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Limit theorems for general functionals of Brownian local times*

Simon Campese † Nicolas Lengert ‡ Mark Podolskij §

Abstract

In this paper, we present the asymptotic theory for integrated functions of increments of Brownian local times in space. Specifically, we determine their first-order limit, along with the asymptotic distribution of the fluctuations. Our key result establishes that a standardized version of our statistic converges stably in law towards a mixed normal distribution. Our contribution builds upon a series of prior works by S. Campese, X. Chen, Y. Hu, W.V. Li, M.B. Markus, D. Nualart and J. Rosen [2, 3, 4, 5, 10, 13, 14], which delved into special cases of the considered problem. Notably, [3, 4, 5, 13, 14] explored quadratic and cubic cases, predominantly utilizing the method of moments technique, Malliavin calculus and Ray-Knight theorems to demonstrate asymptotic mixed normality. Meanwhile, [2] extended the theory to general polynomials under a non-standard centering by exploiting Perkins' semimartingale representation of local time and the Kailath-Segall formula. In contrast to the methodologies employed in [3, 4, 5, 13], our approach relies on infill limit theory for semimartingales, as formulated in [6, 8]. Notably, we establish the limit theorem for general functions that satisfy mild smoothness and growth conditions. This extends the scope beyond the polynomial cases studied in previous works, providing a more comprehensive understanding of the asymptotic properties of the considered functionals.

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1 Introduction

Over the past five decades, the mathematical literature has witnessed a surge in interest regarding the probabilistic and statistical properties of local times. Originating from the structure of a Hamiltonian in a specific polymer model, numerous investigations

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†Department of Mathematics, Hamburg University of Technology. E-mail: simon.campese@tuhh.de
‡Department of Mathematics, University of Luxembourg. E-mail: nicolas.lengert@uni.lu
§Department of Mathematics, University of Luxembourg. E-mail: mark.podolskij@uni.lu

have been dedicated to the asymptotic theory concerning functionals derived from the local time of a Brownian motion. A notable body of work in this domain includes [2, 3, 4, 5, 10, 13]. Recall that the local time $(L^x)_{x\in\mathbb{R}}$ of a Brownian motion $(W_t)_{t\in[0,1]}$ over a time interval $[0,1]$ is defined as the almost sure limit

$$L^x := \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^1 1_{(x-\varepsilon,x+\varepsilon)}(W_s)ds. \tag{1.1}$$

The primary focus of our paper centers around statistics of the form:

$$V(f)^h_{\mathbb{R}} := \int_{\mathbb{R}} f\left(h^{-1/2}(L^{x+h}-L^x)\right)dx, \qquad h>0, \tag{1.2}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a smooth enough function with $f(0)=0$. Our objective is to ascertain the asymptotic behavior of the statistic $V(f)^h_{\mathbb{R}}$ as $h\to 0$. The theorem below summarizes several special cases, extensively explored in the existing literature, that fall within the scope of our investigation.

Theorem 1.1. Let Z be a standard Gaussian random variable independent of the local time $(L^x)_{x\in\mathbb{R}}$.

(i) Chen et al. [3], Hu and Nualart [4], Rosen [14], case $f(x)=x^2$: As $h\to 0$

$$\frac{1}{h^{3/2}}\left(\int_{\mathbb{R}}\left(L^{x+h}-L^x\right)^2dx-4h\right)\xrightarrow{d}\sqrt{\frac{64}{3}\int_{\mathbb{R}}(L^x)^2dx}\times Z.$$

(ii) Hu and Nualart [5], Rosen [13], case $f(x)=x^3$: As $h\to 0$

$$\frac{1}{h^2}\int_{\mathbb{R}}\left(L^{x+h}-L^x\right)^3dx\xrightarrow{d}\sqrt{192\int_{\mathbb{R}}(L^x)^3dx}\times Z.$$

(iii) Campese [2], case $f(x)=x^q$ with $q\in\mathbb{N}_{\geq 2}$: As $h\to 0$

$$\frac{1}{h^{3/2}}\left(\int_{\mathbb{R}}\left(L^{x+h}-L^x\right)^q dx+R_{q,h}\right)\xrightarrow{d}c_q\sqrt{\int_{\mathbb{R}}(L^x)^qdx}\times Z,$$

where the random variable $R_{q,h}$ is given by

$$R_{q,h} := \sum_{k=1}^{\lfloor q/2\rfloor} a_{q,k}\int_{\mathbb{R}}\left(L^{x+h}-L^x\right)^{q-2k}\left(4\int_x^{x+h}L^udu\right)^k dx,$$

and the constants $a_{q,k}$ and c_k are defined as

$$a_{q,k}=\frac{(-1)^kq!}{2^kk!(q-2k)!} \qquad \text{and} \qquad c_q=\sqrt{\frac{2^{2q+1}q!}{q+1}}.$$

The weak convergence established in Theorem 1.1(i) for the quadratic case has been demonstrated via the method of moments in [3]. On the other hand, in [4], techniques from Malliavin calculus, along with a version of the Ray-Knight theorem, were employed to derive the same result. Similar methodologies were applied in [5, 13] to establish the cubic case outlined in Theorem 1.1(ii). The more general outcome of [2], as presented in Theorem 1.1(iii), employs a distinct technique to establish asymptotic mixed normality. The starting point in [2] is the semimartingale representation of the local time $(L^x)_{x\in\mathbb{R}}$, initially proven by Perkins in [11]. This representation, in turn, implies a semimartingale decomposition of the statistic $\int_{\mathbb{R}} \left(L^{x+h} - L^x\right)^q dx$. The somewhat intricate standardization $R_{q,h}$ is derived from the Kailath-Segall formula [15], ensuring that the normalized object is a martingale. In the final step, the asymptotic Ray-Knight theorem is applied to deduce weak convergence.

It is worth noting that Theorem 1.1(iii) extends the results of Theorem 1.1(i) and (ii) due to $R_{2,h} = -4h$ and $R_{3,h} = 0$ (cf. [2]). However, in other cases, the standardization $R_{q,h}$ is somewhat unnatural as it depends on the parameter h. Our main result, presented below, not only extends Theorem 1.1 to general functions but also employs a much more natural standardization. Unless stated otherwise, all random variables are defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 1.2. Let $f \in C(\mathbb{R})$ be a function with polynomial growth satisfying $f(0) = 0$. Define the quantity

$$\rho_u(f) := \mathbb{E}[f(\mathcal{N}(0, u^2))] \qquad \text{for } u \in \mathbb{R}. \tag{1.3}$$

Then it holds that

$$V(f)_{\mathbb{R}}^h \xrightarrow{\mathbb{P}} V(f)_{\mathbb{R}} := \int_{\mathbb{R}} \rho_{\sigma_u}(f) du \qquad \text{where} \qquad \sigma_u := 2\sqrt{L^u} \tag{1.4}$$

as $h \to 0$. If moreover $f \in C^1(\mathbb{R})$ and f, f' have polynomial growth we deduce the stable convergence

$$U(f)_{\mathbb{R}}^h := h^{-1/2} \left(V(f)_{\mathbb{R}}^h - V(f)_{\mathbb{R}}\right) \xrightarrow{d_{st}} U(f)_{\mathbb{R}} := \int_{\mathbb{R}} \sqrt{v_{\sigma_u}^2 - \sigma_u^2 \rho_{\sigma_u}^2(f')}\, dW_u', \tag{1.5}$$

where W' is a Brownian motion defined on an extended probability space and independent of $\mathcal{F}$. The quantity v_x is defined as

$$v_x^2 := 2 \int_0^1 \operatorname{cov}\left(f(xB_1), f(x(B_{s+1} - B_s))\right) ds \tag{1.6}$$

with B being a standard Brownian motion.

