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by

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Born on 11th November 1997 in Barcelona

ARCS, CURVES, AND THE SURFACES THEY LIVE ON

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I hereby confirm that the PhD thesis entitled “Arcs, curves, and the surfaces they live on” has been written independently and without any other sources than cited.

Luxembourg, 20th of December of 2024

Name

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Summary

After a global introduction and background, there will be two main parts of the thesis, ordered by the topological size of the surfaces considered. Chapters 2 and 3 are written so that experts can read them independently.

Part 1

The first part has a purely topological base with combinatorial methods, and with applications to hyperbolic geometry. It presents some results on the discrete counting problem of counting the exact number of closed curves of a given word-length and with a given self-intersection number, all while characterizing the figure-eight curves on the once-punctured torus. This is part of a joint work with Mingkun Liu [FL24].

Next, we will use the combinatorial study of curves to describe the hyperbolic simple length spectrum of the modular torus and reformulate Markov's conjecture in combinatorial terms. This is part of an individual work in [Fis25].

Part 2

In the second part, we present a Riemannian analog to Basmajian's celebrated identity for hyperbolic surfaces, which becomes an inequality involving the volume entropy. We do this by first obtaining an inequality for a special class of metric graphs which encode the set of orthogeodesics of the surface. This is part of a joint work with Florent Balacheff [BF23].

Lastly, there is a chapter about more general perspectives and discussions on the approaches used in this work and a vision of the area, without putting as much effort into the rigor as with the rest of the dissertation, but instead on the intuition and overview.

Introduction and main results

This thesis contains two main parts, that will be later expanded into Chapters 2 and 3. Firstly, we will introduce and motivate the main results historically, and in case of doubt about any definition, the reader should refer to Chapter 1, containing the background.

During this introduction, referenced theorems for historical reasons will be numbered (i.e. Theorem 1, Theorem 2, etc.), whilst all original results will be labeled alphabetically (i.e. Theorem A, Theorem B, etc.).

The main object of study is surfaces, from topological to hyperbolic and Riemannian. The thesis' results will be ordered by increasing the topological size of the surfaces, the first ones being results on the so-called one-holed torus, which is a topological surface of signature $(1, 1)$.

Word-length curve counting on the once-punctured torus

Let us first introduce counting problems. The following classical result, the Prime Geodesic theorem, is sometimes considered the first curve counting result on hyperbolic surfaces. This became a topic in the mid-20th century, and the names of Delsarte, Hejhal, Huber, Margulis, Selberg, and Sarnak are the most prominent. By a curve, we understand a free homotopy class of a loop, or equivalently a conjugation class in the fundamental group. Primitive curves are classes of elements that are not proper powers of nontrivial elements. Essential curves are those not represented by a loop around a boundary component, a puncture, or a point in the surface. Throughout this part, later explained in the background, given a surface S, we will denote by $\mathcal{C}(S), \mathcal{PC}(S), \mathcal{C}^*(S), \mathcal{PC}^*(S)$ to be the set of curves, primitive curves, essential curves and essential primitive curves on S, respectively. For a more detailed definition see Definition 1.2.1.

Theorem 1 ([Hub59, Mar69])**.** *Let S be a finite-type hyperbolic surface, then*

$$|\{\gamma \in \mathcal{PC}(S) \mid \ell(\gamma) \leq L\}| \sim \frac{\mathrm{e}^L}{L}.$$

The estimate in Theorem 1 can be made effective, and the error terms are related to the Laplacian spectrum of X; see, for example, [Ber16, Section 5.4.2], [Bus92, Section 9.6].

A natural next step is to count not only the number of curves with bounded length but also those that share some properties. In this case, we will be interested in those with a given intersection number. We will denote by $\iota : \mathcal{C}(S) \to \mathbb{Z}_{\geq 0}$ the geometric self-intersection number of a curve. For a more detailed definition check Definition 1.2.2.

Among efforts by [Ree81, BS85, MR95b, Riv01], the next result on curve-counting that is especially relevant for us as it is on our same setting is the following, counting the number of curves with no self-intersection and bounded length. We will call the set of curves without self-intersection to be simple curves.

Theorem 2 ([MR95b]). *Let S be a hyperbolic once-punctured torus. Then, there is a constant $C_S > 0$ depending on the hyperbolic structure such that*

$$|\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = 0, \ell(\gamma) \leq L\} \sim C_S \cdot L^2$$

when L grows.

Note that, in the above theorem, one could substitute $\mathcal{PC}(S)$ by $\mathcal{C}(S)$, since all simple curves are primitive.

The next big groundbreaking result was due to Maryam Mirzakhani in [Mir08], who instead of counting all curves of bounded length, counted the ones with a given topological type, and generalized also by them to this next result.

Theorem 3 (Special case of [Mir16]). *For any hyperbolic structure on a surface S of signature (g, n), $X \in \mathcal{T}(S)$, and for any $k \in \mathbb{Z}_{\geq 0}$, there exist explicit constants $C_{g,n,k} > 0$ depending only on g , n, and k, and $B_X > 0$ depending only on the hyperbolic metric X such that*

$$|\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = k, \ell_X(\gamma) \leq L\}| \sim C_{g,n,k} \cdot B_X \cdot L^{6g+6-2n}.$$

After these, many efforts on curve counting have been made. And the biggest generalization of it is by Erlandsson, Parlier, and Souto in [EPS20]. We will however state a later version of the theorem by Erlandsson and Souto because the language and notation used fits better this introduction.

Theorem 4 (Special case of [ES22]). *Let ℓ be any positive, continuous, and homogeneous function on the space of geodesic currents on a surface S of signature (g, n). Then, for any $k \in \mathbb{Z}_{\geq 0}$, there exist positive constants $C_{g,n,k}$ and B_ℓ depending only on g, n, and k, and ℓ, respectively, such that*

$$|\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = k, \ell(\gamma) \leq L\}| \sim C_{g,m,k} \cdot B_\ell \cdot L^{6g-6+2n}.$$

Background on geodesic currents is referenced to [ES22], as will not be used in this the-

sis. However, the set of curves of a surface S can be immersed in the set of geodesic currents, and, moreover, this result generalizes the theorems before to not only hyperbolic length but to any positive continuous and homogeneous function on the space of geodesic currents on the surface.

In particular, we will be interested in one specific length that comes in at this generalization, the *word-length*. Given a surface, fix generators of the fundamental group, then the word-length denoted by $\ell_\omega : \mathcal{C}(S) \to \mathbb{Z}_{\geq}0$ is the minimal number of letters of the words on the generating set representing a given curve. This specific combinatorial length satisfies the conditions in Theorem 4, and hence the asymptotic counting of curves with given intersection is known.

For hyperbolic lengths, aiming to find an exact explicit formula for the number of curves with a given self-intersection and bounded length in terms of some coordinates of the moduli space of hyperbolic metrics seems to be too ambitious. However, taking inspiration on [EPS20], we will try and investigate this question for the word-length.

Let $\Sigma_{1,1}$ be a once-punctured torus, i.e. a surface with signature $(1,1)$. Fix a canonical generating set on the fundamental group and denote it by $\{a,b\}$. Let ℓ_ω be the word-length function on the set of curves with respect to this generating set. Then, in this special case, we do not only want to study the asymptotics on the growth of the number of curves with a given type, but we want to find a closed formula for the following number,

$$|\{\gamma \in \Gamma \subseteq \mathcal{C}(\Sigma_{1,1}) \mid \iota(\gamma) = k, \ell_\omega(\gamma) = L\}, \tag{0.0.1}$$

for $\Gamma \in \{\mathcal{C}(\Sigma_{1,1}), \mathcal{PC}(\Sigma_{1,1}), \mathcal{C}^*(\Sigma_{1,1}), \mathcal{PC}^*(\Sigma_{1,1})\}$.

In terms of counting curves with a given word-length and self-intersection, there are works by Chas, Phillips, Lalley, and McMullen; see [CP10, CL12, Cha15, MCP19]. Many bounds have been found for the general cases and closed formulas for given length-intersection difference, as well as computational experiments and conjectures. Even if being on a different surface, an outstanding result on this topic worth mentioning is the following.

Theorem 5 ([MCP19])**.** *Let S be a surface of signature $(0,3)$. Then, for any $k \geq -1$, there exists a quadratic polynomial $P_k(L)$ such that, for every $L \geq k+4$*

$$P_k(L) = |\{\gamma \in \mathcal{PC}(S) \mid \ell_\omega(\gamma) = L, \iota(\gamma) = k + L\}|.$$

This polynomial is, in general, not explicit. However, given some computations for short curves with small self-intersection by Chas, some low-complexity examples of these polynomials have been made explicit via interpolation.

Hence, back to our surface $\Sigma_{1,1}$, we will now present our main results. The general case for the counting of 0.0.1 is still open because our methods highly rely on combinatorics whose complexity grows exponentially when self-intersection grows. Our main

results are finding closed formulas for the cases with self-intersection 0 and 1 and for any length. We will also study the case with no restriction on the self-intersection. A combinatorial function that will appear often is Euler's totient function, which counts the number of positive integers up to a given number that are relatively prime to this upper bound. We will denote it by φ, see Definition 1.5.5. For simple curves, the counting is already known as a consequence of the correspondence between homotopy and homology on the torus, however, we will provide a new combinatorial proof whose tools will be useful for the case of self-intersection 1. The theorem we will reprove is the following.

Theorem A ([MR95a],[FL24]). *For any $L \in \mathbb{Z}_{\geq 4}$, we have*

$$|\{\gamma \in \mathcal{PC}(\Sigma_{1,1}) \mid \iota(\gamma) = 0,\, \ell_\omega(\gamma) = L\}| = 4\varphi(L),$$

where φ stands for Euler's totient function.

The above counting solves the problem for simple curves. Note also that given Möbius inversion formula (Theorem 1.5.6), one can use primitive counting to count all powers of curves with a given length. For curves with a single self-intersection, also known as figure-eight curves, we have the following counting theorem.

Theorem B ([FL24]). *There are 8 primitive closed curves on $\Sigma_{1,1}$ of length 4 with 1 self-intersection. For any $L \in \mathbb{Z}_{>4}$, we have*

$$|\{\gamma \in \mathcal{PC}(\Sigma_{1,1}) \mid \iota(\gamma) = 1,\, \ell_\omega(\gamma) = L\}| = \begin{cases} 8\,\varphi(L-4) & \text{if } L \text{ is odd,} \\ 8\big(\varphi(L-4) + \varphi(L/2)/2\big) & \text{if } L \text{ is even.} \end{cases}$$

Remark. It was brought to the attention of the authors, after finding this counting as a consequence of the classification in Theorem C, that this result was known by Moira Chas and a different proof was sketched in their Preprint [Cha16, Proposition 3.2].

The main strategy for these proofs is the combinatorial study of the words representing curves with a given self-intersection. For the simple case, there was already an existing classification theorem.

Theorem 6 ([BS88, Theorem 6.2]). *Every simple closed curve on $\Sigma_{1,1}$ can be represented, after suitably renaming the generators, by one of the following words:*

1. a,
2. $aba^{-1}b^{-1}$,
3. $ab^{n_1}ab^{n_2}\cdots ab^{n_r}$, *where $[n_1, \ldots, n_r]$ has small variation.*

Conversely, each of these words is homotopic to a power of a simple closed curve.

Here, a small-variation necklace is a cyclic class of finite sequences of positive integers such that any two blocks of consecutive entries of the same size have sum difference at most 1, see Definition 2.2.2. With this characterization, we study the rigidity of small variation necklaces in terms of their entries and the number of occurrences of their entries, see Proposition 2.2.8. To do this, we apply a reduction of the necklaces by assigning to a necklace a (when necessary) shorter necklace encoding every time that an integer is repeated, see Figure 1. This reduction takes the necklace of integers generated by the exponents and keeps reducing them to the size of runs of consecutive numbers, in a specific way. By proving that this reduction preserves small variation, see Lemma 2.2.9, we have all the necessary ingredients to do the counting and proof Theorem 2.1.5.

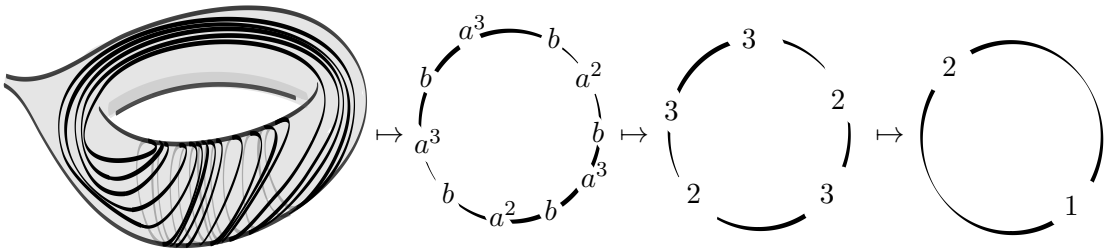


Figure 1

To prove Theorem B with similar methods, there was no characterization of all the words representing curves with a single self-intersection. We prove an analog characterization for these curves. The main ingredient of the proof of the next characterization is Cohen and Lustig's algorithm to determine self-intersection [CL87], finding some exceptional short cases and some general cases.

Theorem C ([FL24]). *A primitive curve in $\mathcal{PC}(\Sigma_{1,1})$ has self-intersection one if and only if, up to renaming the generators in $\{a, b, a^{-1}, b^{-1}\}$, it can be written as one of the following:*

1. a^2b^2, $aba^{-1}b$, $ab^{-1}a^{-1}b^2$, *or*
2. $ab^{-1}a^{-1}ba^{n_1}b\cdots a^{n_k}b$, *or* $ab^{-1}a^{-1}ba^{-n_1}b\cdots a^{-n_k}b$, *where the words* $a^{n_1}b\cdots a^{n_k}b$, *and* $a^{-n_1}b\cdots a^{-n_k}b$ *are uniquely determined representatives of primitive simple curves, or*
3. $a^{n_1}b\cdots a^{n_k}b$, *where* $[n_1,\dots,n_k]$ *satisfies that exists an* $m\in\mathbb{Z}_{\geq 1}$ *such that for all* $i\in\{1,\dots,k\}$, $n_i\in\{m,m+1\}$ *and it is a necklace with* 2*-variation (see Definition 2.3.4), or*
4. $a^mba^{m+2}b$, *for some* $m\in\mathbb{Z}_{\geq 1}$.

For curves of type 2 above, the uniqueness of the representative of the simple curve is proved explicitly in Lemma 2.3.3. For curves of type 3, a 2-variation necklace (Definition

2.3.4), breaks the small-variation condition minimally, meaning that it is a cyclic shift class of finite sequences of positive integers such that there are exactly two blocks of consecutive integers of the same size with sum differing by 2, and the rest of the blocks of the same size have sum difference at most 1.

To prove Theorem B, once the classification is done, the methods are very similar to the methods for the simple case. We study the rigidity of the 2-variation necklaces, i.e. Proposition 2.3.8. These, with the necklace reduction in Lemma 2.2.9 again, make it enough to do the counting.

The combinatorial method with the reduction on the necklaces has proved to be very useful. However, classifying words for the general case of self-intersection k is a very complex question. Indeed, a weaker version of the question that could help to apply the necklace reduction and make the counting for a general case is an open question in research about the number of mapping class group orbits of the number of curves with a given self-intersection. We will now expand on that.

The mapping class group is defined as the group of orientation-preserving homeomorphisms fixing of S modulo isotopy, see Definition 1.2.3. This group acts on the set of curves $\mathcal{C}(S)$ and preserves self-intersection. For simple curves, there is a unique mapping class group orbit of essential simple curves. For primitive curves with self-intersection 1, there are 2 orbits of curves. The general number of orbits is unknown, and some asymptotic bounds and works on it can be found in [AS16, CFP18a].

We are now interested in another natural question, which is the counting when we do not have restrictions on the self-intersection of curves, as on the style of Theorem 1. This counting problem turns out to be purely algebraic, but could not be found in literature. Here is the geometric statement.

Theorem D ([FL24])**.** *There are 4 primitive curves of length 1, 8 of length 2, and for any $L \in \mathbb{Z}_{\geq 3}$, we have the formula*

$$|\{\gamma \in \mathcal{PC}^*(\Sigma_{1,1}) \mid \ell_\omega(\gamma) = L\}| = \frac{1}{L}\sum_{d|L} \mu(d)\, 3^{L/d},$$

where μ is the Möbius function. For not necessarily primitive curves, we have, for any $L \in \mathbb{Z}_{\geq 1}$,

$$|\{\gamma \in \mathcal{C}^*(\Sigma_{1,1}) \mid \ell_\omega(\gamma) = L\}| = \frac{1}{L}\sum_{d|L} \varphi(d)\, 3^{L/d} + \frac{3+(-1)^L}{2}.$$

For the explicit definition of the Möbius function and its relation to Euler's totient function see 1.5.5. This counting is for essential curves, meaning that they do not bound the cusp. However, this only changes by not considering the commutator, and gives a cleaner formula.

Now, consider a classical combinatorial object: necklaces with beads. Given a necklace

with $n > 0$ white beads, one wants to color it with a palette of $k > 0$ colors, painting every bead of a unique color without leaving any white beads. The number of possible colorings is well-known and leads to the following very intriguing remark.

Remark. For $L \in \mathbb{Z}_{\geq 1}$,

$$|\{\gamma \in \mathcal{PC}^*(\Sigma_{1,1}) \mid \ell_\omega(\gamma) = L\}| = \\ = |\{\text{aperiodic necklaces with } L \text{ beads and } 3 \text{ colors}\}| + \delta_{\{1,2\}}(L),$$

where $\delta_{\{1,2\}}(L) = 1$ if $L \in \{1, 2\}$ and vanishes otherwise, and

$$|\{\gamma \in \mathcal{C}^*(\Sigma_{1,1}) \mid \ell_\omega(\gamma) = L\}| = |\{\text{necklaces with } L \text{ beads and } 3 \text{ colors}\}| + \epsilon(L),$$

where $\epsilon(L) = 1$ if L is odd and $\epsilon(L) = 2$ if L is even.

Here, a colored necklace is periodic if it can be realized as the cyclic class of a proper power of a sequence of colors.

Despite observing this numerical coincidence, we could not find any straightforward natural bijection between these sets. Finding such a bijection remains an open question, whose answer would possibly give tools for the counting in higher genus by recharacterizing the problem.

Finally, let us give some consequences of Theorems A and B. These theorems allow us to also count the number of curves with bounded word-length that are not necessarily primitive, and get the asymptotics using the already known asymptotics for Euler's totient function. Recall that a simple multicurve can be defined as a formal integer linear combination of simple curves.

Corollary E ([FL24]). *Let φ be Euler's totient function, and $\Phi(L) = \sum_{i=1}^{L} \varphi(i)$ for $L > 0$, its sum up to a number. Then, for $L \in \mathbb{Z}_{\geq 4}$*

1. *By summing Theorem A we get the counting for bounded length,*

$$|\{\gamma \in \mathcal{PC}(\Sigma_{1,1}) \mid \iota(\gamma) = 0,\ \ell_\omega(\gamma) \leq L\}| = 4\Phi(L) + 2 \sim \frac{12}{\pi^2} L^2.$$

2. *By applying Möbius inversion formula (Theorem 1.5.6) to Theorem A we get the simple multicurves*

$$|\{\gamma \text{ simple multicurve on } \Sigma_{1,1} \mid \iota(\gamma) = 0,\ \ell_\omega(\gamma) = L\}| = 4L.$$

3. *By summing the above we get the counting for bounded length and also the simple multicurves*

$$|\{\gamma \text{ simple multicurve on } \Sigma_{1,1} \mid \iota(\gamma) = 0,\ \ell_\omega(\gamma) \leq L\}| = 2L^2 + 2L.$$

4. *By summing Theorem B we get the asymptotic counting for bounded length*

$$|\{\gamma \in \mathcal{PC}(\Sigma_{1,1}) \mid \iota(\gamma) = 1,\, \ell_\omega(\gamma) \leq L\}| = 8(\Phi(L-4) + \Phi(\lfloor L/2 \rfloor)/2) \sim \frac{27}{\pi^2} L^2.$$

5. *By summing Theorem D we get the asymptotic counting for bounded length*

$$|\{\gamma \in \mathcal{PC}^*(\Sigma_{1,1}) \mid \ell_\omega(\gamma) = L\}| \sim \frac{3^L}{L}, \qquad |\{\gamma \in \mathcal{PC}^*(\Sigma_{1,1}) \mid \ell_\omega(\gamma) \leq L\}| \sim \frac{3}{2} \cdot \frac{3^L}{L}.$$

As a consequence of Corollary E and the result of Theorem 4, since quotients of the following kind do not depend on the function choice, the next statement follows.

Corollary F ([FL24]). *Let $X \in \mathcal{T}(S_{1,1})$ be a hyperbolic structure on the once-punctured torus, then*

$$\lim_{L\to\infty} \frac{|\{\gamma \in \mathcal{PC}(\Sigma_{1,1}) \mid \iota(\gamma) = 0,\, \ell_X(\gamma) \leq L\}|}{|\{\gamma \in \mathcal{PC}(\Sigma_{1,1}) \mid \iota(\gamma) = 1,\, \ell_X(\gamma) \leq L\}|} = \frac{4}{9}.$$

This means that the probability of choosing a simple curve over the curves with self-intersection at most 1 is exactly $4/13$.

The last piece of the first part of the thesis will be on how to use this combinatorial study of curves to study hyperbolic spectra, i.e. the sets of lengths of curves with respect to a hyperbolic structure. The main objective will be to describe combinatorially the simple length spectrum of a specific hyperbolic structure on the once-punctured torus to reformulate Markov's conjecture. This is based on an individual work in [Fis25].

A triple of positive integers $x, y, z > 0$ such that

$$x^2 + y^2 + z^2 = 3xyz$$

is called a *Markov triple* and were introduced by Markov in the late XIX century. Frobenius conjectured in [Fro13] that these triples are always determined by their largest number, also called a *Markov number*. This is known as *Markov's uniqueness conjecture*, see [Aig13]. Since then, many partial results have been made for Markov numbers of a given form, such as prime powers or particular linear functions of prime powers, see e.g. [Bar96, Zha07]. However, the general statement remains open.

Define the modular torus $\mathcal{M}$ as the unique hyperbolic structure on the once-punctured torus such that the isometry group is of maximal order, it being of order 12. As proved by Cohn in [Coh71], there is a map from Markov's triples to lengths of simple closed geodesics on the modular torus. This allows us to reformulate Markov's conjecture in geometric terms.

Markov's conjecture is equivalent to the following conjecture on the modular torus.

Conjecture 7 (Geometric Markov's uniqueness conjecture). *For every two simple closed geodesics on $\mathcal{M}$ of the same length, there is an isometry bringing one to the other.*

We refer to [MP08] for more explanation on this equivalence. This has a translation in terms of the multiplicity of the simple length spectrum on the modular torus, $\mathcal{SS}(\mathcal{M})$: *the first two lengths have multiplicity* 6 *and all the rest have multiplicity* 12. Here again, multiplicity refers to the number of geodesics attaining these lengths. This is due to the shortest 12 geodesics being invariant (up to orientation) by one of the isometries, whilst the rest are not.

We will use the combinatorics in the set of curves on the once-punctured torus to reformulate Markov's uniqueness conjecture in combinatorial terms by passing through geometry.

Denote by $\mathcal{N}$ the set of nontrivial primitive necklaces of positive integers with small variation, i.e. cyclic classes of finite sequences of positive integers such that the sum of any two blocks of the same size has difference at most 1 (see Definition 2.2.2). Primitivity refers to not being represented by a power by concatenation of a smaller sequence.

Define the following function on this set of necklaces.

$$\begin{aligned} \Phi : \mathcal{N} &\to \mathbb{Z}_{>0} \\ [n_1, \dots, n_k] &\mapsto \frac{1}{10^k \cdot 2^{n_1+\cdots+n_k}} \sum_{S \subseteq \{1,\dots,k\}} 3^{r(S)} 2^{k-r(S)} (\xi+2)^{|S|} (\overline{\xi}+2)^{|S^c|} \xi^{\sum_{i\in S} n_i} \overline{\xi}^{\sum_{i\in S^c} n_i}, \end{aligned} \tag{0.0.2}$$

for $\xi = 3+\sqrt{5}$ and $\overline{\xi} = 3-\sqrt{5}$, where $r(S) = \sum_{s\in\{\text{runs of } S \text{ and } S^c\}} |s| - 1$ if $S \neq \{1,\dots,k\}$, and $r(\{1,\dots,k\}) = k$. A *run* s of $S \subseteq \{1,\dots,k\}$ is a maximal subset $s \subseteq S$ such that the numbers are (cyclically) consecutive.

We prove an equivalent statement to Markov's uniqueness conjecture as follows.

Theorem G ([Fis25])**.** *Markov's uniqueness conjecture is equivalent to*

The function Φ *is injective in* $\mathcal{N}$.

We prove this via using the trace formula for a specific representation $\rho : \pi_1(S) \to PSL_2(\mathbb{R})$, with the classification of all simple curves in Theorem 6, to compute in combinatorial terms the simple length spectrum of the modular torus via identifying it as the zero-shear hyperbolic structure on the once-punctured torus (see Section 2.5). This will give the expression of the spectrum that will be found in 2.5.1.

Open questions

The first natural open question is to find the closed formula for any self-intersection number. To approach this question, the expected strategy would be to find first the number

of orbits in the mapping class group $\mathrm{Map}(S)$, which is already a known open question (see [AS18, CFP18b]). Then, use the counting and reduction methods explained in Chapter 2 to count the size of every orbit by finding representatives.

A follow-up question is to explore the simple curve counting on a surface of bigger genus. In this regard, we are still missing a complete characterization of all the words representing simple curves for bigger genus surfaces. This is being explored by adapting Cohen and Lustig's algorithm to determine the self-intersection of given words (see [CL87]).

Finally, these kinds of characterizations can be used to study the length spectra of the once-punctured torus, as done for Theorem G. Here, two areas of research appear, one is to study Markov's conjecture in this new form as for subsets of the set of small-variation necklaces that we have parametrized in three integer variables, and the other one is to keep studying other length spectra as a consequence of the combinatorial work already done.

A Basmajian-type inequality for Riemannian surfaces

Let us now enter the second part of the thesis, and so increase the topological size of the surface. From now on, our results will apply to surfaces S with one unique boundary component and finite genus g, i.e., of signature $(g, 1)$.

