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ARITHMETIC APPLICATIONS OF THE TRACE FORMULA AND ASYMPTOTIC ORTHOGONALITY OF TEMPERED REPRESENTATIONS

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Contents

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1 Introduction

This thesis collects two of the three articles which I have completed since the beginning of my Ph.D. studies at the University of Luxembourg. Both appeared as ArXiv pre-prints: Chapter 2 with the title *On C-Algebraic and C-Arithmetic Automorphic Representations*, and Chapter 3 as *On Kazhdan-Yom Din's Asymptotic Orthogonality for K-finite matrix coefficients of tempered representations*, a joint work with Anne-Marie Aubert.

I have decided not to include the article *Splitting fields of $X^n - X - 1$ (particularly for $n = 5$), prime decompositions and modular forms*, published by the journal Expositiones Mathematicae and written in collaboration with Chandrashekar B. Khare and with my Ph.D. advisor, Gabor Wiese, for two reasons: on one hand, I sought some level of unity of content. This thesis is concerned with representation theory of reductive groups; the flavour of the third article is, instead, rather number-theoretical. On the other hand, the central question in the third article and the ideas needed to answer it have been suggested to me by C. B. Khare and G. Wiese: my contribution has been of modest import.

The mathematical objects studied in this thesis are automorphic representations of reductive groups defined over number fields and tempered representations of real reductive groups. Understanding automorphic representations is one of the central themes of the Langlands program: a vast web of deep conjectures the aim of which is establishing powerful correspondences between representations of reductive groups and objects of arithmetic nature.

The tempered representations of real reductive groups are precisely those representations that account for the celebrated Plancherel decomposition theorem: it describes the structure of the right-regular representation of a real reductive group on the space of square-integrable functions defined on the group itself ([36], Chapters 13 and 14).

Because tempered representations are the key elements in the classification of admissible representations of real reductive groups ([29]), they have featured prominently in the context of the Langlands program since its very beginnings. In the field of non-commutative geometry, their importance is best exemplified by the role they cover in the Connes-Kasparov isomorphism theorem ([7]).

This introductory chapter serves two purposes. The first is presenting the main results proved in Chapter 2 and Chapter 3. The second is discussing their meaning within the broader context of modern research: how they relate to known results and open questions in the literature, which directions of investigation they suggest. I will proceed in a chapter-by-chapter fashion, beginning with Chapter 2.

1.1 Rational Structures in Automorphic Representations

The topic of Chapter 2 is a conjecture on the existence of rational structures within automorphic representations of reductive groups defined over number fields. The formulation of the conjecture adopted throughout the text is due to K. Buzzard and T. Gee ([6], Conjecture 5.15), and it reads as follows.

Conjecture 1.1.1. Let G be a connected, reductive algebraic group defined over a number field F and let π be an irreducible automorphic representation of $G(\mathbb{A}_F)$. Then π is C-algebraic if and only if it is C-arithmetic.

The notions of C-algebraic and C-arithmetic automorphic representation are defined precisely in Section 2.2. What is important for the purposes of this introduction is that being C-algebraic is a property of the Archimedean part of the automorphic representation: in fact, it depends only on the infinitesimal character of each Archimedean factor. Being C-arithmetic, instead, is a property of the non-Archimedean part of the automorphic representation. Roughly, it amounts to the existence of a number field which contains all the unramified Hecke eigenvalues of π.

Having introduced the motivating question, the first result from Chapter 2 that I would like to discuss is the following.

Theorem 1.1.2. Let G be a connected, semisimple, anisotropic algebraic group defined over $\mathbb{Q}$ and let $\pi_0 = \pi_{0,\infty} \otimes \pi_0^\infty$ be an irreducible automorphic representation of $G(\mathbb{A}_\mathbb{Q})$. Let K^∞ be a compact open subgroup of $G(\mathbb{A}_\mathbb{Q}^\infty)$ and assume the existence of a function $f \in C_c^\infty(G(\mathbb{R}))$ verifying the following conditions:

(1) There exists a finite family $\mathcal{C}$ of irreducible automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ containing π_0 such that

   (i) If $\pi \in \mathcal{C}$, then $\pi^{\infty,K^\infty} \neq \{0\}$ and $\mathrm{Tr}\pi_\infty(f) = 1$.

   (ii) If $\pi \notin \mathcal{C}$, then $\mathrm{Tr}\pi_\infty(f) = 0$.

(2) There exists a number field E such that, for every conjugacy class $\gamma \in \{G(\mathbb{Q})\}$, the Archimedean orbital integral $\mathcal{O}_\gamma(f)$ belongs to E.

Then the representation π_0 is C-arithmetic.

This theorem is proved as Corollary 2.3.3. In the statement, the symbol $\mathbb{A}_\mathbb{Q}$ denotes the ring of adeles of $\mathbb{Q}$, while $\mathbb{A}_\mathbb{Q}^\infty$ denotes the finite adeles. Given an irreducible, automorphic representation $\pi = \pi_\infty \otimes \pi^\infty$, the symbol π_∞ denotes the Archimedean factor of π, while π^∞ denotes the non-Archimedean part of π. The choice of the Haar measures implicit in the statement are explained in detail in Section 2.3.

The result above is meant as a proof of principle: it is established using the trace formula and it therefore indicates that the trace formula might be an important tool for questions concerning the existence of rational structures within automorphic representations. In Section 2.3, it is used to give a new proof of the following result (Corollary 2.3.4).

Theorem 1.1.3. Let G be a connected, semisimple, anisotropic algebraic group defined over $\mathbb{Q}$. Let $\pi_0 = \pi_{0,\infty} \otimes \pi_0^\infty$ be an irreducible automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with $\pi_{0,\infty}$ a regular discrete series. Then π_0 is C-arithmetic.

To the best of my knowledge, the only other known proof of this result relies on the fact that discrete series representations of $G(\mathbb{R})$, for G as in Theorem 1.1.3 are cohomological in the sense of [6], Definition 5.51, and that cuspidal automorphic representations with a cohomological Archimedean component are C-arithmetic. A proof of the former statement can be found in [5], Theorem 5.3. The proof of the latter fact relies on Matsushima's decomposition of the cohomology of locally symmetric spaces in terms of automorphic representations ([32], Proposition 2.15).

Although cohomological automorphic representations are C-algebraic ([6], Lemma 5.52), there exist C-algebraic automorphic representations which are not cohomological, even if we extend the notion of cohomological representation to include automorphic representations which contribute to the coherent cohomology of Shimura varieties ([16], 4.1.3 and 4.4.1).

These considerations suggest that it may be worthwhile to try to use the trace formula so as to obtain a generalisation of Theorem 1.1.2 powerful enough to treat more general groups and automorphic representations the Archimedean component of which is any irreducible, C-algebraic automorphic representation.

I would like to point out that the condition in Theorem 1.1.2 and Theorem 1.1.3 that G be semisimple is only meant to avoid complications arising from central characters. The condition that G be defined over $\mathbb{Q}$ is imposed since it is always possible to reduce the problem to this case. More precisely, an irreducible automorphic representation of a connected reductive group G defined over a number field F is C-algebraic if and only if it is C-algebraic as a representation of $\mathrm{Res}_{F/\mathbb{Q}}(G)$ ([6], after Definition 5.11). In addition, an irreducible automorphic representation of G is C-arithmetic if and only if it is C-arithmetic as a representation of $\mathrm{Res}_{F/\mathbb{Q}}(G)$ ([6], after Definition 5.13).

Generalising Theorem 1.1.2 so as to treat more general groups and C-algebraic automorphic representations whose Archimedean component is not a regular discrete series will probably require relaxing the condition that f be compactly supported. In view of the method developed in [4] to isolate cuspidal components in the automorphic spectrum, however, by allowing f to be a Schwartz function, it is possible to satisfy condition (i) of Theorem 1.1.2 provided that G is anisotropic. This point is discussed in more detail in Section 2.1, here I only want to point out that, given an irreducible, C-algebraic automorphic representation $\pi = \pi_\infty \otimes \pi^\infty$ of an anisotropic reductive group G, it is enticing to conjecture the existence of a Schwartz function f on $G(\mathbb{R})$ satisfying both (i) and (ii) of Theorem 1.1.2. Indeed, as we remarked above, the meaning of Conjecture 1.1.1 is that the arithmetic behaviour of π is determined by its Archimedean part π_∞ and condition (ii) in Theorem 1.1.2 is a purely Archimedean condition.

To conclude the discussion of Theorem 1.1.2, I believe that a few lines on the ideas that go into its proof are in order.

The idea of using the trace formula to approach questions of rationality of Hecke eigenvalues is inspired by [27] and it is worth to mention that the ideas in [27] led me first to the proof of Theorem 1.1.3: only afterwards I formulated Theorem 1.1.2 by isolating conditions (i) and (ii).

