n(j,\ell)} |\Delta_i^n G|^p - m_p(\phi_{j/\ell}^{p+} - \phi_{(j-1)/\ell}^{p+})\Big) \\
&\quad + m_p\Big(\sum_{j=1}^{\lfloor \ell t\rfloor} |\sigma_{\mathbf{z}((j-1)/\ell)}|^p(\phi_{j/\ell}^{p+} - \phi_{(j-1)/\ell}^{p+}) - \int_0^t |\sigma_{z(s)}|^p d\phi_s^{p+}\Big) \\
&=: A_t(n) + B_t(\ell,n) + C_t(\ell,n) + D_t(\ell),
\end{aligned}$$

where

$$I_n(j,\ell) := \Big\{ i : i\Delta_n \in \Big(\frac{j-1}{\ell}, \frac{j}{\ell}\Big]\Big\}.$$

By the definition of the Riemann–Stieltjes integral, $D_t(\ell) \overset{\text{a.s.}}{\to} 0$ as $\ell \to \infty$. Moreover, by Lemma IV.4.5, we have $C_t(\ell,n) \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ for any $\ell > 0$.
Let us proceed with the term $B_t(\ell,n)$. To check that for any $\varepsilon > 0$,

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \mathbb{P}(|B_t(\ell,n)| > \varepsilon) = 0,$$

we use the bound

$$|B_t(\ell,n)| \le \sum_{j=1}^{\lfloor \ell t\rfloor} \sup_{s\in((j-2)/\ell,j/\ell]} ||\sigma_{\mathbf{z}(s)}|^p - |\sigma_{\mathbf{z}((j-1)/\ell)}|^p| \bigg(\Delta_n^{1-p(1+\alpha)} \sum_{i\in I_n(j,\ell)} |\Delta_i^n G|^p \bigg). \tag{IV.4.6}$$

Now, observe that, by Lemma IV.4.5, the right-hand side of (IV.4.6) converges in probability, as $n\to\infty$, to

$$\sum_{j=1}^{\lfloor \ell t\rfloor} \sup_{s\in((j-2)/\ell,j/\ell]} ||\sigma_{\mathbf{z}(s)}|^p - |\sigma_{\mathbf{z}((j-1)/\ell)}|^p| (\phi^{p+}_{j/\ell} - \phi^{p+}_{(j-1)/\ell})$$
$$\le \sup_{s\in[0,T]} |\phi_s|^p \frac{1}{\ell} \sum_{j=1}^{\lfloor \ell t\rfloor} \sup_{s\in((j-2)/\ell,j/\ell]} ||\sigma_{\mathbf{z}(s)}|^p - |\sigma_{\mathbf{z}((j-1)/\ell)}|^p| \xrightarrow[\ell\to\infty]{\mathbb{P}} 0,$$

where the final convergence follows from the continuity of the intermittency field σ. Finally, we handle the term $A_t(n)$. Due to inequalities $||x|^p - |y|^p| \le p|x-y|(|x|^{p-1} + |y|^{p-1})$ for $p>1$, and $||x|^p - |y|^p| \le |x-y|^p$ for $p\le 1$, $x,y\in\mathbb{R}$, Cauchy-Schwarz inequality and Lemma IV.4.4, it suffices to show that

$$\Delta_n^{1-2(1+\alpha)} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \mathbb{E}\big[\big(\Delta_i^n Y - \sigma_{\mathbf{z}((i-1)\Delta_n)}\Delta_i^n G\big)^2\big] \to 0 \tag{IV.4.7}$$

as $n\to\infty$. We use the decomposition

$$\Delta_i^n Y - \sigma_{\mathbf{z}((i-1)\Delta_n)}\Delta_i^n G = R_n^{(1)} + R_n^{(2)} + R_n^{(3)} \tag{IV.4.8}$$

where

$$R_i^n(1) = \int_{\mathbf{z}((i-1)\Delta_n)}^{\mathbf{z}(i\Delta_n)} g(\mathbf{z}(i\Delta_n)-\mathbf{s})(\sigma_{\mathbf{s}} - \sigma_{\mathbf{z}((i-1)\Delta_n)})W(d\mathbf{s}),$$
$$R_i^n(2) = \int_{\mathcal{B}_i(\boldsymbol{\varepsilon})} (g(\mathbf{z}(i\Delta_n)-\mathbf{s}) - g(\mathbf{z}((i-1)\Delta_n)-\mathbf{s}))\sigma_{\mathbf{s}} W(d\mathbf{s})$$
$$- \sigma_{\mathbf{z}((i-1)\Delta_n)} \int_{\mathcal{B}_i(\boldsymbol{\varepsilon})} (g(\mathbf{z}(i\Delta_n)-\mathbf{s}) - g(\mathbf{z}((i-1)\Delta_n)-\mathbf{s}))W(d\mathbf{s}),$$
$$R_i^n(3) = \int_{\mathcal{C}_i(\boldsymbol{\varepsilon})} (g(\mathbf{z}(i\Delta_n)-\mathbf{s}) - g(\mathbf{z}((i-1)\Delta_n)-\mathbf{s}))\sigma_{\mathbf{s}} W(d\mathbf{s})$$
$$- \sigma_{\mathbf{z}((i-1)\Delta_n)} \int_{\mathcal{C}_i(\boldsymbol{\varepsilon})} (g(\mathbf{z}(i\Delta_n)-\mathbf{s}) - g(\mathbf{z}((i-1)\Delta_n)-\mathbf{s}))W(d\mathbf{s})$$

for some $\boldsymbol{\varepsilon} = (\varepsilon,\varepsilon) > \mathbf{0}$. Here the sets $\mathcal{B}_i(\boldsymbol{\varepsilon})$ and $\mathcal{C}_i(\boldsymbol{\varepsilon})$ are defined as

$$\mathcal{B}_i(\boldsymbol{\varepsilon}) := [\mathbf{z}((i-1)\Delta_n) - \boldsymbol{\varepsilon}, \mathbf{z}(i\Delta_n)] \setminus [\mathbf{z}((i-1)\Delta_n), \mathbf{z}(i\Delta_n)], \tag{IV.4.9}$$

$$\mathcal{C}_i(\boldsymbol{\varepsilon}) := (-\infty, \mathbf{z}(i\Delta_n)] \setminus [\mathbf{z}((i-1)\Delta_n) - \boldsymbol{\varepsilon}, \mathbf{z}(i\Delta_n)]. \tag{IV.4.10}$$

We note that

$$\mathbb{E}[(R_i^n(1))^2] = \int_{\mathbf{z}((i-1)\Delta_n)}^{\mathbf{z}(i\Delta_n)} g_{i,n}^2(\boldsymbol{s})\mathbb{E}[(\sigma_{\mathbf{s}} - \sigma_{\mathbf{z}((i-1)\Delta_n)})^2]d\mathbf{s}$$
$$\lesssim_f \Delta_n^{2\alpha} \int_{\mathbf{z}((i-1)\Delta_n)}^{\mathbf{z}(i\Delta_n)} \mathbb{E}[(\sigma_{\mathbf{s}} - \sigma_{\mathbf{z}((i-1)\Delta_n)})^2]d\mathbf{s}$$
$$\lesssim_f \Delta_n^{2+2\alpha}\mathbb{E}[r(\mathbf{z}((i-1)\Delta_n), C\Delta_n)]$$

where $r(\mathbf{s},\boldsymbol{\eta}) := \sup\{(\sigma_{\mathbf{s}} - \sigma_{\mathbf{t}})^2 | \mathbf{t} \in [\mathbf{s}-\boldsymbol{\eta}, \mathbf{s}+\boldsymbol{\eta}]\}$. In view of continuity of σ, we deduce the convergence

$$\Delta_n^{1-2(1+\alpha)} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\big[(R_i^n(1))^2\big] \to 0.$$

Next, we decompose $R_i^n(2) = R_i^n(2.1) + R_i^n(2.2)$, where

$$R_i^n(2.1) = \int_{\mathcal{B}_i(\boldsymbol{\varepsilon})} (g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s}))(\sigma_{\mathbf{s}} - \sigma_{\mathbf{z}((i-1)\Delta_n)-\boldsymbol{\varepsilon}})W(d\mathbf{s}),$$
$$R_i^n(2.2) = (\sigma_{\mathbf{z}((i-1)\Delta_n)-\boldsymbol{\varepsilon}} - \sigma_{\mathbf{z}((i-1)\Delta_n)}) \int_{\mathcal{B}_i(\boldsymbol{\varepsilon})} (g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s}))W(d\mathbf{s}).$$

Then, we have:

$$\Delta_n^{1-2(1+\alpha)} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\big[(R_i^n(2.1))^2\big] \lesssim_f \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[r(\mathbf{z}((i-1)\Delta_n), \boldsymbol{\varepsilon})],$$
$$\Delta_n^{1-2(1+\alpha)} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\big[(R_i^n(2.2))^2\big] \lesssim_f \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[r(\mathbf{z}((i-1)\Delta_n), \boldsymbol{\varepsilon})^2]^{1/2}.$$

Hence, both terms converge to 0 when we first let $n \to \infty$ and then $\boldsymbol{\varepsilon} \to \mathbf{0}$. Finally, due to the boundedness of σ on compact sets and Aassumption (A4), we obtain similarly to the proof of Lemma IV.4.4 that

$$\mathbb{E}\big[(R_i^n(3))^2\big] \lesssim_\sigma \int_{-\infty}^{\mathbf{z}((i-1)\Delta_n)-\boldsymbol{\varepsilon}} \big[g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s})\big]^2 d\mathbf{s}$$
$$\lesssim_{\sigma,\mathbf{z},f,\alpha} \Delta_n^{2\alpha+2} \int_{[\varepsilon,\infty)^2} \big[\Delta_n \|\boldsymbol{y}\|^\alpha \|\boldsymbol{y}+\mathbf{1}\|^{\alpha+1} \wedge \|\boldsymbol{y}\|^{2\alpha-2}\big]d\boldsymbol{y}$$
$$\lesssim_{\sigma,\mathbf{z},f,\alpha,\varepsilon} \Delta_n^{2\alpha+2}(\Delta_n M_n^{2\alpha+3} + M_n^{2\alpha}) \le \Delta_n^{2\alpha+2}\Delta_n^{-2\alpha/3},$$

where the second inequality follows from bounds derived around (IV.5.16) and the last step chooses $M_n = \Delta_n^{-1/3}$. This leads to

$$\Delta_n^{1-2(1+\alpha)} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\big[(R_i^n(3))^2\big] \lesssim_{\sigma,\mathbf{z},f,\alpha,t,\varepsilon} \Delta_n^{-2\alpha/3},$$

which completes the proof of Theorem IV.3.1(i).

IV.4.2.2 Proof of Theorem IV.3.1(ii)

We start with the decomposition

$$\Delta_i^n Y =: Y_{i,n}^1 + Y_{i,n}^2 \tag{IV.4.11}$$

where

$$Y_{i,n}^1 := \int_{\mathcal{B}_{i,n}\cup\mathcal{B}'_{i,n}} g_{i,n}(\boldsymbol{s})\sigma_{\mathbf{s}}W(d\mathbf{s}),$$

$$Y_{i,n}^2 := \int_{\mathcal{A}_{i,n}\cup\mathcal{C}_{i-1,n}} \left\{g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s})\right\}\sigma_{\mathbf{s}}W(d\mathbf{s}).$$

We will show later that that the term $Y_{i,n}^1$ dominates under conditions of Theorem IV.3.1(ii). This random variable is further approximated via

$$Y'_{i,n} := \int_{\mathcal{B}_{i,n}} g_{i,n}(\boldsymbol{s})\sigma_{s_1,z_2((i-1)\Delta_n)}W(d\mathbf{s}) + \int_{\mathcal{B}'_{i,n}} g_{i,n}(\boldsymbol{s})\sigma_{z_1((i-1)\Delta_n),s_2}W(d\mathbf{s}). \tag{IV.4.12}$$

The next lemma assesses the stochastic order of the terms $Y_{i,n}^1$, $Y_{i,n}^2$ and $Y'_{i,n}$.

Lemma IV.4.6. *Let $p > 0$. Then, under assumptions (A1), (A2), (A4) and $\alpha \in (-1/2, 0)$,*

$$\mathbb{E}\left[|Y_{i,n}^1|^p + |Y'_{i,n}|^p\right] \lesssim_p \Delta_n^{p/2}, \qquad \mathbb{E}\left[|Y_{i,n}^2|^p\right] \lesssim_p \Delta_n^{p(1+\alpha)}.$$

Proof. See Appendix. □

In particular, we conclude from Lemma IV.4.6 that the term $Y_{i,n}^2$ is negligible compared to $Y_{i,n}^1$. The following statement shows that the term $Y_{i,n}^1$ is well approximated by the quantity $Y'_{i,n}$.

