FACULTY OF LAW, ECONOMICS AND FINANCE

Department of Economics and Management

Discussion Paper

2025-01

Economics

Department of Economics and Management
University of Luxembourg

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January 2025

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Limited factors and why optimal growth has led to destruction*

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January 8, 2025

Abstract

We revisit the classical Ramsey (1928) model with time discounting and a linear production function, explicitly accounting for the inevitable limitations of tangible production factors, which must remain both finite and positive. By employing Pontryagin's (1962) maximum principle, we transform state constraints into control constraints and provide a complete solution for all impatience rates under the linear production framework.

While we classify the levels of impatience as established in the existing literature, we show that the behaviors associated with this threshold fundamentally differ when input limitations are considered - a factor previously overlooked.

Our analysis extends beyond the literature's traditional focus on agents with mild impatience, encompassing the entire spectrum of impatience. For highly patient agents, the policymaker prioritizes investment over consumption, ensuring the economy reaches its maximum capital level in finite time. Once this level is attained, consumption stabilizes indefinitely, achieving the golden rule trajectory - an outcome previously deemed unattainable under time discounting. Conversely, beyond the classical impatience threshold, capital and consumption decline over time. For agents with extreme impatience, we identify a second threshold where investment ceases entirely, leading to rapid depletion of capital and output.

Keywords: Economic growth, Optimal Control, Dynamic Programming, Limited resources, Linear Production, Discount.
Journal of Economic Literature: C61, O44, Q15, Q56, R11.

*This work has greatly benefited from insightful discussions with Raouf Boucekkine. Any remaining errors are solely our responsibility.

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1 Introduction

In 1972, the Club of Rome commissioned a study on the future of the global economy, considering factors such as population growth, resource utilization, the finite nature of natural resources, industrial pollution, and technological progress. This effort culminated in the publication of the groundbreaking book "Limits to Growth". The book introduced a computational model that explored various scenarios for the future of the world economy.

In 1974, three independent studies, by Dasgupta and Heal (1974), Solow (1974), and Stiglitz (1974), applied similar methods to examine the same critical question. These works proposed models in which a policymaker seeks to maximize an intertemporal welfare function while accounting for the depletion of finite natural resources. In these models, production relies on labor, physical capital, and natural resources, which can act as substitutes in an economy capable of technological progress. Consequently, an increasing or more efficient stock of labor or physical capital can offset the diminishing availability of natural resources, enabling a trajectory of sustained or even rising per capita consumption.

The issue of natural resource depletion has gained significant attention in both academic circles and the general public. For instance, while the Dasgupta-Heal-Solow-Stiglitz model's optimal trajectory was fully characterized by Benchekroum and Withagen (2011), the stark reality of resource exhaustion is highlighted annually. Since the early 1990s, we have been reminded of the exact date the global economy surpasses Earth's natural resource budget for the year. Notably, while this "Earth Overshoot Day" fell near the end of December in 1971, it has advanced dramatically, occurring as early as August 1st in 2024. [1]

Beyond natural resources, we argue that all tangible production inputs are ultimately physically limited. It is worth noting that we will avoid delving into the topic of intangible capital and the debates it may provoke. While intangible capital, defined as a firm's "know-how" (Corrado et al., 2022), might seem unlimited due to its lack of physical form (Crouzet et al., 2022), it still requires investment and a storage medium. In this analysis, we adopt a worst case scenario, focusing exclusively on a single tangible production input that is physically finite, with no possibility of substitution by other factors.

In the most widely used modern version of the Ramsey (1928) model, time is discounted, and there is a single production factor, physical capital, that generates the final good and accumulates over time to ensure future consumption. Positive values for both capital and consumption are typically assumed, with the positivity of consumption often guaranteed through the choice of the utility function. However, the literature does not adequately address the positivity of capital in a systematic manner. While many models impose a positivity constraint, this is often achieved by restricting the parameter set, which can bias optimal choices and policy recommendations.

More importantly, to the best of our knowledge, no existing work explicitly considers the physical limitations of capital or the necessity of bounding consumption by production capacity. The only exceptions are models incorporating finite productive factors like land or

[1] Source: https://overshoot.footprintnetwork.org/

non-renewable natural resources, where boundedness is a natural assumption (e.g., Dasgupta and Heal, 1974). Nevertheless, even in such cases, the literature often neglects the upper constraint on consumption, i.e., the requirement that consumption must not exceed total output. Historically, it has been assumed that the economy would function smoothly and that a scenario where all output is consumed would not occur.

In this paper, we demonstrate how the absence of explicit constraints on capital and consumption can lead to overconsumption, revealing the critical need for these boundaries in economic modeling.

This paper revisits the Ramsey (1928) model with time discounting, explicitly incorporating the assumptions that physical capital has a maximum limit and must remain positive at all times, and that consumption is naturally bounded by production. Using Dynamic Programming and Pontryagin's (1962) optimal control theory, we derive new results for a linear production function.

As previously noted, neglecting the finiteness of capital in an economy with agents who are considered only mildly impatient leads to the adoption of an optimal trajectory where all variables grow at a constant rate. However, we demonstrate that once the limitedness of capital is accounted for, the standard accepted level of impatience proves to be too high, resulting in catastrophic outcomes for the economy if it follows the traditionally advised consumption trajectory. In contrast to existing literature, we find that for all impatience rates below the established threshold, the policymaker optimally prioritizes investment over consumption, initially driving the economy to reach the maximum capital level. Once this maximum is achieved, consumption stabilizes at its highest sustainable level indefinitely.

Crucially, we show that the economy optimally reaches its golden rule trajectory (see Phelps, 1961), a result previously considered unattainable in models with time discounting. Conventional wisdom suggested that any degree of preference for the present would dissuade policymakers from investing sufficiently to achieve maximum long-term consumption. Here, the accessibility of the golden rule enables the policymaker to prioritize capital investment, provided agents exhibit only slight impatience.