In order to provide a rough sketch of the parts of [27] which are the most relevant for the present exposition, consider an irreducible, cuspidal automorphic representation $\pi_\delta = \pi_{\delta,\infty} \otimes \pi_\delta^\infty$ of $GL_2/\mathbb{Q}$ with $\pi_{\delta,\infty}$ an integrable discrete series. The authors develop a trace formula which allows for test functions with a matrix coefficient of $\pi_{\delta,\infty}$, say ϕ, as Archimedean factor. Langlands proved that if π_∞ is any unitary representation of $G(\mathbb{R})$, then

$$\pi_\infty(\phi) = 0$$

unless $\pi_\infty \cong \pi_{\delta,\infty}$ ([27], Corollary 10.29). This result is used to isolate in the spectral side of the trace formula a finite direct sum of irreducible, cuspidal automorphic representations of $GL_2(\mathbb{A}_\mathbb{Q})$ all having Archimedean part isomorphic to $\pi_{\delta,\infty}$, and to show that the action of any given classical Hecke operator T acting on the space of cusp forms corresponding to this finite direct sum can be computed as the action of an operator of the form

$$R(\phi \otimes h) : L^2([GL_2]) \longrightarrow L^2([GL_2]),$$

where $h \in C_c^\infty(GL_2(\mathbb{A}_\mathbb{Q}^\infty))$ is U-bi-invariant for an appropriate compact subgroup U of $GL_2(\mathbb{A}_\mathbb{Q}^\infty)$, and

$$[GL_2] := \overline{GL_2}(\mathbb{Q})\backslash\overline{GL_2}(\mathbb{A}_\mathbb{Q})$$

with

$$\overline{GL_2} := Z_{GL_2}\backslash GL_2.$$

Explicit computations of the orbital integrals for the test function

$$\phi \otimes h \in C_c^\infty(GL_2(\mathbb{A}_\mathbb{Q}))$$

provide a formula for the trace of the operator T, showing that it is an algebraic number. The observation that T^n is a $\mathbb{Q}$-linear combination of classical Hecke operators allows the authors to apply the trace formula to each of the classical Hecke operator in the linear combination to deduce that, for every $n \in \mathbb{N}$, the trace of T^n is an algebraic number and to conclude, by an application of the Newton-Girard formula ([27], Proposition 28.1), that the Hecke eigenvalues of T are algebraic numbers (algebraic integers, in fact).

I will now adopt the notation of Theorem 1.1.2. The proof of this result does not rely on any explicit computation of the geometric side of the trace formula. The non-Archimedean orbital integral of a $\mathbb{Q}$-valued Hecke operator is a rational number by [3], Theorem 3. The volume-terms are also rational numbers if we equip the centraliser groups appearing on the geometric side of the trace formula with Gross' measure. In order to use the trace formula to compute the trace of

$$(\pi_0^\infty(h))^n,$$

for h a $\mathbb{Q}$-valued Hecke operator and $n \in \mathbb{N}$, I use directly the formula

$$(\pi_0(h))^n = \pi_0(h^{*n}),$$

where h^{*n} is the convolution of h with itself n times. These considerations suffice to apply the Newton-Girard formula to prove that the Hecke eigenvalues are algebraic numbers, but they are not enough to establish the much stronger statement that π_0 is C-arithmetic. This, however, follows from the next result, the proof of which relies on an observation of G. Wiese.

Proposition 1.1.4. Let V be a finite-dimensional, complex vector space of dimension n. Let $\mathcal{T}_\mathbb{Q} := \{T_\alpha : V \longrightarrow V\}_{\alpha\in A}$ be a family of diagonalisable, commuting linear operators, closed under taking finite $\mathbb{Q}$-linear combinations. Assume the existence of a number field E such that the characteristic polynomial $\mathrm{char}_\alpha(x)$ of T_α has coefficients in E for every $\alpha \in A$. Then there exists a number field E' such that the eigenvalues of T_α belong to E' for every $\alpha \in A$.

Proving Theorem 1.1.3 once Theorem 1.1.2 is available is a matter of finding a function $f \in C_c^\infty(G(\mathbb{R}))$ which satisfies conditions (i) and (ii). By [10], Théorème 3 and [2], Corollary 6.2, a pseudo-coefficient ϕ for $\pi_{0,\infty}$ satisfies condition (i). Condition (ii) follows from [11], Theorem 4.1, which computes the orbital integral of ϕ at a semisimple γ as

$$\mathcal{O}_\gamma(f) = d_\gamma \mathrm{Tr}\rho(\gamma),$$

where d_γ is an integer, ρ is a finite-dimensional, irreducible, algebraic representation of $G(\mathbb{R})$, and from the fact that ρ is defined over a number field.

The last section of Chapter 2 is devoted to the proof of the following result (Theorem 2.4.7).

Theorem 1.1.5. Let F be a number field. If every irreducible, C-algebraic, cuspidal automorphic representation of every connected, reductive algebraic group defined over F is C-arithmetic, then every irreducible, C-algebraic automorphic representation of every connected, reductive algebraic group defined over F is C-arithmetic, and every irreducible, L-algebraic automorphic representation of every connected, reductive algebraic group defined over F is L-arithmetic.

The notions of L-algebraic and L-arithmetic automorphic representation are defined in Section 2.2. Given an irreducible automorphic representation $\pi = \pi_\infty \otimes \pi^\infty$ of a group G as in the statement, the notion of being L-algebraic depends only on the infinitesimal character of each factor of π_∞. Roughly, being L-arithmetic amounts to the existence of a number field over which all the Satake parameters are defined. These two notions parallel the notions of C-algebraic and C-arithmetic automorphic representation in a precise sense. I refer the reader to [6] for a discussion of the motivation that led to introduce these four notions. K. Buzzard and T. Gee formulated the following conjecture ([6], Conjecture 5.14).

Conjecture 1.1.6. Let G be a connected, reductive algebraic group defined over a number field F and let π be an irreducible automorphic representation of $G(\mathbb{A}_F)$. Then π is L-algebraic if and only if it is L-arithmetic.

The purpose of Theorem 1.1.5 is reducing the proof of the implications

$$C - \text{algebraic} \implies C - \text{arithmetic}$$

and

$$L - \text{algebraic} \implies L - \text{arithmetic}$$

in Conjecture 1.1.1 and Conjecture 1.1.6, respectively, to the proof of the single statement that every irreducible, C-algebraic, cuspidal automorphic representation of every connected, reductive algebraic group G is C-arithmetic. I was led to its formulation by the heuristic idea discussed above that the trace formula might be the right tool to prove that irreducible, C-algebraic, cuspidal automorphic representations are C-arithmetic, by the main theorem in [30] stating that every irreducible automorphic representation is a subquotient of a representation parabolically induced from an irreducible, cuspidal automorphic representation, and by [32], Lemma 2.9, which studies how the properties of being C-algebraic and of being C-arithmetic behave under unnormalised parabolic induction.

The proof of Theorem 1.1.5 relies heavily on the results in [6] and on [30]. Given an irreducible automorphic representation π of $G(\mathbb{A}_F)$, with G a connected, reductive algebraic group defined over the number field F, I argue in a case-by-case fashion according to the following four possibilities:

(I) The representation π is cuspidal and C-algebraic.

(II) The representation π is cuspidal and L-algebraic.

(III) The representation π is L-algebraic and not cuspidal.

(IV) The representation π is C-algebraic and not cuspidal.

In case (I), Theorem 1.1.5 is true by assumption.

In case (II), by combining some of the results in [6], it follows that there exist a central extension $\tilde{G}$ of G and an irreducible, cuspidal automorphic representation $\tilde{\pi}$ of $\tilde{G}$ such that π is C-algebraic (respectively L-algebraic) if and only if $\tilde{\pi}$ is C-algebraic (respectively L-algebraic), and such that π is C-arithmetic (respectively L-arithmetic) if and only if $\tilde{\pi}$ is C-arithmetic (respectively L-arithmetic). Moreover, the central extension $\tilde{G}$ has the property that every irreducible, C-algebraic automorphic representation can be twisted into an irreducible, L-algebraic automorphic representation (and vice-versa), and that every irreducible, C-arithmetic automorphic representation can be twisted into an irreducible, L-arithmetic automorphic representation (and vice-versa). Hence, if π is L-algebraic and cuspidal, so is $\tilde{\pi}$. We can twist the latter into a cuspidal C-algebraic one. This twist is C-arithmetic by assumption, which shows that $\tilde{\pi}$ is L-arithmetic and, therefore, so is π.

In case (III), proving the statement requires establishing the following counterpart to [32], Lemma 2.9. (Proposition 2.4.6).

Proposition 1.1.7. Let G be a connected, reductive algebraic group defined over a number field F. Let π be an irreducible automorphic representation of $G(\mathbb{A}_F)$. Let P be a proper parabolic subgroup of G with Levi factor M and let σ be an irreducible, cuspidal automorphic representation of $M(\mathbb{A}_F)$ such that π is a constituent of $\mathrm{Ind}_P(\sigma)$. Then the following two statements hold:

(1) The representation π is L-algebraic if and only if σ is L-algebraic.

(2) If σ is L-arithmetic, then π is L-arithmetic.

It is important to remark that in Proposition 1.1.7 is the normalised parabolic induction that is used, as opposed to the unnormalised one in [32], Lemma 2.9. Having established Proposition 1.1.7, and using the main result in [30], which is formulated in terms of normalised parabolic induction, it is possible to reduce the proof in case (III) to that in case (II).

In case (IV), it is tempting to appeal to [32], Lemma 2.9. However, in order to do so, one would need to know that an irreducible, C-algebraic automorphic π can be realised as an irreducible subquotient of the unnormalised parabolic induction of an irreducible, cuspidal, C-algebraic automorphic representation. Since the main result of [30] is formulated in terms of normalised parabolic induction, this is not obvious. I chose a different route: the representation π can be lifted to a C-algebraic representation $\tilde{\pi}$ of $\tilde{G}$. Twisting $\tilde{\pi}$ into an L-algebraic automorphic representation and reasoning as in case (III), this twist can be shown to be L-arithmetic. This shows that $\tilde{\pi}$ is C-arithmetic and, therefore, so is π.

Having exhausted the contents of Chapter 2, the remaining part of this introduction will be dedicated to Chapter 3.