We will study Riemannian surfaces, i.e. surfaces equipped with Riemannian metrics. We will be especially interested in their orthospectrum. We define arcs on a Riemannian surface with boundary S as relative homotopy classes of immersed paths with endpoints on the boundary. Denote it by $\mathcal{A}(S)$. Define then the orthospectrum of S as the set $\mathcal{O}(S)$ of lengths of the set $\mathcal{A}(S)$ with multiplicity. "With multiplicity" refers to that whenever two or more classes have the same lengths, this number is repeated in the set $\mathcal{O}(S)$, as if used as an index set, then every number appears as many times as classes attaining it.

We will study if a classical identity for hyperbolic surfaces relating the orhtospectrum and the length of the boundary admits some analog in the moduli space of Riemannian metrics. That is, once we lose the rigidity of its curvature, how much of this phenomenon can be attributed to it being a Riemannian metric.

There are many results on hyperbolic manifolds that have found their Riemannian analog. We will give three examples of this phenomenon that we find especially notorious.

Let $\mathrm{sys}(S)$ denote the systole of a Riemannian surface, meaning the infimum of the lengths of the non-contractible curves with positive lengths, see Definition 1.4.5. Note that the positivity of curves in this definition is due to our definition of curves as free homotopy classes of loops.

Theorem 8 (Classical, [BS94]). *Let S be a closed hyperbolic surface of genus $g \geq 2$. Then,*

$$\mathrm{sys}(S) \leq 2\log(4g-2),$$

and exists a family of surfaces S_n with growing genus $\{g_n\}_n \to \infty$ such that

$$\mathrm{sys}(S_n) \sim \frac{4}{3}\log g_n.$$

There has been found an analogous behavior for Riemannian metrics split into the two following two results. For the first one, note that a closed hyperbolic surface has always area $4\pi(g-1)$, which recovers a similar growth rate of Theorem 8 for hyperbolic metrics.

Theorem 9 ([KS04]). *Let S be a closed Riemannian surface of genus g. Then,*

$$\frac{\mathrm{Area(S)}}{\mathrm{sys}(S)^2} \gtrsim \pi\frac{g}{\log^2 g} + o(1),$$

when g grows.

And, as for the second example, it will involve the metric invariant called volume entropy. Define the volume entropy of a compact manifold M as the exponential growth rate of the volume of balls with growing radius and fixed centre in its universal cover and denote it by $h(M)$ (see Definition 1.4.1).

Theorem 10 ([Sab06]). *Let S be a closed Riemannian surface of genus g and volume entropy $h(S)$. Then, there is a constant $C_g > 0$ depending on the genus such that,*

$$h(S) \cdot \mathrm{sys}(S) \leq C_g.$$

Note that for closed hyperbolic surfaces the volume entropy is always 1, and therefore we can see the above theorem as an analog of Theorem 8 for Riemannian metrics.

It turns out that many hyperbolic identities and inequalities as we shall see next have found a Riemannian analog by re-scaling by the volume entropy, as a measure of how far asymptotically the lengths are from hyperbolicity. One way to gain intuition on why this happens is by looking at an alternative expression of the volume entropy. The volume entropy of a surface is also the exponential growth rate of the number of curves with bounded length as the length grows (see Proposition 1.4.3).

When exploring Riemannian versions of hyperbolic phenomena, with the loss of rigidity inequalities might become weaker as happened in the last example, or even equalities might become inequalities, as will be the case for the following example. Another classical identity for hyperbolic surfaces is that, as mentioned earlier, for any closed hyperbolic surface S of genus g,

$$Area(S) = 4\pi(g-1),$$

which is a direct result of Gauss-Bonnet's theorem. We can consider Gauss-Bonnet's theorem as a generalization of it where the integral of the curvature measures how far is the surface from being hyperbolic. However, as said before, one can also use the volume entropy for the same purpose, leading to the following classical theorem.

Theorem 11 ([Kat82]). *Let S be a closed hyperbolic surface of genus g and volume entropy $h(S)$, then*

$$h(S)^2 Area(S) \geq 4\pi(g-1).$$

There is a bigger generalization of this theorem by Besson, Courtois, and Gallot that states the following.

Theorem 12 ([BCG95]). *For any two compact, oriented, and connected Riemannian manifolds M, N of the same dimension n. Let $f : M \to N$ be a continuous function of degree $d > 0$. If N is locally symmetric with negative curvature, then*

$$h(M)^n \mathrm{Vol}(M) \geq d \cdot h(N)^n \mathrm{Vol}(N).$$

When $n \geq 3$ these two quantities are equal if and only if a scalar multiple of the metric in M is a covering of N.

The above can be used to reprove Mostow's rigidity Theorem (Theorem 1.3.5), see [BCG96].

In this case, hyperbolic surfaces are the optimal case for the general inequality, as they minimize $h(S)^2 Area(S)$ for a given topology. A last example is the $\log(2k-1)$ theorem, which states as follows, after losing some of the original assumptions in [ACCS96] given later work of [Can08].

Theorem 13 ([ACCS96, Can08]). *] Let $k \geq 1$ be an integer, for a Kleinian group Γ freely generated by elements $\gamma_1, \ldots, \gamma_k$, then for any $\tilde{x} \in \mathbb{H}^3$, the following inequality holds*

$$\sum_{i=1}^{k} \frac{1}{1 + e^{d(\tilde{x}, \gamma_i \tilde{x})}} \geq \frac{1}{2}.$$

This was later generalized to any Riemannian metric by Balacheff and Merlin as follows. This time, the re-scaling involves a generalization of the volume entropy, which is the critical exponent. Let Γ be a subgroup of isometries of a given Riemannian manifold, and define its critical exponent as the exponential growth rate of the number of elements with bounded translation length, see Definition 1.4.4.

Theorem 14 ([BM23]). *For any $\tilde{M}$ simply connected, complete Riemannian manifold, let Γ be a discrete subgroup of isometries freely generated by some elements $\gamma_1, \ldots, \gamma_k$. For any point $\tilde{x} \in \tilde{M}$,*

$$\sum_{i=1}^{k} \frac{1}{1 + e^{\delta(\Gamma) \cdot d(\tilde{x}, \gamma_i \tilde{x})}} \geq \frac{1}{2},$$

being $\delta(\Gamma)$ the critical exponent of Γ.

Note that the critical exponent of the group Γ coincides with the volume entropy $h(M)$ when taking $\tilde{M}$ to be the universal cover of a closed Riemannian manifold M and $\Gamma = \pi_1(M)$. This is a consequence of the already mentioned characterization of the volume entropy in Proposition 1.4.3.

After all these examples, what we will try to generalize will be the celebrated Basmajian's identity, relating the boundary of a compact hyperbolic surface to its orthospectrum. The general identity proved by Basmajian is for n-dimensional hyperbolic manifolds and with possibly different arc sets. The following is the special case of the identity we are interested in.

Theorem 15 ([Bas93])**.** *Let S be a compact hyperbolic surface with boundary. Let $\mathcal{O}(S)$ be its orthospectrum. Then*

$$\ell(\partial S) = 2 \sum_{\ell \in \mathcal{O}(S)} \log \coth(\ell/2).$$

Basmajian's identity can be seen as a result on the rigidity of the orthospectrum, proving that the length of the boundary is determined by the orthospectrum.

The orthospectrum has since been widely studied in hyperbolic geometry. Two examples of it are the following.

Theorem 16 ([Bri11])**.** *Let S be a compact hyperbolic surface with boundary, and $\mathcal{O}(S)$ its orthospectrum, then*

$$\sum_{\ell \in \mathcal{O}(S)} \mathcal{L}(1/\cosh^2(\ell/2)) = \frac{\pi}{4} \mathit{Area}(S),$$

being $\mathcal{L}$ Roger's dilogarithm function.

Note that since the area of a hyperbolic surface is a multiple of its Euler characteristic, this result proves that the orthospectrum determines the possible topologies of the surface. The other example on the rigidity of the orthospectrum, still for hyperbolic surfaces, is the following.

Theorem 17 ([MM22])**.** *Given a surface with a unique boundary component and genus g and a hyperbolically attainable orthospectrum, there are finitely many hyperbolic structures with this orthospectrum.*

In this fashion, we found an analog inequality to Basmajian's identity, this time for Riemannian surfaces with a unique boundary component. The last ingredient needed for the statement is doubled surfaces. The doubled surface of a compact Riemannian surface with geodesic boundary S is the closed Riemannian surface obtained by taking

two copies of the surface and gluing them via the identity map on the boundary (see Definition 1.3.6). We usually denote it by S'.

Theorem H ([BF23]). *Let S be a compact orientable Riemannian surface with negative Euler characteristic and one geodesic boundary component. Then the following holds:*

$$\ell(\partial S) \geq \frac{2}{h(S')} \operatorname{arcsinh}\left(\sum_{\ell \in \mathcal{O}(S)} \frac{1}{1+e^{h(S')\ell}}\right)$$

where $h(S')$ denotes the volume entropy of the doubled surface S'.

As later expanded in Chapter 3, this result is achieved by first finding an inequality on a special family of metric graphs, that will encode the orthospectrum of the surface. Given a compact Riemannian surface S with a unique geodesic boundary component, we will construct the following sequence of metric graphs (see Figure 2). Fix an order on the set of arcs by non-decreasing length, as the number of arcs with the same length is always finite because the Riemannian metric is quasi-isometric to any hyperbolic metric on the surface, by compactness.

Choose a length-minimizing geodesic representative of the first arc and construct the first metric graph by first taking a circle of length equal to the length of the boundary component of S with two vertices splitting it into two parts of lengths equal to the splitting this first orthogeodesic makes to the boundary. Add an edge with endpoints on the two vertices with length equal to the length of the chosen orthogeodesic. Construct the sequence of metric graphs recursively, where the nth step is as follows. Choose a length-minimizing geodesic representative of the nth arc, add to the $(n-1)$th graph two vertices such that the circle length is split into $2n$ segments distributed as the boundary length in S is distributed with the n orthogeodesic representatives fixed for the first n arcs. Add an edge between those two vertices of length equal to the nth arc.

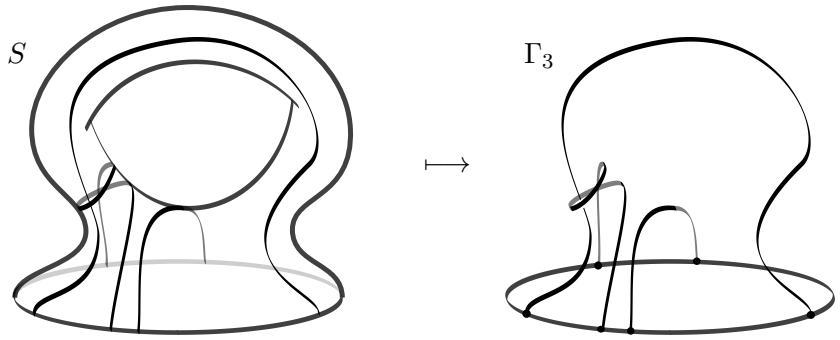


Figure 2

This sequence of graphs satisfies the assumptions of our following theorem.

Theorem I ([BF23]). *Fix $n \geq 1$. Let Γ be a metric graph formed by a circle of length L with $2n$ disjoint vertices on it, and n edges of lengths $\ell_1, \ldots, \ell_n$ joining these vertices by pairs. Then the following holds:*

$$\tanh\left(\frac{h(\Gamma)L}{2}\right) < 2\sum_{i=1}^{n}\frac{1}{1+e^{h(\Gamma)\ell_i}} < \sinh\left(\frac{h(\Gamma)L}{2}\right)$$

where $h(\Gamma)$ denotes the volume entropy of the metric graph Γ.

The main idea of the proof of this theorem is to study the dynamics of this family of graphs given Lim's Theorem in [Lim08]. The final main ingredient to prove Theorem H is to show that for any $n > 0$, with Γ_n being the nth metric graph constructed above, the volume entropy of the doubled surface S' is controlled by the volume entropy of the graph Γ_n (see Proposition 3.3.1), letting us deduce Theorem H from Theorem I.

Open questions

The first question that one can ask is whether the main result is still valid for any number of boundary components. The only part of the argument that is not straightforward generalizable to a multiple boundary case is Theorem I. After many efforts on applying Lim's theorem to the graph that appears in the multiple boundary case, and trying to extract an inequality from the linear system, we were not able to solve it. The combinatorics on the system grow very fast. However, that is a very natural open question that should be solvable even with the same methods.

Many open questions also arise by looking at other hyperbolic phenomena that have not yet been studied in the Riemannian moduli space.

Another celebrated identity, very much related to Basmajian's, is McShane's identity, and that has been generalized in many ways, one of them by Maryam Mirzakhani for any signature in [Mir07]. This one is on the rigidity of the simple length spectrum.

Theorem 18 (McShane's identity, [McS98]). *Let S be a once-punctured torus. Let $X \in \mathcal{T}(S)$ be any hyperbolic structure. Then,*

$$\sum_{\gamma\in\mathcal{C}(S)}\frac{1}{1+e^{\ell_X(\gamma)}} = \frac{1}{2}.$$

It still does not have a proper Riemannian analog, and we did not find any argument against its existence, nor a strategy to find it still. However, it seems to be a natural next step as it was proved by Parlier in [Par20] that there is an interpolation between Basmajian's identity and McShane's by choosing coherent markings of $\mathcal{C}(S)$, being S our fixed surface.

In particular, when the marking is empty, it recovers Basmajian's identity, whilst for the marking being $\mathcal{PC}(S)$, it recovers McShane's identity.

Chapter 1

Background

This chapter contains the necessary background including some basic material. However, its intention is not only to provide background but also to fix the notation and conventions for the rest of the dissertation. To allow independent reading, some of the definitions will be repeated inside the chapters, in more of an article-level style for experts on the topics.

1.1 Graphs and surfaces

For our interest, the lowest-dimensional geometrical object we will care about will be graphs, that will encode in 1 dimension much higher-dimensional information in many ways.

Definition 1.1.1 (Graph)**.** Define a **graph** to be a one-dimensional CW complex Γ. Denote its 0-skeleton by $V(\Gamma)$ and call them **vertices** and call an **edge** the closure of every connected component in $\Gamma \backslash V(\Gamma)$, denoting the set of all edges by $E(\Gamma)$. Note that the set $E(\Gamma)$ corresponds to unoriented edges to which by the CW-structure one can associate an unordered pair of vertices $e_{ij} \mapsto \{v_i, v_j\}$. We will call those the endpoints.

Moreover, every edge is homeomorphic to $[0, 1] \subseteq \mathbb{R}$, admitting two orientations. Hence, there are $2|E(\Gamma)|$ oriented edges in Γ, and each of them has an associated startpoint vertex and an endpoint vertex. Denote the set of oriented edges by $E^*(\Gamma)$.

Even though graphs appear frequently, our focus is on surfaces.

Recall that a **surface** is a two-dimensional manifold. For the rest of the thesis, we will always consider them to be connected and orientable, when not said explicitly otherwise.

Throughout this entire work, S^1 will denote the one-dimensional circle.

Definition 1.1.2 (Fundamental group)**.** As usual, given a path-connected topological space X we will denote its fundamental group as: fixed any basepoint $x \in X$,

$$\pi_1(X) := \pi(X, x) = \{f : [0,1] \to X \text{ continuous map with } f(0) = f(1) = x\}/\text{homotopy},$$

the group of homotopy classes of loops with the concatenation of loops by fixing a common endpoint. This definition is valid, in particular, for X being a graph or a surface. We do not make explicit the basepoint in the notation, as the group will be isomorphic independently of the choice. Another set we will be interested in for a surface S is

$$\{f : [0,1] \to S \text{ continuous map with } f(0), f(1) \in \partial S\}/\text{homotopy rel. to } \partial S,$$

the set of relative homotopy classes of arcs with endpoints on the boundary.

Surfaces with finitely generated fundamental groups are called finite-type surfaces and can be classified up to homeomorphism by their signature. We say that a surface S has signature (g, n) if it is homeomorphic to a surface with genus g and n punctures (see Figure 1.1).

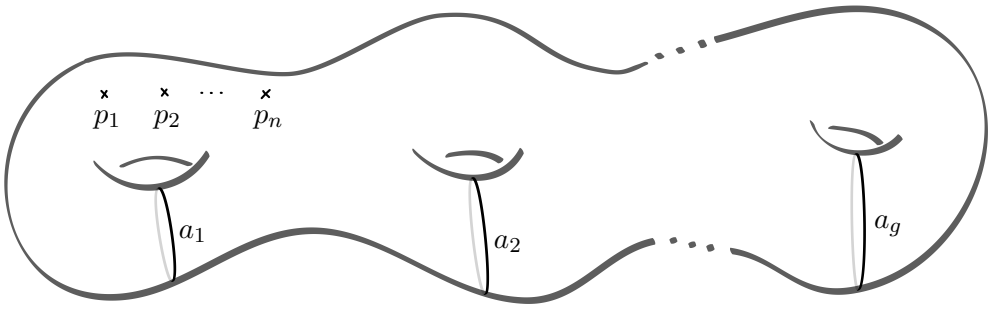
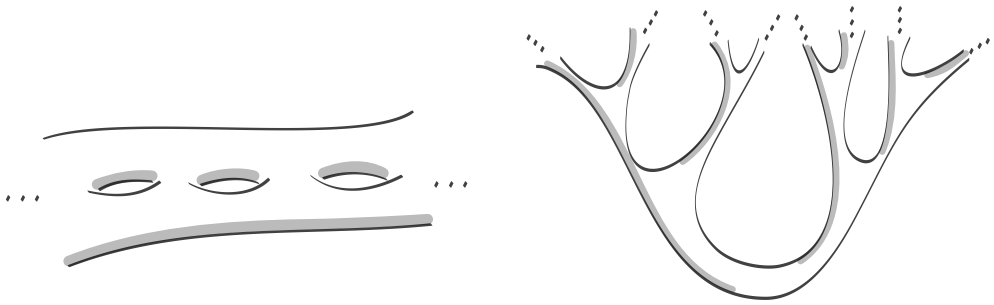


Figure 1.1

Otherwise, when their fundamental group is not finitely generated, we will say that the surface is of infinite type.

Two examples of infinite-type surfaces are the following, the so-called Ladder surface with two ends, and the Cantor Set surface which is planar and with a Cantor set as a space of ends, see Figure 1.2, where there is either geni or punctures accumulating to infinity.

Every original result in this work will be on finite-type surfaces. More specifically, surfaces with signature $(g, 1)$ for some genus $g > 0$.



(a) Ladder surface. (b) Cantor Set surface.

Figure 1.2

1.2 Curves and arcs

Definition 1.2.1. Define a **curve** on a surface S as a free homotopy class of a loop on S, where a loop is a continuous function $c : S^1 \to S$. These are in correspondence with the conjugation classes in $\pi_1(S)$, as it is equivalent to losing the basepoint. Define an **arc** on a surface S as a relative homotopy class of an induced path $\alpha : [0, 1] \to S$ with endpoints on the boundary. Denote the set of curves on a surface S as $\mathcal{C}(S)$ and the set of arcs as $\mathcal{A}(S)$. Denote also by $\mathcal{PC}(S) \subseteq \mathcal{C}(S)$ the set of **primitive curves**, meaning that cannot be written as a power of another non-trivial element in $\pi_1(S)$.

Note that, with our definitions, we are always considering **oriented** curves and arcs.

We say that a curve $\gamma \in \mathcal{C}(S)$ is **essential** if it is not represented by a loop around a puncture or a boundary component or a point. Denote by $\mathcal{C}^*(S)$ the set of essential curves, and by $\mathcal{PC}^*(S)$ the set of primitive essential curves.

Curves on a graph Γ are again free homotopy classes of loops on Γ, which are also in correspondence with the conjugation classes in $\pi_1(\Gamma)$. We will also denote them by $\mathcal{C}(\Gamma)$.

There are two main quantities we will be interested in regarding curves, one is the length which we will define later, and the other one is their intersection number.

Definition 1.2.2 (Intersection number)**.** Let $\alpha, \beta \in \mathcal{C}(S)$ be two curves on S, then their **intersection** number is

$$\iota(\alpha, \beta) = \min |\{(t_1, t_2) \in S^1 \times S^1 \mid a(t_1) = b(t_2)\}|,$$

where the minimum runs over all immersed loops $a, b : S^1 \to S$ representing α and β, respectively.

The **self-intersection** number of a curve is defined as $\iota(\alpha) = \frac{1}{2}\iota(\alpha, \alpha)$, see Figure 1.3.

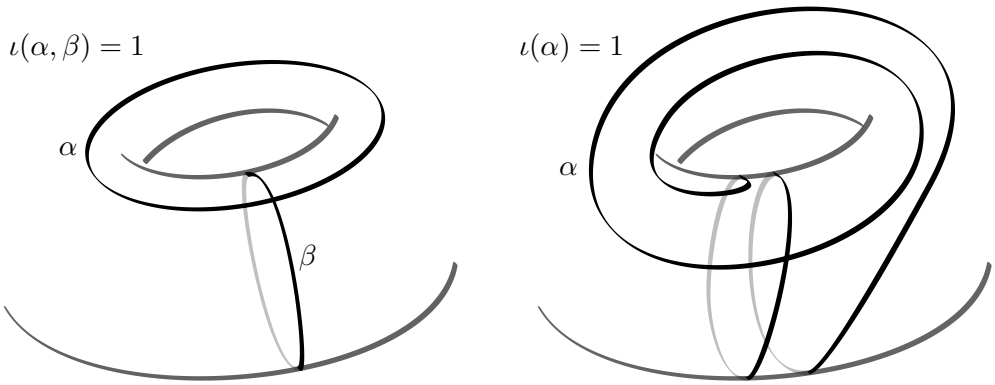


Figure 1.3

Definitions for arcs are analogous by changing S^1 for $[0, 1]$.

Definition 1.2.3 (Mapping class group)**.** For a surface S, its **mapping class group** $\mathrm{Map}(S)$ is the group of orientation preserving homeomorphisms, fixing the boundary of the surface modulo isotopy, i.e.

$$\mathrm{Map}(S) = \mathrm{Homeo}(S, \partial S)^+ / \mathrm{Homeo}_0(S, \partial S).$$

That is, for two homeomorphisms $f, g : S \to S$, define $f \sim g$ if $f \cdot g^{-1}$ is isotopic to the identity, then $\mathrm{Map}(S) = \mathrm{Homeo}(S, \partial S)^+ / \sim$.

This has a natural action on the set of curves, that moreover preserves pairwise intersection numbers.

1.3 Metrics

Definition 1.3.1 (Metric graph)**.** A **metric graph** is a graph Γ equipped with a length function on the edges

$$\ell : E(\Gamma) \to \mathbb{R}_{>0}.$$

This induces a length function on paths and hence on the fundamental group, as the length of an element of the fundamental group is just the sum of the lengths of the edges in a non-backtracking representative.

Definition 1.3.2 (Riemannian surface)**.** A **Riemannian surface** (S, h) is a smooth (see [dC16, page 431]) surface S equipped with a Riemannian metric h, i.e. for each point $p \in S$, assign a positive-definite inner product on the tangent space

$$h_p : T_pS \times T_pS \to \mathbb{R}$$

that varies smoothly on the point p. For the rest of the thesis, we consider every metric to be complete with respect to the exponential map. That is: every geodesic path that will be defined later either continues for infinite time or hits a boundary component.

This leads to the notion of (sectional) curvature at every point (see [dC16, page 448]). We are especially interested in the case where we restrict this curvature.

Definition 1.3.3 (Hyperbolic surface)**.** A **hyperbolic surface** is a Riemannian surface (S, h) with constant sectional curvature -1.

This is equivalent to saying that the universal cover of the surface with the lifted metric is the hyperbolic plane, which we will denote by $\mathbb{H}^2$. The hyperbolic plane can be defined as the only simply connected 2-dimensional manifold with constant curvature -1. Hence, one can take coordinate charts of the surface onto the hyperbolic plane.

By abuse of notation, whenever we do not need to make the metric explicit, we will just say that S is a (hyperbolic) Riemannian surface.

We will now define the space of all hyperbolic structures on a surface. Fix $S_{g,n}$ a surface of signature (g, n). A marking on a hyperbolic surface S is a homeomorphism $f : S_{g,n} \to S$, and we call the pair (S, f) to be a **marked hyperbolic surface**. Moreover, two marked hyperbolic surfaces (S, f) and (S', f') are said to be equivalent if there is an isometry $m : S \to S'$ such that $m \circ f$ and f' are isotopic to each other.

Definition 1.3.4 (Teichmuller space)**.** The **Teichmüller space** of a finite-type surface S admitting a hyperbolic metric is denoted by $\mathcal{T}(S)$ and is the set of equivalence classes of marked finite-area hyperbolic surfaces homeomorphic to S.

As for the intuition on the markings, there is an equivalent definition by changing the markings for canonical generating sets of the fundamental group, which will not be defined here and can be read at [Bus92].

A surface of signature (g, n) admits a hyperbolic metric if and only if its Euler characteristic $\chi(S) = 2 - 2g - n$ is negative (see [Bus92]). Hyperbolic metrics turn out to be much more rigid than Riemannian metrics. For dimension at least 3, hyperbolic metrics are completely determined by their topology.

Theorem 1.3.5 (Mostow's rigidity Theorem [Mos68])**.** *If two hyperbolic, finite-volume, complete manifolds of dimension at least 3 have isomorphic fundamental groups, then they are isometric.*

This implies that the analog of the Teichmüller space defined above for surfaces would contain a unique point for manifolds of higher dimension.

For surfaces, rigidity is weaker, making the Teichmüller space of a surface of signature (g, n) a space of real dimension $6g - 6 + 3n$. One way to understand this phenomenon

is by defining the so-called Fenchel-Nielsen coordinates on the Teichmüller space. A sketch is as follows. Start by decomposing the surface into pairs of pants, i.e., three-holed spheres, by cutting through a maximal set of disjoint simple closed curves. Figure 1.4 is a pants decomposition of a surface of signature $(2, 1)$. Two hyperbolic pairs of pants are isometric if and only if the unordered triple given by the lengths of their three boundaries coincide (see [Bus92, Chapter 3]). Therefore, by decomposing a surface into pairs of pants, all hyperbolic metrics on the surface can be expressed in terms of the lengths of the boundaries of the pants decomposition and how you twist them when you glue them, giving the $6g - 6 + 3n$ dimensions. For further explanation see [Hub16].