Building on the insights presented in [2], our approach begins by leveraging the semimartingale representation of the local time. This transformation allows us to recast the original problem into an asymptotic statistic of a semimartingale. Employing a series of approximation techniques from stochastic analysis, we then apply the limit theory for high-frequency observations of semimartingales, as established in [6]. This application yields the stable convergence result expressed in (1.5). However, it's important to highlight that our framework diverges from classical results established in works such as [1, 7, 9, 12] in several aspects. Firstly, we need to introduce a blocking technique as a necessity to break the correlation in the statistic $V(f)_{\mathbb{R}}^h$. Another notable departure from standard high-frequency theory lies in the assumption regarding the semimartingale property of the diffusion coefficient. This property, crucial for obtaining the necessary smoothness for a stable limit theorem, is absent in our model. Instead, our diffusion coefficient is represented by the process σ_u defined in (1.4), which is not a semimartingale. Consequently, we employ more nuanced techniques to derive the asymptotic theory.

A surprising distinction, in comparison to [9], is observed in the form of the limit at (1.5). Generally, when the function f is not even, the limit typically comprises three terms, revealing an $\mathcal{F}$-conditional bias (cf. [6, 9] and Theorem 2.1 below). However, in our scenario, we obtain a simpler limit denoted as $U(f)_{\mathbb{R}}$, devoid of an $\mathcal{F}$-conditional bias. This holds true regardless of whether the function f is even or not.

The paper is structured as follows. Section 2 provides crucial technical results, including the semimartingale decomposition of the local time $(L^x)_{x\in\mathbb{R}}$ and a functional stable central limit theorem. In Section 3, we delve into the proof of the main result.

Notation

Unless explicitly stated otherwise, all random variables and stochastic processes are defined on a filtered probability space denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. All positive constants are denoted by C (or by C_p if we want to emphasise the dependence on an external parameter p) although they may change from line to line. We use the notation

$$I_t := [\min(0,t), \max(0,t)].$$

We say that a function $f : \mathbb{R} \to \mathbb{R}$ has polynomial growth if it holds that $|f(x)| \leq C(1+|x|^p)$ for some $p \geq 0$. We denote by $\langle X, Y \rangle$ the covariation process of two semimartingales X and Y. For real-valued stochastic processes Y^n and Y, we employ the notation $Y^n \xrightarrow{u.c.p.} Y$ to signify uniform convergence in probability, specifically:

$$\sup_{t\in A} |Y_t^n - Y_t| \xrightarrow{\mathbb{P}} 0$$

for any compact set $A \subset \mathbb{R}$. For a sequence of random variables $(Y^n)_{n\in\mathbb{N}}$ defined on a Polish space $(E, \mathcal{E})$, we say that Y^n converges stably in law towards Y ($Y^n \xrightarrow{d_{st}} Y$), which lives on an extension $(\Omega', \mathbb{F}', \mathbb{P}')$ of the original probability space $(\Omega, \mathbb{F}, \mathbb{P})$, if and only if

$$\lim_{n\to\infty} \mathbb{E}[Fg(Y^n)] = \mathbb{E}'[Fg(Y)]$$

for all bounded $\mathcal{F}$-measurable random variables F and all bounded continuous functions $g : E \to \mathbb{R}$.

2 Definitions and preliminary results

To begin, we utilize the semimartingale representation of the local time process $(L^x)_{x\in\mathbb{R}}$, as derived in [11]. This representation posits the existence of a Brownian motion $(B_t)_{t\in\mathbb{R}}$ such that the local time is expressed as follows:

$$L^x = L^z + \int_z^x \sigma_y dB_y + \int_z^x a_y dy, \qquad x \geq z, \tag{2.1}$$

where the diffusion coefficient σ is defined at (1.4), and the drift coefficient a is a predictable, locally bounded process. This representation, as emphasized in the introduction, serves as a fundamental tool for establishing the stable limit theorem in (1.5). Additionally, we introduce two random times

$$\underline{S} := \inf\{a \leq 0 : \ L^a > 0\}, \qquad \overline{S} := \sup\{a \geq 0 : \ L^a > 0\}, \tag{2.2}$$

and remark that $\underline{S}$ and $\overline{S}$ are stopping times with respect to the filtration generated by L. Note $L^x = 0$ for any $x \notin [\underline{S}, \overline{S}]$.

To establish the results outlined in Theorem 1.2, it is essential to introduce a functional version of the statistic $V(f)_{\mathbb{R}}^h$. For a fixed $T > 0$, we define this functional as follows:

$$V(f)_t^h := \int_{I_t} f\left(h^{-1/2}(L^{x+h} - L^x)\right) dx, \qquad t \in [-T, T]. \tag{2.3}$$

We obtain the following theorem.

Theorem 2.1. Let $f \in C(\mathbb{R})$ be a function with polynomial growth satisfying $f(0) = 0$. Then it holds that

$$V(f)^h \xrightarrow{u.c.p.} V(f) \text{ as } h \to 0 \qquad \text{where} \qquad V(f)_t := \int_{I_t} \rho_{\sigma_u}(f) du. \tag{2.4}$$

Assume moreover that $f \in C^1(\mathbb{R})$ and f, f' have polynomial growth, and define the process

$$U(f)_t^h := h^{-1/2}\left(V(f)_t^h - V(f)_t\right).$$

Then, as $h \to 0$, we obtain the functional stable convergence $U(f)^h \xrightarrow{d_{st}} U(f)$ on $(C([-T,T]), \|\cdot\|_\infty)$, where

$$U(f)_t := \int_{I_t} r_{a_x,\sigma_x} dx + \int_{I_t} w_{\sigma_x} dB_x + \int_{I_t} \sqrt{v_{\sigma_x}^2 - w_{\sigma_x}^2} dW'_x. \tag{2.5}$$

The processes v_u and W' have been introduced in Theorem 1.2, while the quantities w_u and r_{u_1,u_2} are defined as

$$\begin{aligned} w_u &:= u\rho_u(f'), \\ r_{u_1,u_2} &:= u_1\rho_{u_2}(f') + \int_0^1 \mathbb{E}\left[f'(u_2(B_{x+1} - B_x))(B_2^2 - 2)\right] dx. \end{aligned} \tag{2.6}$$

We now demonstrate that the consistency statement in (1.4), as mentioned in Theorem 1.2, follows from the more general results provided in Theorem 2.1. Initially, we observe the identities

$$V(f)_{\mathbb{R}}^h = V(f)_{\underline{S}}^h + V(f)_{\overline{S}}^h + O_{\mathbb{P}}(h), \qquad V(f)_{\mathbb{R}} = V(f)_{\underline{S}} + V(f)_{\overline{S}} \tag{2.7}$$

hold. For any $\varepsilon > 0$, we conclude that

$$\begin{aligned} \mathbb{P}\left(|V(f)_{\underline{S}}^h - V(f)_{\underline{S}}| > \varepsilon\right) \le \mathbb{P}&\left(\sup_{t\in[-T,T]} |V(f)_t^h - V(f)_t| > \varepsilon, |\underline{S}| \le T\right) \\ &+ \mathbb{P}\left(|\underline{S}| > T\right), \end{aligned} \tag{2.8}$$

and a similar estimate holds for the probability $\mathbb{P}(|V(f)_{\overline{S}}^h - V(f)_{\overline{S}}| > \varepsilon)$. Consequently, the uniform convergence in (2.4) implies the statement in (1.4) when we choose T to be sufficiently large and then h to be sufficiently small. This establishes the consistency result in the context of Theorem 1.2.