However, when agents' impatience exceeds the threshold traditionally categorized as mild, the policymaker faces a trade-off between consumption and investment. In such cases, both physical capital and consumption decline over time. If the impatience rate surpasses a second, higher threshold, investment ceases entirely. Under these circumstances, capital stock depletes from the outset, and production and consumption diminish at an accelerated pace. Notably, we fully solve the AK model for all impatience rates, demonstrating that the limitedness of production factors fundamentally alters the policy implications. This contrasts sharply with prior literature, which addressed only a limited range of impatience rates and, as we show, offered welfare-detrimental recommendations when bounded production factors are considered.

The AK model was first introduced in Harrod (1936) and Domar (1946), later enhanced by Frankel (1962) and Arrow (1962) to include knowledge externalities, and subsequently extended by Romer (1986) and others.[2] In the classical AK endogenous growth literature, the time discount rate must be carefully calibrated relative to the economy's growth rate to ensure sustained positive growth. Two threshold values for impatience emerge, underscoring the necessity for agents to exhibit impatience, but only within a mild range, to secure a positive optimal solution.

On the one hand, the discount rate must not be too low. If it is, the present value of future capital could become infinite, violating the transversality condition and leading to over-accumulation of capital. This typically requires the discount rate to exceed the economy's growth rate.[3] Additionally, an excessively low discount rate could cause the economy to engage in excessive borrowing, violating the No-Ponzi Game condition and potentially resulting in financial instability.[4] To avoid such outcomes, the AK model's discount rate must be sufficiently high to ensure that the discounted value of capital does not grow without bound.

On the other hand, the growth rate of consumption must remain positive, which imposes an upper limit on the discount rate. The transversality condition requires that the present value of future capital stocks approaches zero as time approaches infinity. If the discount rate is too high, it could result in negative growth rates, rendering the model unrealistic for long-term economic predictions. When the discount rate falls within these boundaries, the policymaker adopts a balanced growth path where all variables grow at the same rate from the start of the planning period. These constraints are vital for maintaining the predictions of the classical AK model and ensuring that the growth path remains both optimal and feasible over the long term.

However, these constraints on the discount rate limit policymakers' flexibility in setting interest rates and other economic policies, making it difficult to respond to economic shocks or changes in the economic environment. These challenges are particularly significant when capital represents finite resources like land or natural resources. For instance, Dasgupta and Heal (1974) examined the optimal depletion of exhaustible resources, showing that the choice of discount rate critically affects the depletion path. A high discount rate favors present consumption, accelerating resource depletion. Conversely, a lower discount rate supports more gradual depletion, allowing for a smoother transition to alternative resources or technologies. Such a lower discount rate aligns with sustainable approaches by placing greater emphasis on the welfare of future generations.[5]

Let us turn now to the importance and role of the upper bound on capital. Since the

[2] Among the most influential papers employing AK technology, we highlight Lucas (1988), which explores human capital accumulation and transmission; Rebelo (1991), which leverages the model's simplicity to study the impact of public policies on growth; and Acemoglu and Ventura (2002), which examines the terms of trade. For a comprehensive review, see Jones and Manuelli (2005).

[3] For detailed descriptions of the classical AK model, see Acemoglu (2009), Chapter 11, and Barro and Sala-i-Martin (2004), Chapter 4. For a spatial adaptation of the AK model, see Boucekkine et al. (2013).

[4] For more on this, see Groth's Lecture Notes in Macroeconomics (2017), Chapter 11, available at https://web.econ.ku.dk/okocg/VM/VM-general/Material/Chapters-VM.htm.

[5] For similar discussions, see Arrow (1999), Weitzman (1998), Gollier (1999), and debates following the Stern Review (2006), including Dasgupta (2007, 2008), Nordhaus (2007), Weitzman (2007), Yohe (2006), and Fleurbaey and Zuber (2013).

1980s, numerous scholars have sought to develop models capable of replicating sustained long-run growth. While technological progress was a key focus in the 1970s, as emphasized by Stiglitz (1974) and Solow (1974), subsequent works introduced non-convexities to the classical framework. Examples include Romer (1986), who incorporated increasing returns to scale; Lucas (1988), who modeled human capital formation; and Stokey (1988) and Schmitz (1989), who explored learning-by-doing mechanisms.[6] Despite these advancements, the simplest approach to generate endogenous growth remains the use of a linear technology, a method considered by Solow (1956) and pioneered by Gale and Sutherland (1968) and Benveniste (1976) following Harrod (1936) and Domar (1946).

Sustained growth requires an infinite, non-exhaustible inflow of production factors, an impossible condition in reality. Introducing an upper bound for capital would allow existing models to offer more realistic policy insights. Even in standard models with convex technologies, detrended variables eventually reach a steady state in the long run. However, resource exhaustion is inevitable for two reasons: first, while detrended variables stabilize, the underlying stock variables grow indefinitely; second, the steady state is often reached asymptotically or in the distant future, by which point resources may already be depleted.

This paper serves as a warning about the neglect of scarcity in economic models, highlighting the disparity between expected results under assumptions of infinite resources and the actual outcomes when limits are considered.

The rest of the paper is organized as follows. The model is presented in Section 2. Section 3 provides with the optimal analytical solution. Section 4 discusses the two main constraints: one on capital and the other one on consumption. Finally, Section 5 concludes. All proofs are gathered in the Appendix.

2 The bounded economy

As Ramsey (1928), we study an economy made of a representative agent and a unique final good firm, both with an infinite lifespan. Hence, there exists a unique good, which can be used both for consumption and investment. In this setup, there exists a policy maker whose aim is to maximize overall welfare, measured as the discounted sum of instantaneous utilities, which as in Ramsey (1928), depend solely on consumption, C. To fix ideas, and without loss of generality, we can assume that instantaneous utility is a CIES function with parameter σ, satisfying the following assumption

Assumption 1. $0 < \sigma < 1$.