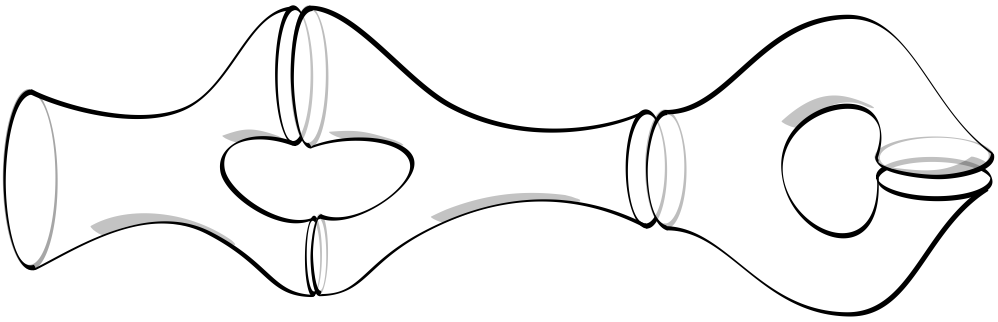


Figure 1.4

Another construction on a surface that will be useful later is the so-called doubled surface.

Definition 1.3.6 (Doubled surface)**.** Let (S, h) be a Riemannian surface with non-empty totally geodesic boundary. Call $(S_1, h_1), (S_2, h_2)$ two copies of (S, h). Let $\gamma_1^i, \dots, \gamma_n^i : S^1 \to S_i$ be pairwise (for $i = 1, 2$) copies of parameterizations of the distinct boundary components of S_i. Define the **doubled surface** (see Figure 1.5) as the surface (S', h') where

$$S' = S_1 \sqcup S_2 / \sim,$$

for $\gamma_j^1(p) \sim \gamma_j^2(p)$ for all $p \in S^1$ and all $j = 1, \dots, n$. The metric h' coincides with h_1 and h_2 on every distinct surface and agrees on the boundary given the totally geodesic condition.

The feature we will be most interested in regarding the metric on a Riemannian surface is that it induces a length function on arcs and loops. Given any Riemannian surface (S, h), one can naturally measure the **length of any loop** on the surface. For a continuous embedding $c : S^1 \to S$, define

$$\ell_h(c) := \int_{S^1} \sqrt{h(\dot{c}(t), \dot{c}(t))} dt.$$

We will denote it by $\ell_X(c)$ whenever we are only considering hyperbolic metrics, with $X \in \mathcal{T}(S)$ a point in the Teichmüller space. The definition for arcs is analogous by

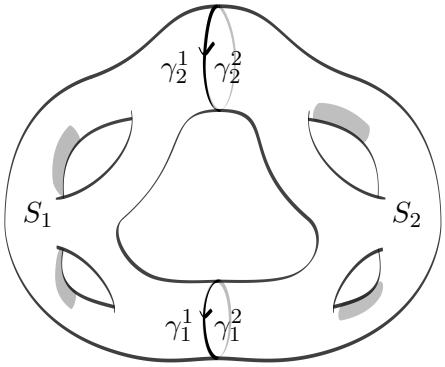


Figure 1.5

changing S^1 for $[0,1] \subseteq \mathbb{R}$. This induces a length function on the fundamental group. A path $a : \mathcal{D} \to S$ with $\mathcal{D}$ a connected domain of $\mathbb{R}$ is called **geodesic** if it is locally distance-minimizing between its points and it is parametrized by its arc-length.

Definition 1.3.7 (Length of curves)**.** Given a Riemannian surface (S, h), we define a length function on its set of curves as

$$\begin{aligned} \ell : \mathcal{C}(S) = \pi_1(S)/\text{conj} &\to \mathbb{R}_{>0} \\ \alpha &\mapsto \inf_{[a]=\alpha} \{\ell_h(a) \mid a : S^1 \to S\}. \end{aligned}$$

The definition for the set of arcs $\mathcal{A}(S)$ is analogous by changing S^1 for $[0,1]$. For a metric graph (Γ, ℓ), by abusing notation calling it also ℓ,

$$\begin{aligned} \ell : \mathcal{C}(\Gamma) = \pi_1(\Gamma)/\text{conj} &\to \mathbb{R}_{>0} \\ \alpha &\mapsto \min_{[a]=\alpha} \{\ell(a) \mid a : S^1 \to \Gamma\}. \end{aligned}$$

For surfaces, whenever the infimum is attained by a loop, it is done by a geodesic loop in its homotopy class. Moreover, in the hyperbolic case, there is a unique geodesic loop (or arc) in any homotopy class due to the surface being modeled on the hyperbolic plane $\mathbb{H}^2$, see e.g. [FM12, Proposition 1.3]. In fact, this phenomenon is true for any metric with strictly negative curvature. In this case, the length of the homotopy class of the curve will be attained by the hyperbolic length of the unique geodesic representing it, see e.g. [Bus92, Theorem 1.5.3].

Definition 1.3.8 ((Simple) length spectrum and orthospectrum)**.** Define the **length spectrum** of a Riemannian surface S to be the set of lengths (with multiplicity) of its curves $\mathcal{C}(S)$, its **simple length spectrum** to be the analog for simple curves, and its **orthospectrum** to be the set of lengths with multiplicity of its arcs $\mathcal{A}(S)$. We denote the length spectrum, simple length spectrum, and orthospectrum by $\mathcal{S}(S), \mathcal{SS}(S), \mathcal{O}(S)$, respectively.

Here, with multiplicity means that if there are $n \geq 1$ curves $\gamma_1, \ldots, \gamma_n \in \mathcal{C}(S)$ with $\ell(\gamma_1) = \cdots = \ell(\gamma_n) = l$, then the positive number l appears n times in $\mathcal{S}(S)$.

One of the main reasons for studying the rigidity of the length spectra is the following classical theorem of Huber.

Theorem 1.3.9 ([Hub59])**.** *Two finite-type hyperbolic surfaces have the same Laplace spectrum if and only if they have the same area and the same length spectrum.*

Also note that, as a consequence of Gauss-Bonet, every hyperbolic surface of signature (g, k) has area $(4g - 4 + 2k)\pi$, giving even more rigidity to hyperbolic structures.

However, that is not the only length we are interested in for the fundamental group. The above is the most naturally associated with a Riemannian metric, however, when only considering topological data, we will also be interested in another type of length.

Definition 1.3.10 (Word-length)**.** Let S be a surface. Fix a generating set $G \subseteq \pi(S)$. Call the **word-length** the following length function on curves,

$$\ell_\omega : \mathcal{C}(S) \to \mathbb{Z}_{\geq 0},$$

such that, for $\gamma \in \mathcal{C}(S)$, $\ell_\omega(\gamma) = n$ if and only if it can be represented by a cyclically reduced word of n letters in $G \cup G^{-1}$. Here, if $G = \{a_1, \ldots, a_n\}$, then $G^{-1} = \{a_1^{-1}, \ldots, a_n^{-1}\}$.

1.4 Some metric invariants

The length function on a metric graph Γ derives a distance function on its vertices $V(\Gamma)$. Two distinct vertices $x, y \in V(\Gamma)$ are at distance $d_\Gamma(x, y) > 0$ if the shortest path joining them has length $d_\Gamma(x, y)$, where the distance of a path is the sum of the lengths of the edges it goes through with multiplicity. One can also measure the distance from a vertex $x \in V(\Gamma)$ to an edge $e \in E(\Gamma)$ by saying that $d_\Gamma(x, e) \leq R \geq 0$ if both endpoints of e are at distance at most R from x.

Consequently, we can define a ball of centre $x \in V(\Gamma)$ and radius $R \geq 0$ as

$$B(x, R) = \{y \in V(\Gamma), e \in E(\Gamma) \mid d_\Gamma(x, y) \leq R, d_\Gamma(x, e) \leq R\}.$$

Define $\ell(B(x, R))$ as the sum of the lengths of the edges in the set.

For Riemannian surfaces, distance is as usual defined as the minimum of the length of the paths joining two points. Then balls are the sets of all points with bounded distance from the central point.

The above allows us to define the exponential volume growth rate of the universal cover of our objects. This will be the main tool for the second part of the thesis.

Definition 1.4.1 (Volume entropy)**.** Let Γ be a metric graph, its **volume entropy** is the quantity $h(\Gamma)$ defined as follows. Fix a point $\tilde{x}$ in $\tilde{\Gamma}$, its universal covering tree with the lifted length. Then,

$$h(\Gamma) := \lim_{R\to\infty} \frac{\log \ell(B(\tilde{x}, R))}{R},$$

where $B(\tilde{x}, R)$ is the ball of radius R centered at $\tilde{x}$.

Let S be a compact Riemannian surface, its **volume entropy** is the quantity $h(S)$ defined as follows. Fix a point $\tilde{p}$ in $\tilde{S}$, the universal cover of S. Then,

$$h(S) = \lim_{R\to\infty} \frac{\log \mathsf{Area}(B(\tilde{p}, R))}{R}.$$

This quantity does not depend on the basepoint chosen (see e.g. [Lim08] and [Man79]). In Figure 1.6 there is a sketch of how the balls on the universal cover grow for a hyperbolic once-punctured torus and a trivalent metric graph.

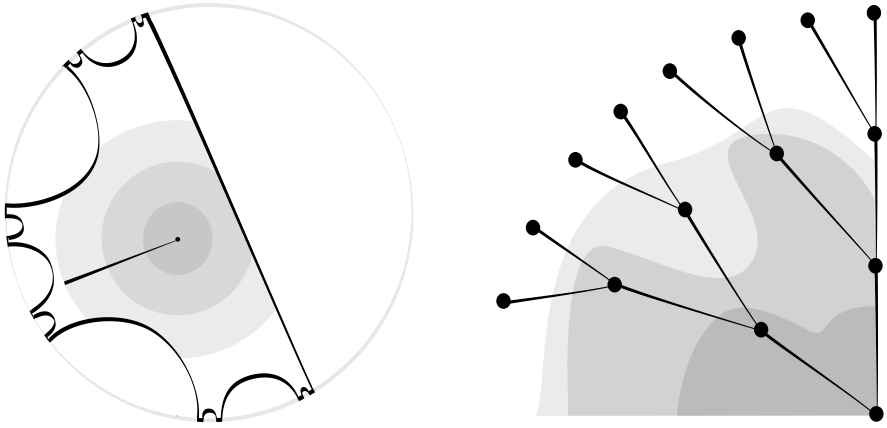


Figure 1.6

This quantity is in general a lower bound for the topological entropy of the geodesic flow, which can explain its name.

Theorem 1.4.2 ([Man79])**.** *Let S be a compact Riemannian surface, $h(S)$ its volume entropy, and $h_{top}(S)$ the topological entropy of its geodesic flow. Then,*

$$h(S) \leq h_{top}(S),$$

and the two are equal when S has non-positive curvature.

The volume entropy can be equally defined for any finite simplicial complex X endowed with a piecewise smooth Riemannian metric, which englobes the definition for surfaces and graphs. This leads to the following characterization of the volume entropy in terms of the fundamental group.

Proposition 1.4.3 ([Sab06]). *Let X be a finite simplicial complex with a piecewise smooth Riemannian metric. For any point $x \in X$,*

$$h(X) = \lim_{R\to\infty} \frac{1}{R} \log \#\{\alpha \in \pi_1(X, x) \mid \ell(\alpha) \leq R\}.$$

This straightforwardly implies that the volume entropy of a surface or graph coincides with the critical exponent of its fundamental group.

Definition 1.4.4 (Critical exponent). Let Γ be a subgroup of isometries of a Riemannian manifold M. Consider the following series for any $s > 0$, and $x \in M$,

$$\sigma_s(x) = \sum_{\gamma\in\Gamma} e^{sd(x,\gamma x)}.$$

The **critical exponent** of Γ is the real number $\delta(\Gamma) \geq 0$ such that $\sigma_s(x)$ converges when $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$. Note that it does not depend on the choice of basepoint.

An alternative definition of the critical exponent is, in the above situation,

$$\delta(\Gamma) = \limsup_{L\to\infty} \frac{1}{L} \log |\{\gamma \in \Gamma \mid d(x, \gamma x)\},$$

where the quantity again does not depend on the choice of the basepoint.

Finally, we will define the so-called systole. It is usually referred to as the length of the shortest curve on a surface.

Definition 1.4.5 (Systole). The **systole** of a Riemannian surface S, denoted $\mathrm{sys}(S)$, is the infimum of the length of the non-contractible curves with positive length, i.e.

$$\mathrm{sys}(S) = \inf_{\alpha\in\mathcal{C}(S)} \{\ell(\alpha) \mid \ell(\alpha) > 0\},$$

This will be attained on compact surfaces by the shortest non-contractible geodesic loop.

In hyperbolic surfaces, there are many phenomena that prevent the systole from growing too much with a fixed topology. For example, the Collar Lemma.

Theorem 1.4.6 (Collar Lemma, [Kee74]). *Let S be a hyperbolic surface. Let $\gamma_1, \gamma_2 \in \mathcal{C}(S)$ with $\iota(\gamma_1) = 0$ and $\iota(\gamma_1, \gamma_2) > 0$. Then*

$$\sinh(\ell(\gamma_1)/2) \cdot \sinh(\ell(\gamma_2)/2) > 1.$$

The hyperbolic rigidity leads to a result on the systole as in Theorem 8. However, nothing prevents the systole from exploding in the moduli space of Riemannian metrics

on a given surface. That is how in Theorem 10 the volume entropy of the surface comes into play, giving a measure of how far away is the Riemannian metric from being hyperbolic.

1.5 Combinatorial objects

Many combinatorial objects will be key to the first part of the thesis. Let us define them and fix the notation.

Definition 1.5.1 (Word and circular word)**.** Given a dictionary, i.e. a finite set $\mathcal{D} = \{a_1, \dots, a_n\}$, we call a **word** ω to be a finite sequence $\omega = (\omega_1, \dots, \omega_n)$ where $\omega_i \in \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$ for all $i = 1, \dots, n$. A **circular word** is a class of words in the quotient by cyclic shifts, denoted by $[\omega] = [\omega_1, \dots, \omega_n]$.

We will be very interested in circular words as when fixing the dictionary to be a generating set of the fundamental group of the once-punctured torus, the cyclically reduced words correspond to closed curves. However, to ease up the notation, whenever we are interested in circular words where the dictionary is made of integers, we will call them necklaces of integers.

Definition 1.5.2 (Necklace)**.** A **necklace of integers**, or simply a **necklace**, is an equivalence class of a finite sequence of positive integers $(n_i)_i$ where two sequences are equivalent if they differ by a circular shift.

We use "$(\dots)$" to denote sequences, and use "$[\dots]$" to denote necklaces. These will appear encoding the circular words of a certain form by only giving their exponents, making them easier to work with.

Let ω be a circular word or necklace. Denote the **size** of ω, i.e. number of symbols in a word or necklace ω, by $|\omega|$, and for any $a \in \mathcal{D}$, the **number of occurrences** of a symbol a in ω by $|\omega|_a$.

Remark 1.5.3. We will often see necklaces as a (finite) sequence of positive numbers written on a circle.

However, there is another classical notion of necklace in combinatorics that will naturally appear in one of the results of the first part.

Definition 1.5.4 (Necklace of beads)**.** A **necklace of beads** is a circular word such that every symbol represents a color in a dictionary of colors $\mathcal{D}$. Note that in this case, every symbol in the word is a color, and no inverses are allowed.

When not written explicitly "necklace of/with beads", "necklace" will always refer to a necklace of integers.

Moreover, we call a circular word or a necklace to be **periodic** if it can be represented by a power by composition of a word or finite sequence, respectively.

Two classical arithmetic functions that will appear often in our counting are the following.

Definition 1.5.5 (Euler's totient function and Möbius function)**.** Let **Euler's totient function** be $\varphi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ the function counting, for a given $n > 0$, how many relatively prime to n natural numbers there are in the set $\{1, \ldots, n\}$. Define also the **Möbius function** $\mu : \mathbb{Z}_{>0} \to \{-1, 0, 1\}$ as, for any $n > 0$,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ different primes} \\ 0 & \text{otherwise.} \end{cases}$$

To give an idea of the growth of Euler's function, a classical bound is the following. For any $\varepsilon > 0$, there is a big enough integer $n > 0$ such that

$$n^{1-\varepsilon} < \varphi(n) < n.$$

The above Möbius function has been proven to be very effective in terms of counting after knowing the non-periodic elements because of the following classical result proved in the first instance by Möbius.

Theorem 1.5.6 (Möbius inversion formula)**.** *Let $f, g : \mathbb{Z}_{>0} \to \mathbb{C}$ be two arithmetic functions satisfying that for any positive integer n, $g(n) = \sum_{d|n} f(d)$. Then, for any positive integer n,*

$$f(n) = \sum_{d|n} \mu(d) g(n/d).$$

In particular, the special case of this identity that relates Möbius function to Euler's totient function and that will be very useful for us later is that, for any integer $n > 0$,

$$\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d}.$$

Chapter 2

Word-length curve counting on the once-punctured torus

2.1 Introduction

This chapter is part of a joint work with Mingkun Liu [FL24]. To facilitate independent reading, we will briefly present again the results we will prove. We classify closed curves on a once-punctured torus with a single self-intersection from a combinatorial perspective. We determine the number of closed curves with a given word-length and with zero, one, and arbitrary self-intersections. That is, in this chapter we will give a new combinatorial proof for the already-known Theorem A, and prove Theorems B, D and C, and Corollaries E and F from the introduction.

Throughout this chapter, let $S = \Sigma_{1,1}$ be a (topological) torus with a puncture. Recall that free homotopy classes of oriented closed curves on S naturally correspond to conjugacy classes of the fundamental group $\pi_1(S)$. We will tactically identify each free homotopy class of closed curves with its representatives, shifting freely between these two viewpoints, and henceforth referring to both as *curves*. Let γ be a curve in S, denoted by $\gamma \in \mathcal{C}(S)$. Recall that γ is *essential* if it is not freely homotopic to a point or a loop around the puncture.

Recall that γ is *primitive* if its free homotopy class is not a proper power of another class, and we denote by $\mathcal{PC}(S) \subset \mathcal{C}(S)$ the set of primitive essential curves, and $\mathcal{PC}^*(S) \subset \mathcal{C}^*(S)$ the set of all primitive curves. The (geometric) intersection number of γ, denoted by $\iota(\gamma)$, is the minimum number of self-intersections among all curves within its free homotopy class:

$$\iota(\gamma) \coloneqq \frac{1}{2} \min_c \{(t_1, t_2) \in \mathbb{S}^1 \times \mathbb{S}^1 \mid c(t_1) = c(t_2), t_1 \neq t_2\}$$

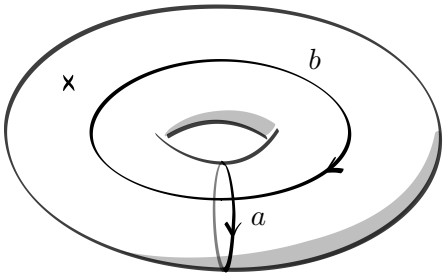


Figure 2.1: A once-punctured torus.

where c runs over the set of loops on S in the free homotopy class γ. Curves with 0 self-intersection are commonly referred to as *simple* curves.

The fundamental group $\pi_1(S)$ is a free group of rank 2. By choosing a pair (a, b) of generators for the fundamental group (see Figure 2.1 for an example), we can identify $\pi_1(S)$ with the free group $\mathbf{F}_2$ generated by a and b. A *word* is a finite sequence $(x_i)_{i=1}^n$ such that $x_i \in \{a, b, a^{-1}, b^{-1}\}$ for all $1 \leq i \leq n$. A *circular word* can be seen as a word written on a circle, or more formally, a circular word is an equivalence class of words where two words are equivalent if they differ by a circular shift. A word $(x_i)_i^n$ is called *reduced* if $x_{i+1} \neq x_i^{-1}$ for all $i = 1, \ldots, n-1$, and $x_1 \neq x_n^{-1}$. A circular word is *reduced* if one (or all) of its representatives are reduced. In this note, we consider only reduced (circular) words. Words can be seen as elements in $\mathbf{F}_2$, and circular words can be seen as conjugacy classes in $\mathbf{F}_2$. Curves in $\mathcal{C}(S)$ are in one-to-one correspondence with reduced circular words. We say that a circular word is *primitive* if none of its representatives is a proper power of another word. Primitive curves correspond to primitive reduced circular words.

We say that a curve $\gamma \in \mathcal{C}(S)$ has *word length* n, denoted by $\ell_\omega(\gamma) = n$, if it can be represented by a reduced word of n letters.

2.1.1 Results

We will again go through the results that will be proved in this chapter. The main result is a characterization of all figure-eight curves on the once-punctured torus, an analog theorem to Buser–Semmler's Theorem 2.2.5 for simple curves.

Theorem 2.1.1 (Theorem C)**.** *A primitive curve in $\mathcal{PC}(S)$ has self-intersection one if and only if, up to renaming the generators in $\{a, b, a^{-1}, b^{-1}\}$, it can be written as one of the following:*

1. a^2b^2, $aba^{-1}b$, $ab^{-1}a^{-1}b^2$, *or*
2. $ab^{-1}a^{-1}ba^{n_1}b\cdots a^{n_k}b$, *or* $ab^{-1}a^{-1}ba^{-n_1}b\cdots a^{-n_k}b$, *where the words* $a^{n_1}b\cdots a^{n_k}b$,

and $a^{-n_1}b\cdots a^{-n_k}b$ *are uniquely determined representatives of primitive simple curves, or*

3. $a^{n_1}b\cdots a^{n_k}b$*, where* $[n_1,\dots,n_k]$ *satisfies that exists an* $m\in\mathbb{Z}_{\geq 1}$ *such that for all* $i\in\{1,\dots,k\}$*,* $n_i\in\{m,m+1\}$ *and it is a necklace with* 2*-variation (see Definition 2.3.4), or*

4. $a^mba^{m+2}b$*, for some* $m\in\mathbb{Z}_{\geq 1}$*.*

The non-necessarily primitive curves can be found by adding the primitive curves and the squares of simple curves. The above classification allows us to do the exact counting of all curves with given word-length and with a single self-intersection.

Theorem 2.1.2 (Theorem B)**.** *There are* 8 *primitive closed curves on* S *of length* 4 *with* 1 *self-intersection. For any* $L\in\mathbb{Z}_{>4}$*, we have*

$$|\{\gamma\in\mathcal{PC}(S)\mid \iota(\gamma)=1,\,\ell_\omega(\gamma)=L\}|=\begin{cases}8\,\varphi(L-4) & \text{if } L \text{ is odd,}\\ 8\big(\varphi(L-4)+\varphi(L/2)/2\big) & \text{if } L \text{ is even,}\end{cases}$$

where φ *stands for Euler's totient function.*

Remark 2.1.3. It was brought to the attention of the authors, after finding this counting as a consequence of the classification, that this result was known by Moira Chas and a different proof was sketched in their Preprint [Cha16, Proposition 3.2].

Corollary 2.1.4 (Corollary E)**.** *Under the same assumptions, the expressions sum up to*

$$|\{\gamma\in\mathcal{PC}(S)\mid \iota(\gamma)=1,\,\ell_\omega(\gamma)\leq L\}|=8(\Phi(L-4)+\Phi(\lfloor L/2\rfloor)/2),$$

where $\Phi(L)$ *denotes the summation up to* L *of Euler's totient function, and, asymptotically,*

$$|\{\gamma\in\mathcal{PC}(S)\mid \iota(\gamma)=1,\,\ell_\omega(\gamma)\leq L\}|\sim\frac{27}{\pi^2}L^2.$$

Proving Theorem 2.1.2 from 2.1.1 is done via developing combinatorial tools on the necklaces of positive integers appearing in the classification of curves. These tools will be developed by doing the same procedure for the case with no self-intersection. Theorem 2.1.1 is an analog to Buser–Semmler's Theorem 2.2.5 for simple curves. We will elaborate from it a new combinatorial proof of the counting theorem for curves with no self-intersection (Theorem 2.1.5). This theorem was already known by experts as a consequence of the homotopy-homology correspondence on the once-punctured torus (see e.g. [MR95a]), our contribution will only be a new complete proof of combinatorial nature.

Theorem 2.1.5 (Theorem A)**.** *For any* $L\in\mathbb{Z}_{\geq 4}$*, we have*

$$|\{\gamma\in\mathcal{PC}(S)\mid \iota(\gamma)=0,\,\ell_\omega(\gamma)=L\}|=4\varphi(L),$$

where φ stands for Euler's totient function.

Again, the result also allows the counting up to a given bounded length.

Corollary 2.1.6 (Corollary E)**.** *For $L \in \mathbb{Z}_{\geq 4}$, we have*

$$|\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = 0,\, \ell_\omega(\gamma) \leq L\}| = 4\Phi(L) + 2, \quad \textit{where} \quad \Phi(L) \coloneqq \sum_{n=1}^{L} \varphi(n),$$

and asymptotically,

$$|\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = 0,\, \ell_\omega(\gamma) \leq L\}| = \frac{12}{\pi^2}\, L^2 + \mathrm{O}(L(\log L)^{\frac{2}{3}}(\log\log L)^{\frac{4}{3}}).$$

This estimate for the error term is the best known and is due to Walfisz, for further study on this see [OS20, Theorem 1.3].

A simple multicurve is a formal sum of pairwise non-homotopic primitive simple curves with positive integer coefficients. On S, the set of multicurves is nothing but $\mathcal{C}(S)$. The last result allows us to count multicurves on S, which the authors found missing in the literature.

Corollary 2.1.7 (Corollary E)**.** *For $L \in \mathbb{Z}_{\geq 4}$, we have*

$$|\{\gamma \in \mathcal{C}(S) \mid \iota(\gamma) = 0,\, \ell_\omega(\gamma) = L\}| = 4L$$

and

$$|\{\gamma \in \mathcal{C}(S) \mid \iota(\gamma) = 0,\, \ell_\omega(\gamma) \leq L\}| = 2L^2 + 2L.$$

The above countings can be put together into the following consequence.

Corollary 2.1.8 (Corollary F)**.** *We have*

$$\lim_{L\to\infty} \frac{|\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = 0,\, \ell_\omega(\gamma) \leq L\}|}{|\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = 1,\, \ell_\omega(\gamma) \leq L\}|} = \frac{4}{9}.$$

Here the word length ℓ_ω can be replaced by the hyperbolic length induced by any complete hyperbolic metric on S (see the section "Related work"), and this corollary can be interpreted as follows: on S, the probability that a random curve with at most one self-intersection has one self-intersection is $9/13$.

Lastly, with different methods, we count the curves with given word length without restriction on the self-intersection.