1.2 Asymptotic Schur's Orthogonality

Unless otherwise stated, let G be a connected, semisimple, real Lie group with finite centre, let $\mathfrak{g}$ denote its Lie algebra, and let K be a fixed maximal compact subgroup of G. We fix a Haar measure, dg, on G.

If (π, H) is an irreducible Hilbert representation of G (Section 4.2), we say that (π, H) is tempered if there exist $v, w \in H$ such that

$$\int_G |\langle \pi(g)v, w\rangle|^{2+\epsilon}\, dg < \infty$$

for every $\epsilon > 0$.

Consider the function

$$\mathbf{r} : G \longrightarrow \mathbb{R}_{>0},\ \mathbf{r}(g) := \log(\max\{\|\mathrm{Ad}(g)\|_{op}, \|\mathrm{Ad}(g^{-1})\|_{op}\}),$$

where $\|\cdot\|_{op}$ denotes the operator norm taken with respect to any chosen norm on $\mathfrak{g}$. Chapter 3 is devoted to the proof of the following result (Theorem 3.4.4).

Theorem 1.2.1. Let G be a connected, semisimple real Lie group with finite centre and let (π, H) be a tempered, irreducible Hilbert representation of G. Then there exist $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$ and $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all K-finite vectors $v_1, v_2, v_3, v_4 \in H$, we have

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle},$$

where, for $r \in \mathbb{R}_{>0}$, the set $G_{<r}$ is defined as

$$G_{<r} := \{g \in G | \mathbf{r}(g) < r\}.$$

The immediate motivation behind Theorem 1.2.1 was to complete the Archimedean part of Theorem 1.7 in [23], which we recall immediately below. At the end of this section, I will suggest how this result points toward a deeper understanding of the structure of the right-regular representation of G.

Theorem 1.2.2. Let (π, H) be a tempered, irreducible, unitary representation of a connected, semisimple group G defined over a local field let and K be a maximal compact subgroup of G. Then there exists $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$ such that:

(1) If G is non-Archimedean, there exists $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all K-finite vectors $v_1, v_2, v_3, v_4 \in H$, we have

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

(2) If G is Archimedean, for any given non-zero K-finite vectors $v_1, v_2 \in H$, there exists $C(v_1, v_2) > 0$ such that

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} |\langle \pi(g)v_1, v_2\rangle|^2\, dg = C(v_1, v_2).$$

The proof of the first statement in the result above occupies a few lines only, while the proof of Theorem 1.2.1 constitutes most of this thesis. One explanation is that the proof offered here may not be the shortest possible. However, there is a significant asymmetry between the Archimedean and the non-Archimedean setting in this context. The space of K-finite vectors of an irreducible, unitary representation of a semisimple group G defined over a non-Archimedean field affords a smooth representation of G. This is not the case in the Archimedean setting: the space of K-finite vectors is a $(\mathfrak{g}, K)$-module (Definition 3.2.6); in general, it is not stable under the action of G.

A detailed explanation of the strategy adopted to prove Theorem 1.2.1 is given in Section 4.1, which begins with an overview of the main parts of the proof and an explanation of how they fit together, and ends with a more accurate account of each part organised in a section-by-section fashion. For this reason, here I will only stress two points about the proof. The first is that I have relied heavily on the techniques to obtain the asymptotic expansion of matrix coefficients relative to standard parabolic subgroups of G developed in [24], Chapter VIII, Section 12. I found the account in loc. cit. to be the most useful for the problem at hand and several of the ideas needed for the proof of Theorem 1.2.1 have been inspired by its reading. The second is that although many of the intermediate results proved in the final part of Section 4.3 appear in some form in loc. cit., I had to provide complete proofs for two reasons. First, I needed to make sure that they apply in the situation at hand, which is not obvious as it will become apparent by reading Section 4.1 and Section 4.3. Second, even in the form in which these statements appear in loc. cit., their proofs are either only sketched or entirely absent.

To conclude this chapter, I would like to propose some questions arising from [23] and from Theorem 1.2.1. In order to do so, I will begin by recalling Schur's orthogonality for discrete series ([27], Proposition 10.25) and Plancherel's theorem.

Theorem 1.2.3. Let G be a connected, semisimple real Lie group with finite centre. Let (π_0, H_0) and (π, H) be discrete series representations of G. Let $v_0, w_0 \in H_0$ and $v, w \in H$. If (π_0, H_0) is equivalent to (π, H), then

$$\int_G \langle \pi_0(g)v_0, w_0\rangle \overline{\langle \pi(g)v, w\rangle}\, dg = \frac{1}{\mathbf{f}(\pi)} \langle v_0, v\rangle \overline{\langle w_0, w\rangle}.$$

If (π_0, H_0) is not equivalent to (π, H), then

$$\int_G \langle \pi_0(g)v_0, w_0\rangle \overline{\langle \pi(g)v, w\rangle}\, dg = 0.$$

The notation $\mathbf{f}(\pi)$ is not ambiguous: if (π, H) is a discrete series, then Theorem 1.2.1 specialises to Theorem 1.2.3. Moreover, still assuming that (π, H) is a discrete series, the quantity $\mathbf{d}(\pi)$ admits an elegant interpretation in the context of Plancherel's theorem. To recall it, let $\hat{G}$ denote the set of equivalence classes of irreducible, unitary representations of G.

Theorem 1.2.4. Let G be a connected, semisimple real Lie group with finite centre. Then there exists a unique measure η on $\hat{G}$ such that, for every $\phi \in L^1(G) \cap L^2(G)$ and for every $\pi \in \hat{G}$, the operator

$$\pi(\phi) : H \longrightarrow H$$

is trace-class and

$$\int_G |\phi(g)|^2\, dg = \int_{\hat{G}} \mathrm{Tr}(\rho(\phi)\rho(\phi)^*)\, d\eta(\rho),$$

where $\rho(\phi)^*$ denotes the adjoint of the operator $\rho(\phi)$.

The measure η is known as the Plancherel measure, and a non-trivial consequence of the Plancherel theorem is the following ([12], Proposition 18.8.5).

Corollary 1.2.5. Let G be a connected, semisimple real Lie group with finite centre and let (π, H) be a discrete series representation of G. Then the singleton $\{\pi\} \subset \hat{G}$ has positive Plancherel measure and we have

$$\eta(\{\pi\}) = \mathbf{f}(\pi).$$

This is a reflection of the fact that every discrete series (π, H) of G is equivalent to an irreducible direct summand of the right-regular representation $(R, L^2(G))$ ([27], Theorem 10.19). For this reason, if (π, H) is tempered but not a discrete series, one should not expect such a simple characterisation of $\mathbf{f}(\pi)$. Nevertheless, considering a K-finite $v \in H$ of norm 1, setting

$$\phi_r(g) := \langle \pi(g)v, v\rangle \mathbb{1}_{G_{<r}}$$

for $r \in \mathbb{R}_{>0}$ and applying Theorem 1.2.4, it follows that

$$\int_{G_{<r}} |\langle \pi(g)v, v\rangle|^2 \, dg = \int_{\hat{G}} \operatorname{Tr}(\rho(\phi_r)\rho(\phi_r)^*) \, d\eta(\rho),$$

and upon multiplying by $\frac{1}{r^{\mathbf{d}(\pi)}}$ and taking the limit as $r \to \infty$, an application of Theorem 1.2.1 yields

$$\frac{1}{\mathbf{f}(\pi)} = \lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{\hat{G}} \operatorname{Tr}(\rho(\phi_r)\rho(\phi_r)^*) \, d\eta(\rho).$$

This may be seen as a reflection of the fact that a tempered (π, H) embeds asymptotically in the right-regular representation $(R, L^2(G))$ of G ([23], Corollary 3.13).

These considerations suggest to investigate the meaning of $\mathbf{d}(\pi)$ and $\mathbf{f}(\pi)$ in the context of the Plancherel decomposition of $(R, L^2(G))$, a question already raised in [23], and it has been proposed by A-M. Aubert that they may also have an interpretation in the framework of the conjectures of K. Hiraga, A. Ichino and T. Ikeda ([22]). Hopefully, this will be the topic of a future work.

2 On C-Algebraic and C-Arithmetic Representations

2.1 Introduction

Understanding the arithmetic of Hecke operators acting on automorphic representations is, by now, a classical problem. In their article [6], K. Buzzard and T. Gee introduced the notion of C-algebraic automorphic representation: this is an Archimedean notion, in the sense that it depends only on the infinitesimal character of the Archimedean component of the automorphic representation. They conjectured that C-algebraic automorphic representations are C-arithmetic; that is: there exists a number field that contains all the unramified Hecke eigenvalues (see Section 2 for precise definitions).

To the best of our knowledge, the only automorphic representations which are known to be C-arithmetic are those whose non-Archimedean part contributes to the cohomology of a locally symmetric space or to the coherent cohomology of a Shimura variety.

Many C-algebraic automorphic representations do not admit such a geometric realisation. It is for this reason that we begin an investigation of the conjecture of K. Buzzard and T. Gee based on the trace formula.

It seems that the only attempt to employ the trace formula to study this kind of question has been made by A. Knightly and C. Li in [27]: they showed that the Hecke eigenvalues of automorphic representations of GL_2 whose Archimedean component is an integrable discrete series are algebraic integers. Their method requires an explicit computation of the geometric side of the trace formula and it does not establish the C-arithmetic property.

