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HIGH-FREQUENCY STATISTICS OF SMOOTH GAUSSIAN RANDOM WAVES

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Abstract

This thesis examines the geometric behavior of smooth Gaussian random waves in the high-frequency limit. We focus, in particular, on a variation of the celebrated Berry's Random Wave model, which plays a central role in various conjectures and results within the theory of Quantum Chaos. Our contribution to this field is a study of a variant we refer to as the Two-Energy Berry's Random Wave model. We confirm that the corresponding nodal number exhibits some classical behaviors, such as asymptotic Gaussianity, while we also identify several new phenomena, including some non-universal features of variance asymptotic and novel variations of the so-called full correlation phenomena.

Keywords: Berry's Random Waves, Bessel Functions, Central Limit Theorem, Fourth Moment Theorem, Gaussian Random Waves, High-Frequency Limit, Isotropy, Kac-Rice Formula, Monochromatic Random Waves, Nodal Length, Nodal Lines, Nodal Number, Nodal Points, Nodal Volumes, Phase Singularities, Quantum Chaos, Random Laplace Eigenfunctions, Random Plane Waves, Random Spherical Harmonics, Reduction Principle, Semiclassical Analysis, Stationarity, Universality, White Noise, Wiener Chaos.

Chapter 1 Introduction

Broadly speaking, this thesis focuses on the statistical analysis of geometric quantities associated with well-behaved Gaussian random waves. A core class of processes considered here consists of almost surely smooth Gaussian random fields, indexed by points on a Riemannian manifold $(\mathcal{M}, g)$, with a covariance function determined by a differential equation. Arguably, the most important—or at least the most studied—objects of this type are the Random Laplace-Beltrami eigenfunctions. Among these models, a particularly notable example is associated with the Euclidean space of dimension two. The unique (in distribution) centered, unit-variance, stationary, and isotropic Gaussian Laplace eigenfunction on $\mathbb{R}^2$ is known as the Random Plane Wave, Monochromatic Random Wave, or, alternatively, the Berry's Random Wave. The last of these names honors the significant contributions of physicist M. V. Berry, who, in 1977, introduced this model as a conjectured universal limit associated with quantum billiards, where the corresponding classical motion is chaotic. Of particular significance to our work is the fact that, for a dynamical system lacking time-reversal symmetry, the Berry's Random Wave must be replaced by two independent copies of the system, or equivalently, by the complex Berry's Random Wave. The bulk of the new results presented in this thesis concerns a gentle modification of the complex version of this model, obtained by breaking the symmetry between the real and imaginary parts of the complex wave sampled from this ensemble. Specifically, we allow the energies (or, equivalently, the wavenumbers) of each component to differ, and we explore the resulting high-frequency limit (equivalent here to the semiclassical limit). Our findings provide a complete characterization of the key statistics, namely the number of complex zeros (also referred to as nodal points or phase singularities), based on the identification of three novel asymptotic parameters. We recover several well-known properties, such as the cancellation phenomenon [10] and asymptotic normality [70], while also uncovering new and surprising results regarding the exact form of the asymptotic variance and full correlation phenomena.

1.1 The Berry's Random Wave Model

The Berry's Random Wave $b_k = \{b_k(x) : x \in \mathbb{R}^d\}$, $d \geq 2$, with frequency $k > 0$, is the centred Gaussian random field with covariance function

$$C(x-y) = \int_{S^{d-1}} e^{ik\langle x-y,w\rangle} d\sigma_{S^{d-1}}(w) \tag{1.1}$$

$$= \rho_{\frac{d}{2}-1}(k||x-y||), \tag{1.2}$$

where $x, y \in \mathbb{R}^d$. The right-hand side of (1.1) is the Fourier transform, computed in $k(x-y)$, of the normalised uniform measure $\sigma_{S^{d-1}}$ on the $(d-1)$-dimensional unit sphere S^{d-1}. The expression (1.2) is a rewriting of the above formula in terms of the normalised $(\frac{d}{2}-1)$-order Bessel function of the first kind.

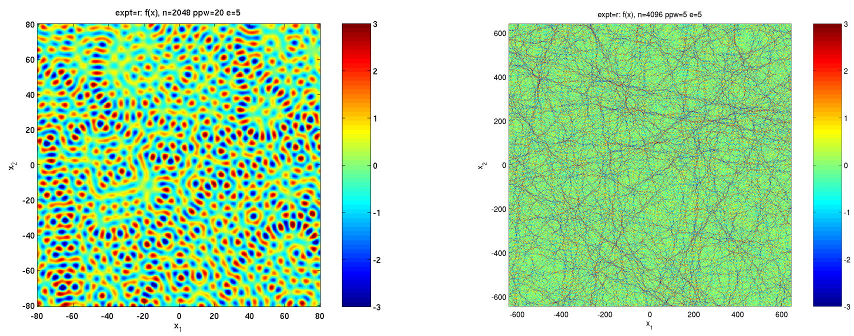

Figure 1.1: Simulation of Berry's Random Waves on the plane. The left panel shows a box size of 25 wavelengths, and the right panel shows a box size of 200 wavelengths. Credit: Alex Barnett, included with permission, https://users.flatironinstitute.org/~ahb/rpws/

For every $\alpha \geq 0$, the normalised α-order Bessel function of the first kind, denoted ρ_α, is defined by the formula

$$\rho_\alpha(t) := \sum_{m=0}^{\infty} (-1)^m \cdot \frac{\Gamma(\alpha+1)}{m!\Gamma(m+\alpha+1)} \cdot \frac{t^{2m}}{2^{2m}} \tag{1.3}$$

$$= 2^\alpha \Gamma(\alpha+1) \cdot \frac{J_\alpha(t)}{t^\alpha}, \tag{1.4}$$

where $t \in \mathbb{R}$. The symbol J_α in (1.4) denotes the standard α-order Bessel function of the first kind, see [23, Eq. (10.25.2)], with the value of (1.4) at $t=0$ being set to 1 by a smooth extension. The properties of the covariance function (1.1) impose that b_k is a random field which is stationary, isotropic, a.s. smooth, and that each realisation of b_k is an eigenfunction of the Laplace operator on $\mathbb{R}^d$ with eigenvalue k^2, that is:

$$\triangle b_k(x) + k^2 \cdot b_k(x) = 0. \tag{1.5}$$

It is easily checked (using e.g. [2, Theorem 5.7.2]) that ever Gaussian random field verifying the above properties, has necessarily the covariance (1.1). The field b_k has been named in honor of M. V. Berry, who introduced it in order to study the local behavior of high-frequency eigenstates in quantum billiards [9, 10]. More precisely, Berry considered the trigonometric random waves

$$\frac{1}{\sqrt{N}} \sum_{i \leq N} \cos(k\langle x, w_i\rangle + \theta_i) \qquad x \in \mathbb{R}^d \tag{1.6}$$

where w_i are independent and uniformly distributed on S^{d-1} and θ_i are independent and uniformly distributed on $[0, 2\pi)$. The Central Limit Theorem implies that as $N \to \infty$, this process converges, in the sense of finite-dimensional distributions, to the process b_k defined above. The spectral measure μ of the process b_k is simply Dirac's delta δ_k in relation to which this process is also sometimes referred to as the *Monochromatic Random Wave* [25]. A further interesting property of the fields $\{b_k\}$ is that, thanks to a classical result known as the Schoenberg theorem [2, p. 116, Theorem 5.7.2], a continuous function $C : \mathbb{R}^d \to \mathbb{R}$ is a covariance function of an isotropic and stationary random field on $\mathbb{R}^d$ if and only if

$$C(x-y) = \int_0^\infty \rho_{d/2-1}(k||x-y||)d\mu(k), \tag{1.7}$$

for some finite Borel measure μ on $[0,\infty)$; this shows that a mean-square continuous Gaussian random field is stationary and isotropic if and only if it is mixture of monochromatic waves.

Our particular focus on this model is justified by its vast connections to various branches of mathematics (Differential Geometry, Semiclassical Analysis, Percolation Theory), and of science (Quantum Chaos, Brain Imaging, Analysis of Cosmological Data), where it is known to appear either as a basic example or as a universal limit. Some of these topics are discussed in more detail in Chapter 2.

1.2 Summary of the Contributions of the Thesis

The new results presented in this thesis have been obtained by the author in [85] and are currently submitted for publication. They concern a specific functional of a complex Berry's Random Wave, the non-negative random integer known as *nodal number*

$$\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}) := |\{x \in \mathcal{D} : b_k(x) = \hat{b}_K(x) = 0\}|,$$

where b_k and $\hat{b}_K$ are two independent real-valued planar Berry's Random Waves, with frequencies k and K respectively, and $|\cdot|$ denotes the cardinality of a set. Here, $\mathcal{D}$ is a fixed, sufficiently well-behaved domain (see Definition 3.4). The basic regularity properties of the nodal number are discussed in Subsection 3.4.3.

In [70], Nourdin, Peccati and Rossi characterised high-wavenumber ($k \to \infty$) fluctuations of the nodal number in the one-wavenumber ($k \equiv K$) planar BRWM. Our inquiry revolves around a natural question: how does the model's behavior change with the introduction of a second parameter ($k \not\equiv K$)? In order to study this question we will consider sequences of pairs of wave-numbers $(k_n, K_n)_{n\in\mathbb{N}}$ such that $2 \le k_n \le K_n < \infty$, and analyse fluctuations of the corresponding nodal numbers

$$\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}) = \left|\{x \in \mathcal{D} : b_{k_n}(x) = \hat{b}_{K_n}(x) = 0\}\right|.$$

under the assumption that $k_n \to \infty$ (so that $K_n \to \infty$ as well).

The following list provides a short summary of our results:

1. **(Mean and variance asymptotics)** In Theorem 4.1, we compute the expectation

$$\mathbb{E}\left[\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})\right] = \frac{\text{area}(\mathcal{D})}{4\pi} \cdot (k_n \cdot K_n),$$

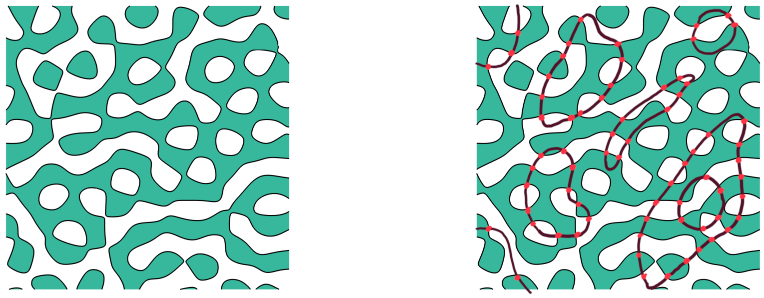

Figure 1.2: Left: Small-scale simulation of the nodal domain of the Berry's Random Wave, credit: Dmitry Belyaev, included with permission, https://people.maths.ox.ac.uk/belyaev/. Right: The same image with a superimposed drawing representing nodal lines from an independent copy of the Berry's Random Wave, along with the resulting intersection points.

in line with natural prediction. The above formula is valid for all n, without a need to pass to the limit. In order to obtain a concise characterization of the asymptotic variance we introduce the asymptotic parameters

$$r^{log} := \lim_n \frac{\ln k_n}{\ln K_n}, \qquad r := \lim_n \frac{k_n}{K_n}, \qquad r^{exp} := 1 - \lim_n \frac{\ln(1 + (K_n - k_n))}{\ln K_n}.$$

They are guaranteed to exist after, and depend on, the choice of a subsequence (see Subsection 4.1). Using these parameters, we establish in Theorem 4.1 the exact asymptotic variance formula

$$\lim_{n\to\infty} \frac{\mathrm{Var}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}))}{\mathrm{area}(\mathcal{D}) \cdot C_\infty \cdot K_n^2 \ln K_n} = 1,$$

where

$$C_\infty := \frac{r^{log} + 36r + r^2 + 50r^{exp}}{512\pi^3}. \tag{1.8}$$

We note that r^{log}, r and r^{exp} are all finite parameters, more precisely $r^{log} \in (0, 1]$ whereas $r, r^{exp} \in [0, 1]$ (for further details see paragraphs following Definition 4.1). Consequently, the quantity C_∞ is a strictly positive finite constant belonging to the interval $(\frac{1}{512\pi^3}, \frac{88}{512\pi^3}]$. We stress that the entire interior of this interval can be attained as well as its right-endpoint, but not its left endpoint. The above result extends the results of Nourdin, Peccati and Rossi who provided analogous formulas for the one-energy model [70, p. 103, Theorem 1.4]. Indeed, the one-energy scenario can be recovered from our formulas by setting $k_n \equiv K_n$ for all $n \in \mathbb{N}$. In this case we obtain that $r^{log} = r = r^2 = r^{exp} = 1$ and C_∞ takes the value $\frac{88}{512\pi^3} = \frac{11}{64\pi^3}$. Since $K_n \equiv 2\pi\sqrt{E_n}$ it follows that $K_n^2 \ln K_n = 4\pi^2 E_n \ln(2\pi\sqrt{E_n}) \sim 2\pi^2 E_n \ln E_n$. This yields that $C_\infty \cdot K_n^2 \ln K_n = \frac{11}{64\pi^3} \cdot 2\pi^2 E_n \ln E_n = \frac{11}{32\pi} E_n \ln E_n$ exactly as in [70, p. 103, Eq. (1.16)]. (See also [10, p. 3036, Eq. (50)] and [70, p. 102, Eq. (1.8)].)

2. **(Domination of the 4-th chaos)** In Theorem 4.2 we show that for some numerical constant $L > 0$, we have

$$\left\| \frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}) - \mathbb{E}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}} - \frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]}} \right\|_{L^2(\mathbb{P})} \leq \frac{L \cdot \gamma_n}{\sqrt{\ln K_n}},$$

where the right-hand side of the inequality converges to 0 with n. Moreover, as a corollary, we show that

$$\mathrm{Corr}\left(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}), \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]\right) \geq \frac{1}{1 + \frac{L \cdot \gamma_n}{\sqrt{\ln K_n}}} \longrightarrow 1,$$

where L is the same numerical constant as before. Here, $\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]$ denotes the 4-th chaotic projection (see (5.36)) and the $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence of asymptotically bounded constants that we define in (4.6). This theorem extends and quantifies the result of Nourdin, Peccati and Rossi who had shown that the domination of the 4-th chaotic projection holds in the one-energy model [70, p. 110, Eq. (2.29)].

3. **(Univariate Central Limit Theorem)** In Theorem 4.2 we prove the convergence in law

$$\frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}) - \mathbb{E}\left[\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})\right]}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}} \xrightarrow{d} Z \sim \mathcal{N}(0,1),$$

and, more precisely, we provide the following inequality in the 1-Wasserstein distance (see **N.4**)

$$W_1\left(\frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}) - \mathbb{E}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}}, Z\right) \leq \frac{L \cdot \gamma_n}{\sqrt{\ln K_n}},$$

where $L > 0$ is some strictly positive numerical constant. An analogous qualitative CLT has been established before in the one-energy model by Nourdin, Peccati and Rossi [70, p. 103, Theorem 1.4].

4. **(Multivariate Central Limit Theorem)** We extend the preceding result to the multivariate setting. Denote

$$\mathbf{Y}_n = \left(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}_1), \ldots, \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}_m)\right),$$

let Σ be a matrix defined by $\Sigma_{ij} = \mathrm{area}(\mathcal{D}_i \cap \mathcal{D}_j)$. In Theorem 4.3 we establish that

$$\frac{\mathbf{Y}_n - \mathbb{E}\mathbf{Y}_n}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}} \xrightarrow{d} \mathbf{Z} \sim \mathcal{N}_m(0, \Sigma),$$

where C_∞ is the same finite constant as in (1.8). We quantify this convergence in C^2 and 1-Wasserstein distances (see **N.4**) as

$$d_{C^2}\left(\frac{\mathbf{Y_n}-\mathbb{E}\mathbf{Y_n}}{\sqrt{C_\infty\cdot K_n^2\ln K_n}},\mathbf{Z}\right)\leq\frac{\widetilde{L}\cdot(1+\sum_{i=1}^m\operatorname{area}(\mathcal{D}_i))}{\sqrt{C_\infty\cdot\ln K_n}},$$
$$\mathbf{W}_1\left(\frac{\mathbf{Y_n}-\mathbb{E}\mathbf{Y_n}}{\sqrt{C_\infty\cdot K_n^2\ln K_n}},\mathbf{Z}\right)\leq\frac{\widetilde{L}\cdot(1+m^{3/2})\cdot\left(1+\sum_{i=1}^m\operatorname{area}(\mathcal{D}_i)^2\right)}{\sqrt{C_\infty\cdot\ln K_n}}+M_n,$$

where $\widetilde{L}$ is some strictly positive numerical constant and the sequence $(M_n)_{n\in\mathbb{N}}$ is defined in (4.13) and vanishes in the limit ($M_n\to 0$) provided that Σ is strictly positive definite. Our result extends and quantifies the multivariate CLT for the one-energy model provided by Vidotto in [78, p. 1000, Theorem 3.2] and relies heavily on some crucial arguments presented herein.

5. **(White Noise Limit)** In Teorem 4.4 we provide an extension of the aforementioned multivariate CLT to the infinite-dimensional setting. That is, we define a random signed measure

$$\mu_n(A)=\frac{\mathcal{N}(b_{k_n},\hat{b}_{K_n},A)-\mathbb{E}\left[\mathcal{N}(b_{k_n},\hat{b}_{K_n},A)\right]}{\sqrt{C_\infty\cdot K_n^2\ln K_n}},\qquad A\in\mathcal{B}([0,1]^2),$$

and show that, in the sense of random generalised functions on $[0,1]^2$ (see Appendix 3.4.5), we have a convergence in law

$$\mu_n(dt_1dt_2)\xrightarrow{d}W(dt_1dt_2).$$

Here, W denotes the White Noise on $[0,1]^2$. This result extends [67, p. 97, Proposition 1.3] established for nodal length by Notarnicola, Peccati and Vidotto.

6. **(Full correlations)** In Theorem 4.5, we establish the Reduction Principle,

$$\left\|\frac{\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})-\mathbb{E}\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})}{\sqrt{\operatorname{Var}\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})}}-Y_{r^{log},r,r^{exp}}\right\|_{L^2(\mathbb{P})}\longrightarrow 0,$$
$$\operatorname{Corr}\left(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D}),Y_{r^{log},r,r^{exp}}\right)\longrightarrow 1,$$

where $Y_{r^{log},r,r^{exp}}$ is the following sum of random integrals (sometimes called polyspectra)

$$-\frac{K_n^2}{192\pi}\Bigg(r^{log}\int_D H_4(b_{k_n}(x))dx+r\cdot\int_D H_4(\hat{b}_{K_n}(x))+\frac{3}{2}H_2(b_{k_n}(x))H_2(\hat{b}_{K_n}(x))dx$$
$$+12r^{exp}\int_D H_2(\widetilde{\partial}_1 b_{k_n}(x))H_2(\hat{b}_{K_n}(x))+H_2(b_{k_n}(x))H_2(\widetilde{\partial}_2\hat{b}_{K_n}(x))dx\Bigg).\tag{1.9}$$

Here, $\widetilde{\partial}_1 b_{k_n}(x)$ and $\widetilde{\partial}_2 \hat{b}_{K_n}(x)$ denote the normalised derivatives defined in (4.25). This result extends the Reduction Principle for the nodal length of a real Berry's Random Wave, established by Vidotto [92, p. 3, Theorem 1.1]. The Reduction Principle for a nodal number of a complex Berry's Random Wave was previously unknown even in the one-energy ($k_n \equiv K_n$) model (it is covered here by the scenario $r^{exp} > 0$). Our Reduction Principle provides simplification beyond what is afforded by the domination of the 4-th chaotic projection given by Theorem 4.2. Observe that the fourth chaotic projection $\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]$ consists of 22 terms (see Lemma 5.2), while (1.9) only contains 5 summands. An unexpected part of this result is the transition from the case $r^{exp} = 0$ to the case $r^{exp} > 0$ and the corresponding necessity to include a term containing derivatives

$$\int_D H_2(\widetilde{\partial}_1 b_{k_n}(x))H_2(\hat{b}_{K_n}(x)) + H_2(b_{k_n}(x))H_2(\widetilde{\partial}_2 \hat{b}_{K_n}(x))dx.$$

Previously discovered reduction principles (in analogous situations) [92, 59, 11] required only an involvement of polyspectra depending directly on the relevant random field and not on its derivative processes. Thus, our result adds a new element to the growing body of research concerning full correlations for the geometric quantities associated with models of random Laplace eigenfunctions [93, 11, 12, 13, 80, 50, 59, 56, 58, 28, 90, 81, 7].

1.3 Framework and Notations

We will use the following standard conventions.

N.1 We will write $a_n \to a$ to denote convergence of a numerical sequence a_n to the number a. Here, and always unless stated otherwise, n will be a non-negative integer and by convergence we will mean the limit as $n \to \infty$. For any two sequences of strictly positive numbers a_n, b_n, we will write $a_n \sim b_n$ if $\frac{a_n}{b_n} \to 1$. For a finite set A we will write $|\mathrm{A}|$ to denote the number of its elements. If A is an infinite Borel measurable set then $|A|$ will denote its Lebesgue measure. Given any set A, the symbol $\mathbf{1}_{\mathrm{A}}$ will denote the characteristic function of the set A, that is, $\mathbf{1}_{\mathrm{A}}(b) = 1$ if $b \in A$ and $\mathbf{1}_{\mathrm{A}}(b) = 0$ if $b \notin \mathrm{A}$. We will write $\delta_a(b)$ for the Kronecker's delta symbol, that is: $\delta_a(b) = 1$ if $a = b$ and $\delta_a(b) = 0$ if $a \neq b$.

N.2 We will write $X_n \xrightarrow{d} X$ to denote the convergence in distribution of a sequence of random variables X_n, to the random variable X. All considered random variables will be defined on the same standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}$ denoting expectation with respect to $\mathbb{P}$ and $L^2(\mathbb{P})$ the corresponding L^2 space. We will write dt or ds and dx or dy to denote integration with respect to the 1-and-2 dimensional Lebesgue measures. By $\mathcal{B}([0,1]^2)$ we will denote the Borel σ-algebra on $[0,1]^2$. Given a non-negative definite $m \times m$ matrix Σ, we will write $\mathbf{Z} \sim \mathcal{N}_m(0, \Sigma)$, $\mathbf{Z} = (Z_1, \dots, Z_m)$, to denote the m-dimensional centred Gaussian random vector with covariance matrix Σ.

N.3 For any non-trivial square-integrable random variables X, Y, the symbol $\mathrm{Corr}(X, Y)$

will denote the standard correlation coefficient. That is,

$$\mathrm{Corr}(X,Y) := \frac{\mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)]}{\sqrt{\mathrm{Var}X}\cdot\sqrt{\mathrm{Var}Y}}, \qquad X,Y \in L^2(\mathbb{P}). \tag{1.10}$$

N.4 By W_1 and $\mathbf{W}_1$ we will denote 1-Wasserstein distance for, respectively, real-valued and vector-valued integrable random variables and by d_{C^2} the distance induced by the separating class of C^2 functions with 1-Lipschitz second partial derivatives. That is,

$$W_1(X,Y) := \sup_{|h'|_\infty \le 1} \mathbb{E}[h(X)] - \mathbb{E}[h(Y)], \tag{1.11}$$

$$\mathbf{W}_1(\mathbf{X},\mathbf{Y}) := \sup_{||f'||_\infty \le 1} \mathbb{E}[f(\mathbf{X})] - \mathbb{E}[f(\mathbf{Y})], \tag{1.12}$$

$$d_{C^2}(\mathbf{X},\mathbf{Y}) := \sup_{g\in C^2(\mathbb{R}^m), ||g''||\le 1} \mathbb{E}[g(\mathbf{X})] - \mathbb{E}[g(\mathbf{Y})], \tag{1.13}$$

where $\mathbb{E}|X|, \mathbb{E}|Y| < \infty$ and $\mathbb{E}||\mathbf{X}||, \mathbb{E}||\mathbf{Y}|| < \infty$. Here, the suprema run, respectively: over all 1-Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$, all 1-Lipschitz functions $f : \mathbb{R} \to \mathbb{R}^m$ and all $g \in C^2(\mathbb{R}^m)$ s.t. $||g''||_\infty \le 1$. Here, $|\cdot|$ denotes absolute value, $||\cdot||$ denotes standard Euclidean norm on $\mathbb{R}^m$, and

$$|h'|_\infty := \sup_{x,y\in\mathbb{R},\ x\neq y} \frac{|h(x)-h(y)|}{|x-y|}, \qquad ||f'||_\infty := \sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^m,\ \mathbf{x}\neq\mathbf{y}} \frac{||f(\mathbf{x})-f(\mathbf{y})||}{||\mathbf{x}-\mathbf{y}||},$$
$$||g''||_\infty := \sup_{\mathbf{x}\in\mathbb{R}^m} \max_{1\le i,j\le m} \left|\frac{\partial^2}{\partial x_i \partial x_j} g(\mathbf{x})\right|. \tag{1.14}$$

N.5 The symbol $||\cdot||_{op}$ stands for the operator norm defined, for any positive-definite matrix A, as $||\mathrm{A}||_{\mathrm{op}} := \sup\{||\mathrm{A}x|| : x \in \mathbb{R}^m, ||x|| \le 1\}$. The symbol $||\cdot||_{HS}$ stands for the Hilbert-Schmidt norm defined, for any positive-definite matrix A, as $||A||_{HS} := \sqrt{\mathrm{trace}(AA^{tr})}$ with A^{tr} being a transpose of the matrix A.

N.6 Given a domain $\mathcal{D} \subset \mathbb{R}^2$ we will write

$$\mathrm{diam}(\mathcal{D}) := \sup_{x,y\in\mathcal{D}} ||x-y||. \tag{1.15}$$

N.7 Let $q \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be q-times everywhere differentiable function. We set

$$||f||_{\mathcal{C}^q} := ||f||_\infty + \ldots + ||f^{(q)}||_\infty \tag{1.16}$$

where $||\cdot||_\infty$ is the usual supremum norm and $f^{(q)}$ denotes the q-th derivative of f.

N.8 For $a \in \mathbb{R}$ and $k \in \mathbb{Z} \cap [0,\infty)$, $(a)_k$ will denote the *Pochammer symbol* that is $(a)_0 = 1$ and for $k > 0$ we have $(a)_k = a(a+1)\ldots(a+k-1)$. For $l \in \mathbb{Z}$, $l \ge 1$, and $a \in \mathbb{R}$ we will use convention $\Gamma_l(a) := \prod_{i=1}^{l} \Gamma(a+(i-1)/2)$ where Γ denotes standard Gamma function.

1.4 Plan of the Thesis

The rest of the thesis is organized as follows. In Chapter 2, we review several topics that motivate the specific and extensive focus on the model studied in this thesis. We also provide an overview of the existing results that are most closely related to our research. In Chapter 3, we offer background on the three types of resources utilized in our work: classical analytical concepts, elements of Gaussian analysis, the essential technique known as the Malliavin-Stein method — including one of its key elements, the Quantitative Fourth Moment Theorem on the Wiener Chaos, and some technical concepts used to describe the geometry of smooth random fields. Chapter 4 contains full statements of our results. Chapters 5-7 contain the proofs of all our new results. In the final Chapter 8, we briefly discuss ongoing work on higher-dimensional analogues and mention several open problems, whose solutions would naturally complement the results obtained in this thesis. This thesis contains also several figures.

Chapter 2 Motivating Problems and Existing Results

In this chapter, we will explore the central topic of our work: *random nodal volumes*, defined as the volumes of the zero sets associated with random eigenfunctions of Laplace-Beltrami operators. This problem has a long and rich history, and we will begin with a brief overview of its deterministic version. Following that, we will delve into the randomized version, with particular emphasis on M. V. Berry's work in the theory of Quantum Chaos. From a focused perspective, one could argue that studying a variation of his model is the main subject of this thesis. As we will explain later in this chapter, this model exhibits significant universality properties.

2.1 Motivating Problems

2.1.1 Laplace-Beltrami Operator on Generic Manifold

In this subsection we will be drawing from the very accessible lecture notes by Canzani [97]. The Laplace operator on the Euclidean space $\mathbb{R}^d$ can be defined as the differential operator

$$\triangle\varphi(x) := \sum_{i=1}^{d} \frac{\partial^2 \varphi}{\partial x_i^2}(x), \tag{2.1}$$

or, equivalently, with the formula

$$\triangle\varphi(x) := \operatorname{div}(\nabla\varphi(x)), \tag{2.2}$$

valid for every $\varphi \in C^2(\mathbb{R}^d)$ at every point $x \in \mathbb{R}^d$. On the Riemannian manifold $(\mathcal{M}, g)$, the Laplacian in local coordinates is given by

$$\triangle_g := \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{|\det g|} \frac{\partial}{\partial x_j} \right), \tag{2.3}$$

or, again in terms of the divergence operator, as

$$\triangle_g = \operatorname{div}_g \nabla_g. \tag{2.4}$$

Here, as usual $g = (g_{ij})$, $g_{ij}(x) := \langle \left(\frac{\partial}{\partial x^i}\right)_x, \left(\frac{\partial}{\partial x^j}\right)_x \rangle_{g(x)}$ and $g^{ij}(x) := (g(x)^{-1})_{ij}$. Below, we consider standard instances of (2.3).

Example 2.1. *Let us use a standard parametrization of the sphere* $\mathbb{S}^2$*, that is*

$$T(\theta, \phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

where $(\theta, \phi) \in (0, \pi) \times (0, 2\pi)$*, and the corresponding standard metric*

$$g_{\mathbb{S}^2}(\theta, \phi) = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}.$$

Then, the Laplacian in local coordinates becomes

$$\triangle_{g_{\mathbb{S}^2}} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{2.5}$$

The Laplace operator on the Hyperbolic (Poincaré) half-plane

$$\mathbb{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\}, \qquad g_{\mathbb{H}} := \frac{1}{y^2}\mathrm{Id}_2, \tag{2.6}$$

is given in local coordinates as

$$\triangle_{g_{\mathbb{H}}} = -y^2\frac{\partial^2}{\partial x^2} - y^2\frac{\partial^2}{\partial y^2}. \tag{2.7}$$

2.1.2 Yau's Conjecture on the Nodal Volumes

In this subsection we will provide only the most basic information on the topic; for a comprehensive and recent review see [49]. Yau's conjecture [98] states the following. Let $(\mathcal{M}, g)$ be a d-dimensional C^∞-smooth closed Riemannian manifold, compact and without boundary. There exist strictly positive constants c, C, depending only the choice of the manifold M and of the Riemannian metric g, such that the following two-sided bound on the Hausdorff measure of the nodal set holds for every eigenfunction φ_λ of the Laplace operator $\triangle_g$ (i.e., for every solution of $\triangle\varphi_\lambda + \lambda\varphi_\lambda = 0$):

$$c\sqrt{\lambda} \le \mathcal{H}^{d-1}(\{\varphi_\lambda = 0\}) \le C\sqrt{\lambda}. \tag{2.8}$$

In a celebrated work [24], Donnelly and Fefferman have shown that the conjecture is true provided that the manifold is real analytic. As of the time of writing, the full conjecture remains unsettled but a major progress has recently been made by Malinnikova and Logunov [47, 48]. A more recent but closely related Quasi-symmetry conjecture [41, 6] postulates that for some strictly positive constants k, K, independent of the choice of the eigenfunction φ_λ, we should have

$$k < \frac{\mathcal{H}^d(\{\varphi_\lambda > 0\})}{\mathcal{H}^d(\{\varphi_\lambda < 0\})} < K. \tag{2.9}$$

Here, φ_λ is assumed to be non-constant and as before $(\mathcal{M}, g)$ is a C^∞ Riemannian manifold. We note that both Yau's conjecture and Quasi-symmetry conjecture are in particular known to hold for the spherical harmonics [24, 49].

2.1.3 Semiclassical Analysis and Quantum Chaos

The study of Quantum Chaos and its relation to various branches of mathematics is a subject which has attained considerable prominence in the last decades (as evidenced, for instance, by the ICM2018 plenary talk by Nalini Anantharaman [3] or the 2024 Shaw Prize awarded to Peter Sarnak). Given the stochastic nature of Quantum Mechanics it is of no surprise that Quantum Chaos has certain connections to the subject of this thesis. In what follows we shall summarise some basic information on this topic drawing substantially on the review due to Zelditch [100] and lecture notes by Nonnenmacher [63]. One of our main goals here is to go far enough so that we can state formulas (2.27)-(2.28) which are relevant in context of the upcoming Section 2.1.4.

Classical Mechanics: we recall the equations of motion (time evolution) on a phase space $t \to (x_t, \xi_t) \in T^*\mathbb{R}^n$ (= cotangent bundle of $\mathbb{R}^n$) in the classical Hamiltonian system

$$\begin{cases} \frac{d}{dt}x_t = \frac{\partial H}{\partial \xi}(x_t, \xi_t) \\ \frac{d}{dt}\xi_t = -\frac{\partial H}{\partial x}(x_t, \xi_t). \end{cases} \tag{2.10}$$

The standard choice of the Hamiltonian function is

$$H(x, \xi) := |\xi|^2 + V(x) : T^*\mathbb{R}^n \mapsto \mathbb{R}. \tag{2.11}$$

When considering a compact Riemannian manifold $(\mathcal{M}, g)$ of dimension d, with Laplace-Beltrami operator locally given as

$$\triangle_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} g^{ij} \sqrt{\det g} \frac{\partial}{\partial x_j} \tag{2.12}$$

the Euclidean square-norm $|\xi|^2$ in the Hamiltonian needs to be replaced by the norm on cotangent bundle

$$|\xi|_g^2 = \sum_{i,j=1}^{d} g^{ij}(x)\xi_i\xi_j : T^*M \mapsto \mathbb{R}_+. \tag{2.13}$$

Let us and this paragraph with some discussion that we will need in the next paragraph. It is often convenient to choose the Hamiltonian $H(x, \xi) = |\xi|_g$ and work with the corresponding half-wave group

$$U(t) = \exp(it\sqrt{-\triangle_g}). \tag{2.14}$$

(In our own work we use an analogous trick. We parametrise the random waves using their wavenumber $k = 2\pi\sqrt{E}$ rather than their energy level E, which streamlines approximately all computations in Chapters 6.1-6.2.) Of particular interest is dynamical flow

$$G^t : S^*M \to S^*M \tag{2.15}$$

on the cotangent unit bundle

$$S^*M = \{\xi \in T^*M : |\xi|_g^2 = 1\}, \tag{2.16}$$

which is well-defined as a restriction of (2.10) because the Hamiltonian flow always preserves the energy level surface $\Sigma_E = \{(x, \xi) \in T^*M : H(x, \xi) = E\}$.

Quantum Mechanics: In Quantum Mechanics the state of the system is represented by a vector ψ in a Hilbert space $\mathcal{H}$. (Normally, $\mathcal{H}$ would be a separable Hilbert space over $\mathbb{C}$.) The evolution of the state $\psi \in \mathcal{H}$ is given by the *time-dependent Schrödinger equation*

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = \hat{H}\psi(t,x), \qquad \psi(0,x) = \psi_0(x), \tag{2.17}$$

where $\hat{H}$ is a bounded Hermitian operator. The states ψ_j which solve the *time-independent Schrödinger equation*

$$\hat{H}\psi_j(x) = E_j\psi_j(x), \tag{2.18}$$

that is, the eigenvectors of $\hat{H}$, are called the stationary states and the corresponding eigenvalue E_j is called an Energy level. If the state $\psi_0(x)$ is stationary then the solution of the time-dependent Schrödinger equation is given by

$$\psi(t,x) = e^{-\frac{itE(\hbar)}{\hbar}}\psi_0(x). \tag{2.19}$$

The most classical example of the time-independent Schrödinger operator is given by the choice $\hat{H} = \hbar^2\triangle_g + V$, where $\hbar$ is the Planck's constant, $\triangle_g$ is the Laplace-Beltrami operator on the Riemannian manifold $(\mathcal{M}, g)$ and V is the potential operator (the multiplication operator $V(f)(x) = V(x)f(x)$ be the function V also called potential). The most familiar and historically first example is that of the Hydrogen atom, where $\mathcal{H} = L^2(\mathbb{R}^3, dx)$, $\triangle_g = -\sum_{j=1}^3 \frac{\partial^2}{\partial x^2}$ is the standard Laplacian on the Euclidean space and $V(x) = -\frac{1}{|x|}$ yielding

$$(\hat{H}\psi)(x) := -\hbar^2 \sum_{j=1}^{3} \frac{\partial^2\psi}{\partial x^2}(x) - \frac{\psi(x)}{|x|}. \tag{2.20}$$

The following remark will come in handy in the next paragraph.

Remark 2.1. *Let us note that if the potential is absent i.e. $V \equiv 0$, then the semiclassical limit $\hbar \to 0$ is equivalent to the high-frequency limit $\lambda_j \to \infty$. Indeed, the equations*

$$\triangle_g \phi_j = \lambda_j^2 \phi_j, \qquad \hbar^2 \triangle_g \varphi = \varphi \tag{2.21}$$

are equivalent if we set $\hbar = 1/\lambda_j$.

Semiclassical Analysis In what follows, by an *observable* we will mean a quantity which has a direct physical significance, that is, a quantity that, at least in principle, could be measured (in a laboratory experiment, by telescope, etc.) In classical mechanics observables are regular functions $a = a(x,\xi)$ on the phase space T^*M and in Quantum Mechanics each observable $\mathcal{O}$ can be identified with a Hermitian operator A on the state space $\mathcal{H}$ (which can be an arbitrary separable Hilbert space). In our context the state space is simply $\mathcal{H} = L^2(M)$. *Quantization* is the process of associating to classical observables, say $a, b \in C^\infty(T^*M)$, operators $Op_\hbar(a), Op_\hbar(b) \in (L^2(M))^*$. It has to be done in such a way that the commutators

$$[Op_\hbar(a), Op_\hbar(b)] := Op_\hbar(a)Op_\hbar(b) - Op_\hbar(b)Op_\hbar(a), \tag{2.22}$$

are, up to a small error term, equal to the multiples of the Poisson brackets

$$\{a, b\} := \sum_j \left(\frac{\partial a}{\partial x_j} \frac{\partial b}{\partial \xi_j} - \frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} \right). \tag{2.23}$$

More precisely, we require that for every $a, b \in C^\infty(T^*M)$, we have

$$[Op_\hbar(a), Op_\hbar(b)] = \frac{\hbar}{i}\{a, b\} + O(\hbar^2), \tag{2.24}$$

see [100, Paragraph 'What is "quantization"?']. Speaking in vague terms, under this condition (and some additional that we will not discuss here) Quantum Mechanics should tend to the Classical Mechanics, if we let $\hbar \downarrow 0$. That, of course, invites the question of determining in which sense such a convergence should take place. In Quantum Mechanics, any observable $\mathcal{O}$, with attached bounded hermitian operator A, can be observed only by probing the corresponding expectation values

$$\langle A\phi, \phi\rangle, \qquad \phi \in \mathcal{H}, \tag{2.25}$$

see [88, Chapter 2 'Basics of Non-relativistic Quantum Mechanics']. We note that with our choice of the state space $\mathcal{H} = L^2(M, \mathrm{vol}_g)$ (2.25) can be written as

$$\int_M (A\phi)(x)\phi(x)\mathrm{Vol}_g(dx). \tag{2.26}$$

However, in Quantum Mechanics we also have the following principle: 'When measuring (by an experiment) the value of the observable $\mathcal{O}$ for a system in state $|\alpha\rangle$, the value obtained is always one of the eigenvalues λ_j of the Hermitian operator A associated to $\mathcal{O}$, see [88, p. 24, Principle 3].' Thus, it is natural to focus our attention on the case where $\phi = \phi_j$ i.e. to analyze the behavior (2.25) (i.e, in our case, of (2.26)) when ϕ is an eigenfunction of the operator A. Now, as a fundamental example, let $(\phi_j)_j$ be the sequence of eigenfunctions of the Laplace-Beltrami operator forming an orthonormal eigenbasis of $L^2(M, \mathrm{vol}_g)$ and assume that we are working in a setting where there the potential is zero on the whole space. Then the Quantum Mechanics tending to Classical Mechanics can be interpreted in terms of the behavior of

$$\langle A\phi_j, \phi_j\rangle \tag{2.27}$$

as $j \to \infty$, i.e., in the high-frequency limit (see [100, Paragraph 'Quantization, observables and expectation values']). It is known [100] that a useful asymptotic will not hold for an arbitrary choice of the test operator A, i.e., it is not enough that the operator A is bounded and Hermitian. In the Schrödinger representation the classical choice is *the space* $\Psi^0(M)$ *of pseudo-differential operators of order zero on* $L^2(M, \mathrm{vol}_g)$, see [100]. Fixing the choice of the quantizations $a \to Op_{\hbar_j}(a)$, $\hbar_j = 1/\lambda_j$, we arrive at an associated fundamental notion of the Wigner distribution $d\Phi_j$ on S^*M which satisfies

$$\langle Op_{\hbar_{\hbar_j}}(a)\phi_j, \phi_j\rangle = \int_{S^*M} a(x, \xi) d\Phi_j, \qquad \hbar_j = 1/\lambda_j, \tag{2.28}$$

and is induced by a positive linear functional

$$a \in C^{\infty}(S^*M) \mapsto Op_{\hbar_j}(a) \in \Psi^0(M) \mapsto \langle Op_{\hbar_j}(a)\phi_j, \phi_j \rangle. \tag{2.29}$$

Then, the problem of understanding the semiclassical limit, can be posed in terms of studying the weak limit of the sequence of measures $d\Phi_j$ with natural questions such as existence, uniqueness and identification of an eventual limiting probability measures. Our motivation for mentioning Wigner distribution is the content of the next section.

Remark 2.2. *While the field of semiclassical analysis studies the behavior under general Hamiltonian as $\hbar \to 0$, the theory of Quantum Chaos focuses specifically on the situation where the flow generated by said Hamiltonian is classically chaotic. It is useful to know that, a priori, the term 'chaotic' is meant in an intuitive and informal sense and various rigorous notions can be found in the literature e.g. ergodic, mixing, hyperbolic, Bernoulli. As should be clear from the above discussion, the classical chaotic dynamic and its quantum analog are only meaningfully connected in the limit, i.e. not for $\hbar$ constant but as $\hbar \to 0$. Of even greater interest (see [100]) is the behaviour of the system under joint asymptotic $\hbar \to 0$ and $t \to \infty$. That is, large energy in a long time horizon.*

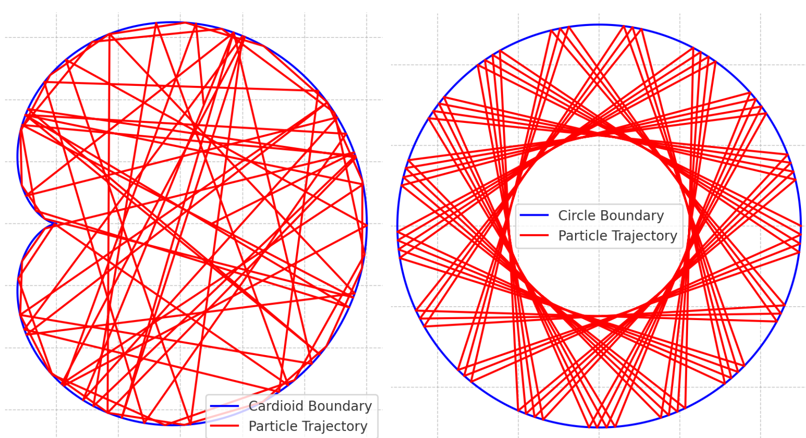

Figure 2.1: Left: Chaotic trajectory of a classical free particle within a cardioid billiard. Right: Regular trajectory of a classical free particle within a circular billiard.

2.1.4 What is the Berry's Random Wave Conjecture?

In the influential work [9], physicist M. V. Berry proposed a celebrated conjecture about the behavior of wavefunctions $\psi(\mathbf{q})$ of individual eigenstates in the semiclassical limit, where the Planck constant $\hbar$ approaches zero ($\hbar \to 0$). According to Berry, this behavior would depend on whether ψ represents what is termed a 'regular' or 'irregular' state. While these classifications lack fully rigorous definitions, they can be understood heuristically: a regular state is associated with a classically completely integrable system, whereas an irregular state corresponds to a classically chaotic system. Since Berry's conjecture, the topic has generated significant interest in both the physics and mathematics communities. Despite ongoing efforts spanning 46 years, a universally accepted formalization remains elusive. As noted in [38], "the ambiguous comparison between a deterministic system and a stochastic field has given rise to many different interpretations." Consequently, we must address the subject in somewhat vague terms. In this section, we will review key concepts from Berry's original paper, and in Subsection 2.1.6, we will explore some modern mathematical formalizations and theorems related to this topic.

Berry's analysis begins with the concept of a *local average*, defined for a 'nice' function $f = f(q_1, \dots, q_n)$ as

$$\mathbb{E} f(q_1 + \varepsilon_1, \dots, q_n + \varepsilon_n) \tag{2.30}$$

where $\varepsilon_1, \dots, \varepsilon_n$ are independent and identically distributed (i.i.d.) random variables uniformly distributed over the interval $[-\delta(\hbar), \delta(\hbar)]$. He postulated the following conditions to ensure that averages are taken over many oscillations of the wavefunction:

$$\lim_{\hbar \to 0} \delta(\hbar) = 0, \qquad \lim_{\hbar \to 0} \frac{\hbar}{\delta(\hbar)} = 0. \tag{2.31}$$

A simple example fulfilling these criteria is $\delta(\hbar) := \hbar^\theta$ with a fixed $\theta \in (0, 1)$. The two statistics of particular interest to Berry were the *local average probability density*, defined as in (2.30):

$$\mathbb{E}|\psi(q_1 + \varepsilon_1, \dots, q_n + \varepsilon_n)|^2, \tag{2.32}$$

and the *autocorrelation function*, also defined through the average as in (2.30):

$$C_{q_1,\dots,q_N}(x_1, \dots, x_N) := \tag{2.33}$$
$$\frac{\mathbb{E}\left[\psi(q_1 + x_1 + \varepsilon_1, \dots, q_n + x_n + \varepsilon_n)\psi^*(q_1 - x_1 + \varepsilon_1, \dots, q_n + x_n - \varepsilon_n)\right]}{\mathbb{E}|\psi(q_1 + \varepsilon_1, \dots, q_N + \varepsilon_N)|^2}, \tag{2.34}$$

where $*$ denotes complex conjugation.

To compute (2.33) for different systems, Berry employed the so-called *Wigner function*:

$$\Psi(q_1, \dots, q_n, p_1, \dots, p_n) := \tag{2.35}$$
$$\frac{1}{\hbar^n} \int_{\mathbb{R}^n} dx_1 \dots dx_n e^{-\frac{2i}{\hbar}\langle p_1,\dots,p_n,x_1,\dots,x_n\rangle} \psi(q_1 + x_1, \dots, q_n + x_n)\psi^*(q_1 - x_1, \dots, q_n - x_n). \tag{2.36}$$

Although it is called a function, the Wigner function is actually a distribution. Therefore, the existence and precise interpretation of (2.35) is a subtle matter (see [88]) that we will not delve into here, as it would take us too far from our main focus. The expression (2.35) can also be compared with (2.28). The crucial point is that, according to a heuristic used in [9], one can compute the local average probability density of ψ by integrating out the p_i variables in the Wigner function:

$$\mathbb{E} f(q_1 + \varepsilon_1, \dots, q_n + \varepsilon_n) = \int dp_1 \dots dp_n \Psi(q_1, \dots, q_n, p_1, \dots, p_n), \tag{2.37}$$

Furthermore, and more relevant to this subsection, one can compute the autocorrelation function as:

$$
\begin{aligned}
&Cq_1,\ldots,q_n(x_1,\ldots,x_n) = \\
&\int_{\mathbb{R}^n} dp_1 \ldots dp_n e^{\frac{2i}{\hbar}\langle p_1,\ldots,p_n,x_1,\ldots,x_n\rangle} \frac{\mathbb{E}\Psi(q_1+\varepsilon_1,\ldots,q_n+\varepsilon_n,p_1,\ldots,p_n)}{\mathbb{E}f(q_1+\varepsilon_1,\ldots,q_n+\varepsilon_n)}
\end{aligned}
\tag{2.38}
$$

Using what he termed ‘the crudest classical approximation’, Berry then obtained that for a classically integrable system

$$
\mathbb{E}\Psi(q_1+\varepsilon,\ldots,q_n+\varepsilon,p_1,\ldots,p_n) = \frac{\delta(I(q_1,\ldots,q_n,p_1,\ldots,p_n)-I_\psi)}{(2\pi)^n}, \tag{2.39}
$$

and for a classically chaotic system

$$
\begin{aligned}
&\mathbb{E}\psi(q_1+\varepsilon_1,\ldots,q_n+\varepsilon_n,p_1,\ldots,p_n) = \\
&\frac{\delta(E-H(q_1,\ldots,q_n,p_1,\ldots,p_n))}{\int dq_1\ldots dq_n \int dp_1\ldots dp_n \delta(E-H(q_1,\ldots,q_n,p_1,\ldots,p_n))}.
\end{aligned}
\tag{2.40}
$$

Proceeding further, Berry considered a classical Hamiltonian

$$
H(q_1,\ldots,q_n,p_1,\ldots,p_n) := \frac{p_1^2+\ldots+p_n^2}{2m} + V(q_1,\ldots,q_n) \tag{2.41}
$$

where m is the mass of the system and $V(q_1,\ldots,q_n)$ denotes the potential in which it moves. Then, using (2.39) and (2.40), Berry arrived at his celebrated conclusions. In particular, for the ‘irregular states’ (corresponding to classically chaotic dynamics), the autocorrelation function (2.38) should be given as

$$
C_{q_1,\ldots,q_n}(x) = \frac{1}{|S^{d-1}|}\cdot\int_{S^{d-1}} \exp\left(i\langle x,w\rangle\cdot\frac{\sqrt{2m(E-V(\mathbf{q}))}}{\hbar}\right)dw \tag{2.42}
$$

$$
= \Gamma(d/2)\cdot\frac{J_{\frac{d}{2}-1}\left(||x||\cdot\frac{\sqrt{2m(E-V(\mathbf{q}))}}{\hbar}\right)}{\left(||x||\cdot\frac{\sqrt{2m(E-V(\mathbf{q}))}}{2\hbar}\right)^{\frac{d}{2}-1}}, \tag{2.43}
$$

where $x \in \mathbb{R}^d$. We can recognize that (2.42) is the same as the covariance function of Berry’s Random Wave model, which was defined in (1.1).

Remark 2.3. *In this section, we aim to explain the foundational work of M.V. Berry as originally presented in [9]. This work, published in Journal of Physics A: Mathematical and Theoretical, is a clear example of theoretical physics rather than pure mathematics. Consequently, the arguments in [9] are understandably centered on deriving practical approximations to develop effective physical models. Notably, Berry’s Random Wave model has proven to be highly successful in empirical sense (see the discussion in [40, p. 17]). Remaining within the framework of physical modeling rather than a purely mathematical approach, a concise summary* *of Berry's Random Wave Conjecture can be found in [40, p. 18]*

'Berry went on to conjecture that all the statistical properties of chaotic systems are described by a superposition of waves with fixed wavenumber and random phases'

On the mathematical side, a caution is needed. For instance, as noted in [100, p. 1414] 'The term "chaotic" is suggestive rather than precise and can be taken to mean ergodic, mixing, hyperbolic, Bernoulli, or some more specific type of chaotic or unpredictable dynamical behavior.'

2.1.5 Berry's Cancellation Phenomena

In an influential paper [10] M. V. Berry studied behaviour of a superposition of random waves

$$\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\cos(x\cos\theta_j + y\sin\theta_j + \phi_j), \qquad (x,y)\in\mathbb{R}^2, \tag{2.44}$$

where θ_j are random directions and ϕ_j are random phases, both distributed independently and uniformly on $[0, 2\pi)$. One of the interesting observations made by Berry was the occurrence of the unexpectedly small - logarithmic terms in the asymptotic variance of nodal length and nodal number (associated respectively with one or two waves). Analogous phenomena were since observed in a number of related models by various authors. Collectively, we will refer to these phenomena (which can differ a bit in a precise formulation) as the *Berry Cancellation Phenomena* (which is, by now, a standard convention on the literature). In a recent review article [95, p. 17] Wigman formulated the following principle/metatheorem:

'under appropriate assumptions on F (or μ) its functionals are susceptible to Berry's cancellation if and only if F is monochromatic.'

A very general rigorous result of this type was recently established by Notarnicola [65, p. 1140, Theorem 2.5].

2.1.6 Universality of Berry's Random Wave Model

In this section we will discuss some of recent mathematical results which put the Berry's intuition about the generic role of his model on a rigorous footing. The results we will about to discuss are due to Canzani and Hanin [14] and to Dierickx, Nourdin, Peccati and Rossi [22]. (For an alternative discussion of the topic of the formalisation of the Berry's conjecture see the work by Ingremeau and Rivera [38] and also [37, 1, 31].) The *Monochromatic Random Wave* on the compact Riemannian manifold $(\mathcal{M}, g)$ without boundary is defined as

$$\phi_\lambda := \frac{1}{\sqrt{\dim(H_\lambda)}}\sum_{\lambda_j\in[\lambda,\lambda+1]} a_j\phi_j \tag{2.45}$$

where ϕ_j is the eigenfunction of the Laplace-Beltrami operator $\triangle_g$ on $(\mathcal{M}, g)$, with eigenvalue λ_j, normalised so that $||\phi_j||_{L^2(\mathcal{M},g)} = 1$, $(a_j)_j$ are i.i.d. standard Gaussian random variables and

$$H_\lambda := \bigoplus_{\lambda_j \in [\lambda, \lambda+1]} \ker(\triangle_g - \lambda_j^2). \tag{2.46}$$

Note that, in the above, the eigenvalues are tacitly counted with multiplicity. We will be interested in the following two statistics

$$Z_\lambda(\psi) := \int_{\phi_\lambda^{-1}(0)} \psi(x) d\mathcal{H}^{d-1}(x), \qquad \mathrm{Crit}_\lambda(\psi) := \sum_{d\phi_\lambda(x)=0} \psi(x), \tag{2.47}$$

where ψ can be any bounded measurable function. We denote

$$\Pi_\lambda(x, y) := \mathrm{Cov}(\phi_\lambda(x), \phi_\lambda(y)) = \frac{1}{\dim H_\lambda} \sum_{\lambda_j \in [\lambda, \lambda+1]} \varphi_j(x) \varphi_j(y) \tag{2.48}$$

and introduce the notation for the asymptotic covariance

$$\Pi_\infty^{x_0}(u, v) = (2\pi)^{d/2} \frac{J_{\frac{d}{2}-1}(|u-v|_{g_{x_0}})}{|u-v|_{g_{x_0}}^{\frac{d}{2}-1}} = \int_{S_{x_0} M} e^{i\langle u-v, w\rangle_{g_{x_0}}} dw \tag{2.49}$$

Here, g_{x_0} denotes the constant coefficient metric obtained by 'freezing' x_0, $S_{x_0}\mathcal{M}$ denotes the unit sphere in tangent space $T_{x_0}\mathcal{M}$ with respect to g_{x_0} and dw is hypersurface measure. An important role will be played by the following *short range correlation* assumption

$$\sup_{x,y \in M : d_g(x,y) \geq \lambda^{-1+\varepsilon}} |\nabla_x^\alpha \nabla_y^\beta \Pi_\lambda(x, y)| = o(\lambda^{\alpha+\beta}). \tag{2.50}$$

Definition 2.1 ([14]). *A point $x \in M$ is a point of isotropic scaling, denoted $x \in IS(\mathcal{M}, g)$, if for every non-negative function r_λ satisfying $r_\lambda = o(\lambda)$ as $\lambda \to \infty$ and all $\alpha, \beta \in \mathbb{N}^d$, we have*

$$\sup_{u,v \in B_{r_\lambda}} |\partial_u^\alpha \partial_v^\beta [\Pi_\lambda^x(u, v) - \Pi_\infty^x(u, v)]| = o_{\alpha,\beta}(1) \tag{2.51}$$

as $\lambda \to \infty$, where the rate of convergence depends on α, β and B_R denotes a ball of radius R centered at $0 \in T_x\mathcal{M}$. We also say that $\mathcal{M}$ is a manifold of isotropic scaling if $\mathcal{M} = IS(\mathcal{M}, g)$ and if the convergence is uniform over $x \in \mathcal{M}$ for each $\alpha, \beta \in \mathbb{N}^d$.