Theorem 2.1.9 (Theorem D)**.** *There are 4 primtive curves of length 1, 8 of length 2, and*

for any $L \in \mathbb{Z}_{\geq 3}$, we have formula

$$|\{\gamma \in \mathcal{PC}^*(S) \mid \ell_\omega(\gamma) = L\}| = \frac{1}{L} \sum_{d|L} \mu(d)\, 3^{L/d},$$

where μ is the Möbius function. For not necessarily primitive curves, we have for any $L \in \mathbb{Z}_{\geq 1}$,

$$|\{\gamma \in \mathcal{C}^*(S) \mid \ell_\omega(\gamma) = L\}| = \frac{1}{L} \sum_{d|L} \varphi(d)\, 3^{L/d} + \frac{3 + (-1)^L}{2}.$$

Corollary 2.1.10 (Corollary E)**.** *We have*

$$|\{\gamma \in \mathcal{PC}^*(S) \mid \ell_\omega(\gamma) = L\}| \sim \frac{3^L}{L}, \qquad |\{\gamma \in \mathcal{PC}^*(S) \mid \ell_\omega(\gamma) \leq L\}| \sim \frac{3}{2} \cdot \frac{3^L}{L}$$

and the same holds true if $\mathcal{PC}^*$ is replaced by $\mathcal{C}^*$.

Remark 2.1.11. Hence, for $L \in \mathbb{Z}_{\geq 1}$,

$$|\{\gamma \in \mathcal{PC}^*(S) \mid \ell_\omega(\gamma) = L\}| = \\ = |\{\text{aperiodic necklaces with } L \text{ beads and } 3 \text{ colors}\}| + \delta_{\{1,2\}}(L),$$

where $\delta_{\{1,2\}}(L) = 1$ if $L \in \{1, 2\}$ and vanishes otherwise, and

$$|\{\gamma \in \mathcal{C}^*(S) \mid \ell_\omega(\gamma) = L\}| = |\{\text{necklaces with } L \text{ beads and } 3 \text{ colors}\}| + \epsilon(L),$$

where $\epsilon(L) = 1$ if L is odd and $\epsilon(L) = 2$ if L is even.

Even when knowing this numerical coincidence, the authors could not find straightforwardly any natural bijection between these sets. However, it remains an open question, whose answer would possibly give automatically the counting on higher genus.

2.1.2 Related work

Curve counting problems have been extensively studied, especially in the context of hyperbolic geometry. More precisely, given a complete hyperbolic metric X on a topological surface $\Sigma_{g,n}$ of genus g with n punctures, in each free homotopy class of essential curves, there exists a unique geodesic representative. Therefore, rather than using the word length, we can also define the length of a free homotopy class $\gamma \in \mathcal{C}(\Sigma_{g,n})$ by the length $\ell_X(\gamma)$ of its geodesic representative induced by X. The famous prime number theorem for geodesics asserts that

$$|\{\gamma \in \mathcal{PC}(\Sigma_{g,n}) \mid \ell_X(\gamma) \leq L\}| \sim \frac{\mathrm{e}^L}{L}. \tag{2.1.1}$$

This was an achievement initiated in the mid-20th century, and the names of Delsarte, Hejhal, Huber, Margulis, Selberg, and Sarnak feature most strongly. The estimate (2.1.1) can be made effective, and the error terms are related to the Laplacian spectrum of X; see, for example, [Ber16, Section 5.4.2], [Bus92, Section 9.6].

One can also count curves under more topological constraints. For example, one may restrict consideration to curves with no self-intersection, or a given number of self-intersections. Efforts in this direction (see for example [Ree81, BS85, MR95b, Riv01]) reached their greatest heights with Mirzakhani's groundbreaking work [Mir08]. As a consequence of her findings, there exist explicit constants $C_{g,n} > 0$ depending only on g and n and $B_X > 0$ depending only on the hyperbolic metric X such that

$$|\{\gamma \in \mathcal{PC}(\Sigma_{g,n}) \mid \iota(\gamma) = 0,\, \ell_X(\gamma) \leq L\}| \sim C_{g,n} \cdot B_X \cdot L^{6g-6+2n}. \tag{2.1.2}$$

This result was extended to the non-simple case by Mirzakhani twelve years later [Mir16]: for any $k \in \mathbb{Z}_{\geq 0}$ there exists a constant $C_{g,n;k}$, such that (2.1.2) remains valid when replacing "$\iota(\gamma) = 0$" by "$\iota(\gamma) = k$" and $C_{g,n}$ by $C_{g,n;k}$.

It turns out that the length function $\ell_X : \mathcal{PC}(\Sigma_{g,n}) \to \mathbb{R}$ defined by the hyperbolic metric X is not essential: Erlandson and Souto [ES22, Theorem 1.2] proved that a very similar result holds for any "positive, continuous and homogeneous function on the space of geodesic currents on $\Sigma_{g,n}$". More precisely, for any such a function ℓ, which can be taken as ℓ_ω (see [Erl19]) or ℓ_X, there exist constants $C_{g,n;k}$ depending only on g, n and k, and B_ℓ depending only on the function ℓ, such that

$$|\{\gamma \in \mathcal{PC}(\Sigma_{g,n}) \mid \iota(\gamma) = k,\, \ell(\gamma) \leq L\}| \sim C_{g,n;k} \cdot B_\ell \cdot L^{6g-6+2n}. \tag{2.1.3}$$

As a result, the following limit

$$\lim_{L\to\infty} \frac{|\{\gamma \in \mathcal{PC}(\Sigma_{g,n}) \mid \iota(\gamma) = 0,\, \ell(\gamma) \leq L\}|}{|\{\gamma \in \mathcal{PC}(\Sigma_{g,n}) \mid \iota(\gamma) = 1,\, \ell(\gamma) \leq L\}|}$$

exists and does not depend on the choice of length function ℓ. Finally, let us mention that the estimate (2.1.2) can also be made effective; see [MR95b, AEM22, AH21].

In terms of counting curves with a given word-length and self-intersection, there are many works by Chas, Phillips, Lalley, and McMullen; see [CP10, CL12, Cha15, MCP19]. Many bounds have been found for the general cases and closed formulas for given length-intersection difference. The main question in this regard is to classify all words of a given self-intersection and find a closed formula for

$$|\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = k, \ell_\omega(\gamma) = L\}|.$$

Organization of the chapter

Section 2.2 of the paper will first introduce in our notation the characterization of simple closed curves by [BS88], study its combinatorial rigidity, and then prove Theorem 2.1.5 with it. Section 2.3 will start by doing the analogous characterization of curves, but this time targeting self-intersection one, hence proving Theorem 2.1.1. The last part of the section will prove Theorem 2.1.2 by using the combinatorics already studied in Section 2.2. Finally, in Section 2.4 we use different methods to prove Theorem 2.1.9, which will be analytic combinatorics, giving the broadest image of our counting.

2.2 Simple curves

In this section, we study simple curves, namely closed curves without self-intersection, on the once-punctured torus S. As an application of the techniques we are about to develop, we provide an alternative proof for Theorem 2.1.5.

2.2.1 Curves as necklaces

As we will soon see, the following seemingly unrelated definition will simplify our discussion.

Definition 2.2.1 (Necklace)**.** A *necklace of integers*, or simply a *necklace*, which can be seen as a (finite) sequence of positive numbers written on a circle, is an equivalence class of a finite sequence of positive integers $(n_i)_i$ where two sequences are equivalent if they differ by a circular shift.

We use "$(\dots)$" to denote sequences, and use "$[\dots]$" to denote necklaces. For instance, $(1,2,3)$, $(2,3,1)$, and $(3,1,2)$ are different sequences but represent the same necklace $[1,2,3]$.

Definition 2.2.2 (Small variation)**.** A necklace $[n_i]_i$ has *small variation* if, for any $s \in \mathbb{Z}_{\geq 1}$, sums of s consecutive elements never differ by more than ± 1. In symbols, this means

$$\left| \sum_{j=1}^{s} n_{i_1+j} - \sum_{j=1}^{s} n_{i_2+j} \right| \leq 1 \tag{2.2.1}$$

for all i_1, i_2, and indices are taken modulo k.

For example, the necklaces $[5,5,5,5]$, $[5,5,5,4]$, and $[5,4,5,4]$ have small variation, but $[5,5,3]$, $[5,5,4,4]$, and $[5,5,4,5,5,5,4,5,4]$ do not.

The following definitions are standard. Let w be a necklace and $m \in \mathbb{Z}$. We denote by $|w|_m$ the number of occurrences of m in w. A sequence is called a *block* of w if it can be found as a contiguous subsequence within a sequential representation of w. For example, $(1, 2)$ and $(4, 1)$ are blocks of $[1, 2, 3, 4]$, but $(1, 4)$ is not. A *run* is a constant block that is not properly contained in any constant block. For instance, there are 3 runs of 2 in $[2, 1, 2, 1, 2, 1, 2, 2]$ (two of length 1 and one of length 3).

Remark 2.2.3. Let $w = [n_i]_i$ be a non-constant necklace. If (2.2.1) holds for $s = 1$, then there exists $m \in \mathbb{Z}$ such that $n_i \in \{m, m + 1\}$ for all index i. If (2.2.1) holds for $s = 2$ too, then m (resp. $m + 1$) cannot appear consecutively if $|w|_m \leq |w|_{m+1}$ (resp. $|w|_{m+1} \leq |w|_m$).

Lemma 2.2.4. *Let $m, x, y \geq 1$, and let w be a necklace with small variation that contains exactly x occurrences of the number m, and y occurrences of the number $m + 1$. Write $q \coloneqq \max(x, y)/\min(x, y)$. If $x \geq y$, then all runs of m in w have size $\lfloor q \rfloor$ or $\lceil q \rceil$, and all runs of $m + 1$ in w have size 1. If $x \leq y$, then all runs of $m + 1$ have size $\lfloor q \rfloor$ or $\lceil q \rceil$, and all runs of m have size 1.*

Proof. Without loss of generality, we assume that $x \geq y$. It follows from (2.2.1) by taking $s = 2$ that all runs of $m + 1$ have size 1. If there exists a run of m of size at most $\lfloor q \rfloor - 1$, then there is a run of m of size at least $\lfloor q \rfloor + 1$, and it follows from (2.2.1) by taking $s = \lfloor q \rfloor + 1$ the necklace does not have small variation. Similarly, if there exists a run of m of size at least $\lceil q \rceil + 1$, then the necklace does not have small variation. □

One of the main reasons why we are interested in necklaces with small variation is the following.

Theorem 2.2.5 ([BS88, Theorem 6.2])**.** *Every simple closed curve on S can be represented, after suitably renaming the generators, by one of the following words:*

1. a,
2. $aba^{-1}b^{-1}$,
3. $ab^{n_1}ab^{n_2} \cdots ab^{n_r}$, *where $[n_1, \ldots, n_r]$ has small variation.*

Conversely, each of these words is homotopic to a power of a simple closed curve.

Remark 2.2.6. Finite words can be seen as periodic infinite words. The small variation condition on the exponents is nothing but the balance condition on words that defines finite Sturmian words in the literature (see for example [Vui03, GJ09]).

We say that a simple closed curve on S has *general type* if it falls into the third case described in Theorem 2.2.5. For such a curve, we associate it with the necklace

$[n_1, \ldots, n_r]$, which we will refer to as its *exponent necklace*. See Figure 2.2a and 2.2b for an example.

Write $M(L)$ for the set of closed multicurves of general type on S of length L, and $N(L)$ for the set of necklaces $[n_1, \ldots, n_r]$ such that $r + \sum_i n_i = L$.

Corollary 2.2.7. *For L odd, $M(L)$ and $N(L)$ are in eight-to-one correspondence. For L even, we have $|M(L)| = 8|N(L)| - 4$.*

Proof. First, it is not hard to check that a power of a necklace (the square of $[1, 2, 3]$ is $[1, 2, 3, 1, 2, 3]$) with small variation has small variation. Thus every power of a (primitive) simple curve of general type is of general type. The eight-to-one correspondence in the case where L is odd arises from the 8 possible renamings of the generators. This correspondence breaks down when L is even because the necklace $[1, \ldots, 1]$ corresponds to only 4 curves (for example, $ab \cdots ab$ and $ba \cdots ba$ give the same curve). □

Therefore, counting simple closed curves on S with respect to the word length can be boiled down to the enumeration of necklaces with small variation. In the next section, we will prove a rigidity result for such necklaces, which allows us to further reduce our curve counting problem to a problem of lattice point counting.

2.2.2 Necklaces with small variation

The following is the main result of the section. That is, from the periodic Sturmian word perspective, the equivalence in definitions as balanced words mentioned before in Remark 2.2.6 and as cutting sequences on the integer grid for an infinite ray starting at the origin with rational slope. This fact is well-known to the experts, however, as far as the knowledge of the authors gets, there seems not to be any good reference on this fact and a proper bijection. In what follows we give a self-contained elementary proof.

Proposition 2.2.8. *Given $m \in \mathbb{Z}_{\geq 1}$, and x,y $\in \mathbb{Z}_{\geq 0}$, there exists a unique necklace with small variation that contains exactly x occurrences of the number m, and y occurrences of the number $m + 1$.*

The proof proceeds by induction, where the operations on necklaces we are defining now play an important role.

Let us start with the following automorphisms of $\mathbf{F}_2$ defined for any $m \in \mathbb{Z}_{\geq 1}$ by

$$\alpha_m : \begin{array}{ll} a^m b & \mapsto b, \\ a^{m+1} b & \mapsto a, \end{array} \qquad \tilde{\alpha}_m : \begin{array}{ll} a^m b & \mapsto a, \\ a^{m+1} b & \mapsto b. \end{array} \tag{2.2.2}$$

They are well-defined because $a^m b$ and $a^{m+1} b$ form a basis of $\mathbf{F}_2$.

The following elementary lemma will be important for our purposes.

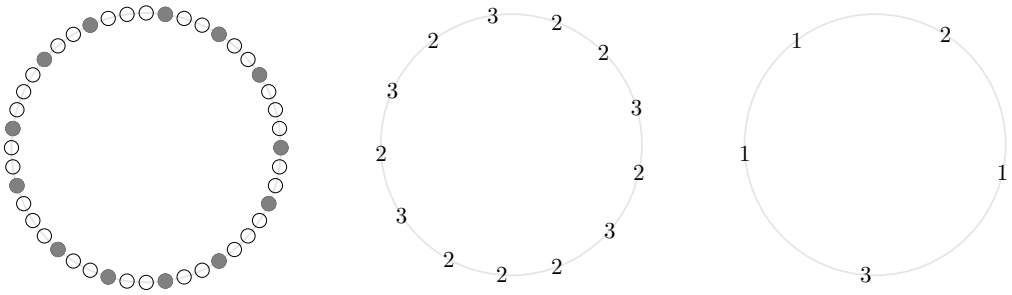
Lemma 2.2.9. *For any word w and any $m \in \mathbb{Z}_{\geq 1}$, the curves represented by w, $\alpha_m(w)$, $\tilde{\alpha}_m(w)$, $\alpha_m^{-1}(w)$, and $\tilde{\alpha}_m^{-1}(w)$ have the same self-intersection number.*

Proof. The automorphisms α_m and $\tilde{\alpha}_m$ preserve the set of conjugacy classes that correspond to the puncture (the conjugacy class of $aba^{-1}b^{-1}$ and the conjugacy class of its inverse $bab^{-1}a^{-1}$). Now the Dehn–Nielsen–Baer theorem implies that the actions of α_m and $\tilde{\alpha}_m$ on conjugacy classes of $\mathbf{F}_2 \simeq \pi_1(S)$ (circular words and free homotopy classes of curves) are induced by some self-homeomorphisms of S. The lemma follows. □

Let w be a necklace with small variation. Recall that we write $|w|_m$ for the number of occurrences of m in w. We define a new necklace $A(w)$ as follows. By Remark 2.2.3, there exists $m \in \mathbb{Z}_{\geq 1}$ such that $n_i \in \{m, m+1\}$ for all i. If $|w|_m \leq |w|_{m+1}$ (resp. $|w|_{m+1} \leq |w|_m$), we define $A(w)$ to be the necklace obtained by removing all the m's (resp. $(m+1)$'s) from w and replacing each run of $m+1$ (resp. m) by the length of the run. For example, $A[4,5,5,4,5,5,5,4,5,5] = [2,3,2]$ and $A[4,5,5,5,4,5,4,5,5] = [3,1,2]$. Note that if $|w|_m = |w|_{m+1}$, then $A(w)$ is a constant necklace $[1,\dots,1]$ of size $|w|_m = |w|_{m+1}$.

This operation can be defined equivalently as follows. Let $w = [n_1,\dots,n_r]$, and consider the word $\omega = a^{n_1}b\cdots a^{n_r}b$. If $|w|_m \leq |w|_{m+1}$ (resp. $|w|_{m+1} \leq |w|_m$), then $A(w)$ is the exponent necklace of $\alpha_m(\omega)$ (resp. $\tilde{\alpha}_m(\omega)$).

With certain supplementary information, the map A can be reversed. We define $B_m(w)$ (resp. $\tilde{B}_m(w)$) to be the necklace obtained by replacing each n_i by a run of $m+1$ (resp. m) of length n_i and inserting a m (resp. $m+1$) between every two consecutive runs of $m+1$ (resp. m). For example, $B_3[1,1,3,2] = [4,3,4,3,4,4,4,3,4,4,3]$. In other words, if $|w|_m \leq |w|_{m+1}$ (resp. $|w|_{m+1} \leq |w|_m$), then $B_m(w)$ is the exponent necklace of $\alpha_m^{-1}(\omega)$ (resp. $\tilde{\alpha}_m^{-1}(\omega)$). Now, by construction, if w is a necklace with small variation, then $B_m(A(w)) = w$ if $|w|_m \leq |w|_{m+1}$, and $\tilde{B}_m(A(w)) = w$ if $|w|_m \geq |w|_{m+1}$.



(a) A circular word (b) Its exponent necklace (c) Image of A

A necklace $w = [n_i]_i$ is said to have *profile* (m,x,y), if $n_i \in \{m, m+1\}$ for all i, $|w|_m = x$, and $|w|_{m+1} = y$.

Note now that by Lemma 2.2.4, if $\min(x,y)$ divides $\max(x,y)$, then Proposition 2.2.8 follows immediately.

Lemma 2.2.10. *Let $m, x, y \in \mathbb{Z}_{\geq 1}$, and w_1, w_2 be two necklaces with the same profile (m, x, y). Then $A(w_1)$ and $A(w_2)$ have the same profile (m', x', y') where*

$$\begin{aligned} m' &= \lfloor \max(x,y)/\min(x,y) \rfloor, \\ x' &= \min(x,y) - \max(x,y) + \min(x,y)\lfloor \max(x,y)/\min(x,y) \rfloor, \\ y' &= \max(x,y) - \min(x,y)\lfloor \max(x,y)/\min(x,y) \rfloor, \end{aligned} \tag{2.2.3}$$

which depends only on (x, y). Moreover, we have $0 \leq \min(x', y') < \min(x, y)$.

Proof. Let (m', x', y') be the profile of $A(w_1)$. Note that m' is determined by Lemma 2.2.4, the total size of $A(w_1)$ is given by construction by $\min(x, y)$ and the sum of all the elements in $A(w_1)$ is given by construction by $\max(x, y)$. Hence, (m', x', y') is determined by the system

$$m' = \lfloor \max(x,y)/\min(x,y) \rfloor, \quad x' + y' = \min(x,y), \quad x'm' + y'(m'+1) = \max(x,y).$$

These equations have a unique solution given by (2.2.3). A direct computation shows that

$$x' \leq \min(x,y), \qquad y' \leq \min(x,y)$$

and $x' = \min(x, y)$ if and only if $\min(x, y)$ divides $\max(x, y)$, in which case $y' = 0$ and hence $y' < \min(x, y)$. The lemma follows. □

Now, we are ready for the main result of the section.

Proof of Proposition 2.2.8. The assertion is evident if $\min(x, y) = 0$. Assume $x, y \geq 1$. We proceed by induction on $\min(x, y)$. Suppose that the proposition holds for all triples $(m, x, y) \in \mathbb{Z}^3_{\geq 1}$ where $\min(x, y) \leq k$. We will prove that the proposition holds for all $(m, x, y) \in \mathbb{Z}^3_{\geq 1}$ such that $\min(x, y) = k + 1$.

By Lemma 2.2.10 and the induction hypothesis, there exist a unique necklace w' with profile (m', x', y'), defined by (2.2.3). By Lemma 2.2.9, if $x \geq y$ (resp. $x \leq y$), then $B_m(w')$ represents a simple curve with profile (m, x, y). This proves the existence. The uniqueness follows from Lemma 2.2.10 the fact that A is injective (by Lemma 2.2.9). This completes the proof. □

2.2.3 Counting

We will start by counting the curves of general type, which means it can be represented by a word of the form $ab^{n_1}ab^{n_2}\cdots ab^{n_r}$, and then at the rest of the curves. Note that such a curve is determined by its exponent necklace $w = [n_1, \ldots, n_r]$.

Now, if $n_i \in \{m, m+1\}$, the curve $\gamma = ab^{n_1}ab^{n_2}\cdots ab^{n_r}$ has word length $|w|_m(m+1) + |w|_{m+1}(m+2)$.

Proposition 2.2.11. *For any positive integer L, there are exactly $4(L-1)$ simple multicurves of general type and length L.*

To prove so we first prove a proposition on our Diophantine equations.

Proposition 2.2.12. *Consider the equation*

$$x(m+1)+y(m+2)=L \tag{2.2.4}$$

where $L \in \mathbb{Z}_{\geq 1}$ is given, and $x, m \in \mathbb{Z}_{\geq 1}$, $y \in \mathbb{Z}_{\geq 0}$ are unknown. Then, there are exactly $L/2$ solutions (x,y,m) if L is even, and $(L-1)/2$ solutions if L is odd.

Proof. Let $S(L)$ denote the set of triples $(m,x,y) \in \mathbb{Z}_{\geq 1}^2 \times \mathbb{Z}_{\geq 0}$ satisfying (2.2.4). Since $|S(2)| = 1$, it suffices to prove that for any $L \in \mathbb{Z}_{\geq 2}$, $|S(L+1)| = |S(L)|$ if L is even, and $|S(L+1)| = |S(L)|+1$ if L is odd. The strategy is to show that there exists a map $\Lambda_+ \colon S(L) \to S(L+1)$ such that, when L is even, Λ_+ is a bijection, and when L is odd, Λ_+ is an injection and $|\Lambda_+(S(L))| = |S(L+1)| - 1$.

Let us define $\Lambda_+ \colon \mathbb{Z}^3 \to \mathbb{Z}^3$ by the formula

$$\Lambda_+(x,y,m) \coloneqq \begin{cases} (x-1, y+1, m) & \text{if } x \geq 2, \\ (y+1, 0, m+1) & \text{if } x = 1. \end{cases}$$

A direct computation shows that $\Lambda_+(S(L)) \subset S(L+1)$. Now consider the map $\Lambda_- \colon \mathbb{Z}^3 \to \mathbb{Z}^3$ defined by

$$\Lambda_-(x,y,m) \coloneqq \begin{cases} (x+1, y-1, m) & \text{if } y \geq 1, \\ (1, x-1, m-1) & \text{if } y = 0. \end{cases}$$

Again, a straightforward computation shows that $\Lambda_-(S(L+1)) \subset S(L)$, and the composition $\Lambda_- \circ \Lambda_+ \colon S(L) \to S(L)$ is the identity. In particular, the restriction of Λ_+ on $S(L)$ is an injection. Now, observe that $(x,y,m) \in S(L+1)$ and $\Lambda_-(x,y,m) \notin S(L)$ if and only if $(y,m) = (0,1)$, and there exists $x \in \mathbb{Z}_{\geq 1}$ such that $(x,0,1) \in S(L+1)$ if and only if $L+1$ is even. This completes the proof. □

Proof of Proposition 2.2.11. This follows immediately from Corollary 2.2.7, Proposition 2.2.8 and 2.2.12. □

Now we are ready to prove the simple curve counting result.

Proof of Theorem 2.1.5. Define

$$M(n) \coloneqq |\{\gamma \in \mathcal{C}(S) \mid \iota(\gamma) = 0,\ \gamma \text{ is of general type},\ \ell_\omega(\gamma) = n\}|. \tag{2.2.5}$$

By Proposition 2.2.11, $m(n) = 4(n-1)$. Define also

$$P(n) \coloneqq |\{\gamma \in \mathcal{PC}(S) \mid \iota(\gamma) = 0,\ \gamma \text{ is of general type},\ \ell_\omega(\gamma) = n\}|.$$

By definition, $M(n) = \sum_{d|n} P(d)$. Applying the Möbius inversion formula we obtain

$$P(n) = \sum_{d|n} \mu(d)\, M(n/d)$$

where $\mu(d)$ stands for the Möbius function. Using the arithmetic identities

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\varphi(n)}{n}, \qquad \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2, \end{cases}$$

where φ is Euler's totient function, we obtain for $n \geq 1$,

$$P(n) = 4\varphi(n) - 4\delta_{\{1\}}(n),$$

where $\delta_{\{1\}}(n) = 0$ for $n > 1$ and $\delta_{\{1\}}(1) = 1$. After summing rewrites as:

$$\sum_{n=1}^{L} P(n) = 4\Phi(L) - 4.$$

Adding the four essential simple curves of non-general type: a, b, a^{-1}, b^{-1}, the first part of the theorem follows.

For the second part, by Proposition 2.2.11, we know that the number of powers of simple closed curves of general type with word length exactly L is $M(L) = 4(L-1)$. Hence,

$$\sum_{n=1}^{L} M(n) = \sum_{n=1}^{L} 4(n-1) = 2(L^2 - L).$$

Adding the words of type a^k, b^k, a^{-k}, b^{-k} for $k = 1, \ldots, L$, the theorem follows. □

2.3 Self-intersection one

In this section, we classify self-intersection one curves and derive a formula for the number of such curves of a given length.

2.3.1 Characterization

The aim of this section is to prove Theorem 2.1.1.

One of the main tools for the section will be the algorithm to determine self-intersection proposed by Cohen and Lustig [CL87] built on the work of Birman and Series [BS84]. We will shortly introduce our notation on it right now.