Lemma IV.4.7. *Let $p > 0$ and recall the notation $f_p(x) = |x|^p$. Then, under assumptions (A1), (A2), (A4) and $\alpha \in (-1/2, 0)$, as $n \to \infty$:*

$$\Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(f_p(\Delta_n^{-1/2}Y_{i,n}^1) - f_p(\Delta_n^{-1/2}Y'_{i,n})\right) \stackrel{u.c.p.}{\longrightarrow} 0.$$

Proof. See Appendix. □

Now, let us apply the previous statements. First of all, due to Lemma IV.4.6, we readily deduce the convergence

$$\Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f_p(\Delta_n^{-1/2}Y_{i,n}^2) \stackrel{\text{u.c.p.}}{\longrightarrow} 0$$

by Markov inequality. Hence, using Lemma IV.4.7, we conclude that

$$\Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(f_p(\Delta_n^{-1/2}\Delta_i^n Y) - f_p(\Delta_n^{-1/2}Y'_{i,n})\right) \stackrel{\text{u.c.p.}}{\longrightarrow} 0.$$

As a consequence, we are left to showing the convergence

$$\Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f_p(\Delta_n^{-1/2} Y'_{i,n}) \xrightarrow{\text{u.c.p.}} m_p \int_0^t |w_s|^p ds.$$

Applying martingale techniques, the latter follows if we prove the following statements:

$$\Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[f_p(\Delta_n^{-1/2} Y'_{i,n}) | \mathcal{F}_{(i-1)\Delta_n}\right] \xrightarrow{\text{u.c.p.}} m_p \int_0^t |w_s|^p ds \tag{IV.4.13}$$

$$\Delta_n^2 \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}\left[f_p^2(\Delta_n^{-1/2} Y'_{i,n}) | \mathcal{F}_{(i-1)\Delta_n}\right] \xrightarrow{\mathbb{P}} 0. \tag{IV.4.14}$$

The convergence at (IV.4.14) follows immediately from Lemma IV.4.6. To prove (IV.4.13), we recall the definition (IV.4.1) of the filtration $\mathcal{F}_t$, and notice that, conditionally on $\mathcal{F}_{(i-1)\Delta_n}$, the random variable $Y'_{i,n}$ is Gaussian with conditional variance

$$\begin{aligned}\mathbb{E}\left[\left(Y'_{i,n}\right)^2 | \mathcal{F}_{(i-1)\Delta_n}\right] &= \int_{\mathcal{B}_{i,n}} g_{i,n}^2(\boldsymbol{s}) \sigma^2_{s_1, z_2((i-1)\Delta_n)} ds \\ &\quad + \int_{\mathcal{B}'_{i,n}} g_{i,n}^2(\boldsymbol{s}) \sigma^2_{z_1((i-1)\Delta_n), s_2} ds =: \Delta_n w_{i,n}^2.\end{aligned}$$

The convergence at (IV.4.13), and hence the statement of Theorem IV.3.1(ii), now follows from the following lemma.

Lemma IV.4.8. *Under assumptions of Theorem IV.3.1(ii), it holds that*

$$\Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left| w^p_{(i-1)\Delta_n} - w^p_{i,n} \right| \xrightarrow{u.c.p.} 0. \tag{IV.4.15}$$

Proof. See Appendix. □

IV.4.3 Proof of the central limit theorem

IV.4.3.1 Auxiliary lemmas

Lemma IV.4.9. *Let $\alpha \in (-1, -3/4)$ and $\Delta \in (0, 1)$. Under Assumptions (A1) and (A2), we obtain that*

$$\sup_{t \in [0,T]} |\Delta^{-(2\alpha+2)} v_{t,t+\Delta} - \phi_t^2| \lesssim_{\alpha, f, T, \boldsymbol{z}} \Delta^\beta \tag{IV.4.16}$$

for some $\beta > 1/2$.

Proof. We proceed as in the proof of Lemma IV.4.3(i) and note that several error terms are $\mathcal{O}(\Delta)$, which mainly stem from (IV.5.9). In this part, we only consider the remaining terms.

Let us first examine the components of $v_{t,t+\Delta}(1)$. In particular, concerning the term I_2 introduced in Section IV.5.2, we have

$$\begin{aligned}\sup_{t\in[0,T]} I_2 &\lesssim_{\alpha,f,T,\boldsymbol{z}} \Delta \int_{(1,M]\times(0,1]} \|\boldsymbol{y}\|^{2\alpha+1} d\boldsymbol{y} + \int_{[M,\infty)\times(0,1]} \|\boldsymbol{y}\|^{2\alpha} d\boldsymbol{y} \\ &\lesssim_{\alpha,f,T,\boldsymbol{z}} \Delta \int_{(1,M]\times(0,1]} y_1^{2\alpha+1} dy_1 dy_2 + \int_{[M,\infty)\times(0,1]} y_1^{2\alpha} dy_1 dy_2 \\ &\lesssim_{\alpha,f,T,\boldsymbol{z}} \Delta \frac{M^{2\alpha+2}}{2\alpha+2} - \frac{M^{2\alpha+1}}{2\alpha+1}.\end{aligned}$$

Now, let us pick $\beta \in (1/2, -2\alpha-1)$, which is guaranteed by $\alpha < -3/4$. Now, we choose M such that $M^{2\alpha+1} = \Delta^\beta$. Then, we have that

$$\sup_{t\in[0,T]} I_2 \lesssim_{\alpha,f,T,\boldsymbol{z}} \Delta^\beta.$$

The term I_3 of $v_{t,t+\Delta}(1)$ (see again Section IV.5.2), and parts of $v_{t,t+\Delta}(2)$ are shown analogously. □

Next, we come back to the computation of the covariance function $\gamma_n(i,j) = \mathrm{cov}(\Delta_i^n G, \Delta_j^n G)$.

Lemma IV.4.10. *Let us fix an integer $l \geq 1$. Then it holds that*

$$\sup_{i=1,\dots,\lfloor T/\Delta_n\rfloor} |\Delta_n^{-2\alpha-2}\gamma_n(i,i+l) - \phi^2_{(i-1)\Delta_n}\rho(l)| \to 0 \qquad \text{as } n\to\infty,$$

where $\rho(t,l)$ is defined as

$$\rho(l) := \frac{1}{2}\left[(l-1)^{2\alpha+2} + (l+1)^{2\alpha+2} - 2l^{2\alpha+2}\right]. \tag{IV.4.17}$$

Proof. We represent the covariances via the polarisation identity:

$$\gamma_n(i,i+l) = \frac{1}{2}\left[v_{i\Delta_n,(i+l-1)\Delta_n} + v_{(i-1)\Delta_n,(i+l)\Delta_n} - v_{i\Delta_n,(i+l)\Delta_n} - v_{(i-1)\Delta_n,(i+l-1)\Delta_n}\right].$$

Then, in view of Lemma IV.4.3(i), (ii), we deduce the claim

$$\sup_{i=1,\dots,\lfloor T/\Delta_n\rfloor} |\Delta_n^{-2\alpha-2}\gamma_n(i,i+l) - \rho((i-1)\Delta_n, l)| \qquad \text{as } n\to\infty, \tag{IV.4.18}$$

where

$$\rho((i-1)\Delta_n, l) = \frac{\phi^2_{(i-1)\Delta_n}}{2}\left[(l-1)^{2\alpha+2} + (l+1)^{2\alpha+2} - 2l^{2\alpha+2}\right].$$

□

Next, we will show the associated functional central limit theorem. For this purpose we recall the Hermite expansion of the function $f_p(x) = |x|^p - m_p$:

$$f_p(x) = \sum_{k=2}^{\infty} \lambda_k H_k(x).$$

We note that $\sum_{k=2}^\infty \lambda_k^2 k! < \infty$ for any $p \geq 0$ due to $\mathbb{E}[f_{2p}(\mathcal{N}(0,1))] < \infty$.

Proposition IV.4.11. *Under the assumptions of Theorem IV.3.5(i), we deduce the functional stable convergence on $D([0,T])$:*

$$\Delta_n^{-1/2}\left(\Delta_n^{1-p(1+\alpha)}V(G,p)_t^n - m_p\int_0^t |\phi_s|^p ds\right) \xrightarrow{st} \int_0^t \kappa_s dB_s, \quad as\ n\to\infty,$$

where B is a Brownian motion independent of $\mathcal{F}$ and κ_s^2 is given as

$$\kappa_s^2 = \phi_s^{2p}\sum_{k=2}^{\infty}\lambda_k^2 k!\left(1+2\sum_{l=1}^{\infty}\rho(l)^k\right).$$

Proof. It suffices to prove the finite dimensional weak convergence and the tightness of the statistic.
We start with the first claim and pick $0\le t_1<\ldots<t_d$ with $d\in\mathbb{N}$. Let us denote $S_n = (S_n(t_1),\ldots,S_n(t_d))$, where

$$S_n(t) = \Delta_n^{-1/2}\left(\Delta_n^{1-p(1+\alpha)}V(G,p)_t^n - m_p\int_0^t |\phi_s|^p ds\right). \tag{IV.4.19}$$

We use the decomposition $S_n(t) = S_n^{(1)}(t)+S_n^{(2)}(t)+S_n^{(3)}(t)$, where

$$S_n^{(1)}(t) := \Delta_n^{1/2}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\phi_{(i-1)\Delta_n}^p\left(\left|\frac{\Delta_i^n G}{\sqrt{v_{(i-1)\Delta_n}}}\right|^p - m_p\right),$$

$$S_n^{(2)}(t) := \Delta_n^{1/2}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left((\Delta_n^{-2\alpha-2}v_{(i-1)\Delta_n})^p - \phi_{(i-1)\Delta_n}^p\right)\left|\frac{\Delta_i^n G}{\sqrt{v_{(i-1)\Delta_n}}}\right|^p,$$

$$S_n^{(3)}(t) := \Delta_n^{1/2}m_p\int_0^t\left(\phi_{[s/\Delta_n]\Delta_n}^p - \phi_s^p\right)ds.$$

In view of Lemma IV.4.9, we conclude that $|S_n^{(2)}(t)| \xrightarrow{\text{u.c.p.}} 0$. On the other hand, the statement (IV.5.17) and the condition $p>1/2$ imply the convergence $|S_n^{(3)}(t)| \xrightarrow{\text{u.c.p.}} 0$. Next, we deal with the main part $S_n^{(1)}$. Here we heavily use techniques from Malliavin calculus. In particular, the Gaussian process $(G_t)_{t\ge 0}$ can be interpreted as an isonormal Gaussian family, to which we can associate a separable Hilbert space $\mathbb{H}$. The scalar product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ on $\mathbb{H}$ is induced by the covariance kernel of G. A detailed discussion of notions used in the arguments below can be found in the monograph [117].
Recalling $|x|^p - m_p = \sum_{k=2}^{\infty}\lambda_k H_k(x)$ and noting the following identity for iterated Wiener integrals

$$H_k\left(\frac{\Delta_i^n G}{\sqrt{v_{(i-1)\Delta_n}}}\right) = I_k\left(\left(\frac{\Delta_i^n G}{\sqrt{v_{(i-1)\Delta_n}}}\right)^{\otimes k}\right),$$

we obtain the following expansion:

$$S_n^{(1)}(t_q) = \sum_{k=2}^{\infty} I_k(f_k^n(q)) \tag{IV.4.20}$$

where

$$f_k^n(q) = \lambda_k \Delta_n^{1/2} \sum_{i=1}^{\lfloor t_q/\Delta_n \rfloor} \phi^p_{(i-1)\Delta_n} \left(\frac{\Delta_i^n G}{\sqrt{v_{(i-1)\Delta_n}}} \right)^{\otimes k}.$$

We will show that

$$\left(S_n^{(1)}(t_1), \ldots, S_n^{(1)}(t_d)\right) \xrightarrow{d} \left(\int_0^{t_1} \kappa_s dB_s, \ldots, \int_0^{t_d} \kappa_s dB_s \right). \tag{IV.4.21}$$

In view of [11, Theorem 5], a sufficient condition for the convergence (IV.4.21) is the fulfillment of the following three conditions:

(i) For any $q \in \{1, \ldots, d\}$, we have

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=m}^{\infty} k! ||f_k^n(q)||^2_{\mathbb{H}^{\otimes k}} = 0.$$

(ii) There exists a sequence $\Sigma, \Sigma_2, \Sigma_3, \ldots$ of positive semidefinite $d \times d$ matrices such that for any $(q_1, q_2) \in \{1, \ldots, d\}^2$ and $k \geq 2$ it holds that

$$\lim_{n\to\infty} k! \langle f_k^n(q_1), f_k^n(q_2) \rangle_{\mathbb{H}^{\otimes k}} = \Sigma_k(q_1, q_2),$$

and $\Sigma = \sum_{k=2}^{\infty} \Sigma_k$.