A starting point of analysis is the following theorem.

Theorem 2.1 ([14]). *Let $(\mathcal{M}, g)$ be a smooth, Riemannian manifold of dimension $d \geq 2$ with no boundary. Let ϕ_λ be as in (2.45) and suppose that M is a manifold* *of isotropic scaling. Then, for any bounded measurable function* $\psi : M \mapsto \mathbb{R}$,

$$\lim_{\lambda\to\infty} \mathbb{E}[\lambda^{-1} Z_\lambda(\psi)] = \frac{1}{\sqrt{\pi d}} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \cdot \int_M \psi(x) dv_g(x) \tag{2.52}$$

and

$$\lim_{\lambda\to\infty} \mathbb{E}[\lambda^{-d} \mathrm{Crit}_\lambda(\psi)] = C_d \cdot \int_M \psi(x) dv_g(x), \tag{2.53}$$

where C_d *is a positive constant that depends only on* d. *Suppose further that* ϕ_λ *has short-range correlations (see (2.50)). Then*

$$\mathrm{Var}\left[\lambda^{-1} Z_\lambda(\psi)\right] = O\left(\lambda^{-\frac{d-1}{2}}\right) \tag{2.54}$$

and

$$\mathrm{Var}\left[\lambda^{-d} \mathrm{Crit}_\lambda(\psi)\right] = O\left(\lambda^{-\frac{d-1}{2}}\right), \tag{2.55}$$

as $\lambda \to \infty$.

For $x_0 \in M$, the *Gaussian Pullback Random Wave* $\phi_\lambda^{x_0}$ on the tangent space $T_{x_0}\mathcal{M}$ to the manifold at x_0 is defined via the exponential map

$$\phi_\lambda^{x_0}(u) := \phi_\lambda\left(\exp_{x_0}\left(\frac{u}{\lambda}\right)\right), \qquad u, v \in T_{x_0}\mathcal{M}, \tag{2.56}$$

where ϕ_λ is as defined in (2.45). The next technical result is a recent achievement of Keeler. It is an important tool used in the proof of the theorem that follows afterwards. We set

$$K_\lambda^{x_0}(u, v) := \Pi_\lambda(\phi_\lambda^{x_0}(u), \phi_\lambda^{x_0}(v)), \qquad u, v \in T_{x_0}\mathcal{M}. \tag{2.57}$$

Lemma 2.1 ([43])**.** *Let* $(\mathcal{M}, g)$ *be a smooth, compact, Riemannian manifold of dimension two without conjugate points. Then, as* $\lambda \to \infty$, *for any multiindices* $\alpha, \beta \in \mathbb{N}^2$

$$\sup_{u,v\in B(r_\lambda)} |\partial^\alpha \partial^\beta \{K_\lambda^{x_0}(u, v) - (2\pi) J_0(||u - v||)\}| = O\left(\frac{1}{\log \lambda}\right) \tag{2.58}$$

whenever $r_\lambda = O\left(\sqrt{\frac{\lambda}{\log \lambda}}\right)$. *Here the implicit constant in the O-notation depends on the choice of* $x_0 \in M$ *and* r_λ, *and on the order of differentiation.*

We note that $B(r_\lambda)$ corresponds to a shrinking ball or radius $\frac{r_\lambda}{\lambda} = O\left(\frac{1}{\sqrt{\lambda \log \lambda}}\right)$ on $\mathcal{M}$. The following result, known as the *Small Scale CLT for Gaussian Pullback Random Waves*, demonstrates that if r_λ grows more slowly than $(\log \lambda)^{1/25}$ as $\lambda \to \infty$, then the nodal length $L(\phi_\lambda^{x_0}; r_\lambda)$ of Gaussian Pullback Random Waves within a ball of radius r_λ

satisfies a Central Limit Theorem, featuring the same scaling as in Berry's Random Wave Model.

Theorem 2.2 ([22]). *Assume that $(\mathcal{M}, g)$ is a compact, smooth, Riemannian surface without boundary and without conjugate points. Then, for every $x_0 \in M$ and every function $\lambda \mapsto r_\lambda$ such that $r_\lambda \to \infty$ and, as $\lambda \to \infty$*

$$\frac{r_\lambda^{25}}{(\log r_\lambda)^4} = o(\log \lambda) \tag{2.59}$$

one has that

$$\mathbb{E}[L(\phi_\lambda^{x_0}; r_\lambda)] \sim \frac{\pi r_\lambda^2}{2\sqrt{2}}, \qquad \mathrm{Var}(L(\phi_\lambda^{x_0}; r_\lambda)) \sim \frac{r_\lambda^2 \log r_\lambda}{256} \tag{2.60}$$

and

$$\frac{L(\phi_\lambda^{x_0}) - \mathbb{E}[L(\phi_\lambda^{x_0})]}{\mathrm{Var}(L(\phi_\lambda^{x_0}))^{1/2}} \xrightarrow{law} Z \sim \mathcal{N}(0,1). \tag{2.61}$$

2.1.7 Lesion-Symptom Mapping and Stochastic Geometry

Since its inception, lesion-symptom mapping has been a cornerstone of neuroscience [42], primarily facilitated by natural experiments such as traumatic head injuries. Over time, data from these injuries have largely been supplanted by insights gained from cardiovascular accidents. As one neurologist aptly put it: “our knowledge of the brain proceeds stroke by stroke” [15]. This somewhat unfortunate reliance on stroke data has only begun to change over the past few decades with the advent of advanced techniques like Magnetic Resonance Imaging (MRI), Magnetic Resonance Spectroscopy (MRS), Positron Emission Tomography (PET), and Electroencephalography. However, perhaps the most significant advancement from a statistician's perspective has come even more recently: the emergence of Functional Magnetic Resonance Imaging (fMRI). This technique fundamentally relies on the coupling between brain activity, interpreted as information processing, and local blood oxygen levels [79]. The importance of fMRI lies in its ability to observe brain activity in real time, such as when a subject engages in a specific task (e.g., a war veteran with Post-Traumatic Stress Disorder listening to a range of test sounds).

This development, however, introduced a significant challenge: how to analyze this new type of data. The difficulty is especially pronounced because many natural questions are considerably more complex than standard classification or prediction tasks for random variables residing in Euclidean space $\mathbb{R}^n$. In machine-learning terminology, we might describe this as a failure of supervised learning due to the absence of a clear ”ground truth.” A key example of this issue is the study of cerebellar dysfunction, where we seek to understand lesion-symptom mapping. Specifically, the target variable of interest is the ”quality” of the cerebellum as a biological movement optimizer, see [86, 21, 83].

These types of issues have spurred a surge of mathematical research [30, 96, 4, 89]. From our perspective, a particularly noteworthy point—and the reason for including this brief section in the current chapter—is that the same mathematical framework central to the study of Random Laplace Eigenfunctions has been pivotal in these efforts. (Worsley and his collaborators, Adler and Taylor, have played a crucial role and are the authors of one of the seminal reference texts in random geometry [2].) Nonetheless, it remains evident that these problems are still largely unresolved and are the focus of ongoing, intensive research [46, 44]. Current methods continue to face numerous challenges, contributing to issues such as replication failure [26] and the weak predictive power of natural statistics (e.g., lesion size) for symptom severity [16].

2.2 Existing Results

As previously discussed in detail, Berry's seminal work [9] introduced several conjectures connecting Quantum Chaos theory with models of random Laplace-Beltrami eigenfunctions. To rigorously explore these conjectures, the mathematical literature has since developed various models of (approximate) Random Laplace eigenfunctions [99, 81, 94, 70]. In this section, we review existing results on the fluctuations of nodal length and number for selected models and highlight their connection to our findings.

2.2.1 Random Waves on the Euclidean Space $\mathbb{R}^d$

We begin this section by examining the real-valued case of the planar Berry's Random Wave model. As outlined in Sections 1.1-1.2, for any 'well-behaved' domain $\mathcal{D}$, the nodal set of the Berry's Random Wave b_k for $k > 0$ almost surely forms a collection of smooth, disjoint curves with finite nodal length, defined as

$$\mathcal{L}(b_k, \mathcal{D}) := \text{length}\left(b_k^{-1}(\{0\}) \cap \mathcal{D}\right). \tag{2.62}$$

Physicist M. V. Berry [10] derived the exact formula for the expectation and provided the asymptotic expression for the variance:

$$\mathbb{E}\left[\mathcal{L}(b_k, \mathcal{D})\right] = \text{area}(\mathcal{D}) \cdot \frac{k}{2\sqrt{2}}, \tag{2.63}$$

$$\text{Var}\left(\mathcal{L}(b_k, \mathcal{D})\right) \sim \text{area}(\mathcal{D}) \cdot \frac{\ln k}{256\pi}. \tag{2.64}$$

Subsequently, Nourdin, Peccati, and Rossi [70] also obtained the same formula and further demonstrated that the asymptotic fluctuations of the nodal length $\mathcal{L}(b_k, \mathcal{D})$ follow a Gaussian distribution.

Theorem 2.3 ([70]). *Let b_k be a real-valued planar Berry's Random Wave with wavenumber $k > 0$, and let $\mathcal{D}$ be a convex, compact planar domain with non-empty interior and a piecewise $\mathcal{C}^1$ boundary $\partial\mathcal{D}$. As $k \to \infty$,*

$$\frac{\mathcal{L}(b_k, \mathcal{D}) - \mathbb{E}\mathcal{L}(b_k, \mathcal{D})}{\sqrt{\text{Var}\left(\mathcal{L}(b_k, \mathcal{D})\right)}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1), \tag{2.65}$$

where d denotes convergence in distribution.

In the same study, M. V. Berry also examined the complex analog of this problem, which involves counting the nodal points (i.e., points in the zero set) of two independent real Berry's Random Waves (See sections 1.1-1.2):

$$\mathcal{N}(b_k, \hat{b}_k, \mathcal{D}) := \Big|\{x \in \mathcal{D} : b_k(x) = \hat{b}_k(x) = 0.\}\Big| \tag{2.66}$$

M.V. Berry established [10] the exact formulas for the expectation and the asymptotic variance:

$$\mathbb{E}\left[\mathcal{N}(b_k, \hat{b}_k, \mathcal{D})\right] = \text{area}(\mathcal{D}) \cdot \frac{k^2}{4\pi}, \tag{2.67}$$

$$\text{Var}\left(\mathcal{N}(b_k, \hat{b}_k, \mathcal{D})\right) \sim \text{area}(\mathcal{D}) \cdot \frac{11}{64\pi^3} \cdot k^2 \ln k. \tag{2.68}$$

As with the previous case, this result was later confirmed by Nourdin, Peccati, and Rossi in [70], who also showed that the corresponding asymptotic fluctuations are Gaussian.

Theorem 2.4 ([70])**.** *Let b_k and $\hat{b}_k$ be independent real-valued planar Berry's Random Waves with wavenumber $k > 0$, and let $\mathcal{D}$ be a convex, compact planar domain with non-empty interior and a piecewise $\mathcal{C}^1$ boundary $\partial\mathcal{D}$. As $k \to \infty$,*

$$\frac{\mathcal{N}(b_k, \hat{b}_k, \mathcal{D}) - \mathbb{E}\mathcal{N}(b_k, \hat{b}_k, \mathcal{D})}{\sqrt{\text{Var}\left(\mathcal{N}(b_k, \hat{b}_k, \mathcal{D})\right)}} \xrightarrow{d} Z \sim \mathcal{N}(0,1), \tag{2.69}$$

where d denotes convergence in distribution.

Consider now l independent Berry's Random Waves $b_k^1, \dots, b_k^l$ on $\mathbb{R}^d$ where $1 \le l \le d$, $d \ge 3$, $k > 0$. The corresponding nodal volume

$$\mathcal{H}^{(d-l)}\left(\{x \in \mathcal{D} : b_k^1(x) = \ldots = b_k^l(x) = 0\}\right) \tag{2.70}$$

has been considered by Notarnicola [65], where again $\mathcal{D}$ is a 'well-behaved' domain. Therein, Notarnicola computed the expected value of the nodal volume

$$\mathbb{E}\left[\mathcal{H}^{(d-l)}\left(\{x \in \mathcal{D} : b_k^1(x) = \ldots = b_k^l(x) = 0\}\right)\right] = \alpha(l,d) \cdot \text{Vol}(\mathcal{D}) \cdot k^l. \tag{2.71}$$

Here, see [65, p. 1134, Eq. (1.8)], we have that

$$\alpha(l,d) := \frac{(d)_l \kappa_d}{(2\pi)^{l/2} \kappa_{d-l}}, \tag{2.72}$$

where $(d)_l := d!/(d-l)!$ and $\kappa_d := \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ stands for the volume of the unit ball in $\mathbb{R}^d$. We note that expectation (2.71), in the case $d = 3$ and $l = 2$, was also computed by Dalmao, Estrade and León [19]. Furthermore, the same authors (and again for $d = 3$ and $l = 2$) established corresponding variance bounds and Central Limit Theorem (see [19, p. 384, Proposition 3.4]), with the lower bound being due to Dalmao (see [18, p. 1094, Theorem 2.1]).

Theorem 2.5 ([19, 18]). *Let b_k, $\hat{b}_k$ denote independent real-valued Berry Random Waves on $\mathbb{R}^3$ with wavenumber $k > 0$ and choose any bounded domain $\mathcal{D} \subset \mathbb{R}^3$ with $\mathrm{Vol}(\mathcal{D}) \geq 1$. Then, as $k \to \infty$, we have*

$$c \cdot k \leq Var(\mathcal{N}(b_k, \hat{b}_k, \mathcal{D})) \leq C \cdot k, \tag{2.73}$$

for some deterministic strictly positive constants $C \geq c > 0$ (independent of k and $\mathcal{D}$). Furthermore, as $k \to \infty$, we have

$$\frac{\mathcal{N}(b_k, \hat{b}_k, \mathcal{D}) - \mathbb{E}\mathcal{N}(b_k, \hat{b}_k, \mathcal{D})}{k^{1/2}} \xrightarrow{d} Z \sim \mathcal{N}(0, V), \tag{2.74}$$

where $V \in (0, \infty)$ is a strictly positive and finite constant.

Let us make the following additional remark. The authors of [19, 18] made a choice to grow the domain, instead of increasing the wavenumber. Due to the fact that the Berry's Random Waves are stationary, it is an equivalent formulation. For the sake completeness, we state the same result in this below, in this different formulation. We remark that the assumption $\mathrm{Vol}(\mathcal{D}) \geq 1$ shown below is just to avoid other asymptotic extrema, i.e., to exclude possibility that $\mathrm{Vol}(\mathcal{D}) \to 0$.

Theorem 2.6 ([19, 18]). *Let b_1, $\hat{b}_1$ denote independent real-valued Berry Random Waves on $\mathbb{R}^3$ with fixed wavenumber equal to 1. There exist numerical constants $\widetilde{C} \geq \widetilde{c} > 0$ such that for every bounded domain $\mathcal{D} \subset \mathbb{R}^3$ with $\mathrm{Vol}(\mathcal{D}) \geq 1$ and for every $k \geq 1$, we have*

$$\widetilde{c} \cdot \mathrm{Vol}(k\mathcal{D}) \leq Var(\mathcal{N}(b_1, \hat{b}_1, k\mathcal{D})) \leq \widetilde{C} \cdot \mathrm{Vol}(k\mathcal{D}). \tag{2.75}$$

Furthermore, if we fix $\mathcal{D}$ and let $k \to \infty$, then we have

$$\frac{\mathcal{N}(b_1, \hat{b}_1, k \cdot \mathcal{D}) - \mathbb{E}\mathcal{N}(b_1, \hat{b}_1, k \cdot \mathcal{D})}{k^{3/2}\,(\mathrm{Vol}(\mathcal{D}))^{3/2}} \xrightarrow{d} Z \sim \mathcal{N}(0, \widetilde{V}), \tag{2.76}$$

where $\widetilde{V} \in (0, \infty)$ is a strictly positive and finite constant.

A discussion related to equivalence between formulation as in Theorem 2.5 and Theorem as in 2.6 can be found in [64, p. 180-181]. The next result, obtained by Grotto, Maini, and Todino in [33], is significant in our context, as explained below.

Theorem 2.7 ([33]). *Let b_k be the real-valued Berry's Random Wave on $\mathbb{R}^d$ with $d \geq 2$, and let H_q denote the Hermite polynomial of order $q \geq 1$. As $k \to \infty$, there* *exists a strictly positive constant $c_q > 0$ such that the following asymptotics hold:*

$$Var\left(\int_{B(0,1)} H_q(b_k(x))\,dx\right) \sim c_q \cdot \begin{cases} k^{1-d} & \text{if } q = 2, \\ k^{-2}\log k & \text{if } q = 4 \text{ and } d = 2, \\ k^{-d} & \text{for all other } d \geq 2 \text{ and } q \geq 3 \end{cases} . \tag{2.77}$$

Using the above theorem and taking into account the cancellation phenomena in the second chaos, the identical asymptotic order of the functions $\rho_{\frac{d}{2}-1}$, $\rho_{\frac{d}{2}}$, and $\rho_{\frac{d}{2}+1}$, and applying chaos expansion as in [70], we can anticipate that for $d \geq 3$,

$$\text{Var}\left(\mathcal{L}(b_k, \mathcal{D})\right) \sim \text{const} \times k^{-(d-2)}, \tag{2.78}$$

$$\text{Var}\left(\mathcal{N}(b_k, \hat{b}_k, \mathcal{D})\right) \sim \text{const} \times k^{-(d-4)}, \tag{2.79}$$

and that for the nodal volume associated with $l \leq d$ waves, the variance order is expected to be $k^{-(d-l^2)}$. Notably, for $d = 3$, formula (2.79) aligns with the result established in (2.73).

2.2.2 Random Waves on the Hypersphere $\mathbb{S}^d$

The *Random Spherical Harmonics* (RSH) are a model of random Laplace eigenfunctions on the d-dimensional unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. It has been extensively studied by various authors [60, 61, 11, 12, 59, 90, 95]. Of particular interest is the most-studied case $d = 2$. We recall that the Laplace eigenvalues on the two-dimensional unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ are the non-negative integers of the form $\lambda_l^2 = l(l+1)$. Here $l \in \mathbb{Z}_{\geq 0}$ and we note that the eigenspace corresponding to λ_l^2 has dimension $2l+1$. Fix for a moment an arbitrary L^2-orthonormal basis $\eta_1, \ldots, \eta_{2l+1}$ of the eigenspace associated with λ_l. We define the *Random Spherical Harmonics* (RSH) of degree $\in \mathbb{N}$, as a random field

$$f_l(x) = \frac{1}{\sqrt{2l+1}} \cdot \sum_{k=1}^{2l+1} a_k \eta_k(x), \qquad x \in \mathbb{S}^2, \tag{2.80}$$

where $a_1, \ldots, a_{2l+1}$ are i.i.d. standard Gaussian random variables. It is not too difficult to check that the law of the Random Spherical Harmonics is independent of the choice of the basis $\eta_1, \ldots, \eta_{2l+1}$. The significance of this model in our context is explained in the following remark.

Remark 2.4. *It is well-known that, locally and after appropriate rescaling, the covariance function of Random Spherical Harmonics on two-dimensional sphere $\mathbb{S}^2$ converges, as $l \to \infty$, to the covariance function J_0 of the real Berry's Random Wave b_1 with wave-number $k = 1$ (as a consequence of the Hilb's asymptotic), see for instance [95, p. 17-18, Section 3.3].*

The expected nodal length of Random Spherical Harmonics on two-dimensional sphere $\mathbb{S}^2$ was computed by Berard [8]

$$\mathbb{E}\left[\mathcal{L}_\lambda\right] = \pi\sqrt{2} \cdot \sqrt{\lambda(\lambda+1)}, \tag{2.81}$$

(see attribution given in [94, p. 3, Eq. (8)]). The asymptotic variance formula was established by Wigman [94, p. 5, Thm. 1.1].

Theorem 2.8 ([94]). *Let $\mathcal{L}_l$ denote the nodal length of Random Spherical Harmonics of degree l on the two-dimensional sphere $\mathbb{S}^2$. Then, the following holds:*

$$Var(\mathcal{L}_\lambda) = \frac{\log \lambda}{32} + O(1), \tag{2.82}$$

where $O(1)$ represents a term that is asymptotically bounded by an absolute constant.

The Random Hyperspehrical Harmonics on $\mathbb{S}^d$, $d \geq 3$, are defined in a manner analogous to 2.80, see [57]. The expected nodal volume associated with Random Spherical Harmonics on the hypersphere $\mathbb{S}^d$, $d \geq 3$, was also was comuted by Berard [8] to be

$$\mathbb{E}\left[\mathcal{L}_\lambda\right] = \frac{(2\pi)^{d/2}}{\sqrt{d} \cdot \Gamma(\frac{d}{2})} \cdot \sqrt{\lambda(\lambda + d - 1)} \tag{2.83}$$

where $E_{\lambda;d} := \lambda(\lambda + d - 1)$ is the λ-th eigenvalue of the Laplace-Beltrami operator on the d-dimensional hypersphere. The current best upper bound on the asymptotic variance was recently established by Marinucci, Rossi and Todino [57, p. 10, Thm. 3.8] and, as stated therein, it is believed to be sharp.

Theorem 2.9 ([57]). *Let $\mathcal{L}_l$ denote the nodal volume of Random Spherical Harmonics of degree l on $\mathbb{S}^d$ for $d \geq 3$. As $l \to \infty$, the following holds:*

$$Var(\mathcal{L}_l) = O\left(l^{-(d-2)}\right). \tag{2.84}$$

As mentioned earlier, Berry's Random Waves are expected to exhibit behavior similar to that of Random Hyperspherical Harmonics. We observe that equation (2.84) is consistent with equation (2.78). In [59], Marinucci, Rossi, and Wigman established the full correlation between the *sample trispectrum* and the nodal length, from which they derived a quantitative Central Limit Theorem as a corollary. To help us state this result, we set

$$\widetilde{\mathcal{L}}_l := \frac{\mathcal{L}_l - \mathbb{E}\mathcal{L}_l}{\sqrt{\mathrm{Var}\left(\mathcal{L}_l\right)}}, \qquad \widetilde{\mathcal{M}}_l := \frac{\mathcal{M}_l}{\sqrt{\mathrm{Var}\left(\mathcal{M}_l\right)}}. \tag{2.85}$$

Theorem 2.10 ([59]). *Let $\mathcal{L}_l$ and $\mathcal{M}_l$ denote the normalized nodal length and normalized trispectrum of Random Spherical Harmonics of degree l on the two-dimensional sphere $\mathbb{S}^2$. As $l \to \infty$, the following holds:*

$$\mathbb{E}\left[\{\widetilde{\mathcal{L}}_l - \widetilde{\mathcal{M}}_l\}^2\right] = O\left(\frac{1}{\log l}\right). \tag{2.86}$$

Furthermore, as $l \to \infty$, we have

$$W_1\left(\widetilde{\mathcal{L}}_l, Z\right) = O\left(\frac{1}{\sqrt{\log l}}\right), \tag{2.87}$$

where Z is a standard Gaussian random variable and W_1 denotes the Wasserstein distance (see Subsection 3.3.1).

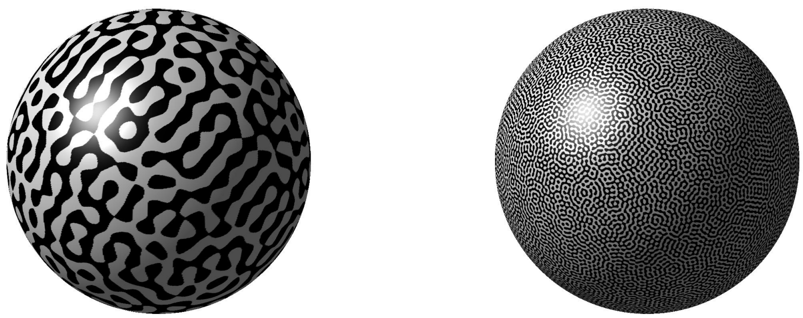

Figure 2.2: Left: Nodal domains of random spherical harmonics of degree 40. Right: Nodal domains of random spherical harmonics of degree 200. Credit: Alex Barnett, used with permission. https://users.flatironinstitute.org/~ahb/rpws/

2.2.3 Random Waves on the Flat Torus $\mathbb{T}^d$

The *Arithmetic Random Waves* (ARW) are the well-known model of random Laplace eigenfunctions on the multidimensional flat torus and associated nodal volumes have been a subject of extensive study [74, 81, 82, 45, 55, 20]. We define

$$\mathcal{S} = \{n \in \mathbb{Z} : n = a^2 + b^2 \text{ for some } a, b \in \mathbb{Z}\}. \tag{2.88}$$

Given a positive integer $n \in \mathcal{S}$ the *Arithmetic Random Wave* T_n is defined as a centred Gaussian field with the covariance function

$$\mathbb{E}[T_n(x) \cdot T_n(y)] = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi\langle \lambda, x - y \rangle), \qquad x, y \in \mathbb{T}^2, \tag{2.89}$$

where $\Lambda_n = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \lambda_1^2 + \lambda_2^2 = n\}$ and $\mathcal{N}_n$ denotes the cardinality of the set Λ_n. The Arithmetic Random Waves $(T_n)_{n \in \mathcal{S}}$ are random Laplace eigenfunctions on the standard flat 2-torus $\mathbb{T} := \mathbb{R}^2 \setminus \mathbb{Z}^2$, i.e.,

$$\triangle T_n(x) + E_n T_n(x) = 0, \qquad x \in \mathbb{T} \tag{2.90}$$

where

$$E_n := 4\pi^2 n. \tag{2.91}$$

The nodal volume of $\mathcal{L}_n$ is almost surely a union of smooth curves and we can write

$$\mathcal{L}_n := \text{length}(T_n^{-1}\{0\}). \tag{2.92}$$

The expectation of the nodal volume $\mathcal{L}_n$ has been computed by Rudnick and Wigman [81].

Theorem 2.11 ([81])**.** *Let $s \in \mathcal{S}$ and let $\mathcal{L}_n$ be the nodal length of the Arithmetic Random Wave T_n. Then*

$$\mathbb{E}\mathcal{L}_n = \frac{\sqrt{E_n}}{2\sqrt{2}}. \tag{2.93}$$

Furthermore, we define the probability measure on $\mathbb{S}^1$ as

$$\mu_n := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\frac{\lambda}{\sqrt{n}}}, \tag{2.94}$$

for $k \in \mathbb{Z}$ we denote $\mu_n(k)$ to be the corresponding Fourier coefficient

$$\hat{\mu}_n(k) := \int_{S^1} z^{-k} d\mu_n(z). \tag{2.95}$$

and we write:

$$c_n := \frac{1 + \widehat{\mu}_n(4)^2}{512}. \tag{2.96}$$

In [45] Krishnapur, Kurlberg and Wigman established the following theorem.

Theorem 2.12 ([45])**.** *If $(n_i)_{i\geq 1}$ is any sequence of elements in $\mathcal{S}$ such that $\mathcal{N}_{n_i} \to \infty$, then*

$$\text{Var}(\mathcal{L}_{n_i}) = c_{n_i} \cdot \frac{E_{n_i}}{\mathcal{N}_{n_i}^2}(1 + o(1)). \tag{2.97}$$

Further, given any $c \in [1/512, 1/256]$, there exists a sequence $(n_i)_{i\geq 1}$ of elements in $\mathcal{S}$ such that as $i \to \infty$, we have $\mathcal{N}_{n_i} \to \infty$ together with $c_{n_i} \to c$ so that

$$\text{Var}(\mathcal{L}_{n_i}) = c \cdot \frac{E_{n_i}}{\mathcal{N}_{n_i}^2}(1 + o(1)). \tag{2.98}$$

Furthermore, in [55] Marinucci, Peccati, Rossi and Wigman proved the following non-central limit theorem.

Theorem 2.13 ([55])**.** *Let $\{\eta_{n_j}\} \subset \mathcal{S}$ be a subsequence of $\mathcal{S}$ satisfying $\mathcal{N}_{n_j} \to \infty$, such that the sequence $\{|\widehat{\mu}_{n_j}(4)| : j \geq 1\}$ of non-negative numbers converges, that* *is:*

$$\left| |\widehat{\mu}_{n_j}(4)| \right| \to \eta, \tag{2.99}$$

for some $\eta \in [0,1]$*. Then*

$$\frac{\mathcal{L}_{n_j} - \mathbb{E}\mathcal{L}_{n_j}}{\sqrt{\operatorname{Var}(\mathcal{L}_{n_j})}} \xrightarrow{d} \mathcal{M}_\eta, \tag{2.100}$$

where

$$\mathcal{M}_\eta := \frac{1}{2\sqrt{1+\eta^2}} \left(2 - (1+\eta)X_1^2 - (1-\eta)X_2^2\right), \tag{2.101}$$

where $X = (X_1, X_2)$ *are independent standard Gaussian.*

For $n \in \mathcal{S}$, let T_n and $\hat{T}_n$ be two independent Arithmetic Random Waves. The volume of the intersection of their nodal sets is described by the random integer

$$I_n := \left| \{x \in \mathbb{T}^2 : T_n(x) = \hat{T}_n(x) = 0\} \right|, \tag{2.102}$$

where $|\cdot|$ denotes the cardinality of the set. Dalmao, Nourdin, Peccati, and Rossi present a thorough characterization of its fluctuations in [20, p. 4, Theorem 1.2], detailing non-universal variance asymptotics, along with non-universal and non-central limit theorem.

Theorem 2.14 ([20])**.** ***1. (Finiteness and mean)*** *With probability one, for every* $n \in \mathcal{S}$ *the set of nodal points is a finite collection of isolated points, and*

$$\mathbb{E}[I_n] = \frac{E_n}{4\pi} = \pi n. \tag{2.103}$$

2. (Non-universal variance asymptotics) *As* $\mathcal{N}_n \to \infty$*,*

$$\operatorname{Var}(I_n) = d_n \times \frac{E_n^2}{\mathcal{N}_n^2}(1 + o(1)) = V_n(1 + o(1)), \tag{2.104}$$

where

$$d_n := \frac{3\widehat{\mu}_n(4)^2 + 5}{128\pi^2}, \qquad \text{and } V_n := d_n \times \frac{E_n^2}{\mathcal{N}_n^2}. \tag{2.105}$$

3. (Universal law of large numbers) *Let* $\{n_j\} \subset \mathcal{S}$ *be a subsequence such that* $\mathcal{N}_{n_j} \to +\infty$*. Then, for every sequence* $\{\varepsilon_{n_j}\}$ *such that* $\varepsilon_{n_j}\mathcal{N}_{n_j} \to \infty$*, one has that*

$$\mathbb{P}\left(\left| \frac{I_{n_j}}{\pi n_j} - 1 \right| > \varepsilon_{n_j} \right) \to 0. \tag{2.106}$$

4. (Non-universal and non-central second order fluctuations) *Let* $\{n_j\} \subset$

$\mathcal{S}$ be such that $\mathcal{N}_{n_j} \to +\infty$ and $|\widehat{\mu}_{n_j}(4)| \to \eta \in [0,1]$. Then,

$$\begin{aligned} \widetilde{I}_{n_j} &:= \frac{I_{n_j} - \mathbb{E}I_{n_j}}{V_{n_j}^{1/2}} \\ &\xrightarrow{d} \frac{1}{2\sqrt{10+6\eta^2}} \left(\frac{1+\eta}{2}A + \frac{1-\eta}{2}B - 2(C-2) \right) \end{aligned} \tag{2.107}$$

with A, B, C independent random variables such that $A \overset{d}{=} B \overset{d}{=} 2X_1^2 + 2X_2^2 - 4X_3^2$ and $C \overset{d}{=} X_1^2 + X_2^2$, where (x_1, X_2, X_3) is standard Gaussian vector of $\mathbb{R}^3$.

Since T_n and $\hat{T}_n$ have the same energies, a direct comparison with our results is not possible. Nevertheless, we would like to note the following. The non-universality in [20, p.4, Theorem 1.2] is controlled by the Fourier coefficient

$$\hat{\mu}_n(4) = \int_{S^1} z^{-4} d\mu(z). \tag{2.108}$$

Using corresponding angle measure defined by the condition

$$v_n(\theta) = \mu_n(\cos\theta, \sin\theta), \qquad \theta \in [0, 2\pi), \tag{2.109}$$

the formula (2.108) can be re-written as

$$\hat{\mu}_n(4) = 1 - \frac{8}{|\Lambda_n|} \int_0^{2\pi} \cos^2\theta \sin^2\theta dv_n(\theta). \tag{2.110}$$

We note that, on a purely heuristic level, the expression (2.110) arises from computations which are somewhat similar to the computations that are leading to appearance of the parameter r^{exp} in Theorem 4.1 (evaluation of the integrals in (6.58) through Lemma 6.9).

2.2.4 Intersections Against Deterministic Submanifolds

The intersections of the zero set of 2-and 3 dimensional Arithmetic Random Waves (ARW) *against a smooth reference curve or hypersuface* (respectively) has been studied in [82, 80, 51], see also [95, Sections 1.4 and 4.3]. Here, we want to make a related simple and heuristic observation. As described in the forthcoming Remark 4.1, our upcoming Theorem 4.1 (see also Section 1.2) can be extended to include the scenario $r^{log} = 0$. In this situation, it is still necessary that $k_n \to \infty$ but this divergence can be arbitrarily slower than the divergence $K_n \to \infty$. We could think of the possibility $k_n \to k < \infty$, $K_n \to \infty$, as a limiting case of such scenario. In this situation, fixing the randomness associated with $b_{k_n} := b_1(k_n\cdot)$ and letting $n \to \infty$ should give a scenario where we intersect the nodal lines of the process $\hat{b}_{K_n}$ against a curve $\mathcal{C}_{k_n}$ which is essentially constant - as it approaches the limit curve $\mathcal{C}_k$. Here,

$$\mathcal{C}_{k_n} := \{x \in \mathcal{D} : b_{k_n}(x) = 0\}, \qquad \mathcal{C}_k := \{x \in \mathcal{D} : b_k(x) = 0\}.$$

Let $\tau_n : [0, \mathcal{L}(b_{k_n}, \mathcal{D})) \to \mathbb{R}^2$, $\tau : [0, \mathcal{L}(b_k, \mathcal{D})) \to \mathbb{R}^2$ be a (well-behaved) parametrisations of the curves $\mathcal{C}_{k_n}$ and $\mathcal{C} : k$, respectively. The above considerations suggest that (conditionally on the randomness of the process b_k) we should have

$$\begin{aligned} \mathcal{N}(b_k, \hat{b}_{K_n}, \mathcal{D}) &= |\{0 \leq t < \mathcal{L}(b_{k_n}, \mathcal{D})) : \hat{b}_{K_n}(\tau_n(t)) = 0\}| \\ &\approx |\{0 \leq t < \mathcal{L}(b_k, \mathcal{D})) : \hat{b}_{K_n}(\tau(t)) = 0\}|, \end{aligned} \tag{2.111}$$

which is a number of intersection against a deterministic curve.

Chapter 3 Preliminaries

This chapter contains well-known results and so the reader might want to omit it at first reading.

3.1 Deterministic Toolbox

3.1.1 Normalised Bessel Functions of the First Kind

We will introduce now the classical Bessel functions of the first kind. They will play a central role throughout the rest of the thesis.

Definition 3.1. *The Bessel function of the first kind and real order α is a (particular) solution to Bessel's differential equation (see [23, 10.2 (i) Bessel's equation, Eq. 10.2.1])*

$$t^2 f''(t) + tf'(t) + (t^2 - \alpha^2) f(t) = 0, \tag{3.1}$$

which takes the form

$$J_\alpha(r) = \sum_{k=0}^{+\infty} (-1)^k \frac{(r/2)^{(2k+\alpha)}}{k!\Gamma(\alpha+1+k)} = \frac{(r/2)^\alpha}{\Gamma(\alpha+1)} - \frac{(r/2)^{2+\alpha}}{2\Gamma(\alpha+2)} + \dots, \quad \alpha, r \in \mathbb{R}, \tag{3.2}$$

(see [23, 10.2(ii) Bessel Function of the First Kind, Eq. 10.2.2]) and where Γ denotes the standard Euler Gamma function (see [23, 5.2 (i) Gamma and Psi Functions, Eq. 5.2.1]).

The preceding plot highlights a key property of Bessel functions of the first kind: their slow decay to zero at infinity, coupled with oscillatory, trigonometric-like behavior. Formal details follow in the next remark.

Remark 3.1. *The Bessel functions J_α described in last definition enjoy the following properties:*

1. *(Uniform bound) If $\alpha \geq -1/2$ then for some constant $K(\alpha)$ (depending only*

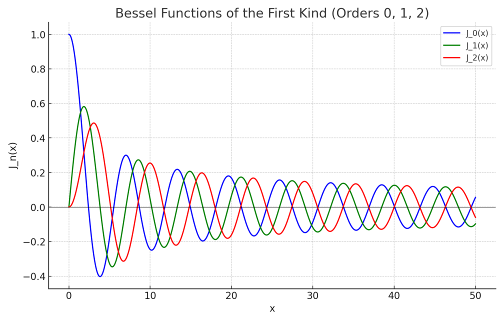

Figure 3.1: Numerical plot of selected Bessel functions of the first kind.

on α) and all $r > 0$ we have

$$|J_\alpha(r)| \leq r^{-1/2} K(\alpha), \tag{3.3}$$

(see [87, p. 167, Theorem 7.31.2]).

2. *(Asymptotic forms) We have*

$$J_\alpha(r) \sim \frac{(r/2)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \neq -1, -2, \ldots, \quad r \downarrow 0, \tag{3.4}$$

$$J_\alpha(r) = r^{-1/2}\sqrt{\frac{\pi}{2}}\left[\cos\left(r - \frac{2\alpha+1}{4}\pi\right) + o(1)\right], \quad \alpha \in \mathbb{R}, \quad r \uparrow \infty, \tag{3.5}$$

where $o(1)$ denotes remainder converging to zero as r diverges to infinity (see [23, 10.7 Limiting forms, Eq. 10.7.3, and Eq. 10.7.8, first form]).

3. *(Recurrence relations for derivatives) We have for every $\alpha \geq 0$*

$$\frac{\partial}{\partial r} J_\alpha(r) = -J_{\alpha+1}(r) + \frac{\alpha}{r} J_\alpha(r), \quad r \in \mathbb{R}, \tag{3.6}$$

(see [23, 10.6(i) Recurrence relations and derivatives, Eq. 10.6.2, second form]) and where the case $r = 0$ should be understood by taking an appropriate limit, which exists thanks to the asymptotic form of J_α at zero (as discussed in the preceding point).

We recall that the normalised Bessel functions of the first kind were defined in (1.3)-(1.4). The following easy lemma will be very useful in simplifying computations involving derivatives as it takes advantage of recurrences inherited from standard Bessel functions J_α.

Lemma 3.1. *Let $\alpha \geq 0$ and $\rho_\alpha(r)$ be a normalised Bessel function of the first* *kind. Then, for $z \in \mathbb{R}^d \setminus \{0\}$ and $i, j \in \{1, ..., d\}$ we have*

$$\begin{aligned} \rho'_\alpha(r) &= \frac{(-r)}{2(\alpha+1)}\rho_{\alpha+1}(r), \qquad \rho''_\alpha(r) = \frac{r^2}{4(\alpha+1)(\alpha+2)}\rho_{\alpha+2}(r), \\ \partial_i \rho_\alpha(|z|) &= \frac{(-z_i)}{2(\alpha+1)}\rho_{\alpha+1}(|z|), \\ \partial_{ij}\rho_\alpha(|z|) &= \frac{-\delta_{ij}}{2(\alpha+1)}\rho_{\alpha+1}(|z|) + \frac{z_i z_j}{4(\alpha+1)(\alpha+2)}\rho_{\alpha+2}(|z|). \end{aligned} \tag{3.7}$$

Proof. By the standard recurrence property of the Gamma function we have that $\Gamma(\alpha+1) = \frac{\Gamma(\alpha+2)}{\alpha+1}$ (see [23, Eq. 10.29.2]). Moreover, $\left[J_\alpha(r)r^{-\alpha}\right]' = J'_\alpha(r)r^{-\alpha} - \alpha J_\alpha(r)r^{-(\alpha+1)} = -\left[\frac{\alpha}{r}J_\alpha(r) - J'_\alpha(r)\right]r^{-\alpha} = -r\left[J_{\alpha+1}(r)r^{-(\alpha+1)}\right]$, with the last equality following by plugging in the recurrence relationship for the derivatives of Bessel functions that we recorded in Point 3 of Remark 3.1. The first requested formula follows now by combining these two observations. The remaining expressions follow immediately by a repeated application of the one already proved, in conjunction with chain rule, product rule and formula $\partial_i|z| = \frac{z_i}{|z|}$. □

3.1.2 Hermite Polynomials

The well-known (probabilistic) Hermite polynomials H_n are defined by the formula

$$\begin{aligned} &H_0(x) = 1, \\ &H_n(x) = -\partial_x H_{n-1}(x) + x H_{n-1}(x), \qquad n = 1, 2, \dots \end{aligned} \tag{3.8}$$

see for example [69, p. 13]. In particular, we have

$$\begin{aligned} &H_0(x) = 1, \qquad H_1(x) = x, \qquad H_2(x) = x^2 - 1, \\ &H_3(x) = x^3 - 3x, \qquad H_4(x) = x^4 - 6x^2 + 3. \end{aligned} \tag{3.9}$$

Some relevant properties of the (probabilistic) Hermite polynomials are the following:

(i) For every $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$H_n(x) = (-1)^n H_n(-x). \tag{3.10}$$

(ii) For every $k \in \mathbb{N}$,

$$H_{2k+1}(0) = 0, \qquad H_{2k}(0) = (-1)^k (2k-1)!! \tag{3.11}$$

(iii) Consider the Gaussian L^2 space

$$L^2_{\mathbb{R}}(\mathbb{R}^{\mathbb{N}},\ \mathcal{B}(\mathbb{R}^{\mathbb{N}}),\ d\gamma), \tag{3.12}$$

where γ is the law of a countable collection of i.i.d. Gaussian r.v.s. Then the products of Hermite polynomials

$$\prod_{l=1}^{n} H_{j_l}(x_l), \qquad n \in \mathbb{N}, \qquad j_l \in \mathbb{N}, \qquad x_l \in \mathbb{R}, \tag{3.13}$$

form an orthogonal basis of this L^2 space and satisfy the property

$$\left\| \prod_{l=1}^{n} H_{j_l}(x_l) \right\|^2_{L^2_{\mathbb{R}}(\mathbb{R}^{\mathbb{N}},\mathcal{B}(\mathbb{R}^{\mathbb{N}}),\ d\gamma)} = \prod_{l=1}^{n} j_l! \tag{3.14}$$

3.1.3 On the Level Sets of Deterministic Functions

The following lemma is a standard result which is equally valid with many other choices of the approximation of the Dirac's delta function, see for instance [2, p. 269, Theorem 11.2.3].

Theorem 3.1. *Let $d, m \geq 1$ be strictly positive integers, let $f = (f_1, \ldots, f_d) : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $g : \mathbb{R}^d \mapsto \mathbb{R}^m$ be C^1 functions over their respective domains. Let $T \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^m$ be, respectively, a closed and an open set. Suppose that for $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ we have that $\partial T \cap f^{-1}(u)$ is empty and that if $f(t) = u$ then both $\det \nabla f(t) \neq 0$ and $g(t) \notin \partial B$. Then, there is a finite number N_u of points $t \in T$ such that $f(t) = u$ and $g(t) \in B$. Furthermore,*

$$N_u = \lim_{\varepsilon \downarrow 0} \frac{1}{(2\varepsilon)^d} \int_T \prod_{i=1}^{d} \mathbf{1}_{\{|f_i(t)-u_i| \leq \varepsilon\}} \mathbf{1}_B(g(t)) |\det \nabla f(t)| dt, \tag{3.15}$$

where $\nabla f(t)$ denotes the matrix of the first partial derivatives of f.

3.2 Elements of Gaussian Analysis

A real-valued Gaussian random variable with mean μ and variance σ^2 is defined as a random variable $X \sim \mathcal{N}(\mu, \sigma)$ with characteristic function

$$\mathbb{E}e^{itX} = \exp\left(it\mu - \frac{\sigma^2 t^2}{2}\right), \qquad t \in \mathbb{R}. \tag{3.16}$$

A Gaussian random vector $X \sim \mathcal{N}(\mu, \Sigma)$ with mean $\mu = (\mu_1, \ldots, \mu_d)$ and non-negative definite covariance matrix Σ is defined as a random vector with characteristic function

$$\mathbb{E}\exp\left(i\langle t, X\rangle\right) = \exp\left(i\langle t, \mu\rangle - \frac{1}{2}\langle t, \Sigma t\rangle\right) \qquad t = (t_1, \ldots, t_d) \in \mathbb{R}^d, \tag{3.17}$$

where $\langle \cdot, \cdot \rangle$ stands for standard Euclidean inner product. A Gaussian random variable with mean μ and variance $\sigma^2 > 0$ has density

$$\gamma_{\mu,\sigma}(t) := \frac{1}{\sqrt{2\pi}\sigma} e^{-(t-\mu)^2/2\sigma^2}, \qquad t \in \mathbb{R}. \tag{3.18}$$

Furthermore, a Gaussian vector with mean $m \in \mathbb{R}^d$ and strictly positive-definite covariance matrix Σ has density

$$\gamma_{m,\Sigma}(x) := \frac{\exp\left(-\frac{1}{2}(x-m)^{tr}\Sigma^{-1}(x-m)\right)}{(2\pi)^{d/2}\sqrt{\det \Sigma}}, \qquad x \in \mathbb{R}^d. \tag{3.19}$$

A Gaussian random field $(X_t)_{t\in T}$ indexed by the set T is a collection of random variables such that for every $n \in \mathbb{N}$ and any choice of distinct points $t_1, \dots, t_n \in T$, the random vector $(X_{t_1}, \dots, X_{t_n})$ has a Gaussian distribution on $\mathbb{R}^n$.

3.2.1 Gaussian Conditioning Formulas

The following result is standard, elementary and remarkably useful, see [5, p. 18, Proposition 1.2]. In its formulation we will use the following well-known convention: let X, Y the square-integrable random vectors on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then, $\mathrm{Var}(Y)$ is a $m \times m$ non-negative definite matrix defined by

$$\mathrm{Var}(Y)_{ij} := \mathrm{Cov}(Y_i, Y_j), \qquad i, j \in \{1, \dots, n\}, \tag{3.20}$$

and the matrix $\mathrm{Cov}(X, Y)$ is the $n \times m$ matrix defined by

$$\mathrm{Cov}(X, Y)_{ij} := \mathrm{Cov}(X_i, Y_j), \tag{3.21}$$

where $i = 1, \dots, n$ and $j = 1, \dots, m$.

Lemma 3.2. *Let $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ be two Gaussian vectors such that $(X, Y) \in \mathbb{R}^{n+m}$ is also a Gaussian vector and suppose $\mathrm{Var}(Y)$ is non-singular. Then, for any bounded measurable function $f : \mathbb{R}^n \mapsto \mathbb{R}$, we have*

$$\mathbb{E}(f(X)|Y = y) = \mathbb{E}(f(Z + Cy)), \tag{3.22}$$

where

$$C := \mathrm{Cov}(X, Y)[\mathrm{Var}(Y)]^{-1}, \qquad Z \sim \mathcal{N}(\mu, \Sigma), \tag{3.23}$$

with

$$\begin{aligned} \mu &:= \mathbb{E}(X) - C\mathbb{E}(Y), \\ \Sigma &:= \mathrm{Var}(X) - \mathrm{Cov}(X, Y)[\mathrm{Var}(Y)]^{-1}\mathrm{Cov}(X, Y)^{tr}. \end{aligned} \tag{3.24}$$

We will be using the formulas provided in the above lemma when working with centered Gaussian vectors and conditioning on $y = 0$ (conditioning on being in a nodal set). Thus the conditioned Gaussian vector will be centered and we will be focusing on the analysis of the variance term

$$\Sigma = \mathrm{Var}(X) - \mathrm{Cov}(X, Z)[\mathrm{Var}(Y)]^{-1}\mathrm{Cov}(X, Y)^{tr}.$$

We will have $\mathrm{Var}(X) = \mathrm{Id}_d$ and $\mathrm{Cov}(X, Y)[\mathrm{Var}(Z)]^{-1}\mathrm{Cov}(X, Y)^{tr} = o(1)$ will be a small perturbation which vanishes in the high-frequency limit.

3.2.2 Wick Theorem for Hermite Polynomials

We borrow the next standard result from [54, p. 98, Proposition 4.15 (Diagram formulae for Hermite Polynomials)]. We recall that for a strictly positive integer n we note $[n] := \{1, \dots, n\}$. Furthermore, we recall that a flat diagram $G = G(l_1, \dots, l_m)$ associated with an ordered list of strictly positive integers $l_1, \dots, l_m$ is a collection of pairs $\{p, q\}$ such that $p \in (l_i, [l_i])$, $q \in (l_j, [l_j])$ for some $i \neq j$ and such that each (l_i, n), $n \in [l_i]$, appears in exactly one element of G. For a flat diagram G and each $i \neq j$ we will write q_{ij} for the number of pairs belonging to G constructed only out of the elements of (l_i, n_i), (l_j, n_j), $n_i \in [l_i]$, $n_j \in [l_j]$. We call q_{ij} the number of unordered edges between i and j.

Theorem 3.2. *Let $(Z_1, \dots, Z_m)$ be a centered Gaussian vector and set $\Sigma_{ij} := \mathbb{E}[Z_i Z_j]$. Let $H_{l_1}, \dots, H_{l_m}$ be Hermite polynomials of strictly positive degrees $l_1, \dots, l_m$. Then,*

$$\mathbb{E}\left[\prod_{j=1}^{m} H_{l_j}(Z_j)\right] = \sum_{G} \prod_{i<j} \Sigma_{ij}^{q_{ij}} \tag{3.25}$$

where the sum is over all diagrams $G = G(l_1, \dots, l_m)$ with no flat edges and where q_{ij} denotes the number of unordered edges between i and j.

3.3 Elements of the Malliavin-Stein Technique

This chapter is devoted to introducing some of the methods which play a central role throughout the thesis. The title of this section reflects their modern placement within the broader mathematical framework. However, we should note that, despite the title, we do not discuss here standard topics such as Stein equation and Malliavin derivatives. For these we refer to the standard references [69, 34, 71, 53], each of which seems to have some unique strong sides.

3.3.1 Probabilistic Distances Induced by Separating Classes

For the sake of completeness, we will recall now some standard material with [70, p. 209-214, Appendix C] serving as our main reference. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{H}$ be a collection of measurable functions $h : \mathbb{R}^d \to \mathbb{R}$. We say that $\mathcal{H}$ is a *separating class* if, given any random variables $F, G : \Omega \to \mathbb{R}^d$, the equation

$$\mathbb{E}h(F) = \mathbb{E}h(G),$$

holding for every $h \in \mathcal{H}$ with $h(F), h(G) \in L^1(\Omega)$ implies that

$$\text{Law}(F) \equiv \text{Law}(G).$$

The most standard example of a separating class consists of the collection of all bounded continuous functions. A simple but crucial observation is that any separating class $\mathcal{H}$ induces a distance:

$$d(F, G) = \sup_{h \in \mathcal{H}} \mathbb{E}h(F) - \mathbb{E}h(G).$$

There are three classical examples that have been exploited extensively in the Malliavin-Stein method: the *Kolmogorov distance*

$$d_{kol}(F, G) := \sup_{t_1, \dots, t_d \in \mathbb{R}} |\mathbb{P}(F_1 \leq t_1, \dots, F_d \leq t_d) - \mathbb{P}(G_1 \leq t_1, \dots, G_d \leq t_d)|,$$

the *Total Variation distance*

$$d_{TV}(F,G) := \sup_{B\in\mathcal{B}(\mathbb{R}^d)} |\mathbb{P}(F\in B) - \mathbb{P}(G\in B)|,$$

and the Wasserstein distance defined as

$$d_W(F,G) := \sup_h \mathbb{E}h(F) - \mathbb{E}h(G),$$

where the supremum is taken over all 1-Lipschitz functions $h:\mathbb{R}^d\to\mathbb{R}$. The Kolmogorov and Total Variation distances seem too stringent to be effectively used in the proofs related to the nodal volumes. Instead, the central role in our work will be played by the Wasserstein distance.

3.3.2 Wiener Isometry

Let $X=(X_t)_{t\in T}$ be a separable infinite-dimensional centred Gaussian process indexed by T. Let $L^2(\Omega,\mathcal{F}_X,\mathbb{P})$ be the associated L^2 space where $\mathcal{F}_X$ is the σ-field generated by the process X. Then, $L^2_{\mathbb{R}}(\Omega,\mathcal{F}_X,\mathbb{P})$ is a separable Hilbert space and, as described in Section 5.5, we can decompose $L^2(\Omega,\mathcal{F}_X,\mathbb{P}) = \oplus_{q=0}^{\infty}\mathcal{H}_q$. Here, for each non-negative integer q, $\mathcal{H}_q$ is the q-th Wiener chaos associated with the process X (see Subsection 5.5 for the definition).

It is convenient to encode the isometric properties of Wiener chaos in terms of L^2 spaces associated with Euclidean hyperrectangles with increasing dimensions. For each positive integer q we set $L^2([0,1]^q) := L^2([0,1]^q,\mathcal{B}([0,1]^q), dt_1\dots dt_q)$ and we set $L^2_s([0,1]^q)\subset L^2([0,1]^q)$ to be a subspace consisting of a.e. symmetric functions. That is, $f\in L^2_s([0,1]^q)$ if and only if $f\in L^2([0,1]^q)$ and for a.e. choice of arguments $0\le t_1,\dots,t_q\le 1$ and for every permutation $\sigma\in S_q$ we have $f(t_1,\dots,t_q)=f(t_{\sigma(1)},\dots,t_{\sigma(q)})$. We endow $L^2_s([0,1]^q)$ with rescaled norm $||\cdot||_{L^2_s([0,1]^q)} = q!||\cdot||_{L^2([0,1]^q)}$

Now choose any orthonormal basis $(f_l)_{l\in\mathbb{N}}$ of $L^2([0,1])$ and fix a sequence $(\xi_l)_{l\in\mathbb{N}}$ of i.i.d. standard Gaussian random variables constituting a basis of the first Wiener Chaos $\mathcal{H}_1$ generated by the process X. For every integer $q\ge 1$ we will now define a bijective isometry I_q from $L^2_s([0,1]^q)$ onto $\mathcal{H}_q$. For any integer $1\le l\le q$, for any collection of positive integers $(q_k)_{1\le k\le l}$ such that $\sum_{k=1}^l q_k = q$, and for any choice of distinct indices $i_1,\dots,i_l\in\mathbb{N}$, we set

$$f_{i_1^{q_1},\dots,i_l^{q_l}}(t_1,\dots,t_q) := \prod_{k=1}^{l}\prod_{m=1}^{q_k} f_k(t_{q_1+\ldots+q_{l-1}+m}), \tag{3.26a}$$

$$\widetilde{f}_{i_1^{q_1},\dots,i_l^{q_l}}(t_1,\dots,t_q) := \frac{1}{q!}\cdot\sum_{\sigma\in S_q}\prod_{k=1}^{l}\prod_{m=1}^{q_k} f_k(t_{\sigma(q_1+\ldots+q_{l-1}+m)}), \tag{3.26b}$$

$$I_q(\widetilde{f}_{i_1^{q_1},\dots,i_l^{q_l}}) := H_{q_1}(\xi_{i_1})\cdot H_{q_2}(\xi_{i_2})\cdots H_{q_l}(\xi_{i_l}). \tag{3.26c}$$

We note that the functions in (3.26a) span $L^2([0,1]^q)$, while the functions in (3.26b) span $L^2_s([0,1]^q)$. The function defined in (3.26b) is called the symmetrization of the function recorded in (3.26a). The simplest case of (3.26c) is that $I_1(f_l)=\xi_l$. Moreover, it follows from the definition (5.26), that products of the type appearing on the right of (3.26c) are dense in $\mathcal{H}_q$. Finally, it is not too difficult to check (taking advantage of the independence) that

$$||\widetilde{f}_{i_1^{q_1},\dots,i_l^{q_l}}||^2_{L_s^2([0,1]^q)} = q_1!q_2!\cdots q_l! = \mathrm{Var}(H_{q_1}(\xi_{i_1})\cdot H_{q_2}(\xi_{i_2})\cdots H_{q_l}(\xi_{i_l})). \tag{3.27}$$

Combining the above observations, it is clear that extending linearly (3.26c) yields an isometry with the postulated properties.