Subject to an additional smoothness condition on the function f, the expression for the limit $U(f)_t$ simplifies as demonstrated in the following proposition.

Proposition 2.2. Assume that $f(0) = 0$, $f \in C^3(\mathbb{R})$, and f and its first three derivatives have polynomial growth. Define the function

$$G(u) := \int_0^u \rho_{2\sqrt{x}}(f')dx, \qquad u \ge 0. \tag{2.9}$$

Then we obtain the identity

$$U(f)_t = G(L^t) - G(L^0) + \int_{I_t} \sqrt{v_{\sigma_x}^2 - w_{\sigma_x}^2} dW'_x. \tag{2.10}$$

Proof. First of all, we note that $\rho_u(g) < \infty$ when the function g has polynomial growth. Observing the semimartingale decomposition at (2.1), an application of Itô formula gives

$$F(L^b) = F(L^a) + \int_a^b F'(L^u)dL^u + 2\int_a^b F''(L^u)L^u du,$$

for any $F \in C^2(\mathbb{R})$ and any $b > a$. A twofold application of an integration by parts formula implies that

$$\int_0^1 \mathbb{E}\left[f'(u_2(B_{x+1}-B_x))(B_2^2-2)\right]dx = u_2^2\rho_{u_2}(f''').$$

According to definition (2.9) it holds that

$$G'(u) = \rho_{2\sqrt{u}}(f') \qquad \text{and} \qquad G''(u) = 2\rho_{2\sqrt{u}}(f''').$$

Consequently, we deduce the identity

$$\begin{aligned}\int_{I_t} r_{a_x,\sigma_x}dx + \int_{I_t} w_{\sigma_x}dB_x &= \int_{I_t}\rho_{\sigma_u}(f')dL^u + \int_{I_t}\sigma_u^2\rho_{\sigma_u}(f''')du \\ &= \int_{I_t} G'(\sigma_u^2/4)dL^u + \frac{1}{2}\int_{I_t}\sigma_u^2 G''(\sigma_u^2/4)du \\ &= \int_{I_t} G'(L^u)dL^u + 2\int_{I_t} L_u^2 G''(L^u)du = G(L^t) - G(L^0).\end{aligned}$$

This completes the proof of the proposition. □

As a consequence of Proposition 2.2 and the identity $L^x = 0$ for $x \notin [\underline{S}, \overline{S}]$, we infer that

$$U(f)_{\mathbb{R}} = U(f)_{\underline{S}} + U(f)_{\overline{S}} = \int_{\mathbb{R}}\sqrt{v_{\sigma_x}^2 - w_{\sigma_x}^2}\,dW'_x, \tag{2.11}$$

provided the function f satisfies the conditions outlined in Proposition 2.2. Therefore, the stable convergence asserted in Theorem 1.2 follows from Proposition 2.2 when accompanied by a suitable approximation argument. The details of this argument will be explained in Section 3.4.

Example 2.3. Here, we illustrate that Theorem 1.2 includes the results of Theorem 1.1(i) and (ii) as specific cases.

(i) Consider the quadratic case $f(x) = x^2$ and note the identities $\rho_u(f) = u^2$, $\rho_u(f') = 0$. We immediately conclude that

$$V(f)_{\mathbb{R}} = \int_{\mathbb{R}}\sigma_u^2 du = 4\int_{\mathbb{R}} L^u du = 4,$$

where the last equality follows from the occupation time formula. We also deduce that

$$v_x^2 = 2x^4\int_0^1 \mathrm{cov}\left(B_1^2, (B_{s+1}-B_s)^2\right)ds = 4x^4\int_0^1 \mathrm{cov}\left(B_1, B_{s+1}-B_s\right)^2 ds = \frac{4}{3}x^4,$$

and consequently we get $v_{\sigma_u}^2 = \frac{64}{3}(L^u)^2$. Thus we recover the statement of Theorem 1.1(i).

(ii) Now, consider the cubic setting $f(x) = x^3$. In this scenario we deduce the identities $\rho_u(f) = 0$ and $\rho_u(f') = 3u^2$. Consequently, $V(f)_{\mathbb{R}} = 0$ and $w_u^2 = 9u^6$. A straightforward computation shows that

$$v_x^2 = 2\int_0^1 \mathrm{cov}\left(f(xB_1), f(x(B_{s+1}-B_s))\right)ds = 12x^6.$$

Thus we deduce the identity $v_{\sigma_u}^2 - w_{\sigma_u}^2 = 192(L^u)^3$ and we recover the statement of Theorem 1.1(ii). □

3 Proofs

3.1 Preliminary results

To simplify our analysis, we begin by establishing stronger assumptions on the involved stochastic processes. Analogous to the reasoning provided in (2.8), we can perform all proofs on the set $\{\underline{S}, \overline{S} \in [-T,T]\}$ for some $T>0$. Provided $L^z > 0$ for all $z \in [y,x]$, we will also use the identity

$$\sigma_z = \sigma_y + 2(B_z - B_y) + \int_y^z \tilde{a}_u \cdot (L^u)^{-1/2} dy, \tag{3.1}$$

where $\tilde{a}$ is a locally bounded process. This identity follows from $\sigma_x = 2\sqrt{L^x}$ and an application of the Itô formula to (2.1).

Additionally, given that $\tilde{a}, a$ and σ are locally bounded processes, we may, without loss of generality, assume that

$$\sup_{\omega \in \Omega,\ x \in [-T,T]} (|\tilde{a}_x(\omega)| + |a_x(\omega)| + |\sigma_x(\omega)|) \le C$$

by employing a standard localization argument (cf. [1]).

Due to the boundedness of coefficients a and σ we deduce from Burkholder inequality for any $a<b$ and $p>0$:

$$\mathbb{E}\left[\sup_{x\in[a,b]} |L^x - L^a|^p\right] \le C_p |b-a|^{p/2}. \tag{3.2}$$

Consequently, due to definition in (1.4), we also deduce the inequality

$$\mathbb{E}\left[\sup_{x\in[a,b]} |\sigma_x - \sigma_a|^p\right] \le C_p |b-a|^{p/4}. \tag{3.3}$$

Furthermore, for any function $g : \mathbb{R} \to \mathbb{R}$ with polynomial growth we have that

$$\mathbb{E}\left[g\left(h^{-1/2}(L^{x+h} - L^x)\right)\right] \le C, \tag{3.4}$$

which follows directly from (3.2).

We will often use the following lemmata, which are well known results.

Lemma 3.1. Consider the process $Y_t^n = \sum_{i=1}^{\lfloor nt \rfloor} \chi_i^n$, $t \in [0,T]$, where the random variables are χ_i^n are $\mathcal{F}_{i/n}$-measurable and square integrable. Assume that

$$\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}\left[\chi_i^n | \mathcal{F}_{(i-1)/n}\right] \xrightarrow{u.c.p.} Y_t \quad \text{and} \quad \sum_{i=1}^{\lfloor nT \rfloor} \mathbb{E}\left[(\chi_i^n)^2 | \mathcal{F}_{(i-1)/n}\right] \xrightarrow{\mathbb{P}} 0.$$

Then $Y^n \xrightarrow{u.c.p.} Y$ as $n \to \infty$.

Lemma 3.2. Consider a sequence of stochastic processes Y^n and $Y^{n,m}$. Assume that

$$Y^{n,m} \xrightarrow{d_{st}} Z^m \quad \text{as } n \to \infty, \qquad Z^m \xrightarrow{d_{st}} Y \quad \text{as } m \to \infty, \qquad \text{and}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |Y_t^{n,m} - Y_t^n| > \varepsilon\right) = 0 \qquad \text{for any } \varepsilon > 0.$$

Then it holds $Y^n \xrightarrow{d_{st}} Y$ as $n \to \infty$.