In the context of connected, semisimple anisotropic algebraic groups defined over $\mathbb{Q}$, we establish the following criterion (see Corollary 2.3.3, sections 2 and 3 for notation and for the relevant choices of Haar measures):

Theorem 2.1.1. Let G be a connected, semisimple, anisotropic algebraic group defined over $\mathbb{Q}$ and let $\pi_0 = \pi_{0,\infty} \otimes \pi_0^\infty$ be an irreducible, automorphic representation of $G(\mathbb{A}_\mathbb{Q})$. Let K^∞ be a compact open subgroup of $G(\mathbb{A}_\mathbb{Q}^\infty)$ and assume the existence of a function $f \in C_c^\infty(G(\mathbb{R}))$ verifying the following conditions:

(1) There exists a finite family $\mathcal{C}$ of irreducible, automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ containing π_0 such that

  (i) If $\pi \in \mathcal{C}$, then $\pi^{\infty, K^\infty} \neq \{0\}$ and $\mathrm{Tr}\pi_\infty(f) = 1$.

  (ii) If $\pi \notin \mathcal{C}$, then $\mathrm{Tr}\pi_\infty(f) = 0$.

(2) There exists a number field E such that, for every conjugacy class $\gamma \in \{G(\mathbb{Q})\}$, the Archimedean orbital integral $\mathcal{O}_\gamma(f)$ belongs to E.

Then there exists a number field E' and a finite set of places, S, containing ∞ and all the places at which π_0 ramifies, such that for every $p \notin S$ and for every $\mathbb{Q}$-valued Hecke operator $h \in \mathcal{H}_\mathbb{Q}(G(\mathbb{Q}_p), K_p)$, the eigenvalue of the operator

$$\pi_{0,p}(h) : \pi_{0,p}^{K_p} \longrightarrow \pi_{0,p}^{K_p}$$

is contained in E'; that is: the representation π_0 is C-arithmetic.

The result above leads to a simple, non-cohomological proof that automorphic representations of connected, semisimple, anisotropic algebraic groups defined over $\mathbb{Q}$ whose Archimedean component is a regular discrete series are C-arithmetic (see Corollary 2.3.4).

To treat more general automorphic representations, the requirement that the test function in the criterion be smooth and compactly supported is too restrictive. In view of the results in [4], we should probably allow Schwartz functions. However, for us, the main point at this stage is making the case that the trace formula can be used to investigate arithmetic properties of automorphic representations. Since smooth compactly supported functions are enough for the application in this article, we decided to state our criterion in this form.

Concerning (1), the article [4] provides a very powerful method to isolate cuspidal automorphic representations and we are currently exploring the possibility of employing it in our context. One difficulty is that the multipliers constructed in [4] are global objects: roughly, it is needed to modify a given test function by introducing Hecke operators which are, a priori, not $\mathbb{Q}$-valued (see, for example, [4] Proposition 3.17, already in the case $M = G$). We cannot afford this freedom in our context: we can only use Archimedean multipliers. For anisotropic groups, however, combining Lemma 2.18 and the 'Archimedean part' of Proposition 3.17, it seems possible to satisfy (1) of the criterion.

Concerning (2), we don't feel confident enough to formulate a precise conjecture, but, at least in this simplified picture, it could indicate why a purely Archimedean condition (C-algebraicity) might constrain the behaviour of the non-Archimedean part of the automorphic representation.

The proof that the Hecke eigenvalues are algebraic (Proposition 2.3.1) is inspired by the method in [27]. To establish the C-arithmetic property we exploit an observation of G. Wiese (Proposition 2.3.2).

Having in mind the idea of employing the trace formula to treat non-cuspidal automorphic representations of more general reductive groups, in Section 4 we reduce the conjecture of K. Buzzard and T. Gee that C-algebraic automorphic representations are C-arithmetic to the analogous statement for cuspidal representations (Theorem 2.4.7).

2.2 Background

Let us begin by recalling some notions introduced in [6]. Our treatment follows [14], as well. The notions of L-algebraic and L-arithmetic automorphic representation are needed in Section 4 only.

Let G be a connected reductive algebraic group defined over a number field F. Let v be an Archimedean place of F, and F_v be the completion of F at v. We identify the algebraic closure $\overline{F}_v$ of F_v with $\mathbb{C}$. Let T_v be a maximal torus of $G_{\mathbb{C}}$ and B_v a Borel subgroup containing T_v. Let ρ_{B_v} denote half of the sum of the positive roots corresponding to this choice, and write $X^*(T_v)$ for the group of characters of T_v. Finally, let $\mathfrak{t}_v$ denote the Lie algebra of T_v and write $\mathfrak{t}_v^{\mathbb{C}} := \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 2.2.1. An irreducible admissible representation π of $G(F_v)$ with infinitesimal character λ_π is C-algebraic if $\lambda_\pi - \rho_{B_v} \in X^*(T_v)$.

Definition 2.2.2. An irreducible admissible representation π of $G(F_v)$ with infinitesimal character λ_π is L-algebraic if $\lambda_\pi \in X^*(T_v)$.

The reader familiar with these notions will realise that they are not formulated as in [6]. However, as it is explained in loc. cit. and in Lemma 12.8.1 in [14], our formulation is equivalent to the one in [6]. Also, these notions are independent of the choice of B_v.

Let $\mathbb{A}_F$ denote the adèle ring of F and $\mathbb{A}_F^\infty$ (resp. $\mathbb{A}_{F,\infty}$) the ring of finite adèles (resp. the Archimedean part of the ring of adèles). Fix a compact open subgroup $K^\infty = \Pi_v K_v$ of $G(\mathbb{A}_F^\infty)$ with K_v a hyperspecial maximal compact subgroup equal to $G(\mathcal{O}_v)$ for all v at which $G(F_v)$ is unramified. Let K_∞ denote a maximal compact subgroup of $G(\mathbb{A}_{F,\infty})$ and form the compact subgroup
$K := K_\infty \times K^\infty$ of $G(\mathbb{A}_F)$. The local definitions above lead to the following global notions:

Definition 2.2.3. An irreducible automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A}_F)$ is C-algebraic if π_v is C-algebraic for every Archimedean place v of F.

Definition 2.2.4. An irreducible automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A}_F)$ is L-algebraic if π_v is L-algebraic for every Archimedean place v of F.

We refer the reader to [6] for a detailed discussion of the rationale for introducing these notions and for how they are related to other aspects of the Langlands Program. We proceed to introduce the purely non-Archimedean notions of C-arithmetic and L-arithmetic automorphic representations.

Let v be a non-Archimedean place of F at which $G(F_v)$ is unramified. Let $\mathcal{H}_{\mathbb{C}}(G(F_v), K_v)$ denote the $\mathbb{C}$-algebra of bi-K_v-invariant, compactly supported, complex-valued functions on $G(F_v)$, and $\mathcal{H}_{\mathbb{Q}}(G(F_v), K_v)$ denote the $\mathbb{Q}$-subalgebra of $\mathbb{Q}$-valued elements in $\mathcal{H}_{\mathbb{C}}(G(F_v), K_v)$. It is well-known that if π is a smooth irreducible admissible representation of $G(F_v)$ with non-trivial K_v-invariant vectors, then $\mathcal{H}_{\mathbb{C}}(G(F_v), K_v)$ acts by a character

$$\mathcal{H}_{\mathbb{C}}(G(F_v), K_v) \longrightarrow \mathbb{C}$$

on the 1-dimensional space of K_v-invariants π^{K_v}.

Definition 2.2.5. Let E be a number field. Let v be a non-Archimedean place of F at which $G(F_v)$ is unramified. We say that a smooth irreducible admissible representation π of $G(F_v)$ is defined over E if the image of the map

$$\mathcal{H}_{\mathbb{Q}}(G(F_v), K_v) \longrightarrow \mathbb{C}$$

induced by the character $\mathcal{H}_{\mathbb{C}}(G(F_v), K_v) \longrightarrow \mathbb{C}$ is contained in E.

Definition 2.2.6. An irreducible automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A}_F)$ is C-arithmetic if there exist a number field E and a finite set of places, S, containing all the Archimedean places and all the places at which π_v ramifies, such that π_v is defined over E for all $v \notin S$.

It only remains to introduce the notion of L-arithmetic automorphic representation.

Again, let v be a place of F at which $G(F_v)$ is unramified. Let T_v be a maximal torus of $G(F_v)$ and B_v a Borel subgroup containing T_v. Recall that the Satake isomorphism identifies $\mathcal{H}_{\mathbb{C}}(G(F_v), K_v)$ with $\mathbb{C}\left[X_*(T_{v,d})\right]^{W_{v,d}}$, where $X_*(T_{v,d})$ denotes the maximal split subtorus of T_v and $W_{v,d}$ is the subgroup of the Weyl group of $G(F_v)$ consisting of the elements leaving $T_{v,d}$ stable. Given a smooth irreducible admissible representation π of $G(F_v)$, we thus obtain a map

$$\mathbb{C}\left[X_*(T_{v,d})\right]^{W_{v,d}} \longrightarrow \mathbb{C}$$

by pre-composing the character $\mathcal{H}_{\mathbb{C}}(G(F_v), K_v) \longrightarrow \mathbb{C}$ with the Satake isomorphism.

Definition 2.2.7. Let E be a number field. Let v be a non-Archimedean place of F at which $G(F_v)$ is unramified and π be a smooth irreducible admissible representation of $G(F_v)$. We say that the Satake parameter of π is defined over E if the image of the map

$$\mathbb{Q}\left[X_*(T_{v,d})\right]^{W_{v,d}} \longrightarrow \mathbb{C}$$

induced by the map $\mathbb{C}\left[X_*(T_{v,d})\right]^{W_{v,d}} \longrightarrow \mathbb{C}$ is contained in E.