3.3.3 Fourth Moment Theorem on Wiener Chaos

The following standard concept is a crucial tool used for proving CLTs for random sequences belonging to a fixed Wiener Chaos.

Definition 3.2. *Let $q \geq r \geq 1$ be integers and $f, g \in L_s^2([0,1]^q)$. Then, the r-contraction $f \otimes_r g \in L^2([0,1]^{2q-2r})$ is defined by*

$$\begin{aligned} &f \otimes_r g(t_1,\dots,t_{q-r},s_1,\dots,s_{q-r}) \\ &\qquad = \int\limits_{[0,1]^r} f(t_1,\dots,t_{q-r},u_1,\dots,u_r)\cdot g(s_1,\dots,s_{q-r},u_1,\dots,u_r)du_1\dots du_r. \end{aligned} \tag{3.28}$$

We will adopt the standard convention that, whenever $q, r, f \otimes_r g$ are as in the above definition then the symmetrisation of $z := f \otimes_r g$ will be denoted by $\widetilde{z} = f\widetilde{\otimes}_r g$. The next result is a crucial technical tool we will need in the proof of Theorem 4.1. For the definition of Total Variation, Kolmogorov and Wasserstein distances used we refer to Section 3.3.1 and [69, p. 209-214, Appendix C].

Theorem 3.3 ([69]). *Let $q \geq 2$ be an integer, let $\mathbf{X} = (X_t)_{t\in T}$ be a centred infinite-dimensional separable Gaussian process with an index set T. Let Z be a standard Gaussian random variable and d denote either Total Variation, Kolmogorov or Wasserstein distance. Let I_q denote the Wiener isometry as defined in (3.26b) and (3.26c). Then, there exists a combinatorial constant $C_q > 0$ such that, for every function $f \in L_s^2([0,1]^q)$ with $||f||_{L^2([0,1]^q)} = 1$, we have*

$$d(I_q(f), Z) \leq C_q \cdot \max_{1\leq r\leq q-1} \left|\left| f \otimes_r f \right|\right|_{L^2([0,1]^{2(q-r)})}. \tag{3.29}$$

Proof. See [69, p. 99, Theorem 5.2.6] and [69, p. 95-96, Eq. (5.2.6) in Lemma 5.2.4]. □

An immediate consequence of the above theorem is that, for variables belonging to the Wiener chaos of order $q \geq 2$, convergence of contractions to zero implies convergence in distribution to the Gaussian law. We will also need the following generalisation of the above theorem.

Theorem 3.4 ([69]). *Let $q \geq 2$ be an integer and let $\mathbf{X} = (X_t)_{t \in T}$ be a centered separable infinite-dimensional Gaussian process indexed by the set T. Let $\mathbf{f} = (f_1, \dots, f_m)$ be a vector of functions $f_i \in L^2_{\text{sym}}([0,1]^q)$, and let $Z_{\mathbf{f}} \sim \mathcal{N}_m(0, \Sigma)$ be a centred Gaussian random vector with covariance matrix Σ defined by*

$$\Sigma_{ij} = \int\limits_{[0,1]^q} f_i(t_1, \dots, t_q) \cdot f_j(t_1, \dots, t_q) dt_1 \dots dt_q, \tag{3.30}$$

where $1 \leq i, j \leq m$. Let I_q denote the Wiener isometry as defined in (3.26b) and (3.26c) and denote $I_q(\mathbf{f}) = (I_q(f_1), \dots, I_q(f_m))$. Then, the following inequality holds for each real-valued function $h \in C^2(\mathbb{R}^m)$

$$|\mathbb{E}[h(I_q(\mathbf{f}))] - \mathbb{E}[h(\mathbf{Z_f})]| \leq C_q \cdot ||h''||_\infty \cdot \sum_{i=1}^{m} \sum_{r=1}^{q-1} ||f_i \otimes_r f_i||_{L^2([0,1]^{2q-2r})}, \tag{3.31}$$

with the norm $||h''||_\infty$ defined in (1.14). Moreover, we can find a combinatorial constant $C_q > 0$ such that, if Σ is a strictly positive definite matrix, then we have

$$\mathbf{W}_1(I_q(\mathbf{f}), Z_{\mathbf{f}}) \leq C_q \cdot m^{3/2} \cdot ||\Sigma^{-1}||_{\text{op}} ||\Sigma||_{\text{op}}^{1/2} \cdot \sum_{i=1}^{m} \sum_{r=1}^{q-1} ||f_i \otimes_r f_i||_{L^2([0,1]^{2q-2r})}, \tag{3.32}$$

where $C_q > 0$ is a combinatorial constant.

Proof. In [69, p. 121, Theorem 6.2.2] a more general statement is considered with different integers $q_1, \dots, q_m$. Our claim follows immediately by specialising it to the case $q_1 = \dots = q_m = q$. ☐

The following simple observation will be useful in the proof of Theorem 4.3.

Remark 3.2. *Let $(\mathbf{f}_n)_{n \in \mathbb{N}}$, $\mathbf{f}_n = (f_n^1, \dots, f_n^m)$, $f_n^i \in L^2_s([0,1]^q)$, be a sequence of vectors such that the right-hand side of (3.31) is converging to zero. Suppose also that for each $1 \leq i, j \leq m$ the following limit exists*

$$\int\limits_{[0,1]^q} f_n^i(t_1, \dots, t_q) \cdot f_n^j(t_1, \dots, t_q) dt_1 \dots dt_q \longrightarrow \Sigma_{ij}. \tag{3.33}$$

Then,

$$I_q(\mathbf{f}_n) \xrightarrow{d} \mathbf{Z}, \tag{3.34}$$

where $\mathbf{Z} \sim \mathcal{N}_m(0, \Sigma)$ is a centred Gaussian vector with covariance matrix Σ defined via (3.33). (This implication is made possible by a simple observation that the real and imaginary parts of the characteristic functions $x \to e^{i\langle \lambda, x \rangle}$ are of C^∞ class and that the suprema of their second partial derivatives are bounded by $|\lambda|^2$.)

The next remark explains the title of the current section, in relation to Theorems 3.3 and 3.4.

Remark 3.3. *It is convenient to formulate Theorems 3.3 and 3.4 using the contraction norms* $||f \otimes_r f||_{L^2([0,1]^{2(q-r)})}$. *However, note the following important equivalence. If* $X_n = I_q(f_n)$ *is a sequence of random variables as in Theorem 3.3, the convergence to zero of the right-hand side of (3.29) is equivalent to the convergence of the 4th moment to that of a standard Gaussian random variable, specifically:*

$$\mathbb{E}X_n^4 \to 3, \tag{3.35}$$

as shown in [69, p. 99, Theorem 5.2.7]. Similarly, if $(X_n^1, \ldots, X_n^m) = (I_q(f_n^1), \ldots, I_q(f_n^m))$ *is a sequence of random vectors as in Theorem 3.4, then the convergence of the right-hand side of (3.31) or (3.32) to zero (depending on whether the limit covariance matrix is strictly positive definite or not) is equivalent to:*

$$\max_{i=1,\ldots,m} \left|\mathbb{E}(X_n^i)^4 - 3\right| \to 0, \tag{3.36}$$

as shown in [69, p. 121, Theorem 6.2.2].

Let us end this section with some bibliographical remarks. Theorem 3.3 is due to Nualart and Peccati [73] and Theorem 3.4 was established by Peccati and Tudor [77]. The connection between the Fourth Moment Theorem and the Malliavin calculus has been first observed by Ortiz-Latorre and Nualart [72]. The very fruitful combination of the Malliavin calculus and Stein method, which is behind the currently used quantitative forms of these theorems and their multiple extensions, has been introduced by Nourdin and Peccati [68]. As noted by Hairer in [34, p. 21] these theorems provide "an incredibly strong form of central limit theorem." The recent results of Herry, Malicet and Poly [35, p. 1164, Theorem 1 and 2] provide another very strong form of central limit theorem on a Wiener Chaos of fixed degree, denotes as 'superconvergence'. As recalled therein, an important fact to keep in mind is that, every non-constant random variable belonging to a finite sum of Wiener Chaoses admits a density with respect to the Lebesgue measure. We also recall that the Fortet–Mourier distance (known also as the bounded Wasserstein distance) metrizes the topology of convergence in distribution and is defined by taking the separating class of functions $h : \mathbb{R}^d \to \mathbb{R}$ which satisfy $||h||_{Lip} + ||h||_\infty \leq 1$ with $||h||_{Lip}$ denoting the Lipschitz norm.

Theorem 3.5 ([35]). *Let* d *be a strictly positive integer and* $q \in \mathbb{N}$. *There exists* $\delta = \delta_{q,d} > 0$ *and* $C = C_{q,d} > 0$ *such that for all* F *in the Wiener chaos of degree* d, *with density* f, *we have*

$$d_{FM}(F, \mathcal{N}(0,1)) \leq \delta \implies [f \in \mathcal{C}^q \text{ and } ||f||_{\mathcal{C}_q} \leq C], \tag{3.37}$$

where d_{FM} *denotes the Fortet–Mourier distance. Moreover, if* $(F_n)_n$ *is a sequence of random variables in a Wiener chaos of a fixed strictly positive degree, with* *respective densities* $(f_n)_n$*, then*

$$F_n \xrightarrow{law} \mathcal{N}(0,1) \quad \text{if and only if for every } q \in \mathbb{N} \quad ||f_n^{(q)} - \gamma^{(q)}||_\infty \to 0. \tag{3.38}$$

Here, the q*-th derivative* $f_n^{(q)}$ *is well-defined for* n *large enough and* γ *denotes the density of the standard Gaussian random variable.*

3.4 Crash-Course on the Geometry of Random Fields

This sections contains some standard results concerning limited range of situations that will be of interest throghout the rest of the thesis.

3.4.1 Regularity of Gaussian Fields

We start with the following basic definition adapted here from [62, p. 263, Definition A.3].

Definition 3.3. *Let* $V \subset \mathbb{R}^d$ *be an open set and let* $K : V \times V \mapsto \mathbb{R}$ *be a function such that* $K(z_1, z_2) = K(z_2, z_1)$ *for every* $z_1, z_2 \in V$*. We say that* K *is of class* $C^{k,k}(V \times V)$ *if*

$$\frac{\partial^{2l} K(z)}{\partial z_{i_{2l}} \dots \partial z_{i_1}} \in C^0(V \times V), \tag{3.39}$$

whenever at most k *of* i_j *are smaller or equal to* d *and at most* k *of* i_j *are strictly bigger than* d*. Here,* $z = (z_1; z_2) = (z_1, \dots, z_d; z_{d+1}, \dots, z_{2d})$.

An immediate consequence of the continuity assumption is that for $K \in C^{k,k}(V \times V)$ the derivative of the form (3.39) does not depend on the order of differentiation. In particular, $\partial_x^\alpha \partial_y^\beta K(x,y) = \partial_y^\beta \partial_x^\alpha K(x,y)$ for every $x, y \in V$ and multiindices $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha|, |\beta| \le k$. Here, $|\cdot|$ denotes the l_1 norm.

The next result is a very useful extension of the classical Kolmogorov´s continuity theorem, which adapts standard arguments to a setting where main emphasis is not just on continuity but also on differentiability.

Theorem 3.6 ([62])**.** *Let* $k \in \mathbb{N}$ *and* $V \subset \mathbb{R}^d$ *be an open set. Suppose that* $K : V \times V \to \mathbb{R}$ *is a positive definite symmetric functions of class* $C^{k,k}(V \times V)$*, and that*

$$\max_{|\alpha|,|\beta| \le k} \sup_{x,y \in V} |\partial_x^\alpha \partial_y^\beta K(x,y)| < \infty. \tag{3.40}$$

Then, there exists a unique in distribution C^{k-1} *Gaussian random field* f *on* V *with the covariance kernel* K*.*

We note that if $K(x,y) \equiv C(x-y)$, then

$$\partial_x^\alpha \partial_y^\beta K(x,y) = (-1)^{|\beta|}(\partial^{\alpha+\beta} C)(x-y). \tag{3.41}$$

Therefore, K is in $C^{k,k}(\mathbb{R}^m \times \mathbb{R}^m)$ if and only if $C \in C^{2k}(\mathbb{R}^m)$. This in turn can be shown to hold if and only if

$$\int_{\mathbb{R}^d} |\lambda|^{2k} d\mu(\lambda) < \infty, \tag{3.42}$$

with μ denoting the spectral measure associated with C (see Section 3.4.2).

3.4.2 Stationarity and Isotropy

The class of centered smooth Gaussian random fields, even just on the Euclidean space, is obviously very large. In order to make the situation manageable it is natural to restrict the attention to a subcollection of fields defined by invariance of the covariance function with respect to an action of some chosen group. The simplest example one can consider is *stationarity* (i.e. the covariance function is independent of location $K(x,y) = C(x-y)$) and *isotropy* (i.e. the covariance remains unchanged under rotation i.e. $K(x,y) = K(\theta x, \theta y)$ for any rotation θ). We will be focused in this work on the situations when both of these assumptions are fulfilled, for an example of work which abandons the second of these assumptions see [27]. The following starting result is well-known.

Theorem 3.7 ([2]). *A continuous function $C : \mathbb{R}^d \mapsto \mathbb{R}$ is a covariance function of a stationary random field on $\mathbb{R}^d$ if and only if there exists a finite measure v on the Borel σ field $\mathcal{B}(\mathbb{R}^d)$ such that*

$$C(x) = \int_{\mathbb{R}^d} e^{i\langle x,\lambda\rangle} v(d\lambda) \tag{3.43}$$

for all $x \in \mathbb{R}^d$.

The measure v in (3.43) is called the *spectral measure*. When restricting ourself to the isotropic situation and integrating out over the S^{d-1} one obtains the following standard result often referred to as the Schoenberg theorem .

Theorem 3.8 ([2]). *For C to be the covariance function of a mean-square continuous, isotropic, stationary random field on $\mathbb{R}^d$ is it necessary and sufficient that*

$$C(x) = \int_0^\infty \frac{J_{\frac{d}{2}-1}(\lambda||x||)}{(\lambda||x||)^{\frac{d}{2}-1}} \mu(d\lambda) \tag{3.44}$$

for some finite measure μ on $[0,\infty)$.

We will refer to the measure μ in (3.44) as the *isotropic spectral measure*. We note that the spectral point of view has recently brought in a major breakthrough in the study of smooth Gaussian random fields [52].

3.4.3 Some Functionals of Smooth Random Fields

The two most basic geometric objects in the random field theory are: the *excursion set* of a measurable real-valued function $f : T \mapsto \mathbb{R}$ above the *threshold level* $u \in \mathbb{R}$ defined as

$$A_u(f) := \{t \in T : f(t) \geq u\}, \tag{3.45}$$

and the *level set* corresponding to u defined as

$$B_u(f) := \{t \in T : f(t) = u\}. \tag{3.46}$$

More generally, one can study the volume of tubes starting from an arbitrary measurable set. For a metric space (T, τ) the *tube of radius* ρ *around* A, where $A \subset T$, is defined as

$$\text{Tube}(A, \rho) := \{x \in T : \tau(x, A) \leq \rho\} = \bigcup_{y \in A} B_\tau(y, \rho), \tag{3.47}$$

where

$$\tau(x, A) := \inf_{y \in A} \tau(x, y).$$

In our work, the primary role is played by the specific levels set $\{f = 0\}$ which is usually referred to as the *Nodal set*. In the next chapter we will give many examples of the prominent role played by this level and justify special interest given to it. Even more precisely speaking, the result presented in this thesis concern a specific type of nodal volume. We introduce it below.

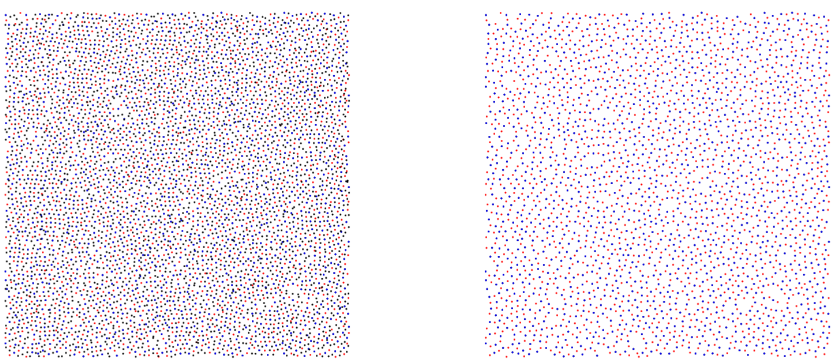

Figure 3.2: Left: Large-scale sample of critical points of the Berry's Random Wave. Right: Restriction to extrema. Credit: Dmitry Belyaev, included with permission, https://people.maths.ox.ac.uk/belyaev/.

Definition 3.4. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let b_k, $\hat{b}_K$ be two independent Berry's*

Random Waves with wave-numbers $0 < k \leq K < \infty$*. We define the corresponding nodal number as the random variable*

$$\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}) := \left| \{x \in \mathcal{D} : b_k(x) = \hat{b}_K(x) = 0\} \right|. \tag{3.48}$$

The following lemma is the starting point of our analysis. It can be seen as a random analog of deterministic Theorem 3.1.

Lemma 3.3. *Let* $\mathcal{D}$ *be a convex compact domain of the plane, with non-empty interior and piecewise* C^1 *boundary* $\partial\mathcal{D}$*. Let* b_k*,* $\hat{b}_K$ *be two independent Berry's Random Waves with wave-numbers* $2 \leq k \leq K < \infty$*. Then, the corresponding nodal number* $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})$ *is an a.s. finite r.v. with finite variance. Moreover, the boundary* $\partial\mathcal{D}$ *does not contribute to the nodal number, that is*

$$\mathbb{P}(\exists x \in \partial\mathcal{D} : b_k(x) = \hat{b}_K(x) = 0) = 0. \tag{3.49}$$

Furthermore, if we set

$$N^\varepsilon(b_k, \hat{b}_K, \mathcal{D}) = \frac{1}{(2\varepsilon)^2} \int_{\mathcal{D}} \mathbf{1}_{\{|b_k(x)| \leq \varepsilon\}} \cdot \mathbf{1}_{\{|\hat{b}_K(x)| \leq \varepsilon\}} \cdot \left| \det \begin{bmatrix} \partial_1 b_k(x) & \partial_2 b_k(x) \\ \partial_1 \hat{b}_K(x) & \partial_2 \hat{b}_K(x) \end{bmatrix} \right| dx, \tag{3.50}$$

then a.s. and in $L^2(\mathbb{P})$ *we have*

$$\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}) = \lim_{\varepsilon \downarrow 0} N^\varepsilon(b_k, \hat{b}_K, \mathcal{D}). \tag{3.51}$$

The proof of the above lemma is given in Section 5.2.

3.4.4 The Kac-Rice Formula

We begin with the following classical observation ([2, p. 277, Lemma 11.2.10]), which enables the extension of Theorem 3.1 to a probabilistic framework. This result, often called the *Bulinskaya lemma*, has multiple variations. Notably, when applied to a pair of independent real planar Berry's Random Waves $(b_k, \hat{b}_K)$, $k, K > 0$, over $T = \partial\mathcal{D}$, $d = 1$, it demonstrates that the boundary does not influence the nodal number (refer to the proof of Lemma 3.3 in Section 5.2).

Theorem 3.9. *Let* $T \subset \mathbb{R}^{d+1}$ *be a compact set of Hausdorff dimension* $d \geq 1$*. Let* $X : T \times \Omega \to \mathbb{R}^{d+1}$ *be a random function such that for almost every* $\omega \in \Omega$ *the path* $t \to X_t(\omega)$ *is of class* $C^1_b(T)$*. Consider a value* $u \in \mathbb{R}^{d+1}$ *and suppose that there is an open neighborhood* $\mathcal{U}$ *of* u *such that the univariate probability densities of* X_t *are bounded on* $\mathcal{U}$ *uniformly over* T*. Then, almost surely, the value* u *is not* *attained by X on T, that is*

$$\mathbb{P}\left(\bigcap_{t\in T}\{X_t \neq u\}\right) = 1. \tag{3.52}$$

The next result is known as the Kac-Rice formula is perhaps the most important tool for understanding the fluctuations of the level sets of smooth random fields. We present it here in a form that combines [2, p. 266-267, An Expectation Metatheorem] and [2, p. 284, Theorem 11.5.1]. For $z \in \mathbb{C}$ we define the quantities

$$(z)_0 := z, \qquad (z)_k := z(z-1)\dots(z-k+1), \qquad k \in \mathbb{N}\setminus\{0\},$$

known as partial factorials (of z).

Theorem 3.10 ([2]). *Let $T \subset \mathbb{R}^d$ be a compact set whose boundary has finite Hausdorff measure of the boundary $\mathcal{H}_{d-1}(\partial T) < \infty$, let $B \subset \mathbb{R}^m$ be an open set with $\dim_{\mathcal{H}}(\partial B) \equiv \dim_{\mathcal{H}}(\bar{B}\setminus B) = m-1$. Let $f = (f^1,\dots,f^d) : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $g = (g^1,\dots,g^m) : \mathbb{R}^d \mapsto \mathbb{R}^m$ be almost surely continuously differentiable random fields and assume that f, ∇f and g have finite variances on T. Suppose also that*

$$\sup_{t\in T}\max_{1\le i,j\le d}\mathbb{E}\left[\left|\frac{\partial f^j}{\partial x_i}(t)\right|^N\right] < \infty.$$

Suppose that for every $\varepsilon > 0$ the Euclidean moduli of continuity w_h associated with the functions $h \in \{f, \nabla f, g\}$ all satisfy a bound of the form $\mathbb{P}(w_h(\eta) > \varepsilon) = o(\eta^d)$. Let $k \geq 0$ be a non-negative integer and suppose that there exists a neigbourhood $\mathcal{O}\times\mathcal{U} \subset \mathbb{R}^{kd^2}\times\mathbb{R}^k$ of $(0^{\otimes kd^2}, u^{\otimes k})$ on which, for every choice of $t_1,\dots,t_k \in T$ all different, the following functions are continuous and bounded:

(i) the density $(x_1,\dots,x_k) \mapsto p_{t_1,\dots,t_k}(x_1,\dots,x_k)$ of $f(t_1),\dots,f(t_k)$,

(ii) the density of $f(t_1),\dots,f(t_k)$ given $g(t_1),\dots,g(t_k)$ and $\nabla f(t_1),\dots,\nabla f(t_k)$,

(iii) the density of $\nabla f(t_1),\dots,\nabla f(t_k)$ given $f(t_1),\dots,f(t_k)$.

Then,

$$\mathbb{E}[(N_u)_k] =$$

$$\int_{T^k}\mathbb{E}\left[\prod_{j=1}^k|\det\nabla f(t_j)|1_B(g(t_j))|f(t_1) = \ldots = f(t_k) = u\right]p_{t_1,\dots,t_k}(u^{\otimes k})dt_1\dots dt_k. \tag{3.53}$$

3.4.5 Random Generalised Functions

We gather here basic information about the notion of random generalised functions which will be needed for us to describe the meaning of the convergence to the White Noise.

1. We start by recalling some background material using [88, p. 698-703, Appendix L] as a reference. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of the infinitely differentiable functions such that all their partial derivatives vanish at infinity faster than the reciprocal of any polynomial. In other words, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ if $\varphi \in C^\infty(\mathbb{R}^n)$ and if all its semi-norms

$$||\varphi||_k = \sup_{x\in\mathbb{R}^n} (1+|x|)^k \sum_{|\alpha|\leq k} \left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \varphi(x) \right| \tag{3.54}$$

are finite. Real-valued linear functional on the Schwartz space is called generalised function (or tempered distribution) if it is a continuous (equivalently, bounded) operator for one of the semi-norms $||\cdot||_k$. In other words, generalised functions are elements of the topological dual $\mathcal{S}(\mathbb{R}^n)'$ to the Schwartz space (equipped with one of the semi-norms defined above).

2. Random distributions are random variables in the sense of classical but very general definition given in [29, p. 14, Def. I.4.1]. As explained after [29, p. 60-61, Definition III.4.1], in our case this definition means simply that $(\omega,\varphi)\mapsto X(\omega,\varphi)$ is a random distribution if and only if each map $\omega\mapsto X(\omega,\varphi)$ is a real-valued random variable and if each functional $\varphi\mapsto X(\omega,\varphi)$ is a tempered distribution. This definition depends tacitly on the topology chosen for the dual space $S(\mathbb{R}^2)'$. The weak topology on $S(\mathbb{R}^2)'$ is determined by a pointwise convergence for each test function (that is, $T_n \to T$ weakly if $T_n(\varphi) \to T_n$ for every Schwartz test function φ). The strong topology is determined by condition that this convergence is uniform over every bounded set of test functions B, see [84, p. 71, 3 L'espace topolgique des distributions]. As follows from [84, p. 69, Thm. IV], boundedness of the set B of test functions is equivalent to two simple conditions. The first one is that every $\varphi \in B$ has support contained in the same compact domain K. The second one is that for each $m\in\mathbb{N}$ we can find a finite constant L_m such that

$$\sup_{x,\varphi,\alpha} \left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \varphi(x) \right| \leq L_m, \tag{3.55}$$

where $\varphi\in B$, $x\in\mathbb{R}^n$ and multi-indices α have norms $\leq m$.

3. The definition of a probability distribution for a random generalised function and the corresponding notion of convergence in law is given in a way which is completely analogous to the standard notions. We refer the reader to [29, p. 21, I.6.2 Convergence etroite] and [29, p. 61, III.4.2 Lois de distributions aleatoires] for technical details.

4. By white noise we mean a random distribution W such that for any $\varphi_1,\cdots,\varphi_n \in \mathcal{S}(\mathbb{R}^d)$ the random vector $\langle W,\varphi_1\rangle,\cdots,\langle W,\varphi_n\rangle$ has a centred Gaussian distribution with covariance function

$$\begin{aligned}\mathbb{E}[\langle W,\varphi_i\rangle\langle W,\varphi_j\rangle] &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_i(x)\varphi_j(y)\delta(x-y)dxdy \\ &= \int_{\mathbb{R}^d} \varphi_i(x)\varphi_j(x)dx,\end{aligned} \tag{3.56}$$

see [32, p. 288-289, 4.8 Gaussian processes with independent values at every point]. As we exploit in the proof of Theorem 4.4, white noise can be seen as a random distributional derivative of the Wiener sheet [32, p. 257, 2.4 Derivatives of generalised gaussian processes].

Chapter 4 Full Statements of Our Results

4.1 Parameter Identification

We start by putting forward a class of ancillary parameters that will play a crucial role in our analysis. In anticipation, we note that the superscripts 'log' and 'exp' in the definition below are suggestive of the transformation of scale/re-parametrisation of k_n and K_n which, in relevant cases, allows one to detect the fine details of fluctuations. However, we stress that always $r^{log} \neq \ln r$ and that typically $r^{exp} \neq e^r$.

Definition 4.1. *For a sequence of pairs of numbers $(k_n, K_n)_{n\in\mathbb{N}}$ s.t. $2 \leq k_n \leq K_n < \infty$ we will write*

$$r_n^{log} := \frac{\ln k_n}{\ln K_n}, \qquad r_n := \frac{k_n}{K_n}, \qquad r_n^{exp} := 1 - \frac{\ln(1 + (K_n - k_n))}{\ln K_n}, \tag{4.1}$$

and provided that the corresponding limits exist

$$r^{log} := \lim_{n\to\infty} r_n^{log}, \qquad r := \lim_{n\to\infty} r_n, \qquad r^{exp} := \lim_{n\to\infty} r_n^{exp}., \tag{4.2}$$

Since, by definition, we have $0 \leq r_n^{log}, r_n, r_n^{exp} \leq 1$, it is always possible to choose a sub-sequence of $(k_n, K_n)_{n\geq 1}$ for which limits r^{log}, r, r^{exp} from the above definition exist.

We note also that $r < 1$ implies $r^{exp} = 0$. Indeed, if $r < 1$, then for all n sufficiently large we have $r_n < 1$. Thus, we can write

$$r_n^{exp} = -\frac{\ln((1-r_n) + K_n^{-1})}{\ln K_n} \sim -\frac{\ln(1-r_n)}{\ln K_n} \longrightarrow 0.$$

On the other hand, if $r = 1$, then r^{exp} can take any value in the interval $[0,1]$. Indeed, for any $r^{exp} \in [0,1]$ we can find a sequence of numbers $(\beta_n)_{n\in\mathbb{N}} \subset (0,1)$ such that $\beta_n \to r^{exp}$ and such that $n^{\beta_n} \to \infty$. Then, for n large enough, we can set $K_n = n$ and $k_n = n - n^{1-\beta_n} + 1$. This yields $K_n - k_n = n^{1-\beta_n} - 1$ and further

$$r_n^{exp} = 1 - \frac{\ln(1 + (K_n - k_n))}{\ln n} = 1 - \frac{\ln(n^{1-\beta_n})}{\ln n} = \beta_n \to r^{exp}.$$

For example, to reach $r^{exp} = 0$ we might set $\beta_n = \frac{1}{\sqrt{\ln n}}$ and note that

$$n^{\beta_n} = n^{\frac{1}{\sqrt{\ln n}}} = \exp\left(\ln n^{\frac{1}{\sqrt{\ln n}}}\right) = \exp\left(\sqrt{\ln n}\right) \longrightarrow \infty.$$

In order to obtain any $r^{exp} \in (0,1)$ it suffices to set $\beta_n \equiv r^{exp}$ and to obtain $r^{exp} = 1$ using this scheme we can simply put $\beta_n = 1 - 1/n$.

We note also that $r > 0$ implies $r^{log} = 1$ since we can rewrite $r_n^{log} = 1 + \frac{\ln r_n}{\ln K_n}$.

4.2 Scaling and Fluctuations

The following theorem is one of our main results.

Theorem 4.1. *Let D be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary ∂D. Consider a sequence of pairs of numbers $(k_n, K_n)_{n\in\mathbb{N}}$ s.t. $2 \le k_n \le K_n < \infty$ and let $b_{k_n}, \hat{b}_{K_n}$ be two independent Berry's Random Waves with wave-numbers k_n and K_n respectively. Then, we have*

$$\mathbb{E}\left[\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})\right] = \frac{area(\mathcal{D})}{4\pi} \cdot (k_n \cdot K_n). \tag{4.3}$$

Furthermore, suppose that $k_n \to \infty$ and that the asymptotic parameters r^{log}, r, r^{exp} defined in (4.2) exist and $r^{log} > 0$. Then, we have that $r^{log} \in (0,1]$ and $r, r^{exp} \in [0,1]$. Moreover

$$\lim_{n\to\infty} \frac{\operatorname{Var}\left(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})\right)}{\operatorname{area}(\mathcal{D}) \cdot C_\infty \cdot K_n^2 \ln K_n} = 1, \tag{4.4}$$

where C_∞ is a strictly positive finite constant defined as:

$$C_\infty := \frac{r^{log} + 36r + r^2 + 50r^{exp}}{512\pi^3}. \tag{4.5}$$

We recall that in [70] Peccati, Nourdin and Rossi studied the nodal number of a pair of independent Berry's Random Waves $b_{k_n}, \hat{b}_{K_n}$ under the assumption that $k_n = K_n$ for all n. It is easy to check that the case $r = 1$ and $r^{exp} = 1$ of the above theorem recovers [70, Theorem 1.4, p. 103].

Remark 4.1. *The case $r^{log} = 0$ was excluded from the above theorem only for expository purposes. All our results can be extended to an arbitrary (non-linear) relationship between $\ln k_n$ and $\ln K_n$. In particular, if $r_n^{log} \to 0$ then (4.4) remains true provided that we replace r^{log} with r_n^{log}. This generalisation is straightforward since, as will be shown later, if $r = 0$ then the nodal number $\mathcal{N}(b_{k_n}, \hat{b}_{k_n}, \mathcal{D})$ is asymptotically $L^2(\mathbb{P})$ equivalent to a deterministic rescaling of a r.v. known as nodal length $\mathcal{L}(b_{k_n}, \mathcal{D})$ (see Subsection 5.7 below for its definition and basic properties).*

4.3 Distributional Limits

For a m-dimensional random vector $\mathbf{Y}_n = (Y_n^1, \dots, Y_n^m)$ we write $\mathbb{E}\mathbf{Y}_n$ to denote the m-dimensional vector $(\mathbb{E}Y_n^1, \mathbb{E}Y_n^2, \dots, \mathbb{E}Y_n^m)$. The definition of the 1-Wasserstein distance W_1 used herein is recalled in (1.11).

Theorem 4.2. *Let D be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary ∂D. Let $(k_n, K_n)_{n\in\mathbb{N}}$ be a sequence of pairs of numbers s.t. $2 \le k_n \le K_n < \infty$, $k_n \to \infty$, and s.t. the asymptotic parameters r^{log}, r, r^{exp} defined in (4.2) exist and $r^{log} > 0$. Suppose also that b_{k_n}, $\hat{b}_{K_n}$ are two independent Berry's Random Waves with wave-numbers k_n, K_n respectively and let $Y_n := \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})$ denote the corresponding nodal number (see (5.36)). Then, there exists a numerical constant $L > 0$ such that if we define*

$$\delta_n^2 := \frac{K_n^2 \ln K_n}{\mathrm{Var} Y_n[4]}, \qquad \gamma_n := \delta_n (1 + \delta_n) \cdot \left(1 + \mathrm{diam}(\mathcal{D})^2\right), \tag{4.6}$$

then $\delta_n^2 \to \mathrm{area}(\mathcal{D}) \cdot C_\infty$ (see (4.5)) and

$$\sqrt{\mathbb{E}\left(\frac{Y_n - \mathbb{E}Y_n}{\sqrt{\mathrm{Var} Y_n}} - \frac{Y_n[4]}{\sqrt{\mathrm{Var} Y_n[4]}}\right)^2} \le \frac{L \cdot \gamma_n}{\sqrt{\ln K_n}}, \qquad \mathrm{Corr}(Y_n, Y_n[4]) \ge \frac{1}{1 + \frac{L\cdot\gamma_n}{\sqrt{\ln K_n}}}. \tag{4.7}$$

Moreover, we have

$$W_1\left(\frac{Y_n[4]}{\sqrt{\mathrm{Var} Y_n[4]}}, Z\right) \le \frac{L \cdot \gamma_n}{\sqrt{\ln K_n}}, \qquad W_1\left(\frac{Y_n - \mathbb{E}Y_n}{\sqrt{\mathrm{Var} Y_n}}, Z\right) \le \frac{L \cdot \gamma_n}{\sqrt{\ln K_n}}, \tag{4.8}$$

*where Z denotes a standard Gaussian random variable and W_1 the 1-Wasserstein distance (see **N.4**)*

The 1-Wasserstein distance in the case of the 4th chaotic projection of the nodal number can be replaced by the Total Variation or Kolmogorov distances (see [69, p. 210, Definition C.2.1] for their definition). However, with our technique, the same is not possible for the nodal number itself. This is because of our use of $L^2(\mathbb{P})$ distance to bound W_1. The following theorem provides a natural extension of the previous result to the multivariate setting. The definition of the distances $\mathbf{W}_1$ and d_{C^2} used in the next theorem is recalled respectively in (1.12) and in (1.13), see also **N.5** for the definition of $||\cdot||_{op}$ and of $||\cdot||_{HS}$.

Theorem 4.3. *Let $\mathcal{D}_1, \dots, \mathcal{D}_m$ be a convex and compact planar domains, with non-empty interiors and piecewise C^1 boundaries $\partial\mathcal{D}_i$. Let $(k_n, K_n)_{n\in\mathbb{N}}$ be a sequence of pairs of numbers such that $2 \le k_n \le K_n < \infty$, $k_n \to \infty$, and such that the asymptotic parameters r^{log}, r, r^{exp}, defined in (4.2) exist and $r^{log} > 0$. Let b_{k_n}, $\hat{b}_{K_n}$ denote independent Berry's Random Waves with wave-numbers k_n, K_n respectively* *and let $\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})$ be the corresponding nodal number. We write*

$$Y_i^n := \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}_i), \qquad \mathbf{Y}_n := (Y_n^1, \dots, Y_n^m), \tag{4.9}$$

where $i = 1, \dots, m$, and we write Σ^n, Σ for matrices defined by

$$\Sigma_{ij}^n := \operatorname{Cov}(Y_n^i, Y_n^j), \qquad \Sigma_{ij} := area(\mathcal{D}_i \cap \mathcal{D}_j), \qquad 1 \le i, j \le m, \tag{4.10}$$

and we let $\mathbf{Z} = (Z_1, \dots, Z_m) \sim \mathcal{N}_m(0, \Sigma)$ denote a centered Gaussian vector with covariance matrix Σ. Then, for some numerical constant $\widetilde{L} > 0$, the following inequalities hold

$$d_{C^2}\left(\frac{\mathbf{Y_n} - \mathbb{E}\mathbf{Y_n}}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}\right) \le \frac{\widetilde{L} \cdot \left(1 + \sum_{i=1}^m \operatorname{diam}(D_i)^2\right)}{\sqrt{C_\infty \cdot \ln K_n}}, \tag{4.11}$$

$$\mathbf{W}_1\left(\frac{\mathbf{Y_n} - \mathbb{E}\mathbf{Y_n}}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}\right) \le \frac{\widetilde{L} \cdot (1 + m^{3/2}) \cdot \left(1 + \sum_{i=1}^m \operatorname{diam}(\mathcal{D}_i)^2\right)}{\sqrt{C_\infty \cdot \ln K_n}} + M_n, \tag{4.12}$$

where

$$M_n := \sqrt{m} \cdot \min\left\{||(\Sigma^n)^{-1}||_{\text{op}} \cdot ||\Sigma^n||_{\text{op}}^{1/2}, ||\Sigma^{-1}||_{\text{op}} \cdot ||\Sigma||_{\text{op}}^{1/2}\right\} \cdot ||\Sigma^n - \Sigma||_{HS}, \tag{4.13}$$

*with the convention that $M_n = \infty$ if either Σ or Σ^n is not invertible (see **N.4** for definition of distances $\mathbf{W}_1$ and d_{C^2}). Futhermore, if the matrix Σ is strictly positive definite, then, for all sufficiently large n, the matrix Σ^n is strictly positive definite and $M_n \to 0$.*

This theorem will be established in Subsection 7.2. For a discussion of the conditions on the domains $\mathcal{D}_1, \dots, \mathcal{D}_m$, which guarantee that the limiting matrix Σ in the above theorem is positive-definite, see [91, p. 63, Remark 7.3]. We recall that the Wiener sheet $[0,1]^2 \ni (t_1, t_2) \mapsto B_{t_1,t_2} \in \mathbb{R}$ is a real-valued, continuous-path, Gaussian, centred stochastic process on $[0,1]^2$, determined by the covariance function

$$\mathbb{E}[B_{t_1,t_2} \cdot B_{s_1,s_2}] = \min(t_1, s_1) \cdot \min(t_2, s_2). \tag{4.14}$$

Let $(k_n, K_n)_{n \in \mathbb{N}}$ be a sequence of pairs of numbers s.t. $2 \le k_n \le K_n < \infty$, $k_n \to \infty$, and the asymptotic parameters r^{log}, r, r^{exp}, defined in (4.2) exist and $r^{log} > 0$. Theorem 4.3 implies in particular that if we define

$$B_{t_1,t_2}^n := \frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, [0,t_1] \times [0,t_2]) - \frac{k_n K_n}{4\pi} \cdot t_1 t_2}{\sqrt{\frac{r^{log} + 36r + r^2 + 50r^{exp}}{512\pi^3} \cdot K_n^2 \ln K_n}}, \tag{4.15}$$

then we have a convergence of stochastic processes in the sense of finite-dimensional distributions

$$(B_{t_1,t_2}^n)_{0 \le t_1, t_2 \le 1} \xrightarrow{d} (B_{t_1,t_2})_{0 \le t_1, t_2 \le 1}, \tag{4.16}$$

which means that for every choice of $m \in \mathbb{N}$ and $0 \le t_1, t_2, \dots, t_{2m-1}, t_{2m} \le 1$, we have a convergence in distribution of random vectors

$$(B^n_{t_1,t_2}, B^n_{t_1,t_2}, \dots, B^n_{t_{2m-1},t_{2m}}) \xrightarrow{d} (B_{t_1,t_2}, B_{t_1,t_2}, \dots, B_{t_{2m-1},t_{2m}}). \tag{4.17}$$

Indeed, it is enough to use Theorem 4.3 with a choice of domains

$$D_1 = [0, t_1] \times [0, t_2], D_2 = [0, t_3] \times [0, t_4], \dots, D_m = [0, t_{2m-1}] \times [0, t_{2m}].$$

An interesting question whether this convergence can be lifted to functional form is beyond the scope of this article. However, the next result is a natural extension of this re-writing if we interpret the white noise as a random distributional derivative of the Wiener sheet. (The necessary technical notions are recalled in Section 3.4.5 for the sake of completeness.) Similar results have been shown before for the planar nodal length [93, p. 4, Proposition 1.3]. Our result covers the extension to the nodal number which was suggested in [93, p. 11, Remark 2.18].

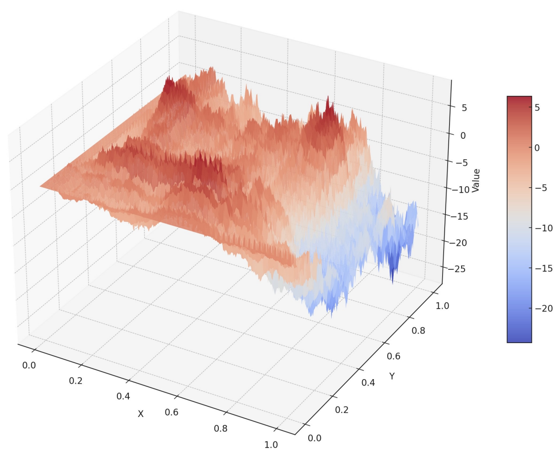

Figure 4.1: Numerical simulation of the Wiener sheet process over the unit square.

Theorem 4.4. *Let D be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary ∂D. Let $(k_n, K_n)_{n\in\mathbb{N}}$ be a sequence of pairs of numbers s.t. $2 \le k_n \le K_n < \infty$, $k_n \to \infty$, and s.t. the asymptotic parameters r^{log}, r, r^{exp} defined in (4.2) exist and $r^{log} > 0$. Suppose also that b_{k_n}, $\hat{b}_{K_n}$ are two independent Berry's Random Waves of the wave-numbers k_n, K_n respectively. Let μ_n denote the random signed measure defined by*

$$\mu_n(A) = \frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, A) - \frac{k_n \cdot K_n}{4\pi} \cdot \text{area}(A)}{\sqrt{\frac{r^{log}+36r+r^2+50r^{exp}}{512\pi^3} \cdot K_n^2 \ln K_n}}, \qquad A \in \mathcal{B}([0,1]^2), \tag{4.18}$$

where $\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, A)$ denotes the corresponding nodal number. Then, in the sense *of random generalised functions on* $[0,1]^2$*, we have convergence in distribution*

$$\mu_n(dt_1dt_2) \xrightarrow{d} W(dt_1dt_2) \tag{4.19}$$

where W *denotes the White Noise on* $[0,1]^2$.

The proof of the above theorem is given in Subsection 7.3. The notion of convergence in law used in this theorem depends tacitly on the topology used to define the dual $(C_c^\infty([0,1]^2))'$. However, whether we choose weak or strong topology the result remains true regardless. We note that, for every $\varphi \in C_c^\infty([0,1]^2)$, we have

$$\int_0^1\int_0^1 \varphi(t_1,t_2)\mu_n(dt_1dt_2) = \frac{\sum_{(t_1,t_2)\in B}\varphi(t_1,t_2) - \frac{k_n\cdot K_n}{4\pi}\int_0^1\int_0^1\varphi(t_1,t_2)dt_1dt_2}{\sqrt{\frac{r^{log}+36r+r^2+50r^{exp}}{512\pi^3}}\cdot K_n^2\ln K_n}, \tag{4.20}$$

where

$$B = \{(t_1,t_2)\in[0,1]^2 : b_{k_n}(t_1,t_2) = \hat{b}_{K_n}(t_1,t_2) = 0\}.$$

Moreover, the above theorem implies that for every collection of test functions $\varphi_1,\ldots,\varphi_m \in C_c^\infty([0,1]^2)$, the random vector

$$(\int_0^1\int_0^1 \varphi_1(t_1,t_2)\mu_n(dt_1dt_2),\ldots,\int_0^1\int_0^1 \varphi_m(t_1,t_2)\mu_n(dt_1dt_2)) \tag{4.21}$$

converges in distribution to a centred Gaussian random vector with covariance matrix $\Sigma\varphi$. Here, the matrix Σ^φ is given as

$$\Sigma_{ij}^\varphi = \int_0^1\int_0^1 \varphi_i(t_1,t_2)\varphi_j(t_1,t_2)dt_1dt_2, \qquad 1\le i,j\le m, \tag{4.22}$$

which is identical to the covariance matrix of the White Noise on $[0,1]^2$.

4.4 Reduction Principle

It has been observed time and again that various interesting geometric functionals of smooth Gaussian random waves can be explained using simpler quantities known as *polyspectra* [93, 11, 12, 13, 80, 50, 59, 56, 58, 28, 90, 81, 7]. In the first result directly related to our setting [59, p. 376, Theorem 1.2], Marinucci, Rossi and Wigman demonstrated that the centred nodal length $\mathcal{L}_l - \mathbb{E}\mathcal{L}_l$ of the random spherical harmonics $f_l(x)$ is asymptotically L^2 equivalent to the *sample trispectrum*

$$\mathcal{M}_l := -\sqrt{2}\cdot\frac{\sqrt{l(l+1)}}{192}\int_{\mathbb{S}^2} H_4(f_l(x))dx. \tag{4.23}$$

In the case of the planar Berry's Random Wave model, Vidotto [92, p. 3, Theorem 1.1] proved that the centred nodal length $\mathcal{L}(b_E,\mathcal{D})-\mathbb{E}\mathcal{L}(b_E,\mathcal{D})$ is asymptotically L^2 equivalent to the *sample trispectrum*

$$-\sqrt{2}\cdot\frac{2\pi\sqrt{E}}{192}\int_D H_4(b_E(x))dx. \tag{4.24}$$

To the best of our knowledge, no similar result had been obtained before for the nodal number, including in the three standard domains (euclidean, spherical, toral). The following result, which will be proved in Section 7.4, provides a complete characterisation of the full correlations for the nodal number in the two-energy complex Berry's Random Wave model. Here, we will use the notation of *normalised derivatives*

$$\widetilde{\partial}_i b_k(x) := \frac{\sqrt{2}}{k} \cdot \partial_i b_k(x), \qquad i \in \{1,2\}, \qquad k > 0, \qquad x \in \mathbb{R}^2, \tag{4.25}$$

for which one has that $\mathrm{Var}(\widetilde{\partial}_i b_k(x)) \equiv 1$.

Theorem 4.5. *Let D be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary ∂D. Let $(k_n, K_n)_{n\in\mathbb{N}}$ be a sequence of pairs of numbers s.t. $2 \le k_n \le K_n < \infty$, $k_n \to \infty$, and s.t. the asymptotic parameters r, r^{log}, r^{exp} defined in (4.2) exist and $r^{log} > 0$. Suppose also that b_{k_n}, $\hat{b}_{K_n}$ are two independent Berry's Random Waves of the wave-numbers k_n, K_n respectively and let $\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})$ denote the corresponding nodal number. Then*

$$\begin{aligned} \mathbb{E}\left[\left(\frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}) - \mathbb{E}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}{\sqrt{\mathrm{Var}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}))}} - Y_{r^{log}, r, r^{exp}}\right)^2\right] &\longrightarrow 0, \\ \mathrm{Corr}\left(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}), Y_{r^{log}, r, r^{exp}}\right) &\longrightarrow 1, \end{aligned} \tag{4.26}$$

where the random variable $Y_{r^{log}, r, r^{exp}}$ is defined as

$$\begin{aligned} Y_{r^{log}, r, r^{exp}} = -\frac{K_n^2}{192\pi}\Bigg(& r^{log} \int_D H_4(b_{k_n}(x))dx \\ & + r \cdot \int_D H_4(\hat{b}_{K_n}(x)) + \frac{3}{2} H_2(b_{k_n}(x)) H_2(\hat{b}_{K_n}(x))dx \\ & + 12 r^{exp} \int_D H_2(\widetilde{\partial}_1 b_{k_n}(x)) H_2(\hat{b}_{K_n}(x)) + H_2(b_{k_n}(x)) H_2(\widetilde{\partial}_2 \hat{b}_{K_n}(x))dx\Bigg), \end{aligned} \tag{4.27}$$

with $\widetilde{\partial}_i b_{k_n}(x)$ and $\widetilde{\partial}_j \hat{b}_{K_n}(x)$ denoting the normalised derivatives given in (4.25).

We note that (4.27) is a substantial refinement of the domination of $\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]$ proved in Theorem 4.2. The reduction is from 22 terms to at most 5, see Lemma 5.2. As discussed in the introduction, the most interesting phenomenon here is the behavior with respect to the parameter r^{exp}.

4.5 Recurrence Trick

The next lemma generalizes observations that had been made before in similar settings, but only on the level of particular chaotic projections, see for example [70, p. 117, Lem.

4.2] or [70, p. 125, Eq. (6.79)]. We start with some necessary definition.

Definition 4.2. *Let D be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary ∂D. Let $2 \le k \le K < \infty$, let $b_k, \hat{b}_K$ be independent Berry's Random Waves of the wave-numbers k, K respectively, and let $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})$ denote the corresponding nodal number. We define the random variable*

$$\mathit{Cross}\left(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})\right) = \sum_{q=0}^{\infty} \mathit{Cross}\left(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2q]\right), \tag{4.28}$$

through the formulas

$$\begin{aligned}
&\mathrm{Cross}\left(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[0]\right) = -\mathbb{E}\left[\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})\right], \\
&\mathrm{Cross}\left(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2q]\right) = \\
&(k \cdot K) \sum_{j_1+\ldots+j_6=2q} \mathbf{1}_{\{j_1+j_2+j_3>0\}} \mathbf{1}_{\{j_4+j_5+j_6>0\}} \cdot c_{j_1,\ldots,j_6} \\
&\int_D H_{j_1}(b_k(x)) H_{j_2}(\widetilde{\partial}_1 b_k(x)) H_{j_3}(\widetilde{\partial}_2 b_k(x)) H_{j_4}(\hat{b}_K(x)) H_{j_5}(\widetilde{\partial}_1 \hat{b}_K(x)) H_{j_6}(\widetilde{\partial}_2 \hat{b}_K(x)) dx,
\end{aligned} \tag{4.29}$$

where $q \ge 1$. Here, $H_{j_1}, \ldots, H_{j_6}$ denote the probabilistic Hermite polynomials, the constants $c_{j_1,\ldots,j_6}$ are deterministic and given by (5.37), and for $i, j \in \{1, 2\}$ the $\widetilde{\partial}_i b_{k_n}(x)$, $\widetilde{\partial}_j \hat{b}_{K_n}(x)$ are the normalised derivatives defined in (4.25).

Lemma 4.1. *Let D be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary ∂D. Let $2 \le k \le K < \infty$, let $b_k, \hat{b}_K$ be independent Berry's Random Waves of the wave-numbers k, K respectively, and let $\mathcal{L}(b_k, \mathcal{D})$, $\mathcal{L}(\hat{b}_K, \mathcal{D})$, $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})$ denote the corresponding nodal lengths (defined in (5.46)) and nodal number (defined in (3.48)). Then, the following equality holds in $L^2(\mathbb{P})$*

$$\begin{aligned}
\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2q] &= \frac{K}{\pi\sqrt{2}} \mathcal{L}(b_k, \mathcal{D})[2q] + \frac{k}{\pi\sqrt{2}} \mathcal{L}(\hat{b}_K, \mathcal{D})[2q] + \mathrm{Cross}(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2q]), \\
\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}) &= \frac{K}{\pi\sqrt{2}} \mathcal{L}(b_k, \mathcal{D}) + \frac{k}{\pi\sqrt{2}} \mathcal{L}(\hat{b}_K, \mathcal{D}) + \mathrm{Cross}(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})),
\end{aligned} \tag{4.30}$$

where $q = 0, 1, 2, \ldots$ and the three terms in each sum are uncorrelated.

The proof of the above lemma will be given in Section 5.8.

Chapter 5 Proofs: Part I

This chapter, the first our of three devoted to proofs, is concerned with some basic results that are necessary to even begin to tackle any of the problems. One exception from this is the material of next section – which contains a convenient indexation scheme that will play an important role in the next chapter.

As noted in [95, p. 29] a major difficulty in controlling variance of the functionals associated the Berry's Random Waves (or similar models) is arising from a need to evaluate the measure of the set of tuples

$$\{(x,y) \in B(R) \times B(R) : ||x-y|| = \psi\},$$

which, as noted therein, does not have a simple or elegant answer. Another major difficulty, and a problem to which we hope to have made a certain contribution, is to control the combinatorial explosion which is associated with the use of the Wiener Chaos Decompositions (see Section 5.5 and [57, 66, 76]).

5.1 Symmetric Indexation

Below, we will introduce an alternative notation which, due to its symmetrical nature, serves as a convenient tool for completing various technical computations needed in the upcoming sections. This notation plays particularly important role in the proof of Lemma 6.3, in Lemma 6.8 (where it provides for concise formulas (6.58), (6.59), (6.60)), and in the computation of exact constants of asymptotic variance (Theorem 4.1 and the input of Lemma 6.10 towards its proof). Lastly, it is helpful with establishing the Reduction Principle - Lemma 4.5.

This technical variation will help us controlling the combinatorial explosion which quickly takes hold when increasing the number of parameters while using Wiener-Itô Chaos Decomposition (see Subsection 5.5). In our situation it is driven by merely replacing one-energy ($k_n \equiv K_n$) with energies which are not necessarily identical ($k_n \not\equiv K_n$). Similar difficulties arise when considering the Berry's Random Wave model on $\mathbb{R}^3$ instead of on $\mathbb{R}^2$ (see the work of Dalmao, Estrade and León [19], of Dalmao [18], and another approach in related context due to Notarnicola [66]). These difficulties are also apparent in the study of the nodal volumes associated with the Random Spherical Harmonics in arbitrary dimension [57]. An alternative tactic would be to provide only main intermediate computations (e.g. [70, p. 141-148, Appendix B]) or to exploit some form of explicit recursion as in the work of Notarnicola [65, p. 1161-1172, Appendices A and B]. (Possible future extensions of our work to the Berry's Random Wave model on $\mathbb{R}^n$ and $1 \leq l \leq n$

distinct energies would likely require combination of all aforementioned approaches. This problem can also be largely avoided by restricting oneself to a study of more qualitative versions of the same problems.)

S.1 We will write $\{k_{-1}, k_1\}$ to denote an unordered pair of strictly positive wave-numbers and (k, K) for the corresponding ordered pair, that is

$$k := \min_{p\in\{-1,1\}} k_p, \qquad K := \max_{p\in\{-1,1\}} k_p. \tag{5.1}$$

S.2 In a complete analogy with **S.1**, given a sequence $\{k^n_{-1}, k^n_1\}_{n\in\mathbb{N}}$ of an unordered pairs of strictly positive wave-numbers, we will write $(k_n, K_n)_{n\in\mathbb{N}}$ for the corresponding sequence of the ordered pairs of wave-numbers, that is

$$k_n := \min_{p\in\{-1,1\}} k^n_p, \qquad K_n := \max_{p\in\{-1,1\}} k^n_p. \tag{5.2}$$

S.3 When considering unordered pairs $\{k_{-1}, k_1\}$ the symbols $b_{k_{-1}}$, b_{k_1} will always indicate independent BRWs with wave-numbers k_{-1} and k_1 respectively. The notation used in Chapter 1 can be recovered by setting

$$u = \text{argmin}_{p\in\{-1,1\}} k_p, \qquad v = \text{argmax}_{p\in\{-1,1\}} k_p, \tag{5.3}$$

and subsequently

$$b_{k_u} := b_k, \qquad b_{k_v} := \hat{b}_K. \tag{5.4}$$

When $k = K$, the selection between 'argmax' and 'argmin' is arbitrary but must remain constant within a given argument. To remain consistent, we will also write $\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})$ and $N^\varepsilon(b_{k_{-1}}, b_{k_1}, \mathcal{D})$ to denote respectively the nodal number and its ε-approximation (see Definition 3.48 and formula (3.50)). These conventions will be naturally extended to sequences of complex Berry's Random Waves.