Let us further assume, without loss of generality, that population is constant and equal to one at all times. The policy maker problem can be stated as

$$\max_{C} \int_0^\infty \frac{C(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \tag{1}$$

[6] Recent contributions include Cervellati et al. (2023) on unified growth theory, the World Bank's Long Term Growth Model (Loayza and Pennings, 2022), and Ahmed and Elfaki (2023) on green technological progress for sustainable growth.

where $\rho > 0$ is the time discount rate. The decisions of the policy maker are subject to the standard law of accumulation of physical capital, which states that at all times, the change in the stock of capital equals production minus consumption minus depreciated capital:

$$\dot{K}(t) = F(K(t)) - \delta K(t) - C(t), \tag{2}$$

where δ is the depreciation rate of capital and F is the final good production. We assume that production is linear in capital, that is, $F(K) = AK$, with A a positive constant. Regarding A, we assume that it is larger than depreciation:

Assumption 2. $0 < \delta < A$.

Furthermore, and as announced, we assume that K must remain positive at all times and that there exists a maximum level of attainable physical capital, $\bar{K}$:

$$0 \leq K(t) \leq \bar{K}, \quad \forall t \geq 0. \tag{3}$$

As usual, $K(0) = K_0$ is known to the policy maker. Note that not only is physical capital bounded from above, but also consumption cannot exceed production at any point in time:

$$0 \leq C(t) \leq AK(t). \tag{4}$$

In the classical AK model, a No-Ponzi Game condition is typically introduced to prevent excessive borrowing by imposing constraints on the discount rate. Here, under the explicit consumption constraint (4), consumption can never exceed total output. This constraint inherently prevents excessive borrowing, thus satisfying the No-Ponzi Game condition without needing any additional constraint.

3 Optimal solution

Following Pontryagin *et al.* (1962), we show next how to take into account possible optimal solutions in which $C(t) = 0$ or $C(t) = AK(t)$ for some t. Worth noting, these are not corner solutions along which consumption remains at a boundary value forever, but solutions which will eventually take those values at a given single t or during some time interval.

Let us define $k = \frac{K}{\bar{K}}$ and $c = \frac{C}{\bar{K}}$. The optimization problem (1)-(4) is equivalent to

$$\max_{c} \int_0^{\infty} \frac{c\,(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \tag{5}$$

subject to

$$\dot{k} = f\,(x, c) \equiv (A - \delta)k - c \qquad \text{for } t > 0, \qquad k\,(0) = k_0, \tag{6}$$

with the state constraint $k(t) \in [0, 1]$ and the control constraint $c(t) \in [0, Ak(t)]$ for all $t \geq 0$. k_0 is naturally defined as $k_0 = \frac{K(0)}{\bar{K}}$.

Then, as Pontryagin *et al.* (1962), we convert the state constraint into a control constraint by introducing the functions

$$g_1(k) = -k, \qquad g_2(k) = k - 1,$$

and

$$\begin{aligned} h_1(k,c) &= g_1'(k) f(k,c) = -(A-\delta)k + c, \\ h_2(k,c) &= g_2'(k) f(k,c) = (A-\delta)k - c. \end{aligned}$$

The control constraints are

$$\begin{aligned} h_1(k,c) &= 0, \qquad \text{if } k = 0 \quad \text{for a period of time,} \\ h_2(k,c) &= 0, \qquad \text{if } k = 1 \quad \text{for a period of time.} \end{aligned}$$

Specifically,

$$c = \begin{cases} 0, & \text{if } k = 0 \quad \text{for a period of time,} \\ A - \delta, & \text{if } k = 1 \quad \text{for a period of time.} \end{cases} \tag{7}$$

The HJB equation. We use dynamic programming to derive the Hamilton-Jacobi-Bellman equation (HJB hereafter). First, we define as usual the Hamiltonian function H associated with the optimization problem (5)-(6)

$$H(k,c,p) = \frac{c^{1-\sigma}}{1-\sigma} + p\left[(A-\delta)k - c\right],$$

with p, the co-state variable associated to k. Using dynamic programming and letting V denote the value function, we obtain

$$\rho V(k) = \max_{c \in S(k)} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + V'(k)\left[(A-\delta)k - c\right] \right\} \tag{8}$$

where $S(k)$ is the admissible control set. Specifically, $S(k) = [0, Ak]$ if $k \in (0,1)$, with $S(0) = \{0\}$ and $S(1) = \{A - \delta\}$. That is, if relative capital k is zero, then no investment in future capital is possible. The economy cannot get indebted. If on the contrary, relative capital is equal to 1, and physical capital equals its upper bound, then capital will optimally remain at the upper boundary in the future. Otherwise, future capital can take any other interior value between 0 and the upper bound.

The optimal control c^* is defined as usual as

$$c^*(k) = \arg\max_{c \in [0,Ak]} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + V'(k)\left[(A-\delta)k - c\right] \right\}$$

for $k \in (0,1)$, and as such, it verifies that

$$c^*(k) = \begin{cases} V'(k)^{-1/\sigma} & \text{if } V'(k) > (Ak)^{-\sigma}, \\ Ak & \text{if } V'(k) \leq (Ak)^{-\sigma} \end{cases} \tag{9}$$

provided that the value function is strictly monotonically increasing with k, that is, $V'(k) > 0$.

Then, substituting for c^* into (8), we have that the HJB equation is defined as

$$\rho V(k) = \begin{cases} \sigma V'(k)^{1-1/\sigma} / (1-\sigma) + (A-\delta)V'(k)\,k & \text{if } V'(k) > (Ak)^{-\sigma}, \\ (Ak)^{1-\sigma} / (1-\sigma) - \delta k V'(k) & \text{if } V'(k) \leq (Ak)^{-\sigma} \end{cases} \tag{10}$$

for any $k \in (0,1)$.

The dynamical system. We define next a two-dimensional dynamic system in k and p. Here we will make use of the fact that p is actually equal to the derivative of the value function as a function of k, that is, $p(t) = V'(k(t))$. This auxiliary system proves itself extremely convenient, providing with a useful tool to solve our problem analytically, and a full description of the optimal solution. As we know, the initial value $k(0)$ is given but this is obviously not the case of $p(0)$. Indeed, p is the derivative of the value function, which is yet to be found, and as part of the problem, it may be a function of the parameter set. In this regard, we will later need to divide the range of the time discount rate in three subintervals.