(iii) For any $q \in \{1, \ldots, d\}$, $k \geq 2$ and $r \in \{1, \ldots, k-1\}$ it holds that

$$\lim_{n\to\infty} ||f_k^n(q) \otimes_r f_k^n(q)||^2_{\mathbb{H}^{\otimes 2(k-r)}} = 0.$$

First, we prove (i) and (ii). To this end, for any $q \in \{1, \ldots, d\}$, we note that

$$\begin{aligned} k!||f_k^n(q)||^2_{H^{\otimes k}} &= k! \lambda_k^2 \Delta_n \sum_{i=1}^{\lfloor t_q/\Delta_n \rfloor} \sum_{j=1}^{\lfloor t_q/\Delta_n \rfloor} \phi^p_{(i-1)\Delta_n} \phi^p_{(j-1)\Delta_n} \left\langle \Big(\frac{\Delta_i^n G}{\sqrt{v_{(i-1)\Delta_n}}} \Big)^{\otimes k}, \Big(\frac{\Delta_j^n G}{\sqrt{v_{(j-1)\Delta_n}}} \Big)^{\otimes k} \right\rangle_{\mathbb{H}^{\otimes k}} \\ &= k! \lambda_k^2 \Delta_n \left(\sum_{i=1}^{\lfloor t_q/\Delta_n \rfloor} \phi^{2p}_{(i-1)\Delta_n} + 2 \sum_{i=1}^{\lfloor t_q/\Delta_n \rfloor} \sum_{l=1}^{\lfloor t_q/\Delta_n \rfloor - i} \phi^p_{(i-1)\Delta_n} \phi^p_{(i+l-1)\Delta_n} r_n(i, i+l)^k \right). \end{aligned}$$

Combining Lemma IV.4.1 and Lemma IV.4.3(iii) leads to

$$|r_n(i,j)| \leq C \bar{\rho}(|i-j|). \tag{IV.4.22}$$

For any $k \geq 2$, this yields $|r_n(i,j)^k| \leq C\bar{\rho}(|i-j|)^2$ due to $|r_n(i,j)| \leq 1$. Using this result together with uniform boundedness of ϕ on compact intervals yields

$$0 \leq \limsup_{n\to\infty} k! ||f_k^n(q)||^2_{H^{\otimes k}} \leq C \sum_{k=m}^{\infty} k! \lambda_k^2 \left(1 + 2 \sum_{i=1}^{\infty} \bar{\rho}(i)^2 \right).$$

Since $\sum_{i=1}^{\infty} \bar{\rho}(i)^2 < \infty$ and $\sum_{k=2}^{\infty} k! \lambda_k^2 < \infty$, we immediately deduce part (i).
Concerning part (ii), we only consider the case $q =: q_1 = q_2$; the scenario $q_1 \neq q_2$ is handled similarly. For any $k \geq 2$ note that

$$k! ||f_k^n(q)||^2_{H^{\otimes k}} = k! \lambda_k^2 M_k^n(q) + 2k! \lambda_k^2 R_k^n(q),$$

where

$$M_k^n(q) = \Delta_n \sum_{i=1}^{\lfloor t_q/\Delta_n \rfloor} \phi_{(i-1)\Delta_n}^{2p} \left(1 + 2 \sum_{l=1}^{[t_q/\Delta_n]-i} \left(\frac{\rho((i-1)\Delta_n, l)}{\phi^2_{(i-1)\Delta_n}}\right)^k\right),$$

$$R_k^n(q) = \Delta_n \sum_{i=1}^{\lfloor t_q/\Delta_n \rfloor} \sum_{l=1}^{\lfloor t_q/\Delta_n \rfloor - i} \phi_{(i-1)\Delta_n}^{p} \phi_{(i+l-1)\Delta_n}^{p} r_n(i, i+l)^k - \phi_{(i-1)\Delta_n}^{2p} \left(\frac{\rho((i-1)\Delta_n, l)}{\phi^2_{(i-1)\Delta_n}}\right)^k.$$

Regarding the main term, we deduce that

$$\lim_{n\to\infty} k!\lambda_k^2 M_k^n(q) = k!\lambda_k^2 \int_0^{t_q} \phi_s^{2p}\Big(1 + 2\sum_{l=1}^{\infty} \rho(l)^k\Big) ds. \tag{IV.4.23}$$

To deal with the term $R_k^n(q)$, we first notice that for each $l \geq 1$, the continuity of ϕ and Lemma IV.4.10 imply that

$$\sup_{i=1,\ldots,\lfloor T/\Delta_n \rfloor} \left| r_n(i, i+l) - \phi^{-2}_{(i-1)\Delta_n} \rho((i-1)\Delta_n, l) \right| \to 0 \qquad \text{as } n \to \infty. \tag{IV.4.24}$$

Due to Lemma IV.4.1, Lemma IV.4.10 and the dominated convergence theorem, we deduce that

$$\sup_{i=1,\ldots,\lfloor T/\Delta_n \rfloor} \sum_{l=1}^{\infty} \left| \phi_{(i-1)\Delta_n}^{p} \phi_{(i+l-1)\Delta_n}^{p} r_n(i, i+l)^k - \phi_{(i-1)\Delta_n}^{2p} \left(\frac{\rho((i-1)\Delta_n, l)}{\phi^2_{(i-1)\Delta_n}}\right)^k \right| \to 0$$

as $n \to \infty$, which implies that $\lim_{n\to\infty} 2k!\lambda_k^2 R_k^n(q) = 0$. As a result, (ii) is verified. Next, we proceed to (iii). For any $q \in \{1, \ldots, d\}$, $k \geq 2$ and $r \in \{1, \ldots, k-1\}$, we set $C_k^n(q, r) = ||f_k^n(q) \otimes_r f_k^n(q)||^2_{H^{\otimes 2(k-r)}}$. We have that

$$\begin{aligned} C_k^n(q, r) =& \lambda_k^4 \Delta_n^2 \sum_{i_1,i_2,i_3,i_4=1}^{\lfloor t_q/\Delta_n \rfloor} \Bigg(\phi^p_{(i_1-1)\Delta_n} \phi^p_{(i_2-1)\Delta_n} \phi^p_{(i_3-1)\Delta_n} \phi^p_{(i_4-1)\Delta_n} \\ & \times r_n(i_1, i_2)^r r_n(i_3, i_4)^r r_n(i_1, i_3)^{k-r} r_n(i_2, i_4)^{k-r} \Bigg) \\ \leq& C\lambda_k^4 \Delta_n^2 \sum_{i_1,i_2,i_3,i_4=1}^{\lfloor t_q/\Delta_n \rfloor} \bar\rho(|i_1 - i_2|)^r \bar\rho(|i_3 - i_4|)^r \bar\rho(|i_1 - i_3|)^{k-r} \bar\rho(|i_2 - i_4|)^{k-r}. \end{aligned}$$

Note that $\bar\rho(|i_1 - i_2|)^r \bar\rho(|i_1 - i_3|)^{k-r} \leq \bar\rho(|i_1 - i_2|)^k + \bar\rho(|i_1 - i_3|)^k$. Using this with multiple applications of the inequality $\sum_{i=1}^{\lfloor t_q/\Delta_n \rfloor} \bar\rho(|i-j|)^q \leq 2\sum_{i=0}^{\lfloor t_q/\Delta_n \rfloor - 1} \bar\rho(i)^v$ for any $v \geq 1$ and $j \in \{1, \ldots, \lfloor t_q/\Delta_n \rfloor\}$ yields

$$\begin{aligned} C_k^n(q, r) &\leq 16C\lambda_k^4 \Delta_n \sum_{j_1=0}^{\lfloor t_q/\Delta_n \rfloor - 1} \bar\rho(j_1)^r \sum_{j_2=0}^{\lfloor t_q/\Delta_n \rfloor - 1} \bar\rho(j_2)^{k-r} \sum_{j_3=0}^{\lfloor t_q/\Delta_n \rfloor - 1} \bar\rho(j_3)^k \\ &= 16C\lambda_k^4 \Delta_n^{1-\frac{r}{k}} \sum_{j_1=0}^{\lfloor t_q/\Delta_n \rfloor - 1} \bar\rho(j_1)^r \Delta_n^{1-\frac{k-r}{k}} \sum_{j_2=0}^{\lfloor t_q/\Delta_n \rfloor - 1} \bar\rho(j_2)^{k-r} \sum_{j_3=0}^{\lfloor t_q/\Delta_n \rfloor - 1} \bar\rho(j_3)^k. \end{aligned}$$

At this stage, since $k \geq 2$ and $\bar{\rho}(i) \leq 1$, we observe that

$$\sum_{i=0}^{\infty} \bar{\rho}(i)^k \leq \sum_{i=0}^{\infty} \bar{\rho}(i)^2 < \infty,$$

which leads with Hölder's inequality to $\Delta_n^{1-\frac{p}{k}} \sum_{i=0}^{\lfloor t_q/\Delta_n \rfloor -1} \bar{\rho}(i)^v \to 0$ for both $v = r$ and $v = k - r$. As a result, we conclude (iii) with $\lim_{n\to\infty} C_k^n(q,r) = 0$.
Next, we show tightness of the sequence $(S_n^{(1)}(t))_{t\in[0,T]}$. To show the latter we use the recent result of [114, Theorem 1.1]. Therein tightness for the partial sum process of functions of stationary Gaussian sequence has been obtained. We slightly modify their arguments to account for the non-stationary case. Using a standard tightness criterium it suffices to prove the inequality

$$\mathbb{E}\left[\left|S_n^{(1)}(t) - S_n^{(1)}(s)\right|^q\right]^{1/q} \lesssim_q (\Delta_n(\lfloor t/\Delta_n \rfloor - \lfloor s/\Delta_n \rfloor))^{1/2} \qquad \text{(IV.4.25)}$$

for some $q > 2$ and $0 \leq s \leq t \leq T$ (cf. [114, Lemma 3.1]). Observing that the function $f(x) = |x|^p - m_p$ has Hermite rank 2, and following the same arguments as displayed in [114, pages 9–10], we conclude that

$$\mathbb{E}\left[\left|S_n^{(1)}(t) - S_n^{(1)}(s)\right|^q\right]^{1/q} \lesssim_{q,p} \sum_{k=0}^{2} R_k \quad \text{and} \quad R_k \lesssim_{k,p,\phi} \left(\Delta_n \sum_{i,j=\lfloor s/\Delta_n \rfloor}^{\lfloor t/\Delta_n \rfloor -1} |\gamma_n(i,j)|^{k+2}\right)^{1/2}$$

Applying Lemma IV.4.1 and recalling that $\sum_{j=1}^{\infty} \overline{\rho}(j)^2 < \infty$, we deduce that

$$R_k \lesssim (\Delta_n(\lfloor t/\Delta_n \rfloor - \lfloor s/\Delta_n \rfloor))^{1/2} \qquad \text{for } k = 0, 1, 2.$$

Hence, condition (IV.4.25) holds and we obtain tightness of the sequence $(S_n^{(1)}(t))_{t\in[0,T]}$.
□

IV.4.3.2 Proof of Theorem IV.3.5

By fixing $\ell > 0$, we use the following decomposition:

$$\Delta_n^{-1/2}\left(\Delta_n^{1-p(1+\alpha)} V(Y,p)_t^n - m_p \int_0^t |\phi_s \sigma_{\mathbf{z}(s)}|^p ds\right) = \widetilde{A}_t(n) + \widetilde{B}_t(\ell,n) + \widetilde{C}_t(\ell,n) + \widetilde{D}_t(n)$$

where

$$\widetilde{A}_t(n) = \Delta_n^{1/2-p(1+\alpha)} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(|\Delta_i^n Y|^p - |\sigma_{\mathbf{z}((i-1)\Delta_n)} \Delta_i^n G|^p\right)$$

$$\widetilde{B}_t(\ell,n) = \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\Delta_n^{1-p(1+\alpha)} |\sigma_{\mathbf{z}((i-1)\Delta_n)} \Delta_i^n G|^p - m_p |\sigma_{\mathbf{z}((i-1)\Delta_n)}|^p \left(\phi_{i\Delta_n}^{p+} - \phi_{(i-1)\Delta_n}^{p+}\right)\right)$$
$$- \Delta_n^{-1/2} \sum_{j=1}^{\lfloor t\ell \rfloor} |\sigma_{\mathbf{z}((j-1)/\ell)}|^p \left(\Delta_n^{1-p(1+\alpha)} \sum_{i\in I_n(j,\ell)} |\Delta_i^n G|^p - m_p(\phi_{j/\ell}^{p+} - \phi_{(j-1)/\ell}^{p+})\right)$$

$$\widetilde{C}_t(\ell,n) = \Delta_n^{-1/2} \sum_{j=1}^{\lfloor t\ell \rfloor} |\sigma_{\mathbf{z}((j-1)/\ell)}|^p \Bigg(\Delta_n^{1-p(1+\alpha)} \sum_{i\in I_n(j,\ell)} |\Delta_i^n G|^p - m_p(\phi_{j/\ell}^{p+} - \phi_{(j-1)/\ell}^{p+}) \Bigg)$$

$$\widetilde{D}_t(n) = m_p \Delta_n^{-1/2} \Bigg(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\sigma_{\mathbf{z}((i-1)\Delta_n)}|^p (\phi_{i\Delta_n}^{p+} - \phi_{(i-1)\Delta_n}^{p+}) - \int_0^t |\sigma_{\mathbf{z}(s)}|^p d\phi_s^{p+} \Bigg)$$

with

$$I_n(j,\ell) := \left\{ i : i\Delta_n \in \left(\frac{j-1}{\ell}, \frac{j}{\ell} \right] \right\}.$$

It turns out that all terms except $\widetilde{C}_t(\ell,n)$ are negligible. We start by handling the main term $\widetilde{C}_t(\ell,n)$. In view of Lemma IV.4.11 and the properties of stable convergence, for each fixed $\ell > 0$, as $n \to \infty$, we obtain the functional stable convergence

$$\widetilde{C}_t(\ell,n) \xrightarrow{st} \sum_{j=1}^{\lfloor t\ell \rfloor} |\sigma_{\mathbf{z}((j-1)/\ell)}|^p \int_{(j-1)/\ell}^{j\ell} \kappa_s dB_s \qquad \text{on } D([0,T]).$$

Then, as $\ell \to \infty$, we deduce the convergence

$$\sum_{j=1}^{\lfloor t\ell \rfloor} |\sigma_{\mathbf{z}((j-1)/\ell)}|^p \int_{(j-1)/\ell}^{j\ell} \kappa_s dB_s \xrightarrow{\text{u.c.p.}} \int_0^t \kappa_s |\sigma_{\mathbf{z}(s)}|^p dB_s. \tag{IV.4.26}$$