Let $\omega = x_1 \cdots x_n$, where $x_i \in \{a, b, a^{-1}, b^{-1}\}$ be a reduced word. Consider all of its circular shifts $\{\omega_i\}_{i=1}^n$ where $\omega_i = x_i x_{i+1} \cdots x_n x_1 x_2 \cdots x_{i-1}$ and the cyclic lexicographic ordering induced by $a < b < a^{-1} < b^{-1}$: two words $\alpha = x_1 \cdots x_n$ and $\beta = y_1 \cdots y_n$ satisfy $\alpha < \beta$ if either $x_1 < y_1$, or for some $1 < t \leq n$, we have $x_i = y_i$ for $1 \leq i \leq t-1$, and $x_t < y_t$ under the new ordering obtained by cyclically shifting the original one until it starts with x_{t-1}^{-1}.

For example, $baab^{-1} < baab$. We will call $(i, j) \in \{1, \ldots, n\}^2$ a *linking pair* if $\omega_i < \omega_j < \omega_i^{-1} < \omega_j^{-1}$ or $\omega_i < \omega_j^{-1} < \omega_i^{-1} < \omega_j$. Consider the set of linking pairs with the equivalence relation induced by $(i, j) \sim (i+1, j+1)$ if ω_i and ω_j start with the same letter and same sign, and $(i+1, j) \sim (i, j+1)$ if ω_i and ω_j start with the same letter and opposite sign. Cohen and Lustig proved that the self-intersection number of the curve represented by ω equals the number of equivalence classes of linking pairs found this way. We refer the reader to [CL87] for more details.

Proposition 2.3.1. *Let $\omega = a^{n_1} b^{m_1} \cdots a^{n_k} b^{n_k}$ be a reduced word representing a curve with a single self-intersection. If there exist $i, j \in \{1, \ldots, k\}$ such that $|n_i|, |m_j| \geq 2$, then up to renaming of the generators and circular shift, the only possible ω is $a^2 b^2$.*

Proof. Take a word that writes reduced as $\omega_1 a^{n_1} \omega_2 b^{n_2} \omega_3$ with $n_1 = n_2 = 2$ and $\omega_1, \ldots, \omega_3$ being subwords, at least one of them being non-empty. This word is the image by the "cross-corner surgery" described in [CP10] of a word $\omega_1^{-1} a b^{-1} \omega_2^{-1} a^{-1} b \omega_3$. By [CP10, Proposition 2.2] this surgery increases self-intersection by at least one, and by Theorem 2.2.5, the initial word did not represent a simple curve, hence, self-intersection of the image is at least two. Same proceeding applies for $n_i = -2$ just by switching a for a^{-1}, b for b^{-1} or both. □

Hence, from now on we can assume that one of the generators has only exponents $\{-1, 1\}$. Following then,

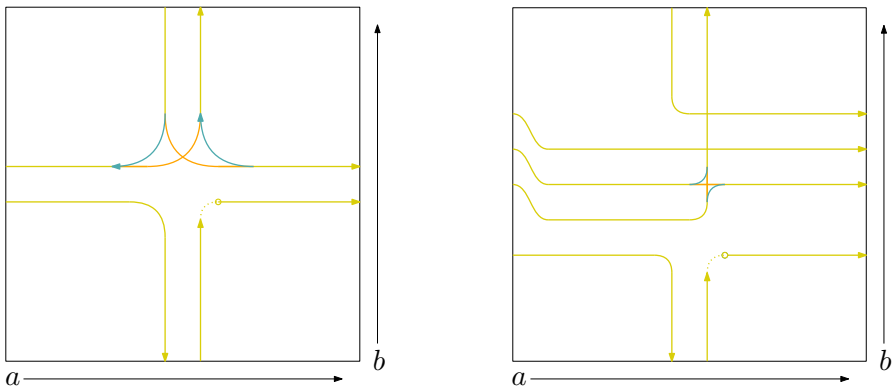
Proposition 2.3.2. *Let ω be a word representing a curve with a single self-intersection of the form $a^{n_1} b^{\epsilon_1} \cdots a^{n_k} b^{\epsilon_k}$ where $\epsilon_i \in \{-1, 1\}$ for all i and $m_i m_j = -1$ for some i, j. Then, ω is, up to renaming the generators and circular shift, of the form*

$$ab^{-1}a^{-1}b \cdot a^{m_1} b \cdots a^{m_r} b \qquad \text{or} \qquad ab^{-1}a^{-1}b \cdot a^{-m_1} b \cdots a^{-m_r} b \tag{2.3.1}$$

where $a^{m_1} b \cdots a^{m_r} b$ represents a primitive simple curve, with the short exceptional cases $ab^{-1}ab, b^{-1}a^{-1}ba^2$. Conversely, every word representing a primitive simple curve

of the form $a^{m_1}b\cdots a^{m_r}b$ can be circularly shifted to a unique word σ such that $ab^{-1}a^{-1}b\cdot\sigma$ represents a curve with a single self-intersection.

Proof. We will characterize all possible changes of sign. Start by considering a reduced word of the form $ab^{-1}a^ibaw$, with $i \geq 1$ and ω being a subword. By applying a surgery as in Figure 2.3b, if $i \geq 2$, the word will lose one self-intersection and become $ab^{-1}a^{i-1}ba\omega$



(a) $i = 1$ (b) $i \geq 2$

Figure 2.3: Surgery on a word of the form $ab^{-1}a^ibaw$: remove the orange part and add the blue part.

(coming in the linking pair notation as losing the linking pair given by the cyclic shifts of the word: $\omega_1 = ba\omega ab^{-1}a^i$ and $\omega_2 = aba\omega ab^{-1}a^{i-1}$). And, for $i = 1$, the word also loses a self-intersection and becomes $ab^{-1}a^{-1}b\omega$ (in linking pairs notation loses the linking pair given by the two cyclic shifts of the word: $\omega_1 = ab^{-1}a^iba\omega$ and $\omega_2 = aba\omega ab^{-1}a^{i-1}$). Now, by the classification of simple closed curves, this word is simple if and only if $\omega = \emptyset$, hence we find that the only candidate of the form $ab^{-1}a^iba\omega$, with $i \geq 1$, for self-intersection one is $ab^{-1}aba$, which indeed can be checked by the algorithm in [CL87] to have a single self-intersection and will correspond to one of the exceptional short cases.

For the rest of the cases, one can assume that when there is a change of sign in b, there is also a change of sign in a. In particular, since renaming all a for a^{-1} (and respectively with b) does not change the self-intersection number, we will consider words of the form $ab^{-1}a^{-i}b\omega$, with $i \geq 2$, and ω being a subword.

Again, by applying a surgery as in Figure 2.4, if $i \geq 2$, the word will lose one self-intersection and become $ab^{-1}a^{-i+1}ba\omega$ (corresponding to losing the linking pair coming from the permutations of the word $\omega_1 = b\omega ab^{-1}a^{-i}$ and $\omega_2 = a^{-1}b\omega ab^{-1}a^{-i+1}$). Once again, by the classification of simple words, if $\omega \neq \emptyset$ this will never be simple and hence the initial word does not have a single self-intersection, whilst if $\omega = \emptyset$, this will be simple if and only if $i = 2$, in which case the only possible candidate for a single self-intersection is the word $ab^{-1}a^{-2}b$, which can be checked by the algorithm in [CL87] to

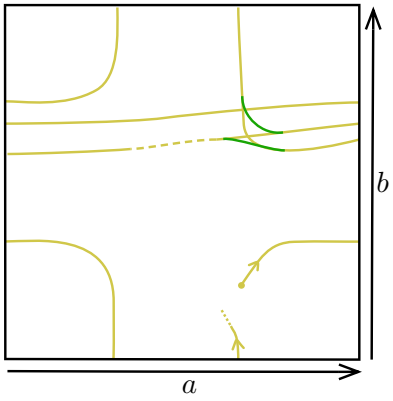


Figure 2.4: Surgery on a word of the form $ab^{-1}a^{-i}b\omega$ with $i \geq 2$.

be indeed of self-intersection one, giving the other short exceptional case.

Hence, we can restrict to the case where the changes of sign come from subwords of the kind $ab^{-1}a^{-1}b$ (up to renaming a for a^{-1} or b for b^{-1}). Hence, take a reduced word $ab^{-1}a^{-1}b\omega$ with ω a subword such that the exponents on b are ± 1. Note that if ω starts with an a, up to homotopy, one finds the first situation in Figure 2.5, meaning that the surface is divided into two regions with the startpoint of the curve in one of them and the continuation in the other one. Thus, the curve will intersect at some point the arcs given by the commutator $ab^{-1}a^{-1}b$, and so $\iota(\omega) \leq \iota(ab^{-1}a^{-1}b\omega) - 1$. Therefore, ω has to represent a simple word starting with a. Moreover, ω has to be of the form $a^{n_1}b\cdots a^{n_k}b$, as if it was of the form $a^{n_1}b^{-1}\cdots a^{n_k}b^{-1}$ the curve $ab^{-1}a^{-1}b\omega$ would contain the subword $ba^{n_1}b^{-1}a^{n_2}$ and the only such a word with self-intersection one is $bab^{-1}a$ as proved above, or if $\omega = ab^{-1}a^{-1}b$, the word would not be primitive.

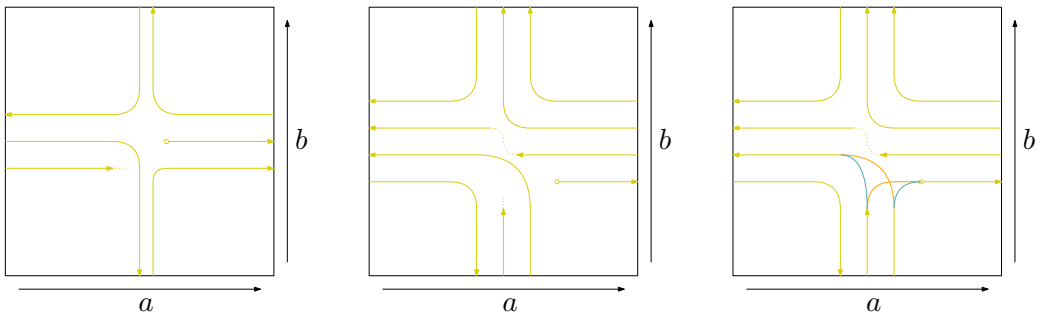


Figure 2.5: Possible words starting by $ab^{-1}a^{-1}b$

On the other hand, if $\omega = a^{-i}b^{\epsilon}\omega'$ with $i \geq 1$ and $\epsilon = \pm 1$, note first again that for $\epsilon = -1$ it is proved above that the only case containing a subword $a^{-1}ba^{-i}b^{-1}$ with a single self-intersection is $a^{-1}ba^{-1}b^{-1}$ and this is not the case. Therefore, $\epsilon = 1$, leading to the second case in Figure 2.5. In this case, all ω has to be of the form $a^{-n_1}b\cdots a^{-n_k}b$, as any change of sign would lead again to the non-possible subword. Therefore, at the

last step, one can perform a surgery as in the third part of Figure 2.5, transforming at one component $ab^{-1}ab\omega$ into ω and lowering the self-intersection by at least 1, hence ω must represent a simple curve.

Lemma 2.3.3. *Let $ab^{-1}a^{-1}b\omega$ be a word with self-intersection one, being ω a simple word of the form $a^{n_1}b\cdots a^{n_k}b$. Then, it will always write as*

$$ab^{-1}a^{-1}ba^{m+1}b(a^mb)^{t_1}\cdots a^{m+1}b(a^mb)^{t_s}$$

with $t_1 < t_s$ if $m > 0$, and as

$$ab^{-1}a^{-1}b(a^{m+1}b)^{t_1}a^mb\cdots(a^{m+1}b)^{t_s}a^mb$$

with $t_1 > t_s$ if $m < -1$.

Proof. Start by writing the word as $ab^{-1}a^{-1}b\omega$ with $\omega = a^{n_1}b\dots a^{n_k}b$ with n_i being positive for all i. The word $ab^{-1}a^{-1}b\omega$ instantly gives a linking pair given the cyclic shifts $\omega_1 = bab^{-1}a^{-1}ba^{n_1}b\cdots a^{n_k}$ and $\omega_2 = ba^{n_1}b\cdots a^{n_k}bab^{-1}a^{-1}$, satisfying $\omega_2^{-1} < \omega_1 < \omega_2 < \omega_1^{-1}$.

Assume now that $n_i \in \{m, m+1\}$ for some $m \geq 1$, as it represents a simple word, and $k > 1$, as otherwise the result is trivial.

Note first that if the curve has self-intersection one, then $n_1 = m+1$ and $n_k = m$: if $n_1 = m$, then exists $j \in \{1,\dots,k\}$ such that $n_j = m+1$ giving another linking pair associated to the permutations $\omega_1 = ba^{n_j}b\cdots a^{n_k}bab^{-1}a^{-1}ba^{n_1}b\cdots a^{n_{j-1}}$ and $\omega_2 = ba^{n_1}b\cdots a^{n_k}bab^{-1}a^{-1}$, rising the total self-intersection to 2. Similarly, if $n_k = m+1$, there is some $j \in \{1,\dots,k\}$ such that $n_j = m$ and so we find a linking pair with the permutations $\omega_1 = a^{n_k-1}bab^{-1}a^{-1}ba^{n_1}b\cdots ba$ and $\omega_2 = a^{n_j}b\cdots a^{n_k}bab^{-1}a^{-1}ba^{n_1}b\cdots a^{n_{j-1}}b$.

Write now the word as $ab^{-1}a^{-1}ba^{m+1}b(a^mb)^{t_1}\cdots a^{m+1}b(a^mb)^{t_s}$.

Finally, $t_1 < t_s$, as otherwise a new linking pair arises with the permutations $\omega_1 = b(a^mb)^{t_s}ab^{-1}a^{-1}ba^{m+1}b(a^mb)^{t_1}\cdots a^{m+1}$ and $\omega_2 = b(a^mb)^{t_1}\cdots a^{m+1}b(a^mb)^{t_s}ab^{-1}a^{-1}ba^{m+1}$.

The negative case can be proved by noting that after renaming generators and shifting cyclically, $ab^{-1}a^{-1}ba^{n_1}b\cdots a^{n_k}b$ can be rewritten as $ab^{-1}a^{-1}bab^{n_1}\cdots ab^{n_k}$ and there is an analog proof for the case with the exponents on b. □

Note now that the maps $\alpha_m, \tilde{\alpha}_m$ defined in (2.2.2) fix the set of conjugacy classes $\{[ab^{-1}a^{-1}b], [(ab^{-1}a^{-1}b)^{-1}]\}$. Hence, considering a reduced circular word of the kind $ab^{-1}a^{-1}b\omega$, such that it has a single self-intersection and such that $\omega = a^{n_1}b\cdots a^{n_k}b$ with $n_i \in \{m, m+1\}$ for some $m \neq 0, 1$ represents a primitive simple word, we can apply the reduction $\alpha_m(ab^{-1}a^{-1}b\omega) = \omega'\alpha_m(\omega)$ if $|\{i \mid n_i = m\}| > |\{i \mid n_i = m+1\}|$ (or with $\tilde{\alpha}_m$ otherwise), where ω' is a cyclically reduced representative of the conjugacy classes of the commutators of a, b. As we are assuming that ω represents a simple word, applying finitely many times maps of the family α_m, $\tilde{\alpha}_m$, this reduction acts as A

on the necklace $[n_1, \dots, n_k]$ defined in Figure 2.2c, and it will reduce to a word of the kind $A^n(\omega) = a^{n'}b$ up to renaming the generators. Therefore, by inversing A with the appropriate B_m's, and given Lemma 2.3.3, after renaming properly the generators such that the representative of the commutator is of the form $ab^{-1}a^{-1}b$, we find that there is a unique permutation of every simple word ω of the form $a^{n_1}b\cdots a^{n_k}b$ or $a^{-n_1}b\cdots a^{-n_k}b$ making $ab^{-1}a^{-1}b\omega$ a self-intersection one curve.

Moreover, applying this reduction to Lemma 2.3.3, one sees that, in order to obtain a self-intersection one curve from a simple one of the form $a^{n_1}b\cdots a^{n_k}b$ with $n_i \in \{m, m+1\}$ for some $m \neq 0, -1$, the commutator $ab^{-1}a^{-1}b$, has to be inserted before a block $a^{n_i}b$ such that $n_i = m+1$, $n_{i-1} = m$ and maximizing as much as possible the beginning of the chain $\{\sum_{j=1}^{s} |n_{i+\epsilon j}|\}_{s\geq 1}$ with $\epsilon = 1$ for positive m and -1 for negative (e.g. see Figure 2.6).

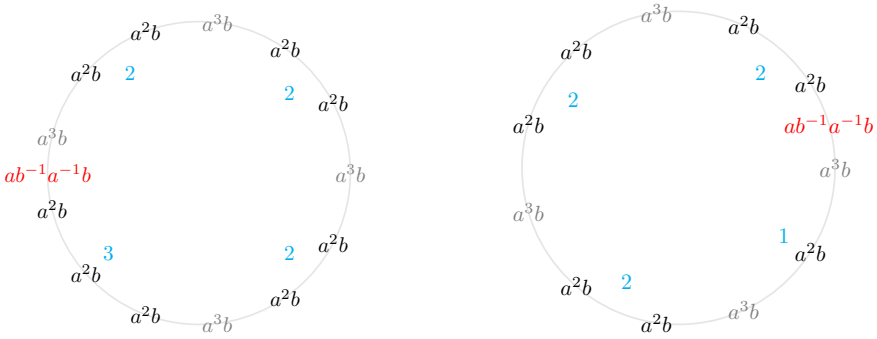


Figure 2.6: Words representing a curve with self-intersection one

Finally, up to generator renaming and circular shifting $ab^{-1}a^{-1}ba^{n_1}b\cdots a^{n_k}b$ generates the same curves as $ab^{-1}a^{-1}ab^{-n_1}\cdots ab^{-n_k}$, whilst $ab^{-1}a^{-1}ab^{n_1}\cdots ab^{n_k}$ generates the same curves as $ab^{-1}a^{-1}ba^{-n_1}b\cdots a^{-n_k}b$, leaving us with the two cases in the statement of this proposition. This concludes the proof. □

Note also that if all the exponents of the letter b are of the same sign, it has been proved already that the only possible word with self-intersection one and with a change of signs in the exponents of a is, up to renaming of the generators, $a^{-1}bab$, which was already considered as an exceptional short case. Thus, now we can assume that there is no change of signs in the exponents and that the exponents of all b's are 1. Hence, up to renaming, we are left with words of the form

$$a^{n_1}ba^{n_2}b\cdots a^{n_k}b$$

with all exponents being positive. For the next proposition, we will need an extra definition, which is analogous to Definition 2.2.2 for a single self-intersection.

Definition 2.3.4 (2-variation)**.** Let $m \in \mathbb{Z}_{\geq 1}$ and $w = [n_i]_i$ be a necklace with $n_i \in$

$\{m, m+1\}$ for all i. We say that a pair of blocks of w of the same size is an *essential pair* if one block is $(m, x_2, x_3, \ldots, x_{k-1}, m)$ and the other block is $(m+1, x_2, x_3, \ldots, x_{k-1}, m+1)$ where $x_i \in \{m, m+1\}$. We say that w has 2-*variation* if, among all pairs of blocks of w, only one is essential.

Remark 2.3.5. Such a name is given after Buser–Semmler's small variation since we consider this to be the smallest case without small variation, hence variation 2: an essential pair of blocks will always look like

$$B_1 = (m, x_2, x_3, \ldots, x_{k-1}, m), \qquad B_2 = (m+1, x_2, x_3, \ldots, x_{k-1}, m+1),$$

with $x_i \in \{m, m+1\}$. Therefore,

$$\left| \sum_{S_1} n_i - \sum_{S_2} n_j \right| = 2,$$

and whenever the small variation condition is broken, there will always exist at least one essential pair of blocks.

Proposition 2.3.6. *The only circular words representing a curve with a single self-intersection of the form $a^{n_1}b\cdots a^{n_k}b$ with all exponents being positive are*

- $a^m b\, a^{m+2} b$ *where* $m \in \mathbb{Z}_{\geq 1}$,
- $a^{n_1}b\cdots a^{n_k}b$ *where the exponent necklace* $[n_1, \ldots, n_k]$ *has* 2*-variation.*

Proof. Let us start by assuming that there are at least two exponents of a with a difference of 3, i.e. there are $i, j, m \geq 1$ such that $n_i = m$, $n_j \geq m+3$. Take then the cyclic shifts of the word $\omega_1 = a^{n_i}b\cdots$, $\omega_2 = a^{n_j-1}ba^{n_{j+1}}\cdots ba$, and $\omega_3 = a^{n_j-2}ba^{n_{j+1}}\cdots ba^2$. By the lexicographic ordering, one has

$$\omega_k < \omega_1 < \omega_k^{-1} < \omega_1^{-1} \quad \text{for} \quad k \in \{2, 3\},$$

which in [CL87] gives 2 linking pairs associated to the pairs (ω_2, ω_1), and (ω_3, ω_1), in different classes, and hence self-intersection of at least 2.

Let us now move to the case where there are $i, j, m \geq 1$ such that $n_i = m$, $n_j = m+2$. Note that, from the existence of such a pair, we can straightforwardly take two cyclic shifts of the word: $\omega_1 = a^{n_i}b\cdots a^{n_{i-1}}b$ and $\omega_2 = a^{n_j-1}ba^{n_{j+1}}b\cdots ba$, given by the same ordering as above $\omega_2 < \omega_1 < \omega_2^{-1} < \omega_1^{-1}$, hence one linking pair giving one self-intersection. Now, if there was a $1 \leq k \neq i, j$ such that $n_k \neq m+1$, by the same procedure we would find an extra self-intersection, giving at least two. Thus, assume now the word is of the form

$$a^{n_1}b\cdots a^{n_k}ba^m ba^{m_1}b\cdots a^{m_{k'}}ba^{m+2}b,$$

with $n_i, m_j = m+1$. If $k' \neq 0$, choosing the cyclic shifts $\omega_1 = a^m b a^{m_1} \cdots a^{n_k} b$ and $\omega_2 = a^{m_{k'}-1} b a^{m+2} \cdots ba$, again one has $\omega_2 < \omega_1 < \omega_2^{-1} < \omega_1^{-1}$ giving another linking pair in a different class, and hence an extra self-intersection, i.e. at least 2. If $k' = 0$ and $k \neq 0$, one can find analogously an extra intersection. Implying then that if there are two exponents with a difference of two, the only candidate with a single self-intersection (up to renaming and cyclic shift) is when $k = k' = 0$, that is $ab^m ab^{m+2}$, and by [CP10, Proposition A.1], indeed has a single self-intersection for every $m \in \mathbb{Z}_{\geq 1}$.

Let us now move to the case where we have a word of the form $a^{n_1} b \cdots a^{n_k} b$ with all $n_i \in \{m, m+1\}$ for some $m \in \mathbb{Z}_{\geq 1}$. We want to prove that it has self-intersection one if and only if the necklace of positive integers $[n_1, \ldots, n_k]$ has 2-variation.

It is enough to see that in this case there is a $1:1$ correspondence between classes of linking pairs in [CL87] algorithm and essential pairs of blocks in the necklace $[n_1, \ldots, n_k]$. We will start by constructing a class of linking pairs in the word given an essential pair of blocks in the associated necklace.

Recall from Remark 2.3.5 such a pair of blocks is always going to be of the form

$$S_1 = \{m, n_{i_1+2}, \ldots, n_{i_1+s-1}, m\}, \qquad S_2 = \{m+1, n_{i_2+2}, \ldots, n_{i_2+s-1}, m+1\},$$

with $n_{i_1+j} = n_{i_2+j} \in \{m, m+1\}$ for $j = 2, \ldots, s-1$. Note that every such pair gives rise to a linking pair on the word $a^{n_1} b \cdots a^{n_k} b$ by taking $\omega_1 = a^m b a^{n_{i_1+2}} b \cdots b$, $\omega_2 = a^m b a^{n_{i_2+2}} b \cdots ba$ with $\omega_2 < \omega_1 < \omega_2^{-1} < \omega_1^{-1}$ with respect to the lexicographic order $a < b < a^{-1} < b^{-1}$. Note also that by this construction every pair of blocks gives rise to a different class of linking pairs, given by the difference in first and last number, and that two linking pairs $(i, j) \sim (i+1, j+1)$ are in the same class if and only if the ith and jth letters of the word representative start by the same letter, being the only equivalence between linking pairs possible in this case. Hence, we found an injection from the set of essential pairs of blocks into self-intersections of the curve represented by the word.

The converse injection is given by the following: take a linking pair (i_1, i_2), that is we have two permutations of the initial word giving

$$\omega_{i_1} < \omega_{i_2} < \omega_{i_1}^{-1} < \omega_{i_2}^{-1}. \tag{2.3.2}$$

Assume that both start with a, then they are of the form $\omega_{i_t} = a^{l_t} b a^{n_{j_t+1}} b \cdots$ and $\omega_{i_t}^{-1} = a^{-(n_{j_t}-l_t)} b^{-1} a^{-n_{j_t-1}} b^{-1} \cdots$ for $t = 1, 2$. Now, first inequality of Equation (2.3.2) implies that either $l_1 \geq l_2 + 1$, or $l_1 = l_2$ and $n_{j_1+r} = n_{j_2+r}$ for $r = 1, \ldots, s-1$ for some s and $n_{j_1+s} = n_{j_2+s} + 1$. Similarly, the last inequality of the equation gives for the other side $n_{j_1} - l_1 \geq n_{j_2} - l_2 + 1$ or $n_{j_1} - l_1 = n_{j_2} - l_2$ and $n_{j_1-r} = n_{j_2-r}$ for $r = 1, \ldots, s'-1$ for some s' and $n_{j_1-s'} = n_{j_2-s'} + 1$. Note that since all $n_i \in \{m, m+1\}$, at most $l_1 = l_2 + 1$ and so in all cases one finds at the extremes two subwords of the form $ba^{m+1} ba^{m_1} b \cdots ba^{m_s} ba^{m+1} b$, $ba^m ba^{m_1} b \cdots ba^{m_s} ba^m b$, that give an essential pair of blocks on the necklace of positive integers.