Let $P(k)$ denote $V'(k)$. By differentiation of the two sides of (10) with respect to k, we obtain

$$(\rho - A + \delta) P(k) = \begin{cases} \left[(A-\delta)k - P(k)^{-1/\sigma}\right] P'(k) & \text{if } P(k) > (Ak)^{-\sigma}, \\ A\left[(Ak)^{-\sigma} - P(k)\right] - \delta P'(k)\,k & \text{if } P(k) \leq (Ak)^{-\sigma}. \end{cases} \tag{11}$$

Using the equation of the dynamics in (6) with $c = c^*$ given by (9), the above equation can be written as

$$P'(k)\,\dot{k} = \begin{cases} (\rho - A + \delta) P(k) & \text{if } P(k) > (Ak)^{-\sigma}, \\ (\rho - A + \delta) P(k) - A\left[(Ak)^{-\sigma} - P(k)\right] & \text{if } P(k) \leq (Ak)^{-\sigma}. \end{cases} \tag{12}$$

Let $p(t)$ denote $P(k(t))$. We obtain the auxiliary dynamical system

$$\begin{aligned} \dot{k} &= (A-\delta)k - \min\left\{p^{-1/\sigma}, Ak\right\}, \\ \dot{p} &= (\rho - A + \delta)\,p - A\ \max\left\{(Ak)^{-\sigma} - p, 0\right\} \end{aligned} \tag{13}$$

for $(k,p) \in \mathbb{R}^+$.

Let us start by the case of a patient agent. We say that the time discount rate is small if

$$0 < \rho < A - \delta. \tag{14}$$

This inequality condition is commonly assumed in the neoclassical AK model to ensure a positive growth rate of both consumption and capital. We prove

Proposition 1 (Small time discount). *Let Assumptions 1 and 2 hold. Suppose that (14) holds and that $A-\delta$ and K_0 are positive constants. Then the optimal control $c^*(t)$ is given by*

$$c^*(t) = \begin{cases} \mu e^{gt}, & \text{for } 0 < t < T, \\ A-\delta, & \text{for } t \geq T \end{cases} \tag{15}$$

where

$$\mu = \frac{(A-\delta-g)\left(1-k_0 e^{(A-\delta)T}\right)}{e^{gT} - e^{(A-\delta)T}}, \tag{16}$$

and T is the unique solution to Eq. (17):

$$(A-\delta)\, e^{-gT} - g e^{-(A-\delta)T} = k_0 (A-\delta-g)\,, \tag{17}$$

with $k_0 = K_0/\bar{K}$ and $g = \frac{A-\delta-\rho}{\sigma} > 0$. The economy reaches the maximum value of physical capital when $t = T$, and is given by

$$k(t) = \begin{cases} \frac{\mu}{A-\delta-g} e^{gt} + \left(k_0 - \frac{\mu}{A-\delta-g}\right) e^{(A-\delta)t}, & \text{for } 0 < t < T, \\ 1 & \text{for } t \geq T. \end{cases} \tag{18}$$

Furthermore, the value function is given by $V(0) = 0$ and

$$V(k) = \frac{1}{\rho}\left[\frac{\sigma}{1-\sigma} P(k)^{1-1/\sigma} + (A-\delta) P(k)\, k\right] \qquad \text{for } 0 < k \leq 1,$$

where $P(k) = V'(k)$ is the solution to the differential equation

$$\left[(A-\delta)k - P(k)^{-1/\sigma}\right] P'(k) = (\rho - (A-\delta))\, P(k) \tag{19}$$

with the terminal condition

$$P(1) = (A-\delta)^{-\sigma}. \tag{20}$$

Proof. See Appendix A.1. □

In few words, Proposition 1 proves that the policy maker can (and will) optimally implement a policy whose first goal is to attain the maximum of capital in finite time, leading then to a second moment in which capital remains at that maximum forever after, allowing this way for the highest consumption. To achieve this goal, the policy maker chooses an increasing trajectory for consumption, culminating at the maximum attainable value for consumption in finite time. Using a well-known concept in the literature, we can say that the economy reaches its golden rule solution in finite time, and this, despite the positive time discount. Two facts make possible the reversal of the general wisdom about the unattainability of the golden rule solution, namely, that there exists a maximum within reach, and that agents are patient enough so as to wait the necessary time to reach it.

Proposition 1 partially has its counterpart in the neoclassical growth literature, where inequality (14) usually ensures a positive growth rate for consumption and capital. However, two fundamental differences arise from the assumption that capital is bounded from above. First, recall that the existing literature typically requires a lower bound on the discount rate ρ, $\rho > (A-\delta)(1-\sigma)$, to satisfy both the transversality condition and bounded attainable utility[7]. Unlike them, our model does not need this lower bound because assumption (3) makes the policy maker pursue and attain the golden rule in finite time, naturally ensuring both conditions. Indeed, equation (17) has one and only one finite solution $T > 0$ under assumption (3), and this regardless of whether the old boundary holds or not.

The second difference is that, in the classical AK-model model consumption and capital grow monotonically at the same rate $g = \frac{A-\delta-\rho}{\sigma}$ from $t = 0$. Before reaching its maximum at time T, consumption indeed also grows here at this rate, but that is not the case anymore for capital. When $t < T$, equation (18) clearly states that capital is made of the sum of two terms. The first term parallels the classical literature and it grows at the same rate as consumption, provided $\rho > (A-\delta)(1-\sigma)$. The second term is the extra bit which ensures that capital adjusts from its initial value K_0 to $\overline{K}$ over the initial phase. Indeed, rewriting the optimal solution for capital in (18) when $0 < t < T$:

$$k(t) = \frac{\mu}{A-\delta-g}\left[e^{gt} - e^{(A-\delta)t}\right] + k_0 e^{(A-\delta)t}.$$

Capital is strictly increasing over time until reaching the upper-bound in finite time, T, the unique solution of equation (17).

The following graphs illustrate the above results. Figure 1 presents the value function $V(k)$ (left) and its derivative $V'(k)$. As can be seen, $V(k)$ is strictly concave and increasing, with an infinite slope at $k = 0$; and the marginal value $V'(k)$ is strictly decreasing.