S.4 For $x \in \mathbb{R}^2$ we will occasionally use indexation $x = (x_{-1}, x_1)$ instead of $x = (x_1, x_2)$.

S.5 Combining the conventions **S.1**, **S.3** and **S.4** we will relabel the *normalised derivatives* defined in (4.25) by setting

$$\widetilde{\partial}_i b_{k_p}(x) := \left(\frac{\sqrt{2}}{k_p}\right)^{|i|} \cdot \partial_i b_{k_p}(x), \tag{5.5}$$

where $p \in \{-1, 1\}$, $i \in \{-1, 0, 1\}$, $x \in \mathbb{R}^2$, and where we use notation $\partial_0 b_{k_p}(x) := b_{k_p}(x)$. Let us stress that ∂_{-1} denotes differentiation with respect to the component x_{-1}, and ∂_1 denotes differentiation with respect to x_1. In contrast, ∂_0 indicates that no differentiation occurs, serving as the identity operator. Obviously, as in (4.25), the variance-normalisation property holds

$$\text{Var}\left(\widetilde{\partial}_i b_{k_p}(x)\right) = 1, \tag{5.6}$$

for every $p \in \{-1, 1\}$, $i \in \{-1, 0, 1\}$ and $x \in \mathbb{R}^2$.

S.6 For any element $\mathbf{j} \in \mathbb{N}^6$, we will use an indexation scheme

$$\begin{aligned} \mathbf{j} &= (j_{-1}; j_1) \\ &= (j_{-1,-1}; j_{-1,0}; j_{-1,1}; j_{1,-1}; j_{1,0}; j_{1,1}), \\ j_{-1} &= (j_{-1,-1}; j_{-1,0}; j_{-1,1}), \\ j_1 &= (j_{1,-1}; j_{1,0}; j_{1,1}), \end{aligned} \tag{5.7}$$

and, the corresponding l_1 norm will be denoted with $|\cdot|$, i.e.

$$\begin{aligned} |\mathbf{j}| &= |j_{-1}| + |j_1| \\ &= j_{-1,-1} + j_{-1,0} + j_{-1,1} + j_{1,-1} + j_{1,0} + j_{1,1}, \\ |j_{-1}| &= j_{-1,-1} + j_{-1,0} + j_{-1,1}, \\ |j_{-1}| &= j_{1,-1} + j_{1,0} + j_{1,1}. \end{aligned} \tag{5.8}$$

Whether this or more standard notation is being used should always be clear from the context and we will frequently leave pointers to the individual elements of the above list, as an additional check. Whenever possible, the statements of theorems or lemmas are given using standard notation and **S.1-S.6** is preferred in corresponding proofs.

5.2 Proof of the Basic Regularity Lemma

Proof of Lemma 3.3. Let (s_{-1}, s_1) be any point in $\mathbb{R}^2$. In a complete analogy with the nodal number, we define

$$N_{s_{-1},s_1}(b_{k_{-1}}, b_{k_1}, \mathcal{D}) = \left|\{x \in \mathcal{D} : b_{k_{-1}}(x) = s_{-1},\ b_{k_1}(x) = s_1\}\right|. \tag{5.9}$$

Since the boundary $\partial\mathcal{D}$ has Hausdorff dimension 1, we deduce using [2, Theorem 11.2.10, p. 277] that the pre-image

$$b_{k_{-1}}^{-1}(\{s_{-1}\}) \cap b_{k_1}^{-1}(\{s_1\}) \cap \partial\mathcal{D}$$

is a.s. empty. Furthermore, since $b_{k_{-1}}$, b_{k_1} are $C^\infty(\mathbb{R}^2)$ independent Gaussian fields, it follows by [5, p. 169, Proposition 6.5] that (s_{-1}, s_1) is a.s. non-singular value on $\mathcal{D}$. That is,

$$\mathbb{P}(\exists x \in \mathcal{D} : \quad b_{k_{-1}}(x) = s_{-1},\ b_{k_1}(x) = s_1, \quad \det \begin{bmatrix} \partial_{-1} b_{k_{-1}}(x) & \partial_1 b_{k_{-1}}(x) \\ \partial_{-1} b_{k_1}(x) & \partial_1 b_{k_1}(x) \end{bmatrix} = 0) = 0. \tag{5.10}$$

Then, using a compactness argument exactly as in [5, p. 162, lines 6-14 in the proof of Proposition 6.1], we deduce from (5.10) and from local inversion theorem that

$$N_{s_{-1},s_1}(b_{k_{-1}}, b_{k_1}, \mathcal{D})$$

is a.s. finite. The postulated approximation formula (3.51) holds almost surely, and in fact is exact for ε small enough (depending on randomness), as a straightforward consequence of the local inversion theorem. It can be proved quickly by reducing to the case $\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D}) = 1$ and detailed argument for almost identical problem can be found in [2, p. 269-270, Theorem 11.2.3]. Since we already have a.s. convergence, to show $L^2(\mathbb{P})$ convergence it is enough to prove convergence of the moments

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}[N^\varepsilon(b_{k_{-1}},b_{k_1},\mathcal{D})^2]=\mathbb{E}[\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D})^2], \tag{5.11}$$

including finitness of the right-hand side. Note that by the standard area formula [5, p. 161, Proposition 6.1] we have

$$N^\varepsilon(b_{k_{-1}},b_{k_1},\mathcal{D})=\frac{1}{(2\varepsilon)^2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}N_{s_{-1},s_1}(b_{k_{-1}},b_{k_1},\mathcal{D})ds_{-1}ds_1. \tag{5.12}$$

Thus, using Fatou's lemma and Jensen's inequality, we can obtain

$$\begin{aligned}\mathbb{E}[\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D})^2]&\le\limsup_{\varepsilon\downarrow 0}\mathbb{E}[\frac{1}{(2\varepsilon)^2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}N_{s_{-1},s_1}(b_{k_{-1}},b_{k_1},\mathcal{D})ds_{-1}ds_1]^2\\&\le\limsup_{\varepsilon\downarrow 0}\frac{1}{(2\varepsilon)^2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\mathbb{E}[N_{s_{-1},s_1}(b_{k_{-1}},b_{k_1},\mathcal{D})^2]ds_{-1}ds_1.\end{aligned} \tag{5.13}$$

To conclude it is enough to conclude that the application

$$(s_{-1},s_1)\to\mathbb{E}N_{s_{-1},s_1}(b_{k_{-1}},b_{k_1},\mathcal{D})^2$$

is continuous (and bounded) at zero. This can be proved using the standard Kac-Rice formulas [5, p. 163-164, Theorems 6.2 and 6.3] and the same strategy as in [70, p. 141]. □

5.3 The Covariance Functions

The following definition will be frequently in use.

Definition 5.1. *Let b_1 be the real Berry's Random Wave with the wave-number $k=1$. For each $i,j\in\{-1,0,1\}$, the covariance function r_{ij} at the point $z=(z_{-1},z_1)\in\mathbb{R}^2$ is defined as*

$$r_{ij}(z):=\mathbb{E}\left[\widetilde{\partial}_i b_1(z)\cdot\widetilde{\partial}_j b_1(0)\right]. \tag{5.14}$$

We use the following shorthand notation for the special cases of the above definition

$$r_{-1}(z):=r_{-10}(z),\qquad r(z):=r_{00}(z),\qquad r_1(z):=r_{10}(z). \tag{5.15}$$

The following result provides basic properties of these covariance functions.

Lemma 5.1. *For each $i,j\in\{-1,0,1\}$, let r_{ij} be a covariance function associated with the real-valued Berry's Random Wave b_1. Then, for every choice of $p,q\in$*

$\{-1,1\}$ and for every $z=(z_{-1},z_1)\in\mathbb{R}^2\setminus\{0\}$, we have

$$\begin{aligned} r(z)&=J_0(|z|), \qquad r_p(z)=-\sqrt{2}\cdot\frac{z_p}{|z|}\cdot J_1(|z|),\\ r_{pq}(z)&=\delta_{pq}\cdot 2\cdot\frac{J_1(|z|)}{|z|}-2\cdot\frac{z_p}{|z_p|}\cdot\frac{z_q}{|z_q|}\cdot J_2(|z|), \end{aligned} \tag{5.16}$$

with continuous extensions at $z=0$ s.t.

$$r(0)=1, \qquad r_p(0)=0, \qquad r_{pq}(0)=\delta_{pq}. \tag{5.17}$$

Here, the J_0, J_1, J_2 denote the Bessel functions of the first kind. Furthermore, there exists a numerical constant $C>0$ s.t. for every $i,j\in\{-1,0,1\}$ and $z\in\mathbb{R}^2\setminus\{0\}$, we have

$$|r_{ij}(z)|\le\frac{C}{|z|^{1/2}}. \tag{5.18}$$

The proof of the above lemma relies on the standard arguments and is therefore omitted. Later on we will make a use of the following simple observation: the equation (5.17) implies that, for every fixed point $z\in\mathbb{R}^2$, the collection $\{\widetilde{\partial}_{-1}b(z),b(z),\widetilde{\partial}_1b(z)\}$ consists of three independent standard Gaussian random variables.

Definition 5.2. *We define the covariance functions associated with the random field b and its derivatives by the formulas*

$$r(z):=\mathbb{E}[b(z)b(0)], \quad r_i(z):=\sqrt{d}\,\mathbb{E}[\partial_i b(z)b(0)], \quad r_{ij}(z):=d\,\mathbb{E}[\partial_i b(z)\partial_j b(0)], \tag{5.19}$$

where $z\in\mathbb{R}^d$ and $i,j\in\{1,...,d\}$.

Remark 5.1. *Since $\rho_\alpha(|x-y|)$ is a $C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ function, it follows by an extension of classical Kolmogorov's continuity condition [62, p. 263, A.9. Kolmogorov's theorem] that b is almost surely smooth on $\mathbb{R}^n$. This in turn implies that the expectation can be exchanged with the differentiation*

$$\mathbb{E}[\frac{\partial^{|\alpha|}}{\partial^\alpha x}b(x)\cdot\frac{\partial^{|\beta|}}{\partial^\beta y}b(y)]=\frac{\partial^{|\alpha|+|\beta|}}{\partial^\alpha x\partial^\beta y}\mathbb{E}[b(x)\cdot b(y)]=\frac{\partial^{|\alpha|+|\beta|}}{\partial^\alpha x\partial^\beta y}\rho_\alpha(|x-y|) \tag{5.20}$$

for any multi-indices α,β, [62, p. 253-254, A.3. Positive-definite kernels]. Thus, by Lemma 3.1 the covariance functions described in the preceding Definition 5.2 are given by formulas

$$r\left(z\right)=\rho_{\frac{(d-2)}{2}}\left(|z|\right), \quad r_i\left(z\right)=\frac{(-z_i)}{\sqrt{d}}\rho_{\frac{d}{2}}\left(|z|\right), \quad r_{ij}\left(z\right)=\delta_{ij}\rho_{\frac{d}{2}}\left(|z|\right)-\frac{z_iz_j}{d+2}\rho_{\frac{d+2}{2}}\left(|z|\right), \tag{5.21}$$

where δ_{ij} denotes Kronecker's delta. When deriving these formulas it's important to note that, while $\sqrt{d}\,\mathbb{E}[\partial_i b\left(z\right)b\left(0\right)]=\partial_i[\rho_{\frac{(d-2)}{2}}\left(|z|\right)]$, we have a change of sign in *the last case*

$$d\,\mathbb{E}[\partial_i b\,(z)\,\partial_j b\,(z)] = -\partial_{ij}[\rho_{\frac{(d-2)}{2}}(|z|)].$$

5.4 Wiener Chaos

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which the two independent planar Berry's Random Waves $b_{k_{-1}}$, b_{k_1}, are defined. Given a subset A of random variables in $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$ we will write $\mathrm{VectSp}(A)$ for the smallest $\mathbb{R}$-linear vector space containing A, that is

$$\mathrm{VectSp}(A) = \left\{ \sum_{l=1}^{n} a_l X_l : \quad n \in \mathbb{N}, \quad a_l \in \mathbb{R}, \quad X_l \in A \right\}. \tag{5.22}$$

We will now define a sequence of closed linear subspaces $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$ using (probabilistic) Hermite polynomials introduced in Section 3.1.2. We start by setting $\mathcal{H}_0 = \mathbb{R}$, and

$$\begin{aligned} \mathrm{A}_1 &= \left\{ b_p(x) : p \in \{-1, 1\}, \quad x \in \mathbb{R}^2 \right\}, \\ \mathcal{H}_1 &= \overline{\mathrm{VectSp}(A_1)}, \end{aligned} \tag{5.23}$$

where the closure, denoted by the horizontal bar, is taken in the space $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$ (such a notational convention is adopted throughout the paper). We remark that the derivatives of the field belong to $\mathcal{H}_1$, that is:

$$\forall p, p' \in \{-1, 1\}, \quad \forall x \in \mathbb{R}^2, \quad \partial_p b_{p'}(x) \in \mathcal{H}_1. \tag{5.24}$$

For $q = 2, 3, 4, \ldots$ we first introduce the notation

$$\begin{aligned} \mathrm{A}_{q,n} &= \left\{ \prod_{l=1}^{n} H_{j_l}(X_l) : \quad \sum_{l=1}^{n} j_l = q, \quad X_l \in \mathcal{H}_1, \quad \mathbb{E} X_l X_m = \delta_{lm}, \quad l, m \in \{1, \ldots, m\} \right\}, \\ \mathrm{A}_q &= \bigcup_{n=1}^{q} \mathrm{A}_{q,n}, \end{aligned} \tag{5.25}$$

and then we set

$$\mathcal{H}_q = \overline{\mathrm{VectSp}(\mathrm{A}_q)}, \tag{5.26}$$

where each H_{j_l} denotes the probabilistic Hermite polynomial of the order $j_l \in \mathbb{N}$ and where δ_{ml} is the Kronecker's delta symbol. For each $q = 0, 1, 2, \ldots$ the closed linear subspace $\mathcal{H}_q$ is known as the *q-th Wiener Chaos generated by* $\mathcal{H}_1$ (see [69, 76, 71, 53, 34] for general results on the spaces $\mathcal{H}_q$).

5.5 Wiener Chaos Decomposition

This section introduces material related to Sections 3.3.2-3.3.3. Suppose that $\mathcal{F}$ is the σ-field generated by the two independent Berry Random Waves $b_{k_{-1}}$, b_{k_1}. That is to say

$$\mathcal{F} = \sigma(\{b_{k_p}(x) : \quad p \in \{-1, 1\}, \quad x \in \mathbb{R}^2\}), \tag{5.27}$$

where, for the set A of random variables, $\sigma(A)$ denotes the smallest σ-algebra with respect to which all the random variables in A are measurable. It is a well-known consequence of the aforementioned properties of the (probabilistic) Hermite polynomials (see e.g. [69, p. 26-28, 2.2 Wiener Chaos]) that the following L^2-orthogonal Wiener-Itô chaos decomposition holds

$$L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q, \tag{5.28}$$

meaning that for every $X \in L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$, we have the equality

$$X = \sum_{q=0}^{\infty} X[q], \tag{5.29}$$

where

$$X[q] = \mathrm{Proj}(X \big| \mathcal{H}_q), \tag{5.30}$$

and both the projection and the sum are in the sense of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$. We note that for every $X \in L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$ and $q, q' \in \mathbb{N}$ we have

$$\mathbb{E}[X[q] \cdot X[q']] = \begin{cases} \mathbb{E}X[q]^2 & \text{if } q = q' \\ 0 & \text{if } q \neq q' \end{cases}, \tag{5.31}$$

and moreover

$$X[0] = \mathbb{E}X, \qquad \mathbb{E}X[q] = 0, \qquad q \geq 1. \tag{5.32}$$

5.6 Chaos Decomposition of the Nodal Number

We recall that for a fixed x, the normalised derivatives $\widetilde{\partial}_i b_{k_p}(x)$, $p \in \{-1, 1\}$, $i \in \{-1, 0, 1\}$, are independent standard Gaussian random variables belonging to $\mathcal{H}_1$ (see (5.17) and (5.24)). Thus we have the implication

$$\text{if} \quad |\mathbf{j}| \equiv \sum_{p=\pm 1} \sum_{i \in \{-1,0,1\}} j_{p,i} = q \qquad \text{then} \qquad \left(\prod_{p=\pm 1} \prod_{i \in \{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p}(x)) \right) \in \mathcal{H}_q. \tag{5.33}$$

Moreover, since each $\mathcal{H}_q$ is a closed linear subspace of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}_b, \mathbb{P})$, a standard approximation argument yields that, for $|\mathbf{j}|$ as in (5.33)

$$\left(\int_{\mathcal{D}} \prod_{p=\pm 1} \prod_{i \in \{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p}(x)) dx \right) \in \mathcal{H}_q, \tag{5.34}$$

for every compact domain $\mathcal{D}$.

The following statement provides the explicit form of the Wiener-Itô Chaos expansion (defined in (5.29)) for the nodal number $\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})$ (defined in (3.48), see also **S.3-S.6** for other notation used in the statement of the next theorem).

Lemma 5.2. *Let $\mathcal{D}$ be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let $\{k_{-1}, k_1\}$ be an unordered pair of strictly positive wave-numbers and let $b_{k_{-1}}$, b_{k_1}, be a corresponding pair of independent real Berry's Random Waves. Then, the nodal number $\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})$ admits the Wiener-Itô chaos decomposition*

$$\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D}) = \sum_{q=0}^{\infty} \mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})[2q], \tag{5.35}$$

where

$$\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})[2q] = (k_{-1} \cdot k_1) \cdot \sum_{\mathbf{j} \in \mathbb{N}^6, |\mathbf{j}|=2q} c_{\mathbf{j}} \int_{\mathcal{D}} \prod_{p \in \{-1,1\}} \prod_{i \in \{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p}(x)) dx. \tag{5.36}$$

Here, the sum runs over all vectors $\mathbf{j} \in \mathbb{N}^6$, $\mathbf{j} = (j_{-1,-1}, j_{-1,0}, j_{-1,1}, j_{1,-1}, j_{1,0}, j_{1,1})$, with

$$|\mathbf{j}| \equiv \sum_{p \in \{-1,1\}} \sum_{i \in \{-1,0,1\}} j_{p,i} = 2q,$$

and the constants $c_{\mathbf{j}}$ are defined as

$$\begin{aligned} c_{\mathbf{j}} :=& \rho(\mathbf{j}) \cdot \frac{(-1)^{\frac{j_{-1,0}+j_{1,0}}{2}} \prod_{p \in \{-1,1\}} (j_{p,0} - 1)!!}{4\pi \prod_{p \in \{-1,1\}} \prod_{i \in \{-1,0,1\}} j_{p,i}!} \\ &\cdot \mathbb{E}\left[\left| \det \begin{bmatrix} Z_{-1,-1} & Z_{-1,1} \\ Z_{-1,1} & Z_{1,1} \end{bmatrix} \right| \cdot \prod_{p,q \in \{-1,1\}} H_{j_{p,q}}(Z_{p,q}) \right]. \end{aligned} \tag{5.37}$$

Here we denote by $\{Z_{p,q} : p, q \in \{-1,1\}\}$ a collection of four independent standard Gaussian random variables and use the notation $\rho(\mathbf{j}) = 1$ if the following conditions are simultaneously satisfied:

1. *For every $p \in \{-1,1\}$ the index $j_{p,0}$ is even,*
2. *For every combination of $p, q \in \{-1,1\}$ either:*
   - *(i) all the indices $j_{p,q}$ are even or,*
   - *(ii) all the indices $j_{p,q}$ are odd.*

If either of the above conditions is not satisfied, then $\rho(\mathbf{j}) = 0$.

Proof. We are going to use a standard strategy, that is we will start with an L^2 approximation formula

$$\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D}) = \lim_{\varepsilon \downarrow 0} N^{\varepsilon}(b_{k_{-1}}, b_{k_1}, \mathcal{D}),$$

provided by Lemma 3.3. We recall that

$$N^{\varepsilon}(b_{k_{-1}}, b_{k_1}, \mathcal{D}) = \int_{\mathcal{D}} \left(\prod_{p\in\{-1,1\}} \frac{\mathbf{1}_{\{|b_{k_p}(x)|\le\varepsilon\}}}{2\varepsilon} \right) \cdot \left| \det \begin{bmatrix} \partial_{-1} b_{k_{-1}}(x) & \partial_1 b_{k_{-1}}(x) \\ \partial_{-1} b_{k_1}(x) & \partial_1 b_{k_1}(x) \end{bmatrix} \right| dx, \tag{5.38}$$

and we start by finding the Wiener-Itô chaotic decomposition of the integrand function, which we first rewrite as

$$\frac{k_{-1}\cdot k_1}{2} \cdot \prod_{p\in\{-1,1\}} \frac{1}{2\varepsilon} \mathbf{1}_{\{|b_{k_p}(x)|\le\varepsilon\}} \cdot \left| \det \begin{bmatrix} \widetilde{\partial}_{-1} b_{k_{-1}}(x) & \widetilde{\partial}_1 b_{k_{-1}}(x) \\ \widetilde{\partial}_{-1} b_{k_1}(x) & \widetilde{\partial}_1 b_{k_1}(x) \end{bmatrix} \right|. \tag{5.39}$$

We obtain

$$\begin{aligned} &\frac{k_{-1}\cdot k_1}{2} \cdot \left(\prod_{p\in\{-1,1\}} \frac{\mathbf{1}_{\{|b_{k_p}(x)|\le\varepsilon\}}}{2\varepsilon} \right) \cdot \left| \det \begin{bmatrix} \widetilde{\partial}_{-1} b_{k_{-1}}(x) & \widetilde{\partial}_1 b_{k_{-1}}(x) \\ \widetilde{\partial}_{-1} b_{k_1}(x) & \widetilde{\partial}_1 b_{k_1}(x) \end{bmatrix} \right| \\ &= \frac{k_{-1}\cdot k_1}{2} \cdot \sum_{q=0}^{\infty} \left(\prod_{p\in\{-1,1\}} \frac{\mathbf{1}_{\{|b_{k_p}(x)|\le\varepsilon\}}}{2\varepsilon} \cdot \left| \det \begin{bmatrix} \widetilde{\partial}_{-1} b_{k_{-1}}(x) & \widetilde{\partial}_1 b_{k_{-1}}(x) \\ \widetilde{\partial}_{-1} b_{k_1}(x) & \widetilde{\partial}_1 b_{k_1}(x) \end{bmatrix} \right| \right)[q] \\ &= (k_{-1}\cdot k_1) \sum_{q=0}^{+\infty} \left(\sum_{\mathbf{j}\in\mathbb{N}^6, |\mathbf{j}|=q} c^{\varepsilon}_{\mathbf{j}} \prod_{p\in\{-1,1\}} \prod_{i\in\{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p}(x)) \right), \end{aligned} \tag{5.40}$$

where

$$\begin{aligned} c^{\varepsilon}_{\mathbf{j}} =& \frac{1/2}{\prod_{p\in\{-1,1\}} \prod_{i\in\{-1,0,1\}} j_{p,i}!} \cdot \mathbb{E}\left[\left(\prod_{p\in\{-1,1\}} \frac{\mathbf{1}_{\{|b_{k_p}(x)|\le\varepsilon\}}}{2\varepsilon} \right) \cdot \left| \det \begin{bmatrix} \widetilde{\partial}_{-1} b_{k_{-1}}(x) & \widetilde{\partial}_1 b_{k_{-1}}(x) \\ \widetilde{\partial}_{-1} b_{k_1}(x) & \widetilde{\partial}_1 b_{k_1}(x) \end{bmatrix} \right| \right. \\ &\left. \cdot \prod_{p\in\{-1,1\}} \prod_{i\in\{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p}(x)) \right] \\ =& \frac{1/2}{\prod_{p\in\{-1,1\}} \prod_{i\in\{-1,0,1\}} j_{p,i}!} \cdot \left(\prod_{p\in\{-1,1\}} \frac{1}{2\varepsilon} \mathbb{E}[\mathbf{1}_{\{|b_{k_p}(x)|\le\varepsilon\}} H_{j_{p,0}}(b_{k_p}(x))] \right) \\ &\cdot \mathbb{E}\left[\left| \det \begin{bmatrix} \widetilde{\partial}_{-1} b_{k_{-1}}(x) & \widetilde{\partial}_1 b_{k_{-1}}(x) \\ \widetilde{\partial}_{-1} b_{k_1}(x) & \widetilde{\partial}_1 b_{k_1}(x) \end{bmatrix} \right| \cdot \prod_{p,q\in\{-1,1\}} H_{j_{p,q}}(\widetilde{\partial}_p b_{k_q}(x)) \right]. \end{aligned} \tag{5.41}$$

Here, the factor $(k_{-1}\cdot k_1)/2$ appears as the inverse of the normalisation factor for derivatives and the product of factorials

$$\prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} j_{p,i}!$$

is needed to normalise the Hermite basis

$$\prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p}(x)),$$

see (3.14). We have

$$\frac{1}{2\varepsilon}\mathbb{E}[\mathbf{1}_{\{|b_{k_p}(x)|\le\varepsilon\}}H_{j_{p,0}}(b_{k_p}(x))] = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} H_{j_{p,0}}(t)\cdot\frac{e^{-t^2/2}}{\sqrt{2\pi}}dt \overset{\varepsilon\downarrow 0}{\to} \frac{1}{\sqrt{2\pi}}H_{j_{p,0}}(0). \tag{5.42}$$

We recall from (3.11) that for every $k\in\mathbb{N}$ we have $H_{2k+1}(0)=0$ and $H_{2k}(0)=(-1)^k(2k-1)!!$, and so the both sides of the above expression vanish if j_{p0} is odd. Furthermore we note that, using parity argument based on (3.10), it has been shown in [20, p. 17, Lemma 3.2] that if there exist $p,q,u,v\in\{-1,1\}$ s.t. $j_{p,q}$ is even and $j_{u,v}$ is odd, then

$$\mathbb{E}\left[\left|\det\begin{bmatrix}\widetilde{\partial}_{-1}b_{k_{-1}} & \widetilde{\partial}_1 b_{k_{-1}}\\ \widetilde{\partial}_{-1}b_{k_1} & \widetilde{\partial}_1 b_{k_1}\end{bmatrix}\right|\cdot\prod_{p,q\in\{-1,1\}}H_{j_{p,q}}(\widetilde{\partial}_p b_{k_q}(x))\right]=0. \tag{5.43}$$

These observations imply that for every $q=0,1,2,\ldots$ we have an implication

$$\text{if}\qquad |\mathbf{j}|\equiv\sum_{p\in\{-1,1\}}\sum_{i\in\{-1,0,1\}}j_{p,i}=2q+1\qquad\text{then}\qquad c_{\mathbf{j}}^{\varepsilon}=0, \tag{5.44}$$

and further that for every $q=0,1,2,\ldots$ the corresponding chaotic projection vanishes

$$\left(\prod_{p\in\{-1,1\}}\frac{\mathbf{1}_{\{|b_{k_p}(x)|<\varepsilon\}}}{2\varepsilon}\cdot\left|\det\begin{bmatrix}\widetilde{\partial}_{-1}b_{k_{-1}}(x) & \widetilde{\partial}_1 b_{k_{-1}}(x)\\ \widetilde{\partial}_{-1}b_{k_1}(x) & \widetilde{\partial}_1 b_{k_1}(x)\end{bmatrix}\right|\right)[2q+1]=0. \tag{5.45}$$

To conclude, we integrate over the domain $\mathcal{D}$ and pass to the limit $\varepsilon\downarrow 0$. We note that the constants $c_{\mathbf{j}}$ are given by $c_{\mathbf{j}}=\lim_{\varepsilon\downarrow 0}c_{\mathbf{j}}^{\varepsilon}$. □

5.7 Nodal Length of the Planar Berry's Random Wave

In this subsection we will recall some known results about the asymptotic ($k\to\infty$) fluctuations of the nodal length $\mathcal{L}(b_k,\mathcal{D})$ of the real planar Berry's Random Wave b_k. This will provide us with a convenient reference, to be used in the upcoming sections. (Lemma 5.3 and Theorem 5.1 below are due to Nourdin, Peccati and Rossi [70], and were established following computations of Berry [10].)

Definition 5.3. *Let $\mathcal{D}$ be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let b_k be the real Berry's Random Wave with wave-number $k>0$. We define the corresponding nodal length as the random variable*

$$\mathcal{L}(b_k,\mathcal{D})=\mathit{length}(\{x\in\mathcal{D}:b_k(x)=0\}). \tag{5.46}$$

Implicit in the above definition is the fact that the random set $\{x\in\mathcal{D}:b_k(x)=0\}$ consists of a finite sum of disjoint rectifiable curves (see [70, p. 137, Lemma 8.4]). Moreover, it is known that the nodal length $\mathcal{L}(b_k,\mathcal{D})$ has a finite variance ([70, p. 113, Lemma 3.3]) and corresponding Wiener-Itô chaos decomposition

$$\mathcal{L}(b_k,\mathcal{D})=\sum_{q=0}^{\infty}\mathcal{L}(b_k,\mathcal{D})[2q], \tag{5.47}$$

has been computed in [70, p. 115, Proposition 3.6]. For the sake of completeness, we reproduce it below using conventions **S.1–S.6**.

Lemma 5.3 ([70]). *Let $\mathcal{D}$ be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let b_k be the real planar Berry's Random Wave with wave-number $k > 0$. Then, the nodal length $\mathcal{L}(b_k, \mathcal{D})$ has a Wiener-Itô chaos expansion*

$$\mathcal{L}(b_k, \mathcal{D}) = \sum_{q=0}^{\infty} \mathcal{L}(b_k, \mathcal{D})[2q], \tag{5.48}$$

where

$$\mathcal{L}(b_k, \mathcal{D})[2q] = k \cdot \sum_{j\in\mathbb{N}^3, |j|=2q} \hat{c}_j \int_{\mathcal{D}} \prod_{i\in\{-1,0,1\}} H_{j_i}(\widetilde{\partial}_i b_k(x))dx. \tag{5.49}$$

Here, the sum runs over all vectors $j = (j_{-1}, j_0, j_1) \in \mathbb{N}^3$, with $|j| = j_{-1} + j_0 + j_1 = 2q$ and the constants $\hat{c}_j$ are defined as

$$\hat{c}_j = \hat{\rho}(j) \cdot \frac{(-1)^{j_0}(j_0 - 1)!!}{\sqrt{2} j_{-1}! j_0! j_1!} \cdot \mathbb{E}\left[\sqrt{Z_{-1}^2 + Z_1^2} \cdot H_{j_{-1}}(Z_{-1}) H_{j_1}(Z_1)\right], \tag{5.50}$$

with Z_{-1}, Z_1 being an independent standard Gaussian random variables. Moreover, we use here the notation $\hat{\rho}(j) = 1$ if the following conditions are simultaneously satisfied:

1. *The index j_0 is even,*
2. *Either:*
   (i) both of the indices j_{-1}, j_1, are even or,
   (ii) both of the indices j_{-1}, j_1, are odd.

If the above conditions are not satisfied then, we set $\hat{\rho}(j) = 0$.

Proof. See [70, p. 115, Proposition 3.6]. □

We will use the information contained in the next theorem to simplify the proof of the asymptotic variance formula (Theorem 4.1) through the application of the Recurrence Representation (Lemma 4.1). (We recall that the correlation coefficient was defined in **N.3** and Wiener-Itô chaotic projections were described in Subsection 5.5.)

Theorem 5.1 ([70]). *Let $\mathcal{D}$ be a convex compact planar domain, with non-empty interior and piecewise C^1 boundary. Let b_k be the real planar Berry's Random Wave with wave-number $k > 0$ and $\mathcal{L}(b_k, \mathcal{D})$ the associated nodal length. Then, we* *have*

$$\mathbb{E}\left[\mathcal{L}(b_k,\mathcal{D})\right] = \textit{area}(\mathcal{D}) \cdot \frac{k}{2\sqrt{2}}. \tag{5.51}$$

Moreover, it holds that

$$\lim_{k\to\infty} \operatorname{Corr}(\mathcal{L}(b_k,\mathcal{D}),\mathcal{L}(b_k,\mathcal{D})[4]) = 1, \qquad \lim_{k\to\infty} \frac{\operatorname{Var}(\mathcal{L}(b_k,\mathcal{D}))}{\frac{\text{area}(\mathcal{D})}{256\pi}\cdot \ln k} = 1, \tag{5.52}$$

where $\mathcal{L}(b_k,\mathcal{D})[4]$ denotes the 4-th Wiener-Itô chaos projection of the nodal length.

Proof. See [70, p. 103, Theorem 1.1] and [70, p. 110, Eq. (2.29)]. □

5.8 Density of Zeros and 2nd Chaotic Projection

The following lemma is a first step in characterising fluctuations of the nodal number.

Lemma 5.4. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let (k,K) be a pair of strictly positive wave-numbers and b_k, $\hat{b}_K$, a corresponding real Berry's Random Waves. Then, the expected value of the nodal number $\mathcal{N}(b_k,\hat{b}_K,\mathcal{D})$ is given by the formula*

$$\mathbb{E}\mathcal{N}(b_k,\hat{b}_K,\mathcal{D}) = \frac{\textit{area}(\mathcal{D})}{4\pi}\cdot(k\cdot K). \tag{5.53}$$

Proof. In this proof, we will use the notation introduced in **S.1**-**S.6**, in particular replacing the ordered pair of wave-numbers (k,K) with the unordered pair $\{k_{-1},k_1\}$. Using basic properties of the Wiener-Itô chaos decomposition (Eq. (5.32)) and explicit chaos decomposition for the nodal number established in Lemma 5.2 we can compute

$$\begin{aligned}
\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}) &= \mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D})[0] \\
&= (k_{-1}\cdot k_1)\cdot c_{\mathbf{0}} \int_{\mathcal{D}} 1dx \\
&= \frac{\text{area}(\mathcal{D})}{4\pi}\cdot(k_{-1}\cdot k_1)\cdot \mathbb{E}\left|\det\begin{bmatrix} Z_{-1,-1} & Z_{-1,1} \\ Z_{1,-1} & Z_{1,1}\end{bmatrix}\right|,
\end{aligned} \tag{5.54}$$

where $\{Z_{p,q} : p,q\in\{-1,1\}\}$ denotes a collection of four independent standard Gaussian random variables and $\mathbf{0} = (0,0,0,0,0,0)\in\mathbb{N}^6$. The proof is completed by observing that

$$\mathbb{E}\left|\det\begin{bmatrix} Z_{-1,-1} & Z_{-1,1} \\ Z_{1,-1} & Z_{1,1}\end{bmatrix}\right| = 1,$$

see [64, p. 73, Lem. II.B.3] with notation of [64, p. 39, Rem. II.1.2]. □

The forthcoming lemma will be used later to show that the second chaotic projection is asymptotically negligible. Note that the proof of Lemma 5.5 uses Lemma 4.1, and that

Lemma 5.5 is **not** used in the proof of Lemma 4.1. Moreover, we stress that the inequality (5.55) holds for every $K \geq k \geq 2$ without additional restrictions because inequality (5.58) used in its proof is non-asymptotic. See also [70, p. 117, Eq. (4.61)] for comparison.

Lemma 5.5. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let (k, K) be a pair of wave-numbers s.t. $2 \leq k \leq K$ and b_k, $\hat{b}_K$ be a corresponding Berry's Random Waves. Then, the variance of the second chaotic projection of the nodal number $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2]$ satisfies the following bound*

$$\operatorname{Var}\left(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2]\right) \leq \frac{diam(\mathcal{D})^2}{128\pi^2} \cdot (k^2 + K^2). \tag{5.55}$$

Proof of Lemma 5.5. Here, we are going to adopt the notation introduced in **S.1-S.6**, in particular replacing the ordered pair of wave-numbers (k, K) with the unordered pair $\{k_{-1}, k_1\}$. We will use the recurrence representation established in Lemma 4.1 and the explicit chaos decomposition for the nodal number established in Lemma 5.2. We observe first that (by using the notation (4.28)-(4.29))

$$\operatorname{Cross}(\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})[2]) = 0, \tag{5.56}$$

yielding

$$\begin{aligned}\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})[2] &= \frac{1}{\pi\sqrt{2}} \cdot k_{-1} \cdot \mathcal{L}(b_{k_1}, \mathcal{D})[2] + \frac{1}{\pi\sqrt{2}} \cdot k_1 \cdot \mathcal{L}(b_{k_{-1}}, \mathcal{D})[2] \\ &= \frac{1}{\pi\sqrt{2}} \sum_{p\in\{-1,1\}} k_{-p} \cdot \mathcal{L}(b_{k_p}, \mathcal{D})[2].\end{aligned} \tag{5.57}$$

To see why k_{-p} appears in front of $\mathcal{L}(b_{k_p}, \mathcal{D})[2]$ note the following. The chaotic decomposition of the nodal number $\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})$ includes a multiplicative factor $(k_{-1} \cdot k_1)$ (see (5.36)). The chaotic decomposition of the nodal length $\mathcal{L}(b_{k_p}, \mathcal{D})$ includes a multiplicative factor k_p (see (5.49)). Thus, when we use recurrence representation (4.30), the wavenumber k_p gets absorbed into expression $\mathcal{L}(b_{k_p}, \mathcal{D})[2]$ while the term k_{-p} remains as a multiplicative factor. By [70, p. 117, Proof of Lemma 4.1] for each $\lambda > 0$ we have

$$\operatorname{Var}(\mathcal{L}(b_\lambda, \mathcal{D})[2]) \leq \frac{\operatorname{perimeter}(\mathcal{D})^2}{64}. \tag{5.58}$$

Since $\mathcal{D}$ is convex and planar the perimeter is at most 6 times longer than the diameter which completes the proof. □

The next remark contains the first lower bound on the asymptotic variance of the nodal number. This bound turns out to be of the correct order.

Remark 5.2. *Consider a sequence $\{k^n_{-1}, k^n_1\}_{n\in\mathbb{N}}$ of unordered pairs of wave-numbers s.t. $k^n_{-1}, k^n_1 \to \infty$ and let $b_{k^n_{-1}}$, $b_{k^n_1}$ be the two corresponding independent*

Berry's Random Waves. Combining Lemma 4.1 with Theorem 5.1 yields

$$\begin{aligned}
&\mathrm{Var}\left(\mathcal{N}(b_{k^n_{-1}}, b_{k^n_1}, \mathcal{D})\right) \\
&= \mathrm{Var}\left(\sum_{p\in\{-1,1\}} \frac{k^n_{-p}}{\pi\sqrt{2}} \cdot \mathcal{L}(b_{k^n_p}, \mathcal{D}) + \mathrm{Cross}(\mathcal{N}(b_{k^n_{-1}}, b_{k^n_1}, \mathcal{D}))\right) \\
&= \sum_{p\in\{-1,1\}} \frac{(k^n_{-p})^2}{2\pi^2} \cdot \mathrm{Var}(\mathcal{L}(b_{k^n_p}, \mathcal{D})) + \mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k^n_{-1}}, b_{k^n_1}, \mathcal{D}))) \\
&\geq \frac{K_n^2}{2\pi^2} \mathrm{Var}(\mathcal{L}(b_{k_n}, \mathcal{D})) \\
&\sim \frac{\mathrm{area}(\mathcal{D})}{512\pi^3} \cdot K_n^2 \ln k_n \\
&\sim r^{log} \cdot \frac{\mathrm{area}(\mathcal{D})}{512\pi^3} \cdot K_n^2 \ln K_n,
\end{aligned} \tag{5.59}$$

where, in the last two lines we have replaced the unordered pair of the wave-numbers $\{k^n_{-1}, k^n_1\}$ *with its ordered equivalent* (k_n, K_n)*. Moreover, we have tacitly assumed that* $r^{log} = \lim_n \frac{\ln k_n}{\ln K_n}$ *exists and* $r^{log} > 0$*. Thus, we have obtained a lower bound consistent with, and in a form of, Theorem 4.1.*

Chapter 6 Proofs: Part II

6.1 Proof of the Domination of the 4th Chaos

This section is devoted to the proof of the following crucial lemma.

Lemma 6.1. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Then, there exists a numerical constant $L > 0$ s.t., for every pair (k, K) of wave-numbers s.t. $2 \leq k \leq K < \infty$, we have*

$$\sum_{q \neq 2} \mathrm{Var}\left(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2q]\right) \leq L \cdot \left(1 + diam(\mathcal{D})^4\right) \cdot K^2. \tag{6.1}$$

Here, b_k, $\hat{b}_K$ denote independent real Berry's Random Waves with wave-numbers k and K respectively, $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2q]$ denotes the $2q$-th chaotic projection of the nodal number and $\mathrm{diam}(\mathcal{D})$ denotes the diameter of the domain $\mathcal{D}$.

We stress that the inequality (6.1) is fully non-asymptotic and the dependency of its left-hand side on the smaller wavenumber k is fully controlled on the right-hand side using the larger wavenumber K. These properties are inherited from inequalities (6.11) and (6.12) which are used in the proof.

Our proof of Lemma 6.1 is based on a variation of the well-known decomposition into singular and non-singular pairs of cells as prescribed by the next definition. (See for comparison [74, p. 318-321, Section 6.1], [70, p. 127-128, Definition 7.2] or [20, p. 26, Definition 5.1].)

Definition 6.1. *Let $\mathcal{D}$ be a compact planar domain with non-empty interior. Fix a pair of wave-numbers (k, K) s.t. $2 \leq k \leq K < \infty$. Let $\{Q_l\}_l$ be a collection of $\lceil k \rceil^2$ closed squares s.t. the following conditions are satisfied:*

1. *The collection $\{Q_l\}_l$ covers $\mathcal{D}$, that is*

$$\mathcal{D} \subseteq Q_1 \cup Q_2 \cup \ldots \cup Q_{\lceil k \rceil^2}.$$

2. *For every $1 \leq l \leq \lceil k \rceil^2$, we have*

$$area(Q_l) = \left(\frac{diam(\mathcal{D})}{\lceil k \rceil}\right)^2.$$

3. *For every* $1 \leq l, m \leq \lceil k \rceil^2$ *with* $l \neq m$, *we have*

$$area(Q_l \cap Q_m) = 0.$$

For every $1 \leq l \leq \lceil k \rceil^2$, *we set* $\mathcal{D}_l := Q_l \cap \mathcal{D}$ *and, for each* $1 \leq m, l \leq \lceil k \rceil^2$, *we say that the ordered pair* $(\mathcal{D}_l, \mathcal{D}_m)$ *is singular if*

$$\max_{p \in \{-1,1\}} \max_{i,j \in \{-1,0,1\}} \sup_{(x,y) \in \mathcal{D}_l \times \mathcal{D}_m} |r_{ij}(k_p(x-y))| > \frac{1}{1000}. \tag{6.2}$$

Otherwise, we say that the ordered pair $(\mathcal{D}_l, \mathcal{D}_m)$ *is non-singular. Here,* r_{ij} *denote the covariance functions associated with the real Berry's Random Wave with wave-number* $k = 1$ *(see Subsection 5.3).*

The constant $1/1000$ in (6.2) holds no particular significance. It is simply a fixed numerical value chosen to be sufficiently small for the arguments presented later in this section to work.

To any collection of pairs $\{(\mathcal{D}_l, \mathcal{D}_m)\}_{l,m}$, $1 \leq l, m \leq \lceil k \rceil^2$, as in Definition 6.1, we will refer to as 'the decomposition of $\mathcal{D} \times \mathcal{D}$ into singular and non-singular pairs of cells.' This allows us to write

$$\begin{aligned} &\sum_{q=3}^{\infty} \operatorname{Var}\left(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}\mathcal{D})[2q]\right) \\ &= \sum_{q=3}^{\infty} \sum_{(\mathcal{D}_l,\mathcal{D}_m)\ \text{singular}} \operatorname{Cov}\left(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}_l)[2q], \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}_m)[2q]\right) \\ &+ \sum_{q=3}^{\infty} \sum_{(\mathcal{D}_l,\mathcal{D}_m)\ \text{non-singular}} \operatorname{Cov}\left(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}_l)[2q], \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}_m)[2q]\right), \end{aligned} \tag{6.3}$$

and we will bound each term in this sum using a different strategy. The main difficulty is in bounding the sum over the singular pairs of cells $(\mathcal{D}_l, \mathcal{D}_m)$ and it arises due to the lack of control on the decay of the covariance functions $r_{ij}(k_p(x-y))$ as $k_p \to \infty$. To circumvent this problem we will take advantage of the next lemma which shows that there are relatively few singular pairs of cells $(\mathcal{D}_l, \mathcal{D}_m)$. (We note that the total number of cells $\mathcal{D}_l$ in the construction described above is $\lceil k \rceil^2$ and so the total number of pairs $(\mathcal{D}_l, \mathcal{D}_m)$ is $\lceil k \rceil^4$.)

Lemma 6.2. *There exists a numerical constant* $C > 0$ *such that, the following inequality holds:*

$$|\{(l,m) : (\mathcal{D}_l, \mathcal{D}_m) \text{ is singular}\}| \leq C \cdot \left(1 + \frac{1}{diam(\mathcal{D})^2}\right) \cdot k^2, \tag{6.4}$$

regardless of the choice of associated parameters. These parameters are: the selection of a compact planar domain $\mathcal{D}$ *with a non-empty interior, a pair* (k, K) *of wave numbers where* $2 \leq k \leq K < \infty$, *and a decomposition* $\{(\mathcal{D}_l, \mathcal{D}_m)\}_{l,m}$, *with* $1 \leq l, m \leq \lceil k \rceil^2$, *of* $\mathcal{D} \times \mathcal{D}$ *into singular and non-singular pairs of cells* $(\mathcal{D}_l, \mathcal{D}_m)$.

Proof. By definition, if the pair of cells $(\mathcal{D}_l, \mathcal{D}_m)$ is singular then we can find $x \in \mathcal{D}_l$, $y \in \mathcal{D}_m$, $i, j \in \{-1, 1\}$ and $p \in \{-1, 1\}$, such that

$$|r_{ij}(k_p(x-y))| > \frac{1}{1000}. \tag{6.5}$$

As a consequence of the inequality (5.18), for some positive numerical constant $L > 0$ (independent of x and y), we have

$$|x-y| \leq \frac{L}{k_p} \leq \frac{L}{k} = \frac{L}{\lceil k \rceil} \cdot \frac{\lceil k \rceil}{k} \leq \frac{2L}{\lceil k \rceil} = \frac{L\sqrt{2}}{\operatorname{diam}(\mathcal{D})} \cdot \left(\sqrt{2} \cdot \frac{\operatorname{diam}(\mathcal{D})}{\lceil k \rceil} \right). \tag{6.6}$$

This obviously shows that, if the pair $(\mathcal{D}_l, \mathcal{D}_m)$ is singular, then

$$\operatorname{dist}(\mathcal{D}_l, \mathcal{D}_m) = \inf_{x \in \mathcal{D}_l, y \in \mathcal{D}_m} |x-y| \leq \frac{L\sqrt{2}}{\operatorname{diam}(\mathcal{D})} \cdot \left(\sqrt{2} \cdot \frac{\operatorname{diam}(\mathcal{D})}{\lceil k \rceil} \right). \tag{6.7}$$

Since

$$\max_{1 \leq l \leq \lceil k \rceil^2} \operatorname{diam}(\mathcal{D}_l) \leq \sqrt{2} \cdot \frac{\operatorname{diam}(\mathcal{D})}{\lceil k \rceil}, \tag{6.8}$$

it follows that for every $1 \leq l \leq \lceil k \rceil^2$, we have

$$|\{m : (\mathcal{D}_l, \mathcal{D}_m) \text{ is singular}\}| \leq C \cdot \left(1 + \frac{1}{\operatorname{diam}(\mathcal{D})^2} \right), \tag{6.9}$$

where $C > 0$ is a some another numerical constant. Therefore, using also that $\frac{\lceil k \rceil}{k} \leq 2$, we obtain

$$\begin{aligned} |\{(l,m) : (\mathcal{D}_l, \mathcal{D}_m) \text{ is singular}\}| &= \sum_{l=1}^{\lceil k \rceil^2} |\{m : (\mathcal{D}_l, \mathcal{D}_m) \text{ is singular}\}| \\ &\leq \lceil k \rceil^2 \cdot \max_{1 \leq l \leq \lceil k \rceil^2} |\{m : (\mathcal{D}_l, \mathcal{D}_m) \text{ is singular}\}| \\ &\leq 4C \cdot \left(1 + \frac{1}{\operatorname{diam}(\mathcal{D})^2} \right) \cdot k^2, \end{aligned} \tag{6.10}$$

which yields the postulated inequality. □

The following lemma allows one to asses the singular sum in the statement of Lemma 6.1.

Lemma 6.3. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Then, there exists a numerical constant $C > 0$ s.t., for every pair (k, K) of wave-numbers with $2 \leq k \leq K < \infty$, and for every decomposition $\{(\mathcal{D}_l, \mathcal{D}_m)\}_{l,m}$, $1 \leq l, m \leq \lceil k \rceil^2$, of $\mathcal{D} \times \mathcal{D}$ into singular and* *non-singular pairs of cells* $(\mathcal{D}_l, \mathcal{D}_m)$*, we have*

$$\begin{aligned}\sum_{q=3}^{\infty} \Bigg| \sum_{(\mathcal{D}_l, \mathcal{D}_m) \ singular} & \mathrm{Cov}\left(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}_l)[2q], \mathcal{N}(b_k, \hat{b}_K, \mathcal{D}_m)[2q]\right) \Bigg| \\ & \leq C \cdot \left(1 + diam(\mathcal{D})^4\right) \cdot K^2.\end{aligned} \tag{6.11}$$

Here, b_k*,* $\hat{b}_K$ *are independent real Berry's Random Waves with wave-numbers* k *and* K *respectively,* $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}_l)[2q]$ *denotes the* $2q$*-th Wiener Chaos projection of the nodal number* $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}_l)$ *and* $\mathrm{diam}(\mathcal{D})$ *denotes the diameter of the domain* $\mathcal{D}$*.*

The next lemma provides a bound on the non-singular sum featuring in Lemma 6.1.

Lemma 6.4. *Let* $\mathcal{D}$ *be a convex compact domain of the plane, with non-empty interior and piecewise* C^1 *boundary* $\partial\mathcal{D}$*. Then, there exists a numerical constant* $C > 0$ *s.t., for every pair* (k, K) *of wave-numbers with* $2 \leq k \leq K < \infty$*, and for every decomposition* $\{(\mathcal{D}_l, \mathcal{D}_m)\}_{l,m}$*,* $1 \leq l, m \leq \lceil k \rceil^2$*, of* $\mathcal{D} \times \mathcal{D}$ *into singular and non-singular pairs of cells* $(\mathcal{D}_l, \mathcal{D}_m)$*, we have*

$$\begin{aligned}\sum_{q=3}^{\infty} \Bigg| \sum_{(\mathcal{D}_l, \mathcal{D}_n) \ non\text{-}sing.} & \mathrm{Cov}\left(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}_l)[2q], \mathcal{N}(b_k, \hat{b}_K, \mathcal{D}_m)[2q]\right) \Bigg| \\ & \leq C \cdot \left(1 + diam(\mathcal{D})^2\right) \cdot K^2.\end{aligned} \tag{6.12}$$

Here, b_k*,* $\hat{b}_K$ *are independent real Berry's Random Waves with wave-numbers* k *and* K *respectively,* $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}_l)[2q]$ *denotes the* $2q$*-th Wiener Chaos projection of the nodal number* $\mathcal{N}(b_k, \hat{b}_K, \mathcal{D}_l)$ *and* $\mathrm{diam}(\mathcal{D})$ *denotes the diameter of the domain* $\mathcal{D}$*.*

Proof of Lemma 6.3. The main idea of this proof is classical: we use Kac-Rice formula, the asymptotic properties of Bessel functions and take advantage of the fact that each subdomain $\mathcal{D}_i$ is small and there are not too many singular pairs $(\mathcal{D}_i, \mathcal{D}_j)$ of such domains. In this proof, we will use the notation introduced in **S.1**-**S.6**. In particular, $b_{k_{-1}}$, b_{k_1} will denote a pair of independent Berry's Random Waves with wave-numbers k_{-1} and k_1 which are ≥ 2. We split the argument into four parts:

Step 1. In this step, we apply the classical Kac-Rice formula to derive the integral, which we will aim to control throughout the rest of the lemma. Let Q_1 be one of the covering squares for $\mathcal{D}$, described in the Definition 6.1. Using the bound on a number of singular pair of cells from Lemma 6.2, the Cauchy-Schwarz inequality and Lemma 5.4 for the value of the expectation, we obtain

$$\begin{aligned}
&\sum_{q=3}^{\infty}\left|\sum_{(\mathcal{D}_l,\mathcal{D}_m)\text{ sing.}} \operatorname{Cov}\left(\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_l)[2q],\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_m)[2q]\right)\right| \\
&\leq \sum_{(\mathcal{D}_l,\mathcal{D}_m)\text{ sing.}}\ \sum_{q=3}^{\infty}\mathbb{E}\left|\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_l)[2q]\cdot\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_m)[2q]\right| \\
&\leq C\cdot\left(1+\frac{1}{\operatorname{diam}(\mathcal{D})^2}\right)\cdot\min(k_{-1},k_1)^2\cdot\mathbb{E}\left[\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)^2\right] \\
&= C\cdot\left(1+\frac{1}{\cdot\operatorname{diam}(\mathcal{D})^2}\right)\cdot\min(k_{-1},k_1)^2\cdot\Big(\mathbb{E}\left[\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)\right] \\
&\qquad +\mathbb{E}[\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)(\mathcal{N}(b_{k_{-1}},\hat{b}_{k_1},Q_1)-1)]\Big) \\
&\leq C\cdot\left(1+\frac{1}{\operatorname{diam}(\mathcal{D})^2}\right)\cdot\min(k_{-1},k_1)^2\cdot\Big(\frac{\operatorname{diam}(\mathcal{D})^2}{4\pi\lceil\min(k_{-1},k_1)\rceil^2}\cdot k_{-1}\cdot k_1 \\
&\qquad +\mathbb{E}\left[\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)(\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)-1)\right]\Big) \\
&\leq C\cdot\left(1+\operatorname{diam}(\mathcal{D})^2\right)\cdot\max(k_{-1},k_1)^2 \\
&\qquad +C\cdot\left(1+\frac{1}{\operatorname{diam}(\mathcal{D})^2}\right)\cdot\min(k_{-1},k_1)^2\cdot\mathbb{E}[\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)(\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)-1)],
\end{aligned} \tag{6.13}$$

where $C>0$ is a numerical constant taken from Lemma 6.2. Denoting with $\varphi_{(b_{k_p}(x),b_{k_p}(y))}(0,0)$ the (Gaussian) density of the vector $(b_{k_p}(x),b_{k_p}(y))$ at the point zero, and using the Kac-Rice formula (see [5, p. 164, Theorem 6.3 (Rice Formula for the k-th Moment)]) we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{N}(b_{k_{-1}},b_{k_1},Q_l)(\mathcal{N}(b_{k_{-1}},b_{k_1},Q_l)-1)\right]= \\
&=\int_{Q_1\times Q_1}\mathbb{E}\Bigg[\left|\det\begin{bmatrix}\partial_{-1}b_{k_{-1}}(x) & \partial_1 b_{k_{-1}}(x)\\ \partial_{-1}b_{k_1}(x) & \partial_1 b_{k_1}(x)\end{bmatrix}\right|\cdot\left|\det\begin{bmatrix}\partial_{-1}b_{k_{-1}}(y) & \partial_1 b_{k_{-1}}(y)\\ \partial_{-1}b_{k_1}(y) & \partial_1 b_{k_1}(y)\end{bmatrix}\right| \\
&\qquad \Bigg| b_{k_{-1}}(x)=b_{k_1}(x)=b_{k_{-1}}(y)=b_{k_1}(y)=0\Bigg]\cdot\Big(\prod_{p\in\{-1,1\}}\varphi_{(b_{k_p}(x),b_{k_p}(y))}(0,0)\Big)dxdy \\
&\leq\frac{1}{4}\cdot(k_{-1}^2\cdot k_1^2)\cdot\operatorname{area}(Q_1)\int_{Q_1-Q_1}(1-J_0(k_{-1}|z|)^2)^{-1/2}\cdot(1-J_0(k_1|z|)^2)^{-1/2} \\
&\qquad\cdot\mathbb{E}\left[\left(\det\begin{bmatrix}\widetilde{\partial}_{-1}b_{k_{-1}}(z) & \widetilde{\partial}_1 b_{k_{-1}}(z)\\ \widetilde{\partial}_{-1}b_{k_1}(z) & \widetilde{\partial}_1 b_{k_1}(z)\end{bmatrix}\right)^2\Bigg| b_{k_{-1}}(z)=b_{k_1}(z)=b_{k_{-1}}(0)=b_{k_1}(0)=0\right]dz \\
&\leq\frac{\operatorname{diam}(\mathcal{D})^2}{64\pi^3}\cdot\max(k_{-1},k_1)^2\cdot\int_{Q_1-Q_1}(1-J_0(k_{-1}|z|)^2)^{-1/2}\cdot(1-J_0(k_1|z|)^2)^{-1/2} \\
&\qquad\cdot\mathbb{E}\left[\left(\det\begin{bmatrix}\widetilde{\partial}_{-1}b_{k_{-1}}(z) & \widetilde{\partial}_1 b_{k_{-1}}(z)\\ \widetilde{\partial}_{-1}b_{k_1}(z) & \widetilde{\partial}_1 b_{k_1}(z)\end{bmatrix}\right)^2\Bigg| b_{k_{-1}}(z)=b_{k_1}(z)=b_{k_{-1}}(0)=b_{k_1}(0)=0\right]dz,
\end{aligned} \tag{6.14}$$

where to get the penultimate inequality we have used the conditional Cauchy-Schwarz inequality and the stationarity of the field $(b_{k_{-1}}, b_{k_1})$.