Moreover, when both words start with a b, then the same argument for $l_1 = l_2 = 0$ applies. Finally, assume they start with different letters. By Equation (2.3.2), ω_{i_1} starts with a and ω_{i_2} with b, i.e. $\omega_{i_1} = a^l b a^{n_{j_1}+1} b \cdots$ for some $0 < l < n_{j_1}$ and $\omega_{i_2} = b a^{n_{j_2}+1} b \cdots$. In this case, $\omega_{i_1}^{-1} = a^{-(n_{j_1}-l)} b^{-1} a^{n_{j_1-1}} b \cdots$ and $\omega_{i_2}^{-1} = a^{-n_{j_2}} b^{-1} a^{n_{j_1-1}} b \cdots$.The first two inequalities of Equation (2.3.2) are automatically true. The third inequality implies $n_{j_1} - l \geq n_{j_2}$, and since they can only differ by one it can only happen if $l = 1$, $n_{j_1} = m+1$, and $n_{j_2} = m$. Then, all $n_{j_1-r} = n_{j_2-r}$ for $r = 1, \ldots, s-1$ until some $n_{j_1-s} = m+1$ and $n_{j_2-s} = m$, for which we get again an essential pair of blocks in the necklace of positive integers. Moreover, note that by the construction above all the linking pairs giving the same essential pair of blocks are in the same class, as again the only possible equivalence is $(i, j) \sim (i+1, j+1)$ are in the same class if and only if the ith and jth letters of the word representative start by the same letter. □

Remark 2.3.7. The proofs of the above propositions have inside all the steps to prove that Theorem 2.2.5 follows from [CL87] algorithm.

2.3.2 Counting

This section is dedicated to counting primitive curves with self-intersection one, i.e. the following proof.

Proof of Theorem 2.1.2. The exceptional cases are for length 4 and 5. For length 4 there are two cases: a^2b^2 and $aba^{-1}b$, which after renaming of generators give us 8 conjugacy classes, i.e. $|\{\gamma \in \mathcal{PC}_1(S \mid \iota(\gamma) = 1,\, \ell_\omega(\gamma) = 4\}| = 8$, and for length 5 we have only the words of type $ab^{-1}a^{-1}b^2$ which give $|\{\gamma \in \mathcal{PC}_1(S) \mid \iota(\gamma) = 1,\, \ell_\omega(\gamma) = 5\}| = 8$.

For the general case of length $L \geq 6$, name $P_1(L)$ the number of words arising from Proposition 2.3.2, $P_2(L)$ the number of words arising from Proposition 2.3.6 of the form $ab^m ab^{m+2}$, and $P_3(L)$ the number of the rest words arising from Proposition 2.3.6. Then, for $L \geq 6$, we will have

$$|\{\gamma \in \mathcal{PC}_1(S) \mid \iota(\gamma) = 1,\, \ell_\omega(\gamma) = L\}| = 8 \cdot (P_1(L) + P_2(L) + P_3(L)),$$

since each of these words will give different conjugacy classes after all the possible renamings of generators $\{a, b, a^{-1}, b^{-1}\}$.

It is straightforward from the proposition that $P_1(L) = 2 \cdot |\{$aperiodic necklaces of positive integers $[n_1, \ldots, n_k]$ with small variation such that $k + \sum_{i=1}^{k} n_i = L - 4\}|$. Hence, from the proof of Theorem 2.1.5, that is

$$P_1(L) = 2 \cdot \sum_{d|(L-4)} \mu(d) \left\lfloor \frac{L-4}{2d} \right\rfloor = \varphi(L-4),$$

where again μ and φ denote the Möbius function and Euler's totient function, respectively.

Now, $P_2(L) = 1$ for even L and vanishes otherwise. Lastly, to count $P_3(L)$, we need to count the number of 2-variation necklaces of positive integers that give a word of length L.

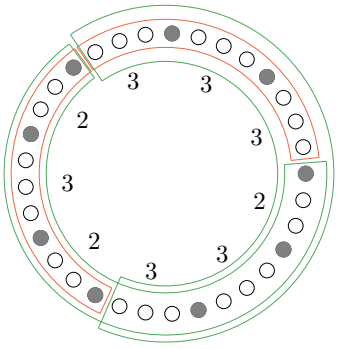
Proposition 2.3.8. *Let $m, x, y \in \mathbb{Z}_{\geq 1}$. If $\gcd(x, y) = 2$, then there exists a unique necklace of integers with 2-variation that contains exactly x occurrences of the number m, and y occurrences of the number $m + 1$. Otherwise, no such necklaces exist.*

Proof. Without loss of generality, we assume throughout the proof that $x \leq y$. Let $k \in \mathbb{Z}_{\geq 1}$, and let $[n_i]_i = [n_1, \ldots, n_k]$ be a necklace such that $n_i \in \{m, m+1\}$ for all $1 \leq i \leq k$, $|\{i \mid n_i = m\}| = x$, and $|\{i \mid n_i = m+1\}| = y$.

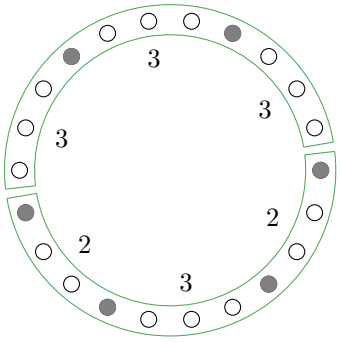
First, we prove that if $x \mid y$ then the necklace $[n_i]_i$ does not have variation 2. If every run of $m+1$ has size y/x, then $[n_i]_i$ has small variation. If there exist two runs of $m+1$ with sizes differing by at least 3 (for example, $y/x - 1$ and $y/x + 2$), then $[n_i]_i$ does not have variation 2. Thus, if $[n_i]_i$ has variation 2, then the sizes of its runs of $m+1$ can only take values in $\{y/x - 1, y/x, y/x + 1\}$, as the sum of the runs have to sum y and there are x of them.

Further, if there is one run of size $y/x + 1$, then there is at least one run of size $y/x - 1$ and there can only be one as otherwise, these give immediately two essential pairs of blocks. Therefore, for a necklace with 2-variation, there is one run of size $y/x - 1$, one of size $y/x + 1$, and $x - 2$ runs of size y/x.

If $x > 2$, then the necklace cannot have 2-variation, as one essential pair of blocks is given straight by the runs with difference 2, i.e. $\{m, m+1, \overset{y/x-1}{\ldots}, m+1, m\}$ and $\{m+1, \overset{y/x+1}{\ldots}, m+1\}$, and another one by extending these sets to an adjacent gap of size y/x (e.g. see Figure 2.7a). When $x = 2$, there are only two runs, of sizes $y/x - 1$ and $y/x + 1$, and one unique possible configuration with these, giving always variation 2 (see Figure 2.7b).



(a) $x = 3, y = 6$,



(b) $x = 2, y = 4$.

Assume from now on $x \nmid y$. First note that for a necklace with variation 2 the sizes of its runs have to take values in $\{\lfloor y/x \rfloor, \lceil y/x \rceil\}$: the existence of a run of size at least $\lceil y/x \rceil + 1$ implies that there should be at most one run of size $\lfloor y/x \rfloor$, as every couple of runs with difference two gives rise to an essential pair of blocks, be of size $\lceil y/x \rceil$. However, there are x runs and so $y = (\lceil y/x \rceil + 1) + \lfloor y/x \rfloor + (x-2)\lceil y/x \rceil = x\lceil y/x \rceil$, contradicting $x \nmid y$. The case for the existence of a run of size at most $\lfloor y/x \rfloor - 1$ is symmetric.

Moreover, the necklace $[n_i]_i$ has 2-variation if and only if the associated run necklace $A[n_i]_i$ described in Figure 2.2c has. This comes naturally from the map A as two blocks of the form $S_1 = \{m, n_{j+2}, \dots, n_{j+s-1}, m\}$ and $S_2 = \{m+1, n_{j+2}, \dots, n_{j+s-1}, m+1\}$ will map to the same sequence of runs with the first and last one being bigger by one at the second case. Conversely, if there is such a sequence of runs, finding the associated pair of blocks in $[n_i]_i$ is straightforward.

Finally, note that by the same computations in Lemma 2.2.10, a necklace of profile (m, x, y) maps to a necklace of profile $(\lfloor y/x \rfloor, x - y + x\lfloor y/x \rfloor, y - x\lfloor y/x \rfloor)$, and by elementary properties $\gcd(x, y) = \gcd(x - y + x\lfloor y/x \rfloor, y - x\lfloor y/x \rfloor)$. Therefore, as the number of appearances in the profile keeps decreasing while they do not divide each other, this will only stop when they do, and as the $\gcd$ is maintained, that will happen when the minimum reaches $\gcd(x, y)$, and as the dividing case has already been proved, there will exist a 2-variation necklace if and only if $\gcd(x, y) = 2$. □

Hence, since a word $ab^{n_1} \cdots ab^{n_k}$ has length $k + \sum_{i=1}^{k} n_i$, $P_3(L)$ will be exactly the number of solutions to the following equation.

Proposition 2.3.9. *Consider the equation*

$$x(m+1) + y(m+2) = L \tag{2.3.3}$$

where $L \in \mathbb{Z}_{\geq 1}$ is given, and $x, y, m \in \mathbb{Z}_{\geq 1}$ are unknown such that $\gcd(x, y) = 2$. Then for even L, there are exactly $\lceil \varphi(L/2)/2 \rceil - 1$ solutions, and none for odd L.

Proof. Let us start by introducing some notation. Define

$$\begin{aligned} S(L) &\coloneqq \{(x, y, m) \in \mathbb{Z}_{\geq 1}^3 \mid x(m+1) + y(m+2) = L\}, \\ S_{>1}(L) &\coloneqq \{(x, y, m) \in S(L) \mid \gcd(x, y) > 1\}, \\ S_1(L) &\coloneqq \{(x, y, m) \in S(L) \mid \gcd(x, y) = 1\}, \\ S_2(L) &\coloneqq \{(x, y, m) \in S(L) \mid \gcd(x, y) = 2\}. \end{aligned}$$

Our objective is to determine $|S_2(L)|$. The set of solutions $S_2(L)$ is empty if L is odd. When L is even, the mapping $(x, y) \mapsto (x/2, y/2)$ defines a bijection from $S_2(L)$ to $S_1(L)$, and hence, we have $|S_2(L)| = |S_1(L/2)|$ for L even. By Proposition 2.2.12, we have

$$|S_{>1}(L)| + |S_1(L)| = \lfloor L/2 \rfloor - \sigma_0(L) + 1,$$

where $\sigma_0(L)$ denotes the number of divisors of L. (Here, we exclude solutions where $y = 0$.) Since any solution $(x, y, m) \in S_{>1}(L)$ corresponds to a solution in $S_1(L/\gcd(x, y))$, we have

$$|S_{>1}| = \sum_{d|L, d\neq 1} |S_1(L/d)|.$$

Therefore, we have

$$\sum_{d|L} |S_1(L/d)| = \lfloor L/2 \rfloor - \sigma_0(L) + 1, \tag{2.3.4}$$

and in particular, we have

$$|S_1(p)| = \lfloor p/2 \rfloor, \qquad \text{for any prime } p. \tag{2.3.5}$$

Now, note that since $S_1(0) = 1$, $|S_1(L)|$ can be uniquely determined by (2.3.4) and (2.3.5) for any $L \in \mathbb{Z}_{\geq 1}$. On the other hand, a direct computation shows that the function defined by $L \mapsto \lceil \varphi(L)/2 \rceil - 1$ maps 0 to 1, and satisfies (2.3.4) and (2.3.5). Hence, for any $L \in \mathbb{Z}_{\geq 1}$, we have

$$|S_1(L)| = \lceil \varphi(L)/2 \rceil - 1$$

and therefore, for any L even, we have

$$|S_2(L)| = |S_1(L/2)| = \lceil \varphi(L/2)/2 \rceil - 1.$$

The proposition follows. □

Concluding, Proposition 2.3.9 implies that $P_3(L) = \lceil \varphi(L/2)/2 \rceil - 1$ for even L, and zero otherwise, hence one gets for $L \geq 4$,

$$\begin{aligned} |\{\gamma \in \pi_1(S) \mid \ell_\omega(\gamma) = L, \iota(\gamma) = 1\}| &= 8 \cdot (P_1(L) + P_2(L) + P_3(L)) \\ &= \begin{cases} 8\,\varphi(L-4) & \text{for odd } L, \\ 8\,(\varphi(L-4) + 1 + \lceil \varphi(L/2)/2 \rceil - 1) & \text{for even } L. \end{cases} \end{aligned}$$

by summing the general cases and checking that it coincides for $L = 4, 5$ with the convention $\varphi(0) = 0$. □

2.4 All curves

In this section, we shall count all closed curves on S of given word length, regardless of their self-intersection numbers.

This result (Theorem 2.1.9) is expected to be known because of its elementary nature. However, despite our search in the literature, we couldn't find a reference. Hence, we provide a complete proof here.

Proof of Theorem 2.1.9. Let $n \in \mathbb{Z}_{\geq 1}$. Let us denote by w_n the number of reduced words in $\{a, b, a^{-1}, b^{-1}\}$ of length n. Define generating functions

$$W(t) \coloneqq \sum_{n=1}^{\infty} w_n t^n.$$

Every word ω under consideration can be written in the form

$$x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

where for all i, $x_i \in \{a, a^{-1}, b, b^{-1}\}$, $n_i \in \mathbb{Z}_{\geq 1}$, and $x_{i+1} \notin \{x_i, x_i^{-1}\}$. We discuss based on the parity of k. If k is even, then the last letter x_k must be b or b^{-1}. Thus

$$\sum_{n=1}^{\infty} w_{2n} t^{2n} = 4 \sum_{i=1}^{\infty} \frac{t}{1-t} \left(\frac{2t}{1-t} \right)^{2i-1} = \frac{8t^2}{-3t^2 - 2t + 1}.$$

This arises from the following reasoning. We assume $x_1 = a$. The exponent n_1 can be any positive integer, giving a factor $t + t^2 + \cdots = t/(1-t)$. Next, n_2 can also be any positive integer, and x_2 can be chosen between b and b^{-1}. This gives a factor of $2t/(1-t)$, and so on and so forth. Finally, x_1 can also be a^{-1}, b, or b^{-1}, which gives a factor of 4.

If k is odd and $k \neq 1$, then we have $x_k = x_1$. So by a similar argument, we have

$$\sum_{n=1}^{\infty} w_{2n+1} t^{2n+1} = 4 \sum_{i=1}^{\infty} \frac{t}{1-t} \left(\frac{2t}{1-t} \right)^{2i-1} \frac{t}{1-t} = \frac{8t^3}{(1-t)(-3t^2 - 2t + 1)}.$$

Therefore,

$$W(t) = \frac{8t^2}{-3t^2 - 2t + 1} + \frac{8t^3}{(1-t)(-3t^2 - 2t + 1)} + \frac{4t}{1-t} = \frac{4t - 12t^3}{(1-t)(-3t^2 - 2t + 1)}$$

where the term $1/(1-t)$ corresponds to the case $k = 1$. This can be rewritten as

$$W(t) = -4 + \frac{2}{1-t} + \frac{1}{1+t} + \frac{1}{1-3t} = \sum_{n=1}^{\infty} \left(2 + (-1)^n + 3^n\right) t^n,$$

and hence, for any $n \in \mathbb{Z}_{\geq 1}$, we have

$$w_n = 2 + (-1)^n + 3^n.$$

Now, the Möbius inversion formula implies that the number of primitive reduced words of length n is equal to

$$\sum_{d \mid n} \mu(d) \left(2 + (-1)^{n/d} + 3^{n/d}\right),$$

and therefore, as primitive words of length n and primitive words of length n are in n-to-1 correspondence, the number of primitive reduced necklaces in $\{a, b, a^{-1}, b^{-1}\}$ is

$$\frac{1}{n}\sum_{d|n}\mu(d)\big(2+(-1)^{n/d}+3^{n/d}\big)=\begin{cases}4 & \text{if } n=1,2,\\ \dfrac{1}{n}\displaystyle\sum_{d|n}\mu(d)\,3^{n/d} & \text{if } n\geq 3\end{cases} \tag{2.4.1}$$

where we have used the arithmetic identities

$$\sum_{d|n}\mu(d)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n\geq 2,\end{cases} \quad \text{and} \quad \sum_{d|n}\mu(d)(-1)^{n/d}=\begin{cases}-1 & \text{if } n=1,\\ 2 & \text{if } n=2,\\ 0 & \text{if } n\geq 3.\end{cases}$$

Now, (2.4.1) imp+lies that (by summing over all factors of n) the number of (not necessarily primitive) reduced necklaces of length n is

$$\frac{3+(-1)^n}{2}+\frac{1}{n}\sum_{d|n}\varphi(d)\,3^{n/d}.$$

This completes the proof. □

2.5 From combinatorics into hyperbolic geometry: Markov's uniqueness conjecture

The aim of this section is to give a glimpse into passing from combinatorial to hyperbolic length and prove Theorem G from the introduction. This is based on an individual work in [Fis25].

In 1913 Fröbenius stated a big conjecture on the set of solutions to a specific Diophantine equation. After many years of effort, the general statement is still open.

Conjecture 2.5.1 (Markov's uniqueness conjecture)**.** Call a triple of positive integers $(a, b, c) \in \mathbb{Z}_{\geq 1}$ a Markov triple if

$$a^2+b^2+c^2=3abc.$$

Then, there are no distinct triples with the same largest number.

There is a well-known geometric conjecture analogous to this one. We should first define shear coordinates on Teichmüller space of the once-punctured torus.

The once-punctured torus can be constructed as a gluing of two ideal triangles. Since all hyperbolic ideal triangles are isometric (see [Bus92]), every hyperbolic metric on

the torus can be described in terms of the gluing of these two triangles. These are called **shear coordinates** and were introduced by Thurston in [Thu98]. Fix an edge e_i on the gluing and fix an orientation on it. On the two triangles T_1, T_2 glued by e_i, consider the orthogonal geodesic going from the vertex at infinity opposite to e_i and hitting orthogonally e_i. Then define the shear coordinate s_i as the signed hyperbolic distance separating the hitting points (see Figure 2.8).

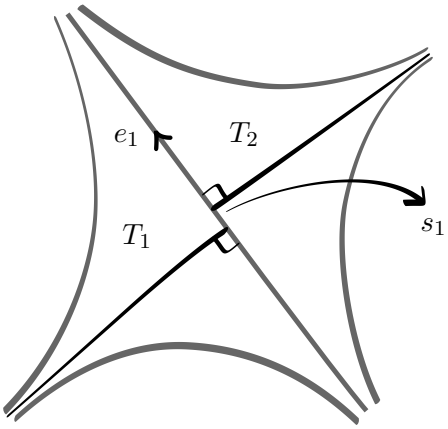


Figure 2.8

The modular torus $\mathcal{M}$ is defined as the hyperbolic once-punctured torus with zero shear coordinates, which is the only hyperbolic once-punctured torus with isometry group of maximal order, being order 12. There is a well-known equivalent conjecture to Markov's uniqueness conjecture in terms of the multiplicity of curves in the modular torus. For further reading on this equivalence, see [MP10]. Recall that $\mathcal{SS}(M)$ denotes the simple length spectrum (Definition 1.3.8).

Conjecture 2.5.2 (Equivalent to Markov's uniqueness conjecture)**.** Let $\ell_1 < \ell_2$ be the two smallest lengths in $\mathcal{SS}(\mathcal{M})$. Then, ℓ_1 and ℓ_2 have multiplicity 6 in $\mathcal{SS}(\mathcal{M})$, and for all $\ell \in \mathcal{SS}(\mathcal{M})_{>\ell_2}$, there are exactly 12 curves attaining this length.

To the given triangulation from the shear coordinates construction, there is an associated dual graph embedded on $\mathcal{M}$ with two vertices of valence 3 and 3 edges. This graph is a spine of the surface, and any closed curve on the surface can then be represented by a circular class of sequences of left/right moves on the graph, following the retraction of the curve to the graph. Starting at any basepoint, at every vertex of the graph the path turns either left or right. This gives a correspondence between curves in $\mathcal{C}(M)$ and circular words in $\{L, R\}$. For example, the curve represented in Figure 2.9 has $[LR]$ as its associated circular word. Note that the choice of basepoint translates to a circular shift in the left/right word.

There is a well-known trace formula in $\mathcal{M}$ relating hyperbolic length and the left/right

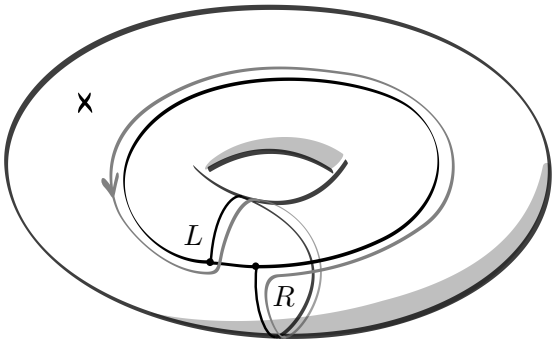


Figure 2.9

words (see e.g. [MS16]). Fix

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let $\gamma \in \mathcal{C}(\mathcal{M})$ and $\omega(\gamma)$ its associated left/right circular word in $\{L, R\}$. Consider $\omega(\gamma)$ as a matrix by the product of the L, R matrices above. Then,

$$\ell_{\mathcal{M}}(\gamma) = 2\,\mathrm{arccosh}(\mathrm{Tr}(\omega(\gamma))/2).$$

Recall that we have fixed canonical generators of the fundamental group on $\mathcal{M}$ as in Figure 2.1. By Buser and Semmler's Theorem 2.2.5, other than the commutators that have zero hyperbolic length and the curves a and b (and their inverses) that correspond to the circular word LR, the rest of the curves are coming from small variation necklaces $[n_1, \ldots, n_k]$. For $[n_1, \ldots, n_k] \neq [1]$, each of these gives rise to eight different curves, which are the following and their inverses:

$$ab^{n_1} \cdots ab^{n_k},\ ba^{n_1} \cdots ba^{n_k},\ ab^{-n_1} \cdots ab^{-n_k}, \text{ and } ba^{-n_1} \cdots ba^{-n_k}.$$

These correspond to the circular words

$$L(LR)^{n_1-1}R \cdots L(LR)^{n_k-1}R,\ R(RL)^{n_1-1}L \cdots R(RL)^{n_k-1}L,$$
$$L(LR)^{n_1}R \cdots L(LR)^{n_k}R, \text{ and } R(RL)^{n_1}L \cdots R(RL)^{n_k}L,$$

from which the first two will have the same trace, and so will the second two. Similarly, for the exceptional necklace $[1]$ there are only four closed curves via renaming the generators, which are ab, ab^{-1} and their inverses, corresponding to the cyclic words in the graph LR and $LLRR$, respectively.

Denote by $\mathcal{N}$ the set of primitive small variation necklaces of positive integers different from $[1]$, and define the real-valued function $\theta(\_) = 2 \cdot \mathrm{acosh}(\mathrm{Trace}(\_)/2)$ on $SL_2(\mathbb{Z})$. We

have

$$\mathcal{SS}(\mathcal{M}) = 6\{\theta(LR)\} \cup 2\{\theta(LLRR)\} \cup 4\{\theta(L(LR)^{n_1}R\cdots L(LR)^{n_k}R) \mid [n_1,\dots,n_k]\in\mathcal{N}\} \\ \cup\, 4\{\theta(L(LR)^{n_1-1}R\cdots L(LR)^{n_k-1}R) \mid [n_1,\dots,n_k]\in\mathcal{N}\},$$

where the scalar multiples denote multiplicity of the set.

The small variation condition is invariant under the choice of the two consecutive integers appearing. Define $\mathcal{N}^*$ as the set of primitive necklaces with all elements in $\{0,1\}$ with small variation, excluding the two simplest necklaces $[0]$ and $[1]$. It follows that

$$\begin{aligned}\mathcal{SS}(\mathcal{M}) =& 8\{\theta(L(LR)^{n_1}R\cdots L(LR)^{n_k}R) \mid [n_1,\dots,n_k]\in\mathcal{N}\}\cup \\ & 4\{\theta(L(LR)^{n_1}R\cdots L(LR)^{n_k}R) \mid [n_1,\dots,n_k]\in\mathcal{N}^*\}\cup \\ & 6\{\theta(LR)\}\cup 6\{\theta(LLRR)\}.\end{aligned} \tag{2.5.1}$$

Moreover, we can prove that a natural bijection between $\mathcal{N}$ and $\mathcal{N}^*$ preserves the trace of the associated word in "L, R".

Lemma 2.5.3. *Let $\eta = [\eta_1,\dots,\eta_l]\in\mathcal{N}^*$. Write it uniquely (up to cyclic shifting) as*

$$\eta = [0,\overset{n_1-1}{\dots},0,1,0,\overset{n_2-1}{\dots},0,1,\dots,0,\overset{n_k-1}{\dots},0,1].$$

Then,

$$\mathrm{Tr}(L(LR)^{\eta_1}R\cdots L(LR)^{\eta_l}R) = \mathrm{Tr}(L(LR)^{n_1}R\cdots L(LR)^{n_k}R).$$

Proof. The lemma follows from elementary properties of the trace. We will write it for convenience of the reader. First of all,

$$L(LR)^{\eta_1}R\cdots L(LR)^{\eta_l}R = (LR)^{n_1-1}LLRR\cdot(LR)^{n_2-1}LLRR\cdot\cdots\cdot(LR)^{n_k-1}LLRR.$$

Applying a double cyclic shift and regrouping one finds

$$RR(LR)^{n_1-1}LLRR\cdot(LR)^{n_2-1}LLRR\cdot\cdots\cdot(LR)^{n_k-1}LL = \\ = R(RL)^{n_1}L\cdot R(RL)^{n_2}L\cdots R(RL)^{n_k}L,$$

which has the same trace as $L(LR)^{n_1}R\cdots L(LR)^{n_k}R$. □

Note that in the bijection

$$\Theta : \{\text{necklace of positive integers}\}\longrightarrow\{\text{necklace of integers in }\{0,1\}\} \\ \eta=[n_1,\dots,n_k]\longmapsto[0,\overset{n_1-1}{\dots},0,1,0,\overset{n_2-1}{\dots},0,1,\dots,0,\overset{n_k-1}{\dots},0,1],$$

it follows from Lemma 2.2.9 that a necklace of positive integers η has small variation if and only if $\Theta(\eta)$ has, as η is the necklace of runs of its image plus 1, ensuring positivity.

Hence, to compute the simple length spectrum of $\mathcal{M}$, we are left with finding what is the trace of $L(LR)^{n_1}R\cdots L(LR)^{n_k}R$, where $[n_1,\dots,n_k]$ are small variation necklaces.