Next, we move on to the term $\widetilde{D}_t(n)$. Due to Hölder continuity and the boundedness of σ on compact sets, we obtain that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\widetilde{D}_t(n)| \right] \lesssim \Delta_n^{-1/2+\gamma(p\wedge 1)}.$$

Since we assumed that $\gamma(p \wedge 1) > 1/2$, we conclude the convergence

$$\widetilde{D}(n) \xrightarrow{\text{u.c.p.}} 0. \tag{IV.4.27}$$

Now, we handle the term $\widetilde{B}_t(\ell,n)$. In view of the results in [53], for any $\varepsilon > 0$, we obtain that

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |\widetilde{B}_t(\ell,n)| > \varepsilon \right) = 0. \tag{IV.4.28}$$

Finally, we handle the term $\widetilde{A}_t(n)$. The inequality $||x|^p - |y|^p| \le p|x-y|(|x|^{p-1} + |y|^{p-1})$ for $p > 1$ combined with the Cauchy-Schwarz inequality, $||x|^p - |y|^p| \le |x-y|^p$ for $p \in (0,1]$, and Lemma IV.4.4 lead to

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\widetilde{A}_t(n)| \right] \lesssim \Delta_n^{1/2-(1+\alpha)(p\wedge 1)} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \left(\mathbb{E}\left[|\Delta_i^n Y - \sigma_{\mathbf{z}((i-1)\Delta_n)} \Delta_i^n G|^2 \right] \right)^{(p\wedge 1)/2} \tag{IV.4.29}$$

To deal with the terms in above sum, we consider the decomposition

$$\Delta_i^n Y - \sigma_{\mathbf{z}((i-1)\Delta_n)} \Delta_i^n G = R_i^n(1) + R_i^n(2) + \sum_{j=1}^{m} \widetilde{R}_i^n(j) \qquad \text{(IV.4.30)}$$

with

$$R_i^n(1) = \int_{\mathbf{z}((i-1)\Delta_n)}^{\mathbf{z}(i\Delta_n)} g_{i,n}(\boldsymbol{s})(\sigma_{\mathbf{s}} - \sigma_{\mathbf{z}((i-1)\Delta_n)})W(d\mathbf{s}),$$

$$R_i^n(2) = \int_{\mathcal{B}_i(\boldsymbol{\varepsilon}^{(1)})} (g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s}))\sigma_{\mathbf{s}} W(d\mathbf{s})$$

$$- \sigma_{\mathbf{z}((i-1)\Delta_n)} \int_{\mathcal{B}_i(\boldsymbol{\varepsilon}^{(1)})} (g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s}))W(d\mathbf{s}),$$

$$\widetilde{R}_i^n(j) = \int_{\mathcal{B}_i(\boldsymbol{\varepsilon}^{(j+1)})\setminus(\cup_{r=1}^{j}\mathcal{B}_i(\boldsymbol{\varepsilon}^{r})\cup[\mathbf{z}((i-1)\Delta_n),\mathbf{z}(i\Delta_n)])} (g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s}))\sigma_{\mathbf{s}} W(d\mathbf{s})$$

$$- \sigma_{\mathbf{z}((i-1)\Delta_n)} \int_{\mathcal{B}_i(\boldsymbol{\varepsilon}^{(j+1)})\setminus(\cup_{r=1}^{j}\mathcal{B}_i(\boldsymbol{\varepsilon}^{r})\cup[\mathbf{z}((i-1)\Delta_n),\mathbf{z}(i\Delta_n)])} (g_{i,n}(\boldsymbol{s}) - g_{i-1,n}(\boldsymbol{s}))W(d\mathbf{s}),$$

where $\boldsymbol{\varepsilon}_n^{(j)} = (\varepsilon_n^{(j)}, \varepsilon_n^{(j)})$, $0 < \varepsilon_n^{(1)} < \varepsilon_n^{(2)} < \ldots < \varepsilon_n^{(m)} < \varepsilon_n^{(m+1)} = \infty$, and the sets $\mathcal{B}_i(\boldsymbol{\varepsilon}^{(j)})$ have been introduced in (IV.4.9). Due to Lemma IV.4.2 we deduce the inequalities

$$\mathbb{E}[|R_i^n(1)|^2] \lesssim \Delta_n^{2+2\alpha+2\gamma}, \qquad \mathbb{E}[|R_i^n(2)|^2] \lesssim \Delta_n^2(\varepsilon_n^{(1)})^{2\alpha+2\gamma}, \qquad \mathbb{E}[|\widetilde{R}_i^n(m)|^2] \lesssim \Delta_n^2(\varepsilon_n^{(m)})^{2\alpha},$$

$$\mathbb{E}[|\widetilde{R}_i^n(j)|^2] \lesssim \Delta_n^2(\varepsilon_n^{(j+1)})^{2\gamma}(\varepsilon_n^{(j)})^{2\alpha}, \qquad j = 1, \ldots, m-1.$$

In view of (IV.4.29), the decomposition (IV.4.30), [12, Lemma] and the arguments in [12, pp. 1191-1192], we conclude that there exist an $m \geq 1$ and sequences $\varepsilon_n^{(j)} = \Delta_n^{a_j}$ with $1 > a_1 > \ldots > a_m > 0$ such that

$$\widetilde{A}(n) \xrightarrow{\text{u.c.p.}} 0. \qquad \text{(IV.4.31)}$$

In view of (IV.4.26)-(IV.4.31), the proof of Theorem IV.3.5 is complete.

IV.5 Appendix

IV.5.1 Proof of Lemma IV.4.1

We show first two auxiliary lemmata. The proof of the first lemma is omitted as it is straightforward.

Lemma IV.5.1. *Under Assumption (A2) and $\alpha < 0$, for all $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}_+^2$ and $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}_+^2$ we have that*

$$|g(\boldsymbol{x}) - g(\boldsymbol{y})| \lesssim_f \|(x_1 \wedge y_1, x_2 \wedge y_2)\|^{\alpha-1}\|\boldsymbol{x} - \boldsymbol{y}\|.$$

Lemma IV.5.2. *For any $a > 0$ and $b > 0$, the following asymptotic statements hold:*

(i) *If* $\beta \in (-1,0)$ *and* $\gamma < 0$, *then*

$$\int_{(0,a]^2} \|\boldsymbol{x}\|^\beta \|\boldsymbol{x} + h\mathbf{1}\|^\gamma d\boldsymbol{x} \sim \sqrt{2} h^\gamma \int_{(0,a]^2} \|\boldsymbol{x}\|^\beta d\boldsymbol{x}, \quad h \to \infty.$$

(ii) *If* $\beta \in (-1,0)$ *and* $\beta + \gamma < -1$, *then*

$$\int_{(0,a]\times(b,\infty)} \|\boldsymbol{x}\|^\beta \|\boldsymbol{x} + h\mathbf{1}\|^\gamma d\boldsymbol{x} = \mathcal{O}(h^{\beta+\gamma+1}), \quad h \to \infty.$$

(iii) *If* $\beta \in (-2,0)$ *and* $\beta + \gamma < -2$, *then*

$$\int_{(b,\infty)^2} \|\boldsymbol{x}\|^\beta \|\boldsymbol{x} + h\mathbf{1}\|^\gamma d\boldsymbol{x} = \mathcal{O}(h^{\beta+\gamma+2}), \quad h \to \infty.$$

Proof. Throughout the proof, we consider $h > 0$.

(i) We have

$$\int_{(0,a]^2} \|\boldsymbol{x}\|^\beta \|\boldsymbol{x} + h\mathbf{1}\|^\gamma d\boldsymbol{x} = h^\gamma \int_{(0,a]^2} \|\boldsymbol{x}\|^\beta \|h^{-1}\boldsymbol{x} + \mathbf{1}\|^\gamma d\boldsymbol{x},$$

where $\|h^{-1}\boldsymbol{x} + \mathbf{1}\|^\gamma \leq 2^{\frac{\gamma}{2}}$ for any $h > 0$ and $\boldsymbol{x} \in (0,a]^2$. By the dominated convergence theorem,

$$h^{-\gamma} \int_{(0,a]^2} \|\boldsymbol{x}\|^\beta \|\boldsymbol{x} + h\mathbf{1}\|^\gamma d\boldsymbol{x} = \int_{(0,a]^2} \|\boldsymbol{x}\|^\beta \|h^{-1}\boldsymbol{x} + \mathbf{1}\|^\gamma d\boldsymbol{x} \xrightarrow[h\to\infty]{} \sqrt{2} \int_{(0,a]^2} \|\boldsymbol{x}\|^\beta d\boldsymbol{x}.$$

(ii) Since $\|(x_1,x_2)\| \geq x_2$ for any $(x_1,x_2) \in [0,\infty)^2$, by Tonelli's theorem we arrive at the bound

$$\begin{aligned}\int_{(0,a]\times(b,\infty)} \|\boldsymbol{x}\|^\beta \|\boldsymbol{x} + h\mathbf{1}\|^\gamma d\boldsymbol{x} &= \int_0^a \int_b^\infty \|(x_1,x_2)\|^\beta \|(x_1+h, x_2+h)\|^\gamma dx_1 dx_2 \\ &\leq a \int_b^\infty x^\beta (x+h)^\gamma dx,\end{aligned}$$

since under the present assumptions, both $\beta < 0$ and $\gamma < 0$. By substituting $u = x/h$, we get

$$\int_b^\infty x^\beta (x+h)^\gamma dx = h^\gamma \int_b^\infty x^\beta \Big(\frac{x}{h} + 1\Big)^\gamma dx = h^{\beta+\gamma+1} \int_{b/h}^\infty u^\beta (u+1)^\gamma du, \quad \text{(IV.5.1)}$$

where

$$\int_{b/h}^\infty u^\beta (u+1)^\gamma du \xrightarrow[h\to\infty]{} \int_0^\infty u^\beta (u+1)^\gamma du < \infty, \quad \text{(IV.5.2)}$$

since $\beta \in (-1,0)$ and $\beta + \gamma < -1$. Hence,

$$\limsup_{h\to\infty} h^{-(\beta+\gamma+1)} \int_{(0,a]\times(b,\infty)} \|\boldsymbol{x}\|^\beta \|\boldsymbol{x} + h\mathbf{1}\|^\gamma d\boldsymbol{x} \leq a \int_0^\infty u^\beta (u+1)^\gamma du < \infty.$$

(iii) Note first that we have again $\beta < 0$ and $\gamma < 0$. Due to the elementary inequality $\|(x_1, x_2)\| \geq \sqrt{x_1 x_2}$, valid for any $(x_1, x_2) \in [0, \infty)^2$, and Tonelli's theorem, we find that

$$\int_{(b,\infty)^2} \|\boldsymbol{x}\|^{\beta} \|\boldsymbol{x} + h\mathbf{1}\|^{\gamma} d\boldsymbol{x} \leq \left(\int_b^{\infty} x^{\beta/2} (x + h)^{\gamma/2} dx \right)^2,$$

where $\int_b^{\infty} x^{\beta/2}(x+h)^{\gamma/2}dx < \infty$ since $\beta/2 + \gamma/2 = (\beta+\gamma)/2 < -1$. Note that, additionally, $\beta/2 \in (-1, 0)$, so we can reuse (IV.5.1) and (IV.5.2), with obvious changes, to deduce that

$$\int_b^{\infty} x^{\beta/2}(x+h)^{\gamma/2} dx \sim h^{\beta/2+\gamma/2+1} \int_0^{\infty} x^{\beta/2}(x+1)^{\gamma/2} dx, \quad h \to \infty.$$

Thus,

$$\limsup_{h\to\infty} h^{-(\beta+\gamma+2)} \int_{(b,\infty)^2} \|\boldsymbol{x}\|^{\beta} \|\boldsymbol{x} + h\mathbf{1}\|^{\gamma} d\boldsymbol{x} \leq \left(\int_0^{\infty} x^{\beta/2}(x+1)^{\gamma/2} dx \right)^2 < \infty,$$

which concludes the proof. □

Proof of Lemma IV.4.1. Suppose that $i \geq 0$ and $\lfloor T/\Delta_n \rfloor \geq j \geq i+1$. Since W is a Gaussian white noise process, we can then express the covariance $\gamma_n(i,j)$ as

$$\begin{aligned}
\gamma_n(i,j) &= \int_{\mathcal{A}_{i,n}} g_{i,n}(\boldsymbol{x})\big(g_{j,n}(\boldsymbol{x}) - g_{j-1,n}(\boldsymbol{x})\big) d\boldsymbol{x} \\
&\quad + \int_{\mathcal{B}_{i,n}} g_{i,n}(\boldsymbol{x})\big(g_{j,n}(\boldsymbol{x}) - g_{j-1,n}(\boldsymbol{x})\big) d\boldsymbol{x} \\
&\quad + \int_{\mathcal{B}'_{i,n}} g_{i,n}(\boldsymbol{x})\big(g_{j,n}(\boldsymbol{x}) - g_{j-1,n}(\boldsymbol{x})\big) d\boldsymbol{x} \\
&\quad + \int_{\mathcal{A}_{i-1,n}} \big(g_{i,n}(\boldsymbol{x}) - g_{i-1,n}(\boldsymbol{x})\big)\big(g_{j,n}(\boldsymbol{x}) - g_{j-1,n}(\boldsymbol{x})\big) d\boldsymbol{x} \\
&\quad + \int_{\mathcal{B}_{i-1,n}} \big(g_{i,n}(\boldsymbol{x}) - g_{i-1,n}(\boldsymbol{x})\big)\big(g_{j,n}(\boldsymbol{x}) - g_{j-1,n}(\boldsymbol{x})\big) d\boldsymbol{x} \\
&\quad + \int_{\mathcal{B}'_{i-1,n}} \big(g_{i,n}(\boldsymbol{x}) - g_{i-1,n}(\boldsymbol{x})\big)\big(g_{j,n}(\boldsymbol{x}) - g_{j-1,n}(\boldsymbol{x})\big) d\boldsymbol{x} \\
&\quad + \int_{\mathcal{C}_{i-2,n}} \big(g_{i,n}(\boldsymbol{x}) - g_{i-1,n}(\boldsymbol{x})\big)\big(g_{j,n}(\boldsymbol{x}) - g_{j-1,n}(\boldsymbol{x})\big) d\boldsymbol{x} \\
&=: a_n(i,j) + b_n(i,j) + b'_n(i,j) + \tilde{a}_n(i,j) + \tilde{b}_n(i,j) + \tilde{b}'_n(i,j) + c_n(i,j).
\end{aligned}$$