Step 2. In this step, we carry out precise Gaussian computations to obtain a simplified formula where, importantly, the roles of k_{-1} and k_1 are decoupled. We observe that

$$\begin{aligned}
&\left(\det\begin{bmatrix}\widetilde{\partial}_{-1} b_{k_{-1}}(z) & \widetilde{\partial}_1 b_{k_{-1}}(z)\\ \widetilde{\partial}_{-1} b_{k_1}(z) & \widetilde{\partial}_1 b_{k_1}(z)\end{bmatrix}\right)^2\\
&\quad = \left(\widetilde{\partial}_{-1} b_{k_{-1}}(z)\widetilde{\partial}_1 b_{k_1}(z) - \widetilde{\partial}_{-1} b_{k_1}(z)\widetilde{\partial}_1 b_{k_{-1}}(z)\right)^2\\
&\quad = \sum_{q\in\{-1,1\}}\prod_{p\in\{-1,1\}}\left(\widetilde{\partial}_{(pq)} b_{k_p}(z)\right)^2 - 2\cdot\prod_{p,q\in\{-1,1\}}\widetilde{\partial}_q b_{k_p}(z),
\end{aligned}\tag{6.15}$$

where $\widetilde{\partial}_{(pq)}$ should be understood as $\widetilde{\partial}_v$ where $v := pq$. Using standard conditioning formulas for the Gaussian vectors ([5, p. 18, Proposition 1.2]) and Lemma 5.1 we have that for each $p = \pm 1$ and for any choice of $u, v \in \{-1, +1\}$,

$$\begin{aligned}
&\mathbb{E}\left[\widetilde{\partial}_u b_{k_p}(z)\cdot\widetilde{\partial}_v b_{k_p}(z)\middle| b_{k_{-1}}(z) = b_{k_1}(z) = b_{k_{-1}}(0) = b_{k_1}(0) = 0\right]\\
&\quad = \left(\delta_{uv} - \frac{r_u(k_p z) r_v(k_p z)}{1 - r(k_p z)^2}\right).
\end{aligned}\tag{6.16}$$

Writing

$$r_{(pq)} := r_v, \qquad v := pq, \qquad p, q \in \{-1, 1\},\tag{6.17}$$

and combining (6.15) with (6.16), one can deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left(\det\begin{bmatrix}\widetilde{\partial}_{-1} b_{k_{-1}}(z) & \widetilde{\partial}_1 b_{k_{-1}}(z)\\ \widetilde{\partial}_{-1} b_{k_1}(z) & \widetilde{\partial}_1 b_{k_1}(z)\end{bmatrix}\right)^2 \middle| b_{k_{-1}}(z) = b_{k_1}(z) = b_{k_{-1}}(0) = b_{k_1}(0) = 0\right]\\
&= \sum_{q\in\{-1,1\}}\mathbb{E}\left[\prod_{p\in\{-1,1\}}\left(\widetilde{\partial}_{(pq)} b_{k_p}(z)\right)^2 \middle| b_{k_{-1}}(z) = b_{k_1}(z) = b_{k_{-1}}(0) = b_{k_1}(0) = 0\right]\\
&\qquad - 2\mathbb{E}\left[\prod_{p,q\in\{-1,1\}}\widetilde{\partial}_q b_{k_p}(z)\middle| b_{k_{-1}}(z) = b_{k_1}(z) = b_{k_{-1}}(0) = b_{k_1}(0) = 0\right]\\
&= \sum_{q\in\{-1,1\}}\prod_{p\in\{-1,1\}}\mathbb{E}\left[\left(\widetilde{\partial}_{(pq)} b_{k_p}(z)\right)^2 \middle| b_{k_p}(z) = b_{k_p}(0) = 0\right]\\
&\qquad - 2\prod_{p\in\{-1,1\}}\mathbb{E}\left[\prod_{q\in\{-1,1\}}\widetilde{\partial}_q b_{k_p}(z)\middle| b_{k_p}(z) = b_{k_p}(z) = 0\right].
\end{aligned}\tag{6.18}$$

Now we rewrite (6.18) as

$$
\begin{aligned}
&= \sum_{q\in\{-1,1\}} \prod_{p\in\{-1,1\}} \left(1 - \frac{r_{(pq)}(k_p z)^2}{1-r(k_p z)^2}\right) - \frac{2\prod_{p,q\in\{-1,1\}} r_q(k_p z)}{\prod_{p\in\{-1,1\}}(1-r(k_p z)^2)} \\
&= \sum_{q\in\{-1,1\}} \left(1 - \sum_{p\in\{-1,1\}} \frac{r_{(pq)}(k_p z)^2}{1-r(k_p z)^2} + \frac{\prod_{p\in\{-1,1\}} r_{(pq)}(k_p z)}{\prod_{p\in\{-1,1\}}(1-r(k_p z)^2)}\right) \\
&\quad - \frac{2\prod_{p,q\in\{-1,1\}} r_q(k_p z)}{\prod_{p\in\{-1,1\}}(1-r(k_p z)^2)} \\
&= 2 - \left(\sum_{p\in\{-1,1\}} \frac{\sum_{q\in\{-1,1\}} r_q(k_p z)^2}{1-r(k_p z)^2}\right) \\
&\quad + \frac{\sum_{q\in\{-1,1\}} \prod_{p\in\{-1,1\}} r_{(pq)}(k_p z)^2 - 2\prod_{p,q\in\{-1,1\}} r_q(k_p z)}{\prod_{p\in\{-1,1\}}(1-r(k_p z)^2)}.
\end{aligned}
\tag{6.19}
$$

Before proceeding, we note the following: choose $p, q \in \{-1, 1\}$ and define $v := pq$. We then adopt the convention $r_{(pq)} \equiv r_v$. We observe that, for each fixed $q \in \{-1, 1\}$, we have

$$
\begin{aligned}
\prod_{p\in\{-1,1\}} r_{(pq)}(k_p z)^2 &= \prod_{p\in\{-1,1\}} \left(2 \cdot \frac{z_{(pq)}^2}{|z|^2} \cdot J_1(k_p|z|)^2\right) \\
&= 4 \cdot \frac{\prod_{v\in\{-1,1\}} z_v^2}{|z|^4} \cdot \left(\prod_{p\in\{-1,1\}} J_1(k_p|z|)^2\right) \\
&= \prod_{v,p\in\{-1,1\}} \left(\sqrt{2} \cdot \frac{z_v}{|z|} \cdot J_1(k_p|z|)\right) \\
&= \prod_{v,p\in\{-1,1\}} r_v(k_p z),
\end{aligned}
\tag{6.20}
$$

and we would like to highlight that we have obtained a quantity independent of the choice of q. The equality obtained in (6.20):

$$
\prod_{p\in\{-1,1\}} r_{(pq)}(k_p z)^2 = \prod_{v,p\in\{-1,1\}} r_v(k_p z) \tag{6.21}
$$

implies the following cancellation

$$
\begin{aligned}
&\sum_{q\in\{-1,1\}} \prod_{p\in\{-1,1\}} r_{(pq)}(k_p z)^2 - 2 \prod_{p,q\in\{-1,1\}} r_q(k_p z) \\
&= \sum_{q\in\{-1,1\}} \left(\prod_{v,p\in\{-1,1\}} r_v(k_p z)\right) - 2 \prod_{p,q\in\{-1,1\}} r_q(k_p z) \\
&= 2 \prod_{v,p\in\{-1,1\}} r_v(k_p z) - 2 \prod_{p,q\in\{-1,1\}} r_q(k_p z) = 0.
\end{aligned}
\tag{6.22}
$$

Applying (6.22) in the last line of (6.19) shows that (6.19) can be reduced to the following terms:

$$2-\sum_{p\in\{-1,1\}}\left(\frac{\sum_{q\in\{-1,1\}} r_q(k_p z)^2}{1-r(k_p z)^2}\right). \tag{6.23}$$

Furthermore, we observe that

$$\begin{aligned} 2-\sum_{p\in\{-1,1\}}\left(\frac{\sum_{q\in\{-1,1\}} r_q(k_p z)^2}{1-r(k_p z)^2}\right) &= 2\left(1-\sum_{p\in\{-1,1\}}\frac{\sum_{q\in\{-1,1\}}\frac{z_q^2}{|z|^2}J_1(k_p|z|)^2}{1-r(k_p z)^2}\right)\\ &= 2\sum_{p\in\{-1,1\}}\left(1/2-\frac{J_1(k_p|z|)^2}{1-J_0(k_p|z|)^2}\right). \end{aligned} \tag{6.24}$$

Thus, finally, we see that

$$\begin{aligned} &\mathbb{E}\left[\left(\det\begin{bmatrix}\widetilde{\partial}_{-1}b_{k_{-1}}(z) & \widetilde{\partial}_{1}b_{k_{-1}}(z)\\ \widetilde{\partial}_{-1}b_{k_{1}}(z) & \widetilde{\partial}_{1}b_{k_{1}}(z)\end{bmatrix}\right)^2 \middle| b_{k_{-1}}(z)=b_{k_1}(z)=b_{k_{-1}}(0)=b_{k_1}(0)=0\right]\\ &=2\sum_{p\in\{-1,1\}}\left(1/2-\frac{J_1(k_p|z|)^2}{1-J_0(k_p|z|)^2}\right). \end{aligned} \tag{6.25}$$

Step 3. In this step, we apply the simplified form (6.25) to (6.14) and use the properties of Bessel functions to derive the main upper bound of this lemma. In complete analogy with (4.2) we set

$$r:=\frac{\min_{p\in\{-1,1\}} k_p}{\max_{p\in\{-1,1\}} k_p}. \tag{6.26}$$

We use (6.25) and the change of variables $x := kz$ to bound the integral in the last line of (6.14) by

$$\begin{aligned} &\int_{Q_1-Q_1}\left(1-J_0(k_{-1}|z|)^2\right)^{-1/2}\cdot\left(1-J_0(k_1|z|)^2\right)^{-1/2}\\ &\qquad\cdot\mathbb{E}\left[\left(\det\begin{bmatrix}\widetilde{\partial}_{-1}b_{k_{-1}}(z) & \widetilde{\partial}_{1}b_{k_{-1}}(z)\\ \widetilde{\partial}_{-1}b_{k_{1}}(z) & \widetilde{\partial}_{1}b_{k_{1}}(z)\end{bmatrix}\right)^2 \middle| b_{k_{-1}}(z)=b_{k_1}(z)=b_{k_{-1}}(0)=b_{k_1}(0)=0\right]dz\\ &=\frac{2}{k^2}\cdot\int_{k(Q_1-Q_1)}\frac{\left(1/2-\frac{J_1(|x|)^2}{1-J_0(|x|)^2}\right)+\left(1/2-\frac{J_1(r^{-1}|x|)^2}{1-J_0(r^{-1}|x|)^2}\right)}{\sqrt{1-J_0(|x|)^2}\cdot\sqrt{1-J_0(r^{-1}|x|)^2}}dx\\ &\le\frac{2}{k^2}\cdot\int_{\lceil k\rceil(Q_1-Q_1)}\frac{\left(1/2-\frac{J_1(|x|)^2}{1-J_0(|x|)^2}\right)+\left(1/2-\frac{J_1(r^{-1}|x|)^2}{1-J_0(r^{-1}|x|)^2}\right)}{\sqrt{1-J_0(|x|)^2}\cdot\sqrt{1-J_0(r^{-1}|x|)^2}}dx\\ &\le\frac{2}{k^2}\cdot\int_{B(0,2\cdot\operatorname{diam}(\mathcal{D}))}\frac{\left(1/2-\frac{J_1(|x|)^2}{1-J_0(|x|)^2}\right)+\left(1/2-\frac{J_1(r^{-1}|x|)^2}{1-J_0(r^{-1}|x|)^2}\right)}{\sqrt{1-J_0(|x|)^2}\cdot\sqrt{1-J_0(r^{-1}|x|)^2}}dx. \end{aligned} \tag{6.27}$$

It is easy to verify that

$$\forall t > 0, \qquad 0 \leq \frac{1/2 - \frac{J_1(t)^2}{1-J_0(t)^2}}{\sqrt{1-J_0(t)^2}} \leq 1, \tag{6.28}$$

using which we can bound the last line of (6.27) by

$$\begin{aligned}
&\frac{2}{k^2} \cdot \int_{B(0,2\cdot\mathrm{diam}(\mathcal{D}))} \frac{1}{\sqrt{1-J_0(|x|)^2}} \cdot \frac{1/2 - \frac{J_1(r^{-1}|x|)^2}{1-J_0(r^{-1}|x|)^2}}{\sqrt{1-J_0(r^{-1}|x|)^2}} dx \\
&\qquad + \frac{2}{k^2} \cdot \int_{B(0,2\cdot\mathrm{diam}(\mathcal{D}))} \frac{1}{\sqrt{1-J_0(r^{-1}|x|)^2}} \cdot \frac{1/2 - \frac{J_1(|x|)^2}{1-J_0(|x|)^2}}{\sqrt{1-J_0(|x|)^2}} dx \\
&\leq \frac{2}{k^2} \cdot \int_{B(0,2\cdot\mathrm{diam}(\mathcal{D}))} \frac{dx}{\sqrt{1-J_0(|x|)^2}} + \frac{2}{k^2} \cdot \int_{B(0,2\cdot\mathrm{diam}(\mathcal{D}))} \frac{dx}{\sqrt{1-J_0(r^{-1}|x|)^2}} \\
&\leq \frac{4}{k^2} \cdot \left(\int_{B(0,1)} \frac{1}{\sqrt{1-J_0(|x|)^2}} dx + \frac{4\pi}{\sqrt{1-J_0(1)^2}} \mathrm{diam}(\mathcal{D})^2 \right) \\
&\leq \frac{4}{k^2} \cdot \left(2 \int_{B(0,1)} \frac{1}{|x|} dx + 8\pi \cdot \mathrm{diam}(\mathcal{D})^2 \right) \\
&\leq \frac{32\pi}{k^2} \cdot \left(1 + \mathrm{diam}(\mathcal{D})^2\right).
\end{aligned} \tag{6.29}$$

Here, we have also used the following simple observations

$$\begin{aligned}
&\forall t > 0, \qquad |J_0(t)| \leq 1, \qquad |J_1(t)| \leq 1, \\
&\forall t > 1, \qquad J_0(t)^2 < J_0(1)^2, \\
&t \mapsto J_0(t)^2 \text{ is non-increasing on the interval } [0,1].
\end{aligned} \tag{6.30}$$

Step 4. We combine (6.13) and (6.14) with (6.29) to obtain that for some strictly positive numerical constants C_1, C_2, C_3, C_4, we have

$$\begin{aligned}
&\sum_{q=3}^{\infty}\left|\sum_{(\mathcal{D}_l,\mathcal{D}_m)\text{ sing.}} \operatorname{Cov}\left(\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_l)[2q],\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_m)[2q]\right)\right| \\
&\le C_1(1+\operatorname{diam}(\mathcal{D})^2)\cdot\max(k_{-1},k_1)^2 \\
&\qquad + C_1(1+\frac{1}{\operatorname{diam}(\mathcal{D})^2})\cdot\min(k_{-1},k_1)^2\cdot\mathbb{E}[\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)(\mathcal{N}(b_{k_{-1}},b_{k_1},Q_1)-1)] \\
&\le C_1(1+\operatorname{diam}(\mathcal{D})^2)\cdot\max(k_{-1},k_1)^2 \\
&\qquad + C_2(1+\operatorname{diam}(\mathcal{D})^2)\cdot k_{-1}^2\cdot k_1^2\cdot\int_{Q_1-Q_1}(1-J_0(k_{-1}|z|)^2)^{-1/2}\cdot(1-J_0(k_1|z|)^2)^{-1/2} \\
&\qquad \cdot\mathbb{E}[(\det\begin{bmatrix}\widetilde{\partial}_{-1}b_{k_{-1}}(z) & \widetilde{\partial}_1 b_{k_{-1}}(z)\\ \widetilde{\partial}_{-1}b_{k_1}(z) & \widetilde{\partial}_1 b_{k_1}(z)\end{bmatrix})^2\Big|b_{k_{-1}}(z)=b_{k_1}(z)=b_{k_{-1}}(0)=b_{k_1}(0)=0]dz \\
&\le C_1(1+\operatorname{diam}(\mathcal{D})^2)\cdot\max(k_{-1},k_1)^2+C_3(1+\operatorname{diam}(\mathcal{D})^4)\cdot\max(k_{-1},k_1)^2 \\
&\le C_4(1+\operatorname{diam}(\mathcal{D})^4)\cdot\max(k_{-1},k_1)^2.
\end{aligned} \tag{6.31}$$

This is the postulated inequality and the proof is therefore concluded. ☐

Proof of Lemma 6.4. The argument presented here is standard and uses in a crucial way the geometric decay of $|r_{ij}(k_p(x-y))|^{2q}$ on the set of non-singular pairs of cells to control the contribution associated with $2q$-th chaotic projections. For this proof, we are going to adopt the notation introduced in **S.1**-**S.6**. In particular, we will write k_{-1},k_1 to denote an unordered pair of wave-numbers corresponding to the ordered pair k,K and $b_{k_{-1}}$, b_{k_1} to denote a pair of independent BRWs with wave-numbers k_{-1} and k_1 respectively.

Step 1. We recall that an explicit formula for the Wiener Chaos decomposition of the nodal number $\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D})$ has been established in Lemma 5.2 and that for every $q\ge 1$ the projection $\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D})[q]$ on the q-th Wiener chaos has zero mean (see (5.32)). This yields

$$\begin{aligned}
&\sum_{q=3}^{\infty}\left|\sum_{(\mathcal{D}_l,\mathcal{D}_m)\text{ non sing.}} \operatorname{Cov}\left(\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_l)[2q],\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_m)[2q]\right)\right| \\
&=\sum_{q=3}^{\infty}\left|\sum_{(\mathcal{D}_l,\mathcal{D}_m)\text{ non sing.}} \mathbb{E}(\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_l)[2q]\cdot\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_m)[2q])\right| \\
&=\sum_{q=3}^{\infty}\Bigg|\sum_{(\mathcal{D}_l,\mathcal{D}_m)\text{ non sing.}}(k_{-1}\cdot k_1)^2\sum_{\mathbf{i}\in\mathbb{N}^6,|\mathbf{i}|=2q}\ \sum_{\mathbf{j}\in\mathbb{N}^6,|\mathbf{j}|=2q}c_{\mathbf{i}}c_{\mathbf{j}} \\
&\qquad\cdot\int_{\mathcal{D}_l\times\mathcal{D}_m}\prod_{p\in\{-1,1\}}\mathbb{E}\left[\prod_{v\in\{-1,0,1\}}H_{i_{p,v}}(\widetilde{\partial}_v b_{k_p}(x))\cdot H_{j_{p,v}}(\widetilde{\partial}_v b_{k_p}(y))\right]dxdy\Bigg|,
\end{aligned} \tag{6.32}$$

where $\mathbf{i},\mathbf{j} \in \mathbb{N}^6$ are indexed as

$$\begin{aligned}\mathbf{i} &:= (i_{-1,-1}; i_{-1,0}; i_{-1,1}; i_{1,-1}; i_{1,0}; i_{1,1}),\\ \mathbf{j} &:= (j_{-1,-1}; j_{-1,0}; j_{-1,1}; j_{1,-1}; j_{1,0}; j_{1,1}),\end{aligned}$$

and where we use notation

$$\begin{aligned}|\mathbf{i}| &:= i_{-1,-1} + i_{-1,0} + i_{-1,1} + i_{1,-1} + i_{1,0} + i_{1,1},\\ |\mathbf{j}| &:= j_{-1,-1} + j_{-1,0} + j_{-1,1} + j_{1,-1} + j_{1,0} + j_{1,1}.\end{aligned}$$

Step 2. In this step, we will demonstrate how to bound the contribution to (6.32) coming from the deterministic constants $c_{\mathbf{j}}$, which arise from the chaotic decomposition of the determinant. Let $\{Z_{p,q} : p, q \in \{-1, 1\}\}$ be a collection of four independent standard Gaussian random variables. Using the defining formula (5.37) and the Cauchy-Schwartz inequality we observe that for every $\mathbf{j} \in \mathbb{N}^6$ s.t. $c_{\mathbf{j}} \neq 0$, we have

$$\begin{aligned}|c_{\mathbf{j}}| &= \frac{1}{2} \cdot \left(\prod_{p=\pm 1} \frac{1}{j_{p,0}!!} \right) \cdot \left(\prod_{p,q\in\{-1,1\}} \frac{1}{j_{p,q}!} \right)\\ &\quad \cdot \left| \mathbb{E}\left[\left| \det \begin{bmatrix} Z_{-1,-1} & Z_{-1,1} \\ Z_{1,-1} & Z_{1,1} \end{bmatrix} \right| \cdot \prod_{p,q\in\{-1,1\}} H_{j_{p,q}}(Z_{p,q}) \right] \right|\\ &\le \frac{1}{2} \cdot \left(\prod_{p=\pm 1} \frac{1}{j_{p,0}!!} \right) \cdot \left(\prod_{p,q\in\{-1,1\}} \frac{1}{j_{p,q}!} \right)\\ &\quad \cdot \sqrt{\mathbb{E}\left[\left(\det \begin{bmatrix} Z_{-1,-1} & Z_{-1,1} \\ Z_{1,-1} & Z_{1,1} \end{bmatrix} \right)^2 \right]} \cdot \sqrt{\mathbb{E}\left[\prod_{p,q\in\{-1,1\}} H_{j_{p,q}}(Z_{p,q})^2 \right]}.\end{aligned} \tag{6.33}$$

Since

$$\mathbb{E}\left[\left(\det \begin{bmatrix} Z_{-1,-1} & Z_{-1,1} \\ Z_{1,-1} & Z_{1,1} \end{bmatrix} \right)^2 \right] = \sum_{\varepsilon=\pm 1} \mathbb{E}\left[\prod_{p=\pm 1} Z_{p,(\varepsilon p)}^2 \right] - 2\mathbb{E}\left[\prod_{p,q=\pm 1} Z_{p,q} \right], \tag{6.34}$$

we can conclude that

$$\begin{aligned}|c_{\mathbf{j}}| &\le \frac{1}{\sqrt{2}} \cdot \left(\prod_{p=\pm 1} \frac{1}{j_{p,0}!!} \right) \cdot \left(\prod_{p,q=\pm 1} \frac{1}{\sqrt{j_{p,q}!}} \right)\\ &\le \frac{1}{\sqrt{2}} \cdot \left(\prod_{p=\pm 1} \prod_{i\in\{-1,0,1\}} \frac{1}{\sqrt{j_{p,i}!}} \right)\\ &\le \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{6^{\sum_{p=\pm 1} \sum_{i\in\{-1,0,1\}} j_{p,i}}}{\left(\sum_{p=\pm 1} \sum_{i\in\{-1,0,1\}} j_{p,i} \right)!}}.\end{aligned} \tag{6.35}$$

We have deduced the last inequality above by comparing with the probability mass function of a random vector $(L_1, \dots, L_n)$ having a multinomial distribution such that

$$\mathbb{P}(L_1 = l_1, \dots, L_n = l_n) = \frac{(l_1 + \dots + l_n)!}{l_1! \dots l_m!} l_1^{p_1} \dots l_m^{p_m}, \qquad l_1, \dots, l_m \in \mathbb{N}, \tag{6.36}$$

with $p_1 = \dots = p_n = 1/m$ and $m = 6$. We note that, in particular, if $|\mathbf{i}| = |\mathbf{j}| = 2q$ then (6.35) reduces to

$$|c_{\mathbf{i}} c_{\mathbf{j}}| \leq \frac{1}{2} \cdot \frac{6^{2q}}{(2q)!}. \tag{6.37}$$

We note that, for every $q \geq 1$ we have a trivial bound

$$|\{ \mathbf{j} \in \mathbb{N}^6 : |\mathbf{j}| = 2q\}| \leq (q+1)^6 \leq e^{6q}. \tag{6.38}$$

Step 3. Here, we will show how to bound the contribution to (6.32) coming from the expectations involving the Hermite polynomials, expressed in terms of the covariance functions r_{ij}. We consider now $i, j \in \mathbb{N}^3$, and write

$$i := (i_{-1}, i_0, i_1), \qquad j := (j_{-1}, j_0, j_1),$$

as well as

$$|i| := i_{-1} + i_0 + i_1, \qquad |j| := j_{-1} + j_0 + j_1.$$

Thanks to the classical Diagram formulae for Hermite polynomials (see Section 3.2.2) we can observe that, if $|i| \neq |j|$, then

$$\mathbb{E}\left[\prod_{v \in \{-1,0,1\}} H_{i_v}(\widetilde{\partial}_v b_k(x)) H_{j_v}(\widetilde{\partial}_v b_k(y)) \right] = 0,$$

and if $|i| = |j|$ then

$$\left| \mathbb{E}\left[\prod_{v \in \{-1,0,1\}} H_{i_v}(\widetilde{\partial}_v b_k(x)) H_{j_v}(\widetilde{\partial}_v b_k(y)) \right] \right| \leq |j|! \cdot \max_{i,j\{-1,0,1\}} |r_{ij}(k(x-y))|^{|j|}. \tag{6.39}$$

Here, $k > 0$ is any fixed positive wave-number and b_k is a corresponding real Berry's Random Wave.

Step 4. In this step, we will combine the results from the two preceding steps to obtain an estimate that behaves favorably with respect to summation over q. According to [70, p. 128, Lemma 7.6], there exists a numerical constant $C > 0$, such that for all $i, j \in \{-1, 0, 1\}$ and $k \geq 2$, we have

$$\int_{\mathcal{D}} \int_{\mathcal{D}} r_{ij}(k(x-y))^6 dx dy \leq \frac{C \cdot (1 + \operatorname{diam}(\mathcal{D})^2)}{k^2}. \tag{6.40}$$

Now, let us fix $\mathbf{i}, \mathbf{j} \in \mathbb{N}^6$ such that $c_{\mathbf{i}}, c_{\mathbf{j}} \neq 0$ and $|\mathbf{i}| = |\mathbf{j}| = 2q$. Using (6.39) and Hölder inequality we can write

$$\begin{aligned}
&\left| \sum_{(\mathcal{D}_l, \mathcal{D}_m) \text{ non-sing.}} \int_{\mathcal{D}_l \times \mathcal{D}_m} \prod_{p \in \{-1,1\}} \mathbb{E}\left[\prod_{v \in \{-1,0,1\}} H_{i_{p,v}}(\widetilde{\partial}_i b_{k_p}(x)) H_{j_{p,v}}(\widetilde{\partial}_i b_{k_p}(y)) \right] dxdy \right| \\
&\le \sum_{(\mathcal{D}_l, \mathcal{D}_m) \text{ non-sing.}} \left| \int_{\mathcal{D}_l \times \mathcal{D}_m} \prod_{p \in \{-1,1\}} \mathbb{E}\left[\prod_{v \in \{-1,0,1\}} H_{i_{p,v}}(\widetilde{\partial}_v b_{k_p}(x)) H_{j_{p,v}}(\widetilde{\partial}_v b_{k_p}(y)) \right] dxdy \right| \\
&\le \sum_{(\mathcal{D}_l, \mathcal{D}_m) \text{ non-sing.}} \int_{\mathcal{D}_l \times \mathcal{D}_m} \prod_{p \in \{-1,1\}} \left(|j_p|! \cdot \max_{i,j \in \{-1,0,1\}} |r_{ij}(k_p(x-y))|^{|j_p|} \right) dxdy \\
&\le (2q)! \cdot \int_{(\mathcal{D}_l, \mathcal{D}_m) \text{ non-sing.}} \prod_{p \in \{-1,1\}} \max_{i,j \in \{-1,0,1\}} |r_{ij}(k_p(x-y))|^{|j_p|} dxdy \\
&\le (2q)! \cdot \prod_{p \in \{-1,1\}} \left(\int_{(\mathcal{D}_l, \mathcal{D}_m) \text{ non-sing.}} \max_{i,j \in \{-1,0,1\}} |r_{ij}(k_p(x-y))|^{2q} dxdy \right)^{\frac{|j_p|}{2q}} \\
&\le 9 \cdot (2q)! \cdot \prod_{p \in \{-1,1\}} \max_{i,j \in \{-1,0,1\}} \left(\int_{(\mathcal{D}_l, \mathcal{D}_m) \text{ non-sing.}} |r_{ij}(k_p(x-y))|^{2q} dxdy \right)^{\frac{|j_p|}{2q}} .
\end{aligned} \tag{6.41}$$

Thanks to (6.40) we can find a numerical constant $C > 0$ such that the last line of (6.41) can be upper-bounded by

$$\begin{aligned}
&\le 9 \cdot \frac{(2q)!}{(1000)^{2q-6}} \cdot \prod_{p \in \{-1,1\}} \max_{i,j \in \{-1,0,1\}} \left(\int_{(\mathcal{D}_l, \mathcal{D}_m) \text{ non-sing.}} |r_{ij}(k_p(x-y))|^{6} dxdy \right)^{\frac{|j_p|}{2q}} \\
&\le C \cdot \left(1 + \operatorname{diam}(\mathcal{D})^2\right) \cdot \frac{(2q)!}{(1000)^{2q-6}} \cdot \left(\prod_{p \in \{-1,1\}} k_p^{-\frac{|j_p|}{2q}} \right) \\
&\le \frac{(2q)!}{(1000)^{2q-6}} \cdot \frac{C \cdot (1 + \operatorname{diam}(\mathcal{D})^2)}{\min(k_{-1}, k_1)^2}.
\end{aligned} \tag{6.42}$$

Step 5. We apply inequalities (6.37), (6.38) and (6.42) to (6.32) and conclude that for another numerical constant $L > 0$ we have

$$\begin{aligned}&\sum_{q=3}^{\infty}\left|\sum_{(\mathcal{D}_l,\mathcal{D}_m)\text{ non-sing.}}\operatorname{Cov}\left(\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_l)[2q],\mathcal{N}(b_{k_{-1}},b_{k_1},\mathcal{D}_m)[2q]\right)\right|\\&\le L\cdot(1+\operatorname{diam}(\mathcal{D})^2)\cdot\max(k_{-1},k_1)^2\cdot\sum_{q=3}^{+\infty}\left(\frac{6e^3}{1000}\right)^{2q}.\end{aligned}\tag{6.43}$$

Since the series on the right of the above inequality is convergent we conclude that the proof of our lemma is complete. □

We are finally ready to achieve the main goal of this section.

Proof of Lemma 6.1. We choose any decomposition $\{(\mathcal{D}_l,\mathcal{D}_m)\}_{l,m}$, $1 \le l,m \le \lceil k\rceil^2$ of $\mathcal{D}\times\mathcal{D}$ into singular and non-singular pairs of cells $(\mathcal{D}_l,\mathcal{D}_m)$, as described in Definition 6.1. Then, thanks to Lemmas 6.3 and 6.4, we can find a numerical constants $C_1, C_2 > 0$ such that

$$\begin{aligned}&\sum_{q=3}^{\infty}\operatorname{Var}\left(\mathcal{N}(b_k,\hat{b}_K,\mathcal{D})[2q]\right)\\&\le\sum_{q=3}^{\infty}\left|\sum_{(\mathcal{D}_l,\mathcal{D}_n)\text{ singular}}\operatorname{Cov}\left(\mathcal{N}(b_k,\hat{b}_K,\mathcal{D}_l)[2q],\mathcal{N}(b_k,\hat{b}_K,\mathcal{D}_n)[2q]\right)\right|\\&\quad+\sum_{q=3}^{\infty}\left|\sum_{(\mathcal{D}_l,\mathcal{D}_n)\text{ non singular}}\operatorname{Cov}\left(\mathcal{N}(b_k,\hat{b}_K,\mathcal{D}_l)[2q],\mathcal{N}(b_k,\hat{b}_K,\mathcal{D}_n)[2q]\right)\right|\\&\le C_1(1+\operatorname{diam}(\mathcal{D})^2)\cdot K^2+C_2(1+\operatorname{diam}(\mathcal{D})^4)\cdot K^2\\&\le C_3(1+\operatorname{diam}(\mathcal{D})^4)\cdot K^2,\end{aligned}\tag{6.44}$$

with C_3 being another numerical constant. We combine the above bound with Lemma 5.5 and we obtain

$$\begin{aligned}\sum_{q\neq 2}\operatorname{Var}\left(\mathcal{N}(b_k,\hat{b}_K,\mathcal{D})[2q]\right)&=\operatorname{Var}\left(\mathcal{N}(b_k,\hat{b}_K,\mathcal{D})[2]\right)+\sum_{q=3}^{\infty}\operatorname{Var}\left(\mathcal{N}(b_k,\hat{b}_{K_n},\mathcal{D})[2q]\right)\\&\le\frac{\operatorname{diam}(\mathcal{D})^2}{64\pi^2}\cdot K^2+\sum_{q=3}^{\infty}\operatorname{Var}\left(\mathcal{N}(b_k,\hat{b}_K,\mathcal{D})[2q]\right)\\&\le\frac{\operatorname{diam}(\mathcal{D})^2}{64\pi^2}\cdot K^2+C_3(1+\operatorname{diam}(\mathcal{D})^4)\cdot K^2\\&\le C_4\cdot(1+\operatorname{diam}(\mathcal{D})^4)\cdot K^2,\end{aligned}\tag{6.45}$$

where C_4 is the final numerical constant whose existence was postulated in the statement of our lemma. □

6.2 Computation of the Asymptotic Variance Formula

This section is devoted to the proof of the variance formula stated in Theorem 4.1 (the expectation has already been computed in Lemma 5.4). First, we summarise the information acquired in the preceding sections. Suppose that the assumptions of Theorem 4.1 are fulfilled and admit the notation used therein. Combining Remark 5.2 with Lemma 6.1 we can already see that

$$\begin{aligned}\lim_{n\to\infty}\frac{\mathrm{Var}\left(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]\right)}{\mathrm{Var}\left(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})\right)}&=1,\\ \lim_{n\to\infty}\mathrm{Corr}\left(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D}),\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]\right)&=1.\end{aligned}\tag{6.46}$$

Using Lemma 4.1 and Remark 6.1, we immediately see that

$$\begin{aligned}&\mathrm{Var}\left(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]\right)\\ &=\mathrm{Var}\left(\frac{k_n}{\pi\sqrt{2}}\cdot\mathcal{L}(b_{k_n},\mathcal{D})[4]+\frac{K_n}{\pi\sqrt{2}}\cdot\mathcal{L}(\hat{b}_{K_n},\mathcal{D})+\mathrm{Cross}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])\right)\\ &=\frac{K_n^2}{2\pi^2}\cdot\mathrm{Var}(\mathcal{L}(b_{k_n},\mathcal{D})[4])+\frac{k_n^2}{2\pi^2}\cdot\mathrm{Var}(\mathcal{L}(\hat{b}_{K_n},\mathcal{D})[4])+\mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]))\\ &=\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot(K_n^2\ln k_n+k_n^2\ln K_n)+\mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])),\end{aligned}\tag{6.47}$$

where we have used Theorem 5.1 established by Nourdin, Peccati and Rossi in [70]. Combining (6.46) and (6.47) we see that there is only one more step needed in order to achieve the goal of this section. That is, we need to characterise the asymptotic contribution to the variance which comes from the cross-term. Our strategy will vary according to the value of the asymptotic ratio r which was defined in (4.2). The case $r = 0$ will be treated in Lemma 6.5 and we will show that in this scenario the contribution of the cross-term is negligible. The case $r > 0$ is the subject of Lemma 6.6 and, there, the cross-term will bring a meaningful contribution. In the next lemma we use notation introduced in (4.1) and in (4.2).

Lemma 6.5. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let $(k_n, K_n)_{n\in\mathbb{N}}$ be a sequence of pairs of wave-numbers such that $2 \le k_n \le K_n < \infty$ and $k_n \to \infty$. Suppose also that the asymptotic parameters r^{log} and r defined in (4.2) exist and satisfy $r = 0$ and $r^{log} > 0$. Then,*

$$\lim_{n\to\infty}\frac{\mathrm{Var}\left(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})\right)}{\frac{area(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}=1.\tag{6.48}$$

Moreover, we have

$$\lim_{n\to\infty} \left\| \frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}) - \mathbb{E}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}{\sqrt{\operatorname{Var}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}))}} - \frac{\mathcal{L}(b_{k_n}, \mathcal{D}) - \mathbb{E}\mathcal{L}(b_{k_n}, \mathcal{D})}{\sqrt{\operatorname{Var}(\mathcal{L}(b_{k_n}, \mathcal{D}))}} \right\|_{L^2(\mathbb{P})} = 0, \tag{6.49}$$

as well as,

$$\lim_{n\to\infty} Corr\Big(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}), \mathcal{L}(b_{k_n}, \mathcal{D})\Big) = 1. \tag{6.50}$$

Here, b_{k_n} and $\hat{b}_{K_n}$ denote independent real Berry's Random Waves with wave-numbers k_n and K_n respectively, while $\mathcal{L}(b_{k_n}, \mathcal{D})$ and $\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})$ are the associated nodal length and nodal number, respectively.

The proof of the above lemma will be given in Subsection 6.2.3. Now we want to highlight its possible heuristic interpretation.

Remark 6.1. *Suppose that $K_n \to \infty$ much faster than $k_n \to \infty$, in a sense that $r = 0$. Thanks to a stationarity argument, we can always use a construction driven by a single pair of Berry's Random Waves with wave-numbers $k = K = 1$, that is we can set $b_{k_n}(\cdot) := b_1(k_n\cdot)$ and $\hat{b}_{K_n}(\cdot) := \hat{b}_1(K_n\cdot)$. Then, conditionally on the randomness of the field b_1, the nodal lines of b_{k_n} can be seen as essentially constant - relatively to the nodal lines associated with $\hat{b}_{K_n}$, which would cover $\mathcal{D}$ uniformly. The uniform covering is a consequence of the fact, shown in [67, p. 97, Proposition 1.3], that after an apropriate deterministic rescaling, the nodal length $\mathcal{L}(\hat{b}_{K_n}, \mathcal{D})$ converges, as a random distribution , to the White Noise. Thus, heuristically speaking, the number of nodal intersections $\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})$ should be asymptotically proportional to the nodal length $\mathcal{L}(b_{k_n}, \mathcal{D})$. The conclusion of Lemma 6.5 is consistent with this intuition.*

As before, in the next lemma we will use notation introduced in (4.1) and in (4.2).

Lemma 6.6. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let $(k_n, K_n)_{n\in\mathbb{N}}$ be a sequence of pairs of wave-numbers such that $2 \le k_n \le K_n < \infty$ and $k_n \to \infty$. Suppose also that the limits r^{log}, r, r^{exp} defined in (4.2) exist and $r > 0$. Then*

$$\lim_{n\to\infty} \frac{\operatorname{Var}\Big(Cross(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4])\Big)}{\frac{area(\mathcal{D})}{512\pi^3} \cdot r \cdot (36 + 50 r^{exp}) \cdot K_n^2 \ln K_n} = 1, \tag{6.51}$$

where b_{k_n} and $\hat{b}_{K_n}$ are independent real Berry's Random Waves with wave-numbers k_n and K_n, respectively. Here, $\operatorname{Cross}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4])$ denotes the 4-th cross-term of the corresponding nodal number (see Definition 4.2).

The proof of the above lemma will be given in Subsection 6.2.4. Thanks to Lemmas 6.5 and 6.6 we are now in position to easily achieve the goal of this section.

Proof of Theorem 4.1. The expectation was computed in Lemma 5.4 and we are left with a task of finding the formula for asymptotic variance. The case $r = 0$ is entirely and directly covered by Lemma 6.5 and consequently we can assume from now on that $r > 0$. We recall that, as noted in Subsection 4.1, if $r > 0$ then $r^{log} = 1$ and that if $r < 1$ then $r^{exp} = 0$. We proceed as in (6.47) and observe that

$$\begin{aligned}
&\mathrm{Var}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})) \\
&\sim \mathrm{Var}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]) \\
&\sim \frac{\mathrm{area}(\mathcal{D})}{512\pi^3} \cdot (K_n^2 \ln k_n + k_n^2 \ln K_n) + \mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4])) \\
&\sim \frac{\mathrm{area}(\mathcal{D})}{512\pi^3} \cdot (1 + r^2) \cdot K_n^2 \ln K_n + \mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4])) \\
&\sim \frac{\mathrm{area}(\mathcal{D})}{512\pi^3} \cdot (1 + r^2) \cdot K_n^2 \ln K_n + \frac{\mathrm{area}(\mathcal{D})}{512\pi^3} \cdot r(36 + 50 r^{exp}) \cdot K_n^2 \ln K_n \cdot \\
&= \frac{\mathrm{area}(\mathcal{D})}{512\pi^3} \cdot (1 + 36r + r^2 + 50 r^{exp}) \cdot K_n^2 \ln K_n,
\end{aligned} \tag{6.52}$$

where to obtain the penultimate expression we have used Lemma 6.6. □

6.2.1 Wiener Chaos Decomposition of the Cross-Term

We begin preparing for the proofs of Lemmas 6.5 and 6.6 by deriving an explicit expression for the fourth cross-term of the nodal number. Our analysis focuses solely on the fourth chaos because, as demonstrated in the previous section, the fourth chaotic projection is asymptotically dominant. We further narrow our attention to $\mathrm{Cross}(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[4])$, since the other two terms in decomposition (4.1) are merely rescaled versions of nodal lengths. Their contributions can thus be directly controlled using the results from [70], which we summarized in Section 5.7.

We recall that $\mathrm{Cross}(\mathcal{N}(b_k, \hat{b}_K, \mathcal{D})[2q])$, $q \in \mathbb{N}$, was described in Definition 4.2. It will be convenient to formulate this result using the notation introduced in **S.1-S.6**. In particular, we will replace an ordered pair of wave-numbers (k, K), $k \leq K$, with the corresponding unordered pair k_{-1}, k_1, where $k \equiv \min_{p \in \{-1,1\}} k_p$ and $K \equiv \max_{p \in \{-1,1\}} k_p$.

Lemma 6.7. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let $b_{k_{-1}}$ and b_{k_1} be independent real Berry's Random Waves with strictly positive wave-numbers k_{-1} and k_1, respectively. Then, the corresponding cross-term (defined in (4.29)), is given by the formula*

$$\mathrm{Cross}(\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})[4]) = \frac{(k_{-1} \cdot k_1)^2}{128\pi} \cdot \sum_{\mathbf{j} \in \{-1,0,1\}^{\otimes 2} \cup \{*\}} \eta_{\mathbf{j}} \cdot Y_{\mathbf{j}}, \tag{6.53}$$

where

$$Y_{\mathbf{j}} := \int_{\mathcal{D}} \prod_{p\in\{-1,1\}} H_2(\widetilde{\partial}_{j_p} b_{k_p}(x))dx, \qquad \mathbf{j} \neq *$$
$$Y_* := \int_{\mathcal{D}} \prod_{p,q\in\{-1,1\}} \widetilde{\partial}_p b_{k_q}(x)dx. \tag{6.54}$$

Here, for $\mathbf{j} \neq *$ *we write* $\mathbf{j} = (j_{-1}, j_1)$ *and the constants* $\eta_{\mathbf{j}}$ *are given by the table*

condition on $\mathbf{j}$	$j_{-1} = j_1 = 0$	$\lvert j_{-1}\rvert + \lvert j_1\rvert = 1$	$j_{-1} \cdot j_1 = -1$	$j_{-1} \cdot j_1 = +1$	$\mathbf{j} = *$
value of $\eta_{\mathbf{j}}$	8	-4	5	-1	-12

(6.55)

Proof. It is enough to note that the constants $c_{\mathbf{j}}$ in the chaotic decomposition of the nodal number $\mathcal{N}(b_{k_{-1}}, b_{k_1}, \mathcal{D})$ (as established in Lemma 5.2) do not depend on the wave-numbers k_n, K_n. Thus, we can reuse the values established in [70, p. 116, lines -6, -5] in the context of the one-energy ($k_n \equiv K_n$) complex Berry's Random Wave model. □

Remark 6.2. *We note that the results in [70], which we have used in the proof of the above lemma are based directly on [55, p. 939, Lemma 3.4] and on [20, p. 17, Lemmas 3.2-3.3]. We want to also highlight an interesting alternative route to the proof of Lemma 6.7. Namely, we could have used the elegant formulas established in [65]. More precisely speaking: [64, p. 77, Proposition II.B.5 (i), (iv), (v)], [64, p. 69, Proposition II.B.5 (i), (iv), (v)] and [64, p. 70, Proposition II.B.5 (i), (iv), (v)], to be read with notation introduced in [64, p. 72, Proposition II.B.5 (i), (iv), (v)] and in [64, p. 39, Proposition II.B.5 (i), (iv), (v)].*

6.2.2 Asymptotic Integrals of the Covariance Functions

The next lemma is a straightforward generalization of the following crucial results proved in [70, p. 119, Proposition 5.1] and [70, p. 122, Proposition 5.2]. The concise form in which we state it, is based on the notation introduced in **S.1**-**S.6**. We note that the approximation to be given in (6.58) is meaningful thanks to the fact, to be shown in Lemma 6.9, that (at least in the cases of interest) the term $C_3^n(\mathbf{q})$ has an asymptotic order $\ln K_n$.

Lemma 6.8. *Let* $\mathcal{D}$ *be a convex and compact domain of the plane, with non-empty interior and piecewise* C^1 *boundary* $\partial\mathcal{D}$. *Let* $\{k_{-1}^n, k_1^n\}_{n\in\mathbb{N}}$ *be a sequence of pairs of wave-numbers such that* $2 \leq k_{-1}^n, k_1^n < \infty$ *and* $k_{-1}^n, k_1^n \to \infty$. *Let* $b_{k_{-1}^n}$, $b_{k_1^n}$ *denote independent real Berry's Random Waves with wave-numbers* k_{-1}^n *and* k_1^n, *respectively. For each* $i, j \in \{-1, 0, 1\}$, *let* r_{ij} *denote the covariance function defined in (5.14). For each* $p \in \{-1, 1\}$, *let* $q(p) \in \mathbb{N}^9$ *be a vector of non-negative* *integers indexed as*

$$q(p) := (q(p)_{-1,-1}; q(p)_{-1,0}; q(p)_{-1,1}; q(p)_{0,-1}; q(p)_{0,0}; q(p)_{0,1}; q(p)_{1,-1}; q(p)_{1,0}; q(p)_{1,1}), \tag{6.56}$$

and such that $|q(p)| = 2$, *where*

$$|q(p)| := \sum_{i,j\in\{-1,0,1\}} q(p)_{i,j}, \tag{6.57}$$

and set $\mathbf{q} := (q(-1), q(1)) \in \mathbb{N}^{18}$. *Then, it holds that*

$$\begin{aligned}
&\int_{\mathcal{D}\times\mathcal{D}} \prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} r_{ij}(k_p^n(x-y))^{q(p)_{ij}} dxdy = \\
&area(\mathcal{D}) \int_0^{diam(\mathcal{D})} \left(\int_0^{2\pi} \prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} r_{ij}((k_p^n \phi)\cdot(\cos\theta, \sin\theta))^{q(p)_{ij}} d\theta \right) \cdot \phi d\phi \\
&\qquad + O\left(\frac{1 + diam(\mathcal{D})^2}{k_{-1}^n \cdot k_1^n} \right) = \\
&\frac{4}{\pi^2} \cdot \frac{area(\mathcal{D})}{k_{-1}^n \cdot k_1^n} \cdot C_1(\mathbf{q}) \cdot C_2(\mathbf{q}) \cdot C_3^n(\mathbf{q}) + O\left(\frac{1 + diam(\mathcal{D})^2}{k_{-1}^n \cdot k_1^n} \right).
\end{aligned} \tag{6.58}$$

Here, we have

$$\begin{aligned}
C_1(\mathbf{q}) &:= \prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} v_{ij}^{q(p)_{ij}}, \\
C_2(\mathbf{q}) &:= \int_0^{2\pi} \prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} h_{ij}(\theta)^{q(p)_{ij}} d\theta, \\
C_3^n(\mathbf{q}) &:= \int_1^{\max(k_{-1}^n, k_1^n)} \prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} g_{ij}\left(\frac{k_p^n \phi}{\max(k_{-1}^n, k_1^n)} \right)^{q(p)_{ij}} \frac{d\phi}{\phi},
\end{aligned} \tag{6.59}$$

where for each $i, j \in \{-1, 0, 1\}$ *we define*

$$\begin{aligned}
v_{ij} &:= (\sqrt{2})^{|i|+|j|}, \\
h_{ij}(\theta) &:= \cos^{\delta_{-1}(i)+\delta_{-1}(j)}(\theta) \cdot \sin^{\delta_1(i)+\delta_1(j)}(\theta), \\
g_{ij}(\phi) &:= \cos^{(1-\delta_1(|i|+|j|))}(2\pi\phi - \pi/4) \cdot \sin^{\delta_1(|i|+|j|)}(2\pi\phi - \pi/4).
\end{aligned} \tag{6.60}$$

For the sake of clarity, let us note that the constants involved in the 'O' notation in (6.58) are independent of the choice of the domain $\mathcal{D}$, and of the choice of the sequence of pairs of wave-numbers $\{k_{-1}^n, k_1^n\}_{n\in\mathbb{N}}$.

Proof. It is sufficient to provide minor modifications to the proofs of [70, p. 119, Proposition 5.1] and of [70, p. 122, Proposition 5.2]. These propositions concern the one-energy ($k_{-1}^n \equiv k_1^n$) scenario and correspond respectively to the first and the second equality postulated in (6.58).

The proof of the first proposition in question is based on the co-area formula, on the Steiner formula for convex sets and on the uniform bound on the first-kind Bessel functions J_0, J_1, J_2 (see (3.2)). Only the application of this last element needs to be adapted, and it is through inequalities of the form

$$\max_{i,j\in\{-1,0,1\}} |r_{ij}(k_p^n z)| \leq \frac{C}{\sqrt{k_p^n |z|}}, \qquad z \in \mathbb{R}^2 \setminus \{0\}, \tag{6.61}$$

where C is a numerical constant. To arrive at the desired conclusion it is enough to count multiplicity with which each of the wave-numbers k_{-1}^n and k_1^n will appear in relevant expressions.

In order to provide the postulated extension of the second of aforementioned propositions we rewrite the approximations formulas for the covariance functions given in [70, p. 121-122, Eq. (5.69), (5.70)] using our notation **S.1**-**S.6**. That is, we note that the covariance functions r_{ij}, $i,j \in \{-1,0,1\}^2$, defined in (5.14) can be approximated using (6.60) as:

$$r_{ij}(\phi(\cos\theta, \sin\theta)) = \sqrt{\frac{2}{\pi}} \cdot v_{ij} \cdot h_{ij}(\theta) \cdot g_{ij}(\phi) + O(\phi^{-\frac{3}{2}}), \qquad \phi > 0, \qquad \theta \in [0, 2\pi), \tag{6.62}$$

where the numerical constant involved in the 'O'-notation is independent of φ and θ.

Finally, we note that, even thought the error rate was not provided in the beforementioned propositions, it can be deduced immediately by careful analysis of the original proofs. □

6.2.3 Case $r = 0$: Full Correlation with the Nodal Length

Proof of Lemma 6.5. The essence of the argument that follows is the observation that, when the growth of wavenumbers is unbalanced (in the sense that $r = 0$), relatively crude estimates are sufficient to demonstrate full correlation of the nodal number with the nodal length of the slower evolving wave (i.e., the wave with smaller energy). This also relies on the dominance of the 4th chaotic projection, as established in the preceding section, as well as on the Recurrence Representation from Lemma 2.5. We divide the proof into two parts.