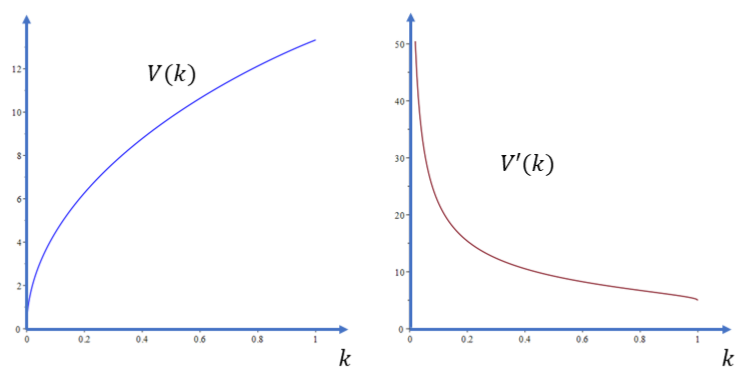

Figure 1: The value function (left) and its derivative (right) with small time discount.

Figure 2 shows the time-evolution of the capital $k(t)$ and the consumption $c(t)$. As the figure shows, both $k(t)$ and $c(t)$ are strictly increasing until $k(t)$ reaches its maximum.

[7]See again Chapter 4 in Barro and Sala-Martin (2004), or Boucekkine et al. (2013).

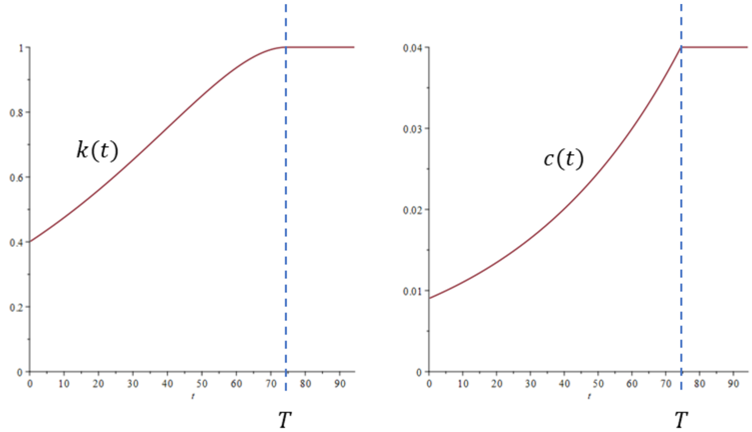

Figure 2: The changes of capital (left) and consumption (right) with time with small time discount.

We next consider the case where agents are slightly more impatient, namely

$$A - \delta \leq \rho < A - \delta + \sigma\delta. \tag{21}$$

Obviously, the situation in (21) has not been investigated yet in the literature, given that it would yield a negative growth rate for consumption. Exception being Dasgupta and Heal (1974, Proposition 8), who partially studied the case where $\rho > A - \delta$ when investigating the problem of optimal depletion of exhaustible resources. In their setting both consumption and capital converge to nil in the long-run as time goes to infinite.

Proposition 2 (Moderate time discount)**.** *Let Assumptions 1 and 2 hold. Suppose that (21) holds and that A and K_0 are positive constants. Then the optimal control is given by*

$$c^*(t) = \frac{\sigma k_0}{[\rho - (A - \delta)(1 - \sigma)]} e^{gt} \qquad \text{for } t > 0, \tag{22}$$

where $g = \frac{A-\delta-\rho}{\sigma} < 0$. With this control, the optimal stock of capital follows

$$k(t) = k_0 e^{gt} \qquad \text{for } t > 0 \tag{23}$$

and the value function is given by

$$V(k) = \frac{k^{1-\sigma}}{1-\sigma}\left\{\frac{\sigma}{\rho - (A - \delta)(1 - \sigma)}\right\}^{\sigma}. \tag{24}$$

Proof. See Appendix A.2. □

Proposition 2 depicts a complete different picture from that in Proposition 1, where consumption and capital were monotonically increasing functions, growing at different rates during an initial period, then becoming constant. We can distinguish two sub-cases when the time discount rate is moderate as in (21). In the case where $\rho = A - \delta$, both $c^*(t)$ and

$K(t)$ are time-independent. When $\rho > A - \delta$, both are decreasing exponential functions at the same rate $g(< 0)$. Consumption $c^*(t)$ is always positive and proportional to the initial minimum-maximum capital ratio $\frac{K_0}{K}(\leq 1)$. Here obviously the policy maker privileges current consumption in detriment of future consumption. In other words, consumption starts taking its maximum value and it decreases thereafter.

Although capital decreases monotonically, the vanishing of capital only occurs asymptotically. However, if there was a poverty threshold in terms of living standards, or if our setting included renewable resources and that a minimum survival level was essential, it is for sure that those thresholds would be crossed in finite time.

Like in models with Bang-bang controls, the upper threshold for the discount rate in equation (21)

$$\rho^* = A - \delta + \sigma\delta$$

establishes the frontier between economies with $\rho < \rho^*$, which never consume all output, $c^* < Ak$, and those beyond the threshold which do consume all output and for which $c^* = Ak$.[8]

Finally, we consider the case of extremely impatient agents where

$$\rho \geq \rho^* = A - (1 - \sigma)\,\delta. \tag{25}$$

We prove in this case

Proposition 3 (Large time discount)**.** *Let Assumptions 1 and 2 hold. Suppose that (25) holds, and that A and K_0 are positive constants. There is no investment at any time, consumption and capital decrease with time at rate δ. Specifically, the optimal pair is given by*

$$c^*(t) = Ak_0 e^{-\delta t}, \quad k(t) = k_0 e^{-\delta t} \qquad \text{for } t > 0, \tag{26}$$

Proof. See Appendix A.3. □

Recall that as in the case of moderate impatient agents, consumption and capital decrease with time. However, in this case with even more impatient agents, there is no investment in new capital at any point in time, the stock of physical capital continuously shrinks with time at the rate of depreciation. Production and consumption also necessarily decrease at the same rate δ, that is, the economy heads towards extinction at the same rate as capital wears down. Consumption and physical capital are at their maximum at the initial date.