Note that, under Assumption (A2), we have

$$|g_{i,n}(\boldsymbol{x})| \lesssim_f \|\boldsymbol{z}(i\Delta_n) - \boldsymbol{x}\|^{\alpha}, \qquad \boldsymbol{x} \in \mathcal{C}_{i,n}, \tag{IV.5.3}$$

$$|g_{i,n}(\boldsymbol{x}) - g_{i-1,n}(\boldsymbol{x})| \lesssim_f \big\|\boldsymbol{z}\big((i-1)\Delta_n\big) - \boldsymbol{x}\big\|^{\alpha}, \qquad \boldsymbol{x} \in \mathcal{C}_{i-1,n}. \tag{IV.5.4}$$

Moreover, by Lemma IV.5.1,

$$|g_{i,n}(\boldsymbol{x}) - g_{i-1,n}(\boldsymbol{x})| \lesssim_{\alpha,f,\boldsymbol{z}} \Delta_n \big\|\boldsymbol{z}\big((i-1)\Delta_n\big) - \boldsymbol{x}\big\|^{\alpha-1}, \quad \boldsymbol{x} \in \mathcal{C}_{i-1,n}. \tag{IV.5.5}$$

Applying (IV.5.3) and (IV.5.5) to $a_n(i,j)$ we get

$$
\begin{aligned}
|a_n(i,j)| &\lesssim_{\alpha,f,\boldsymbol{z}} \Delta_n \int_{\mathcal{A}_{i,n}} \|\boldsymbol{z}(i\Delta_n)-\boldsymbol{x}\|^{\alpha} \big\|\boldsymbol{z}\big((j-1)\Delta_n\big)-\boldsymbol{x}\big\|^{\alpha-1} d\boldsymbol{x} \\
&\overset{(a)}{=} \Delta_n \int\limits_{\substack{(0,z_1(i\Delta_n)-z_1((i-1)\Delta_n)] \\ \times(0,z_2(i\Delta_n)-z_2((i-1)\Delta_n)]}} \|\boldsymbol{y}\|^{\alpha} \big\|\boldsymbol{z}\big((j-1)\Delta_n\big)-\boldsymbol{z}(i\Delta_n)+\boldsymbol{y}\big\|^{\alpha-1} d\boldsymbol{y} \\
&\overset{(b)}{\leq} \Delta_n \int\limits_{(0,\Delta_n \overline{z}'_T]^2} \|\boldsymbol{y}\|^{\alpha} \big\|\Delta_n \underline{z}'_T (j-i-1)\mathbf{1}+\boldsymbol{y}\big\|^{\alpha-1} d\boldsymbol{y} \\
&\overset{(c)}{=} \Delta_n^{2\alpha+2} \underbrace{\int\limits_{(0,\overline{z}'_T]^2} \|\tilde{\boldsymbol{y}}\|^{\alpha} \big\|\underline{z}'_T (j-i-1)\mathbf{1}+\tilde{\boldsymbol{y}}\big\|^{\alpha-1} d\tilde{\boldsymbol{y}},}_{=:\overline{a}(j-i-1)}
\end{aligned}
\tag{IV.5.6}
$$

where in step (a) we substitute $\boldsymbol{y}=\boldsymbol{z}(i\Delta_n)-\boldsymbol{x}$, in (b) we use (IV.4.4), and finally in (c) we substitute $\tilde{\boldsymbol{y}}=\Delta_n^{-1}\boldsymbol{y}$. Additionally, by applying (IV.5.4) and (IV.5.5) to $\tilde{a}_n(i,j)$ and using estimates analogous to (IV.5.6), we can show that $|\tilde{a}_n(i,j)| \lesssim_{\alpha,f,\boldsymbol{z}} \overline{a}(j-i)$.

Next, we apply (IV.5.3) and (IV.5.5) to $b_n(i,j)$, which enables us to deduce, adapting (IV.5.6), that

$$
\begin{aligned}
|b_n(i,j)| &\lesssim_{\alpha,f,\boldsymbol{z}} \Delta_n \int_{\mathcal{B}_{i,n}} \|\boldsymbol{z}(i\Delta_n)-\boldsymbol{x}\|^{\alpha} \big\|\boldsymbol{z}\big((j-1)\Delta_n\big)-\boldsymbol{x}\big\|^{\alpha-1} d\boldsymbol{x} \\
&= \Delta_n \int\limits_{\substack{(z_1(i\Delta_n)-z_1((i-1)\Delta_n),\infty) \\ \times(0,z_2(i\Delta_n)-z_2((i-1)\Delta_n)]}} \|\boldsymbol{y}\|^{\alpha} \big\|\boldsymbol{z}\big((j-1)\Delta_n\big)-\boldsymbol{z}(i\Delta_n)+\boldsymbol{y}\big\|^{\alpha-1} d\boldsymbol{y} \\
&\leq \Delta_n \int\limits_{(\Delta_n \underline{z}'_T,\infty)\times(0,\Delta_n \overline{z}'_T]} \|\boldsymbol{y}\|^{\alpha} \big\|\Delta_n \underline{z}'_T (j-i-1)\mathbf{1}+\boldsymbol{y}\big\|^{\alpha-1} d\boldsymbol{y} \\
&= \Delta_n^{2\alpha+2} \underbrace{\int\limits_{(\underline{z}'_T,\infty)\times(0,\overline{z}'_T]} \|\tilde{\boldsymbol{y}}\|^{\alpha} \big\|\underline{z}'_T (j-i-1)\mathbf{1}+\tilde{\boldsymbol{y}}\big\|^{\alpha-1} d\tilde{\boldsymbol{y}}\,.}_{=:\overline{b}(j-i-1)}
\end{aligned}
\tag{IV.5.7}
$$

Swapping the components, we also find that $|b'_n(i,j)| \lesssim_{\alpha,f,\boldsymbol{z}} \overline{b}(j-i-1)$. Further, using (IV.5.4) and (IV.5.5), and mimicking (IV.5.7) we can prove the estimate $|\tilde{b}_n(i,j)|+|\tilde{b}'_n(i,j)| \leq \overline{b}(j-i)$.

Finally, by (IV.5.5), we get

$$
\begin{aligned}
|c_n(i,j)| &\lesssim_{\alpha,f,\boldsymbol{z}} \Delta_n^2 \int_{\mathcal{C}_{i-1,n}} \big\|\boldsymbol{z}\big((i-1)\Delta_n\big) - \boldsymbol{x}\big\|^{\alpha-1} \big\|\boldsymbol{z}\big((j-1)\Delta_n\big) - \boldsymbol{x}\big\|^{\alpha-1} d\boldsymbol{x} \\
&= \Delta_n^2 \int\limits_{\substack{(z_1((i-1)\Delta_n)-z_1((i-2)\Delta_n),\infty) \\ \times(z_2((i-1)\Delta_n)-z_2((i-2)\Delta_n),\infty)}} \|\boldsymbol{y}\|^{\alpha-1} \big\|\boldsymbol{z}\big((j-1)\Delta_n\big) - \boldsymbol{z}\big((i-1)\Delta_n\big) + \boldsymbol{y}\big\|^{\alpha-1} d\boldsymbol{y} \\
&\leq \Delta_n^2 \int\limits_{(\Delta_n \underline{z}_T',\infty)^2} \|\boldsymbol{y}\|^{\alpha-1} \big\|\Delta_n \underline{z}_T'(j-i)\mathbf{1} + \boldsymbol{y}\big\|^{\alpha-1} d\boldsymbol{y} \\
&= \Delta_n^{2\alpha+2} \underbrace{\int\limits_{(\underline{z}_T',\infty)^2} \|\tilde{\boldsymbol{y}}\|^{\alpha-1} \big\|\underline{z}_T'(j-i)\mathbf{1} + \tilde{\boldsymbol{y}}\big\|^{\alpha-1} d\tilde{\boldsymbol{y}}}_{=:\overline{c}(j-i)},
\end{aligned}
$$

where the steps are analogous to those in (IV.5.6) and (IV.5.7).
To summarize, we have shown that

$$
\begin{aligned}
|\gamma_n(i,j)| &\lesssim_{\alpha,f,\boldsymbol{z}} \Delta_n^{2\alpha+2}\big(\overline{a}(j-i-1) + \overline{a}(j-i) + \overline{b}(j-i-1) + \overline{b}(j-i) + \overline{c}(j-i)\big) \\
&\lesssim \Delta_n^{2\alpha+2}\big(\overline{a}(j-i-1) + \overline{b}(j-i-1) + \overline{c}(j-i)\big),
\end{aligned}
$$

which motivates us to define

$$\overline{\rho}(k) := \overline{a}(k-1) + \overline{b}(k-1) + \overline{c}(k), \quad k \geq 2.$$

Additionally, we may set

$$\overline{\rho}(k) := \sup_{n\in\mathbb{N}} \sup_{\substack{0\leq i,j\leq [T/\Delta_n] \\ |i-j|=k}} \Delta_n^{-(2\alpha+2)} |\gamma_n(i,j)|, \quad k = 0, 1,$$

since $\overline{\rho}(0) < \infty$ and $\overline{\rho}(1) < \infty$ then by Lemma IV.4.3. It remains to note that

$$
\begin{aligned}
\overline{a}(k) &= \mathcal{O}(k^{\alpha-1}), && \text{by Lemma IV.5.2(i) with } \beta = \alpha \text{ and } \gamma = \alpha - 1, \\
\overline{b}(k) &= \mathcal{O}(k^{2\alpha}), && \text{by Lemma IV.5.2(ii) with } \beta = \alpha \text{ and } \gamma = \alpha - 1, \\
\overline{c}(k) &= \mathcal{O}(k^{2\alpha}), && \text{by Lemma IV.5.2(iii) with } \beta = \alpha - 1 \text{ and } \gamma = \alpha - 1.
\end{aligned}
$$

Thus, $\sum_{k=0}^{\infty} \overline{\rho}(k)^2 < \infty$ as long as $\alpha < -1/4$, which concludes the proof. □

IV.5.2 Proof of Lemma IV.4.3

Throughout the proof we assume, without loss of generality, that $\Delta \in (0,1]$. By the mean value theorem, for any $t \in [0,T]$ and $\Delta \in (0,1]$ there exist $\xi_{t,\Delta}^1$, $\xi_{t,\Delta}^2 \in [t, t+\Delta]$ such that

$$z_1(t+\Delta) - z_1(t) = \Delta z_1'(\xi_{t,\Delta}^1) \quad \text{and} \quad z_2(t+\Delta) - z_2(t) = \Delta z_2'(\xi_{t,\Delta}^2). \tag{IV.5.8}$$

For the sake of simpler notation, we write $\boldsymbol{z}_\Delta'(t) := \big(z_1'(\xi_{t,\Delta}^1), z_2'(\xi_{t,\Delta}^2)\big)$. Since $\boldsymbol{z}$ is C^2, we have

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_\Delta'(t) - \boldsymbol{z}'(t)\| \lesssim_{\boldsymbol{z},T} \Delta. \tag{IV.5.9}$$

It is also useful to note that, by Assumption (A1), for any $\Delta \in (0,1]$, $t \in [0,T]$, and $\boldsymbol{x} \in \mathbb{R}_+^2$,

$$\|\boldsymbol{z}'_\Delta(t) \circ \boldsymbol{x}\| \gtrsim_{T,\boldsymbol{z}} \|\boldsymbol{x}\|, \quad \|\boldsymbol{z}'(t) \circ \boldsymbol{x}\| \gtrsim_{T,\boldsymbol{z}} \|\boldsymbol{x}\|. \tag{IV.5.10}$$

(i) We can write

$$v_{t,t+\Delta} = v_{t,t+\Delta}(1) + v_{t,t+\Delta}(2)$$

where

$$v_{t,t+\Delta}(1) = \int\limits_{(-\boldsymbol{\infty},\boldsymbol{z}(t+\Delta))\setminus(-\boldsymbol{\infty},\boldsymbol{z}(t))} g(\boldsymbol{z}(t+\Delta)-\boldsymbol{x})^2 d\boldsymbol{x}$$

$$v_{t,t+\Delta}(2) = \int\limits_{(-\boldsymbol{\infty},\boldsymbol{z}(t)]} \big(g(\boldsymbol{z}(t+\Delta)-\boldsymbol{x}) - g(\boldsymbol{z}(t)-\boldsymbol{x})\big)^2 d\boldsymbol{x}.$$