The following estimate is important for the mathematical arguments below.

Proposition 3.3. It holds that

$$\sup_{x\in[-T,T]} \mathbb{P}\left(x \in (\underline{S}, \overline{S}),\ L^x \in [0,\varepsilon)\right) \le C_T \varepsilon.$$

Proof. Let τ_x be the first time the Brownian motion W hits the level $x \in \mathbb{R}$. For $x \in (\underline{S}, \overline{S})$, it must satisfy $\tau_x < 1$. Now, applying the strong Markov property of Brownian motion, we introduce a new process $\widetilde{W}_t := W_{t+\tau_x} - W_{\tau_x}$, where $t \ge 0$. Consequently, $\widetilde{W}$ is a new Brownian motion independent of τ_x. Let $L_t^x(W)$ denote the local time of W at point x up to time t. This leads to the relation $L_1^x(W) = L^0_{1-\tau_x}(\widetilde{W})$. A well-known result asserts that $L_u^0(\widetilde{W}) \overset{d}{=} |\widetilde{W}_u|$ for any fixed u. Thus, by conditioning on τ_x, we infer that

$$\begin{aligned}\mathbb{P}\left(x \in (\underline{S}, \overline{S}),\ L^x \in [0,\varepsilon)\right) &\le \mathbb{P}\left(\tau_x < 1,\ L^x \in [0,\varepsilon)\right) \\ &= \mathbb{P}\left(\tau_x < 1,\ \sqrt{1-\tau_x} \cdot |\mathcal{N}(0,1)| \in [0,\varepsilon)\right) \\ &\le C\varepsilon \mathbb{E}\left[1_{\{\tau_x<1\}}(1-\tau_x)^{-1/2}\right].\end{aligned}$$

The density of τ_x is given by $p(u) = (2\pi)^{-1/2}|x|u^{-3/2}\exp(-x^2/2u)1_{\{u>0\}}$. Hence, we conclude that

$$\mathbb{E}\left[1_{\{\tau_x<1\}}(1-\tau_x)^{-1/2}\right] < \infty,$$

which completes the proof. □

3.2 Law of large numbers

In this section we show the uniform convergence in probability as stated in (2.4). An application of (2.8) implies the statement (1.4).

The basic idea of all proofs is to consider the approximation

$$h^{-1/2}(L^{x+h} - L^x) \approx h^{-1/2}\sigma_x(B_{x+h} - B_x).$$

Observing this approximation we see that the two increments $h^{-1/2}(L^{x+h} - L^x)$ and $h^{-1/2}(L^{y+h} - L^y)$ are asymptotically correlated when $|x-y| < h$. To break this dependence we use a classical blocking technique. For $i \ge 0$ we introduce the sets

$$A_i(m) = [i(m+1)h, i(m+1)h + mh],$$

$$B_i(m) = [i(m+1)h + mh, i(m+1)h + (m+1)h].$$

Note that the length of $A_i(m)$ is mh (big block) while $B_i(m)$ has the length h (small block). In the first step we obtain the following decomposition:

$$V(f)_t^h = Z_t^{h,m}(f) + R_t^{h,m}(f) + D_t^{h,m}(f),$$

where

$$Z_t^{h,m}(f) := \sum_{i\in\mathbb{N}: i(m+1)h+mh\in I_t} \int_{A_i(m)} f\left(h^{-1/2}(L^{x+h} - L^x)\right) dx,$$

$$R_t^{h,m}(f) := \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \int_{B_i(m)} f\left(h^{-1/2}(L^{x+h} - L^x)\right) dx,$$

and $D_t^{h,m}(f)$ comprises the edge terms and satisfies

$$D^{h,m}(f) \overset{u.c.p.}{\longrightarrow} 0 \qquad \text{as } h \to 0 \tag{3.5}$$

due to (3.4) and the polynomial growth of f. Next, we will analyse the asymptotic behaviour of the processes $Z^{h,m}(f)$ and $R^{h,m}(f)$.

(a) *Negligibility of $R_t^{h,m}(f)$:* First, we observe the inequality

$$\sup_{t\in[-T,T]} |R_t^{h,m}(f)| \le R_T^{h,m}(|f|) + R_{-T}^{h,m}(|f|)$$

Since f has polynomial growth we deduce that $|f(x)| \le C(1+|x|^p)$ for some $p>0$. Due to inequality (3.2) we get

$$\mathbb{E}\left[R_T^{h,m}(|f|) + R_{-T}^{h,m}(|f|)\right] \le Cm^{-1}.$$

Thus, we conclude that

$$\lim_{m\to\infty}\limsup_{h\to 0} \mathbb{P}\left(\sup_{t\in[-T,T]} |R_t^{h,m}(f)| > \varepsilon\right) = 0, \tag{3.6}$$

for any $\varepsilon > 0$. This proves the negligibility of the term $R^{h,m}(f)$. □

(b) *Law of large numbers for the approximation:* We introduce the following approximation of the statistic $Z_t^{h,m}(f)$:

$$\overline{Z}_t^{h,m}(f) := \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \alpha_i^h(m), \tag{3.7}$$

$$\alpha_i^h(m) := \int_{A_i(m)} f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right)dx,$$

where $t_i^h(m) = i(m+1)h$ is the left boundary of the interval $A_i(m)$. Due to Riemann integrability we deduce that

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[\alpha_i^h(m)|\mathcal{F}_{t_i^h(m)}] = mh \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \rho_{\sigma_{t_i^h(m)}}(f) \xrightarrow{u.c.p.} \frac{m}{m+1}V(f)_t$$

as $h\to 0$ and $m/(m+1)V(f) \xrightarrow{u.c.p.} V(f)$ as $m\to\infty$. By Lemma 3.1 it suffices to prove that

$$\sum_{i\in\mathbb{N}:\le i(m+1)h+mh\in[-T,T]} \mathbb{E}\left[\left|\alpha_i^h(m)\right|^2|\mathcal{F}_{t_i^h(m)}\right] \xrightarrow{\mathbb{P}} 0 \quad \text{as } h\to 0.$$

By (3.4) we readily deduce that

$$\mathbb{E}[|\alpha_i^h(m)|^2|\mathcal{F}_{t_i^h(m)}] \le C(mh)^2.$$

Hence, we conclude

$$\overline{Z}^{h,m}(f) \xrightarrow{u.c.p.} \frac{m}{m+1}V(f) \text{ as } h\to 0, \qquad \text{and} \qquad \frac{m}{m+1}V(f) \xrightarrow{u.c.p.} V(f) \text{ as } m\to\infty. \tag{3.8}$$

□

(c) In view of steps (a) and (b) we are left to proving the statement

$$\overline{Z}^{h,m}(f) - Z^{h,m}(f) \xrightarrow{u.c.p.} 0. \tag{3.9}$$

Since f has polynomial growth we have the following inequality for $\varepsilon, A > 0$:

$$|f(x)-f(y)| \le C\left(w_f(A,\varepsilon) + (1+|x|^p+|y|^p)(1_{\{|x|>A\}} + 1_{\{|y|>A\}} + 1_{\{|x-y|>\varepsilon\}})\right),$$

where $w_f(A,\varepsilon) := \sup\{|f(x)-f(y)| :\ |x|,|y| \le A,\ |x-y| \le \varepsilon\}$ is the modulus of continuity of f. Using this inequality and (3.2), and also $1_{\{|x|>A\}} \le A^{-1}|x|$, $1_{\{|x-y|>\varepsilon\}} \le \varepsilon^{-1}|x-y|$, we conclude that

$$\mathbb{E}\left[\sup_{t\in[-T,T]} \left|\overline{Z}_t^{h,m}(f) - Z_t^{h,m}(f)\right|\right] \le C\Big(w_f(A,\varepsilon) + A^{-1}$$

$$+\, \varepsilon^{-1} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in[-T,T]} \int_{A_i(m)} \left(h^{1/2} + h^{-1/2}\mathbb{E}\left[\int_x^{x+h} |\sigma_u - \sigma_{t_i^h(m)}|^2 du\right]^{1/2}\right) dx.$$