4 Discussion of the two constraints

Compared to the classical AK-model, this study's main difference lies in constraints (3) and (4), which distinguish it from the classical models. In the original AK-model, these upper

[8]See Caputo (2005) for a detailed description of Bang-bang solutions to optimal control problems. In particular, the reader will find there an example on managing a renewable resource, such as a fishery, which is constrained by natural conditions. In this example, the control variable is the harvesting rate. The optimal strategy might involve switching between maximum harvesting (to maximize immediate profit) and no harvesting (to allow the resource to replenish), depending on the resource stock level.

constraints are not imposed.

However, in real-world scenarios, there may be physical limits to capital accumulation. For instance, finite natural resources or limited space for infrastructure may cap how much capital can grow.

Furthermore, while the AK-model assumes constant returns to scale in capital, in a broader economic context, other factors such as labor and technology might exhibit diminishing returns. This can indirectly impose a bound on capital because, beyond a certain threshold, additional capital does not significantly increase output due to limitations in other inputs.

Additionally, technological advancements and institutional frameworks can also restrict capital accumulation. Inefficiencies in financial markets, regulatory constraints, or technological ceilings can prevent unbounded capital growth.

Environmental and sustainability considerations provide further justification capital constraint. Excessive capital accumulation can lead to environmental degradation, potentially stifling further economic growth and limiting capital expansion. Finally, government policies, such as taxation, investment regulations, and capital controls, may also impose upper limits on capital accumulation to prevent economic imbalances or overheating.

These factors highlight that while the AK model offers a simplified view of economic growth driven by capital accumulation, real-world constraints often necessitate considering bounds on capital. However, when capital takes the form of human capital, knowledge-based capital or, more broadly speaking, intangible capital, it is possible for there might not be a clear upper bound.

Furthermore, control constraint (4) only takes effect with high discount rates (higher than the second threshold). Its primary impact is to slow the rate of capital decline. Without this constraint, capital still decreases exponentially but at a faster rate. Thus, for lower or moderate discount rates, this control constraint is ineffective, aligning with the classical AK model, where the discount rate's upper bound aligns with the first threshold, rendering this constraint redundant.

5 Conclusion

In conclusion, our analysis of the Ramsey (1928) model with time discounting and explicitly bounded, positive production factors offers significant new insights. By transforming state constraints into control constraints using Pontryagin's (1962) technique, we provided a comprehensive solution for all impatience rates under a linear production function. Our findings highlight that accounting for the finite nature of capital fundamentally reshapes the behaviors associated with the established threshold of impatience.

Previous models primarily focused on mildly impatient agents, neglecting the broader spectrum of impatience. We addressed this gap by demonstrating that highly patient agents efficiently drive the economy to its maximum capital level, achieving sustained consumption growth. Conversely, higher impatience rates result in stark declines in both capital and consumption, with extreme impatience halting investment altogether.

This study challenges the assumptions of traditional models, revealing that neglecting resource limitations can dangerously underestimate the risks of overconsumption and resource depletion. Our findings emphasize the urgent need to incorporate production factor constraints into economic modeling to provide more accurate and sustainable policy guidance for long-term economic stability.

A Appendix

A.1 Proof of Proposition 1

Value function. We first construct a value function $V(k)$. In the case where $k = 1$, by (7), $c = A - \delta$. Thus $\dot{k} = 0$ for all $t > 0$. This implies that $k = 1$ and $c = A - \delta$ for all $t > 0$. In this case

$$V(1) = \int_0^\infty \frac{(A-\delta)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt = \frac{(A-\delta)^{1-\sigma}}{(1-\sigma)\,\rho}. \tag{27}$$

From (10) we find that $P(k) \equiv V'(k)$ satisfies the terminal condition (20) and the differential equation (19) in a neighborhood of $k = 1$. To solve, we let

$$Q(k) = (A-\delta)k - P(k)^{-1/\sigma}.$$

Then, by (19) and (20),

$$Q'(k) = (A-\delta) + g - \frac{(A-\delta)gk}{Q(k)} \qquad \text{for } k \in (0,1), \qquad Q(1) = 0 \tag{28}$$

where

$$g = \frac{(A-\delta)-\rho}{\sigma}. \tag{29}$$

Let

$$z(k) = \left[\frac{Q(k)}{k}\right]^2. \tag{30}$$

Then, the initial value problem becomes

$$z' = \frac{2}{k}\left[((A-\delta)+g)\sqrt{z} - z - (A-\delta)g\right] \qquad \text{for } k \in (0,1), \qquad z(1) = 0.$$

This is a separable equation which can be solved explicitly, although the closed form solution is tedious. Luckily, we do not need its explicit solution. But if interested, it is easy to compute the solution numerically.

We show that z exists and is positive for $k \in (0,1)$. Note that $z'(1) = -2(A-\delta)g < 0$. Thus the solution exists and is positive in a neighborhood of $x = 1$. If there is $k^* \in (0,1)$ such that $z(k^*) = 0$, we may assume that k^* is the largest of such k. From the above equation we find

$$z'(k^*) = -\frac{2(A-\delta)g}{k^*} < 0.$$

However, this is impossible since it implies that $z(k) < 0$ for some $k > k^*$. Thus $z(k) > 0$ at all $k \in (0,1)$ such that it exists. The existence of $z(k)$ for the entire interval then follows from the existence and uniqueness theorem.

Since $Q(k) > 0$ for all $k \in (0,1)$, it follows that

$$P(k)^{-1/\sigma} < (A-\delta)k < Ak \qquad \text{for } k \in (0,1).$$

Hence, the solution

$$P(k) = [(A-\delta)k - Q(k)]^{-\sigma}$$

is valid on $(0,1)$.