Concerning the first term, we have that

$$\begin{aligned} v_{t,t+\Delta}(1) &= \Delta^2 z'_1(\xi^1_{t,\Delta}) z'_2(\xi^2_{t,\Delta}) \int_{\mathbb{R}_+^2\setminus(\boldsymbol{1},\boldsymbol{\infty})} g(\Delta \boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y})^2 d\boldsymbol{y} \\ &= \Delta^{2\alpha+2} z'_1(\xi^1_{t,\Delta}) z'_2(\xi^2_{t,\Delta}) \int_{\mathbb{R}_+^2\setminus(\boldsymbol{1},\boldsymbol{\infty})} f(\Delta \boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y})^2 \|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^{2\alpha} d\boldsymbol{y}, \end{aligned}$$

where we have substituted $\boldsymbol{y} = \big(\frac{z_1(t+\Delta)-x_1}{z_1(t+\Delta)-z_1(t)}, \frac{z_2(t+\Delta)-x_2}{z_2(t+\Delta)-z_2(t)}\big)$ and used (IV.5.8). It follows from (IV.5.9) that the factor $z'_1(\xi^1_{t,\Delta}) z'_2(\xi^2_{t,\Delta})$ converges to $z'_1(t)z'_2(t)$ as $\Delta \to 0$ uniformly in $t \in [0,T]$. Thus it is enough to show that

$$\lim_{\Delta\to 0} \sup_{t\in[0,T]} \int_{\mathbb{R}_+^2\setminus(1,\infty)^2} |f(\Delta \boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y})^2 \|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^{2\alpha} - f(\boldsymbol{0})^2 \|\boldsymbol{z}'(t)\circ\boldsymbol{y}\|^{2\alpha}| d\boldsymbol{y} = 0. \tag{IV.5.11}$$

The integrand in (IV.5.11), for any $t \in [0,T]$, $\Delta \in (0,1)$, and $\boldsymbol{y} \in \mathbb{R}_+^2$, can be estimated as

$$\begin{aligned} &|f(\Delta \boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y})^2 \|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^{2\alpha} - f(\boldsymbol{0})^2 \|\boldsymbol{z}'(t)\circ\boldsymbol{y}\|^{2\alpha}| \\ &\qquad \le |f(\Delta \boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y})^2 - f(\boldsymbol{0})^2| \|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^{2\alpha} \\ &\qquad\quad + f(\boldsymbol{0})^2 |\|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^{2\alpha} - \|\boldsymbol{z}'(t)\circ\boldsymbol{y}\|^{2\alpha}|. \end{aligned} \tag{IV.5.12}$$

The first term on the right-hand side of (IV.5.12) satisfies

$$|f(\Delta \boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y})^2 - f(\boldsymbol{0})^2| \|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^{2\alpha} \lesssim_f \Delta \|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^{2\alpha+1} \lesssim_{T,\boldsymbol{z}} \Delta \|\boldsymbol{y}\|^{2\alpha+1} \tag{IV.5.13}$$

by the mean value theorem, boundedness of the gradient of f (guaranteed by Assumption (A2)), and (IV.5.10), whilst the second term can be bounded as

$$\begin{aligned} &|\|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^{2\alpha} - \|\boldsymbol{z}'(t)\circ\boldsymbol{y}\|^{2\alpha}| \\ &\qquad \le 2|\alpha| (\|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\| \wedge \|\boldsymbol{z}'(t)\circ\boldsymbol{y}\|)^{2\alpha-1} \|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y} - \boldsymbol{z}'(t)\circ\boldsymbol{y}\| \\ &\qquad \lesssim_{\alpha,T,\boldsymbol{z}} \Delta \|\boldsymbol{y}\|^{2\alpha} \end{aligned} \tag{IV.5.14}$$

by (IV.5.9), (IV.5.10) and the inequality $|x^\gamma - y^\gamma| \le |\gamma|(x \wedge y)^{\gamma-1}|x-y|$, valid for $x>0$, $y>0$, and $\gamma<1$. It is useful to note that the first term on the right-hand side of (IV.5.12) satisfies also an alternative bound

$$|f(\Delta \boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y})^2 - f(\boldsymbol{0})^2| \|\boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y}\|^{2\alpha} \le \big(f(\Delta \boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y})^2 + f(\boldsymbol{0})^2\big) \|\boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y}\|^{2\alpha} \lesssim_{\alpha,f,T,\boldsymbol{z}} \|\boldsymbol{y}\|^{2\alpha} \tag{IV.5.15}$$

by Assumption (A2) and (IV.5.10).
Having established the necessary estimates, we will proceed to splitting the integral in (IV.5.11) into three parts via

$$\mathbb{R}_+^2 \setminus (1,\infty)^2 \subseteq (0,1]^2 \cup (1,\infty)\times(0,1] \cup (0,1]\times(1,\infty)$$

and denote the respective integrals by I_1, I_2 and I_3, respectively.
In view of (IV.5.13) and (IV.5.14), the integration by polar coordinates leads to

$$\begin{aligned}\sup_{t\in[0,T]} I_1 &\lesssim_{\alpha,T,\boldsymbol{z}} \Delta \int_{(0,1]^2} \Big(\|\boldsymbol{y}\|^{2\alpha+1} + \|\boldsymbol{y}\|^{2\alpha}\Big) d\boldsymbol{y} \\ &\lesssim_{\alpha,T,\boldsymbol{z}} \Delta \int_0^{\pi/2}\int_0^2 \Big(r^{2\alpha+2} + r^{2\alpha+1}\Big) dr d\theta = \mathcal{O}(\Delta).\end{aligned}$$

To deal with I_2, fix $M>1$. In view of (IV.5.13), (IV.5.14) and (IV.5.15), we obtain that

$$\begin{aligned}\limsup_{\Delta\to 0}\sup_{t\in[0,T]} I_2 \lesssim_{\alpha,f,T,\boldsymbol{z}} &\limsup_{\Delta\to 0} \Delta \int_{(1,M]\times(0,1]} \|\boldsymbol{y}\|^{2\alpha+1} d\boldsymbol{y} \\ &+ \limsup_{\Delta\to 0} \Delta \int_{(1,M]\times(0,1]} \|\boldsymbol{y}\|^{2\alpha} d\boldsymbol{y} + \limsup_{\Delta\to 0} \int_{(M,\infty)\times(0,1]} \|\boldsymbol{y}\|^{2\alpha} d\boldsymbol{y} \\ &= \int_{(M,\infty)\times(0,1]} \|\boldsymbol{y}\|^{2\alpha} d\boldsymbol{y} \\ &\le \int_{(M,\infty)\times(0,1]} y_1^{2\alpha} dy_1 dy_2 = \mathcal{O}(M^{2\alpha+1}).\end{aligned}$$

Since $M>1$ is arbitrary, it follows that

$$\limsup_{\Delta\to 0}\sup_{t\in[0,T]} I_2 = 0.$$

We deal with I_3 similarly and consequently finish the proof of (IV.5.11).
Next, we study the asymptotic behavior of the second term of $v_{t,t+\Delta}$. In particular, we have that

$$\begin{aligned}v_{t,t+\Delta}(2) =& \Delta^2 z_1'(\xi_{t,\Delta}^1) z_2'(\xi_{t,\Delta}^2) \int_{\mathbb{R}_+^2} \big(g(\Delta \boldsymbol{z}'_\Delta(t)\circ(\boldsymbol{y}+\boldsymbol{1})) - g(\Delta \boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y})\big)^2 d\boldsymbol{y} \\ =& \Delta^{2\alpha+2} z_1'(\xi_{t,\Delta}^1) z_2'(\xi_{t,\Delta}^2) \\ &\times \int_{\mathbb{R}_+^2} \big(f(\Delta \boldsymbol{z}'_\Delta(t)\circ(\boldsymbol{y}+\boldsymbol{1}))\|\boldsymbol{z}'_\Delta(t)\circ(\boldsymbol{y}+\boldsymbol{1})\|^{\alpha} - f(\Delta \boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y})\|\boldsymbol{z}'_\Delta(t)\circ \boldsymbol{y}\|^{\alpha}\big)^2 d\boldsymbol{y},\end{aligned}$$

where we substituted $\boldsymbol{y} = \left(\frac{z_1(t)-x_1}{z_1(t+\Delta)-z_1(t)}, \frac{z_2(t)-x_2}{z_2(t+\Delta)-z_2(t)}\right)$.
Arguing as in (IV.5.11), we are left to show that

$$\lim_{\Delta\to 0} \sup_{t\in[0,T]} \int_{\mathbb{R}^2_+} |A(y)| d\boldsymbol{y} = 0, \tag{IV.5.16}$$

where

$$\begin{aligned} A(y) =& \big(f(\Delta \boldsymbol{z}'_\Delta(t)\circ(\boldsymbol{y}+\mathbf{1}))\|\boldsymbol{z}'_\Delta(t)\circ(\boldsymbol{y}+\mathbf{1})\|^\alpha - f(\Delta\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y})\|\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y}\|^\alpha\big)^2 \\ &- f(\mathbf{0})^2\big(\|\boldsymbol{z}'(t)\circ(\boldsymbol{y}+\mathbf{1})\|^\alpha - \|\boldsymbol{z}'(t)\circ\boldsymbol{y}\|^\alpha\big)^2. \end{aligned}$$

Proceeding as in (IV.5.12), (IV.5.13) and (IV.5.14), we obtain that

$$\begin{aligned} |A(y)| &\lesssim_{\alpha,f,T,\boldsymbol{z}} (\|\boldsymbol{y}+\mathbf{1}\|^\alpha + \|\boldsymbol{y}\|^\alpha)\Delta(\|\boldsymbol{y}+\mathbf{1}\|^{\alpha+1} + \|\boldsymbol{y}\|^{\alpha+1}) \\ &\lesssim_{\alpha,f,T,\boldsymbol{z}} \Delta\|\boldsymbol{y}\|^\alpha\|\boldsymbol{y}+\mathbf{1}\|^{\alpha+1}. \end{aligned}$$

On the other hand, Lemma IV.5.1 and the inequality $|x^\gamma - y^\gamma| \le |\gamma|(x\wedge y)^{\gamma-1}|x-y|$ leads to an alternative bound:

$$\begin{aligned} |A(y)| \le& \Delta^{-2\alpha}\big(g(\Delta\boldsymbol{z}'_\Delta(t)\circ(\boldsymbol{y}+\mathbf{1})) - g(\Delta\boldsymbol{z}'_\Delta(t)\circ\boldsymbol{y})\big)^2 \\ &+ |f(\mathbf{0})^2|\big(\|\boldsymbol{z}'(t)\circ(\boldsymbol{y}+\mathbf{1})\|^\alpha - \|\boldsymbol{z}'(t)\circ\boldsymbol{y}\|^\alpha\big)^2 \\ \lesssim_{\alpha,f,T,\boldsymbol{z}}& \|\boldsymbol{y}\|^{2\alpha-2} \end{aligned}$$

Again, fix $M > 1$. In view of these two estimates, we get that

$$\begin{aligned} \lim_{\Delta\to 0}\sup_{t\in[0,T]}\int_{\mathbb{R}^2_+}|A(y)|d\boldsymbol{y} \lesssim_{\alpha,f,T,\boldsymbol{z}}& \limsup_{\Delta\to 0}\sup_{t\in[0,T]}\Delta\int_{(0,M]^2}\|\boldsymbol{y}\|^\alpha\|\boldsymbol{y}+\mathbf{1}\|^{\alpha+1}d\boldsymbol{y} \\ &+ \limsup_{\Delta\to 0}\sup_{t\in[0,T]}\int_{\mathbb{R}^2_+\setminus(0,M]^2}\|\boldsymbol{y}\|^{2\alpha-2}d\boldsymbol{y} \\ =& \int_{\mathbb{R}^2_+\setminus(0,M]^2}\|\boldsymbol{y}\|^{2\alpha-2}d\boldsymbol{y} = \mathcal{O}(M^{2\alpha}). \end{aligned}$$

Since M can be chosen arbitrarily large, we get (IV.5.16) and finish the proof of (i).