Step 1. We start by verifying that the asymptotic variance formula (6.48) holds. Using Lemma 6.1 we deduce that

$$\begin{aligned}
&\frac{\mathrm{Var}\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})}{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}\\
&=\frac{\sum_{q\neq 2}\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[2q])}{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}+\frac{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])}{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}\\
&=O\left(\frac{1+\mathrm{diam}(\mathcal{D})^2}{\mathrm{area}(\mathcal{D})}\cdot\frac{1}{r^{log}}\cdot\frac{1}{\ln K_n}\right)+\frac{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])}{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}.
\end{aligned}\tag{6.63}$$

We recall from Lemma 4.1 that

$$\begin{aligned}
\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])=\frac{K_n^2}{2\pi^2}\cdot\mathrm{Var}(\mathcal{L}(b_{k_n},\mathcal{D})[4])+\frac{k_n^2}{2\pi^2}\cdot\mathrm{Var}(\mathcal{L}(\hat{b}_{K_n},\mathcal{D})[4])\\
+\mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])).
\end{aligned}\tag{6.64}$$

Using Theorem 5.1 we have that

$$\begin{aligned}
\frac{\frac{K_n^2}{2\pi^2}\cdot\mathrm{Var}(\mathcal{L}(b_{k_n},\mathcal{D})[4])}{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}&=\frac{256\pi\cdot\mathrm{Var}(\mathcal{L}(b_{k_n},\mathcal{D})[4])}{\mathrm{area}(\mathcal{D})\cdot r^{log}\cdot\ln K_n}\sim\frac{1}{r^{log}}\cdot\frac{\ln k_n}{\ln K_n}\to 1,\\
\frac{\frac{k_n^2}{2\pi^2}\cdot\mathrm{Var}(\mathcal{L}(\hat{b}_{K_n},\mathcal{D})[4])}{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}&=\frac{256\pi\cdot\mathrm{Var}(\mathcal{L}(\hat{b}_{K_n},\mathcal{D})[4])}{\mathrm{area}(\mathcal{D})\cdot r^{log}\cdot\ln K_n}\cdot(\frac{k_n}{K_n})^2\sim\frac{1}{r^{log}}\cdot(\frac{k_n}{K_n})^2\to 0.
\end{aligned}\tag{6.65}$$

Thus, combining (6.64) with (6.65) we can see that, in order to establish (6.48), the only fact we still need to show is the convergence

$$\frac{\mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])}{K_n^2\ln K_n}\to 0.$$

We recall Lemma 6.7, and we write $Y_{\mathbf{j}}^n$ to denote random integrals appearing in this lemma, where we take $k_n =: \min_{p\in\{-1,1\}} k_p$ and $K_n =: \max_{p\in\{-1,1\}} k_p$. We conclude that, for some numerical constant $C_1>0$, we have

$$\begin{aligned}
\mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]))&=\mathrm{Var}\left(\frac{k_n\cdot K_n}{128\pi}\cdot\sum_{\mathbf{j}\in\{-1,0,1\}^{\otimes 2}\cup\{*\}}\eta_{\mathbf{j}}Y_{\mathbf{j}}^n\right)\\
&\leq C_1\cdot(k_n\cdot K_n)^2\cdot\max_{\mathbf{j}\in\{-1,0,1\}^{\otimes 2}\cup\{*\}}\mathrm{Var}(Y_{\mathbf{j}}^n).
\end{aligned}\tag{6.66}$$

Let us now write $\mathbf{p},\mathbf{q}\in\mathbb{N}^9$ with indexation

$$\begin{aligned}
\mathbf{p}&=(p_{-1,-1};p_{-1,0};p_{-1,1};p_{0,-1};p_{0,0};p_{0,1};p_{1,-1};p_{1,0};p_{1,1}),\\
\mathbf{q}&=(q_{-1,-1};q_{-1,0};q_{-1,1};q_{0,-1};q_{0,0};q_{0,1};q_{1,-1};q_{1,0};q_{1,1}),
\end{aligned}\tag{6.67}$$

and also

$$|\mathbf{p}|:=\sum_{i,j\in\{-1,0,1\}}p_{i,j},\qquad |\mathbf{q}|:=\sum_{i,j\in\{-1,0,1\}}q_{i,j}.\tag{6.68}$$

We can find a numerical constant $C_2 > 0$ such that, for every choice of $\mathbf{p}, \mathbf{q} \in \mathbb{N}^9$ with $|\mathbf{p}| = |\mathbf{q}| = 2$, we have

$$\begin{aligned}
&\int_{\mathcal{D}\times\mathcal{D}} \prod_{i,j\in\{-1,0,1\}} |r_{ij}(k_n(x-y))|^{q_{ij}} |r_{ij}(K_n(x-y))|^{p_{ij}} dxdy \\
&\le \sqrt{\int_{\mathcal{D}\times\mathcal{D}} \prod_{i,j\in\{-1,0,1\}} |r_{ij}(k_n(x-y))|^{2\cdot q_{ij}} dxdy} \cdot \sqrt{\int_{\mathcal{D}\times\mathcal{D}} \prod_{i,j\in\{-1,0,1\}} |r_{ij}(K_n(x-y))|^{2\cdot p_{ij}} dxdy} \\
&\le C_2 \cdot (1+\mathrm{diam}(\mathcal{D})^2) \cdot (k_n \cdot K_n)^2 \cdot \sqrt{\frac{\ln k_n}{k_n^2}} \cdot \sqrt{\frac{\ln K_n}{K_n^2}} \\
&= C_2 \cdot (1+\mathrm{diam}(\mathcal{D})^2) \cdot (k_n \cdot K_n) \cdot \sqrt{\ln k_n \cdot \ln K_n}.
\end{aligned} \tag{6.69}$$

Here, in order to obtain last inequality in (6.69), we have used (6.58). Combining (6.66) with (6.69) yields that, for some numerical constant $C_3 > 0$, we have

$$\frac{\mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]))}{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3} \cdot r^{log} \cdot K_n^2 \ln K_n} \le \frac{C_3}{r^{log}} \cdot \frac{k_n}{K_n} \cdot \sqrt{\frac{\ln k_n}{\ln K_n}} \to 0. \tag{6.70}$$

Step 2. In this second and final step we will prove L^2 equivalence (6.49) and full-correlation (6.50). Using the triangle inequality we can easily see that

$$\sqrt{\mathbb{E}\left(\frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}) - \mathbb{E}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}} - \frac{\mathcal{L}(b_{k_n}, \mathcal{D}) - \mathbb{E}\mathcal{L}(b_{k_n}, \mathcal{D})}{\sqrt{\mathrm{Var}\mathcal{L}(b_{k_n}, \mathcal{D})}}\right)^2} \le a_n + b_n + c_n, \tag{6.71}$$

where

$$\begin{aligned}
a_n &:= \sqrt{\mathbb{E}\left(\frac{\sum_{q\neq 0,2} \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[2q]}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}}\right)^2}, \\
b_n &:= \sqrt{\mathbb{E}\left(\frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})}} - \frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]}{\sqrt{\mathrm{Var}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4])}}\right)^2}, \\
c_n &:= \sqrt{\mathbb{E}\left(\frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4]}} - \frac{\mathcal{L}(b_{k_n}, \mathcal{D}) - \mathbb{E}\mathcal{L}(b_{k_n}, \mathcal{D})}{\sqrt{\mathrm{Var}\mathcal{L}(b_{k_n}, \mathcal{D})}}\right)^2}.
\end{aligned} \tag{6.72}$$

We will bound each of these terms separately. We start with the following estimate:

$$\begin{aligned} a_n &= \sqrt{\mathbb{E}\left(\frac{\sum_{q\neq 0,2}\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[2q]}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})}}\right)^2} \\ &= \sqrt{\frac{\sum_{q\neq 2}\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[2q])}{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D}))}} \\ &= \sqrt{\frac{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D}))}}\cdot\sqrt{\frac{\sum_{q\neq 2}\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[2q])}{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}} \\ &\leq L_1\cdot\sqrt{\frac{\frac{\mathrm{area}(\mathcal{D})}{512\pi^3}\cdot r^{log}\cdot K_n^2\ln K_n}{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D}))}\cdot\frac{1+\mathrm{diam}(\mathcal{D})^2}{r^{log}\cdot\ln K_n}}\longrightarrow 1, \end{aligned} \tag{6.73}$$

where L_1 is a numerical constants which exists thanks to Lemma 6.1 and the convergence follows by the first (already proved) part of this lemma - formula (6.48). Furthermore, we have

$$\begin{aligned} b_n &= \sqrt{\mathbb{E}\left(\frac{\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})}}-\frac{\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]}{\sqrt{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])}}\right)^2} \\ &= \left|1-\frac{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])}{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D}))}\right|\longrightarrow 0, \end{aligned} \tag{6.74}$$

where we have used (6.48) and (6.63). Finally, using the recurrence representation from Lemma 4.1, we observe that

$$\begin{aligned} c_n &= \sqrt{\mathbb{E}\left(\frac{\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]}{\sqrt{\mathrm{Var}\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]}}-\frac{\mathcal{L}(b_{k_n},\mathcal{D})-\mathbb{E}\mathcal{L}(b_{k_n},\mathcal{D})}{\sqrt{\mathrm{Var}(\mathcal{L}(b_{k_n},\mathcal{D}))}}\right)^2} \\ &\leq \frac{k_n}{\pi\sqrt{2}}\cdot\sqrt{\frac{\mathrm{Var}(\mathcal{L}(\hat{b}_{K_n},\mathcal{D})[4]))}{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])}}+\sqrt{\frac{\mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4]))}{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])}} \\ &\quad+\frac{K_n}{\pi\sqrt{2}}\cdot\sqrt{\mathrm{Var}(\mathcal{L}(b_{k_n},\mathcal{D})[4]))} \\ &\quad\cdot\left|\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])^{-1/2}-\mathrm{Var}(\mathcal{L}(b_{k_n},\mathcal{D})[4])^{-1/2}\right| \\ &= o(1)+\left(1-\sqrt{\frac{\frac{K_n}{\pi\sqrt{2}}\cdot\mathrm{Var}(\mathcal{L}(b_{k_n},\mathcal{D})[4]))}{\mathrm{Var}(\mathcal{N}(b_{k_n},\hat{b}_{K_n},\mathcal{D})[4])}}\right)\longrightarrow 0, \end{aligned} \tag{6.75}$$

where in the last line we have used (6.65) and (6.70). This completes the proof of L^2 equivalence (6.49) and the full-correlation (6.50) follows immediately. □

6.2.4 Case $r > 0$: Asymptotic Variance of the Cross-Term

In the next lemma, we use the notation introduced in formulas (4.1) and (4.2), and in **S.1-S.6**. This lemma allows one to evaluate (6.58) and, hence, plays a crucial role in the computation of the constant term (4.5) which is contributing to the asymptotic variance formula (4.4) (of Theorem 4.1).

Lemma 6.9. *Let $\mathcal{D}$ be a convex and compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let $\{k_{-1}^n, k_1^n\}_{n\in\mathbb{N}}$ be a sequence of pairs of wave-numbers such that $2 \le k_{-1}^n, k_1^n < \infty$ and $k_{-1}^n, k_1^n \to \infty$. Let $b_{k_{-1}^n}$, $b_{k_1^n}$ denote independent real Berry's Random Waves with wave-numbers k_{-1}^n and k_1^n, respectively. For each $p \in \{-1, 1\}$, let $q(p) \in \mathbb{N}^9$ be a vector of non-negative integers indexed as*

$$q(p) := (q(p)_{-1,-1}; q(p)_{-1,0}; q(p)_{-1,1}; q(p)_{0,-1}; q(p)_{0,0}; q(p)_{0,1}; q(p)_{1,-1}; q(p)_{1,0}; q(p)_{1,1}), \tag{6.76}$$

and such that $|q(p)| = 2$, where

$$|q(p)| := \sum_{i,j\in\{-1,0,1\}} q(p)_{i,j}, \tag{6.77}$$

and set $\mathbf{q} := (q(-1), q(1)) \in \mathbb{N}^{18}$. Then, provided that the limits r and r^{exp} defined in (4.2) exist and that $r > 0$, we have

$$\begin{aligned} C_3^n(\mathbf{q}) &:= \int_1^{\max(k_{-1}^n,k_1^n)} \prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} g_{ij}\left(\frac{k_p^n\phi}{\max(k_{-1}^n,k_1^n)}\right)^{q(p)_{ij}} \frac{d\phi}{\phi} \\ &\sim \frac{1}{4}\cdot(1 + r^{exp}\cdot\kappa_{\mathbf{q}})\cdot\ln(\max(k_{-1}^n,k_1^n)), \end{aligned} \tag{6.78}$$

where

$$\kappa_{\mathbf{q}} := \frac{1}{2}\cdot \prod_{p\in\{-1,1\}} (-1)^{\sum_{i,j\in\{-1,0,1\}} \frac{q(p)_{ij}}{2}(|i|+|j|)} \tag{6.79}$$

Here, the functions g_{ij} are as defined in (6.60) and the asymptotic is valid as $n \to \infty$.

Proof. This proof consists of two key components. The first involves the application of standard trigonometric expansions, which, combined with integration by parts, readily reveals the asymptotic order $\ln(\max(k_{-1}^n, k_1^n))$. The second component is the calculation of the constant $\frac{1}{4}(1 + r^{exp}\cdot\kappa_{\mathbf{q}})$, which represents a more intricate step, particularly in the two-energy setting. This part relies on the choice of notation, which helps to efficiently manage and track numerous expressions. We will use the equivalence of notations introduced in **S.1-S.6**:

$$k_n \equiv \min(k_{-1}^n, k_1^n), \qquad K_n \equiv \max(k_{-1}^n, k_1^n), \tag{6.80}$$

and we additionally set

$$r_n := \frac{k_n}{K_n}. \tag{6.81}$$

We split the argument into four parts.

Step 1. In this step, we establish all relevant exact forms that $C_3^n(\mathbf{q})$ can take before proceeding to the asymptotic analysis. Let $\mathbf{q}$ be as defined via (6.76)-(6.77). We define recursively

$$\begin{aligned} \alpha_{q(p)_{ij}} &:= q(p)_{ij}(1-\delta_1(|i|+|j|)), & \beta_{q(p)_{ij}} &:= q(p)_{ij}\delta_1(|i|+|j|), \\ \alpha_{q(p)} &:= \sum_{i,j\in\{-1,1\}} \alpha_{q(p)_{ij}}, & \beta_{q(p)} &:= \sum_{i,j\in\{-1,1\}} \beta_{q(p)_{ij}}, \\ \alpha_{\mathbf{q}} &:= \sum_{p\in\{-1,1\}} \alpha_{q(p)}, & \beta_{\mathbf{q}} &:= \sum_{p\in\{-1,1\}} \beta_{q(p)}, \end{aligned} \tag{6.82}$$

where $p \in \{-1,1\}$ and $i,j \in \{-1,0,1\}$. Directly from the definition of the functions g_{ij} (see (6.60)), we have

$$\begin{aligned} C_3^n(\mathbf{q}) &\equiv \int_1^{K_n} \prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} g_{ij}\left(\frac{k_p^n}{K_n}\cdot\phi\right)^{q(p)_{ij}} \frac{d\phi}{\phi} \\ &= \int_1^{K_n} \prod_{p\in\{-1,1\}} \prod_{i,j\in\{-1,0,1\}} \cos^{\alpha_{q(p)_{ij}}}\left(\frac{2\pi k_p^n\phi}{K_n}-\frac{\pi}{4}\right)\cdot\sin^{\beta_{q(p)_{ij}}}\left(\frac{2\pi k_p^n\phi}{K_n}-\frac{\pi}{4}\right)\frac{d\phi}{\phi} \\ &= \int_1^{K_n} \prod_{p\in\{-1,1\}} \cos^{\alpha_{q(p)}}\left(\frac{2\pi k_p^n\phi}{K_n}-\frac{\pi}{4}\right)\cdot\sin^{\beta_{q(p)}}\left(\frac{2\pi k_p^n\phi}{K_n}-\frac{\pi}{4}\right)\frac{d\phi}{\phi} \\ &= \int_1^{K_n} \prod_{p\in\{-1,1\}} \cos^{\alpha_{q(p)}}\left(\frac{2\pi k_p^n\phi}{K_n}-\frac{\pi}{4}\right)\cdot\sin^{2-\alpha_{q(p)}}\left(\frac{2\pi k_p^n\phi}{K_n}-\frac{\pi}{4}\right)\frac{d\phi}{\phi}. \end{aligned} \tag{6.83}$$

We observe that, for each $p \in \{-1,1\}$, either $\alpha_{q(p)} = 0$ or $\alpha_{q(p)} = 2$ and, consequently, we always have $\alpha_{\mathbf{q}} \in \{0,2,4\}$. This allows us to split the analysis into 3 cases:

(a) if $\alpha_{\mathbf{q}} = 4$, then formula (6.83) reduces to

$$C_3^n(\mathbf{q}) = \int_1^{K_n} \cos^2(2\pi\phi-\pi/4)\cdot\cos^2(2\pi\phi r_n-\pi/4)\frac{d\phi}{\phi}, \tag{6.84}$$

(b) if $\alpha_{\mathbf{q}} = 0$, then formula (6.83) reduces to

$$C_3^n(\mathbf{q}) = \int_1^{K_n} \sin^2(2\pi\phi-\pi/4)\cdot\sin^2(2\pi\phi r_n-\pi/4)\frac{d\phi}{\phi}, \tag{6.85}$$

(c) otherwise, formula (6.83) reduces to one of the following expressions

$$C_3^n(\mathbf{q}) = \int_1^{K_n} \cos^2(2\pi\phi - \pi/4) \cdot \sin^2(2\pi\phi r_n - \pi/4) \frac{d\phi}{\phi}, \tag{6.86}$$

$$C_3^n(\mathbf{q}) = \int_1^{K_n} \cos^2(2\pi\phi r_n - \pi/4) \cdot \sin^2(2\pi\phi - \pi/4) \frac{d\phi}{\phi}. \tag{6.87}$$

Step 2. Using the exact forms of $C_3^n(\mathbf{q})$ established in the previous step (which we expand via standard trigonometric identities), we identify and compute the asymptotics for all necessary integrals. In particular, in point (e) we carry out the computation leading to the emergence of the term r^{exp}. In order to compute the integrals described by the formula (6.83), setting

$$x = x(\phi) := 2\pi\phi - \pi/4, \qquad y = y(\phi) := 2\pi\phi r_n - \pi/4, \tag{6.88}$$

we use the following standard identities:

$$\begin{aligned}
\cos^2(x) \cdot \cos^2(y) &= \frac{1}{4} + \frac{1}{8}\cos(2x + 2y) + \frac{1}{8}\cos(2x - 2y) + \frac{1}{4}\cos(2x) + \frac{1}{4}\cos(2y), \\
\sin^2(x) \cdot \sin^2(y) &= \frac{1}{4} + \frac{1}{8}\cos(2x + 2y) + \frac{1}{8}\cos(2x - 2y) - \frac{1}{4}\cos(2x) - \frac{1}{4}\cos(2y), \\
\cos^2(x) \cdot \sin^2(y) &= \frac{1}{4} - \frac{1}{8}\cos(2x + 2y) - \frac{1}{8}\cos(2x - 2y) - \frac{1}{4}\cos(2x) + \frac{1}{4}\cos(2y), \\
\sin^2(x) \cdot \cos^2(y) &= \frac{1}{4} - \frac{1}{8}\cos(2x + 2y) - \frac{1}{8}\cos(2x - 2y) + \frac{1}{4}\cos(2x) - \frac{1}{4}\cos(2y).
\end{aligned} \tag{6.89}$$

Here, each line corresponds to (6.84), (6.85), (6.86) and (6.87), respectively. The integrals of the elements appearing on the right-hand side of (6.89) can be evaluated using the following estimates:

(a)

$$\begin{aligned}
\left| \int_1^{K_n} \frac{\cos(2x(\phi))}{\phi} d\phi \right| &= \left| \int_1^{K_n} \frac{\cos(4\pi\phi - \pi/2)}{\phi} d\phi \right| \\
&= \left| \frac{\sin(4\pi\phi - \pi/2)}{4\pi\phi} \Big|_{\phi=1}^{\phi=K_n} + \int_1^{K_n} \frac{\sin(4\pi\phi - \pi/2)}{\phi^2} d\phi \right| \\
&\le \frac{1}{4\pi} \left(1 + \frac{1}{K_n} + \int_1^{K_n} \frac{1}{\phi^2} d\phi \right) = \frac{1}{2\pi},
\end{aligned} \tag{6.90}$$

(b)

$$\begin{aligned}\left|\int_1^{K_n} \frac{\cos(2y(\phi))}{\phi} d\phi\right| &= \left|\int_1^{K_n} \frac{\cos(4\pi r_n \phi - \pi/2)}{\phi} d\phi\right| \\ &= \left|\frac{\sin(4\pi r_n \phi - \pi/2)}{4\pi r_n \phi}\Big|_{\phi=1}^{\phi=K_n} + \int_1^{K_n} \frac{\sin(4\pi r_n \phi - \pi/2)}{4\pi r_n \phi^2} d\phi\right| \\ &\le \frac{1}{2\pi r_n} \overset{n\to\infty}{\longrightarrow} \frac{1}{2\pi r},\end{aligned} \tag{6.91}$$

(c)

$$\begin{aligned}\left|\int_1^{K_n} \frac{\cos(2x(\phi) + 2y(\phi))}{\phi} d\phi\right| &= \left|\int_1^{K_n} \frac{\cos(4\pi(1 + r_n)\phi - \pi)}{\phi} d\phi\right| \\ &\le \frac{1}{2\pi(1 + r_n)} \overset{n\to\infty}{\longrightarrow} \frac{1}{2\pi(1 + r)},\end{aligned} \tag{6.92}$$

(d) provided that $r \in (0, 1)$:

$$\begin{aligned}\left|\int_1^{K_n} \frac{\cos(2x(\phi) - 2y(\phi))}{\phi} d\phi\right| &= \left|\int_1^{K_n} \frac{\cos(4\pi(1 - r_n)\phi - \pi)}{\phi} d\phi\right| \\ &\le \frac{1}{2\pi(1 - r_n)} \overset{n\to\infty}{\longrightarrow} \frac{1}{2\pi(1 - r)},\end{aligned} \tag{6.93}$$

(e) provided that $r = 1$: we expand cosine into power series and exchange integration with summation to obtain

$$\begin{aligned}\int_1^{K_n} \frac{\cos(2x(\phi) - 2y(\phi))}{\phi} d\phi &= \int_1^{K_n} \cos(4\pi(1 - r_n)\phi) \frac{d\phi}{\phi} \\ &= \ln K_n + \sum_{l=1}^{\infty} \frac{(-1)^l (4\pi)^{2l} (1 - r_n)^{2l}}{2l(2l)!} \cdot \phi^{2l} \Big|_{\phi=1}^{\phi=K_n} \\ &\sim \ln K_n + \sum_{l=1}^{\infty} \frac{(-1)^l (K_n - k_n)^{2l}}{2l(2l)!} \\ &\sim \ln K_n + \sum_{l=1}^{\infty} \frac{(-1)^l (1 + (K_n - k_n))^{2l}}{2l(2l)!} \\ &\sim r^{exp} \cdot \ln K_n,\end{aligned} \tag{6.94}$$

where we have used the fact that, for every $t > 0$, we have

$$\sum_{l=1}^{\infty} \frac{(-1)^l t^{2l}}{2l(2l)!} = \mathrm{Ci}(t) - \ln t - \gamma. \tag{6.95}$$

Here, γ is the Euler-Mascheroni constant and cosine integral defined as

$$\operatorname{Ci}(t) := -\int_t^\infty \frac{\cos(s)}{s} ds, \tag{6.96}$$

is globally bounded on $[1, \infty)$ (even $\operatorname{Ci}(t) \to 0$ as $t \to \infty$), see [23, 6.2(ii) Sine and Cosine Integrals, Eq. (6.2.11)].

Step 3. In this step, we apply the results of Step 2 to the formulas identified in Step 1. First, we provide a detailed computation for one representative case, followed by the results for the remaining analogous cases. We focus on (6.84) and use expansion (6.89) to obtain

$$\begin{aligned}
&\int_1^{K_n} \cos^2(2\pi\phi - \pi/4) \cdot \cos^2(2\pi\phi r_n - \pi/4) \frac{d\phi}{\phi} \\
&= \frac{1}{4} \int_1^{K_n} \frac{d\phi}{\phi} + \frac{1}{8} \int_1^{K_n} \cos(4\pi(1 - r_n)\phi) \frac{d\phi}{\phi} \\
&\quad + \frac{1}{4} \int_1^{K_n} \cos(4\pi\phi - \pi/2) \frac{d\phi}{\phi} + \frac{1}{4} \int_1^{K_n} \cos(4\pi\phi r_n - \pi/2) \frac{d\phi}{\phi} \\
&\quad + \frac{1}{8} \int_1^{K_n} \cos(4\pi(1 + r_n)\phi - \pi) \frac{d\phi}{\phi} \\
&\sim \frac{1}{4} \cdot \ln K_n + \frac{r^{exp}}{2} \cdot \ln K_n + O(1) \\
&\sim \frac{1}{4} \cdot \left(1 + \frac{r^{exp}}{2}\right) \cdot \ln K_n,
\end{aligned} \tag{6.97}$$

where we have also used the formulas (6.90)–(6.94). The computations in the other cases (6.85)–(6.86) are very similar and yield that

$$\begin{aligned}
\int_1^{K_n} \sin^2(2\pi\phi - \pi/4) \cdot \sin^2(2\pi\phi r_n - \pi/4) \frac{d\phi}{\phi} &\sim \frac{1}{4} \cdot \left(1 + \frac{r^{exp}}{2}\right) \cdot \ln K_n \\
\int_1^{K_n} \cos^2(2\pi\phi - \pi/4) \cdot \sin^2(2\pi\phi r_n - \pi/4) \frac{d\phi}{\phi} &\sim \frac{1}{4} \cdot \left(1 - \frac{r^{exp}}{2}\right) \cdot \ln K_n \\
\int_1^{K_n} \sin^2(2\pi\phi - \pi/4) \cdot \cos^2(2\pi\phi r_n - \pi/4) \frac{d\phi}{\phi} &\sim \frac{1}{4} \cdot \left(1 - \frac{r^{exp}}{2}\right) \cdot \ln K_n.
\end{aligned} \tag{6.98}$$

The change of sign in the last two cases of (6.98) is a consequence of the change of sign in front of the term $\cos(2x - 2y)$ while applying (6.89).

Step 4. The only remaining task is to verify that (6.98) is consistent with the definition of $\kappa_{\mathbf{q}}$ given in (6.79). Using the notation introduced in (6.82) we observe that

$$\begin{aligned} \kappa_{\mathbf{q}} &= \frac{1}{2} \cdot \prod_{p\in\{-1,1\}} (-1)^{\sum\limits_{w,v\in\{-1,1\}} q(p)_{wv} + \sum\limits_{u\in\{-1,1\}} (\frac{q(p)_{0u}}{2} + \frac{q(p)_{u0}}{2})} \\ &= \frac{1}{2} \cdot \prod_{p\in\{-1,1\}} (-1)^{\alpha_{q(p)} - q(p)_{00}} \cdot (-1)^{\beta_{q(p)}/2} \\ &= \frac{1}{2} \cdot (-1)^{\alpha_{\mathbf{q}}} \cdot (-1)^{\beta_{\mathbf{q}}/2} \\ &= \frac{1}{2} \cdot (-1)^{\beta_{\mathbf{q}}/2} \\ &= \begin{cases} -1/2 & \text{if} \quad \beta_{\mathbf{q}} = 2 \\ 1/2 & \text{if} \quad \beta_{\mathbf{q}} \in \{0,4\} \end{cases} \end{aligned} \tag{6.99}$$

where we have used the fact that for each $p \in \{-1,1\}$ we have $q(p)_{00} \in \{0,2\}$ and that $\alpha_{\mathbf{q}} \in \{0,2,4\}$. Thus, combining (6.84)–(6.86), with (6.97)–(6.99) yields the postulated formula (6.78) and concludes the proof. □

In the next lemma we will again use the notation introduced in (4.1) and (4.2), and in **S.1-S.6**. The concise formula this result affords will give us the ability to complete, in a rather straightforward manner, the summation of the terms contributing to the asymptotic variance (in the proof of Lemma 6.6).

Lemma 6.10. *Let $\mathcal{D}$ be a convex compact domain of the plane, with non-empty interior and piecewise C^1 boundary $\partial\mathcal{D}$. Let $\{k_{-1}^n, k_1^n\}_{n\in\mathbb{N}}$ be a sequence of pairs of wave-numbers such that $2 \le k_{-1}^n, k_1^n < \infty$ and $k_{-1}^n, k_1^n \to \infty$. Suppose also that the limits r and r^{exp} defined in (4.2) exist and $r > 0$. Choose any $\mathbf{i}, \mathbf{j} \in \{-1,0,1\}^{\otimes 2} \cup \{*\}$ and let $Y_{\mathbf{i}}^n$, $Y_{\mathbf{j}}^n$, denote the random integrals defined in (6.54). Then, we have the following asymptotic as $n \to \infty$*

$$\begin{aligned} \operatorname{Cov}(Y_{\mathbf{i}}^n, Y_{\mathbf{j}}^n) \sim \operatorname{area}(\mathcal{D}) \cdot \frac{8}{\pi^2} \cdot \frac{\ln(\max(k_{-1}^n, k_1^n))}{\max(k_{-1}^n, k_1^n)^2} \\ \cdot 2^{|\mathbf{i}|+|\mathbf{j}|} \cdot \psi(\gamma(\mathbf{i}) + \gamma(\mathbf{j})) \cdot \frac{2 + r^{exp} \cdot (-1)^{|\mathbf{i}|+|\mathbf{j}|}}{r}. \end{aligned} \tag{6.100}$$

Here, we use notation

$$\begin{aligned} |\mathbf{i}| &:= \sum_{p\in\{-1,1\}} |i_p|, & \gamma(\mathbf{i}) &:= (\sum_{p\in\{-1,1\}} \delta_{-1}(i_p), \sum_{p\in\{-1,1\}} \delta_1(i_p)), \\ |*| &:= 2, & \gamma(*) &:= (1,1), \end{aligned} \tag{6.101}$$

where $\mathbf{i} = (i_{-1}, i_1) \in \{-1,0,1\}^{\otimes 2}$, and we use definition

$$\psi(l,m) := \int_0^{2\pi} \cos^{2l}(\theta) \cdot \sin^{2m}(\theta) d\theta, \qquad l,m \in \mathbb{N}. \tag{6.102}$$

Proof. We will split our proof into three different steps corresponding to the three distinct cases of the formula (6.100):

1. $\mathrm{Cov}(Y^n_{\mathbf{i}}, Y^n_{\mathbf{j}}), \quad \mathbf{i}, \mathbf{j} \in \{-1, 0, 1\}^{\otimes 2}$,
2. $\mathrm{Cov}(Y^n_{\mathbf{i}}, Y^n_{*}), \quad \mathbf{i} \in \{-1, 0, 1\}^{\otimes 2}$,
3. $\mathrm{Var}(Y^n_{*})$.

In each scenario the strategy of the proof is the same. We will start by rewriting the integrand functions as the products of covariance functions $r_{ij}(k^n_p(x-y))$ (see Subsection 5.3). Then, we will regroup the terms using suitably chosen vectors $\mathbf{q}$ of non-negative integers $q(p)_{ij}$, they will count the powers with which each of the covariance functions appears. Then, we will simply apply Lemmas 6.8 and 6.9, and we will compare the result with the appropriate case of (6.100).

Step 1. Let us choose any $\mathbf{i}, \mathbf{j} \in \{-1, 0, 1\}^{\otimes 2}$. Using the standard properties of Hermite polynomials ([69, Proposition 2.2.1, p. 26]) and Lemma 6.8 we obtain

$$\begin{aligned}
&\mathrm{Cov}\left(\int_{\mathcal{D}} \prod_{p\in\{-1,1\}} H_2(\widetilde{\partial}_{i_p} b_{k^n_p}(x))dx, \int_{\mathcal{D}} \prod_{p\in\{-1,1\}} H_2(\widetilde{\partial}_{j_p} b_{k^n_p}(y))dy\right) \\
&= \int_{\mathcal{D}\times\mathcal{D}} \prod_{p\in\{-1,1\}} \mathbb{E}\left[H_2(\widetilde{\partial}_{i_p} b_{k^n_p}(x)) \cdot H_2(\widetilde{\partial}_{j_p} b_{k^n_p}(y))\right] dxdy \\
&= 4\int_{\mathcal{D}\times\mathcal{D}} \prod_{p\in\{-1,1\}} r_{i_p,j_p}(k^n_p(x-y))^2 dxdy \qquad (6.103)\\
&\sim \frac{16}{\pi^2} \cdot \frac{\mathrm{area}(\mathcal{D})}{k^n_{-1}\cdot k^n_1} \cdot \left(\prod_{p\in\{-1,1\}} v^2_{i_p,j_p}\right) \cdot \left(\prod_{p\in\{-1,1\}} h^2_{i_p,j_p}\right) \\
&\quad \cdot \int_1^{\max(k^n_{-1},k^n_1)} \prod_{p\in\{-1,1\}} g_{i_p,j_p}\left(\frac{k^n_p}{\max(k^n_{-1},k^n_1)}\phi\right)^2 \frac{d\phi}{\phi}.
\end{aligned}$$

Using Lemma 6.9 we further rewrite (6.103) as

$$
\begin{aligned}
&= \frac{16}{\pi^2}\cdot\frac{\operatorname{area}(\mathcal{D})}{k_{-1}^n\cdot k_1^n}\cdot(\sqrt{2})^{\sum_{p\in\{-1,1\}}2(|i_p|+|j_p|)}\\
&\quad\cdot\int_0^{2\pi}\cos^{\sum\limits_{p\in\{-1,1\}}2(\delta_{-1}(i_p)+\delta_{-1}(j_p))}(\theta)\cdot\sin^{\sum\limits_{p\in\{-1,1\}}2(\delta_1(i_p)+\delta_1(j_p))}(\theta)d\theta\\
&\quad\cdot\int_1^{\max(k_{-1}^n,k_1^n)}\prod_{p\in\{-1,1\}}\cos^{2(1-\delta_1(|i_p|+|j_p|))}\left(\frac{2\pi k_p^n\phi}{\max(k_{-1}^n,k_1^n)}-\frac{\pi}{4}\right)\\
&\quad\cdot\sin^{2\delta_1(|i_p|+|j_p|)}\left(\frac{2\pi k_p^n\phi}{\max(k_{-1}^n,k_1^n)}-\frac{\pi}{4}\right)\frac{d\phi}{\phi} \qquad (6.104)\\
&\sim\frac{4}{\pi^2}\cdot\frac{\operatorname{area}(\mathcal{D})}{k_{-1}^n\cdot k_1^n}\cdot(\sqrt{2})^{\sum_{p\in\{-1,1\}}2(|i_p|+|j_p|)}\cdot\\
&\quad\cdot\int_0^{2\pi}\cos^{\sum\limits_{p\in\{-1,1\}}2(\delta_{-1}(i_p)+\delta_{-1}(j_p))}(\theta)\cdot\sin^{\sum\limits_{p\in\{-1,1\}}2(\delta_1(i_p)+\delta_1(j_p))}(\theta)d\theta\\
&\quad\cdot\left(1+\frac{r^{exp}}{2}\cdot\prod_{p\in\{-1,1\}}(-1)^{|i_p|+|j_p|}\right)\cdot\ln(\max(k_{-1}^n,k_1^n)),
\end{aligned}
$$

which, written in terms of the notation introduced in (6.101), is the same as (6.100).

Step 2. We start with the following auxiliary observation: if X, Y, Z denote standard Gaussian random variables with Y independent of Z then

$$
\mathbb{E}\left[H_2(X)YZ\right]=2\operatorname{Cov}(X,Y)\operatorname{Cov}(X,Z), \qquad (6.105)
$$

where $H_2(x)=x^2-1$ is the second Hermite polynomial. Indeed, the random variable $(Y+Z)/\sqrt{2}$ has a standard normal distribution and so using the standard properties of Hermite polynomials ([69, Proposition 2.2.1, p. 26]) we have

$$
\begin{aligned}
\mathbb{E}\left[H_2(X)\left(\frac{Y+Z}{\sqrt{2}}\right)^2\right]&=\mathbb{E}\left[H_2(X)H_2\left(\frac{Y+Z}{\sqrt{2}}\right)\right]=2\operatorname{Cov}\left(X,\frac{Y+Z}{\sqrt{2}}\right)^2\\
&=\operatorname{Cov}(X,Y)^2+\operatorname{Cov}(Y,Z)^2+2\operatorname{Cov}(X,Y)\operatorname{Cov}(Y,Z).
\end{aligned} \qquad (6.106)
$$

On the other hand, we have

$$
\begin{aligned}
\mathbb{E}\left[H_2(X)\left(\frac{Y+Z}{\sqrt{2}}\right)^2\right]&=\frac{1}{2}\mathbb{E}\left[H_2(X)\left(Y^2+2YZ+Z^2\right)\right]\\
&=\frac{1}{2}\left(\mathbb{E}[H_2(X)H_2(Y)]+\mathbb{E}[H_2(X)H_2(Z)]+2\mathbb{E}[H_2(X)YZ]\right)\\
&=\operatorname{Cov}(X,Y)^2+\operatorname{Cov}(X,Z)^2+\mathbb{E}[H_2(X)YZ],
\end{aligned} \qquad (6.107)
$$

and (6.105) follows by comparing (6.106) with (6.107). Now we choose any $\mathbf{i} \in \{-1,0,1\}^{\otimes 2}$ and use (6.105) and Lemma 6.8 to we obtain

$$\begin{aligned}
&\mathrm{Cov}\left(\int_{\mathcal{D}} \prod_{p\in\{-1,1\}} H_2(\widetilde{\partial}_{i_p} b_{k_p^n}(x))dx, \int_{\mathcal{D}} \prod_{p,q\in\{-1,1\}} \widetilde{\partial}_q b_{k_p^n}(y))dy\right) \\
&= \int_{\mathcal{D}\times\mathcal{D}} \prod_{p\in\{-1,1\}} \mathbb{E}[H_2(\widetilde{\partial}_{i_p} b_{k_p^n}(x)) \cdot \prod_{q\in\{-1,1\}} \widetilde{\partial}_q b_{k_p^n}(y)]dxdy \\
&= 4\int_{\mathcal{D}\times\mathcal{D}} \prod_{p,q\in\{-1,1\}} r_{i_p,q}(k_p^n(x-y))dxdy \qquad (6.108) \\
&\sim \frac{16}{\pi^2}\cdot\frac{\mathrm{area}(\mathcal{D})}{k_{-1}^n\cdot k_1^n}\cdot\left(\prod_{p,q\in\{-1,1\}} v_{i_p,q}\right)\cdot\left(\prod_{p,q\in\{-1,1\}} h_{i_p,q}\right) \\
&\quad \cdot \int_1^{\max(k_{-1}^n,k_1^n)} \prod_{p,q\in\{-1,1\}} g_{i_p,q}\left(\frac{k_p^n}{\max(k_{-1}^n,k_1^n)}\cdot\phi\right)\frac{d\phi}{\phi}.
\end{aligned}$$

Using Lemma 6.9 we can further rewrite (6.108) as

$$\begin{aligned}
&= \frac{16}{\pi^2}\cdot\frac{\mathrm{area}(\mathcal{D})}{k_{-1}^n\cdot k_1^n}\cdot(\sqrt{2})^{4+2\cdot\sum_{p\in\{-1,1\}}|i_p|}\cdot\int_0^{2\pi}\cos^{2+2\cdot\sum_{p\in\{-1,1\}}\delta_{-1}(i_p)}(\theta)\cdot\sin^{2+2\cdot\sum_{p\in\{-1,1\}}\delta_1(i_p)}(\theta)d\theta \\
&\cdot\int_1^{\max(k_{-1}^n,k_1^n)}\prod_{p\in\{-1,1\}}\cos^{2\delta_1(|i_p|)}\left(\frac{2\pi k_p^n\phi}{\max(k_{-1}^n,k_1^n)}-\frac{\pi}{4}\right)\cdot\sin^{2\delta_0(i_p)}\left(\frac{2\pi k_p^n\phi}{\max(k_{-1}^n,k_1^n)}-\frac{\pi}{4}\right)\frac{d\phi}{\phi} \\
&\sim \frac{4}{\pi^2}\cdot\frac{\mathrm{area}(\mathcal{D})}{k_{-1}^n\cdot k_1^n}\cdot(\sqrt{2})^{4+2\cdot\sum_{p\in\{-1,1\}}|i_p|}\int_0^{2\pi}\cos^{2+2\cdot\sum_{p\in\{-1,1\}}\delta_{-1}(i_p)}(\theta)\cdot\sin^{2+2\cdot\sum_{p\in\{-1,1\}}\delta_1(i_p)}(\theta)d\theta \\
&\cdot\left(1+\frac{r^{exp}}{2}\cdot\prod_{p\in\{-1,1\}}(-1)^{|i_p|}\right)\cdot\ln\max(k_{-1}^n,k_1^n).
\end{aligned} \qquad (6.109)$$

This, written using notation (6.101), recovers the corresponding case of (6.100).

Step 3. We start with the following ancillary observation: for each $p\in\{-1,1\}$ and $x,y\in\mathbb{R}^2$ we have

$$\begin{aligned}
&\mathbb{E}\left[\prod_{q\in\{-1,1\}}\widetilde{\partial}_q b_{k_p^n}(x)\cdot\widetilde{\partial}_q b_{k_p^n}(y)\right] \\
&= \mathbb{E}\left[(\prod_{u\in\{-1,1\}}\widetilde{\partial}_u b_{k_p^n}(x))\cdot(\prod_{v\in\{-1,1\}}\widetilde{\partial}_v b_{k_p^n}(y))\right] \qquad (6.110) \\
&= \prod_{q\in\{-1,1\}}\mathbb{E}\left[\widetilde{\partial}_q b_{k_p^n}(x)\cdot\widetilde{\partial}_q b_{k_p^n}(y)\right]+\left(\mathbb{E}\left[\widetilde{\partial}_{-1} b_{k_p^n}(x)\cdot\widetilde{\partial}_1 b_{k_p^n}(y)\right]\right)^2 \\
&= r_{-1,-1}(k_p^n(x-y))\cdot r_{1,1}(k_p^n(x-y))+r_{-1,1}(k_p^n(x-y))^2.
\end{aligned}$$

Indeed, this is a direct consequence of the classical Wick formulae for moments of a Gaussian products ([76, p. 38, Eq. (3.2.21)]) and of the fact that for each fixed $z \in \mathbb{R}^2$ the random variables $\{\widetilde{\partial}_i b_{k_p^n}(z) : i \in \{-1,0,1\}\}$ form a collection of three independent standard Gaussian random variables. Thus, using subsequently the formula (6.110) and Lemma 6.8 we obtain

$$\begin{aligned}
&\mathrm{Var}\left(\int_{\mathcal{D}} \prod_{p,q\in\{-1,1\}} \widetilde{\partial}_q b_{k_p^n}(x)dx\right) \\
&= \int_{\mathcal{D}\times\mathcal{D}} \prod_{p\in\{-1,1\}} \mathbb{E}\left[\prod_{p'\in\{-1,1\}} \widetilde{\partial}_{p'} b_{k_p^n}(x)\cdot \widetilde{\partial}_{p'} b_{k_p^n}(y)\right] dxdy \\
&= \int_{\mathcal{D}\times\mathcal{D}} \prod_{p\in\{-1,1\}} (r_{-1,-1}(k_p^n(x-y))\cdot r_{1,1}(k_p^n(x-y)) + r_{-1,1}(k_p^n(x-y))^2)dxdy \\
&= \int_{\mathcal{D}\times\mathcal{D}} \prod_{p\in\{-1,1\}} r_{-1,1}(k_p^n(x-y))^2 dxdy \\
&\quad + \int_{\mathcal{D}\times\mathcal{D}} \prod_{p\in\{-1,1\}} r_{-1,-1}(k_p^n(x-y)) r_{1,1}(k_p^n(x-y))dxdy \\
&\quad + \sum_{p\in\{-1,1\}} \int_{\mathcal{D}\times\mathcal{D}} r_{-1,-1}(k_p^n(x-y))\cdot r_{1,1}(k_p^n(x-y))\cdot r_{-1,1}(k_{-p}^n(x-y))^2 dxdy.
\end{aligned} \tag{6.111}$$

Using Lemma 6.9 we can we can further rewrite (6.111) as

$$\begin{aligned}
&\sim \frac{16}{\pi^2}\cdot\frac{\mathrm{area}(\mathcal{D})}{k_{-1}^n\cdot k_1^n}\cdot(\sqrt{2})^8\cdot\int_0^{2\pi}\cos^4\theta\cdot\sin^4\theta d\theta \\
&\quad\cdot\int_1^{\max(k_{-1}^n,k_1^n)}\cos^2\left(2\pi\phi-\frac{\pi}{4}\right)\cdot\cos^2\left(2\pi r_n\phi-\frac{\pi}{4}\right)\frac{d\phi}{\phi} \\
&\sim \frac{4}{\pi^2}\cdot\frac{\mathrm{area}(\mathcal{D})}{k_{-1}^n\cdot k_1^n}\cdot(\sqrt{2})^8\cdot\int_0^{2\pi}\cos^4\theta\cdot\sin^4\theta d\theta\cdot\left(1+\frac{r^{exp}}{2}\right)\cdot\ln\max(k_{-1}^n,k_1^n).
\end{aligned} \tag{6.112}$$

This completes the proof of the formula (6.100). □

We are finally ready to achieve the main goal of this subsection.

Proof of Lemma 6.6. We will use the formula for $\mathrm{Cross}(\mathcal{N}(b_{k_{-1}^n}, b_{k_1^n}, \mathcal{D})[4])$ that was established in Lemma 6.7 with additional notation introduced for the sake of brevity

$$T := \{-1,0,1\}^{\otimes 2}\cup\{*\}, \tag{6.113}$$

where for $\mathbf{i}\in\{-1,0,1\}^{\otimes 2}$ we use indexation $\mathbf{i} = (i_{-1}, i_1)$. On this index set we will use a natural ordering

$$(-1,-1)\le(-1,0)\le(-1,1)\le(0,-1)\le(0,0)\le(0,1)\le(1,-1)\le(1,0)\le(1,1)\le *, \tag{6.114}$$

that is $*$ is the largest element of T and for $\mathbf{i},\mathbf{j} \in \{-1,0,+1\}^2$ we set $\mathbf{i} \leq \mathbf{j}$ if $i_{-1} < j_{-1}$ or if $i_{-1} = j_{-1}$ and $i_1 \leq j_1$. Additionally, we define

$$r_n := \frac{\min(k^n_{-1}, k^n_1)}{\max(k^n_{-1}, k^n_1)} \tag{6.115}$$

and we observe that, thanks to Lemma 6.10, we have

$$\begin{aligned}
&\mathrm{Var}(\mathrm{Cross}(\mathcal{N}(b_{k^n_{-1}}, b_{k^n_1}, \mathcal{D})[4])) = \\
&= \frac{(k^n_{-1} \cdot k^n_1)^2}{(128)^2\pi^2} \cdot \mathrm{Var}(\sum_{\mathbf{i}\in T} \eta_{\mathbf{i}} \cdot Y^n_{\mathbf{i}}) \\
&\sim \frac{\mathrm{area}(\mathcal{D})}{2^{12}\pi^4} \cdot r_n \cdot \max(k^n_{-1}, k^n_1)^2 \ln(\max(k^n_{-1}, k^n_1)) \\
&\quad \cdot \sum_{\mathbf{i},\mathbf{j}\in T} \eta_{\mathbf{i}}\eta_{\mathbf{j}} \cdot 2^{|\mathbf{i}|+|\mathbf{j}|} \cdot \psi(\gamma(\mathbf{i}) + \gamma(\mathbf{j})) \cdot (1 + \delta_1(r) \cdot \frac{r^{exp}}{2} \cdot (-1)^{|\mathbf{i}|+|\mathbf{j}|}) \\
&\sim \frac{\mathrm{area}(\mathcal{D})}{2^{12}\pi^4} \cdot r_n \cdot \max(k^n_{-1}, k^n_1)^2 \ln(\max(k^n_{-1}, k^n_1)) \\
&\quad \cdot \left(\sum_{\mathbf{i},\mathbf{j}\in T} \eta_{\mathbf{i}}\eta_{\mathbf{j}} \cdot 2^{|\mathbf{i}|+|\mathbf{j}|} \cdot \psi(\gamma(\mathbf{i}) + \gamma(\mathbf{j})) + \frac{r^{exp}}{2} \cdot \sum_{\mathbf{i},\mathbf{j}\in T} \eta_{\mathbf{i}}\eta_{\mathbf{j}} \cdot (-2)^{|\mathbf{i}|+|\mathbf{j}|} \cdot \psi(\gamma(\mathbf{i}) + \gamma(\mathbf{j})) \right) \\
&= \frac{\mathrm{area}(\mathcal{D})}{2^{12}\pi^4} \cdot r_n \cdot \max(k^n_{-1}, k^n_1)^2 \ln(\max(k^n_{-1}, k^n_1)) \cdot (K_1 + \frac{r^{exp}}{2} \cdot K_{-1}).
\end{aligned} \tag{6.116}$$

Here, we have used the notation

$$K_\varepsilon := \sum_{\mathbf{i},\mathbf{j}\in T} \eta_{\mathbf{i}}\eta_{\mathbf{j}} \cdot (\varepsilon 2)^{|\mathbf{i}|+|\mathbf{j}|} \cdot \psi(\gamma(\mathbf{i}) + \gamma(\mathbf{j})), \qquad \varepsilon \in \{-1, 1\}. \tag{6.117}$$

We deduce that, in order to complete our computation, we only need to find the constants K_ε. We note that

$$K_\varepsilon = A^{tr}_\varepsilon \Psi A_\varepsilon, \tag{6.118}$$

where

$$A_\varepsilon := ((\varepsilon 2)^{|\mathbf{i}|}\eta_{\mathbf{i}})_{\mathbf{i}\in T}, \qquad \Psi := [\psi(\Gamma_{\mathbf{ij}})]_{\mathbf{i},\mathbf{j}\in T}, \qquad \Gamma_{\mathbf{ij}} := \gamma(\mathbf{i}) + \gamma(\mathbf{j}). \tag{6.119}$$

Using the above notation and writing $\mathrm{diag}(w^{tr})$ for the diagonal matrix corresponding to the vector w, we compute

$$\begin{aligned}
A^{tr}_\varepsilon &= (-1, -4, 5, -4, 8, -4, 5, -4, -1, -12) \\
&\quad \cdot \mathrm{diag}(4, 2, 4, 2, 1, 2, 4, 2, 4, 4) \\
&\quad \cdot \mathrm{diag}(1, \varepsilon, 1, \varepsilon, 1, \varepsilon, 1, \varepsilon, 1, 1) \\
&= 4(-1, -2, 5, -2, 2, -2, 5, -2, -1, -12) \\
&\quad \cdot \mathrm{diag}(1, \varepsilon, 1, \varepsilon, 1, \varepsilon, 1, \varepsilon, 1, 1) \\
&= 4[(-1, 0, 5, 0, 2, 0, 5, 0, -1, -12) - 2\varepsilon(0, 1, 0, 1, 0, 1, 0, 1, 0, 0)] \\
&= 4(u^{tr} - 2\varepsilon v^{tr}),
\end{aligned} \tag{6.120}$$

where we define

$$\begin{aligned} u^{tr} &:= (-1, 0, 5, 0, 2, 0, 5, 0, -1, -12), \\ v^{tr} &:= (0, 1, 0, 1, 0, 1, 0, 1, 0, 0). \end{aligned} \tag{6.121}$$

Following definition (6.119) we set $\Gamma = [\Gamma_{\mathbf{ij}}]_{\mathbf{i},\mathbf{j}}$ where $\mathbf{i}, \mathbf{j} \in T$ and $\Gamma_{\mathbf{ij}} := \gamma(\mathbf{i}) + \gamma(\mathbf{j})$. Here, the function γ is as defined in (6.101). We record that the matrix Γ is equal to

$$\left[\begin{array}{c|cccccccccc}
 & (-,-) & (-,0) & (-,+) & (0,-) & (0,0) & (0,+) & (+,-) & (+,0) & (+,+) & * \\
\hline
(-,-) & (4,0) & (3,0) & (3,1) & (3,0) & (2,0) & (2,1) & (3,1) & (2,1) & (2,2) & (3,1) \\
(-,0) & (3,0) & (2,0) & (2,1) & (2,0) & (1,0) & (1,1) & (2,1) & (1,1) & (1,2) & (2,1) \\
(-,+) & (3,1) & (2,1) & (2,2) & (2,1) & (1,1) & (1,2) & (2,2) & (1,2) & (1,3) & (2,2) \\
(0,-) & (3,0) & (2,0) & (2,1) & (2,0) & (1,0) & (1,1) & (2,1) & (1,1) & (1,2) & (2,1) \\
(0,0) & (2,0) & (1,0) & (1,1) & (1,0) & (0,0) & (0,1) & (1,1) & (0,1) & (0,2) & (1,1) \\
(0,+) & (2,1) & (1,1) & (1,2) & (1,1) & (0,1) & (0,2) & (1,2) & (0,2) & (0,3) & (1,2) \\
(+,-) & (3,1) & (2,1) & (2,2) & (2,1) & (1,1) & (1,2) & (2,2) & (1,2) & (1,3) & (2,2) \\
(+,0) & (2,1) & (1,1) & (1,2) & (1,1) & (0,1) & (0,2) & (1,2) & (0,2) & (0,3) & (1,2) \\
(+,+) & (2,2) & (1,2) & (1,3) & (1,2) & (0,2) & (0,3) & (1,3) & (0,3) & (0,4) & (1,3) \\
* & (3,1) & (2,1) & (2,2) & (2,1) & (1,1) & (1,2) & (2,2) & (1,2) & (1,3) & (2,2)
\end{array}\right]. \tag{6.122}$$

We recall that the function ψ was defined in (6.102), and we evaluate $\psi(\Gamma_{\mathbf{ij}})$ for all distinct arguments $\Gamma_{\mathbf{ij}}$, which yields the values

$$\begin{array}{lll}
\psi(4,0) = 35\pi/2^6 & \psi(3,0) = 5\pi/2^3 & \psi(2,1) = \pi/2^3 \\
\psi(3,1) = 5\pi/2^6 & \psi(2,0) = 3\pi/2^2 & \psi(1,1) = \pi/2^2 \\
\psi(2,2) = 3\pi/2^6 & \psi(1,0) = \pi & \psi(0,0) = 2\pi.
\end{array}$$

Now, in order to facilitate the further elementary computations, for each $l, m \in \mathbb{N}$ we set $\widehat{\psi}(l, m) := \frac{2^6}{\pi} \cdot \psi(l, m)$ and we record the corresponding values

$$\begin{array}{lll}
\widehat{\psi}(4,0) = 35 & \widehat{\psi}(3,0) = 40 & \widehat{\psi}(2,1) = 8 \\
\widehat{\psi}(3,1) = 5 & \widehat{\psi}(2,0) = 48 & \widehat{\psi}(1,1) = 16 \\
\widehat{\psi}(2,2) = 3 & \widehat{\psi}(1,0) = 64 & \widehat{\psi}(0,0) = 128.
\end{array}$$

We continue by defining the matrix $\widehat{\Psi} = [\widehat{\psi}(\Gamma_{\mathbf{ij}})]_{\mathbf{i},\mathbf{j}}$ where $\mathbf{i}, \mathbf{j} \in T$. We compute that the matrix $\widehat{\Psi}$ is equal to

$$\left[\begin{array}{c|cccccccccc}
 & (-,-) & (-,0) & (-,+) & (0,-) & (0,0) & (0,+) & (+,-) & (+,0) & (+,+) & * \\
\hline
(-,-) & 35 & 40 & 5 & 40 & 48 & 8 & 5 & 8 & 3 & 5 \\
(-,0) & 40 & 48 & 8 & 48 & 64 & 16 & 8 & 16 & 8 & 8 \\
(-,+) & 5 & 8 & 3 & 8 & 16 & 8 & 3 & 8 & 5 & 3 \\
(0,-) & 40 & 48 & 8 & 48 & 64 & 16 & 8 & 16 & 8 & 8 \\
(0,0) & 48 & 64 & 16 & 64 & 128 & 64 & 16 & 64 & 48 & 16 \\
(0,+) & 8 & 16 & 8 & 16 & 64 & 48 & 8 & 48 & 40 & 8 \\
(+,-) & 5 & 8 & 3 & 8 & 16 & 8 & 3 & 8 & 5 & 3 \\
(+,0) & 8 & 16 & 8 & 16 & 64 & 48 & 8 & 48 & 40 & 8 \\
(+,+) & 3 & 8 & 5 & 8 & 48 & 40 & 5 & 40 & 35 & 5 \\
* & 5 & 8 & 3 & 8 & 16 & 8 & 3 & 8 & 5 & 3
\end{array}\right]. \tag{6.123}$$

We note that thanks to symmetries of $\widehat{\Psi}$ and of the vectors u^{tr}, v^{tr} (see (6.121)), and for the the purpose of computing relevant inner products, the following is a split into equivalent columns of $\widehat{\Psi}$: $\{1,9\}$, $\{2,4,6,8\}$, $\{3,7,10\}$, $\{5\}$. We take advantage of this and obtain

$$\begin{array}{rclllllllllll} u^{tr}\cdot\widehat{\Psi} &=& (48, & 64, & 16, & 64, & 128, & 64, & 16, & 64, & 48, & 16) \\ &=& 16\cdot(3, & 4, & 1, & 4, & 8, & 4, & 1, & 4, & 3, & 1) \\ v^{tr}\cdot\widehat{\Psi} &=& (96, & 128, & 32, & 128, & 256, & 128, & 32, & 128, & 96, & 32) \\ &=& 32\cdot(3, & 4, & 1, & 4, & 8, & 4, & 1, & 4, & 3, & 1). \end{array}$$

Thus, notably, $u^{tr}\cdot\widehat{\Psi} = 2v^{tr}\cdot\widehat{\Psi}$ and going further

$$u^{tr}\widehat{\Psi}u = 128, \qquad v^{tr}\widehat{\Psi}v = 512 \qquad v^{tr}\widehat{\Psi}u = 256. \tag{6.124}$$

This yields

$$\begin{aligned} K_\varepsilon &= A_\varepsilon^{tr}\widehat{\Psi}A_\varepsilon \\ &= \frac{\pi}{4}(u-2\varepsilon v)^{tr}\widehat{\Psi}(u-2\varepsilon v) \\ &= \frac{\pi}{4}(u\widehat{\Psi}u^{tr} + 4v\widehat{\Psi}v^{tr} - 4\varepsilon v^{tr}\widehat{\Psi}u) \\ &= \frac{\pi}{4}(2176 - \varepsilon\cdot 1024) \\ &= 32\pi(17-8\varepsilon), \end{aligned} \tag{6.125}$$

and in consequence

$$K_{-1} = 2^5\cdot 5^2 = 800\pi, \qquad K_{+1} = 2^5\cdot 9\pi = 288\pi. \tag{6.126}$$

We conclude the proof by plugging the values established in (6.126) into the last line of (6.116). □

Chapter 7 Proofs: Part III

7.1 Proof of the Univariate Central Limit Theorem

Proof of Theorem 4.2. The proof that follows adopts a classical approach. Specifically, we first reduce the problem to the study of the 4th chaotic projection using the variance estimates established in Section 6.1. Next, we express the random integrals comprising the 4th chaotic projection via the Wiener isometry, as outlined in Subsection 3.3.2. Applying the Fourth Moment Theorem (see Theorem 3.3), we then reformulate the task of proving the CLT as an analytic problem involving the bounding of contraction norms. Finally, we accomplish this last goal using a strategy borrowed from [70].