Definition 2.2.8. An irreducible automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A}_F)$ is L-arithmetic if there exist a number field E and a finite set of places, S, containing all the Archimedean places and all the places at which π_v ramifies, such that the Satake parameter of π_v is defined over E for all $v \notin S$.

As explained in [6], the motivation for introducing two notions of arithmetic automorphic representations is the observation that $\mathcal{H}_{\mathbb{Q}}(G(F_v), K_v)$ and $\mathbb{Q}\left[X_*(T_{v,d})\right]^{W_{v,d}}$ provide two $\mathbb{Q}$-structure of $\mathcal{H}_{\mathbb{C}}(G(F_v), K_v)$ which, in general, do not coincide. In loc. cit., the following conjecture is proposed:

Conjecture 2.2.9. Let G be a connected, reductive algebraic group defined over a number field F. An irreducible automorphic representation $\pi = \bigotimes_v \pi_v$ is C-algebraic (respectively, L-algebraic) if and only if it is C-arithmetic (respectively, L-arithmetic).

2.3 Employing the Trace Formula

Let G be a connected, semisimple, anisotropic algebraic group defined over $\mathbb{Q}$. Let $K_0 := K_{0,\infty} \times K_0^\infty$ be a maximal compact subgroup of $G(\mathbb{A}_\mathbb{Q})$, where $K_{0,\infty}$ is a maximal compact subgroup of $G(\mathbb{R})$ and K_0^∞ a compact open subgroup of $G(\mathbb{A}_\mathbb{Q}^\infty)$ such that $K_{0,p}$ is hyperspecial for all p at which G is unramified. In addition, we fix a compact open subgroup K^∞ of $G(\mathbb{A}_\mathbb{Q}^\infty)$ contained in K_0^∞ such that $K_p = K_{0,p}$ for almost all p.

We recall that the right regular representation $(R, L^2([G]))$, where $[G] := G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$, decomposes into a Hilbert direct sum

$$L^2([G]) = \bigoplus_{\pi \in \mathcal{A}(G)} m(\pi)\pi.$$

Here, $\mathcal{A}(G)$ denotes the set of equivalence classes of irreducible automorphic representations of $G(\mathbb{A}_\mathbb{Q})$, the multiplicity $m(\pi)$ is finite for every $\pi \in \mathcal{A}(G)$.

In the following, unless otherwise stated, π will always denote an element in $\mathcal{A}(G)$.

If $f = f_\infty \otimes f^\infty$ is an element in $C_c^\infty(G(\mathbb{A}_\mathbb{Q}))$, then the operator $R(f)$ is trace-class and we have the well known trace formula

$$\sum_{\pi \in \mathcal{A}(G)} m(\pi)\mathrm{Tr}\pi_\infty(f_\infty)\mathrm{Tr}\pi^\infty(f^\infty) = \sum_{\gamma \in \{G(\mathbb{Q})\}} \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}_\mathbb{Q}))\mathcal{O}_\gamma(f),$$

where $\{G(\mathbb{Q})\}$ denotes the set of conjugacy classes of $G(\mathbb{Q})$, the orbital integral $\mathcal{O}_\gamma(f)$ is defined as

$$\mathcal{O}_\gamma(f) := \int_{G_\gamma(\mathbb{A}_\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})} f(x^{-1}\gamma x)\, dx,$$

and $G_\gamma(\mathbb{Q})$ (resp. $G_\gamma(\mathbb{A}_\mathbb{Q})$) denotes the stabiliser of γ in $G(\mathbb{Q})$ (resp. in $G(\mathbb{A}_\mathbb{Q})$).

Proposition 2.3.1 and Corollary 2.3.3 below require Haar measures on $G(\mathbb{A}_\mathbb{Q})$ and on the centralisers $G_\gamma(\mathbb{A}_\mathbb{Q})$ satisfying two conditions:

(M1) For every $\gamma \in G(\mathbb{Q})$, the quantity $\mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}_\mathbb{Q}))$ is a rational number.

(M2) For every non-Archimedean place, the local Haar measure assigns a rational number to every open compact subset. For every non-Archimedean place p at which $G(\mathbb{Q}_p)$ is unramified, the local Haar measure μ_p on $G(\mathbb{Q}_p)$ assigns measure 1 to $K_{0,p}$.

Realising condition (M1) is non-trivial: in Corollary 3.3 below we will choose Gross' measure to fulfil it.

For the first part of condition (M2) we argue as follows. By V.5.2 in [31], $G(\mathbb{Q}_p)$ admits a maximal compact subgroup C_0 with the following property: there exists a neighbourhood basis of the identity, say $\mathcal{B}$, consisting of open compact subgroups and such that every $C \in \mathcal{B}$ is a normal subgroup of C_0. Normalising the Haar measure on $G(\mathbb{Q}_p)$ so that C_0 has measure 1, we obtain a Haar measure that assigns a rational number to every $C \in \mathcal{B}$ and, therefore, it assigns a rational number to every open compact subset. For p such that $G(\mathbb{Q}_p)$ is unramified, we normalise the local Haar measure so that it gives measure 1 to $K_{0,p}$ which, being hyperspecial, satisfies the property above. Therefore the measure so normalised assigns a rational number to every compact open subsets of $G(\mathbb{Q}_p)$.

We begin by proving that, under certain conditions on the automorphic representation we want to study, the trace formula can be used to establish the algebraicity of the eigenvalues of a $\mathbb{Q}$-valued Hecke operator. This is inspired by the method developed in [27]: it relies on the Newton-Girard identities.

Proposition 2.3.1. Let G be a connected, semisimple, anisotropic algebraic group defined over $\mathbb{Q}$. Let $\pi_0 = \pi_{0,\infty} \otimes \pi_0^\infty$ be an irreducible automorphic representation of $G(\mathbb{A}_\mathbb{Q})$. Assume the existence of a function $f \in C_c^\infty(G(\mathbb{R}))$ verifying the following conditions:

(1) There exists a finite family $\mathcal{C}$ of irreducible automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ containing π_0 such that:

  (i) If $\pi \in \mathcal{C}$, then $\pi^{\infty,K^\infty} \neq \{0\}$ and $\mathrm{Tr}\pi_\infty(f) = 1$.

  (ii) If $\pi \notin \mathcal{C}$, then $\mathrm{Tr}\pi_\infty(f) = 0$.

(2) For every $\gamma \in \{G(\mathbb{Q})\}$, the Archimedean orbital integral $\mathcal{O}_\gamma(f)$ is an algebraic number.

Then, for every $h \in \mathcal{H}_\mathbb{Q}(G(\mathbb{A}_\mathbb{Q}^\infty), K^\infty)$, the eigenvalues of the operator

$$\pi_0^\infty(h) : \pi_0^{\infty,K^\infty} \longrightarrow \pi_0^{\infty,K^\infty}$$

are algebraic.

Proof. Let $f \in C_c^\infty(G(\mathbb{R}))$ be as in the statement and consider $h \in \mathcal{H}_\mathbb{Q}(G(\mathbb{A}_\mathbb{Q}^\infty), K^\infty)$. Applying the trace formula to compute the trace of $R(f \otimes h)$, we obtain

$$\begin{aligned}\mathrm{Tr}\left[R(f \otimes h)\right] &= \sum_{\pi \in \mathcal{A}(G)} m(\pi)\mathrm{Tr}\pi_\infty(f)\mathrm{Tr}\pi^\infty(h) \\ &= \sum_{\pi \in \mathcal{C}} m(\pi)\mathrm{Tr}\pi^\infty(h)\end{aligned}$$

on the spectral side, and therefore

$$\sum_{\pi \in \mathcal{C}} m(\pi)\mathrm{Tr}\pi^\infty(h) = \sum_{\gamma \in \{G(\mathbb{Q})\}} \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}))\mathcal{O}_\gamma(f \otimes h).$$

The spectral side is equal to the trace of the operator

$$T := \bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^\infty(h) : \bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty,K^\infty} \longrightarrow \bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty,K^\infty}.$$

The sum on the geometric side is finite by Lemma 9.1 of [1] and the volume terms are rational numbers by condition M1. Each global orbital integral can be factored into a product of an Archimedean orbital integral and a non-Archimedean one. Since the Hecke operator h is $\mathbb{Q}$-valued, the non-Archimedean orbital integrals are rational numbers by condition (M2) and Theorem 3 in [3]. The Archimedean orbital integrals are algebraic numbers by (2). It follows that the trace of the operator T is algebraic. Next, we show that, for every positive integer m less than or equal to the dimension of the vector space

$$\bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty,K^\infty},$$

the trace of T^m is algebraic. First, we observe that

$$T^m = \bigoplus_{\pi \in \mathcal{C}} m(\pi) \left[\pi^\infty(h)\right]^m = \bigoplus_{\pi \in \mathcal{C}} m(\pi) \pi^\infty(h^{*m}),$$

where h^{*m} denotes the convolution of h with itself m times. Using the test function $f \otimes h^{*m}$, and arguing as above, we see that the trace of T^m is algebraic. Applying the Newton-Girard identities (Proposition 28.1 in [27] and the discussion preceeding it), it follows that the coefficients of the characteristic polynomial of T are algebraic numbers, and so must be the eigenvalues of T. Since $\pi_0 \in \mathcal{C}$, the eigenvalues of $\pi_0^\infty(h)$ are in particular eigenvalues of T, and the result follows.