(ii) Since z_1', z_2' are in C^1 and hence Lipschitz continuous on compact intervals, we have

$$\sup_{t\in[0,T]}\big(|z_1'(t+\Delta) - z_1'(t)| + |z_2'(t+\Delta) - z_2'(t)|\big) \lesssim_{T,\boldsymbol{z}} \Delta.$$

Arguing as in (IV.5.14), we also obtain that

$$\begin{aligned} &\sup_{t\in[0,T]}\big|\|\boldsymbol{z}'(t+\Delta)\circ\boldsymbol{x}\|^{2\alpha} - \|\boldsymbol{z}'(t)\circ\boldsymbol{x}\|^{2\alpha}\big| \lesssim_{T,\boldsymbol{z}} \Delta\|\boldsymbol{x}\|^{2\alpha}, \\ &\sup_{t\in[0,T]}\big|\|\boldsymbol{z}'(t+\Delta)\circ(\boldsymbol{x}+\mathbf{1})\|^\alpha - \|\boldsymbol{z}'(t+\Delta)\circ\boldsymbol{x}\|^\alpha \\ &\qquad - \|\boldsymbol{z}'(t)\circ(\boldsymbol{x}+\mathbf{1})\|^\alpha + \|\boldsymbol{z}'(t)\circ\boldsymbol{x}\|^\alpha\big| \lesssim_{T,\boldsymbol{z}} \Delta\|\boldsymbol{x}+\mathbf{1}\|^\alpha + \Delta\|\boldsymbol{x}\|^\alpha, \\ &\sup_{t\in[0,T+1]}\big|\|\boldsymbol{z}'(t)\circ(\boldsymbol{x}+\mathbf{1})\|^\alpha - \|\boldsymbol{z}'(t)\circ\boldsymbol{x}\|^\alpha\big| \lesssim_{T,\boldsymbol{z}} \|\boldsymbol{x}\|^{\alpha-1}\vee\|\boldsymbol{x}\|^\alpha. \end{aligned}$$

Using these identities along with the boundedness of the integral

$$\int_{\mathbb{R}_+^2\setminus(1,\infty)^2} \|\boldsymbol{x}\|^{2\alpha} d\boldsymbol{x} + \int_{\mathbb{R}_+^2} (\|\boldsymbol{x}+\mathbf{1}\|^{\alpha} + \|\boldsymbol{x}\|^{\alpha})(\|\boldsymbol{x}\|^{\alpha-1} \vee \|\boldsymbol{x}\|^{\alpha}) d\boldsymbol{x} < \infty,$$

we deduce that

$$\sup_{t\in[0,T]} |\phi_{t+\Delta} - \phi_t| \lesssim_{\alpha,f,T,\boldsymbol{z}} \Delta. \tag{IV.5.17}$$

(iii) Since $f(\mathbf{0}) \neq 0$ and f is continuous, there exists $\varepsilon \in (0,1)$ such that

$$\inf_{\boldsymbol{x}\in B_{\mathbf{0}}(\varepsilon)} f(\boldsymbol{x})^2 > 0. \tag{IV.5.18}$$

In addition, $\sup_{n\geq 1, t\in[0,T]} \Delta_n \boldsymbol{z}'_{\Delta_n}(t) \leq C$. Combining these results with the fact that z_1', z_2' are bounded away from zero, we conclude that

$$\begin{aligned}
\inf_{n\in\mathbb{N},t\in[0,T]} \Delta_n^{-2\alpha-2} v_{t,t+\Delta_n} &\geq \inf_{n\in\mathbb{N},t\in[0,T]} \Delta_n^{-2\alpha-2} v_{t,t+\Delta_n}(1) \\
&\gtrsim_{T,\boldsymbol{z}} \inf_{n\in\mathbb{N},t\in[0,T]} \int_{\mathbb{R}_+^2\setminus(1,\infty)^2} f(\Delta_n \boldsymbol{z}'_{\Delta_n}(t) \circ \boldsymbol{y})^2 \|\boldsymbol{y}\|^{2\alpha} d\boldsymbol{y} \\
&\gtrsim_{T,\boldsymbol{z}} \inf_{\boldsymbol{x}\in B_{\mathbf{0}}(\varepsilon)} f(\boldsymbol{x})^2 \int_{B_{\mathbf{0}}(\varepsilon/C)} \|\boldsymbol{y}\|^{2\alpha} d\boldsymbol{y} > 0.
\end{aligned}$$

IV.5.3 Proof of Lemma IV.4.6

Introduce Gaussian random variables

$$G_{i,n}^1 := \int_{\mathcal{B}_{i,n}\cup\mathcal{B}'_{i,n}} g_{i,n}(\mathbf{s}) W(d\mathbf{s}),$$

$$G_{i,n}^2 := \int_{\mathcal{A}_{i,n}\cup\mathcal{C}_{i-1,n}} \{g_{i,n}(\mathbf{s}) - g_{i-1,n}(\mathbf{s})\} W(d\mathbf{s}).$$

Due to Lemma IV.4.2 we conclude that

$$\mathbb{E}\left[|G_{i,n}^1|^p\right] \lesssim_p \Delta_n^{p/2}, \qquad \mathbb{E}\left[|G_{i,n}^2|^p\right] \lesssim_p \Delta_n^{p(1+\alpha)}.$$

Following the arguments of the proof of Lemma IV.4.4, we deduce that

$$\mathbb{E}[|Y_{i,n}^1|^p + |Y'_{i,n}|^p] \lesssim_p \left(\Delta_n^{p/2} + \Delta_n^p \mathbb{E}\left[F_{\boldsymbol{z}((i-1)\Delta_n)}^{p/2}\right]\right)$$

and

$$\mathbb{E}[|Y_{i,n}^2|^p] \lesssim_p \left(\Delta_n^{p(1+\alpha)} + \Delta_n^p \mathbb{E}\left[F_{\boldsymbol{z}((i-1)\Delta_n)}^{p/2}\right]\right).$$

This completes the proof since F is bounded on compact sets.

IV.5.4 Proof of Lemma IV.4.7

The arguments follow along the same lines as the proof of Theorem IV.3.1(i). First of all, we note that it suffices to show the convergence

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[(Y^1_{i,n} - Y'_{i,n})^2\right] \to 0.$$

Observing the treatment of the term $R^n_i(1)$ in the proof of Theorem IV.3.1(i), we conclude that

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[(Y^1_{i,n} - Y'_{i,n})^2\right] \leq \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[r(\mathbf{z}((i-1)\Delta_n), C\Delta_n)\right] + o(1),$$

where $r(\mathbf{s},\boldsymbol{\eta}) = \sup\{(\sigma_{\mathbf{s}} - \sigma_{\mathbf{t}})^2 | \mathbf{t} \in [\mathbf{s}-\boldsymbol{\eta}, \mathbf{s}+\boldsymbol{\eta}]\}$. Due to continuity and boundedness of σ on compact sets, the proof is complete.

IV.5.5 Proof of Lemma IV.4.8

We introduce the notation

$$w_{t,t+\Delta_n}(1) := \Delta_n^{-1} \int_{-\infty}^{z_1(t)} \int_{z_2(t)}^{z_2(t+\Delta_n)} g^2(\boldsymbol{z}(t+\Delta_n) - \boldsymbol{s}) \sigma^2_{s_1, z_2(t)} d\boldsymbol{s},$$

$$w_{t,t+\Delta_n}(2) = \Delta_n^{-1} \int_{z_1(t)}^{z_1(t+\Delta_n)} \int_{-\infty}^{z_2(t)} g^2(\boldsymbol{z}(t+\Delta_n) - \boldsymbol{s}) \sigma^2_{z_1(t), s_2} d\boldsymbol{s}.$$

and

$$w_t(1) := z_2'(t) \int_0^\infty |x|^{2\alpha} f^2(x,0) \sigma^2_{z_1(t)-x, z_2(t)} dx,$$

$$w_t(2) := z_1'(t) \int_0^\infty |x|^{2\alpha} f^2(0,x) \sigma^2_{z_1(t), z_2(t)-x} dx.$$

Observe that $w_t^2 = w_t(1) + w_t(2)$. To show Lemma IV.4.8 it suffices to prove the convergence

$$\sup_{t\in[0,T]} |w_{t,t+\Delta_n}(j) - w_t(j)| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } \Delta_n \to 0, \tag{IV.5.19}$$

for $j = 1, 2$. We prove this convergence only for $j = 1$ as the case $j = 2$ is shown similarly. We start with the change of variable $\boldsymbol{y} = \boldsymbol{z}(t+\Delta_n) - \boldsymbol{s}$ to deduce the identity

$$w_{t,t+\Delta_n}(1) = \Delta_n^{-1} \int_{z_1(t+\Delta_n)-z_1(t)}^{\infty} \int_0^{z_2(t+\Delta_n)-z_2(t)} g^2(\boldsymbol{y}) \sigma^2_{z_1(t+\Delta_n)-y_1, z_2(t)} d\boldsymbol{y}.$$

Now, recalling the Assumption (A2) and observing that

$$\sup_{t\in[0,T]} |z_1(t+\Delta_n) - z_1(t)| \to 0 \qquad \text{and} \qquad |\Delta_n^{-1}(z_2(t+\Delta_n) - z_2(t)) - z_2'(t)| \to 0$$

as $\Delta_n \to 0$, we readily obtain the convergence in (IV.5.19) for $j = 1$. This completes the proof of Lemma IV.4.8.

Appendix A

Technical results

A.1 Walsh theory of stochastic integration

In this section we introduce the main concepts of the Walsh theory of stochastic integration for martingale measure. This theory of stochastic integration is used notably to define stochastic integrals in dimension ≥ 2. We refer to Section I.2.2 where such integrals are needed to define a 2-dimensional ambit field.

Most of the results presented below comes from the seminal paper of Walsh [149] and the comprehensive survey of Podolskij [129] on ambit field. We refer e.g. to [56, 57, 94] to explore further the topic. The Walsh theory of stochastic integration generalizes in some sense Itô's stochastic integration theory to dimension higher than 1.

We recall first Definition I.2.23 of a Lévy basis:

Definition A.1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a set and $\mathcal{E}$ a σ-field on $\mathcal{E}$. A **Lévy measure** L on $(E, \mathcal{E})$ is a collection of random variables $(L(B))_{B\in\mathcal{E}}$ such that

(i) $L(\emptyset) = 0$,

(ii) Let $A_1, ..., A_n \in \mathcal{E}$ be mutually disjoint sets. Then the random variables $L(A_1), ..., L(A_n)$ are mutually independent.

(iii) For all $A \in \mathcal{E}$ the random variable $L(A)$ is infinitely divisible.

We consider in the following Lévy basis on a bounded domain. We denote by $\mathcal{B}_b(\mathbb{R}^d)$ the Borel σ-field generated by bounded Borel sets of $\mathbb{R}^d$. Assume that we have a Lévy basis L on $[0, T] \times S$ with $S \in \mathcal{B}_b(\mathbb{R}^d)$. Let $A \subseteq S$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $0 < t \leq T$. We define

$$L_t(A) := L((0, t] \times A).$$

Assume that for all $A \in \mathcal{B}_b(\mathbb{R}^d)$, $L_t(A)$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We need then to define a right continuous filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mathcal{N}$ the $\mathbb{P}$-null sets of $\mathcal{F}$ and define

$$\mathcal{F}_t^0 := \sigma\{L_s(A) \ : \ A \subseteq S, \ A \in \mathcal{B}_b(\mathbb{R}^d), \ 0 < s \leq t\} \wedge \mathcal{N}.$$

We establish the right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ as

$$\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}_s^0 \quad \text{for all } t \geq 0.$$

We can now define the notion of orthogonal martingale measure:

Definition A.1.2. We consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The collection of random variables $(L_t(A))_{t\geq 0, A\subseteq S, A\in\mathcal{B}_b(\mathbb{R}^d)}$ is an **orthogonal $\mathcal{F}_t$-martingale measure** if each of the following conditions are satisfied:

(i) $L_0(A) = 0$, for all $A \subseteq S$, $A \in \mathcal{B}_b(\mathbb{R}^d)$.

(ii) $(L_t(A))_{t\geq 0}$ is a $\mathcal{F}_t$-martingale, for all $A \subseteq S$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, with zero mean.

(iii) $\forall t > 0$, $L_t(\cdot)$ is a σ-finite $L^2(\mathbb{P})$-valued signed measure.

(iv) Let $A, B \subseteq S$, $A, B \in \mathcal{B}_b(\mathbb{R}^d)$. If A and B are disjoint sets, that is $A \cap B = \emptyset$, then $(L_t(A))_{t\geq 0}$ and $(L_t(B))_{t\geq 0}$ are orthogonal martingales, i.e. the random variables $L_t(A)$ and $L_t(B)$ are independent for any $t \geq 0$.

Example A.1.3. The white noise on $\mathbb{R}^d$ can be defined as follows. Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ-field containing all Borel sets of $\mathbb{R}^d$. Let $\dot{W}$ be a process such that

$$\mathbb{E}\left[\dot{W}(A)\right] = 0, \text{ for all } A \in \mathcal{B}(\mathbb{R}^d)$$

and

$$\text{cov}(\dot{W}(A), \dot{W}(B)) = \lambda^d(A \cap B), \text{for all } A, B \in \mathcal{B}(\mathbb{R}^d),$$

where λ^d denotes the Lebesgue measure on $\mathbb{R}^d$. Then the process $\dot{W} := (\dot{W}(A))_{A\in\mathcal{B}(\mathbb{R}^d)}$ is a Gaussian process called the **white noise** process. One can show that the white noise process on $\mathbb{R}^d$ defines an orthogonal martingale measure $(W_t(A))_{t\geq 0, A\subseteq S, A\in\mathcal{B}_b(\mathbb{R}^{d-1})}$ on $\mathbb{R}^{d-1}$.

Following the usual construction of the Itô integral, we define the notion of **elementary functions**:

Definition A.1.4. Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a function. f is said to be **elementary** if

$$f(t, x, \omega) = X(\omega)\mathbb{1}_{(a,b]}(t)\mathbb{1}_A(x)$$

where X is bounded and $\mathcal{F}_a$-measurable and $A \in \mathcal{B}_b(\mathbb{R}^d)$. Linear combinations of elementary functions are called **simple** functions. We denote by $\mathcal{S}$ the space of elementary functions.

We then define the stochastic integral of the simple function f with respect to the martingale measure L by

$$\int_0^t \int_B f(s, x)L(ds, dx) := X\left(L_{t\wedge b}(A \cap B) - L_{t\wedge a}(A \cap B)\right)$$

where $B \in \mathcal{B}_b(\mathbb{R}^d)$, $A, B \subseteq S$ and $0 < t \leq T$.

To extend this result to all predictable processes we use the classical techniques: we would like to use a density argument to extend the integral for stochastic integrands in the class of elementary functions to the class of predictable processes, in this context that is predictable with respect to the σ-algebra generated by elementary functions.