Since σ is continuous and bounded we see that the third term converges to 0 as $h \to 0$. On the other hand, we have that $\lim_{\varepsilon\to 0} w_f(A,\varepsilon) = 0$ for a any fixed A. Hence, we deduce that

$$\overline{Z}^{h,m}(f) - Z^{h,m}(f) \xrightarrow{u.c.p.} 0$$

by letting first $h \to 0$, then $\varepsilon \to 0$ and $A \to \infty$. Due to statements (3.6) and (3.8), we obtain the convergence in (2.4). □

3.3 Stable central limit theorem

Demonstrating the stable central limit theorem as stated in Theorem 2.1 poses a more intricate challenge. Our approach is primarily based upon limit theorems for semimartingales, notably in works such as [1]. It is crucial to highlight that the diffusion coefficient $\sigma_x = 2\sqrt{L^x}$ is not a semimartingale, introducing a heightened level of complexity to the proofs. We will continue to employ the blocking technique introduced in the preceding section.

First of all, we decompose our statistic into several terms:

$$U(f)^h = \sum_{k=1}^{3} Z^{h,m,k}(f) + \sum_{k=1}^{3} R^{h,m,k}(f) + \overline{D}^{h,m}. \tag{3.10}$$

Here the processes $Z^{h,m,k}(f)$, $k = 1,2,3$, are big blocks approximations, which are defined by

$$Z_t^{h,m,1}(f) := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \left(\alpha_i^h(m) - \mathbb{E}[\alpha_i^h(m)|\mathcal{F}_{t_i^h(m)}]\right),$$

$$Z_t^{h,m,2}(f) := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \left(\int_{A_i(m)} f\left(h^{-1/2}(L^{x+h} - L^x)\right) dx - \alpha_i^h(m)\right),$$

$$Z_t^{h,m,3}(f) := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \int_{A_i(m)} \left(\rho_{\sigma_{t_i^h(m)}}(f) - \rho_{\sigma_x}(f)\right) dx.$$

The small block processes $R^{h,m,k}(f)$, $k = 1,2,3$, are introduced in exactly the same way with the set $A_i(m)$ being replaced by $B_i(m)$ in all relevant definitions. Finally, the process $\overline{D}^{h,m}$ comprises all the edge terms. Similarly to the treatment of the term $D^{h,m}$ in (3.5), we immediately conclude that

$$\overline{D}^{h,m} \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to 0. \tag{3.11}$$

In the following subsections we will show that all small blocks terms are negligible in the sense

$$\lim_{m\to\infty} \limsup_{h\to 0} \mathbb{P}\left(\sup_{t\in[-T,T]} |R_t^{h,m,k}(f)| > \varepsilon\right) = 0 \qquad \text{for any } \varepsilon > 0, \tag{3.12}$$

for all $k = 1, 2, 3$. Finally, we will show that

$$Z^{h,m,1}(f) \xrightarrow{d_{st}} U'^m(f), \qquad Z^{h,m,2}(f) \xrightarrow{u.c.p.} U''^m(f), \qquad Z^{h,m,3}(f) \xrightarrow{u.c.p.} 0$$

as $h \to 0$, and moreover

$$U'^m(f) \xrightarrow{d_{st}} U'(f) = \int_0^\cdot w_{\sigma_x} dB_x + \int_0^\cdot \sqrt{v^2_{\sigma_x} - w^2_{\sigma_x}} dW'_x, \tag{3.13}$$

$$U''^m(f) \xrightarrow{u.c.p.} U''(f) = \int_0^\cdot r_{a_x,\sigma_x} dx$$

as $m \to \infty$. Consequently, due to (3.11)-(3.13), an application of Lemma 3.2 and properties of stable convergence imply the statement of Theorem 2.1.

3.3.1 Central limit theorem for the approximation

Recalling the notation from the previous subsection, we set

$$Z_t^{h,m,1}(f) =: \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} X_i^h(m).$$

We now prove the stable central limit theorem for $Z^{h,m,1}(f)$ as $h \to 0$. According to Theorem [8, Theorem IX.7.28] we need to show that

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[|X_i^h(m)|^2|\mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} \int_{I_t} v^2_{\sigma_x}(m)dx \tag{3.14}$$

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[X_i^h(m)(B_{t_i^h(m)+(m+1)h} - B_{t_i^h(m)})|\mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} c_m \int_{I_t} w_{\sigma_x} dx \tag{3.15}$$

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[|X_i^h(m)|^2 1_{\{|X_i^h(m)|>\epsilon\}}|\mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} 0 \quad \forall \epsilon > 0 \tag{3.16}$$

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[X_i^h(m)(N_{t_i^h(m)+(m+1)h} - N_{t_i^h(m)})|\mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} 0 \tag{3.17}$$

where the last statement should hold for all bounded continuous martingales N with $\langle B, N\rangle = 0$, $c_m = m/(m+1)$, and the function $v_u(m)$ will be introduced below.

We start by showing the condition (3.14). A straightforward computation using the substitution $x = hz_1, y = hz_2$ shows that

$$\begin{aligned}
\mathbb{E}[|X_i^h(m)|^2|\mathcal{F}_{t_i^h(m)}] &= h^{-1} \int_{A_i^2(m)} \Big(\mathbb{E}\Big[f\Big(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\Big) \\
&\times f\Big(h^{-1/2}\sigma_{t_i^h(m)}(B_{y+h} - B_y)\Big)\Big|\mathcal{F}_{t_i^h(m)}\Big] - \rho^2_{\sigma_{t_i^h(m)}}(f)\Big) 1_{\{|x-y|<h\}}dxdy \\
&= h \int_{[i(m+1),i(m+1)+m]^2} \Big(\mathbb{E}\Big[f\Big(\sigma_{t_i^h(m)}(B_{z_1+1} - B_{z_1})\Big) \\
&\times f\Big(\sigma_{t_i^h(m)}(B_{z_2+1} - B_{z_2})\Big)\Big|\mathcal{F}_{t_i^h(m)}\Big] - \rho^2_{\sigma_{t_i^h(m)}}(f)\Big) 1_{\{|z_1-z_2|<1\}}dz_1dz_2.
\end{aligned}$$

Hence, by Riemann integrability we deduce that

$$\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \mathbb{E}[|X_i^h(m)|^2|\mathcal{F}_{t_i^h(m)}] \xrightarrow{\mathbb{P}} \int_{I_t} v^2_{\sigma_x}(m)dx,$$

where

$$v_u^2(m) := \frac{1}{m}\int_{[0,m]^2} \text{cov}\left(f\left(u(B_{z_1+1}-B_{z_1})\right), f\left(u(B_{z_2+1}-B_{z_2})\right)\right)$$
$$\times 1_{\{|z_1-z_2|<1\}} dz_1 dz_2.$$

We note that

$$\lim_{m\to\infty} v_u^2(m) = v_u^2, \tag{3.18}$$

where v_u^2 has been introduced in (1.6).