□

The criterion to establish that an automorphic representation is C-arithmetic requires strengthening condition (2) in the proposition above. We will also need the following key result, for which we are grateful to G. Wiese.

Proposition 2.3.2. Let V be a finite-dimensional complex vector space of dimension n. Let $\mathcal{T}_{\mathbb{Q}} := \{T_\alpha : V \longrightarrow V\}_{\alpha \in A}$ be a family of diagonalisable, commuting linear operators, closed under taking finite $\mathbb{Q}$-linear combinations. Assume the existence of a number field E such that the characteristic polynomial $\mathrm{char}_\alpha(x)$ of T_α has coefficients in E for every $\alpha \in A$. Then there exists a number field E' such that the eigenvalues of T_α belong to E' for every $\alpha \in A$.

Proof. For every $\alpha \in A$, the degree of $\mathrm{char}_\alpha(x)$ equals the dimension of V. Let E_α denote the splitting field of T_α over E, then $[E_\alpha : E] \leq n!$. Let M denote the maximum of the set

$$\{m \in \mathbb{N} |\ m = [E_\alpha : E] \text{ for some } \alpha \in A\},$$

and let $T_{\alpha_0} \in \mathcal{T}_{\mathbb{Q}}$ be such that $[E_{\alpha_0} : E] = M$. By assumption, there exists a basis $\{v_1, \ldots, v_n\}$ of V with respect to which the operators in $\mathcal{T}_{\mathbb{Q}}$ can be simultaneously diagonalised. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of T_{α_0} corresponding to the eigenvectors $v_1, \ldots, v_n$. If $T_\beta \in \mathcal{T}_{\mathbb{Q}}$, let $\mu_1, \ldots, \mu_n$ be the eigenvalues of T_β corresponding to the eigenvectors $v_1, \ldots, v_n$. For every $k \in \{1, \ldots, n\}$, there are only finitely many elements $c \in E$ for which $\lambda_k + c\mu_k$ is not a primitive element of $E(\lambda_k, \mu_k)$. We can therefore find an element $r \in \mathbb{Q}$ such that $\lambda_k + r\mu_k$ is a primitive element of $E(\lambda_k, \mu_k)$ for every $k \in \{1, \ldots, n\}$. Now, the operator $T_{\alpha_0} + rT_\beta$ is in $\mathcal{T}_{\mathbb{Q}}$ by assumption, its eigenvalues corresponding to the eigenvectors $v_1, \ldots, v_n$ are $\lambda_1 + r\mu_1, \ldots, \lambda_n + r\mu_n$, and the splitting field over E of its characteristic polynomial is $L = E(\lambda_1 + r\mu_1, \ldots, \lambda_n + r\mu_n)$. By construction, L contains E_{α_0}, hence $[L : E]$ is at least M. Moreover, $[L : E]$ is at most M, since $T_{\alpha_0} + rT_\beta$ is in $\mathcal{T}_{\mathbb{Q}}$. It follows that $L = E_{\alpha_0}$ and, since E_β is contained in L by construction, we have that E_β is contained in E_{α_0}. Since T_β is an arbitrary element in $\mathcal{T}_{\mathbb{Q}}$, the result follows by setting $E' = E_{\alpha_0}$.

□

Corollary 2.3.3. Let G be a connected, semisimple, anisotropic algebraic group defined over $\mathbb{Q}$. Let $\pi_0 = \pi_{0,\infty} \otimes \pi_0^\infty$ be an irreducible automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$. Assume the existence of a function $f \in C_c^\infty(G(\mathbb{R}))$ verifying the following conditions:

(1) There exists a finite family $\mathcal{C}$ of irreducible automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$ containing π_0 such that:

  (i) If $\pi \in \mathcal{C}$, then $\pi^{\infty, K^\infty} \neq \{0\}$ and $\mathrm{Tr}\pi_\infty(f) = 1$.

  (ii) If $\pi \notin \mathcal{C}$, then $\mathrm{Tr}\pi_\infty(f) = 0$.

(2') There exists a number field E such that, for every $\gamma \in \{G(\mathbb{Q})\}$, the Archimedean orbital integral $\mathcal{O}_\gamma(f)$ belongs to E.

Then there exists a number field E' and a finite set of places, S, containing ∞ and all the places at which π_0 ramifies, such that for every $p \notin S$, and for every $h \in \mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p), K_p)$, the eigenvalue of the operator

$$\pi_{0,p}(h) : \pi_{0,p}^{K_p} \longrightarrow \pi_{0,p}^{K_p}$$

is contained in E'; that is: the representation π_0 is C-arithmetic.

Proof. There exists a fnite set of places, S, which contains ∞ and such that, for $p \notin S$, the representation π_p is unramified for every $\pi \in \mathcal{C}$ and $K_p = K_{0,p}$.
For $p \notin S$, and for each element $h \in \mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p), K_p)$, we can form an element in $\mathcal{H}_{\mathbb{Q}}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}), K^{\infty})$, called again h abusing notation, defined by tensoring h with the unit of $\mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_{p'}), K_{p'})$ for every $p' \neq p$. We thus obtain, for every $p \notin S$, and for every $\pi \in \mathcal{C}$, a family of commuting diagonalisable operators

$$\{\pi^{\infty}(h) : \pi^{\infty, K^{\infty}} \longrightarrow \pi^{\infty, K^{\infty}}\}_h$$

indexed by $\mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p), K_p)$. For every $p \notin S$, this gives rise to a family of commuting diagonalisable operators

$$\mathcal{T}_{\mathbb{Q},p} := \{T_h := \bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty, K^{\infty}}(h) : \bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty, K^{\infty}} \longrightarrow \bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty, K^{\infty}}\}_h$$

indexed by $\mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p), K_p)$. Arguing as in the proof of Proposition 2.3.1, but using the stronger condition (2'), it follows that, for every $p \notin S$, the characteristic polynomial of every $T_h \in \mathcal{T}_{\mathbb{Q},p}$ has coefficients in E. We observe that elements belonging to different $\mathcal{T}_{\mathbb{Q},p}$'s commute. Therefore, we form the $\mathbb{Q}$-subspace $\mathcal{T}_{\mathbb{Q}}$ of the endomorphism space of

$$\bigoplus_{\pi \in \mathcal{C}} \pi^{\infty, K^{\infty}}$$

generated by the families $\mathcal{T}_{\mathbb{Q},p}$. It consists of commuting diagonalisable operators. Let $T \in \mathcal{T}_{\mathbb{Q}}$ and write it as

$$T = \sum_{i=1}^{r} a_i T_{h_i}$$

for some $a_1, \dots, a_r \in \mathbb{Q}$ and some $h_1, \dots, h_r \in \bigcup_{p \notin S} \mathcal{T}_{\mathbb{Q},p}$. We observe that T is equal to the operator

$$\bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty, K^{\infty}}(\tilde{h}) : \bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty, K^{\infty}} \longrightarrow \bigoplus_{\pi \in \mathcal{C}} m(\pi)\pi^{\infty, K^{\infty}},$$

where

$$\tilde{h} := \sum_{i=1}^{r} a_i h_i \in \mathcal{H}_{\mathbb{Q}}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}), K^{\infty}).$$

Arguing as in Proposition 2.3.1, and by condition (2'), we conclude that the characteristic polynomial of T has coefficients in E. We can thus apply Proposition 2.3.2 to $\mathcal{T}_{\mathbb{Q}}$ and the result follows since, for every $p \notin S$ and for every $h \in \mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p), K_p)$, the eigenvalue of the operator $\pi_{0,p}(h)$ is an eigenvalue of the operator $T_h \in \mathcal{T}_{\mathbb{Q},p} \subset \mathcal{T}_{\mathbb{Q}}$.

□

To conclude, we apply Corollary 2.3.3 to give a new proof that an automorphic representation with a regular discrete series as Archimedean component is C-arithmetic. These have been studied in [2].

We equip the centraliser groups $G_\gamma(\mathbb{A}_\mathbb{Q})$ with Gross' measure, so that (M1) is satisfied. For every non-Archimedean p at which $G(\mathbb{Q}_p)$ is unramified, we choose local measures μ_p on $G(\mathbb{Q}_p)$ such that $\mu_p(K_{0,p}) = 1$ and, for the remaining places, we normalise the Haar measures so that they satisfy (M2) in the way explained above.

Corollary 2.3.4. Let G be a connected, semisimple, anisotropic algebraic group defined over $\mathbb{Q}$. Let $\pi_0 = \pi_{0,\infty} \otimes \pi_0^\infty$ be an irreducible automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with $\pi_{0,\infty}$ a regular discrete series and $\pi_0^{\infty,K^\infty} \neq \{0\}$. Then π_0 is C-arithmetic.