Step 1. We start by proving inequality (4.7) and we recall the auxiliary notation $Y_n := \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})$ (see (3.48)). Using triangle inequality we obtain

$$\begin{aligned}
&\sqrt{\mathbb{E}\left(\frac{Y_n - \mathbb{E}Y_n}{\sqrt{\mathrm{Var}Y_n}} - \frac{Y_n[4]}{\sqrt{\mathrm{Var}Y_n[4]}}\right)^2} \\
&= \sqrt{\mathbb{E}\left(\frac{\sum_{q\neq 0,2} Y_n[2q]}{\sqrt{\mathrm{Var}Y_n}} + Y_n[4]\left((\mathrm{Var}Y_n)^{-1/2} - (\mathrm{Var}Y_n[4])^{-1/2}\right)\right)^2} \\
&\leq \sqrt{\frac{\sum_{q\neq 2} \mathrm{Var}Y_n[2q]}{\mathrm{Var}Y_n}} + \left(1 - \sqrt{\frac{\mathrm{Var}Y_n[4]}{\mathrm{Var}Y_n}}\right) \\
&\leq \sqrt{\frac{\sum_{q\neq 2} \mathrm{Var}Y_n[2q]}{\mathrm{Var}Y_n}} + \left(1 - \frac{\mathrm{Var}Y_n[4]}{\mathrm{Var}Y_n}\right) \\
&\leq 2\sqrt{\frac{\sum_{q\neq 2} \mathrm{Var}Y_n[2q]}{\mathrm{Var}Y_n}}.
\end{aligned} \tag{7.1}$$

Lemma 6.1 implies that, there exists a numerical constant $L > 0$, such that

$$\begin{aligned}
\sqrt{\frac{\sum_{q\neq 2} \mathrm{Var}Y_n[2q]}{\mathrm{Var}Y_n}} \leq \sqrt{\frac{\sum_{q\neq 2} \mathrm{Var}Y_n[2q]}{\mathrm{Var}Y_n[4]}} &\leq \frac{L \cdot (1 + \mathrm{diam}(\mathcal{D})^2) \cdot K_n}{\sqrt{\mathrm{Var}Y_n[4]}} \\
&= \frac{L \cdot \delta_n \cdot (1 + \mathrm{diam}(\mathcal{D})^2) \cdot}{\sqrt{\ln K_n}},
\end{aligned} \tag{7.2}$$

and comparing with (4.6) yields that we had just proved the first part of (4.7). Using orthogonality between chaotic projections of different orders, we can write

$$\mathrm{Corr}(Y_n, Y_n[4]) = \frac{\mathrm{Cov}(Y_n, Y_n[4])}{\sqrt{\mathrm{Var}Y_n}\cdot\sqrt{\mathrm{Var}Y_n[4]}} = \sum_{q\neq 2}\frac{\mathrm{Cov}(Y_n[2q], Y_n[4])}{\sqrt{\mathrm{Var}Y_n}\cdot\sqrt{\mathrm{Var}Y_n[4]}} = \sqrt{\frac{\mathrm{Var}Y_n[4]}{\mathrm{Var}Y_n}}. \tag{7.3}$$

We note that, for the the same numerical constant L as in (7.2), we have

$$\begin{aligned}\sqrt{\frac{\mathrm{Var}(Y_n[4])}{\mathrm{Var}Y_n}} &= \frac{1}{\sqrt{1+\sum_{q\neq 2}\frac{\mathrm{Var}(Y_n[2q])}{\mathrm{Var}(Y_n[4])}}}\\ &= \frac{1}{\sqrt{1+\delta_n^2\cdot\frac{\sum_{q\neq 2}\mathrm{Var}Y_n[2q]}{K_n^2\ln K_n}}}\\ &\geq \frac{1}{\sqrt{1+L\delta_n^2\gamma_n^2}},\end{aligned} \tag{7.4}$$

which yields the second inequality postulated in (4.7).

Step 2. As anticipated, in this step we will express the elements of the 4th chaotic projection using the Wiener isometry. We recall that, for each $x\in\mathbb{R}^2$ the collection $\{\widetilde{\partial}_i b_{k_p^n}(x) : p\in\{-1,1\}, i\in\{-1,0,1\}\}$ consists of 6 independent standard Gaussian random variables. Thus, with I_1 denoting the Wiener-Itô isometry (see Subsection 3.3.2), we can write for each $p\in\{-1,1\}$, $i\in\{-1,0,1\}$, that

$$\widetilde{\partial}_i b_{k_p^n}(x) = I_1(f_{p,i}(k_p^n x,\cdot)), \tag{7.5}$$

where $f_{p,i}(k_p^n x,\cdot)\in L^2([0,1])$ and the collection $\{f_{p,i}(k_p^n x,\cdot) : p\in\{-1,1\}, i\in\{-1,0,1\}\}$ consists of 6 functions which are $L^2([0,1])$-orthonormal. Let us now introduce a natural generalisation of the notation introduced in (3.26a)-(3.26c). For any collection $\mathbf{j}=\{j_{p,i} : p\in\{-1,1\}, i\in\{-1,0,1\}\}$ of 6 non-negative integers such that $\sum_{p\in\{-1,1\}}\sum_{i\in\{-1,0,1\}} j_{p,i} = 4$ we will denote by $f_n^{\mathbf{j}}(x,\cdot)\in L^2_s([0,1]^4)$ the unique function such that

$$\prod_{p\in\{-1,1\}}\prod_{i\in\{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p^n}(x)) = I_4(f_n^{\mathbf{j}}(x,\cdot)). \tag{7.6}$$

For instance,

$$H_2(\widetilde{\partial}_{-1} b_{k_{-1}^n}(x)) H_2(b_{k_p^n}(x)) = I_4(f_n^{\mathbf{j}}(x,\cdot)), \tag{7.7}$$

where the function $f_n^{\mathbf{j}}(x,\cdot)\in L^2_s([0,1]^4,\mathcal{B}([0,1]^4), dt_1dt_2dt_3dt_4)$ is given by the formula

$$\begin{aligned}&f_n^{\mathbf{j}}(x,t_1,t_2,t_3,t_4) =\\ &\frac{1}{4!}\cdot\sum_{\sigma\in S_4} f_{-1,-1}(k_{-1}^n x, t_{\sigma(1)}) f_{-1,-1}(k_{-1}^n x, t_{\sigma(2)}) f_{1,0}(k_1^n x, t_{\sigma(3)}) f_{1,0}(k_1^n x, t_{\sigma(4)}).\end{aligned} \tag{7.8}$$

Using Lemma 5.2 we can write

$$\begin{aligned} &\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4] = I_4(\tilde{g}_n), \\ &\tilde{g}_n(t_1,t_2,t_3,t_4) = (k_{-1}^n \cdot k_1^n) \sum_{\mathbf{j}\in\mathbb{N}^6, |\mathbf{j}|=4} c_{\mathbf{j}} \int_{\mathcal{D}} f_n^{\mathbf{j}}(x,t_1,t_2,t_3,t_4)dx, \end{aligned} \tag{7.9}$$

where the sum is over $\mathbf{j} = (j_{-1,-1}, j_{-1,0}, j_{-1,1}, j_{1,-1}, j_{1,0}, j_{1,1}) \in \mathbb{N}^6$ such that $j_{-1,-1} + \ldots + j_{1,1} = 4$, and where the numerical constants $c_{\mathbf{j}}$ are as defined in (5.37). We note that $\widetilde{g}_n \in L_s^2([0,1]^4)$.

Step 3. The preceding point has prepared us for the following. Since our goal is to use the 4th Moment Theorem on the Wiener Chaos [69, p. 99, Theorem 5.2.7] (in the form recorded in Theorem 3.3) we need to study the contractions associated with the elements of the 4th Wiener Chaos. We recall that "the contraction maps $\otimes_r$" ($r \geq 0$ an integer) were defined in (3.2). For each $r = 1,2,3$, with $\widetilde{g_n}$ defined in (7.9), we have

$$\tilde{g}_n \otimes_r \tilde{g}_n(t_1,\ldots,t_{4-r},s_1,\ldots,s_{4-r}) = (k_{-1}^n \cdot k_1^n)^2 \sum_{\mathbf{i},\mathbf{j}\in\mathbb{N}^6, |\mathbf{i}|=|\mathbf{j}|=4} c_{\mathbf{i}} c_{\mathbf{j}} \cdot A_{\mathbf{ij}}^n, \tag{7.10}$$

where $A_{\mathbf{ij}}^n$ is the quantity

$$\int_{\mathcal{D}}\int_{\mathcal{D}} \left(\int_{[0,1]^r} f_n^{\mathbf{i}}(x,t_1,\ldots,t_{4-r},u_1,\ldots,u_r) \cdot f_n^{\mathbf{j}}(y,s_1,\ldots,s_{4-r},u_1,\ldots,u_r) du_1 \ldots du_r \right) dxdy. \tag{7.11}$$

Consequently, for some numerical constant $C > 0$, we can write

$$||\widetilde{g}_n \otimes_r \widetilde{g}_n||_{L^2([0,1]^{8-2r})}^2 \leq C \cdot (k_{-1}^n \cdot k_1^n)^4 \cdot \max_{\mathbf{i},\mathbf{j}\in\mathbb{N}^6, |\mathbf{i}|=|\mathbf{j}|=4} B_{\mathbf{ij}}^n, \tag{7.12}$$

where $B_{\mathbf{ij}}^n$ denotes

$$\left|\left| \int_{\mathcal{D}^2} \left(\int_{[0,1]^r} f_n^{\mathbf{i}}(x,u_1,\ldots,u_r,t_{r+1},\ldots,t_4) f_n^{\mathbf{j}}(y,u_1,\ldots,u_r,s_{r+1},\ldots,s_4) du_1 \ldots du_r \right) dxdy \right|\right|. \tag{7.13}$$

Here, the norm $|| \cdot ||$ is in the sense of $L^2([0,1]^{8-2r}, dt_{r+1}\ldots dt_4 ds_{r+1} \ldots ds_4)$. The maximum of $B_{\mathbf{ij}}^n$ can be upper-bounded by the maximum of the quantities $C_{\mathbf{p},\mathbf{q},\mathbf{l},\mathbf{m}}^n$

$$\left|\left| \int_{\mathcal{D}^2} \prod_{v=1}^r \int_0^1 f_{p_v,l_v}(k_{p_v}^n x, u) f_{q_v,m_v}(k_{q_v}^n y, u) du \cdot \prod_{v=r+1}^4 f_{p_v,l_v}(k_{p_v}^n x, t_v) f_{q_v,m_v}(k_{q_v}^n y, z_v) dxdy \right|\right|, \tag{7.14}$$

where $\mathbf{p}, \mathbf{q} \in \{-1,1\}^{\otimes 4}$ and $\mathbf{l}, \mathbf{m} \in \{-1,0,1\}^{\otimes 4}$. Indeed, this follows directly from definition (7.6) (see also example (7.8)). As a consequence of the isometry (7.5), we have

$$\begin{aligned}\prod_{v=1}^{r} \int_0^1 f_{p_v,l_v}(k_{p_v}^n x, u) f_{q_v,m_v}(k_{q_v}^n y, u) du &= \prod_{v=1}^{r} \mathbb{E}\left[\tilde{\partial}_{l_v} b_{k_{p_v}}(x) \cdot \tilde{\partial}_{m_v} b_{k_{q_v}}(y)\right] \\ &= \prod_{v=1}^{r} \delta_{p_v,q_v} \cdot r_{l_v,m_v}(k_{p_v}^n (x-y)).\end{aligned} \tag{7.15}$$

Consequently, $C^n_{\mathbf{p},\mathbf{q},\mathbf{l},\mathbf{m}}$ defined in (7.14) can be bounded by

$$\begin{aligned}D^n_{\mathbf{p},\mathbf{q},\mathbf{l},\mathbf{m}} := \int_{\mathcal{D}^4} &\prod_{v=1}^{4} r_{l_v,m_v}(k_{p_v}^n (x-y)) \cdot r_{l_v,m_v}(k_{q_v}^n (\tilde{x}-\tilde{y})) \\ &\times \prod_{v=r+1}^{4} r_{l_v,l_v}(k_{p_v}^n (x-\tilde{x})) \cdot r_{l_v,l_v}(k_{p_v}^n (y-\tilde{y})) \\ &\times \prod_{v=r+1}^{4} r_{m_v,m_v}(k_{q_v}^n (x-\tilde{x})) \cdot r_{m_v,m_v}(k_{q_v}^n (y-\tilde{y})) dx dy d\tilde{x} d\tilde{y}.\end{aligned} \tag{7.16}$$

Here, for instance, we have also used the fact that

$$\int_{[0,1]^{4-r}} \prod_{v=r+1}^{4} f_{p_v,l_v}(k_{p_v}^n x, t_v) \cdot f_{p_v,l_v}(k_{p_v}^n \tilde{x}, t_v) dt_{r+1} \dots dt_4 = \prod_{v=r+1}^{4} r_{l_v,l_v}(k_{p_v}^n (x-\tilde{x})). \tag{7.17}$$

which holds for the same reasons as (7.15). (We note that, the only reasons for which (7.16) is an upper bound and not an equality, is that we had disregarded the products of the Kronecker's delta symbols.) Finally, we arrive at the bound

$$||\widetilde{g}_n \otimes_r \widetilde{g}_n||^2_{L^2([0,1]^{8-2r})} \leq C \cdot (k_{-1}^n \cdot k_1^n)^4 \cdot L_n, \tag{7.18}$$

where, with $D^n_{\mathbf{p},\mathbf{q},\mathbf{l},\mathbf{m}}$ denoting the integrals given in (7.16), we have defined

$$L_n := \max\{D^n_{\mathbf{p},\mathbf{q},\mathbf{l},\mathbf{m}} : \mathbf{p}, \mathbf{q} \in \{-1,1\}^{\otimes 4}, \quad \mathbf{l}, \mathbf{m} \in \{-1,0,1\}^{\otimes 4}\}. \tag{7.19}$$

Step 4. The preceding point has yielded an estimate on the contraction norms that now we will be able to turn into an upper bound which shows that the properly normalised contraction norms are vanishing in the limit. We observe that, for some numerical constant $C > 0$, the quantity L_n defined in (7.19) satisfies

$$\begin{aligned}L_n \leq C \cdot \max_{p \in \{-1,1\}} \max_{a \in \{0,1,2\}} \int_{\mathcal{D}^4} &|J_a(k_p^n (x-y))|^r \cdot |J_a(k_p^n (\tilde{x}-\tilde{y}))|^r \\ &\cdot |J_a(k_p^n (x-\tilde{x}))|^{4-r} \cdot |J_a(k_p^n (y-\tilde{y}))|^{4-r} dx dy d\tilde{x} d\tilde{y}\end{aligned} \tag{7.20}$$

It has been shown in [70, p. 131, Lemma 8.1] that for some numerical constant $C > 0$ we have for each $a \in \{0,1,2\}$ and $p \in \{-1,1\}$ that

$$\int_{\mathcal{D}^4} |J_a(k_p^n(x-y))|^r \cdot |J_a(k_p^n(\tilde{x}-\tilde{y}))|^r \cdot |J_a(k_p^n(x-\tilde{x}))|^{4-r} \cdot |J_a(k_p^n(y-\tilde{y}))|^{4-r} dx dy d\tilde{x} d\tilde{y} \\ \leq C \cdot (1+\mathrm{diam}(\mathcal{D})^2) \cdot \frac{\ln k_p^n}{(k_p^n)^4}. \tag{7.21}$$

Thus, for some numerical constant $C > 0$ we have

$$||\widetilde{g}_n \otimes_r \widetilde{g}_n||^2_{L^2([0,1]^{8-2r})} \leq C \cdot (1+\mathrm{diam}(\mathcal{D})^2) \cdot K_n^4 \ln K_n. \tag{7.22}$$

Step 5. We recall from (7.9) that $\widetilde{g}_n = I_4(Y_n[4])$ and we let Z denote a standard normal Gaussian random variable. Suppose first that the asymptotic ratio r defined in (4.2) is strictly positive. Using Theorem 3.3 we have that, for some numerical constant $C > 0$, we have

$$\begin{aligned} W_1\left(\frac{Y_n[4]}{\sqrt{\mathrm{Var}(Y_n[4])}}, Z\right) &\leq C \cdot \max_{1\leq r\leq 3} \left|\left| \frac{\widetilde{g}_n}{\sqrt{\mathrm{Var}Y_n[4]}} \otimes_r \frac{\widetilde{g}_n}{\sqrt{\mathrm{Var}Y_n[4]}} \right|\right|_{L^2([0,1]^{8-2r})} \\ &\leq C \cdot (1+\mathrm{diam}(\mathcal{D})) \cdot \frac{K_n^2 \sqrt{\ln K_n}}{\mathrm{Var}Y_n[4]} \\ &= \frac{C \cdot (1+\mathrm{diam}(\mathcal{D})) \cdot \delta_n^2}{\sqrt{\ln K_n}} \end{aligned} \tag{7.23}$$

where we have also used (7.22). Now, if $r = 0$, then the nodal number is fully correlated with the nodal length (see Lemma 6.5) and we can refer to the estimates obtained in [70] which yield analogous bounds.

Step 6. We have now reached the final stage of the proof and have gathered all the ingredients necessary to establish the quantitative CLT. Let Z denote standard Gaussian random variable and recall the auxilliary notation $Y_n = \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})$. We observe that

$$\begin{aligned} W_1\left(\frac{Y_n - \mathbb{E}Y_n}{\sqrt{\mathrm{Var}Y_n}}, Z\right) &\leq W_1\left(\frac{Y_n - \mathbb{E}Y_n}{\sqrt{\mathrm{Var}Y_n}}, \frac{Y_n[4]}{\sqrt{\mathrm{Var}Y_n[4]}}\right) + W_1\left(\frac{Y_n[4]}{\sqrt{\mathrm{Var}Y_n[4]}}, Z\right) \\ &\leq \sqrt{\mathbb{E}\left(\frac{Y_n - \mathbb{E}Y_n}{\sqrt{\mathrm{Var}Y_n}} - \frac{Y_n[4]}{\sqrt{\mathrm{Var}Y_n[4]}}\right)^2} + W_1\left(\frac{Y_n[4]}{\sqrt{\mathrm{Var}Y_n[4]}}, Z\right) \\ &\leq \frac{L \cdot \gamma_n}{\sqrt{\ln K_n}}, \end{aligned} \tag{7.24}$$

where the first term has been bounded using (7.2) and the second term using (7.23). This concludes the proof of Theorem 4.2. □

7.2 Proof of the Multivariate Central Limit Theorem

In the subsequent proof of the multivariate Central Limit Theorem (Theorem 4.3), we incorporate the fundamental components from the univariate CLT's proof (Theorem 4.2). While it necessitates extra technical effort, the core section of the proof is accomplished by leveraging the results contained in [92].

Proof of Theorem 4.3. The proof is structured into four distinct steps. The initial step involves approximating through the 4-chaotic projection. Subsequently, the second step addresses the convergence of covariances. The third step provides a control on the $\mathbf{W}_1$ distance between the centred Gaussian vectors $\mathbf{Z}_n$ and $\mathbf{Z}$ which have covariances Σ^n and Σ, respectively (see (4.10)). In the last step we combine the Multivariate 4-th Moment Theorem on the Wiener Chaos with the preceeding observations and prove (4.11)-(4.12), conluding the argument.

Step 1. We recall that $\mathbf{Z} = (Z_1, \dots, Z_m)$ denotes a centred Gaussian vector such that for each $1 \le i, j \le m$ we have $\mathrm{Cov}(Z_i, Z_j) = \mathrm{area}(\mathcal{D}_i \cap \mathcal{D}_j)$, and that we use the notation

$$Y_n^i = \mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}_i), \qquad \mathbf{Y}_n = (Y_n^1, \dots, Y_n^m), \tag{7.25}$$

where $i = 1, \dots, m$. We note that

$$\begin{aligned}
&\mathbf{W}_1\left(\frac{\mathbf{Y}_n - \mathbb{E}\mathbf{Y}_n}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}\right) \\
&= \mathbf{W}_1\left(\frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}} + \frac{\sum_{q\ge 3}\mathbf{Y}_n[2q]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}\right) \\
&\le \mathbf{W}_1\left(\frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}} + \frac{\sum_{q\ge 3}\mathbf{Y}_n[2q]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}\right) \\
&\qquad + \mathbf{W}_1\left(\frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}\right) \\
&\le \sum_{i=1}^m \mathbb{E}\left|\sum_{q\ge 3}\frac{Y_n^i[2q]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}\right| + \mathbf{W}_1\left(\frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}\right) \\
&\le \frac{\sum_{i=1}^m \sqrt{\sum_{q\ge 3}\mathrm{Var}Y_n^i[2q]}}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}} + \mathbf{W}_1\left(\frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}\right) \\
&\le \frac{L \cdot \sum_{i=1}^m (1 + \mathrm{diam}(\mathcal{D}_i)^2)}{\sqrt{C_\infty \cdot \ln K_n}} + \mathbf{W}_1\left(\frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}\right),
\end{aligned} \tag{7.26}$$

where L is a numerical constant we can obtain thanks to Lemma 6.1.

Step 2. We note the following extension of the Lemma 6.10: for each $1 \leq i, j \leq m$ we have

$$\lim_{n\to\infty} \mathrm{Cov}\left(\frac{Y_n^i[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \frac{Y_n^j[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}\right) = \mathrm{area}(\mathcal{D}_i \cap \mathcal{D}_j). \tag{7.27}$$

This follows by a tedious but straightforward adaptation of the argument given in [78, p. 1006, Proposition 5.1], in [78, p. 1006, Proposition 5.2], and in [78, p. 1011, Proof of Theorem 3.2].

Step 3. We recall that the matrices Σ^n and Σ are defined by setting for each $1 \leq i, j \leq m$

$$\Sigma_{ij}^n = \mathrm{Cov}(Y_i^n, Y_j^n), \qquad \Sigma_{ij} = \mathrm{area}(\mathcal{D}_i \cap \mathcal{D}_j). \tag{7.28}$$

Suppose that the matrix Σ is strictly positive definite. We had already proved that for each $1 \leq i, j \leq m$ we have $\Sigma_{ij}^n \to \Sigma_{ij}$ and so, for n sufficiently large, it must be that Σ^n is strictly positive definite. Then, using [69, p. 126, Eq. (6.4.2)], we have that for two centred Gaussian vectors $\mathbf{Z}^n = (Z_n^1, \dots, Z_n^m) \sim \mathcal{N}_m(0, \Sigma^n)$, $\mathbf{Z} = (Z^1, \dots, Z^m) \sim \mathcal{N}_m(0, \Sigma)$, we have a bound

$$\mathbf{W}_1(\mathbf{Z}_n, \mathbf{Z}) \leq \mathrm{M}(\Sigma^n, \Sigma) \cdot ||\Sigma^n - \Sigma||_{\mathrm{HS}}, \tag{7.29}$$

where

$$\mathrm{M}(\Sigma^n, \Sigma) := \sqrt{m} \cdot \min\left\{||(\Sigma^n)^{-1}||_{\mathrm{op}} \cdot ||\Sigma^n||_{\mathrm{op}}^{1/2}, ||\Sigma^{-1}||_{\mathrm{op}} \cdot ||\Sigma||_{\mathrm{op}}^{1/2}\right\}. \tag{7.30}$$

Here, 'HS' and 'op' stand respectively for 'Hilbert-Schmidt' and 'operator', see **N.5**.

Step 4. In this last element of the proof, we combine observations about the asymptotic covariance structure made in the two preceding points with the information about the deterministic constants associated with each of the relevant random integrals. Since each $Y_n^i[4]$ is an element of the 4-th Wiener Chaos we can find functions $f_n^i \in L_s^2([0,1]^4)$ such that $Y_n^i[4] = I_4(f_n^i)$. Then, we can use the multivariate version of the 4-th Moment Theorem ([69, p. 121, Theorem 6.2.2]), in the form recorded in Theorem 3.4, to obtain

$$\begin{aligned} d_{C^2}\left(\frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}_n\right) &\leq \frac{C_4 \cdot \sum_{i=1}^m \sum_{r=1}^3 ||f_n^i \otimes_r f_n^i||_{L^2([0,1]^{8-2r})}}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}} \\ &\leq \frac{L \cdot \sum_{i=1}^m (1 + \mathrm{diam}(\mathcal{D}_i))}{\sqrt{C_\infty \cdot \ln K_n}}, \end{aligned} \tag{7.31}$$

which, together with (7.26), yields the first of postulated inequalities - (4.11). Similarly, if Σ is strictly positive definite then, for n sufficiently large, Σ_n is strictly positive definite and we can write

$$\begin{aligned} \mathbf{W}_1\left(\frac{\mathbf{Y}_n[4]}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \mathbf{Z}_n\right) &\leq C_4 \cdot m^{3/2} \cdot ||\Sigma_n^{-1}||_{op} \cdot ||\Sigma_n||_{op}^{1/2} \cdot \sum_{i=1}^m \sum_{r=1}^3 ||f_n^i \otimes_r f_n^i||_{L^2([0,1]^{8-2r})} \\ &\leq m^{3/2} \cdot ||(\Sigma^n)^{-1}||_{op} \cdot ||\Sigma^n||_{op}^{1/2} \cdot \frac{L \cdot \sum_{i=1}^m (1 + \mathrm{diam}(\mathcal{D}_i))}{\sqrt{C_\infty \cdot \ln K_n}}, \end{aligned} \tag{7.32}$$

where as before we have used the corresponding 1-dimensional bound (7.23). The second postulated inequality - (4.12) - follows immediately by combining (7.26), (7.29) and (7.32). □

7.3 Proof of the Convergence to The White Noise

Proof of Theorem 4.4. We split the argument into several steps. In the first step, we verify that μ_n is a random generalized function. The proof of convergence to White Noise relies on an abstract result by Fernique, which requires us to verify two conditions. The first condition, checked in Step 2, involves the continuity of the characteristic functional, while Steps 3 and 4 address the second condition, which concerns pointwise convergence when evaluated on test functions.

Step 1. We start by noting that the problem is well-posed. That is, each map

$$\mathcal{S}(\mathbb{R}^2) \ni \varphi \mapsto \int_0^1 \int_0^1 \varphi(t_1,t_2)\mu_n(dt_1dt_2) \tag{7.33}$$

is a.s. a tempered distribution since it is a finite linear combination of the Dirac's delta functions $\delta_{y_l(\omega)}$ (points $y_l(\omega)$ are random) and of the deterministic distribution $\varphi \mapsto \int_0^1\int_0^1 \varphi(t_1,t_2)dt_1dt_2$. Secondly, it is a standard fact that the white noise $W(dt_1dt_2)$ is a random tempered distribution [17, p. 1]. Since $C_c^\infty([0,1]^2) \subset \mathcal{S}(\mathbb{R}^2)$ with topology generated by the same family of semi-norms, it follows that both of these functionals are elements of $(C_c^\infty([0,1]^2))'$.

Step 2. As a consequence of [29, Theorem III.6.5, p. 69] the convergence of random distributions $\mu_n(dt_1dt_2)$ to the white noise $W(dt_1dt_2)$ in the sense of weak and strong topologies is equivalent to the two conditions. The first one is the continuity of the characteristic functional $\varphi \mapsto \mathbb{E}e^{i\langle W,\varphi\rangle}$ on $C_c^\infty([0,1]^2)$. Note first that the convergence of φ_n to φ in the space $C_c^\infty([0,1]^2)$ implies in particular that $\sup_{0\le t_1,t_2\le 1}|\varphi_n(t_1,t_2)-\varphi_n(t_1,t_2)|$ converges to zero. Then, using Gaussianity and isometry $\mathrm{Var}\langle W,\varphi_n\rangle = ||\varphi_n||^2_{L^2([0,1]^2)}$ we obtain as required

$$\mathbb{E}[e^{i\langle W,\varphi_n\rangle}] = e^{-\frac{1}{2}||\varphi_n||^2_{L^2([0,1]^2)}} \xrightarrow{n\to\infty} e^{-\frac{1}{2}||\varphi||^2_{L^2([0,1]^2)}} = \mathbb{E}[e^{i\langle W,\varphi\rangle}]. \tag{7.34}$$

Step 3. The second condition required by [29, p. 69, Thm. III.6.5] is the convergence in law

$$\langle \mu_n(dt_1dt_2),\varphi\rangle \xrightarrow{d} \langle W(dt_1dt_2),\varphi\rangle \tag{7.35}$$

for every test function $\varphi \in C_c^\infty([0,1]^2)$. The construction as in [36, p. 13-21, 2.1 White noise] provides a version of a white noise as a random integral i.e. the postulated convergence can be written as

$$\int_0^1\int_0^1 \varphi(t_1,t_2)\mu_n(dt_1dt_2) \xrightarrow{d} \int_0^1\int_0^1 \varphi(t_1,t_2)W(dt_1dt_2). \tag{7.36}$$

One can use integration by parts for Wiener-Ito integrals as in [36, p. 18, Eq. (2.1.15) and (2.1.16)] to obtain equality in L^2

$$\int_0^1 \int_0^1 B_{t_1,t_2} \cdot \frac{\partial^2}{\partial t_1 \partial t_2} \varphi(t_1,t_2) dt_1 dt_2 = \int_0^1 \int_0^1 \varphi(t_1,t_2) W(dt_1 dt_2), \tag{7.37}$$

where the process B_{t_1,t_2} denotes the Wiener sheet. We adopt the notation

$$B^n_{t_1,t_2} = \frac{\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, [0,t_1] \times [0,t_2]) - \frac{k_n K_n}{4\pi} \cdot t_1 t_2}{\sqrt{C_\infty \cdot K_n^2 \ln K_n}}, \tag{7.38}$$

where C_∞ is the constant defined in (4.5). We observe that the convergence in distribution

$$\int_0^1 \int_0^1 B^n_{t_1,t_2} \cdot \frac{\partial^2}{\partial t_1 \partial t_2} \varphi(t_1,t_2) dt_1 dt_2 \xrightarrow{d} \int_0^1 \int_0^1 B_{t_1,t_2} \cdot \frac{\partial^2}{\partial t_1 \partial t_2} \varphi(t_1,t_2) dt_1 dt_2 \tag{7.39}$$

will follow immediately by [39, p. 20, Theorem 4] once we verify the three corresponding assumptions. The first condition is a convergence of stochastic processes in the sense of finite dimensional distributions

$$(B^n_{t_1,t_2})_{0 \le t_1,t_2 \le 1} \longrightarrow (B_{t_1,t_2})_{0 \le t_1,t_2 \le 1}, \tag{7.40}$$

which means that for every choice of $m \in \mathbb{N}$ and $0 \le t_1, t_2, \ldots, t_{2m-1}, t_{2m} \le 1$, we have a convergence in distribution of random vectors

$$(B^n_{t_1,t_2}, B^n_{t_1,t_2}, \ldots, B^n_{t_{2m-1},t_{2m}}) \xrightarrow{d} (B_{t_1,t_2}, B_{t_1,t_2}, \ldots, B_{t_{2m-1},t_{2m}}). \tag{7.41}$$

This however is a special case of Theorem 4.3 with a choice of domains

$$\mathcal{D}_1 = [0,t_1] \times [0,t_2], \mathcal{D}_2 = [0,t_3] \times [0,t_4], \ldots, \mathcal{D}_m = [0,t_{2m-1}] \times [0,t_{2m}].$$

The second condition is that for each $0 \le t_1, t_2 \le 1$ we have a convergence of second moments $\mathbb{E}(B^n_{t_1,t_2})^2 \to \mathbb{E}B^2_{t_1,t_2}$, which follows from Theorem 4.1 applied to the domain $\mathcal{D} = [0,t_1] \times [0,t_2]$. The last condition is that

$$\lim_{n\to\infty} \sup_{0 \le t_1,t_2 \le 1} \mathbb{E}(B^n_{t_1,t_2})^2 < \infty. \tag{7.42}$$

We note that, using Lemma 6.1 and following the strategy used in the proof of Lemma 6.8, we can find the numerical constants $0 < C_1 \le C_2 \le C_3 \le C_4$ s.t. for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\sup_{0 \le t_1,t_2 \le 1} \mathbb{E}(B^n_{t_1,t_2})^2 \\
&\quad = \frac{\sup_{0 \le t_1,t_2 \le 1} \operatorname{Var}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D}))}{C_\infty \cdot K_n^2 \ln K_n} \\
&\quad \le \frac{C_1}{r^{log} \ln K_n} \cdot \left(1 + \frac{\sup_{0 \le t_1,t_2 \le 1} \operatorname{Var}(\mathcal{N}(b_{k_n}, \hat{b}_{K_n}, \mathcal{D})[4])}{K_n^2}\right) \\
&\quad \le \frac{C_2}{r^{log}} \cdot \left(1 + \max_{i,j \in \{-1,0,1\}} \int_{B(0,2)} r_{ij}(k_n z)^4 dz + \max_{i,j \in \{-1,0,1\}} \int_{B(0,2)} r_{ij}(K_n z)^4 dz\right) \\
&\quad \le \frac{C_3}{r^{log} \ln K_n} \cdot (1 + \ln K_n) \le \frac{C_4}{r^{log}}.
\end{aligned} \tag{7.43}$$

Step 4. Now we will show that the left-hand side of the equation (7.39) is exactly as needed to deduce the convergence postulated in (7.36) from formula (7.37). That is, that we have

$$\begin{aligned}\langle \mu_n(dt_1dt_2), \varphi\rangle &:= \int_0^1\int_0^1 \varphi(t_1,t_2)\mu_n(dt_1dt_2)\\ &= \int_0^1\int_0^1 B^n_{t_1,t_2}\cdot\frac{\partial^2}{\partial t_1\partial t_2}\varphi(t_1,t_2)dt_1dt_2.\end{aligned}\tag{7.44}$$

We denote $B := \{(s_1,s_2)\in[0,1]^2 : b_{k_n}(s_1,s_2) = \hat{b}_{K_n}(s_1,s_2) = 0\}$ and we observe that

$$\begin{aligned}&\int_0^1\int_0^1 B^n_{t_1,t_2}\cdot\frac{\partial^2}{\partial t_1\partial t_2}\varphi(t_1,t_2)dt_1dt_2\\ &= \frac{1}{\sqrt{C_\infty\cdot K_n^2\ln K_n}}\int_0^1\int_0^1\sum_{(s_1,s_2)\in B}\mathbf{1}_{[0,s_1]}(t_1)\mathbf{1}_{[0,s_2]}(t_2)\cdot\frac{\partial^2}{\partial t_1\partial t_2}\varphi(t_1,t_2)dt_1dt_2\\ &\quad-\frac{k_nK_n}{4\pi\sqrt{C_\infty\cdot K_n^2\ln K_n}}\int_0^1\int_0^1 t_1t_2\cdot\frac{\partial^2}{\partial t_1\partial t_2}\varphi(t_1,t_2)dt_1dt_2.\end{aligned}\tag{7.45}$$

We observe that, for every $(s_1,s_2)\in[0,1]$, using the fact that φ has a compact support contained in $(0,1)^2$ and integrating we find that

$$\begin{aligned}\int_0^1\int_0^1\mathbf{1}_{[0,s_1]}(t_1)\mathbf{1}_{[0,s_2]}(t_2)\cdot\frac{\partial^2}{\partial t_1\partial t_2}\varphi(t_1,t_2)dt_1dt_2 &= \int_0^{s_1}\int_0^{s_2}\frac{\partial^2}{\partial t_1\partial t_2}\varphi(t_1,t_2)dt_1dt_2\\ &= \int_0^{s_1}\frac{\partial}{\partial t_1}\varphi(t_1,s_2)-\frac{\partial}{\partial t_1}\varphi(t_1,0)dt_1\\ &= \varphi(s_1,s_2).\end{aligned}\tag{7.46}$$

Similarly,

$$\begin{aligned}&\int_0^1\int_0^1 t_1t_2\cdot\frac{\partial^2}{\partial t_1\partial t_2}\varphi(t_1,t_2)dt_1dt_2\\ &= \int_0^1 t_1t_2\cdot\frac{\partial}{\partial t_1}\varphi(t_1,t_2)\Big|_{t_2=0}^{t_2=1}dt_1 - \int_0^1\int_0^1 t_1\cdot\frac{\partial}{\partial t_1}\varphi(t_1,t_2)dt_1dt_2\\ &= -\int_0^1\int_0^1 t_1\cdot\frac{\partial}{\partial t_1}\varphi(t_1,t_2)dt_1dt_2\\ &= \int_0^1\int_0^1\varphi(t_1,t_2)dt_1dt_2.\end{aligned}\tag{7.47}$$

Comparing (7.45) with (7.46) and (7.47) we obtain (7.44). We conclude that the proof of Theorem 4.4 has been completed. □

7.4 Proof of the Reduction Principle

In this section, we focus on proving Theorem 4.5. The necessary computations for this proof are intimately connected to those we conducted in proving Lemma 6.6.

Proof of Theorem 4.5. We immediately note that the case $r = 0$ is directly derived from Lemma 6.5 and the established Reduction Principle for nodal length, as outlined in [92, p. 3, Theorem 1.1]. Consequently, our focus shifts to the case where r lies in the interval (0,1]. The overall strategy of the proof is straightforward: after reducing the problem to the analysis of the 4th chaotic projection, we treat the remaining integrals as vectors within a finite-dimensional subspace of L^2. This reformulation transforms the problem into a linear algebra exercise focused on the asymptotic covariance matrix. The argument will be divided into four distinct paragraphs for clarity.

Step 1. We recall the notation **S.1-S.6**. It follows by L^2-equivalence (4.7) that we can restrict our analysis to the 4-th chaotic projection $\mathcal{N}(b_{k^n_{-1}}, b^n_{k_1}, \mathcal{D})[4]$. Furthermore, decomposition

$$\mathcal{N}(b_{k^n_{-1}}, b^n_{k_1}, \mathcal{D})[4] = \frac{1}{\pi\sqrt{2}} \cdot \sum_{p\in\{-1,1\}} k^n_{-p} \cdot \mathcal{L}(b_{k^n_p}, \mathcal{D})[4] + \mathrm{Cross}(\mathcal{N}(b_{k^n_{-1}}, b^n_{k_1}, \mathcal{D})[4]), \tag{7.48}$$

splits this projection into 3 uncorrelated parts (see (4.30)). Consequently, and thanks to a linear nature of the problem, we can analyse each of the terms in (7.48) separately. The first two terms on the right of the postulated formula (4.27) correspond to the first two terms in (7.48). As in the case of $r = 0$ discussed above, the full-correlation and L^2-equivalence for these terms is an immediate consequence of the Reduction Principle for the nodal length. Thus, from now on we only need to focus on the $\mathrm{Cross}(\mathcal{N}(b_{k^n_{-1}}, b_{k^n_1}, \mathcal{D})[4])$ and its relationship with remaining 3 terms on the right-hand side of (4.27).

Step 2. As mentioned at the beginning of this proof, we need to examine the asymptotic covariance matrix. In this step, we will demonstrate how, for our purposes, this matrix can be replaced by a simpler one (7.54). We recall that in Lemma 6.10 we have established the formula (6.100) which yields the asymptotic covariances between different terms contributing to $\mathrm{Cross}(\mathcal{N}(b_{k^n_{-1}}, b^n_{k_1}, \mathcal{D})[4])$. We recall the indexation defined in (6.113) and for every $\mathbf{i}, \mathbf{j} \in T = \{-1, 0, 1\}^{\otimes 2} \cup \{*\}$ we set

$$\begin{aligned} \psi^*_{\mathbf{ij}} &:= (-1)^{|\mathbf{i}|+|\mathbf{j}|} \cdot \widehat{\psi}_{\mathbf{ij}}, \\ \chi_{\mathbf{ij}} &:= \widehat{\psi}_{\mathbf{ij}} + \frac{r^{exp}}{2} \cdot \psi^*_{\mathbf{ij}} = \left(1 + \frac{r^{exp}}{2} \cdot (-1)^{|\mathbf{i}|+|\mathbf{j}|}\right) \cdot \widehat{\psi}_{\mathbf{ij}}. \end{aligned} \tag{7.49}$$

Here, we have used the same notation as in (6.123), that is $\widehat{\psi}_{\mathbf{ij}} = (2^6/\pi) \cdot \psi(\gamma(\mathbf{i}) + \gamma(\mathbf{j}))$, where γ and ψ are as defined in (6.101)–(6.102). Comparing (6.100) with (7.49) we see that, up to rescaling, the former is identical to the later. Thus, in order to understand the structure of correlations between different integrals contributing to $\mathrm{Cross}(\mathcal{N}(b_{k^n_{-1}}, b^n_{k_1}, \mathcal{D})[4])$, it is enough to study matrices

$$\mathbf{V} := [\chi_{\mathbf{ij}}]_{\mathbf{i},\mathbf{j}\in T}, \qquad \Psi^* := [\psi^*_{\mathbf{ij}}]_{\mathbf{i},\mathbf{j}\in T}, \qquad \widehat{\Psi} := [\widehat{\psi}_{\mathbf{ij}}]_{\mathbf{i},\mathbf{j}\in T}, \tag{7.50}$$

where we use the ordering defined in (6.114). We can readily see that the matrices $\mathbf{V}$, Ψ^* and $\widehat{\Psi}$, have the same six groups of identical rows (equivalently, columns): $\{1\}$, $\{2, 4\}$, $\{3, 7, 10\}$, $\{5\}$, $\{6, 8\}$ and $\{9\}$ (for $\mathbf{V}$ and $\widehat{\Psi}$ these groups correspond to asymptotically

L^2-equivalent random integrals). Thus, we can focus instead on reduced 6×6 versions of these matrices, provided that for every matrix we have choose the same representative of each row-group. We note that:

(a) If $r^{exp} = 0$, then, the reduced form of matrix $\mathbf{V}$ is equal to the reduced form of the matrix $\widehat{\Psi}$. This yields the matrix

$$\begin{bmatrix} \begin{array}{c|cccccc} & (-,-) & (-,0) & (-,+) & (0,0) & (0,+) & (+,+) \\ \hline (-,-) & 35 & 40 & 5 & 48 & 8 & 3 \\ (-,0) & 40 & 48 & 8 & 64 & 16 & 8 \\ (-,+) & 5 & 8 & 3 & 16 & 8 & 5 \\ (0,0) & 48 & 64 & 16 & 128 & 64 & 48 \\ (0,+) & 8 & 16 & 8 & 64 & 48 & 40 \\ (+,+) & 3 & 8 & 5 & 48 & 40 & 35 \end{array} \end{bmatrix}. \tag{7.51}$$

(b) We compute that the reduced form of matrix Ψ^* is

$$\begin{bmatrix} \begin{array}{c|cccccc} & (-,-) & (-,0) & (-,+) & (0,0) & (0,+) & (+,+) \\ \hline (-,-) & 35 & -40 & 5 & 48 & -8 & 3 \\ (-,0) & -40 & 48 & -8 & -64 & 16 & -8 \\ (-,+) & 5 & -8 & 3 & 16 & -8 & 5 \\ (0,0) & 48 & -64 & 16 & 128 & -64 & 48 \\ (0,+) & -8 & 16 & -8 & -64 & 48 & -40 \\ (+,+) & 3 & -8 & 5 & 48 & -40 & 35 \end{array} \end{bmatrix}. \tag{7.52}$$

(c) We compute that, if $r^{exp} = 1$, then, the reduced form of matrix $\mathbf{V}$ is equal to

$$\frac{1}{2} \times \begin{bmatrix} \begin{array}{c|cccccc} & (-,-) & (-,0) & (-,+) & (0,0) & (0,+) & (+,+) \\ \hline (-,-) & 105 & 40 & 15 & 144 & 8 & 9 \\ (-,0) & 40 & 144 & 8 & 64 & 48 & 8 \\ (-,+) & 15 & 8 & 9 & 48 & 8 & 15 \\ (0,0) & 144 & 64 & 48 & 384 & 64 & 144 \\ (0,+) & 8 & 48 & 8 & 64 & 144 & 40 \\ (+,+) & 9 & 8 & 15 & 144 & 40 & 105 \end{array} \end{bmatrix}. \tag{7.53}$$

(d) Let us for a moment write $t := r^{exp}$ for the sake of visual simplicity. Combining the preceding points, we obtain that, in general (for any $t = r^{exp} \in [0, 1]$), the reduced form of matrix $\mathbf{V}$ is

$$\frac{1}{2} \times \begin{bmatrix} \begin{array}{c|cccccc} & (-,-) & (-,0) & (-,+) & (0,0) & (0,+) & (+,+) \\ \hline (-,-) & 35(2+t) & 40(2-t) & 5(2+t) & 48(2+t) & 8(2-t) & 3(2+t) \\ (-,0) & 40(2-t) & 48(2+t) & 8(2-t) & 64(2-t) & 16(2+t) & 8(2-t) \\ (-,+) & 5(2+t) & 8(2-t) & 3(2+t) & 16(2+t) & 8(2-t) & 5(2+t) \\ (0,0) & 48(2+t) & 64(2-t) & 16(2+t) & 128(2+t) & 64(2-t) & 48(2+t) \\ (0,+) & 8(2-t) & 16(2+t) & 8(2-t) & 64(2-t) & 48(2+t) & 40(2-t) \\ (+,+) & 3(2+t) & 8(2-t) & 5(2+t) & 48(2+t) & 40(2-t) & 35(2+t) \end{array} \end{bmatrix}. \tag{7.54}$$

Step 3. In this step we will study the rank of the reduced matrix $\mathbf{V}$, starting from the cases $r^{exp} = 0$ and $r^{exp} = 1$, and then proceeding to general scenario. In each case and depending on the rank of the matrix, we will fix a basis of corresponding random integrals and find coefficients in this basis which correspond to the remaining random integrals.

(a) The reduced matrix $\mathbf{V}$ in scenario $r^{exp} = 0$ has been evaluated in (7.51). It is not difficult to check, using basic computational tools such, that, if $r^{exp} = 0$, then the matrix **V** has a rank 3 and that as a corresponding basis of random variables one can choose integrals

$$\int_{\mathcal{D}} H_2(\widetilde{\partial}_i b_{k_n}(x)) \cdot H_2(\widetilde{\partial}_i \hat{b}_{K_n}(x))dx, \qquad i \in \{-1, 0, 1\}. \tag{7.55}$$

Solving for linear coefficients yields the matrix

$$\left[\begin{array}{c|ccc} & (-,0) & (-,+) & (0,+) \\ \hline (-,-) & 1/2 & -1/2 & -1/2 \\ (0,0) & 1/2 & 1/2 & 1/2 \\ (+,+) & -1/2 & -1/2 & 1/2 \end{array}\right], \tag{7.56}$$

where each column gives coefficients for one of the linearly dependent variables. For instance, the column labeled $(-,0)$ in the matrix (7.51) is the following weighted sum of the columns labelled $(-,-)$, $(0,0)$, $(+,+)$:

$$\begin{aligned}(40, 48, 8, 64, 16, 8)^{tr} &= 1/2 \cdot (35, 40, 5, 38, 8, 3)^{tr} \\ &+ 1/2 \cdot (48, 64, 16, 128, 64, 48)^{tr} \\ &- 1/2 \cdot (3, 8, 5, 48, 40, 35)^{tr}.\end{aligned}$$

(b) The reduced matrix **V** in scenario $r^{exp} = 1$ has been evaluated in (7.53). Similarly, one can check that, if $r^{exp} = 1$, then **V** has a rank 5 where we can again choose the column labeled $(-,+)$ as the dependent one and where the linear coefficients are as before (supplemented by 0). That is:

$$\begin{aligned}1/2 \cdot (15, 8, 9, 48, 8, 15)^{tr} &= -1/4 \cdot (15, 8, 9, 48, 8, 15)^{tr} \\ &+ 1/4 \cdot (105, 40, 15, 144, 8, 9)^{tr}\ , \\ &- 1/4 \cdot (9, 8, 15, 144, 40, 105)^{tr},\end{aligned}$$

while the remaining 5 columns are linearly independent.

(c) The form of matrix **V** for $r^{exp} \in (0, 1)$ has been given in (7.54). We observe that in this situation the matrix **V** has the same structure of linear dependency as we observed when we had $r^{exp} = 1$. To see this note first that the parameter r^{exp} affects identically columns in each of the groups: $\{1, 3, 5, 6\}$, $\{2, 4\}$. This yields the -1 rank reduction and re-use of the coefficients (as above) for the first group of columns. Going further, it is not too difficult to verify that the corresponding (reduced) 5×5 matrix has zero determinant if and only if $r^{exp} = 0$ (for this computation, it is convenient to divide each row by $2 + r^{exp}$ and parametrise with $s = (2 - r^{exp})/(2 + r^{exp})$).

Step 4. In this last element of the proof, we combine observations about the asymptotic covariance structure made in the two preceding points with the information about the deterministic constants associated with each of the relevant random integrals. Taking into consideration the identical columns (rows) in the matrix **V** and the deterministic coefficients $\eta_{\mathbf{j}}$ (6.55), we obtain

$$\begin{aligned} &\frac{k_{-1}^n \cdot k_1^n}{128\pi} \times \left(\begin{array}{c|c|c|c|c|c} (-,-) & (-,0) & (-,+) & (0,0) & (0,+) & (+,+) \\ \hline -1 & -8 & -2 & 8 & -8 & -1 \end{array} \right) \\ &= -\frac{k_{-1}^n \cdot k_1^n}{192\pi} \times \left(\begin{array}{c|c|c|c|c|c} (-,-) & (-,0) & (-,+) & (0,0) & (0,+) & (+,+) \\ \hline 3/2 & 12 & 3 & -12 & 12 & 3 \end{array} \right). \end{aligned} \tag{7.57}$$

Using linear coefficients (7.56) we obtain the matrix

$$-\frac{k_{-1}^n \cdot k_1^n}{192\pi} \times \left[\begin{array}{c|cccccc} & (-,-) & (0,0) & (+,+) & (-,0) & (-,+) & (0,+) \\ \hline (-,-) & 3/2 & 0 & 0 & 6 & -3/2 & -6 \\ (0,0) & 0 & -12 & 0 & 6 & 3/2 & 6 \\ (+,+) & 0 & 0 & 3/2 & -6 & -3/2 & 6 \end{array} \right], \tag{7.58}$$

where, for each row, the sum over columns yields the final constant that appears in postulated formula, next to the relevant random integral. Similarly, for $r^{exp} \in (0,1]$ we obtain the matrix

$$-\frac{k_{-1}^n \cdot k_1^n}{192\pi} \times \left[\begin{array}{c|cccccc} & (-,-) & (0,0) & (+,+) & (-,0) & (-,+) & (0,+) \\ \hline (-,-) & 3/2 & 0 & 0 & 0 & -3/2 & 0 \\ (0,0) & 0 & -12 & 0 & 0 & 3/2 & 0 \\ (+,+) & 0 & 0 & 3/2 & 0 & -3/2 & 0 \\ (-,0) & 0 & 0 & 0 & 12 & 0 & 0 \\ (0,+) & 0 & 0 & 0 & 0 & 0 & 12 \end{array} \right], \tag{7.59}$$

where the sum of each row plays the same role as in the case of the previous matrix. This completes the proof. □

7.5 Derivation of the Recurrence Trick

The following technical lemma is essential for the proof of Lemma 4.1. It can be applied in the situation where one works with n independent Gaussian random waves on $\mathbb{R}^n$ but we will only use it in the simplest case $n = 2$.

Lemma 7.1. *Let X be a $n \times n$ matrix of independent standard Gaussian random variables and let $\hat{X}$ denote a matrix obtained from the matrix X by removing the first row. Then,*

$$\mathbb{E}\left[|\det X|\right] = \sqrt{\frac{2}{\pi}} \cdot \mathbb{E}\left[\sqrt{\det(\hat{X}\hat{X}^{tr})}\right]. \tag{7.60}$$

Proof. By the Laplace expansion of the determinant,

$$\det X = \sum_{j=1}^{n} (-1)^{1+j} X_{1j} \det M_{1j}^X, \tag{7.61}$$

where M_{1j}^X denotes the matrix created out of the matrix X by removing its first row and its j-th column. Thus, conditionally on the random matrix $\hat{X}$, the sum on the right-side of (7.61) defines a centred Gaussian random variable with variance

$$\sigma^2(\hat{X}) = \sum_{j=1}^{n} (\det M_{1j}^X)^2. \tag{7.62}$$

This implies that

$$\mathbb{E}\left[\left|\sum_{j=1}^{d} (-1)^{1+j} X_{1j} \det M_{1j}^X\right|\right] = \sqrt{\frac{2}{\pi}} \cdot \mathbb{E}[\sigma(\hat{X})]. \tag{7.63}$$

Let $M_j^{\hat{X}}$ denote the matrix formed out of the matrix $\hat{X}$ by removing the j-th column. Using the Cauchy-Binet's identity [65, p. 1166, Eq. (B.2)] and the fact that $M_{1j}^X = M_j^{\hat{X}}$ we obtain

$$\begin{aligned} \sqrt{\det(\hat{X}\hat{X}^{tr})} &= \sqrt{\sum_{j=1}^{n} (\det M_j^{\hat{X}})^2} \\ &= \sqrt{\sum_{j=1}^{n} (\det M_{1j}^X)^2} = \sigma(\hat{X}), \end{aligned} \tag{7.64}$$

which is enough to complete the proof. □

The next proof is written using the notation introduced in **S.1-S.6**.