In the next step we show condition (3.15). By the integration by parts formula we deduce the identity

$$\mathbb{E}[X_i^h(m)(B_{t_i^h(m)+(m+1)h} - B_{t_i^h(m)})|\mathcal{F}_{t_i^h(m)}]$$
$$= \int_{A_i(m)} \mathbb{E}\left[f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right) h^{-1/2}(B_{x+h}-B_x)|\mathcal{F}_{t_i^h(m)}\right] dx$$
$$= (mh)^{-1} w_{\sigma_{t_i^h(m)}}$$

where the function w_u has been defined in (2.6). This implies condition (3.15) by Riemann integrability.

To show condition (3.16), we observe the inequality

$$\mathbb{E}\left[|X_i^h(m)|^2 1_{\{|X_i^h(m)|>\epsilon\}}|\mathcal{F}_{t_i^h(m)}\right] \le \epsilon^{-2}\mathbb{E}\left[|X_i^h(m)|^4|\mathcal{F}_{t_i^h(m)}\right] \le C\epsilon^{-2}m^4h^2.$$

Hence, we deduce the statement of (3.16).

To prove condition (3.17), we apply a martingale representation theorem to deduce the representation

$$X_i^h(m) = \int_{A_i(m)} \eta_{i,x}^{h,m} dB_x,$$

where $\eta_i^{h,m}$ is a predictable square integrable process. Now, applying Itô isometry, we obtain that

$$\mathbb{E}\left[X_i^h(m)(N_{t_i^h(m)+(m+1)h} - N_{t_i^h(m)})|\mathcal{F}_{t_i^h(m)}\right] = \mathbb{E}\left[\int_{A_i(m)} \eta_{i,x}^{h,m} d\langle B, N\rangle_x |\mathcal{F}_{t_i^h(m)}\right] = 0$$

Consequently, we showed condition (3.17).

Now, due to (3.14)-(3.17), we conclude the stable convergence $Z^{h,m,1}(f) \xrightarrow{d_{st}} U'^m(f)$ as $h \to 0$ with

$$U'^m(f)_t := c_m \int_{I_t} w_{\sigma_x} dB_x + \int_{I_t} \sqrt{v_{\sigma_x}^2(m) - c_m^2 w_{\sigma_x}^2} dW'_x.$$

On the other hand, since $c_m \to 1$ and $v_u^2(m) \to v_u^2$ as $m \to \infty$, we obtain that

$$U'^m(f) \xrightarrow{d_{st}} U'(f) = \int_{I_\cdot} w_{\sigma_x} dB_x + \int_{I_\cdot} \sqrt{v_{\sigma_x}^2 - w_{\sigma_x}^2} dW'_x \tag{3.19}$$

as $m \to \infty$.

3.3.2 Negligibility of the small blocks: the martingale term

Here we show that the small block term $R^{h,m,1}(f)$ is negligible. We recall

$$R_t^{h,m,1}(f) = h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \left(\beta_i^h(m) - \mathbb{E}[\beta_i^h(m)|\mathcal{F}_{t_i^h(m)+mh}]\right),$$

where

$$\beta_i^h(m) := \int_{B_i(m)} f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\right) dx.$$

Since $R^{h,m,1}(f)$ is a martingale, f has polynomial growth and σ is bounded, we conclude that

$$\mathbb{E}\left[|R_T^{h,m,1}(f)|^2 + |R_{-T}^{h,m,1}(f)|^2\right] \le Cm^{-1}.$$

Hence, by Lemma 3.1 we obtain that

$$\lim_{m\to\infty} \limsup_{h\to 0} \mathbb{P}\left(\sup_{t\in[-T,T]} |R_t^{h,m,1}(f)| > \varepsilon\right) = 0 \qquad \text{for any } \varepsilon > 0.$$

3.3.3 Riemann sum approximation error

We now consider the Riemann sum approximation error associated with big blocks. We need to show that

$$Z_t^{h,m,3}(f) = h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \int_{A_i(m)} \left(\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f)\right) dx \overset{u.c.p.}{\longrightarrow} 0.$$

(The corresponding statement for the small block term $R^{h,m,3}(f)$ is shown in exactly the same way). For this purpose we introduce the threshold

$$\varepsilon_h = h^r \qquad \text{for some } r \in (1/4, 1/2). \tag{3.20}$$

On each big block $A_i(m)$, we will distinguish two cases according to whether $L^{t_i^h(m)} < \varepsilon_h$ or $L^{t_i^h(m)} \ge \varepsilon_h$.

We start with the first case. Since $f \in C^1(\mathbb{R})$ the map $u \mapsto \rho_u(f)$ is C^1. Also note that $\sup_{u\in A} |\rho'_u(f)|$ is bounded if A is a compact set. Due to mean value theorem and boundedness of σ we have

$$\begin{aligned} &1_{\left\{t_i^h(m)\in(\underline{S},\overline{S}),\ L^{t_i^h(m)}<\varepsilon_h\right\}} \int_{A_i(m)} \left|\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f)\right| dx \\ &\le C 1_{\left\{t_i^h(m)\in(\underline{S},\overline{S}),\ L^{t_i^h(m)}<\varepsilon_h\right\}} \int_{A_i(m)} \left|\sigma_x - \sigma_{t_i^h(m)}\right| dx \end{aligned}$$

Now, we use Proposition 3.3, inequality (3.3) as well as Hölder inequality with conjugates $p, q > 1$, $1/p + 1/q = 1$, to deduce that

$$\mathbb{E}\left[1_{\left\{t_i^h(m)\in(\underline{S},\overline{S}),\ L^{t_i^h(m)}<\varepsilon_h\right\}} \left|\sigma_x - \sigma_{t_i^h(m)}\right|\right] \le Ch^{1/4}\varepsilon_h^{1/q}.$$

Thus, we obtain that

$$\begin{aligned} &h^{-1/2}\mathbb{E}\left[\sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in[-T,T]} 1_{\left\{t_i^h(m)\in(\underline{S},\overline{S}),\ L^{t_i^h(m)}<\varepsilon_h\right\}} \int_{A_i(m)} \left|\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f)\right| dx\right] \\ &\le Ch^{-1/4}\varepsilon_h^{1/q} \to 0, \end{aligned} \tag{3.21}$$

where we use the definition at (3.20) and choose q close enough to 1.

Now, we treat the case $L^{t_i^h(m)} \geq \varepsilon_h$. For a fixed $m \in \mathbb{N}$, we conclude by Borel–Cantelli lemma and $\varepsilon_h = h^r$ for $r < 1/2$ that there exists a $h_0 > 0$ such that $\mathbb{P}$-almost surely $\sup_{x\in A_i(m)} |L^x - L^{t_i^h(m)}| \leq \varepsilon_h/2$ for any $h < h_0$. In the scenario $L^{t_i^h(m)} \geq \varepsilon_h$ the latter implies that

$$\inf_{x\in A_i(m)} |L^x| \geq \varepsilon_h/2 \qquad \mathbb{P}\text{-almost surely,} \tag{3.22}$$

for $h < h_0$. We introduce the following process:

$$Z_t^{h,m,3.1} := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} 1_{\left\{L^{t_i^h(m)}\geq\varepsilon_h\right\}} \int_{A_i(m)} \left(\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f)\right) dx.$$