Definition A.1.5. We denote by $\mathcal{P}$ the σ-algebra generated by all elementary functions. We call $\mathcal{P}$ the **predictable $\boldsymbol{\sigma}$-algebra.**

We introduce the covariance measure:

Definition A.1.6. Let L be an orthogonal martingale measure. The **covariance measure** associated to L is defined as

$$Q([0,t]\times A) = \langle L(A)\rangle_t, \quad A\in\mathcal{B}_b(\mathbb{R}^d),$$

where $\langle\cdot\rangle$ is the conditional quadratic variation, that is the non-decreasing process $\langle X, X\rangle$ such that, if X is a local martingale, $M^2 - \langle M, M\rangle$ is a local martingale. We associated to Q an L^2-norm:

$$\|f\|_Q^2 := \mathbb{E}\left[\int_{[0,T]\times S} f^2(t,x)Q(dt,dx)\right].$$

One can show that the space $(L^2(\Omega\times[0,T]\times S,\mathcal{P},Q),\|\cdot\|_Q)$ is a Hilbert space and that $\mathcal{S}$ is dense in $L^2(\Omega\times[0,T]\times S,\mathcal{P},Q)$. From this, we can approximate any random function $f\in L^2(\Omega\times[0,T]\times S,\mathcal{P},Q)$ by a sequence of functions $f_n\in\mathcal{S}$, such that

$$\|f-f_n\|_Q\to 0 \qquad \text{and}$$
$$\int_0^t\int_B f(s,x)L(ds,dx) := \lim_{n\to\infty}\int_0^t\int_B f_n(s,x)L(ds,dx) \quad \text{in } L^2(\Omega,\mathcal{F},\mathbb{P}).$$

We finish this section by citing two results, useful notably in the proof of the main result of Chapter IV.

Corollary A.1.7. *By construction, the Itô type isometry*

$$\mathbb{E}\left[\left|\int_0^T\int_S f(t,x)L(dt,dx)\right|^2\right] = \|f\|_Q^2$$

is satisfied. In particular, if L is a white noise process W, we obtain

$$\mathbb{E}\left[\left|\int_0^T\int_S f(t,x)W(dt,dx)\right|^2\right] = \mathbb{E}\left[\int_0^T\int_S f^2(t,x)dtdx\right].$$

We also have a Burkholder inequality (see [42]): for all $p\geq 2$ there exists a constant $c_p\in(0,\infty)$ such that for all $f\in L^2(\Omega\times[0,T]\times S,\mathcal{P},Q)$ and all $0<t\leq T$,

$$\mathbb{E}\left[\left|\int_0^t\int_B f(s,x)L(ds,dx)\right|^p\right] \leq c_p\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}^d} f^2(s,x)Q(ds,dx)\right)^{p/2}\right].$$

A.2 Stable convergence of processes

In this section we introduce the concept of stable convergence, a mode of convergence extremely useful notably when dealing with random limit. This section is based on the book [64], although the results presented below are classical and can be found in many different sources.

We start by giving some motivations on why we should define stable convergence. We consider the following simple example: assume that we are given a central limit theorem for some error process of the form

$$\sqrt{n}(X_n - X) \xrightarrow{d} \mathcal{N}(0, V) \tag{A.2.1}$$

where X_n and X are random variables and assume that V is a non-negative random variables. For statistical purpose, this central limit theorem is unusable as the distribution of the mixed normal $\mathcal{N}(0, V)$ can be difficultly tractable, or even untractable. We would like to renormalize the error process $\sqrt{n}(X_n - X)$ by an estimator V_n of the random variance V, using Slutsky's theorem, to obtain a statistically tractable limit, in that case a standard normal distribution.

However, to do so we need the joint convergence in distribution $(V_n, \sqrt{n}(X_n - X)) \xrightarrow{d} (V, \mathcal{N}(0, V))$ which is not guaranteed by the central limit theorem in (A.2.1). We need a notion of weak convergence stronger than the convergence in distribution to allow a stronger version of Slutsky's theorem.

Stable convergence is defined as follows:

Definition A.2.1. Let $(X_n)_{n\geq 1}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in a measurable Polish space $(E, \mathcal{E})$. The sequence $(X_n)_{n\geq 1}$ is said to converge stably in law towards a limit X defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of the original probability space, and we denote it $X_n \xrightarrow{d_{st}} X$, if and only if it holds that

$$\mathbb{E}[g(X_n)Y] \xrightarrow{n\to\infty} \overline{\mathbb{E}}[g(X)Y]$$

for any bounded continuous g and any bounded $\mathcal{F}$-measurable random variable Y.

This definition is general and includes in particular the case of stochastic process, for example when considering $E = D([0, T])$ the space of càdlàg functions, equipped with some Skorokhod topology. It is possible to define equivalently stable convergence by defining it as the convergence of conditional distribution with respect to the original σ-field $\mathcal{F}$. Note that contrary to the convergence in distribution and similarly to the convergence in probability, stable convergence is a characteristic of the random variable and not of its distribution.

Indeed, if we have the convergence in distribution $X_n \xrightarrow{d} X$, the probability space where X_n and X are defined does not matter and can be different, in contrary to the stable convergence where the limiting random variable has to be defined on an extension of the original space.

We give next a lemma exhibiting the different relations between different modes of convergence:

Lemma A.2.2. *Let* $(X_n)_{n\geq 1}$ *be a sequence of random variables defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. It holds that*

(i) The stable convergence $X_n \xrightarrow{d_{st}} X$ *implies the convergence in distribution* $X_n \xrightarrow{d} X$*.*

(ii) The convergence in probability $X_n \xrightarrow{\mathbb{P}} X$ *implies the stable convergence* $X_n \xrightarrow{d_{st}} X$*.*

(iii) If $X_n \xrightarrow{d_{st}} X$ *and* X *is* $\mathcal{F}$*-measurable, then* $X_n \xrightarrow{\mathbb{P}} X$*.*

The next proposition gives us criteria to prove the stable convergence $X_n \xrightarrow{d_{st}} X$:

Proposition A.2.3. *Let* $(X_n)_{n\geq 1}$ *be a sequence of random variables. Then the following properties are equivalent:*

(i) $X_n \xrightarrow{d_{st}} X$*.*

(ii) $(X_n, Y) \xrightarrow{d} (X, Y)$ *for all* $\mathcal{F}$*-measurable random variables* Y*.*

(iii) $(X_n, Y) \xrightarrow{d_{st}} (X, Y)$ *for all* $\mathcal{F}$*-measurable random variables* Y*.*

Condition (ii) is particularly useful in practice to prove stable central limit theorem and is one of the key property used in Chapter III to prove the stable convergence of the error process of interest.

We finally mention the following stable version of Slutsky's theorem:

Theorem A.2.4. *Let* $(X_n)_{n\geq 1}$*,* $(V_n)_{n\geq 1}$ *be two sequences of random variables, defined on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. We assume that*

$$X_n \xrightarrow{d_{st}} X \qquad \text{and} \qquad V_n \xrightarrow{\mathbb{P}} V.$$

Then

$$(X_n, V_n) \xrightarrow{d_{st}} (X, V).$$

A.3 Some elements of Malliavin Calculus

This subsection is an introduction to Malliavin calculus. Originated by Malliavin in 1976 in his seminal paper [105], Malliavin calculus was defined first to get a better understanding on the existence of densities for the distribution of random variables and in particular the existence of densities for stochastic partial differential equations (see e.g. [40] or recently [60]). It allows to define a notion of stochastic differentiation originally on Wiener spaces but it was extended later to Poisson spaces or Wiener-Poisson spaces [75]. We refer to [27, 106, 117, 118] for a comprehensive introduction on Malliavin calculus.

In 2005 Nualart and Peccati in [119] proved the so-called fourth moment theorem: a central limit theorem for sequence of iterated Wiener-Itô integrals that laid out the premises of the Malliavin-Stein method, used to obtain quantitative central limit theorems. We refer to e.g. [115, 116, 122].

We provide the main tools to prove the step (ii), (iii) and (iv) of the methodology presented in Section I.3.3. Most of these results can be find in [117, Chapter 1]. Let H be a real separable Hilbert space equipped with a inner product $\langle \cdot, \cdot \rangle_H$ that induces an associated norm denoted $\|\cdot\|_H$. We start with the definition of an isonormal Gaussian process:

Definition A.3.1. Let $B = \{B(h),\ h \in H\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that B is a centered Gaussian family of random variables. Then B is called an **isonormal Gaussian process** (on H) when

$$\mathbb{E}\left[B(h)B(g)\right] = \langle h, g\rangle_H, \quad \forall h, g \in H.$$

We define the **Hermite polynomials**:

$$\begin{cases} H_n(x) &= (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2/2}\right), \quad n \geq 1, \\ H_0(x) &= 1. \end{cases} \tag{A.3.1}$$

One can show that this polynomial family is orthogonal with respect to the Gaussian distribution, namely:

$$\mathbb{E}[H_n(X)H_m(Y)] = \mathbb{1}_{\{n=m\}} n! \left(\mathbb{E}[XY]\right)^n, \quad \text{for } X, Y \sim \mathcal{N}(0,1) \text{ jointly Gaussian.} \tag{A.3.2}$$

We denote by $\mathcal{G}$ the σ-algebra generated by $\{B(h),\ h \in H\}$ and for all $k \geq 1$ define $\mathcal{H}_k \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ the closed linear subspace

$$\mathcal{H}_k = \overline{\text{span}\{H_k(B(h)),\ h \in H,\ \|h\|_H = 1\}}.$$

$\mathcal{H}_0$ is the set of constants. $\mathcal{H}_k$ is called the **Wiener chaos of order k.** Then one can prove the following theorem [117, Theorem 1.1.1]:

Theorem A.3.2. *Denote by γ the standard normal distribution. Then the Hermite polynomials form a complete orthogonal system in $L^2(\mathbb{R}, \gamma)$. Moreover, the space $L^2(\Omega, \mathcal{G}, \mathbb{P})$ can be decomposed as the orthogonal sum:*

$$L^2(\Omega, \mathcal{G}, \mathbb{P}) = \bigoplus_{k=0}^{\infty} H_n.$$

For any $k \geq 1$ we define $H^{\otimes k}$ as the k-th tensor product of H and $H^{\odot k}$ as the k-th symmetric tensor product of H. We endowed the space $H^{\odot k}$ with the norm $\|\cdot\|_{H^{\odot k}} = \sqrt{k!}\|\cdot\|_{H^{\otimes k}}$.

Let $h = h_1 \otimes ... \otimes h_k,\ g = g_1 \otimes ... \otimes g_k$ be two elements of $H^{\otimes k}$. We define the p-th contraction of h and g, denoted $h \otimes_p g$, as

$$h \otimes_p g = \langle h_{k-p+1}, g_1\rangle_H ... \langle h_k, g_p\rangle_H h_1 \otimes ... \otimes h_{k-p} \otimes g_{p+1} \otimes ... \otimes g_k \in H^{\otimes 2(k-p)}$$

and by $\tilde{h}$ the symmetrization of h defined as

$$\tilde{h} = \frac{1}{k!} \sum_{\sigma \in S_k} h_{\sigma(1)} \otimes ... \otimes h_{\sigma(k)} \in H^{\odot k} \quad \text{with } S_k \text{ the group of permutations of } \{1, ..., k\}.$$

One can show the following [117, Section 1.1.2]:

Theorem A.3.3. *Let $k \geq 1$. There exists a unique linear isometry $I_k : H^{\odot k} \to \mathcal{H}_k$, called the k-th* ***Wiener-Itô integral*** *with respect to B. Moreover, let $h \in H$ with $\|h\|_H = 1$. Then*

$$I_k(h^{\otimes q}) = H_q(B(h)).$$

Theorem A.3.2 is the tool used for the step (ii) while Theorem A.3.3 is necessary for the step (iii).

We state the last result of this subsection, necessary to proceed with step (iv). Let $Y_n = (Y_n^1, ..., Y_n^d)$ be a centered d-valued stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that Y_n admits the chaotic expansion:

$$Y_n^k = \sum_{m=1}^{\infty} I_k(f_{m,n}^k), \quad f_{m,n}^k \in H^{\odot m}.$$

We have the following central limit theorem [11]:

Theorem A.3.4. *If each of the following conditions holds:*

(i) For any $k = 1, ..., d$ we have

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{m=N+1}^{\infty} m! \|f_{m,n}^k\|^2_{H^{\otimes m}} = 0.$$

(ii) For any $m \geq 1$, $k, l = 1, ..., d$ we have constants $C_{k,l}^m$ such that

$$\lim_{n\to\infty} m! \|f_{m,n}^k\|^2_{H^{\otimes m}} = C_{k,k}^m,$$
$$\lim_{n\to\infty} \mathbb{E}\left[I_m(f_{m,n}^k) I_m(f_{m,n}^l)\right] = C_{k,l}^m, \quad k \neq l,$$

and the matrix $C^m = (C_{k,l}^m)_{1\leq k,l\leq d}$ is positive definite for all m.

(iii) $\sum_{m=1}^{\infty} C^m = C \in \mathbb{R}^{d\times d}$.

(iv) For any $m \geq 1$, $k = 1, ..., d$ and $p = 1, ..., m-1$

$$\lim_{n\to\infty} \|f_{m,n}^k \otimes_p f_{m,n}^k\|^2_{H^{\otimes 2(m-p)}} = 0.$$

Then we have the weak convergence

$$Y_n \xrightarrow{d} \mathcal{N}_d(0, C).$$

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