Proof. By Théorème 3 in [10], if ρ is an irreducible, finite-dimensional representation of $G(\mathbb{R})$, there exists a smooth compactly supported function f_ρ such that, for every admissible $(\mathfrak{g}, K_{0,\infty})$-module of finite length σ, we have

$$\mathrm{Tr}\sigma(f_\rho) = \sum_{i=0}^{\infty} (-1)^i \mathrm{dim}\mathrm{H}^i(\mathfrak{g}, K_{0,\infty}, \sigma \otimes \rho).$$

Now, let π_0 be as in the statement, let ρ be an irreducible, finite-dimensional, algebraic representation of $G(\mathbb{R})$ the infinitesimal character of which is the contragredient of the infinitesimal character of $\pi_{0,\infty}$. Then the $(\mathfrak{g}, K_{0,\infty})$-cohomology of $\pi_{0,\infty}$ with respect to ρ is concentrated in one degree by part (b) of Theorem 5.3 in [5] and it is 1-dimensional: we can thus normalise f_μ so that $\mathrm{Tr}\pi_{0,\infty}(f_\rho) = 1$. This will then hold for every discrete series with the same infinitesimal character as $\pi_{0,\infty}$. By the proof of Corollary 6.2 in [2], under the regularity assumption on $\pi_{0,\infty}$, the unitary irreducible representations of $G(\mathbb{R})$ with non-vanishing $(\mathfrak{g}, K_{0,\infty})$-cohomology with respect to ρ are precisely the discrete series with the same infinitesimal character as $\pi_{0,\infty}$. It is well-known that there are only finitely many such discrete series representations, therefore (1) of Corollary 2.3.3 is fulfilled. For (2'), we observe that since G is defined over $\mathbb{Q}$, the representation ρ admits a model over a number field E. Let $\gamma \in \{G(\mathbb{Q})\}$. By Theorem 4.1 in [11], taking into account our normalisation of f_ρ and the choice of Haar measure, the Archimedean orbital integral $\mathcal{O}_\gamma(f_\rho)$ vanishes or we have

$$\mathcal{O}_\gamma(f_\rho) = d_\gamma \mathrm{Tr}\rho(\gamma)$$

where $d_\gamma \in \mathbb{Z}\backslash\{0\}$. Since ρ is defined over E, the orbital integral $\mathcal{O}_\gamma(f_\rho)$ belongs to E. We can now apply Corollary 2.3.3, concluding the proof.

□

2.4 Reduction to Cuspidal Representations

We recall that by the discussion in Section 5.1 of [6], given a connected, reductive algebraic group G defined over a number field F, and a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G' \longrightarrow G \longrightarrow 1,$$

since the map $G'(\mathbb{A}_F) \longrightarrow G(\mathbb{A}_F)$ is surjective, we can identify the irreducible automorphic representations of $G(\mathbb{A}_F)$ with the irreducible automorphic representations of $G'(\mathbb{A}_F)$ which are trivial on the image of $\mathbb{G}_m(\mathbb{A}_F)$ in $G'(\mathbb{A}_F)$. We would like to thank T. Gee for explaining that this identification follows from the results in Section 5 of [28].

Proposition 2.4.1. Let G be a connected, reductive algebraic group defined over a number field F and let

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G' \longrightarrow G \longrightarrow 1$$

be a central extension. Let π be an irreducible automorphic representation of $G(\mathbb{A}_F)$ and let π' denote the irreducible automorphic representation of $G'(\mathbb{A}_F)$ obtained by lifting π along the surjection $G'(\mathbb{A}_F) \twoheadrightarrow G(\mathbb{A}_F)$. Then the following two statements hold:

(1) The representation π is C-algebraic (resp. C-arithmetic, resp. L-algebraic, resp. L-arithmetic) if and only if the representation π' is C-algebraic (resp. C-arithmetic, resp. L-algebraic, resp. L-arithmetic).

(2) If π is cuspidal, then π' is cuspidal.

Proof. See [6], Lemma 5.33, for part (1). Part (2) is [28], Theorem 5.2.1. □

For certain groups, the C and L notions that we are considering are related by a character twist. More precisely:

Theorem 2.4.2. Let G be a connected, reductive algebraic group defined over a number field F. Assume that G admits a twisting element in the sense of Definition 5.34 of [6]. Then there exists a character χ of $G(F)\backslash G(\mathbb{A}_F)$ such that the following statements hold:

(1) An irreducible automorphic representation π of $G(\mathbb{A}_F)$ is C-algebraic if and only if $\pi \otimes \chi$ is L-algebraic.

(2) An irreducible automorphic representation π of $G(\mathbb{A}_F)$ is C-arithmetic if and only if $\pi \otimes \chi$ is L-arithmetic.

Proof. See Proposition 5.35 in [6] for part (1) and Proposition 5.36 in [6] for part (2). □

Remark 2.4.3. A consequence of Theorem 2.4.2, as explained in [6], is that, if G admits a twisting element, we can twist L-algebraic and C-algebraic representations into each other, and we can twist C-arithmetic and L-arithmetic representations into each other. Indeed, if π is L-algebraic, writing it as $(\pi \otimes \chi^{-1}) \otimes \chi$, we obtain that $\pi \otimes \chi^{-1}$ is C-algebraic by part (1) of Theorem 2.4.2. Conversely, if $\pi \otimes \chi^{-1}$ is C-algebraic, then, twisting it by χ and applying part (1) of Theorem 2.4.2, shows that π is L-algebraic. The same reasoning applies for the notions of C-arithmetic and L-arithmetic automorphic representations.

If a group does not admit twisting elements, then there exists a central extension which does.

Theorem 2.4.4. Let G be a connected, reductive algebraic group defined over a number field F. Then there exists a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

such that $\widetilde{G}$ admits a twisting element.

Proof. See part (a) of Proposition 5.37 in [6]. □

Before proceeding, let us recall the fundamental result of R. Langlands establishing that every automorphic representation is a subquotient of the parabolic induction of a cuspidal automorphic representation. Following [30], if P is a parabolic subgroup of G with Levi factor M, and if $\sigma = \bigotimes_v \sigma_v$ is a cuspidal automorphic representation of $M(\mathbb{A}_F)$, we call an irreducible subquotient of $\mathrm{Ind}_P(\sigma) = \bigotimes_v \mathrm{Ind}_{P_v}(\sigma_v)$ a constituent. Recall that there exists a finite set of places, S, such that for $v \notin S$, the representation $\mathrm{Ind}_{P_v}(\sigma_v)$ has exactly one constituent, denoted by π_v°, with non-zero $G(\mathcal{O}_v)$-invariant vectors.

Theorem 2.4.5. Let G be a connected, reductive algebraic group defined over a number field F. Then the following statements hold:

(1) If P is a parabolic subgroup of G with Levi factor M, and $\sigma = \bigotimes_v \sigma_v$ is an irreducible, cuspidal automorphic representation of $M(\mathbb{A}_F)$, then the constituents of $\mathrm{Ind}_P(\sigma)$ are the representations $\pi = \bigotimes_v \pi_v$, where π_v is a constituent of $\mathrm{Ind}_{P_v}(\sigma_v)$, and, for all $v \notin S$, $\pi_v = \pi_v^\circ$.

(2) An irreducible representation π of $G(\mathbb{A}_F)$ is automorphic if and only if there exists a parabolic subgroup P of G with Levi factor M, and an irreducible, cuspidal automorphic representation σ of $M(\mathbb{A}_F)$ such that π is a constituent of $\mathrm{Ind}_P(\sigma)$.

Proof. Part (1) is Lemma 1 in [30], part (2) is Proposition 2 in [30]. □

We will make use of the following result, which we have learnt from Remark 2.10 in [32], and for which we supply a proof.

Proposition 2.4.6. Let G be a connected, reductive group defined over a number field F, and π be an irreducible automorphic representation of $G(\mathbb{A}_F)$. Let P be a proper parabolic subgroup of G with Levi factor M, and σ be an irreducible, cuspidal automorphic representation of $M(\mathbb{A}_F)$ such that π is a constituent of $\mathrm{Ind}_P(\sigma)$. Then the following two statements hold:

(1) The representation π is L-algebraic if and only if σ is L-algebraic.

(2) If σ is L-arithmetic, then π is L-arithmetic.

Proof. Let π_v be an Archimedean component of π. Then π_v is a constituent of $\mathrm{Ind}_{P_v}(\sigma_v)$, and σ_v is an irreducible admissible representation of the Levi factor M_v of P_v. Writing M_v as $M_v^\circ A_v$, with A_v denoting the split component of M_v, then we can write σ_v as the tensor product of an admissible, irreducible representation of M_v° and a character of A_v. With this observation, it follows from Proposition 8.22 in [24] that π_v and σ_v have the same infinitesimal character. Since this is true for all Archimedean places, it follows that σ is L-algebraic if and only if π is L-algebraic: this proves part (1). Part (2) is a special case of Lemma 5.45 in [6]. □

We can now prove the main result of this section.

Theorem 2.4.7. Let F be a number field. If every irreducible, C-algebraic, cuspidal automorphic representation of every connected, reductive algebraic group defined over F is C-arithmetic, then every irreducible, C-algebraic automorphic representation of every connected, reductive algebraic group defined over F is C-arithmetic, and every irreducible, L-algebraic automorphic representation of every connected, reductive algebraic group defined over F is L-arithmetic.

Proof. Let π be an irreducible automorphic representation of $G(\mathbb{A}_F)$, with G a connected, reductive algebraic group defined over F. We distinguish four cases.

(I) The representation π is C-algebraic and cuspidal. Then the result holds by assumption.

(II) The representation π is L-algebraic and cuspidal. Then, by Theorem 2.4.4, there exists a central extension
$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$
such that $\widetilde{G}$ admits a twisting element. Let $\tilde{\pi}$ denote the automorphic representation of $\widetilde{G}(\mathbb{A}_F)$ obtained by lifting π along the surjection $\widetilde{G}(\mathbb{A}_F) \twoheadrightarrow G(\mathbb{A}_F)$. Then, by Proposition 2.4.1, $\tilde{\pi}$ is cuspidal and L-algebraic. By Theorem 2.4.2 we can twist $\tilde{\pi}$ to obtain a cuspidal C-algebraic automorphic representation of $\widetilde{G}(\mathbb{A}_F)$, which is therefore C-arithmetic by assumption. By Theorem 2.4.2, the representation $\tilde{\pi}$ is L-arithmetic, and so is π by Proposition 2.4.1.