To handle the process $Z^{h,m,3.1}$ we will apply the decomposition (3.1). First of all, we use the mean value theorem to deduce that

$$\rho_{\sigma_x}(f) - \rho_{\sigma_{t_i^h(m)}}(f) = \rho'_{\sigma_{t_i^h(m)}}(f)(\sigma_x - \sigma_{t_i^h(m)}) + (\rho'_{\sigma_{x_i^h}}(f) - \rho'_{\sigma_{t_i^h(m)}}(f))(\sigma_x - \sigma_{t_i^h(m)}),$$

where x_i^h is a certain point in the interval $(t_i^h(m), x)$. Now, applying (3.1), we decompose $Z^{h,m,3.1} = Z^{h,m,3.2} + Z^{h,m,3.3} + Z^{h,m,3.4}$ as

$$Z_t^{h,m,3.2} := 2h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} 1_{\left\{L^{t_i^h(m)}\geq\varepsilon_h\right\}} \int_{A_i(m)} \rho'_{\sigma_{t_i^h(m)}}(f) \left(B_x - B_{\sigma_{t_i^h(m)}}\right) dx,$$

$$Z_t^{h,m,3.3} := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} 1_{\left\{L^{t_i^h(m)}\geq\varepsilon_h\right\}} \int_{A_i(m)} \rho'_{\sigma_{t_i^h(m)}}(f) \left(\int_{t_i^h(m)}^x \tilde{a}_y \cdot (L^y)^{-1/2} dy\right) dx,$$

$$Z_t^{h,m,3.4} := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} 1_{\left\{L^{t_i^h(m)}\geq\varepsilon_h\right\}} \int_{A_i(m)} (\rho'_{\sigma_{x_i^h}}(f) - \rho'_{\sigma_{t_i^h(m)}}(f))(\sigma_x - \sigma_{t_i^h(m)}) dx.$$

Since $Z^{h,m,3.2}$ is a martingale, we obtain that

$$\mathbb{E}\left[|Z_T^{h,m,3.2}|^2 + |Z_{-T}^{h,m,3.2}|^2\right] \leq C_T h.$$

By Lemma 3.1 we deduce that

$$Z^{h,m,3.2} \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to 0. \tag{3.23}$$

Due to (3.22) we know that $(L^x)^{-1/2} < 2\varepsilon_h^{-1/2}$ for all $x \in A_i(m)$ and $h < h_0$. Since $\sigma, \tilde{a}$ are bounded, we conclude that

$$\mathbb{E}\left[\sup_{t\in[-T,T]} |Z_t^{h,m,3.3}|\right] \leq C_T h^{1/2} \varepsilon_h^{-1/2} \to 0 \qquad \text{as } h \to 0. \tag{3.24}$$

To handle the last term $Z^{h,m,3.4}$ we apply a similar technique as in step (c) of Section 3.2. Notice that the quantity $\rho'_{\sigma_x}(f)$ is bounded, because σ is bounded. We obtain that

$$\left|\rho'_{\sigma_{x_i^h}}(f) - \rho'_{\sigma_{t_i^h(m)}}(f)\right| \leq C \left(w_{\rho'(f)}(A, \epsilon) + 1_{\{|\sigma_{x_i^h} - \sigma_{t_i^h(m)}| > \epsilon\}}\right).$$

Since $L^x \geq \varepsilon_h/2$ for all $x \in A_i(m)$ and $h < h_0$, we deduce from representation (3.1) that

$$\mathbb{E}\left[|\sigma_x - \sigma_{t_i^h(m)}|^p\right] \leq C\left(h^{p/2} + h^p \varepsilon_h^{-p/2}\right)$$

for any $p > 0$, for all $x \in A_i(m)$ and $h < h_0$. Hence, we now obtain from (3.3) that

$$\mathbb{E}\left[\sup_{t\in[-T,T]} |Z_t^{h,m,3.4}|\right] \leq C\left(w_{\rho'(f)}(A,\varepsilon)\left(1 + h^{1/2}\varepsilon_h^{-1/2}\right) + \epsilon^{-3}h^{1/2}\right)$$

Thus, we conclude that

$$Z^{h,m,3.4} \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to 0. \tag{3.25}$$

A combination of (3.21) and (3.23)-(3.25) implies the statement

$$Z^{h,m,3}(f) \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to \infty.$$

Similarly, $R^{h,m,3}(f) \xrightarrow{u.c.p.} 0$ as $h \to 0$.

3.3.4 The terms $Z^{h,m,2}(f)$ and $R^{h,m,2}(f)$

In view of the previous steps, we are left with handling the terms $Z^{h,m,2}(f)$ and $R^{h,m,2}(f)$. We start with the term $Z^{h,m,2}(f)$. First, we consider an approximation of $Z^{h,m,2}(f)$ given as

$$\overline{Z}_t^{h,m,2}(f) := h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh\in I_t} \int_{A_i(m)} \mathbb{E}\left[f\left(h^{-1/2}(L^{x+h} - L^x)\right)\right.$$

$$\left. - f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\right)|\mathcal{F}_{t_i^h(m)}\right] dx.$$

Applying Lemma 3.1 and following the same arguments as presented in part (c) of Section 3.2 (see the proof of (3.9)), we deduce that

$$\overline{Z}^{h,m,2}(f) - Z^{h,m,2}(f) \xrightarrow{u.c.p.} 0 \qquad \text{as } h \to 0. \tag{3.26}$$

We use again the mean value theorem to obtain the decomposition

$$\mathbb{E}\left[f\left(h^{-1/2}(L^{x+h} - L^x)\right) - f\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\right)|\mathcal{F}_{t_i^h(m)}\right]$$

$$= \mathbb{E}\left[h^{-1/2}f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\right)\left((L^{x+h} - L^x) - \sigma_{t_i^h(m)}(B_{x+h} - B_x)\right)|\mathcal{F}_{t_i^h(m)}\right]$$

$$+ \mathbb{E}\left[h^{-1/2}\left(f'(z_i^h) - f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\right)\right)\right.$$

$$\left.\times\left((L^{x+h} - L^x) - \sigma_{t_i^h(m)}(B_{x+h} - B_x)\right)|\mathcal{F}_{t_i^h(m)}\right],$$

where z_i^h is a point between $h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)$ and $h^{-1/2}(L^{x+h} - L^x)$. As in the previous subsection we need to discuss the cases $L^{t_i^h(m)} \geq \varepsilon_h$ and $L^{t_i^h(m)} < \varepsilon_h$ separately. The easier case $L^{t_i^h(m)} < \varepsilon_h$ is handled in exactly the same way as presented in (3.21), so we focus on the scenario $L^{t_i^h(m)} \geq \varepsilon_h$. Similarly to the treatment of $Z^{h,m,3.4}$ we conclude that

$$h^{-1/2} \sum_{i\in\mathbb{N}:\ i(m+1)h+mh<t} 1_{\{L^{t_i^h(m)}\geq\varepsilon_h\}}$$

$$\int_{A_i(m)} \mathbb{E}\left[h^{-1/2}\left(f'(z_i^h) - f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h} - B_x)\right)\right)\right.$$

$$\left.\times\left((L^{x+h} - L^x) - \sigma_{t_i^h(m)}(B_{x+h} - B_x)\right)|\mathcal{F}_{t_i^h(m)}\right] dx \xrightarrow{u.c.p.} 0$$

as $h \to \infty$. Thus, we need to show that

$$\overline{Z}_t^{h,m,2.1} := h^{-1/2} \sum_{i \in \mathbb{N}:\ i(m+1)h+mh \in I_t} 1_{\{L^{t_i^h(m)} \geq \varepsilon_h\}} \int_{A_i(m)} \mathbb{E}\left[h^{-1/2} f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right) \right.$$
$$\left. \times \left(\int_x^{x+h} (a_u - a_{t_i^h(m)}) du + \int_x^{x+h} \left(\int_{t_i^h(m)}^u \tilde{a}_s \cdot (L^s)^{-1/2} ds \right) dB_u \right) | \mathcal{F}_{t_i^h(m)} \right] dx \xrightarrow{u.c.p.} 0$$