Proof of Lemma 4.1. Our argument is based on a term-by-term comparison of the chaotic decomposition of the nodal number, computed in Lemma 5.2, with the chaotic decomposition of the nodal length, as given in Lemma 5.3. We recall that the corresponding deterministic constant coefficients were denoted $c_{\mathbf{j}}$ in the case of a nodal number $N(b_{k_{-1}}, b_{k_1}, \mathcal{D})$, and, in the case of a nodal length $\mathcal{L}(b_k, \mathcal{D})$, we have used instead $\hat{c}_j$. We will continue this convention here. Given $j = (j_{-1}, j_0, j_1) \in \mathbb{N}^3$ we will write $(j, 0)$ and $(0, j)$ to denote elements of $\mathbb{N}^6$ defined by the formulas

$$\begin{aligned} (j, 0) &:= (i_{-1,-1}; i_{-1,0}; i_{-1,1}; 0; 0; 0), \\ (0, j) &:= (0; 0; 0; i_{1,-1}; i_{1,0}; i_{1,1}), \end{aligned} \tag{7.65}$$

where

$$i_{-1,-1} = i_{1,-1} := j_{-1}, \qquad i_{-1,0} = i_{1,0} := j_0, \qquad i_{-1,1} = i_{1,1} := j_1. \tag{7.66}$$

We will also use the fact that $c_{(j,0)} = c_{(0,j)}$ which is a an immediate consequence of the fact that the constants $c_{\mathbf{j}}$ are independent of the wave-numbers. Let $Z_{-1,-1}$, $Z_{-1,1}$, $Z_{1,-1}$, $Z_{1,1}$ be four independent standard Gaussian random variables. It follows by the case $n = 2$ of Lemma 7.1, that for each $p \in \{-1, 1\}$, we have

$$\begin{aligned} &\mathbb{E}\left[\sqrt{Z_{p,-1}^2 + Z_{p,1}^2} \cdot H_{j_{p,-1}}(Z_{p,-1}) H_{j_{p,-1}}(Z_{p,1})\right] = \sqrt{\frac{\pi}{2}} \times \\ &\mathbb{E}\left[\left|\det \begin{bmatrix} Z_{-1,-1} & Z_{-1,1} \\ Z_{1,-1} & Z_{1,1} \end{bmatrix}\right| \cdot H_{j_{-1,-1}}(Z_{-1,-1}) H_{j_{-1,1}}(Z_{-1,1}) H_{j_{1,-1}}(Z_{1,-1}) H_{j_{1,1}}(Z_{1,1})\right]. \end{aligned} \tag{7.67}$$

Consequently, in the notation (7.65), we have that

$$c_{(j,0)} = \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \hat{c}_j = \frac{1}{\pi\sqrt{2}} \cdot \hat{c}_j. \tag{7.68}$$

Thanks to the preceding computations, we observe that for every $q \geq 1$ we have

$$\begin{aligned}
&N(b_{k_{-1}}, b_{k_1}, \mathcal{D})[2q] \\
&\quad = (k_{-1} \cdot k_1) \cdot \sum_{\mathbf{j} \in \mathbb{N}^6, |\mathbf{j}|=2q} c_{\mathbf{j}} \int_{\mathcal{D}} \prod_{p \in \{-1,1\}} \prod_{i \in \{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p}(x)) dx \\
&\quad = (k_{-1} \cdot k_1) \cdot \sum_{p \in \{-1,1\}} \sum_{j \in \mathbb{N}^3, |j|=2q} c_{(j,0)} \int_{\mathcal{D}} \prod_{i \in \{-1,0,1\}} H_{j_{p,i}}(\widetilde{\partial}_i b_{k_p}(x)) dx \\
&\qquad + \mathrm{Cross}(N(b_{k_{-1}}, b_{k_1}, \mathcal{D})[2q]) \\
&\quad = \frac{1}{\pi\sqrt{2}} \cdot \sum_{p \in \{-1,1\}} k_{-p} \cdot \left(k_p \cdot \sum_{j \in \mathbb{N}^3, |j|=2q} \hat{c}_j \int_{\mathcal{D}} \prod_{i \in \{-1,0,1\}} H_{j_i}(\widetilde{\partial}_i b_{k_p}(x)) dx \right) \\
&\qquad + \mathrm{Cross}(N(b_{k_{-1}}, b_{k_1}, \mathcal{D})[2q]) \\
&\quad = \frac{1}{\pi\sqrt{2}} \cdot \sum_{p \in \{-1,1\}} k_{-p} \cdot \mathcal{L}(b_{k_p}, \mathcal{D})[2q] + \mathrm{Cross}(N(b_{k_{-1}}, b_{k_1}, \mathcal{D})[2q]).
\end{aligned} \tag{7.69}$$

The last step is to determine the value of $\mathrm{Cross}(N(b_{k_{-1}}, b_{k_1}, \mathcal{D})[0])$. We must have

$$\begin{aligned}
\mathrm{Cross}(N(b_{k_{-1}}, b_{k_1}, \mathcal{D})[0]) &= \mathbb{E}\mathrm{Cross}(N(b_{k_{-1}}, b_{k_1}, \mathcal{D})) \\
&= \mathbb{E}N(b_{k_{-1}}, b_{k_1}, \mathcal{D}) - \frac{1}{\pi\sqrt{2}} \sum_{p \in \{-1,1\}} k_{-p} \cdot \mathbb{E}\mathcal{L}(b_{k_p}, \mathcal{D}) \\
&= \frac{\mathrm{area}(\mathcal{D})}{4\pi} \cdot \prod_{p \in \{-1,1\}} k_p - \frac{1}{\pi\sqrt{2}} \cdot \sum_{p \in \{-1,1\}} k_{-p} \cdot \frac{\mathrm{area}(\mathcal{D})}{2\sqrt{2}} k_p \\
&= -\frac{\mathrm{area}(\mathcal{D})}{4\pi} \cdot \prod_{p \in \{-1,1\}} k_p \\
&= -\mathbb{E}N(b_{k_{-1}}, b_{k_1}, \mathcal{D}),
\end{aligned} \tag{7.70}$$

where the expected value of the nodal number is taken from Lemma 5.4 and the expected value of the nodal length is taken from [70, p. 103, Theorem 1.1]. ☐

Chapter 8 Ongoing Work and Some Open Problems

8.1 Overview

The purpose of this chapter is to present some ongoing work, aimed at investigating how the ideas and techniques developed in this thesis might be applicable to related problems (see Chapter 2). Naturally, the extent to which these preliminary insights will prove useful in future research remains to be seen, and this will serve as an important test of the value of the work presented here. In particular, we will focus on Berry's Random Waves in the Euclidean space $\mathbb{R}^d$, $d \geq 3$.

It is also clear that similar questions could also be asked regarding Arithmetic Random Waves, Random Waves on generic compact Riemannian manifolds, or Pullback Random Waves. Perhaps even more important would be to establish a clear and simple description of the relationship between the parameters introduced in this thesis and those used in experimental models of Quantum Chaos or in Brain Imaging studies (see, respectively, Sections 2.1.3–2.1.4 and Section 2.1.7). Both these venues require considerable technical expertise and a deep understanding of physics. As such, they fall outside the scope of this thesis and are left for future collaborative or interdisciplinary efforts.

8.2 Higher Dimensional Euclidean Spaces

As described above, in this work we have focused on the planar case $d = 2$. We are interested in the question of extending this work to arbitrary dimensions $d \geq 3$. It is generally believed that, in principle, the behavior of this model should be easier to analyse than the two-dimensional model. However, as of writing, the only known result in this direction concerns two identical frequency random waves in dimension $d = 3$. Here, the CLT and asymptotic order of variance are known [19, 18]. However, no exact asymptotic of the variance has been established for any $d \geq 3$ with any number of random waves. It is conceivable that a very skilled application of Kac-Rice formula or standard Wiener Chaos expansion might yield definitive answers for questions of this type. Nevertheless, as of writing, it seems unclear if that is actually the case. In our ongoing attempt at making progress on this problem we started studying the one-wave case using a novel technique of matrix-variate Hermite chaos expansion of Gramm determinant introduced by Notarnicola [66] (see also [57], by Marinucci, Rossi and Todino, where generalised Laguerre polynomials are used to tackle similar problem on the $d \geq 3$ dimensional hypersphere).

There are several equivalent definitions of the class of generalized Hermite polynomials that we are using. The classical definition is constructed through the so-called *Zonal* polynomials. However, a more elementary approach is possible (based on the *generalised rodrigues formula* [66, p. 986, Eq. (3.6)]).

Definition 8.1. *We define the vector-variate Hermite polynomials by the formulas*

$$H_0^d \equiv 1, \tag{8.1}$$

$$H_k^d(x) := \frac{e^{||x||^2/2}}{2^k d(d+2)\dots(d+2k-2)} \cdot \Delta^k e^{-||x||^2/2}, \qquad k \geq 1, \tag{8.2}$$

where $x \in \mathbb{R}^d$ *and*

$$\Delta^k := \left(\frac{\partial^2}{\partial^2 x_1} + \dots + \frac{\partial^2}{\partial^2 x_d} \right)^k. \tag{8.3}$$

We record the following formulas provided in [66, p. 988-989, Eq. (3.16)] for later use

$$H_1^d(X) = \frac{1}{2d} \cdot \sum_{j=1}^{d} H_2(X_j), \tag{8.4}$$

$$H_2^d(X) = \frac{1}{4d(d+2)} \cdot \left(\sum_{1 \leq j \leq d} H_4(X_j) + \sum_{1 \leq i \neq j \leq d} H_2(X_i) H_2(X_j) \right). \tag{8.5}$$

The following is a special case of a more general theorem proved in [66].

Theorem 8.1. *The vector-variate Hermite polynomials form an orthogonal basis of* $L^2(\sigma(||X||))$ *where* $X = (X_1, \dots, X_d)^{tr}$ *is a vector of i.i.d. standard Gaussian random variables.*

The following is a special case of a more general theorem proved in [66, p. 992, Corollary 3.8]. Later, we will be applying it with the choice $X := \widetilde{\nabla} b_\lambda(x)$.

Theorem 8.2. *Let* $X = (X_1, \dots, X_d)^{tr}$ *be a vector of i.i.d. standard Gaussian random variables. Then, we have the following orthogonal decomposition in* $L^2(\sigma(||X||))$

$$||X|| = \sum_{k=0}^{\infty} \widehat{F}(k) \cdot H_k^d(X), \tag{8.6}$$

where the deterministic coefficients $\widehat{F}(k)$ are given by the formulas

$$\widehat{F}(0) = \mathbb{E}||X|| = \sqrt{2} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}, \tag{8.7}$$

$$\widehat{F}(k) = (\mathbb{E}||X||) \times (-1)^k d(d+2)\dots(d+2k-2) \times \sum_{s=0}^{k} \frac{g_d(s)}{(k-s)!}, \quad k \geq 1, \tag{8.8}$$

where

$$g_d(0) := 1, \tag{8.9}$$

$$g_d(s) := \frac{(-1)^s}{s!} \cdot \frac{(d+1)\dots(d+2s-1)}{d(d+2)\dots(d+2s-2)}, \qquad s \geq 1. \tag{8.10}$$

We note that, for example

$$\widehat{F}(0) = \sqrt{2} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}, \tag{8.11}$$

$$\widehat{F}(1) = \sqrt{2} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}, \tag{8.12}$$

$$\widehat{F}(2) = \sqrt{2} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \cdot (-1/2). \tag{8.13}$$

The following proposition adapted from [66, p. 988, Theorem 3.2] clarifies the relationship between the decomposition (8.6) and the classical Wiener chaos expansion into univariate Hermite polynomials.

Lemma 8.1. *Consider the Wiener chaos expansion*

$$||X|| = \sum_{q=0}^{\infty} ||X||[2q] \tag{8.14}$$

then,

$$||X||[2q+1] = 0, \tag{8.15}$$

$$||X||[2q] = \sum_{q_1+\dots+q_d=q} \frac{\mathbb{E}[||X||H_{2q_1}(X_1)\dots H_{2q_d}(X_d)]}{(2q_1)!\dots(2q_l)!} H_{2q_1}(X_1)\dots H_{2q_d}(X_d) \tag{8.16}$$

and moreover

$$||X||[2q] = \widehat{F}(q) H_q^d(X). \tag{8.17}$$

Proof. As noted in [66, p. 986, Eq. (3.8)], the vector-variate Hermite polynomials are related to the Laguerre polynomials by means of the formula

$$L_k^{\frac{d}{2}-1}(||X||^2/2) = \frac{(-1)^k}{d(d+2)\dots(d+2k-2)} H_k^d(X) \tag{8.18}$$

We note that, the formula (8.17) provides immediately the following expression for the q-th vector-variate Hermite polynomial

$$\begin{aligned} &H_q^d(X) = \\ &\widehat{F}(q)^{-1} \cdot \sum_{q_1+\dots+q_d=q} \frac{\mathbb{E}\left[||X||H_{2q_1}(X_1)\dots H_{2q_d}(X_d)\right]}{(2q_1)!\dots(2q_d)!} H_{2q_1}(X_1)\dots H_{2q_d}(X_d) \end{aligned} \tag{8.19}$$

and this further implies

$$\begin{aligned} &\mathbb{E}\left[H_{2q_0}(X_0)H_q^d(X)\cdot H_{2q_0'}(Y_0)H_{q'}^d(Y)\right] = \\ &\widehat{F}(q)^{-1}\widehat{F}(q')^{-1} \sum_{q_1\dots q_d=q}\ \sum_{q_1'+\dots q_d'=q'} \\ &\frac{\mathbb{E}\left[||X||H_{2q_1}(X_1)\dots H_{2q_d}(X_d)\right]}{(2q_1)!\dots(2q_d)!} \cdot \frac{\mathbb{E}[||X||H_{2q_1'}(X_1)\dots H_{2q_d'}(X_d)]}{(2q_1')\dots(2q_d')!} \\ &\times \mathbb{E}[H_{2q_0}(X_0)H_{2q_1}(X_1)\dots H_{2q_d}(X_d)\cdot H_{2q_0'}(Y_0)H_{2q_1'}(Y_1)\dots H_{2q_d'}(Y_d)] \end{aligned} \tag{8.20}$$

Now, we can use the classical diagram formula (see Theorem 3.2) to obtain that

$$\begin{aligned} &\mathbb{E}[H_{2q_0}(X_0)H_q^d(X)\cdot H_{2q_0'}(Y_0)H_{q'}^d(Y)] = \\ &\widehat{F}(q)^{-1}\widehat{F}(q')^{-1} \sum_{q_1\dots q_d=q}\ \sum_{q_1'+\dots q_d'=q'} \\ &\mathbb{E}[||X||H_{2q_1}(X_1)\dots H_{2q_d}(X_d)]\cdot\mathbb{E}[||X||H_{2q_1'}(X_1)\dots H_{2q_d'}(X_d)] \\ &\times \sum_{k_{ij}} \frac{\mathbb{E}[X_iY_j]^{k_{ij}}}{k_{ij}!} \end{aligned} \tag{8.21}$$

where the integer coefficient k_{ij} satisfy the equations as in [57, p. 14, Eq. (5.21)]. □

Lemma 8.2. *Let $b_k = \{b_k(x) : x \in \mathbb{R}^d\}$, $d \geq 2$, be the Berry's Random Wave with wavenumber $k > 0$ (see Section 1.1). Then, the nodal nodal volume $\mathcal{L}(b_k) := \mathcal{H}^{d-1}(B(0,1) \cap b_k^{-1}(0))$ has the following orthogonal Wiener chaos decomposition in $L^2(\mathbb{P})$*

$$\mathcal{L}(b_k) = \sum_{q=0}^{\infty} \mathcal{L}(b_k)[2q], \tag{8.22}$$

where

$$\mathcal{L}(b_k)[0] = \mathbb{E}\mathcal{L}(b_k) = \frac{k}{\sqrt{d\pi}} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \cdot \mathrm{Vol}_d(B(0,1)), \tag{8.23}$$

$$\mathcal{L}(b_k)[2q] = \frac{k}{\sqrt{2\pi d}} \sum_{m=0}^{q} \frac{(-1)^m}{(2m)!!} \widehat{F}(q-m) \int_{B(0,1)} H_{2m}(b_k(x)) H^d_{q-m}(\widetilde{\nabla} b_k(x)) dx, \tag{8.24}$$

and the coefficients $\widehat{F}(q-m)$ *are as defined in (8.7)-(8.8).*

The next lemma follows almost immediately from the preceding one. An important point to note is that when applying the vector-variate Hermite expansion, the number of terms in the 4th chaotic projection remains constant. This consistency offers the potential for efficient computation of lower bounds on the variance of the nodal volume in any dimension.

Lemma 8.3. *Let* $b_k = \{b_k(x) : x \in \mathbb{R}^d\}$, $d \geq 2$, *be the Berry's Random Wave with wavenumber* $k > 0$ *(see Section 1.1). Then,* 1*st and* 2*nd Chaotic projections of the nodal volume* $\mathcal{L}(b_k) := \mathcal{H}^{d-1}(B(0,1) \cap b_k^{-1}(0))$ *vanish. Moreover the* 2*nd,* 3*rd and* 4*th chaotic projections of the nodal volume* $\mathcal{L}(b_k)$ *are given by the formulas*

$$\mathcal{L}(b_k)[0] = \sqrt{2} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}, \tag{8.25}$$

$$\mathcal{L}(b_k)[2] = \frac{k}{\sqrt{d\pi}} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \cdot \left(\int_{B(0,1)} H^d_1(\widetilde{\nabla} b_k(x)) dx - \frac{1}{2} \int_{B(0,1)} H_2(b_k(x)) dx \right), \tag{8.26}$$

$$\begin{aligned} \mathcal{L}(b_k)[4] = \frac{k}{\sqrt{\pi d}} \cdot \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \cdot \\ \Big(\frac{1}{8} \int_{B(0,1)} H_4(b_k(x)) dx - \frac{1}{2} \int_{B(0,1)} H_2(b_k(x)) H^d_1(\widetilde{\nabla} b_k(x)) dx \\ - \frac{1}{2} \int_{B(0,1)} H^d_2(\widetilde{\nabla} b_k(x)) dx \Big) \end{aligned} \tag{8.27}$$

The next lemma is a first step towards fulfilling the lower-bound goal.

Lemma 8.4. *Let* $X_0, X_1, \dots, X_d$ *and* $Y_0, \dots, Y_d$ *be jointly Gaussian centered random variables s.t.*

$$\mathbb{E} X_i X_j = \mathbb{E} Y_i Y_j = 0, \qquad i, j \in \{0, 1, \dots, d\},$$

and denote

$$X := (X_1, \dots, X_d)^{tr}, \qquad Y := (Y_1, \dots, Y_d)^{tr}.$$

Then, the following formulas hold

$$\mathbb{E}\left[H_4(X_0)\cdot H_4(Y_0)\right] = 4!\cdot\mathbb{E}[X_0Y_0]^4, \tag{8.28}$$

$$\mathbb{E}\left[H_4(X_0)\cdot H_2^d(Y)\right] = \frac{3}{d(d+2)}\left(\sum_{i=1}^{d}\mathbb{E}[X_0Y_i]^2\right)^2, \tag{8.29}$$

$$\mathbb{E}\left[H_4(X_0)\cdot H_2(Y_0)H_1^d(Y)\right] = \frac{12}{d}\cdot\mathbb{E}[X_0Y_0]^2\cdot\sum_{i=1}^{d}\mathbb{E}[X_0Y_i]^2, \tag{8.30}$$

and, moreover,

$$\begin{aligned}
&\mathbb{E}\left[H_2(X_0)H_1^d(X)\cdot H_2^d(Y)\right] = \\
&\frac{1}{d^2(d+2)}\Bigg(3\sum_{j=1}^{d}\sum_{l=1}^{d}\mathbb{E}[X_0Y_l]^2\mathbb{E}[X_jY_l]^2 + \sum_{j=1}^{d}\sum_{1\le k\neq l\le d}\mathbb{E}[X_0Y_k]^2\mathbb{E}[X_jY_l]^2 \\
&+ 2\sum_{j=1}^{d}\sum_{1\le k\neq l\le d}\mathbb{E}[X_0Y_k]\mathbb{E}[X_0Y_l]\mathbb{E}[X_jY_k]\mathbb{E}[X_jY_l]\Bigg),
\end{aligned} \tag{8.31}$$

$$\begin{aligned}
&\mathbb{E}\left[H_2^d(X)\cdot H_2^d(Y)\right] = \\
&\frac{1}{2d^2(d+2)^2}(3\cdot\sum_{j=1}^{d}\sum_{m=1}^{d}\mathbb{E}[X_jY_m]^4 + 3\sum_{1\le j\le d}\sum_{1\le k\neq l\le d}\mathbb{E}[X_jY_k]^2\mathbb{E}[X_jY_l]^2 \\
&+ 3\sum_{1\le j\le d}\sum_{1\le k\neq l\le d}\mathbb{E}[Y_jX_k]^2\mathbb{E}[Y_jX_l]^2 \\
&+ 2\sum_{1\le i\neq j\le d}\sum_{1\le l\neq k\le d}\mathbb{E}[X_0Y_k]\mathbb{E}[X_0Y_l]\mathbb{E}[X_iY_k]\mathbb{E}[X_jY_l]),
\end{aligned} \tag{8.32}$$

$$\begin{aligned}
&\mathbb{E}\left[H_2(X_0)H_1^d(X)\cdot H_2(Y_0)H_1^d(Y)\right] = \\
&\frac{1}{d^2}\cdot\mathbb{E}[X_0Y_0]^2\left(\sum_{1\le i,j\le d}\mathbb{E}[X_iY_j]^2\right) + \frac{1}{d^2}(\sum_{1\le i\le d}\mathbb{E}[X_iY_0]^2)\left(\sum_{1\le j\le d}\mathbb{E}[X_0Y_j]^2\right) \\
&+ \frac{4}{d^2}\cdot\mathbb{E}[X_0Y_0]\sum_{1\le i,j\le d}\mathbb{E}[X_0Y_j]\mathbb{E}[X_iY_0]\mathbb{E}[X_iY_j].
\end{aligned} \tag{8.33}$$

Proof. The formula (8.28) follows by the standard property of classical Hermite polynomials

$$\mathbb{E}[H_p(X_0)H_q(Y_0)] = \mathbf{1}_{p=q}p!\cdot\mathbb{E}[X_0Y_0]^p. \tag{8.34}$$

(see [69, p. 26, Proposition 2.2.1]). We record that, using the classical diagram formula for Hermite polynomials (see [54, p. 96-99, Section 4.3.1 Diagram formula]) we have

$$\mathbb{E}\left[H_4(X_0)\cdot H_2(Y_i)H_2(Y_j)\right] = 24\cdot\mathbb{E}[X_0Y_i]^2\mathbb{E}[X_0Y_j]^2 \tag{8.35}$$

$$\begin{aligned}\mathbb{E}\left[H_2(X_0)H_2(X_i)\cdot H_2(Y_k)H_2(Y_l)\right] &= 4\cdot\mathbb{E}[X_0Y_k]^2\mathbb{E}[X_iY_l]^2\\ &\quad+4\cdot\mathbb{E}[X_0Y_l]^2\mathbb{E}[X_iY_k]^2\\ &\quad+16\cdot\mathbb{E}[X_0Y_k]\mathbb{E}[X_0Y_l]\mathbb{E}[X_iY_k]\mathbb{E}[X_iY_l].\end{aligned} \tag{8.36}$$

Using first (8.5), and subsequently (8.34) and (8.35) we can compute (8.29):

$$\begin{aligned}&\mathbb{E}\left[H_4(X_0)\cdot H_{(2)}^{(1,d)}(Y)\right]\\ &=\frac{1}{4d(d+2)}(4!\times\sum_{1\le i\le d}\mathbb{E}[X_0Y_i]^4+\sum_{1\le i\ne j\le d}\mathbb{E}[H_4(X_0)H_2(Y_i)H_2(Y_j)])\\ &=\frac{1}{4d(d+2)}(4!\times\sum_{1\le i\le d}\mathbb{E}[X_0Y_i]^4+24\sum_{1\le i\ne j\le d}\mathbb{E}[X_0Y_i]^2\mathbb{E}[X_0Y_j]^2)\\ &=\frac{6}{d(d+2)}\sum_{1\le i,j\le d}\mathbb{E}[X_0Y_i]^2\mathbb{E}[X_0Y_j]^2\\ &=\frac{6}{d(d+2)}(\sum_{i=1}^{d}\mathbb{E}[X_0Y_i]^2)^2.\end{aligned} \tag{8.37}$$

Using first (8.4) and then (8.35) we deduce (8.30):

$$\begin{aligned}\mathbb{E}\left[H_4(X_0)\cdot H_2(Y_0)H_{(1)}^{(1,d)}(Y)\right] &= \frac{1}{2d}\sum_{1\le i\le d}\mathbb{E}[H_4(X_0)H_2(Y_0)H_2(Y_i)]\\ &=\frac{12}{d}\cdot\sum_{1\le i\le d}\mathbb{E}[X_0Y_0]^2\mathbb{E}[X_0Y_i]^2\\ &=\frac{12}{d}\cdot\mathbb{E}[X_0Y_0]^2\times\sum_{i=1}^{d}\mathbb{E}[X_0Y_i]^2.\end{aligned} \tag{8.38}$$

We recall (8.4) and (8.5), and then use (8.35) and (8.36) to evaluate (8.31) as:

$$\begin{aligned}
&\mathbb{E}\left[H_2(X_0)H^{(1,d)}_{(1)}(X)\cdot H^{(1,d)}_{(2)}(Y)\right] = \frac{1}{8d^2(d+2)}\times \\
&(\sum_{j=1}^{d}\sum_{l=1}^{d}\mathbb{E}[H_2(X_0)H_2(X_j)H_4(Y_l)] + \sum_{j=1}^{d}\sum_{1\le k\neq l\le d}\mathbb{E}[H_2(X_0)H_2(X_j)H_2(Y_k)H_2(Y_l)]) \\
&= \frac{1}{8d^2(d+2)}(24\times\sum_{j=1}^{d}\sum_{l=1}^{d}\mathbb{E}[X_0Y_l]^2\mathbb{E}[X_jY_l]^2 + 4\cdot\sum_{j=1}^{d}\sum_{1\le k\neq l\le d}\mathbb{E}[X_0Y_k]^2\mathbb{E}[X_jY_l]^2 \\
&\quad + 4\cdot\sum_{j=1}^{d}\sum_{1\le k\neq l\le d}\mathbb{E}[X_0Y_l]^2\mathbb{E}[X_jY_k]^2 + 16\cdot\mathbb{E}[X_0Y_k]\mathbb{E}[X_0Y_l]\mathbb{E}[X_jY_k]\mathbb{E}[X_jY_l]) \\
&= \frac{1}{8d^2(d+2)}(24\times\sum_{j=1}^{d}\sum_{l=1}^{d}\mathbb{E}[X_0Y_l]^2\mathbb{E}[X_jY_l]^2 + 8\sum_{j=1}^{d}\sum_{1\le k\neq l\le d}\mathbb{E}[X_0Y_k]^2\mathbb{E}[X_jY_l]^2 \\
&\quad + 16\sum_{j=1}^{d}\sum_{1\le k\neq l\le d}\mathbb{E}[X_0Y_k]\mathbb{E}[X_0Y_l]\mathbb{E}[X_jY_l]\mathbb{E}[X_jY_k]).
\end{aligned} \tag{8.39}$$

Finally, to derive (8.32) we first use (8.5) and (8.34) to obtain

$$\begin{aligned}
\mathbb{E}\left[H^{(1,d)}_{(2)}(X)\cdot H^{(1,d)}_{(2)}(Y)\right] = \frac{1}{16d^2(d+2)^2}&\times(4!\cdot\sum_{1\le j,m\le d}\mathbb{E}[X_jY_m]^4 \\
&+\sum_{1\le j\le d}\sum_{1\le k\neq l\le d}\mathbb{E}[H_4(X_j)H_2(Y_k)H_2(Y_l)] \\
&+\sum_{1\le j\le d}\sum_{1\le k\neq l\le d}\mathbb{E}[H_4(Y_j)H_2(X_k)H_2(X_l)] \\
&+\sum_{1\le i\neq j\le d}\sum_{1\le l\neq k\le d}\mathbb{E}[H_2(X_i)H_2(X_j)H_2(Y_k)H_2(Y_l)])
\end{aligned} \tag{8.40}$$

which, using (8.35) can be further rewritten into

$$\begin{aligned}
&\frac{1}{16d^2(d+2)^2}(4!\cdot\sum_{1\le j,m\le d}\mathbb{E}[X_jY_m]^4 \\
&+24\sum_{1\le j\le d}\sum_{1\le k\neq l\le d}\mathbb{E}[X_jY_k]^2\mathbb{E}[X_jY_l]^2 + 24\sum_{1\le j\le d}\sum_{1\le k\neq l\le d}\mathbb{E}[Y_jX_k]^2\mathbb{E}[Y_jX_l]^2 \\
&+\sum_{1\le i\neq j\le d}\sum_{1\le l\neq k\le d}\mathbb{E}[H_2(X_i)H_2(X_j)H_2(Y_k)H_2(Y_l)]),
\end{aligned} \tag{8.41}$$

and finally using (8.36) we obtain

$$\begin{aligned}
&\frac{1}{16d^2(d+2)^2}(24\cdot\sum_{1\le j,m\le d}\mathbb{E}[X_jY_m]^4\\
&+24\sum_{1\le j\le d}\sum_{1\le k\ne l\le d}\mathbb{E}[X_jY_k]^2\mathbb{E}[X_jY_l]^2+24\sum_{1\le j\le d}\sum_{1\le k\ne l\le d}\mathbb{E}[Y_jX_k]^2\mathbb{E}[Y_jX_l]^2\\
&+8\cdot\sum_{1\le i\ne j\le d}\sum_{1\le l\ne k\le d}\mathbb{E}[X_0Y_k]^2\mathbb{E}[X_iY_l]^2\\
&+16\cdot\sum_{1\le i\ne j\le d}\sum_{1\le l\ne k\le d}\mathbb{E}[X_0Y_k]\mathbb{E}[X_0Y_l]\mathbb{E}[X_iY_k]\mathbb{E}[X_iY_l]).
\end{aligned}\tag{8.42}$$

Now, using (8.4) and then (8.36), we infer the identity

$$\begin{aligned}
&\mathbb{E}\left[H_2(X_0)H_{(1)}^{(1,d)}(X)\cdot H_2(Y_0)H_{(1)}^{(1,d)}(Y)\right]=\\
&\frac{1}{d^2}\sum_{1\le i,j\le d}\mathbb{E}[H_2(X_0)H_2(X_i)\cdot H_2(Y_0)H_2(Y_j)]=\\
&\frac{1}{d^2}\sum_{1\le i,j\le d}(\mathbb{E}[X_0Y_0]^2\mathbb{E}[X_iY_j]^2+\mathbb{E}[X_0Y_j]^2\mathbb{E}[X_iY_0]^2\\
&+4\mathbb{E}[X_0Y_0]\mathbb{E}[X_0Y_j]\mathbb{E}[X_iY_0]\mathbb{E}[X_iY_j]),
\end{aligned}\tag{8.43}$$

which proves (8.33) and concludes the proof. □

8.3 Some Computations Towards Spectral Extension

The purpose of the present section is to discuss some computations aimed at extending our results in direction of a wider class of isotropic spectral measures and the main presented here is Lemma 8.7.

Let $b=\{b(x):x\in\mathbb{R}^d\}$ be a real-valued centered Gaussian random field on $\mathbb{R}^d$ which is almost surely $C^3(\mathbb{R}^d)$ and satisfies $\mathbb{E}b(x)=0$ and $\mathrm{Var}(b(x))=1$ for every $x\in\mathbb{R}^d$. Suppose also that b is stationary, isotropic and its (isotropic) spectral measure μ has a finite and strictly positive second moment $0<\int_0^\infty\lambda^2d\mu(\lambda)<\infty$. We define the normalised derivatives of the process b by the formula

$$\widetilde{\partial}_{x_i}b(x):=\sqrt{\frac{d}{\int_0^\infty\lambda^2d\mu(\lambda)}}\cdot\partial_{x_i}b(x),\tag{8.44}$$

where we let $i=1,\dots,d$ and $x\in\mathbb{R}^d$. Similarly, we define the normalised gradient as the random vector

$$\widetilde{\nabla}b(x):=\sqrt{\frac{d}{\int_0^\infty\lambda^2d\mu(\lambda)}}\cdot\nabla b(x)\equiv(\widetilde{\partial}_{x_1}b(x),\dots,\widetilde{\partial}_{x_d}b(x))^{tr}.\tag{8.45}$$

As made clear in the next lemma, this normalisation ensures that $\mathrm{Var}(\widetilde{\partial}_{x_i}b(x))\equiv1$. The following definition will be in frequent use throughout the remainder of this chapter.

Definition 8.2. *Let $b = \{b(x) : x \in \mathbb{R}^d\}$ be an almost surely $C^3(\mathbb{R}^d)$ Gaussian random field such that $\mathbb{E}b(x) = 0$ and $Var(b(x)) = 1$ for every $x \in \mathbb{R}^d$. Suppose also that b is stationary and isotropic with (an isotropic) spectral measure μ which has a finite and strictly positive first moment $0 < \int_0^\infty \lambda^2 d\mu(\lambda) < \infty$. Then, for every $x, y \in \mathbb{R}^d$ and $i, j = 1, \ldots, d$, we set*

$$\begin{aligned} C(x-y) &:= \mathbb{E}\left[b(x) \cdot b(y)\right], \\ C_i(x-y) &:= \mathbb{E}\left[\widetilde{\partial}_{x_i} b(x) \cdot b(y)\right], \\ C_{ij}(x-y) &:= \mathbb{E}\left[\widetilde{\partial}_{x_i} b(x) \cdot \widetilde{\partial}_{y_j} b(y)\right], \end{aligned} \tag{8.46}$$

where the normalised derivatives $\widetilde{\partial}_{x_i}$, $\widetilde{\partial}_{y_j}$ are as defined in (8.44).

In the next lemma, we provide conditions under which the covariance functions defined in (8.46) can be easily related to the behaviour of the underlying (isotropic) spectral measure of the process.

Lemma 8.5. *Let $b = \{b(x) : x \in \mathbb{R}^d\}$ be a centered, unit variance, non-constant, stationary and isotropic Gaussian random field. Suppose any of the following equivalent conditions is satisfied:*

(i) $x \mapsto b(x)$ is almost surely $C^3(\mathbb{R}^d)$,

(ii) $(x, y) \mapsto \mathbb{E}\left[b(x) \cdot b(y)\right]$ is $C^{4,4}(\mathbb{R}^d \times \mathbb{R}^d)$,

(iii) the (isotropic) spectral measure μ associated with the process b has a finite 4-th moment $\int_0^\infty \lambda^4 d\mu(\lambda) < \infty$.

Then, the covariance functions (8.46) are given by the formulas

$$\begin{aligned} C(x-y) &= \int_0^\infty \rho_{\frac{d}{2}-1}(\lambda|x-y|) d\mu(\lambda), \\ C_i(x-y) &= -\frac{(x_i - y_i)}{\sqrt{d \cdot \int_0^\infty \lambda^2 d\mu(\lambda)}} \cdot \int_0^\infty \rho_{\frac{d}{2}}(\lambda||x-y||) \lambda^2 d\mu(\lambda), \\ C_{ij}(x-y) &= \delta_{ij} \cdot \frac{\int_0^\infty \rho_{\frac{d}{2}}(\lambda|x-y|)\lambda^2 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)} \\ &\quad - \frac{(x_i - y_i)(x_j - y_j)}{(d+2) \cdot \int_0^\infty \lambda^2 d\mu(\lambda)} \cdot \int_0^\infty \rho_{\frac{d}{2}+1}(\lambda|x-y|)\lambda^4 d\mu(\lambda), \end{aligned} \tag{8.47}$$

where $i, j \in \{1, \ldots, d\}$, $x, y \in \mathbb{R}^d$ and where the normalised Bessel function ρ_α is as defined in (1.3).

Proof. Firstly, we note that, in the case of the Berry's Random Wave with wavenumber

$k = 1$ (that is, when $\mu = \delta_1$) the covariance functions (8.46) take the form

$$\begin{aligned} C(x-y) &= \rho_{\frac{d}{2}-1}\left(|x-y|\right), \\ C_i(x-y) &= -\frac{(x_i-y_i)}{\sqrt{d}} \cdot \rho_{\frac{d}{2}}\left(|x-y|\right), \\ C_{ij}(x-y) &= \delta_{ij} \cdot \rho_{\frac{d}{2}}\left(|x-y|\right) - \frac{(x_i-y_i)(x_j-y_j)}{d+2} \cdot \rho_{\frac{d}{2}+1}\left(|x-y|\right), \end{aligned} \tag{8.48}$$

see [85, p. 74, Lemma A.1]. We note that for each $\alpha \geq 0$, the function ρ_α is globally bounded and thanks to the existence of the 2nd spectral moment we can compute

$$\begin{aligned} \mathbb{E}\left[\widetilde{\partial}_{x_i} b(x) \cdot b(y)\right] &= \widetilde{\partial}_{x_i} \mathbb{E}\left[b(x) \cdot b(y)\right] \\ &= \widetilde{\partial}_{x_i}\left(\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda|x-y|)d\mu(\lambda)\right) \\ &= \int_0^\infty \widetilde{\partial}_{x_i} \rho_{\frac{d}{2}-1}(\lambda|x-y|)d\mu(\lambda) \\ &= -\frac{(x_i-y_i)}{\sqrt{d \cdot \int_0^\infty \lambda^2 d\mu(\lambda)}} \cdot \int_0^\infty \rho_{\frac{d}{2}}(\lambda|x-y|)\lambda^2 d\mu(\lambda), \end{aligned} \tag{8.49}$$

proving the first postulated formula. Similarly, using the existence of the 4th spectral moment we can compute

$$\begin{aligned} \widetilde{\partial}_{y_j}\widetilde{\partial}_{x_i}\mathbb{E}\left[b(x) \cdot b(y)\right] &= \widetilde{\partial}_{y_j}\left(-\frac{(x_i-y_i)}{\sqrt{d \cdot \int_0^\infty \lambda^2 d\mu(\lambda)}} \cdot \int_0^\infty \rho_{\frac{d}{2}}(\lambda|x-y|)\lambda^2 d\mu(\lambda)\right) \\ &= \delta_{ij} \cdot \frac{\int_0^\infty \rho_{\frac{d}{2}}(\lambda|x-y|)\lambda^2 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)} \\ &\quad - \frac{(x_i-y_i)}{\int_0^\infty \lambda^2 d\mu(\lambda)} \cdot \int_0^\infty \partial_{y_j}\rho_{\frac{d}{2}}(\lambda|x-y|)\lambda^2 d\mu(\lambda) \\ &= \delta_{ij} \cdot \frac{\int_0^\infty \rho_{\frac{d}{2}}(\lambda|x-y|)\lambda^2 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)} \\ &\quad - \frac{(x_i-y_i)(x_j-y_j)}{(d+2) \cdot \int_0^\infty \lambda^2 d\mu(\lambda)} \cdot \int_0^\infty \rho_{\frac{d}{2}+1}(\lambda|x-y|)\lambda^4 d\mu(\lambda), \end{aligned} \tag{8.50}$$

which is the second postulated formula. □

Lemma 8.6. *Let $b = \{b(x) : x \in \mathbb{R}^d\}$ be a centered, unit variance, stationary and isotropic Gaussian random field with isotropic spectral measure μ having strictly positive and finite 4th moment, i.e., $0 < \int_0^\infty \lambda^4 d\mu(\lambda) < \infty$. Let $\widetilde{\nabla} b = \{\widetilde{\nabla} b(x) : x \in$*

$\mathbb{R}^d\}$ be the corresponding normalised gradient field, see (8.44)-(8.45). Then

$$\begin{aligned}
&\mathrm{Var}\left(\widetilde{\nabla} b(x) | b(x) = b(y) = 0\right) \\
&= W_1^{\mu}(||x-y||) \cdot \mathrm{Id}_d \\
&\quad + \mathrm{M}^{x,y} \cdot ||x-y||^2 \cdot \left(-\frac{1}{d+2} \cdot W_2^{\mu}(||x-y||) + \frac{1}{d} \cdot W_3^{\mu}(||x-y||)\right).
\end{aligned} \tag{8.51}$$

where

$$M_{ij}^{x,y} := \left\langle \frac{x_i - y_i}{||x-y||}, \frac{x_j - y_j}{||x-y||} \right\rangle \tag{8.52}$$

and for any $r \geq 0$ we have

$$\begin{aligned}
W_1^{\mu}(r) &:= \frac{\int_0^{\infty} \rho_{\frac{d}{2}}(\lambda r) \lambda^2 d\mu(\lambda)}{\int_0^{\infty} \lambda^2 d\mu(\lambda)} \\
W_2^{\mu}(r) &:= \frac{\int_0^{\infty} \rho_{\frac{d}{2}+1}(\lambda r) \lambda^4 d\mu(\lambda)}{\int_0^{\infty} \lambda^2 d\mu(\lambda)} \\
W_3^{\mu}(r) &:= \frac{\left(\int_0^{\infty} \rho_{\frac{d}{2}}(\lambda r) \lambda^2 d\mu(\lambda)\right)^2}{\left(\int_0^{\infty} \lambda^2 d\mu(\lambda)\right)^2 \cdot \left(1 - \left(\int_0^{\infty} \rho_{\frac{d}{2}-1}(\lambda r) d\mu(\lambda)\right)^2\right)}.
\end{aligned} \tag{8.53}$$

Proof. We will rely on the Gaussian conditioning formulas (see Subsection 3.2.1) and on the formulas established in Lemma 8.5. We note that

$$\begin{aligned}
\mathrm{Var}(\widetilde{\nabla} b(x)) &:= \left[\mathrm{Cov}(\widetilde{\partial}_i b(x), \widetilde{\partial}_j b(x))\right]_{i,j \leq d} \\
&= \frac{\int_0^{\infty} \rho_{\frac{d}{2}}(\lambda ||x-y||) \lambda^2 d\mu(\lambda)}{\int_0^{\infty} \lambda^2 d\mu(\lambda)} \cdot \mathrm{Id}_d \\
&\quad - \frac{1}{d+2} \cdot \frac{\int_0^{\infty} \rho_{\frac{d}{2}+1}(\lambda ||x-y||) \lambda^4 d\mu(\lambda)}{\int_0^{\infty} \lambda^2 d\mu(\lambda)} \cdot M^{x,y} \cdot ||x-y||^2
\end{aligned} \tag{8.54}$$

where the $d \times d$ matrix $M^{x,y}$ is defined as in (8.52). We have

$$\begin{aligned}
&\mathrm{Var}((b(x), b(y))) \\
&:= \begin{bmatrix} \mathrm{Var}(b(x)) & \mathrm{Cov}(b(x), b(y)) \\ \mathrm{Cov}(b(x), b(y)) & \mathrm{Var}(b(y)) \end{bmatrix} \\
&= \begin{bmatrix} 1 & \int_0^{\infty} \rho_{\frac{d}{2}-1}(\lambda ||x-y||) d\mu(\lambda) \\ \int_0^{\infty} \rho_{\frac{d}{2}-1}(\lambda ||x-y||) d\mu(\lambda) & 1 \end{bmatrix},
\end{aligned} \tag{8.55}$$

and in consequence

$$\begin{aligned}
&\mathrm{Var}((b(x), b(y)))^{-1} \\
&\quad = \frac{1}{1 - \left(\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||d\mu(\lambda)\right)^2} \\
&\quad\quad \times \begin{bmatrix} 1 & -\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||d\mu(\lambda)) \\ -\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||d\mu(\lambda)) & 1. \end{bmatrix}
\end{aligned} \tag{8.56}$$

Furthermore,

$$\begin{aligned}
&\mathrm{Cov}\left(\widetilde{\nabla} b(x), (b(x), b(y))\right) \\
&\quad := \begin{bmatrix} \mathrm{Cov}(\widetilde{\partial}_1 b(x), b(x)) & \mathrm{Cov}(\widetilde{\partial}_1 b(x), b(y)) \\ \vdots & \vdots \\ \mathrm{Cov}(\widetilde{\partial}_d b(x), b(x)) & \mathrm{Cov}(\widetilde{\partial}_d b(x), b(y)) \end{bmatrix} \\
&\quad = -\frac{1}{\sqrt{d}} \cdot \frac{\int_0^\infty \rho_{\frac{d}{2}}(\lambda||x-y||)\lambda^2 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)} \cdot ||x-y|| \cdot \begin{bmatrix} 0 & \frac{x_1-y_1}{||x-y||} \\ \vdots & \vdots \\ 0 & \frac{x_d-y_d}{||x-y||} \end{bmatrix}
\end{aligned} \tag{8.57}$$

We observe that

$$\begin{aligned}
&\mathrm{Cov}\left(\widetilde{\nabla} b(x), (b(x), b(y))\right)^{tr} \cdot \mathrm{Var}((b(x), b(y)))^{-1} \cdot \mathrm{Cov}\left(\widetilde{\nabla} b(x), (b(x), b(y))\right) \\
&= \frac{1}{d} \cdot \frac{||x-y||^2 \cdot \left(\int_0^\infty \rho_{\frac{d}{2}}(\lambda||x-y||)\lambda^2 d\mu(\lambda)\right)^2}{\left(\int_0^\infty \lambda^2 d\mu(\lambda)\right)^2 \cdot \left(1 - \left(\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||)d\mu(\lambda)\right)^2\right)} \times \begin{bmatrix} 0 & \frac{x_1-y_1}{||x-y||} \\ \vdots & \vdots \\ 0 & \frac{x_d-y_d}{||x-y||} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} 1 & -\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||)d\mu(\lambda) \\ -\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||)d\mu(\lambda) & 1 \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} 0 & \dots & 0 \\ \frac{x_1-y_1}{||x-y||} & \dots & \frac{x_d-y_d}{||x-y||} \end{bmatrix} \\
&= \frac{1}{d} \cdot \frac{||x-y||^2 \cdot \left(\int_0^\infty \rho_{\frac{d}{2}}(\lambda||x-y||)\lambda^2 d\mu(\lambda)\right)^2}{\left(\int_0^\infty \lambda^2 d\mu(\lambda)\right)^2 \cdot \left(1 - \left(\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||)d\mu(\lambda)\right)^2\right)} \times \begin{bmatrix} 0 & \frac{x_1-y_1}{||x-y||} \\ \vdots & \vdots \\ 0 & \frac{x_d-y_d}{||x-y||} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} -\frac{x_1-y_1}{||x-y||} \cdot \int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||)d\mu(\lambda) & \dots & -\frac{x_d-y_d}{||x-y||} \cdot \int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||)d\mu(\lambda) \\ \frac{x_1-y_1}{||x-y||} & \dots & \frac{x_d-y_d}{||x-y||} \end{bmatrix} \\
&= -\frac{1}{d} \cdot \frac{||x-y||^2 \cdot \left(\int_0^\infty \rho_{\frac{d}{2}}(\lambda||x-y||)\lambda^2 d\mu(\lambda)\right)^2}{\left(\int_0^\infty \lambda^2 d\mu(\lambda)\right)^2 \cdot \left(1 - \left(\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||)d\mu(\lambda)\right)^2\right)} \cdot \mathrm{M}^{x,y},
\end{aligned} \tag{8.58}$$

where, as before, $\mathrm{M}^{x,y}$ is defined by (8.52). $\square$

Regarding the next result, it is noteworthy that its conclusion could be enhanced in three main directions. First, by extending the analysis to include $l = 2, 3, \ldots, d$ waves. Second, by relaxing the conditions on the spectral measure μ, particularly to encompass measures without compact support. Third, by refining the right-hand side of the inequality, at least under specific conditions. Among these, the first extension appears to be the most achievable.

Recall that the isotropic spectral measure μ corresponding to the random field b on $\mathbb{R}^d$ is defined such that

$$\mathbb{E}\left[b(x) \cdot b(y)\right] = \int_0^\infty \rho_{\frac{d}{2}-1}(\lambda||x-y||)\, d\mu(\lambda), \tag{8.59}$$

where $\rho_{\frac{d}{2}-1}$ denotes the normalized Bessel function of the first kind of order $\frac{d}{2} - 1$. Additionally, by

$$\mathrm{supp}(\mu) = [k, K]$$

we mean that $\mu([k, K]) = 1$, and $[k, K]$ is the smallest interval with this property (which could be degenerate if $k = K$).

Lemma 8.7. *Fix a constant $c \in (0, 1]$. There exists a constant $L_{d,c} > 0$ with the following property. For every $K \geq k \geq 2$, $\frac{k}{K} \geq c$, and for every centered, unit variance, stationary and isotropic Gaussian random field on $\mathbb{R}^d$, $d \geq 3$, with isotropic spectral measure μ satisfying $\mathrm{supp}(\mu) = [k, K]$, we have*

$$\mathbb{E}\left[\mathcal{H}^{d-1}\left(\{x \in B(0,1) : b(x) = 0\}\right)^2\right] \leq L_{d,c} \cdot \left(1 + \int_0^\infty \lambda^4 d\mu(\lambda)\right). \tag{8.60}$$

Proof. We begin by noting that the nodal volume under consideration is well-defined. This follows from (8.47), which ensures that for every fixed point $x \in B(0, 1)$, the random vector $(b(x), \widetilde{\nabla} b(x))$ has a non-degenerate Gaussian distribution. Specifically, it is a standard Gaussian vector of dimension $d + 1$. Therefore, the well-definedness is established, for instance, by [75, p. 3, Theorem 1.3]. We split the argument in the three parts. In the first step we will use classical Kac-Rice formula to reduce the problem to bounding of the separate quantities that will be then controlled respectively in steps 2 and 3.

Step 1. Using the Kac-Rice formula (see Subsection 3.4.4) yields

$$\begin{aligned}&\mathbb{E}\left[\mathcal{H}^{d-1}\left(\{x \in B(0,1) : b(x) = 0\}\right)^2\right] \\ &= \int_{B(0,1)} \int_{B(0,1)} \mathbb{E}\left[||\nabla b(x)|| \cdot ||\nabla b(y)|| \Big| b(x) = b(y) = 0\right] \cdot \phi_{b(x),b(y)}(0,0) dx dy,\end{aligned} \tag{8.61}$$

where

$$\begin{aligned}\phi_{b(x),b(y)}(0,0) &= \frac{1}{2\pi\sqrt{1-\mathrm{Cov}(b(x),b(y))^2}} \\ &= \frac{1}{2\pi\sqrt{1-\left(\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda|x-y|)d\mu(\lambda)\right)^2}}\end{aligned} \tag{8.62}$$

is the (Gaussian) density of the random vector $(b(x), b(y))$ evaluated at the point $(0,0)$. Using the normalised-gradient notation (see (8.45)) we can rewrite the right-hand side of (8.61) as

$$\frac{\int_0^\infty \lambda^2 d\mu(\lambda)}{d}\cdot\int_{B(0,1)}\int_{B(0,1)} \frac{\mathbb{E}\Big[||\widetilde{\nabla} b(x)||\cdot||\widetilde{\nabla} b(y)||\Big|b(x)=b(y)=0\Big]}{\sqrt{1-\left(\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda|x-y|)d\mu(\lambda)\right)^2}}\frac{dxdy}{2\pi}. \tag{8.63}$$

Furthermore, using conditional Cauchy-Schwartz inequality and inequality $1/\sqrt{1-t^2} \leq 1/(1-|t|)$ valid for $t \in (-1,1)$ and using convexity of absolute value, we have

$$\frac{1}{2\pi d}\cdot\int_0^\infty \lambda^2 d\mu(\lambda)\cdot\int_{B(0,1)}\int_{B(0,1)} \frac{\mathbb{E}\left[||\widetilde{\nabla} b(x)||^2\Big|b(x)=b(y)=0\right]}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(\lambda|x-y|)|d\mu(\lambda)}dxdy. \tag{8.64}$$

Using stationarity, the above can be further rewritten as

$$\frac{\mathrm{Vol}(B_d(0,1))}{2\pi d}\cdot\int_0^\infty \lambda^2 d\mu(\lambda)\cdot\int_{B(0,2)} \frac{\mathbb{E}\Big[||\widetilde{\nabla} b(z)||^2\Big|b(z)=b(0)=0\Big]}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(\lambda|z|)|d\mu(\lambda)}dxdy. \tag{8.65}$$

Step 2. This step is devoted to the control of

$$\mathbb{E}\Big[||\widetilde{\nabla} b(z)||^2\Big|b(z)=b(0)=0\Big]. \tag{8.66}$$

Combining (8.51) and (8.53) we have that, possibly up to some (irrelevant) constant C_d (depending on the dimension but not on the spectral measure), this task can be accomplished by bounding three suprema. Firstly, we can note that

$$|W_1^\mu(r)| \leq \frac{\int_0^\infty |\rho_{\frac{d}{2}-1}(\lambda r)|\lambda^2 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)} \leq 1, \tag{8.67}$$

since all functions $|\rho_\alpha|$ are globally bounded by 1. Secondly, we observe that

$$|r^2 W_2^\mu(r)| \leq \frac{\int_0^\infty |\rho_{\frac{d}{2}+1}(\lambda r)|r^2\lambda^4 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)} \leq r^2\frac{\int_0^\infty \lambda^4 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)} \tag{8.68}$$

Furthermore,

$$
\begin{aligned}
|r^2 W_3^\mu(r)| &\leq \frac{(\int_0^\infty |\rho_{\frac{d}{2}-1}(\lambda r)| r\lambda^2 d\mu(\lambda))^2}{\int_0^\infty \lambda^2 d\mu(\lambda)\cdot(1-(\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda r)d\mu(\lambda))^2)} \\
&\leq \frac{r^2\cdot\int_0^\infty \lambda^2 d\mu(\lambda)}{1-\int_0^\infty \rho_{\frac{d}{2}-1}(\lambda r)d\mu(\lambda)} \\
&\leq \frac{r^2\int_0^\infty \lambda^2 d\mu(\lambda)}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(\lambda r)|d\mu(\lambda)}.
\end{aligned}
\tag{8.69}
$$

We conclude that if $|z| = r > 0$ then (8.66) can be globally bounded by

$$
C_d\left(1 + r^2\cdot\frac{\int_0^\infty \lambda^4 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)} + r^2\cdot\frac{\int_0^\infty \lambda^2 d\mu(\lambda)}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(\lambda r)|d\mu(\lambda)}\right)
\tag{8.70}
$$

where the constant C_d which is independent of the choice of μ.

Step 3. In this step we will perform a couple of preparatory computations. Using the series expansion recorded in (1.3) we observe that for each $\alpha \geq 0$, we have

$$
\begin{aligned}
\rho_\alpha(s) &= \Gamma(\alpha+1)\cdot\sum_{m=0}^\infty (-1)^m\cdot 2^{-2m}\cdot\frac{s^{2m}}{m!\Gamma(m+\alpha+1)} \\
&= 1 - \frac{s^2}{4(\alpha+1)} + \frac{\Gamma(\alpha+1)}{16}\cdot s^4\cdot\sum_{m=0}^\infty (-1)^m \frac{(s/2)^{2m}}{(m+2)!\Gamma(m+\alpha+2)}.
\end{aligned}
\tag{8.71}
$$

If $0 \leq s \leq 1$ then (8.71) can be written as

$$
\rho_\alpha(s) = 1 - \frac{s^2}{4(\alpha+1)} + s^4\cdot C_{s,\alpha}
\tag{8.72}
$$

where the quantity $C_{s,\alpha}$ satisfies the uniform bound

$$
\sup_{\alpha\geq 0}\sup_{0\leq s\leq 1} |C_{s,\alpha}| \leq 1.
$$

In particular, if $\alpha = \frac{d}{2}-1$ we have

$$
\rho_{\frac{d}{2}-1}(s) = 1 - \frac{s^2}{2d} + s^4\cdot O(1),
\tag{8.73}
$$

and conveniently

$$
|\rho_{\frac{d}{2}-1}(s)| = \rho_{\frac{d}{2}-1}(s)
$$

Step 4. Now we will show how to use (8.70) to obtain the final postulated bound starting from the constant term $C_d \cdot 1$ in (8.70). In this case, it enough to control

$$\begin{aligned} \int_{B(0,2)} \frac{1}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(\lambda|z|)|d\mu(\lambda)}dz &= \widetilde{C}_d \int_0^2 \frac{r^{d-1}}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(r\lambda)|\mu(d\lambda)}dr \\ &= \widetilde{C}_d \int_0^{1/(2dK)} \frac{r^{d-1}}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(r\lambda)|\mu(d\lambda)}dr \\ &\quad + \widetilde{C}_d \int_{1/(2dK)}^2 \frac{r^{d-1}}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(r\lambda)|\mu(d\lambda)}dr. \end{aligned} \tag{8.74}$$

We observe

$$\begin{aligned} \int_0^{1/(2dK)} \frac{r^{d-1}}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(r\lambda)|\mu(d\lambda)}dr &= \int_0^{1/(2dK)} \frac{r^{d-1}}{\int_0^\infty (1-\rho_{\frac{d}{2}-1}(\lambda r))d\mu(\lambda)}dr \\ &= \int_0^{1/(2dK)} \frac{r^{d-1}}{\int_0^\infty \frac{\lambda^2 r^2}{2d} - \lambda^4 r^4 \cdot O(1) d\mu(\lambda)}dr \\ &= \int_0^{1/(2dK)} \frac{r^{d-3}}{\int_0^\infty \frac{\lambda^2}{2d} - \lambda^4 r^2 \cdot O(1) d\mu(\lambda)}dr \\ &= \int_0^{1/(2dK)} \frac{r^{d-3}}{\int_0^\infty \lambda^2(\frac{1}{2d} - \lambda^2 r^2 \cdot O(1))d\mu(\lambda)}dr \\ &\leq \int_0^{1/(2dK)} \frac{r^{d-3}}{\int_0^\infty \lambda^2(\frac{1}{2d} - \frac{1}{(2d)^2})d\mu(\lambda)} \\ &\leq \frac{L_d}{\int_0^\infty \lambda^2 d\mu(\lambda)}, \end{aligned} \tag{8.75}$$

where L_d is a constant depending on the dimension but not on the choice of spectral measure μ. On the other hand

$$\begin{aligned} \int_{1/(2dK)}^2 \frac{r^{d-1}}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(r\lambda)|\mu(d\lambda)}dr &\leq \widetilde{L}_d \cdot \sup_{2\geq r\geq 1/(2dK)} \frac{1}{1-\int_0^\infty |\rho_{\frac{d}{2}-1}(\lambda r)|d\mu(\lambda)} \\ &\leq \widetilde{L}_d \cdot \frac{1}{1-\sup_{s\geq \frac{k}{2dK}} |\rho_{\frac{d}{2}-1}(s)|} \\ &\leq \widetilde{L}_d \cdot \frac{1}{1-\sup_{s\geq \frac{c}{2d}} |\rho_{\frac{d}{2}-1}(s)|} \\ &=: \widetilde{L}_{d,c}. \end{aligned} \tag{8.76}$$

We note that, together with (8.64) this yields the term

$$C_d \left(1 + \int_0^\infty \lambda^2 d\mu(\lambda)\right). \tag{8.77}$$

The case $r^2\frac{\int_0^\infty \lambda^4 d\mu(\lambda)}{\int_0^\infty \lambda^2 d\mu(\lambda)}$ can be upper bounded similarly by

$$C_d\left(1+\int_0^\infty \lambda^4 d\mu(\lambda)\right). \tag{8.78}$$

The last case yields again the term as (8.77). □

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