Proposition 2.5.4. *Let $\xi = 3+\sqrt{5}$ and $\overline{\xi} = 3-\sqrt{5}$. For any $n_1,\dots n_k \in \mathbb{Z}_{\geq 0}$,*

$$\begin{aligned}&\mathrm{Trace}(L(LR)^{n_1}R\cdots L(LR)^{n_k}R) = \\ &\quad = \frac{1}{10^k\cdot 2^{n_1+\cdots+n_k}}\sum_{S\subseteq\{1,\dots,k\}} 3^{r(s)}2^{k-r(S)}(\xi+2)^{|S|}(\overline{\xi}+2)^{|S^c|}\xi^{\sum_{i\in S}n_i}\overline{\xi}^{\sum_{i\in S^c}n_i},\end{aligned}$$

where $r(S) = \sum_{s\in\{\text{runs of } S \text{ and } S^c\}}|s|-1$ if $S\neq\{1,\dots,k\}$, and $r(\{1,\dots,k\})=k$. A run s of $S\subseteq\{1,\dots,k\}$ is a maximal subset $s\subseteq S$ such that the numbers are (cyclically) consecutive.

Proof. Note that

$$LR = \begin{pmatrix}2&1\\1&1\end{pmatrix} = S\cdot\frac{1}{2}\begin{pmatrix}\overline{\xi}&0\\0&\xi\end{pmatrix}\cdot S^{-1},$$

where $\xi = 3+\sqrt{5}$, $\overline{\xi}=3-\sqrt{5}$, $S = \frac{1}{2}\begin{pmatrix}\overline{\xi}-2&\xi-2\\2&2\end{pmatrix}$ and $S^{-1} = \frac{1}{10}\begin{pmatrix}2(\overline{\xi}-3)&\xi+2\\2(\xi-3)&\overline{\xi}+2\end{pmatrix}$.

Hence,

$$L(LR)^nR = \frac{1}{2^n}\cdot\tilde{S}\cdot\begin{pmatrix}\overline{\xi}^n&0\\0&\xi^n\end{pmatrix}\cdot\tilde{\tilde{S}},$$

where $\tilde{S} = \frac{1}{2}\begin{pmatrix}\overline{\xi}&\xi\\2&2\end{pmatrix}$ and $\tilde{\tilde{S}} = \frac{1}{10}\begin{pmatrix}\overline{\xi}+2&\xi+2\\\xi+2&\overline{\xi}+2\end{pmatrix}$.

Thus,

$$\begin{aligned}&\mathrm{Tr}(L(LR)^{n_1}R\cdots L(LR)^{n_k}R) = \\ &\qquad = \frac{1}{10^k\cdot 2^{n_1+\cdots+n_k}}\cdot\mathrm{Tr}\left(M\cdot\begin{pmatrix}\overline{\xi}^{n_1}&0\\0&\xi^{n_1}\end{pmatrix}\cdot\cdots\cdot M\cdot\begin{pmatrix}\overline{\xi}^{n_k}&0\\0&\xi^{n_k}\end{pmatrix}\right),\end{aligned}$$

where $M = 10\cdot\tilde{\tilde{S}}\cdot\tilde{S} = \begin{pmatrix}3(\overline{\xi}+2)&2(\xi+2)\\2(\overline{\xi}+2)&3(\xi+2)\end{pmatrix}$.

Therefore, it is left to study

$$\mathrm{Tr}\left(M\cdot\begin{pmatrix}\overline{\xi}^{n_1}&0\\0&\xi^{n_1}\end{pmatrix}\cdot\cdots\cdot M\cdot\begin{pmatrix}\overline{\xi}^{n_k}&0\\0&\xi^{n_k}\end{pmatrix}\right) = \mathrm{Tr}(\prod_{l=1}^{k}A_l),$$

with $A_l = \begin{pmatrix}3(\overline{\xi}+2)\overline{\xi}^{n_l}&2(\xi+2)\xi^{n_l}\\2(\overline{\xi}+2)\overline{\xi}^{n_l}&3(\xi+2)\xi^{n_l}\end{pmatrix} = (a^l_{i_1,i_2})_{i_1,i_2}$.

Write $a^l_{i_1,i_2} = c_{i_1,i_2} \cdot d^l_{i_2}$ for $i_1, i_2 = 1, 2$, and $l = 1, \dots k$, where

$$c_{i_1,i_2} = \begin{cases} 3 \text{ if } i_1 = i_2 \\ 3 \text{ if } i_1 \neq i_2 \end{cases}, \text{ and } d^l_{i_2} = \begin{cases} (\overline{\xi}+2)\overline{\xi}^n_l \text{ if } i_2 = 1 \\ (\xi+2)\xi^n_l \text{ if } i_2 = 2. \end{cases}$$

Thus,

$$\mathrm{Tr}(\prod_{l=1}^{k} A_l) = \sum_{j_1=1}^{2} \cdots \sum_{j_k=1}^{2} a^1_{j_1,j_2} \cdots a^k_{j_k,j_1} = \sum_{S \subseteq \{1,\dots,k\}} a^1_{j_1,j_2} \cdots a^k_{j_k,j_1},$$

where $j_i = 1$ if $i \in S$ and $j_i = 2$ otherwise.

Hence,

$$\begin{aligned} \mathrm{Tr}(\prod_{l=1}^{k} A_l) &= \sum_{S \subseteq \{1,\dots,k\}} c_{j_1,j_2} \cdots c_{j_k,j_1} \cdot d^1_{j_2} \cdots d^k_{j_1} \\ &= \sum_{S \subseteq \{1,\dots,k\}} 3^{r(s)} 2^{k-r(S)} (\xi+2)^{|S|} (\overline{\xi}+2)^{|S^c|} \xi^{\sum_{i \in S} n_i} \overline{\xi}^{\sum_{i \in S^c} n_i}, \end{aligned}$$

as was to be proven. □

Hence, defining the function $\Phi : \mathcal{N} \to \mathbb{Z}_{>0}$ in the set of nontrivial primitive necklaces of positive integers with small variation $\mathcal{N}$, again as

$$\Phi([n_1, \dots, n_k]) = \frac{1}{10^k \cdot 2^{n_1+\cdots+n_k}} \sum_{S \subseteq \{1,\dots,k\}} 3^{r(S)} 2^{k-r(S)} (\xi+2)^{|S|} (\overline{\xi}+2)^{|S^c|} \xi^{\sum_{i \in S} n_i} \overline{\xi}^{\sum_{i \in S^c} n_i},$$

we have just proven the following reformulation of Markov's conjecture.

Theorem 2.5.5. *Markov's uniqueness conjecture is equivalent to*

The function Φ is injective in $\mathcal{N}$.

We passed through geometry to reformulate Markov's uniqueness conjecture in combinatorial terms.

Chapter 3

A Basmajian-type inequality for Riemannian surfaces

3.1 Introduction

This chapter is based on a joint work with Florent Balacheff (see [BF23]). To facilitate independent reading, we will briefly present again the results we will prove. We explore for compact Riemannian surfaces whose boundary consists of a single closed geodesic the relationship between orthospectrum and boundary length. More precisely, we establish a uniform lower bound on the boundary length in terms of the orthospectrum when fixing a metric invariant of the surface related to the classical notion of volume entropy. This inequality can be thought of as a Riemannian analog of Basmajian's identity for hyperbolic surfaces. That is, in this chapter we will prove Theorems H and I from the introduction.

Let S be a compact orientable Riemannian surface with geodesic boundary ∂S and negative Euler characteristic. Define its orthospectrum $\mathcal{O}(S)$ as the set of (oriented) lengths of homotopy classes relative to ∂S with multiplicity, i.e., the lengths of the set of arcs $\mathcal{A}(S)$. Here, given such a homotopy class η, its length $\ell(\eta)$ is defined as the minimal length $\ell(c)$ over all of its representative arcs c. This minimal length is always realized as the length of a geodesic arc lying in the corresponding class and hitting orthogonally the boundary. See Figure 3.1 for example.

3.1.1 Results

We will again go through the results that will be proved in this chapter.

Our purpose is to study a generalization of Basmajian's identity to the Riemannian

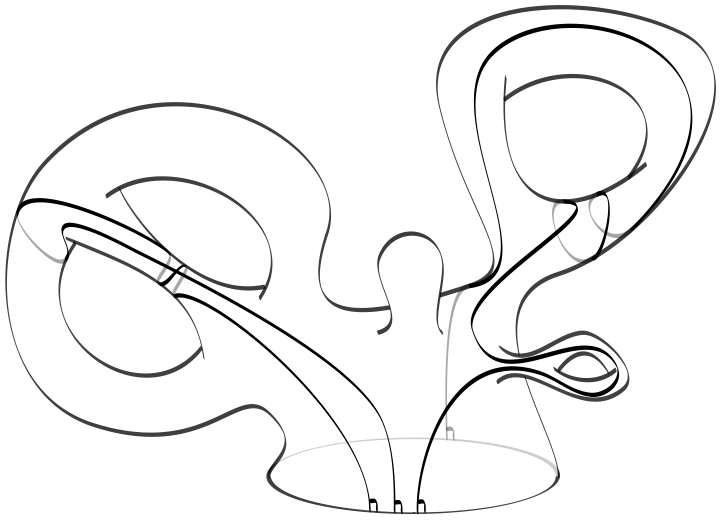


Figure 3.1

world. To do so, we will need to first relax the equality into an inequality, and secondly to involve an auxiliary Riemannian invariant. This auxiliary Riemannian invariant will be constructed using the classical notion of volume entropy for *closed* Riemannian surfaces defined as the exponential growth rate of the volume of large metric balls in their universal cover. More specifically, denoting by S' the Riemannian closed surface obtained by doubling S, that is

$$S' = S \sqcup S / \sim$$

where $\sim$ identifies the boundary of the two copies of S via the identity map. Denote by $\tilde{S}$ the universal Riemannian cover of S'. We will be interested in the *volume entropy* of S' defined as the quantity

$$h(S') := \lim_{R \to \infty} \frac{1}{R} \log \mathsf{Area}_{\tilde{S}} B(\tilde{x}, R),$$

where $\tilde{x} \in \tilde{S}$ and $B(\tilde{x}, R)$ is the ball of radius R centered at $\tilde{x}$ in $\tilde{S}$. This limit always exists and does not depend on the chosen point $\tilde{x}$ (see [Man79]). Observe that closed orientable Riemannian surfaces with negative Euler characteristic always have positive volume entropy, and that their volume entropy will always be equal to 1 in the particular case where the metric is hyperbolic.

Since adding positive curvature can enlarge the orthospectrum without changing the boundary length of the surface, the orthospectrum cannot determine the length of the boundary in the Riemannian moduli space. However, if we also fix the volume entropy of the doubled surface, then the orthospectrum provides the following lower bound on the boundary length.

The main result is the following, which we view as a curvature-free analog of the cel-

ebrated Basmajian's identity for hyperbolic surfaces with one geodesic boundary component.

Theorem 3.1.1 (Theorem H)**.** *Let S be a compact orientable Riemannian surface with negative Euler characteristic and one geodesic boundary component. Then the following holds true:*

$$\ell(\partial S) \geq \frac{2}{h(S')} \operatorname{arcsinh}\left(\sum_{\ell \in \mathcal{O}(S)} \frac{1}{1+e^{h(S')\ell}}\right)$$

where $h(S')$ denotes the volume entropy of the doubled surface S'.

Note that

$$\operatorname{arcsinh}(x) = \log(x + \sqrt{1+x^2}) \geq \log(1+x),$$

and, on the other hand,

$$\log \coth(\ell/2) = \log(1 + \frac{2}{e^{\ell}-1}).$$

Hence, comparing Theorem 3.1.1 and Basmajian's Theorem 3.1.3, as in the hyperbolic case the volume entropy of the doubled surface is always 1, Theorem 3.1.1 recovers asymptotically the linear term of Basmajian's identity when the length of the boundary goes to zero and consequently all the terms in the orthospectrum grow up to infinity.

Theorem 3.1.1 is another evidence of how analogs of hyperbolic identities and inequalities can be found in the Riemannian free-curvature setting by involving Riemannian invariants associated to the volume entropy, like in [BM23] where a curvature-free version of the classic $\log(2k-1)$ Theorem was proved using the notion of critical exponent. So far, the question to know if a Riemannian analog of Basmajian's identity holds for several geodesic boundary components remains open. To prove Theorem 3.1.1, we prove the following result for a special family of metric graphs that will encode the orthospectrum.

Theorem 3.1.2 (Theorem I)**.** *Fix $n \geq 1$. Let Γ be a metric graph formed by a circle of length L with $2n$ disjoint vertices on it, and n edges of lengths $\ell_1, \ldots, \ell_n$ joining these vertices by pairs. Then the following holds true:*

$$\tanh\left(\frac{h(\Gamma)L}{2}\right) < 2\sum_{i=1}^{n} \frac{1}{1+e^{h(\Gamma)\ell_i}} < \sinh\left(\frac{h(\Gamma)L}{2}\right)$$

where $h(\Gamma)$ denotes the volume entropy of the metric graph Γ.

3.1.2 Related work

The orthospectrum has been widely studied in hyperbolic geometry, where each homotopy class admits a unique geodesic representative. There have been some celebrated results on the rigidity of the hyperbolic structures with a given orthospectrum. For example, Basmajian's identity [Bas93] gives an expression of the length of the boundary in terms of the orthospectrum, Bridgeman's identity [Bri11] gives an expression of the area given its orthospectrum, Parker proved in [Par95] that the entropy of the geodesic flow of the surface can be expressed using Poincaré series with a given orthospectrum, and most recently, Masai and McShane [MM22] proved that for a given surface and orthospectrum, there are only finitely many hyperbolic structures. In the present article, we will focus on Basmajian's identity which can be stated as follows.

Theorem 3.1.3 (Special case of [Bas93])**.** *Suppose that the Riemannian surface S is hyperbolic. Then*

$$\ell(\partial S) = 2 \sum_{\ell \in \mathcal{O}(S)} \log \coth(\ell/2).$$

Organization of the chapter

Here is the plan of the chapter. In order to prove the main theorem, i.e., Theorem 3.1.1, we will prove in the first section Theorem 3.1.2 for metric graphs.

In the second section, we will show how to transfer this result from metric graphs to Riemannian surfaces and prove Theorem 3.1.1. The idea is to embed a suitable sequence of metric graphs in our initial surface S whose volume entropies will be controlled by the volume entropy of the doubled surface using a ping-pong map.

3.2 A generalization of Basmajian's identity for metric graphs

To prove Theorem 3.1.2, we start by giving some notation for the graphs we are interested in. These graphs topologically consist of a circle, playing the role of the boundary, with some additional edges playing the role of the orthogeodesics and joining disjoint pairs of disjoint points on the circle. We will consider the various possible metrics on such a graph, and prove a Basmajian-type inequality for them. Afterward, we analyze the optimality of our result.

3.2.1 Notation and proof of Theorem 3.1.2

By a metric graph, we mean a 1-dimensional simplicial complex Γ endowed with a piecewise Riemannian metric denoted by ℓ. We will simply denote by Γ the metric graph (Γ, ℓ) when the metric ℓ is clear out from the context. For such a choice of metric, each 1-simplex (or edge) e turns out to be isometric to the segment $[0, \ell(e)]$ with the standard Euclidean metric for some positive real number $\ell(e)$ called its length. The length of a graph Γ is then the sum of the lengths of its edges, which could possibly be infinite. As any subset $X \subset \Gamma$ which is itself a 1-complex inherits from ℓ an induced metric, its induced length is then well defined and will be denoted by $\ell(X)$. With this notation in mind, the *volume entropy* of a metric graph Γ is then the quantity

$$h(\Gamma) := \lim_{R \to \infty} \frac{\log \ell(B(\tilde{x}, R))}{R}$$

where $B(\tilde{x}, R)$ denotes the ball of radius R centered at some point $\tilde{x}$ in the universal covering tree of Γ endowed with the lifted metric. This limit always exists and does not depend on the chosen point $\tilde{x}$, see [Lim08].

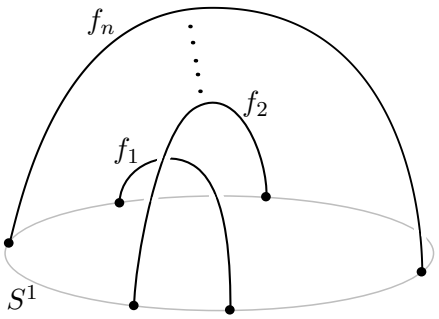


Figure 3.2

Our special metric graphs are constructed as follows. Let $n \geq 1$ be an integer. Consider the circle $\mathbb{S}^1$ with some orientation and fix $2n$ cyclically ordered vertices $v_1, \ldots, v_{2n}$ on $\mathbb{S}^1$. We denote by e_i the edge defined as the portion of the circle between v_{i-1} and v_i for $i = 1, \ldots, 2n$ using a cyclic index. Then fix a decomposition of $\{1, \ldots, 2n\} = \{i_1, j_1\} \cup \ldots \cup \{i_n, j_n\}$ into n pairs of indices. For each $k = 1, \ldots, n$ join v_{i_k} and v_{j_k} through an extra edge denoted by f_k and denote by $\tau_k \in \mathfrak{S}_{2n}$ the transposition permuting i_k and j_k. For latter purposes set $\omega := \tau_1 \ldots \tau_n$. The topological structure of our graph is defined as $\Gamma := \mathbb{S}^1 \cup f_1 \cup \ldots \cup f_n$ which is a finite simplicial 1-complex. Now fix $(L_1, \ldots, L_{2n}) \in (\mathbb{R}_{>0})^{2n}$ and $(\ell_1, \ldots, \ell_n) \in (\mathbb{R}_{>0})^n$, and choose a metric ℓ on our graph Γ such that $\ell(e_i) = L_i$ for $i = 1, \ldots, 2n$, and $\ell(f_k) = \ell_k$ for $k = 1, \ldots, n$. Finally set $L := \sum_{i=1}^{2n} L_i$.

We will now show that these special metric graphs (Γ, ℓ) satisfy the Basmajian's type double inequality already stated in Theorem 3.1.2 that we recall here for reader's con-

venience:

$$\tanh\left(\frac{hL}{2}\right) < 2\sum_{i=1}^{n}\frac{1}{1+e^{h\ell_i}} < \sinh\left(\frac{hL}{2}\right).$$

The idea of the proof is quite simple. By [Lim08, Theorem 4] we can associate to our graph a system of linear equations with $6n$ variables, whose coefficients depend only on the volume entropy of Γ and the lengths of the edges, and such that the system admits a positive solution. We will prove that the existence of such a positive solution implies that the double inequality holds true.

Proof of Theorem 3.1.2. Start by identifying the finite simplicial 1-complex Γ with an unoriented graph, see [Lim08]. Then we associate to each oriented edge of Γ a variable as follows. Using a cyclic index, we denote by

- x_k the variable associated to the oriented edge corresponding to e_k and going from v_{k-1} to v_k for $k = 1, \dots, 2n$ (therefore x_1 is associated to the oriented edge e_1 from v_{2n} to v_1),
- $\overline{x}_1, \dots, \overline{x}_{2n}$ the $2n$ variables associated to the same edges but with opposite orientation,
- y_k the variable associated to the extra oriented edge going from v_k to $v_{\omega(k)}$ for $k = 1, \dots, 2n$ (that is, $f_{k'}$ for some $k' \in \{1, \dots, n\}$).

Finally, define ℓ'_k as the length of the oriented edge associated to the variable y_k. Here the length of an oriented edge is defined as the length of the 1-simplex to which it is naturally associated. We do not need to introduce the variables $\overline{y}_k$ because they would satisfy that $\overline{y}_k = y_{\omega(k)}$. In a similar way, observe that $\ell'_{\omega(k)} = \ell'_k$ for $k = 1, \dots, 2n$ and thus $\sum_{k=1}^{2n} \ell'_k = 2 \cdot \sum_{k=1}^{n} \ell_k$.

We now form the following system of linear equations:

$$\{x_f = \sum_{f' \in E(f)} e^{-h\ell(f')} x_{f'} \mid f \in E^*(\Gamma)\}$$

where h denotes the volume entropy $h(\Gamma)$ of our graph, x_f is the variable associated to an oriented edge f, $E^*(\Gamma)$ denotes the set of oriented edges of Γ and $E(f)$ is the set of oriented edges different from $\overline{f}$ whose startpoint is f's endpoint. By [Lim08, Theorem 4] we know that h is the only positive number such that this system admits a positive solution. Therefore there exist positive real numbers $(X_1, \dots, X_{2n}, \overline{X}_1, \dots, \overline{X}_{2n}, Y_1, \dots, Y_{2n}) \in \mathbb{R}^{6n}_{>0}$ satisfying the following system of equations:

$$\begin{cases} Y_k = e^{-hL_{\omega(k)+1}} X_{\omega(k)+1} + e^{-hL_{\omega(k)}} \overline{X}_{\omega(k)} \\ X_k = e^{-hL_{k+1}} X_{k+1} + e^{-h\ell'_k} Y_k \\ \overline{X}_k = e^{-hL_{k-1}} \overline{X}_{k-1} + e^{-h\ell'_{k-1}} Y_{k-1} \end{cases}$$

where $k = 1 \ldots, 2n$.

By substituting cyclically the equations of the second line of our system as follows:

$$X_k = e^{-h\ell'_k} Y_k + e^{-hL_{k+1}}(e^{-h\ell'_{k+1}} Y_{k+1} + e^{-hL_{k+2}}(e^{-h\ell'_{k+2}} Y_{k+2} + \ldots)),$$

one obtains that

$$X_k = \sum_{i=1}^{2n} \alpha_{i,k} \ \frac{e^{-h\ell'_i}}{1 - e^{-hL}} \ Y_i$$

where $\alpha_{i,k} := e^{-h(\sum_{j=k}^{i} L_j - L_k)}$ is cyclically summed, as $L = \sum_{i=1}^{2n} L_i$. Analogously, starting by substituting cyclically the equations of the third line in our system we obtain that for $k = 1, \ldots, 2n$

$$\overline{X}_k = \sum_{i=1}^{2n} \beta_{i,k} \ \frac{e^{-h\ell'_i}}{1 - e^{-hL}} \ Y_i$$

where $\beta_{i,k} = e^{-h(\sum_{j=i+1}^{k} L_j - L_k)}$ is cyclically summed. Combining both equalities above, one obtains using the equations of the first line in our system that

$$Y_{\omega(k)} = e^{-hL_{k+1}} X_{k+1} + e^{-hL_k} \overline{X}_k = \sum_{i=1}^{2n} e^{-h\ell'_i} \ \frac{e^{-hL_{k+1}} \alpha_{i,k+1} + e^{-hL_k} \beta_{i,k}}{1 - e^{-hL}} \ Y_i$$

for $k = 1, \ldots, 2n$.

Note that, on one hand, we have for $i \neq k$

$$\begin{aligned}
\frac{e^{-hL_{k+1}} \alpha_{i,k+1} + e^{-hL_k} \beta_{i,k}}{1 - e^{-hL}} &= \frac{e^{-h\sum_{j=k+1}^{i} L_j} + e^{-h\sum_{j=i+1}^{k} L_j}}{1 - e^{-hL}} \\
&= \frac{e^{-h\sum_{j=k+1}^{i} L_j} + e^{-hL + h\sum_{j=k+1}^{i} L_j}}{1 - e^{-hL}} \\
&= \frac{e^{hL/2 - h\sum_{j=k+1}^{i} L_j} + e^{-hL/2 + h\sum_{j=k+1}^{i} L_j}}{e^{hL/2} - e^{-hL/2}} \\
&= \frac{\cosh(h(\sum_{j=k+1}^{i} L_j - L/2))}{\sinh(hL/2)},
\end{aligned}$$

and, on the other hand,

$$\frac{e^{-hL_{k+1}} \alpha_{k,k+1} + e^{-hL_k} \beta_{k,k}}{1 - e^{-hL}} = \frac{2e^{-hL}}{1 - e^{-hL}} = \frac{e^{-hL/2}}{\sinh(hL/2)}.$$

Hence

$$Y_{\omega(k)} + e^{-h\ell'_k} Y_k = \sum_{i=1}^{2n} e^{-h\ell'_i} \ \frac{\cosh(h(\sum_{j=k+1}^{i} L_j - L/2))}{\sinh(hL/2)} Y_i,$$

and finally

$$(1+e^{-h\ell'_k})(Y_k+Y_{\omega(k)})$$
$$=\sum_{i=1}^{2n} e^{-h\ell'_i}\cdot\frac{\cosh(h(\sum_{j=k+1}^{i}L_j-L/2))+\cosh(h(\sum_{j=\omega(k)+1}^{i}L_j-L/2))}{\sinh(hL/2)}Y_i$$

for $k=1,\dots,2n$ using that $\ell'_{\omega(k)}=\ell'_k$.

Applying now that for all $k,i\in\{1,\dots,2n\}$ the inequality

$$\cosh(h(\sum_{j=k+1}^{i}L_j-L/2))\leq\cosh(hL/2),$$

one obtains for all $k=1,\dots,2n$,

$$(1+e^{-h\ell'_k})(Y_k+Y_{\omega(k)})\leq\sum_{i=1}^{2n}\frac{2e^{-h\ell'_i}}{\tanh(hL/2)}Y_i=\frac{1}{\tanh(hL/2)}\sum_{i=1}^{2n}e^{-h\ell'_i}(Y_i+Y_{\omega(i)}).$$

With a change of variables to $Z_1,\dots,Z_n\in\mathbb{R}_{>0}$, where $Z_{k'}=(1+e^{-h\ell'_k})(Y_k+Y_{\omega(k)})$, for k' being such that $\ell_{k'}=\ell'_k$, one finds for $k=1,\dots,n$

$$Z_k\leq\frac{2}{\tanh(hL/2)}\sum_{i=1}^{n}\frac{1}{1+e^{h\ell_i}}Z_i.$$

Finally, taking the equation for $Z_{\max}=\max Z_i$, and since $Z_k>0$ for all k, we obtain

$$\tanh\left(\frac{hL}{2}\right)\leq 2\sum_{i=1}^{n}\frac{1}{1+e^{h\ell_i}}.$$

Analogously, applying that for all $k,i\in\{1,\dots,2n\}$, $\cosh(h(\sum_{j=k+1}^{i}L_j-L/2))\geq 1$, and taking the minimum for Z_j at the end, we obtain

$$\sinh\left(\frac{hL}{2}\right)\geq 2\sum_{i=1}^{n}\frac{1}{1+e^{h\ell_i}}.$$

Strictness of both inequalities follows from the fact that it is impossible for any sequence $\{L_1,\dots,L_{2n}\}$ of positive numbers to achieve $\sum_{j=k+1}^{i}L_j=L/2$ for all k,i, or $\sum_{j=k+1}^{i}L_j-L/2=\pm L/2$ for all k,i. □

3.2.2 On the optimality of the Basmajian-type inequality for metric graphs

Observe that if one allows the L_i's to be zero, Theorem 3.1.2 is still true with the inequalities being non-strict by continuity of the volume entropy in terms of the involved lengths. Moreover, the condition

$$\sum_{j=k+1}^{i} L_j - L/2 = \pm L/2$$

$\forall k, i = 1, \dots, 2n$ is then achievable and corresponds to the extremal case where all of the L_i's are zero except one. Even if this kind of graph will not appear as the orthogeodesic graph that will be constructed in the next section from a smooth Riemannian surface with boundary, it proves that the first inequality in Theorem 3.1.2 is optimal. More precisely, we have the following.