Finally, we find the value function by (10), i.e.,

$$V(k) = \frac{1}{\rho}\left[\frac{\sigma}{1-\sigma}P(k)^{1-1/\sigma} + (A-\delta)P(k)\,k\right].$$

By (13), since $\rho < (A-\delta)$, $p(t)$ is decreasing in t. Since $p(t) = P(k(t))$ and $P(k)$ is decreasing in k, it follows that $k(t)$ is increasing in t. As a result, $k(t) = 1$ for all $t > 0$ if $k(0) = 1$. By (7), in this case $u(t) = (A-\delta)$ for all $t > 0$. This implies that

$$V(1) = \frac{(A-\delta)^{1-\sigma}}{1-\sigma}\int_0^\infty e^{-\rho t}dt = \frac{(A-\delta)^{1-\sigma}}{(1-\sigma)\rho}$$

which is the same as (27). Thus the value function constructed before Proposition 1 is valid. As a result,

$$p(t) = P(k(t)) > (Ak(t))^{-\sigma} \qquad \text{for all } t > 0.$$

Thus, the dynamical system (13) takes the form

$$\begin{aligned} \dot{k} &= (A-\delta)k - p^{-1/\sigma}, && k(0) = k_0, \\ \dot{p} &= [\rho - (A-\delta)]\,p, && p(0) = P(k_0) \end{aligned} \tag{31}$$

whenever $(k(t), p(t)) \in (0,1) \times (0,\infty)$. The solution is

$$p(t) = P(k_0)\,e^{(\rho-(A-\delta))t}, \qquad k(t) = \frac{P(k_0)^{-1/\sigma}}{(A-\delta)-g}e^{gt} + \left(k_0 - \frac{P(k_0)^{-1/\sigma}}{(A-\delta)-g}\right)e^{(A-\delta)t} \tag{32}$$

where $k_0 = K_0/\bar{K}$. Since $k(t)$ is a sum of exponential functions, it reaches $k = 1$ in finite time.

The time T at which $k = 1$ is the solution of the equation

$$1 = \frac{P(k_0)^{-1/\sigma}}{(A-\delta)-g}e^{gT} + \left(k_0 - \frac{P(k_0)^{-1/\sigma}}{(A-\delta)-g}\right)e^{(A-\delta)T}.$$

This equivalent to (33).

Finally, by (9) and (7),

$$c(t) = p(t)^{-1/\sigma} = P(k_0)^{-1/\sigma} e^{gt} \qquad \text{for } t < T$$

and $c(t) = A - \delta$ for $t \geq T$.

It remains to show that $P(k_0)e^{-1/\sigma}$ satisfies (16). For any constant $\tau > 0$ we define

$$c(t,\mu) = \begin{cases} \mu(\tau)e^{gt} & \text{for } 0 \leq t < \tau, \\ A - \delta & \text{for } t \geq \tau \end{cases}$$

and

$$k(t,\tau) = \begin{cases} \frac{\mu(\tau)}{A-\delta-g}\left[e^{gt} - e^{(A-\delta)t}\right] + k_0 e^{(A-\delta)t} & \text{if } t < \tau, \\ 1 & \text{if } t \geq \tau \end{cases}$$

where $\mu(\tau)$ is the solution to the equation

$$\frac{\mu(\tau)}{A-\delta-g}\left[e^{g\tau} - e^{(A-\delta)\tau}\right] + k_0 e^{(A-\delta)\tau} = 1. \tag{33}$$

We also define

$$J(k_0,\tau) = \int_0^\infty \frac{c(t,\tau)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt.$$

Clearly,

$$V(k_0) = \max_\tau J(k_0,\tau)$$

and the maximizer is $\tau = T$. Hence, (33) leads to (16). It remains to show that T satisfies (17).

By computation,

$$\begin{aligned} J(k_0,\tau) &= \frac{1}{1-\sigma}\left\{\int_0^\tau \mu(\tau)^{1-\sigma} e^{[(1-\sigma)g-\rho]t} dt + \int_\tau^\infty (A-\delta)^{1-\sigma} e^{-\rho t} dt\right\} \\ &= \frac{\mu(\tau)^{1-\sigma}}{(1-\sigma)[(1-\sigma)g-\rho]}\left[e^{[(1-\sigma)g-\rho]\tau} - 1\right] + \frac{(A-\delta)^{1-\sigma}}{(1-\sigma)\rho} e^{-\rho\tau}. \end{aligned}$$

Note that

$$A - \delta - g = \frac{1}{\sigma}\left[\rho - (1-\sigma)(A-\delta)\right] = \rho - (1-\sigma)g. \tag{34}$$

Hence,

$$J(k_0,\tau) = \frac{\mu(\tau)^{1-\sigma}}{(1-\sigma)(A-\delta-g)}\left[1 - e^{-(A-\delta-g)\tau}\right] + \frac{(A-\delta)^{1-\sigma}}{(1-\sigma)\rho} e^{-\rho\tau}$$

By differentiation,

$$\begin{aligned} J_\tau(k_0,\tau) &= \frac{\mu(\tau)^{-\sigma}\mu'(\tau)}{A-\delta-g}\left[1 - e^{-(A-\delta-g)\tau}\right] + \frac{\mu(\tau)^{1-\sigma}}{1-\sigma} e^{-(A-\delta-g)\tau} \\ &\quad - \frac{(A-\delta)^{1-\sigma} e^{-\rho\tau}}{1-\sigma}. \end{aligned}$$

By (33),

$$\mu(\tau) = \frac{(A-\delta-g)\left(e^{-g\tau} - k_0 e^{(A-\delta-g)\tau}\right)}{1 - e^{(A-\delta-g)\tau}}. \tag{35}$$

It follows that

$$\mu'(\tau) = \frac{(A-\delta-g)\, e^{(A-\delta-g)\tau}}{\left(e^{(A-\delta-g)\tau} - 1\right)^2} \left\{(A-\delta)\, e^{-g\tau} - k_0\,(A-\delta-g) - g e^{-(A-\delta)\tau}\right\}.$$

Therefore, $\mu'(\tau) = 0$ if

$$(A-\delta)\, e^{-g\tau} - g e^{-(A-\delta)\tau} = k_0\,(A-\delta-g).$$

It can be shown that the left-hand side is decreasing (respectively increasing) if $A-\delta-g > 0$ (respectively $A-\delta-g \leq 0$). Furthermore, the left-hand side is less (respectively greater) than the right-hand side at $\tau = 0$, it follows that the above equation has a unique positive solution. In view of (17), that solution is T.