and

$$\overline{Z}_t^{h,m,2.2} := h^{-1/2} \sum_{i \in \mathbb{N}:\ i(m+1)h+mh \in I_t} 1_{\{L^{t_i^h(m)} \geq \varepsilon_h\}} \int_{A_i(m)} \mathbb{E}\left[h^{-1/2} f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right) \right.$$
$$\left. \times \left(h a_{t_i^h(m)} + 2\int_x^{x+h} (B_u - B_{t_i^h(m)}) dB_u \right) | \mathcal{F}_{t_i^h(m)} \right] dx$$
$$\xrightarrow{u.c.p.} \frac{m}{m+1} \int_{I_t} r_{a_x,\sigma_x} dx \quad (3.27)$$

as $h \to 0$. The statement $\overline{Z}^{h,m,2.1} \xrightarrow{u.c.p.} 0$ is obtained along the lines of the arguments presented in the previous subsection. Finally, observe the identities

$$\mathbb{E}\left[f'\left(h^{-1/2}\sigma_{t_i^h(m)}(B_{x+h}-B_x)\right) a_{t_i^h(m)} | \mathcal{F}_{t_i^h(m)} \right] = a_{t_i^h(m)} \rho_{\sigma_{t_i^h(m)}}(f')$$

and

$$2h^{-1} \int_{A_i(m)} \mathbb{E}\left[f'\left(h^{-1/2}u(B_{x+h}-B_x)\right) \int_x^{x+h} (B_u - B_{t_i^h(m)}) dB_u \right] dx$$
$$= 2h^{-1} \int_{A_i(m)} \mathbb{E}\left[f'\left(h^{-1/2}u(B_{x+h}-B_x)\right) \int_x^{x+h} (B_u - B_x) dB_u \right] dx$$
$$= 2mh \int_0^1 \mathbb{E}\left[f'\left(u(B_{y+1}-B_y)\right) \int_y^{y+1} (B_u - B_y) dB_u \right] dy$$
$$= 2mh \int_0^1 \mathbb{E}\left[f'\left(u(B_{y+1}-B_y)\right) \int_0^2 B_u dB_u \right] dy$$
$$= mh \int_0^1 \mathbb{E}\left[f'\left(u(B_{y+1}-B_y)\right)(B_2^2 - 2) \right] dy,$$

where we used the substitution $x = hy$ and the self-similarity of the Brownian motion. Hence, the convergence in (3.27) follows from Riemann integrability.

Following exactly the same arguments we conclude that

$$\lim_{m\to\infty} \limsup_{h\to 0} \mathbb{P}\left(\sup_{t\in[-T,T]} |R_t^{h,m,2}(f)| > \varepsilon \right) = 0 \qquad \text{for any } \varepsilon > 0.$$

This completes the proof of stable convergence $U(f)^h \xrightarrow{d_{st}} U(f)$.

3.4 Proof of Theorem 1.2

Here we prove the statements of Theorem 1.2 via an application of Theorem 2.1 and Proposition 2.2. Recall that we have already shown the convergence at (1.4); see (2.8). Thus we are left to proving the stable central limit theorem presented in Theorem 1.2.

We recall that it suffices to show all convergence results under the restriction $\underline{S}, \overline{S} \in [-T, T]$ for some $T > 0$. Theorem 2.1 states that $U(f)^h \xrightarrow{d_{st}} U(f)$ on $(C([-T, T]), \|\cdot\|_\infty)$. Since the mapping $F : [-T, T]^2 \times C([-T, T]) \to \mathbb{R}$ defined as $F((t_1, t_2), H) := H(t_1) + H(t_2)$ is continuous, we deduce by the properties of stable convergence and (2.7):

$$\begin{aligned} U(f)^h_{\mathbb{R}} &\xrightarrow{d_{st}} U(f)_{\underline{S}} + U(f)_{\overline{S}} \\ &= \int_{\mathbb{R}} r_{a_x,\sigma_x} dx + \int_{\mathbb{R}} w_{\sigma_x} dB_x + \int_{\mathbb{R}} \sqrt{v^2_{\sigma_x} - w^2_{\sigma_x}} dW'_x, \end{aligned}$$

under conditions of Theorem 2.1. Our task now boils down to demonstrating that the sum of the first two terms in the limit are equal to zero. This assertion has already been established in (2.11) under the condition $f(0) = 0$, with $f \in C^3(\mathbb{R})$, and f and its first three derivatives exhibiting polynomial growth. Therefore, our focus shifts to confirming that this statement carries over under the weaker assumptions of Theorem 1.2.

Let $f \in C^1(\mathbb{R})$ be an arbitrary function satisfying the conditions of Theorem 1.2. Then there exists a sequence of functions $(f_n)_{n\geq 1} \in C^3(\mathbb{R})$ that fulfils the conditions $f_n(0) = 0$,

$$|f_n(x)| + |f'_n(x)| + |f''_n(x)| + |f'''_n(x)| \leq C(1 + |x|^p) \qquad \text{for some } p > 0,$$

and

$$\sup_{x\in A} \left(|f_n(x) - f(x)| + |f'_n(x) - f'(x)|\right) \to 0 \qquad \text{as } n \to 0, \tag{3.28}$$

for any compact set $A \subset \mathbb{R}$. In view of Lemma 3.2 it suffices to show that

$$U(f_n)_{\mathbb{R}} \xrightarrow{\mathbb{P}} U(f)_{\mathbb{R}} \qquad \text{as } n \to \infty, \qquad \text{and} \tag{3.29}$$

$$\lim_{n\to\infty} \limsup_{h\to 0} \mathbb{P}\left(\left|U(f_n)^h_{\mathbb{R}} - U(f)^h_{\mathbb{R}}\right| > \varepsilon\right) = 0 \qquad \text{for any } \varepsilon > 0. \tag{3.30}$$

We start by proving the statement (3.29). For this purpose we introduce the notation $r_{a_x,\sigma_x}(f)$, $w_{\sigma_x}(f)$ and $v_{\sigma_x}(f)$ to explicitly denote the dependence of these quantities on the function f. Since $\underline{S}, \overline{S} \in [-T, T]$ it suffices to prove the convergence

$$\mathbb{E}\left[\int_{-T}^{T} |r_{a_x,\sigma_x}(f_n) - r_{a_x,\sigma_x}(f)| + |w_{\sigma_x}(f_n) - w_{\sigma_x}(f)| + |v_{\sigma_x}(f_n) - v_{\sigma_x}(f)| dx\right] \to 0$$

as $n \to \infty$, to conclude (3.29). But the latter follows directly from (3.28) since the processes a and σ are bounded.

Now, we show condition (3.30). Applying Theorem 2.1 we deduce that

$$\begin{aligned} \mathbb{P}\left(\left|U(f_n)^h_{\mathbb{R}} - U(f)^h_{\mathbb{R}}\right| > \varepsilon\right) &\leq \mathbb{P}\left(\sup_{t\in[-T,T]} \left|U(f_n)^h_t - U(f)^h_t\right| > \varepsilon\right) \\ &\to \mathbb{P}\left(\sup_{t\in[-T,T]} |U(f_n)_t - U(f)_t| > \varepsilon\right) \qquad \text{as } h \to 0. \end{aligned}$$

Using Markov and Burkholder inequalities, and the same arguments as in the proof of (3.29), we obtain that

$$\mathbb{P}\left(\sup_{t\in[-T,T]} |U(f_n)_t - U(f)_t| > \varepsilon\right) \to 0 \qquad \text{as } n \to \infty.$$

Thus we deduce (3.30), which completes the proof of Theorem 1.2.

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