Proposition 3.2.1. *Fix $L \in \mathbb{R}_{>0}$ and $(\ell_1, \dots, \ell_n) \in (\mathbb{R}_{>0})^n$. Any metric graph of the type $(\bigvee_{i=0}^{n} \mathbb{S}^1_i, \ell)$ where $\ell(\mathbb{S}^1_0) = L$ and $\ell(\mathbb{S}^1_i) = \ell_i$ for all $i = 1 \dots, n$ satisfies*

$$\tanh\left(\frac{hL}{2}\right) = 2\sum_{i=1}^{n} \frac{1}{1+e^{h\ell_i}}$$

where h denotes its volume entropy.

Proof. It easily follows from the fact that

$$\frac{1}{1+e^{hL}} + \sum_{i=1}^{n} \frac{1}{1+e^{h\ell_i}} = \frac{1}{2}$$

which holds true by [BM23, Lemma 5]. □

A second graph will exemplify that this phenomenon is not rigid.

Example 3.2.2. Take a graph Γ with a circle of length L with two vertices at distance $L/2$, and n edges joining the two vertices of length $\ell_1, \dots, \ell_n$. Again, this is a limit case of the graphs at the statement of Theorem 3.1.2 with $L_1 = L_{n+1} = L/2$ and $L_j = 0$ otherwise.

In this case, the example is simple enough so one can solve it by using Lim's system directly. Associating to every unoriented edge a variable x_f for the left to right orientation and a variable $\overline{x_f}$ for right to left, one gets that Lim's system can be written, after summing equations for opposite orientations, as

$$(x_f + \overline{x_f})(1 + e^{h\ell(f)}) = \sum_{f \in E(\Gamma)} e^{-h\ell(f)}(x_f + \overline{x_f}),$$

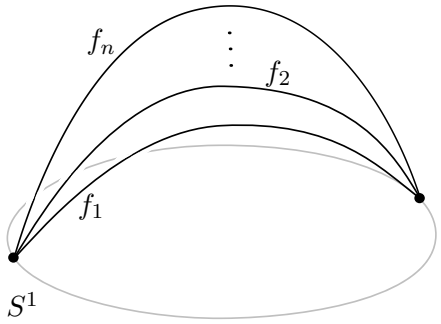


Figure 3.3

where $E(\Gamma)$ here is the set of unoriented edges in Γ. This implies $1 = \sum_{f \in E(\Gamma)} \frac{1}{1+e^{h\ell(f)}}$, that translates to

$$\tanh(hL/4) = \sum_{i=1}^{n} \frac{1}{1+e^{h\ell_i}}.$$

Remark 3.2.3 (Non-sharpness of the upper bound)**.** For the upper bound to be sharp one would need to limit graphs of the form in Theorem 3.1.2 to get $(\cosh(h(\Gamma)\cdot(\sum_{j=k+1}^{i} L_j - L/2)))_{i,k} = (1)_{i,k}$. Let now Γ be a metric graph now consisting of a circle, $N > 0$ vertices on it, and $h > 0$ a positive real number. Denote the lengths of the segments of the circle by $L_1, \ldots, L_N$, and $L := L_1 + \cdots + L_N$. Set, for $i = 1, \ldots, N$,

$$C(i) = \#\{k \in \{1, \ldots, N\} \mid \cosh(h(\sum_{j=k+1}^{i} L_j - L/2)) > \cosh(hL/4)\},$$

where the sum is cyclic. We claim that $\#\{i \in \{1, \ldots, N\} \mid C(i) \geq N/4\} \geq N/2$, hence the bound cannot be sharp.

One can prove it in the following way: rewrite

$$C(i) = \#\{k \in \{1, \ldots, N\} \mid \sum_{j=k+1}^{i} L_j < L/4 \text{ or } \sum_{j=k+1}^{i} L_j > 3L/4\}.$$

For any $\lambda \in [0, 1]$, $\#\{i \in \{1, \ldots, N\} \mid C(i) \geq N/4\} \geq \lambda N$ if and only if $\#\{v \in V(\Gamma) \mid |B(v, L/4)| \geq N/4\} \geq \lambda N$. Assume now the opposite, i.e. $\#\{v \in V(\Gamma) \mid |B(v, L/4)| \geq N/4\} < \lambda N$. This implies the existence of $\lceil (1-\lambda)N \rceil$ vertices such that $|B(v, L/4)| < N/4$. However, each of these balls is a half circle centered at a vertex $v \in V(\Gamma)$, hence at most there are $N/2$ vertices with this property. Therefore, $\lambda > 1/2$ which proves the claim.
This argument shows how the bound is not sharp, however, it only leads to non-explicit expressions that do not improve the behavior other than maintaining a $\sinh$-order bound. On the other hand, every example computed by the authors has a $\tanh$ growth which

makes us think that the behavior is more rigid than the proved statement.

3.3 From metric graphs to Riemannian surfaces with boundary

Let S be a compact orientable Riemannian surface with negative Euler characteristic and one geodesic boundary component of length L that we will denote by γ.

3.3.1 A sequence of special metric graphs

Fix any order $\{\eta_k\}_{k\geq 1}$ on the set of homotopy classes of S relative to $\partial S = \gamma$. For any $k \geq 1$, denote by ℓ_k the length of the homotopy class η_k, and fix any length-minimizing arc c_k in this class. In particular, each arc c_k will be geodesic, have length $\ell(c_k) = \ell_k$, and meet the boundary curve γ orthogonally at two points that we will denote by $w_{k,1}$ and $w_{k,2}$. Furthermore observe that $\{\ell_k \mid k \geq 1\} = \mathcal{O}(S)$, and that all points $\{w_{k,i}\}$ are pairwise disjoint.

We now construct a sequence of special metric graphs $\{\Gamma_n\}_{n\geq 1}$ associated to our Riemannian surface with one boundary geodesic as follows.

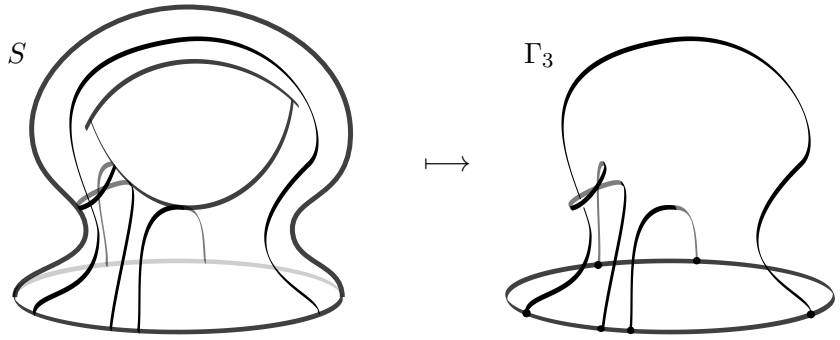


Figure 3.4

Choose an orientation of γ. For each $n \geq 1$, rewrite the set $\{w_{i,j} \mid i = 1, \ldots, n\,,\ j = 1, 2\}$ of intersecting points as a cyclically ordered set along γ of $2n$ pairwise distinct vertices $\{v_1, \ldots, v_{2n}\}$. We have a natural decomposition $\{1, \ldots, 2n\} = \{i_1, j_1\} \cup \ldots \cup \{i_n, j_n\}$ such that for any $k = 1, \ldots, n$ the arc c_k joins v_{i_k} and v_{j_k}. Define L_i as the length of the subarc of γ going from v_{i-1} to v_i for $i = 1, \ldots, 2n$ using a cyclic index. Given a diffeomorphism $\gamma \simeq \mathbb{S}^1$, we can consider the ordered set of $2n$ vertices $\{v_1, \ldots, v_{2n}\}$ constructed above as laying on $\mathbb{S}^1$. We denote by e_i the edge defined as the portion of $\mathbb{S}^1$ between v_{i-1} and v_i for $i = 1, \ldots, 2n$ using a cyclic index. Next, for each $k = 1, \ldots, n$ join v_{i_k} and v_{j_k} through an extra edge denoted by f_k. The topological structure of our

graph Γ_n is then defined as the finite 1-dimensional simplicial complex (see Figure 3.4 for an example of Γ_3 for some Riemannian surface S of signature $(1,1)$)

$$\Gamma_n = \mathbb{S}^1 \cup f_1 \cup \ldots \cup f_n.$$

Now choose any metric ℓ on our graph Γ_n such that $\ell(e_i) = L_i$ for $i = 1, \ldots, 2n$, and $\ell(f_k) = \ell_k$ for $k = 1, \ldots, n$. Denote simply by Γ_n the metric graph (Γ_n, ℓ) thus defined. Observe that each metric graph Γ_n could be viewed as a subgraph of the metric graph Γ_{n+1}. According to Theorem 3.1.2 we have that

$$\ell(\partial S) > \frac{2}{h(\Gamma_n)} \operatorname{arcsinh}\left(2 \sum_{k=1}^{n} \frac{1}{1 + e^{h(\Gamma_n)\ell_k}}\right). \tag{3.3.1}$$

3.3.2 Proof of Theorem 3.1.1 via the doubled surface

Denote by S' the closed Riemannian surface obtained by doubling S, that is

$$S' = S \sqcup S / \sim$$

where $\sim$ identifies the boundary of the two copies of S via the identity map. Denote by $\tilde{S}$ the universal Riemannian cover of S'.

Theorem 3.1.1 will be a direct consequence of the following result.

Proposition 3.3.1. *For any $n \geq 1$, the volume entropy of the metric graph Γ_n is at most equal to the volume entropy of S', that is:*

$$h(\Gamma_n) \leq h(S').$$

Indeed, from Equation (3.3.1) and Proposition 3.3.1, we derive that for any $n \geq 1$

$$\ell(\partial S) > \frac{2}{h(S')} \operatorname{arcsinh}\left(2 \sum_{k=1}^{n} \frac{1}{1 + e^{h(S')\ell_k}}\right),$$

which implies Theorem 3.1.1 by letting $n \to +\infty$ as $\mathcal{O}(S) = \{\ell_k \mid k \geq 1\}$.

Proof. First recall that the volume entropy of a finite simplicial complex X endowed with a piecewise smooth Riemannian metric (such as a metric graph, or a closed Riemannian surface) satisfies the formula

$$h(X) = \lim_{R \to \infty} \frac{1}{R} \log \#\{\alpha \in \pi_1(X, x) \mid \ell(\alpha) \leq R\},$$

for any point $x \in X$, see [Sab06, Lemma 2.3]. Here we have denoted by $\ell(\alpha)$ the length

of a homotopy class $\alpha \in \pi_1(X, x)$ defined as the shortest length of a loop based at x belonging to the class α.

Fix some point $x \in \gamma \simeq \mathbb{S}^1$ distinct from the v_i's. Note first that any homotopy class of the fundamental group of Γ_n based at x is uniquely represented as a length minimizing path of the form $\omega_0 f_{i_1} \omega_1 \cdots f_{i_m} \omega_m$ for some $m \geq 0$, where the letters f_{i_j} stand for the edges associated to the chosen orthogeodesics $c_1, \ldots, c_n$, and the words ω_j's are locally length minimizing paths of the subgraph $\mathbb{S}^1 \subset \Gamma_n$. If $m = 0$ the path is reduced to a closed minimal loop ω_0 of $\mathbb{S}^1 \subset \Gamma_n$.

Now we define a ping-pong map as follows (see Figure 3.5). Denote by $S_1, S_2 \subseteq S'$ the two natural copies of S contained in S', and by $\gamma' = S_1 \cap S_2 \subseteq S'$ their intersection. By construction of Γ_n, there is a natural identification between the closed geodesic $\gamma' \subseteq S'$ and the subgraph $\mathbb{S}^1 \subseteq \Gamma_n$. Denote by x' the point of γ' corresponding to the point x of $\gamma \simeq \mathbb{S}^1$. To a given homotopy class α of Γ_n based at x and represented by the sequence $\omega_0 f_{i_1} \omega_1 \cdots f_{i_m} \omega_m$, we associate the homotopy class $h(\alpha) \in \pi_1(S', x')$ based at the point x' corresponding to the path formed by following first the subarc denoted by ω_0' of γ' corresponding to ω_0, then through the copy f_{i_1}' of the orthogeodesic associated to f_{i_1} and laying in S_1, then through the segment ω_1' associated to ω_1 in γ', following through the orthogeodesic f_{i_2}' associated to f_{i_2} in S_2, and keep alternating S_1 and S_2 until completing the word $\omega_0 f_{i_1} \omega_1 \cdots f_{i_m} \omega_m$ and closing up in a loop $\omega_0' f_{i_1}' \omega_1' \cdots f_{i_m}' \omega_m'$ of S'. The ping-pong map $\varphi_n : \pi_1(\Gamma_n, x) \to \pi_1(S', x')$ thus defined is not a morphism of groups, but satisfies the following property.

Lemma 3.3.2. *The map $\varphi_n : \pi_1(\Gamma_n, x) \to \pi_1(S', x')$ is injective.*

Proof of Lemma 3.3.2. Lift the geodesic γ' to an infinite geodesic $\tilde{\gamma}$ in the universal cover $\tilde{S}$ of S', and the point x' to a point $\tilde{x}$ on $\tilde{\gamma}$. If $p : \tilde{S} \to S'$ denotes the universal covering map, we define $\tilde{S}_1$ as the connected component of $p^{-1}(S_1) \subset \tilde{S}$ whose boundary contains $\tilde{\gamma}$. The boundary of $\tilde{S}_1$ consists of a numerable set of infinite geodesics. There exists an injective correspondence between sequences of the form $\omega_0 f_{i_1}$ and boundary infinite geodesics of $\partial \tilde{S}_1$ described as follows.

First, we lift to $\tilde{S}$ the path ω_0' contained in γ' and corresponding to ω_0 starting from the point $\tilde{x}$ into a geodesic subarc of $\tilde{\gamma}$ denoted by $\tilde{\omega}_0$. Next, we lift the orthogeodesic arc f_{i_1}' of $S_1 \subset S'$ corresponding to f_{i_1} starting from the final point of the arc $\tilde{\omega}_0$ into a geodesic arc of $\tilde{S}_1$ denoted by $\tilde{f}_{i_1}$. By construction, the final point of $\tilde{f}_{i_1}$ belongs to a boundary component of $\tilde{S}_1$ which defines our correspondence. One can see that the final points of two sequences of the form $\omega_0 f_{i_1}$ belong to the same boundary infinite geodesic if and only if sequences are equal: since this statement only depends on the topology of the surface, replace the metric in S' by a hyperbolic metric, and if two different sequences of the form $\omega_0 f_{i_1}$ translated $\tilde{x}$ to the same boundary component, we would have constructed a hyperbolic rectangle which is impossible.

With a fixed $\omega_0 f_{i_1}$, repeat the argument for the next sequence $\omega_1 f_{i_2}$, which will give an

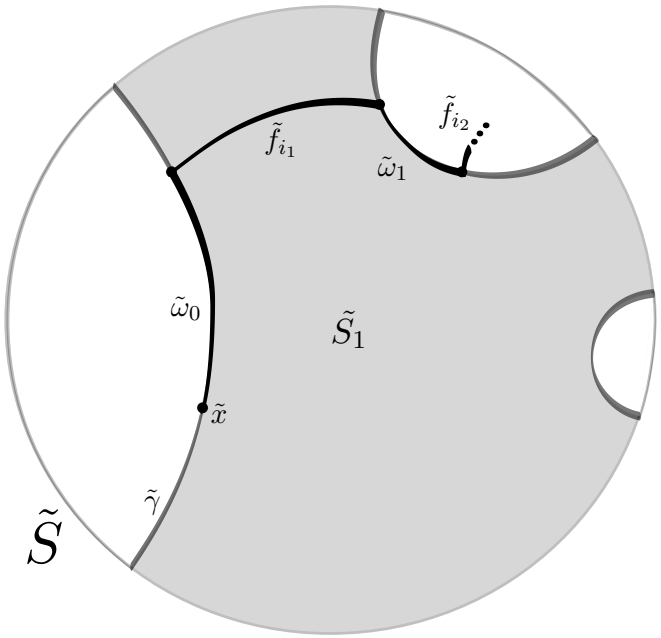


Figure 3.5

injective correspondence between all possible sequences of this kind and the countably many boundaries of the lift of $S_2 \subseteq S'$ bounded by the lift of γ' which contains the endpoint of $\tilde{f}_{i_1}$. Iteratively, one finds that the set of all possible sequences on the graph of the form $\omega_0 f_{i_1} \omega_1 \cdots f_{i_m}$ injects to the set of lifts of γ' lying in the half-space of the universal cover $\tilde{S}$ bounded by $\tilde{\gamma}$ and starting with $\tilde{S}_1$ (see figure above). Moreover, the last letters ω_m give you all possible lifts of x' in the particular lift of γ' corresponding to the sequence $\omega_0 f_{i_1} \omega_1 \cdots f_{i_m}$.

So the image by φ_n of two different homotopy classes of $\pi_1(\Gamma_n, x)$ send x' to two different endpoints, and hence φ_n is injective. □

Now by construction, we have that $\ell(\varphi_n(\alpha)) \leq \ell(\alpha)$ for all $\alpha \in \pi_1(\Gamma_n, x)$. Therefore we find that

$$\#\{\alpha \in \pi_1(\Gamma_n, x) \mid \ell(\alpha) \leqslant R\} \leqslant \#\{\beta \in \pi_1(S', x') \mid \ell(\beta) \leqslant R\}$$

from which we derive the desired inequality $h(\Gamma_n) \leq h(S')$. □

3.3.3 On the double inequality for compact surfaces

Note that, even though we obtained a double inequality for the metric graph case, for a general Riemannian surface, an upper bound for the boundary in terms of the orthospectrum and volume entropy cannot hold, as one can deform the metric such that

the boundary length increases with a controlled orthospectrum and entropy. A further question is whether with additional assumptions on the surface (as requiring the boundary geodesic to be length minimizing in its free homotopy class) this could be possible. With the techniques used in this article, this question translates to the following.

Via the same construction of a sequence of metric graphs Γ_n in 3.3.1, limiting the construction we get an infinite graph Γ encoding the entire set of relative homotopy classes to the boundary. This graph will have an infinite set of trivalent vertices. Referring to [CP20, Lemma 2.3], the entropy of this graph is finite and well-defined, and in fact, by continuity $h(\Gamma) = \lim_{n\to\infty} h(\Gamma_n)$. This follows, for example, from the expression of the volume entropy of a graph as the critical exponent of the fundamental group. Then, the existence of an upper bound in Theorem 3.1.1 would be implied by controlling the volume entropy of the doubled surface in terms of the volume entropy of the graph.

Chapter 4

Perspectives

I would like to use this space to share a more relaxed and heuristic approach to the content and results above, and my personal point of view on the topic.

Studying arcs and curves on manifolds has been a huge topic in one of the most fundamental branches of Mathematics, *Geometry*. Open problems on curves on surfaces are very intricate. On one hand, it being such an elementary topic in nature, makes still open problems hard to approach and often they are solved by impressively technical methods and even passing through more subtle branches of mathematics. However, their simplicity is at times forgotten. One (*our*) big example is how much the combinatorial nature of arcs and curves plays a big role in the geometry of smooth manifolds.

In the second part of the thesis, we are using graphs as the main tool to study the orthogeodesics hitting the boundary on a given Riemannian surface. Basmajian's identity is a perfect example of how by the rigidity of hyperbolic metrics, simple trigonometry can make you deduce something as strong as the length of the boundary only given the lengths of the geodesics hitting it orthogonally. When losing this rigidity and letting the distances in the surface deform, one asks what it is that one is left with. For us the answer to this question was clear: one is left with the boundary, the hitting points of the arcs we are studying, and the lengths of these arcs. Giving us the family of graphs that we are studying, and with only two tasks: understand these graphs, and translate this information back to the surface. Getting the maximum information from these graphs would fulfill the problem in the elementary way we wanted. So far, we achieved sharpness on one bound of Theorem I but not on the other. However, the job of translating the whole world of information that one has on a moduli space of Riemannian metrics into getting just the right combinatorial information to give some answer to the question we had, was successful.

There is still one more question to finish this matter. What about admitting multiple boundary components? The answer is hidden in a very elementary question: *can we find a relation between the lengths of the edges and the volume entropy of the following*

type of graph?

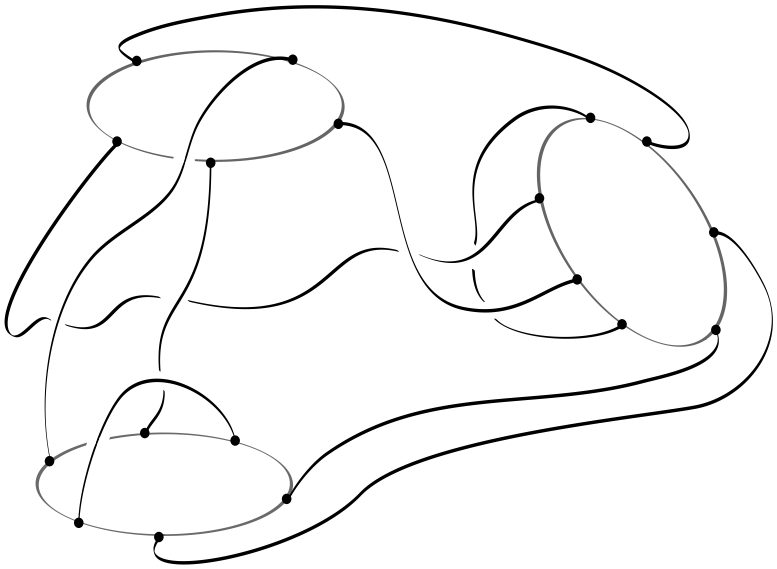


Figure 4.1

The question has been reduced to a very elementary statement on graphs. However, sometimes very elementary questions are the hardest to answer.

On the other hand, in the first part of the thesis, we approached curve counting also in the most elementary way possible. Fixing generators of the fundamental group, can we know explicitly the words representing curves of a given type? And if so, can we count exactly how many curves there are with a given length? This world of only caring about the most combinatorial aspect of curve counting is still full of potential.

On one hand, one would like to get a similar formula for any intersection number and for any genus. To approach the question, we should look at what has been done from a bigger perspective. The explicit classification of all words giving you a certain type of curve is comforting. However, our strategy can be reduced to counting how many types of curves (orbits of the mapping class group) there are with a given condition and counting how the lengths grow within a given type. To address this, we can also try to understand all the recurrences behind it. We can see what kind of transformations of a given word can increase intersection and by how much. With these, for every intersection, there are "pure orbits of self-intersection k", transformations of "pure orbits of self-intersection $k-1$" to increase self-intersection by 1, and, in general, transformations of "pure orbits of self-intersection $k-k'$" to increase self-intersection by k'. An example of the latter is attaching to a certain point of a simple curve a loop around the cusp, making it a transformation of kind 1. These kinds of transformations should all be encoded in Cohen and Lustig's algorithm which has been used before.

The below picture illustrates what are these types in self-intersection 1. On the left, one

has a transformation by concatenation at a specific point of a simple curve a^2ba^3b into a self-intersection 1 curve $ab^{-1}a^{-1}bab^2ab^3$. On the right, one has an example in the "pure orbit of self-intersection 1", that would be a curve whose exponent necklace has 2-variation, as $a^2ba^3ba^3ba^2b$.

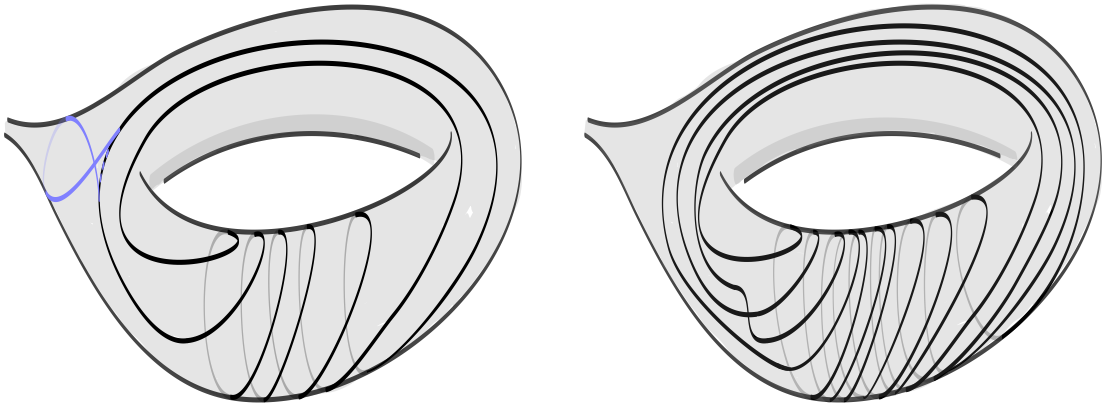


Figure 4.2

After all the explicit work on this topic, there is a way of taking the most elementary and combinatorial part of the problem, which is: *how do word transformations change self-intersection numbers and what recurrences does this build?*

The above leads to applying this work to hyperbolic metrics, where even with just the work already done there are still many consequences to explore from this point of view. This is a purely combinatorial path to study spectra in Tiechmüller spaces. When parametrizing Teichmüller space with shear coordinates, one has a trace formula with the spine given by the dual graph of the chosen triangulation. This spine of the surface is nothing but a way to rewrite words in the generators of the fundamental group into Left/Right/Shear words. This combinatorial way of studying spectra opens up the possibility of rewriting many hyperbolic known and conjectured identities to their combinatorial analogue, giving new tools to explore the easiest solutions to intricate problems.

In conclusion, this entire work follows the idea of trying to study curves and arcs on surfaces by choosing the minimum vital information and making these problems as elementary as possible.

I vet aquí un gos, vet aquí un gat, aquesta tesi s'ha acabat.

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