We show that $J_\tau(k_0, T) = 0$. By (34),

$$\mu(T)^{1-\sigma} e^{-(A-\delta-g)T} = \left[\mu(T)\, e^{gT}\right]^{1-\sigma} e^{-gT}.$$

It suffices to show that $\mu(T)\, e^{gT} = A-\delta$. By (35), the equation is equivalent to

$$(A-\delta-g)\left(1 - k_0 e^{(A-\delta)T}\right) = (A-\delta)\left(1 - e^{(A-\delta-g)T}\right).$$

By multiplying $e^{-(A-\delta)T}$ to the both sides, the equation becomes

$$(A-\delta-g)\left(e^{-(A-\delta)T} - k_0\right) = (A-\delta)\left(e^{-(A-\delta)T} - e^{-gT}\right).$$

It is easy to see that the above equation is equivalent to (17). This proves that $J_\tau(k_0, T) = 0$.

Finally, it can be shown that $J_\tau(k_0, \tau) > 0$ for $\tau < T$. Thus T maximizes $J(k_0, T)$. This proves (15).

The proof is complete.

A.2 Proof of Proposition 2

We seek a solution of (10) in the form $V(k) = ak^{1-\sigma}$ with some constant a, in the case where $V'(k) > (Ak)^{-\sigma}$. Substituting this function into

$$\rho V(k) = \frac{\sigma}{1-\sigma} V'(k)^{1-1/\sigma} + (A-\delta) V'(k)\, k,$$

we obtain

$$\rho a k^{1-\sigma} = \frac{\sigma}{1-\sigma}\left[a\,(1-\sigma)\, k^{-\sigma}\right]^{1-1/\sigma} + (A-\delta) a\,(1-\sigma)\, k^{1-\sigma}.$$

This leads to

$$\rho = \sigma\left[a\,(1-\sigma)\right]^{-1/\sigma} + (A-\delta)\,(1-\sigma).$$

Thus

$$a = \frac{1}{1-\sigma} \left\{ \frac{\sigma}{\rho - (A-\delta)(1-\sigma)} \right\}^{\sigma}$$

and therefore,

$$V(k) = \frac{k^{1-\sigma}}{1-\sigma} \left\{ \frac{\sigma}{\rho - (A-\delta)(1-\sigma)} \right\}^{\sigma}. \tag{36}$$

For this solution to be valid, it is necessary that it complies with the initial assumption in this proof, namely that $V'(k) > (Ak)^{-\sigma}$, which is equivalent in this case to;

$$(1-\sigma)\, a > A^{-\sigma}.$$

Equivalently

$$\rho < (A-\delta) + \sigma\delta. \tag{37}$$

Therefore, the value function V in (36) is valid if (21) holds.

Using the value function in (36),

$$P(k) = \left[\frac{\sigma k}{\rho - (A-\delta)(1-\sigma)} \right]^{-\sigma} > (Ak)^{-\sigma}. \tag{38}$$

Thus, the dynamical system (13) takes the form (31). As a result,

$$p(t) = P(k_0)\, e^{[\rho-(A-\delta)]t} \qquad \text{for } t > 0.$$

$$c(t) = p(t)^{-1/\sigma} = P(k_0)^{-1/\sigma} e^{gt} = \frac{\sigma k_0 e^{gt}}{\rho - (A-\delta)(1-\sigma)}$$

Using the relation $C(t) = \bar{K} c(t)$ and $k_0 = K_0/\bar{K}$, we obtain (22).

To find $K(t)$, we solve the first equation in (31) to obtain

$$k(t) = \frac{P(k_0)^{-1/\sigma}}{(A-\delta)-g} e^{gt} + \left[k_0 - \frac{P(k_0)^{-1/\sigma}}{(A-\delta)-g} \right] e^{(A-\delta)t} \qquad \text{for } t > 0.$$

By (38),

$$P(k_0)^{-1/\sigma} = \frac{\sigma k_0}{\rho - (A-\delta)(1-\sigma)}$$

and by (29),

$$(A-\delta) - g = \frac{1}{\sigma} [\rho - (A-\delta)(1-\sigma)].$$

It follows that

$$\frac{P(k_0)^{-1/\sigma}}{(A-\delta)-g} = k_0.$$

Therefore,

$$k(t) = k_0 e^{gt} \qquad \text{for } t > 0.$$

for all $t > 0$. This is equivalent to (23).

This completes the proof.

A.3 Proof of Proposition 3

In the case where (25) holds, we seek a solution in the form $V(k) = bk^{1-\sigma}$. Substituting this function into

$$\rho V(k) = \frac{(Ak)^{1-\sigma}}{1-\sigma} - \delta k V'(k),$$

we obtain

$$\rho b k^{1-\sigma} = \frac{(Ak)^{1-\sigma}}{1-\sigma} - \delta(1-\sigma) b k^{1-\sigma}.$$

This leads to

$$b = \frac{A^{1-\sigma}}{[\rho + \delta(1-\sigma)](1-\sigma)}$$

and thus

$$V(k) = \frac{(Ak)^{1-\sigma}}{[\rho + \delta(1-\sigma)](1-\sigma)}. \tag{39}$$

This solution is valid if $V'(k) \geq (Ak)^{-\sigma}$, i.e.,

$$\frac{A}{\rho + \delta(1-\sigma)} \geq 1,$$

which is equivalent to (25).

Since by (39)

$$P(x) = \frac{A(Ax)^{-\sigma}}{\rho + \delta(1-\sigma)} \geq (Ax)^{-\sigma}, \tag{40}$$

the dynamical system (13) takes the form

$$\dot{x} = (A-\delta)x - Ax, \qquad \dot{p} = (\rho+\delta)p - A(Ax)^{-\sigma} \qquad \text{for } t > 0. \tag{41}$$

The first equation is equivalent to

$$\dot{x} = -\delta x.$$

This leads to

$$x(t) = x_0 e^{-\delta t} \qquad \text{for } t > 0$$

which is equivalent to the second relation in (26). Also, by (40) and (9),

$$c(t) = Ax(t) = Ax_0 e^{-\delta t}.$$

As a result,

$$C(t) = \bar{K}c(t) = \bar{K}Ax_0 e^{-\delta t},$$

hence the first relation in (26) follows.

The proof is complete.

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