(III) The representation π is L-algebraic and not cuspidal. Then, by part (2) of Theorem 2.4.5, π is a constituent of $\mathrm{Ind}_P(\sigma)$, where P is a proper parabolic subgroup of G with Levi factor M, and σ is a cuspidal automorphic representation of $M(\mathbb{A}_F)$. By (1) of Proposition 2.4.6, σ is L-algebraic and, arguing as in case (II), we obtain that σ is L-arithmetic. By part (2) of Proposition 2.4.6, it follows that π is L-arithmetic.

(IV) The representation π is C-algebraic and not cuspidal. By Theorem 2.4.4, there exists a central extension
$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$
such that $\widetilde{G}$ admits a twisting element. Let $\tilde{\pi}$ denote the automorphic representation of $\widetilde{G}(\mathbb{A}_F)$ obtained by lifting π along the surjection $\widetilde{G}(\mathbb{A}_F) \twoheadrightarrow G(\mathbb{A}_F)$. By part (1) of Proposition 2.4.1, $\tilde{\pi}$ is C-algebraic. By Theorem 2.4.2, we can twist $\tilde{\pi}$ to obtain an L-algebraic automorphic representation of $\widetilde{G}(\mathbb{A}_F)$. Arguing as in case (III), we obtain that this twist of $\tilde{\pi}$ is L-arithmetic. By Theorem 2.4.2, it follows that $\tilde{\pi}$ is C-arithmetic and we conclude that π is C-arithmetic by part (1) of Proposition 2.4.1.

□

3 Asymptotic Orthogonality of Tempered Representations

3.1 Introduction

Let G be a semisimple group over a local field, let K be a maximal compact subgroup of G. We fix a Haar measure on G, denoted dg. If H is the Hilbert space underlying a unitary representation of G, let H_K denote the space of K-finite vectors and H^∞ the space of smooth vectors.

In their recent work [23], D. Kazhdan and A. Yom Din conjectured the validity of an asymptotic version of Schur's orthogonality relations. It should hold for matrix coefficients of tempered irreducible unitary representations of G, generalising Schur's well-known orthogonality relations for discrete series.

Following their article, we fix a norm on the Lie algebra $\mathfrak{g}$ of G. By [23], Claim 5.2, we can choose it so that $\mathrm{Ad}K$ acts unitarily on $\mathfrak{g}$. We define the function

$$\mathbf{r} : G \longrightarrow \mathbb{R}_{\geq 0}, \quad \mathbf{r}(g) = \log\left(\max\{\|\mathrm{Ad}(g)\|_{op}, \|\mathrm{Ad}(g^{-1})\|_{op}\}\right)$$

so that, given $r \in \mathbb{R}_{>0}$, we can introduce the corresponding ball

$$G_{<r} := \{g \in G | \mathbf{r}(g) < r\}.$$

Given this set-up, we are in position to state their conjecture.

Conjecture 3.1.1 (Kazhdan-Yom Din, Asymptotic Schur's Orthogonality Relations)**.** Let G be a semisimple group over a local field and let (π, H) be a tempered irreducible unitary representation of G. Then there exist $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$ and $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H$, the following holds:

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

Assuming that the matrix coefficients involved are K-finite, one has the following result:

Theorem 3.1.2 ([23], Theorem 1.7)**.** Let (π, H) be a tempered, irreducible, unitary representation of G and K be a maximal compact subgroup of G. Then there exists $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$ such that:

(1) If G is non-Archimedean, there exists $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

(2) If G is Archimedean, for any given non-zero $v_1, v_2 \in H_K$, there exists $C(v_1, v_2) > 0$ such that

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} |\langle \pi(g)v_1, v_2\rangle|^2\, dg = C(v_1, v_2).$$

In the non-Archimedean case, the proof of (1) is achieved by first establishing the validity of the analogous version of (2). The polarisation identity allows the authors of [23] to define a form

$$D(\cdot,\cdot,\cdot,\cdot) : H_K \times H_K \times H_K \times H_K \longrightarrow \mathbb{C}$$

via the prescription

$$D(v_1, v_2, v_3, v_4) := \lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg.$$

In [23], Section 4.1, this form is shown to be G-invariant and one would like to invoke an appropriate form of Schur's lemma to argue as in the standard proof of Schur's orthogonality relations. That is, for fixed $v_2, v_4 \in H_K$, one defines the form

$$D(\cdot, v_2, \cdot, v_4) : H_K \times H_K \longrightarrow \mathbb{C}$$

and, for fixed $v_1, v_3 \in H_K$, the form

$$D(v_1, \cdot, v_3, \cdot) : H_K \times H_K \longrightarrow \mathbb{C}.$$

One applies Schur's lemma to these forms, which implies that each such form is a scalar multiple of the inner product on H. Upon comparing them, one obtains the desired orthogonality relations.

The appropriate version of Schur's lemma in the non-Archimedean case is provided by Dixmier's lemma, which can be applied since in the non-Archimedean setting the subspace of K-finite vectors H_K and the subspace of smooth vectors H^∞ coincide, the latter being equipped with the structure of a Fréchet representation of G, which is irreducible since H itself is irreducible.

The purpose of this article is to prove that the analogue of (1) in Theorem 3.1.2 holds in the Archimedean case. As explained in [23], Section 4.2, it suffices to prove the result for real semisimple Lie groups (Theorem 3.4.4). In what follows, all Lie groups will be assumed to be real.

Theorem 3.1.3. Let (π, H) be a tempered, irreducible, Hilbert representation of a connected, semisimple Lie group G with finite centre. Let K be a maximal compact subgroup of G. Then there exists $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

We need to modify the strategy above to account for the fact that the space of K-finite vectors of a Hilbert space representation (π, H) of a real semisimple group does not afford a representation of G. It is, however, an admissible $(\mathfrak{g}, K)$-module.

Our approach relies crucially on the admissibility of irreducible, Hilbert representations of reductive Lie groups, a foundational theorem proved by Harish-Chandra. The theory of admissible $(\mathfrak{g}, K)$-modules then provides us with the appropriate version of Schur's lemma for $(\mathfrak{g}, K)$-invariant forms (Definition 3.2.11).

Hence, we are reduced to verify that $D(\cdot, v_2, \cdot, v_4)$ and $D(v_1, \cdot, v_3, \cdot)$ are, indeed, $(\mathfrak{g}, K)$-invariant. Having established this, to conclude the proof of Theorem 2.3.3, we can argue as in [23], Section 4.

From now on, to make the notation look more compact, given a Hilbert representation (π, H) of G and vectors $v, w \in H$, we set

$$\phi_{v,w}(g) := \langle \pi(g)v, w \rangle.$$

For connected, semisimple Lie groups with finite centre, K-invariance is a consequence of $\mathfrak{g}$-invariance (Proposition 3.2.14). Therefore, the problem is establishing the $\mathfrak{g}$-invariance. Explicitly, we prove the following (Proposition 3.4.2).

Proposition 3.1.4. Let G be a connected, semisimple Lie group with finite centre and let (π, H) be a tempered, irreducible, Hilbert representation of G. Then, for all $X \in \mathfrak{g}$, and for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1,v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg = -\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi}_{\dot{\pi}(X)v_3,v_4}(g)\, dg$$

and

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,\dot{\pi}(X)v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg = -\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi}_{v_3,\dot{\pi}(X)v_4}(g)\, dg.$$

The key observation is that, by exploiting the theory of asymptotic expansions of matrix coefficients of tempered representations both with respect to a minimal parabolic subgroup $P = MAN$ and with respect to the standard (for P) parabolic subgroups of G, the expression

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1,v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg$$

reduces, roughly, to a sum of finitely many terms of the form

$$\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\rangle_{L^2(M_\lambda)}\, dk.$$

Here, M_λ comes from a standard parabolic subgroup $P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0}$ of G. We denote $\mathfrak{m}_\lambda$, $\mathfrak{a}_{\lambda_0}$, $\mathfrak{n}_{\lambda_0}$ the Lie algebras of M_λ, A_{λ_0}, N_{λ_0}, respectively. The functions $\Gamma_{\lambda,l}, \Gamma_{\mu,m}$, as functions of m_λ, are analytic and square-integrable and arise from the asymptotic expansion of the matrix coefficients $\phi_{\dot{\pi}(X)v_1,v_2}$ and ϕ_{v_3,v_4}, respectively, relative to P_λ. The subscript in P_λ is meant to indicate that the parabolic subgroup is obtained, in an appropriate sense, from the datum of λ. Moreover, (λ, l) and (μ, m) are related in a precise way.

We shall elaborate on these points later on. For the moment, let us point out that we reduced the initial problem to showing that, for every $X \in \mathfrak{g}$, and for all relevant pairs (λ, l) and (μ, m), the integral

$$\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\rangle_{L^2(M_\lambda)}\, dk$$

equals

$$-\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)\dot{\pi}(X)v_3, w_4)\rangle_{L^2(M_\lambda)}\, dk.$$

We will prove that, if (λ, l) and (μ, m) satisfy a certain condition (to be explained below), the functions $\Gamma_{\lambda,l}(\cdot, v_1, w_2)$ and $\Gamma_{\mu,m}(\cdot, v_3, w_4)$ are, in fact, $Z(\mathfrak{g}_\mathbb{C})$-finite, with $Z(\mathfrak{g}_\mathbb{C})$ denoting the centre of the universal enveloping algebra of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$, and $K \cap M_\lambda$-finite.

It will then follow from a theorem of Harish-Chandra (Theorem 3.2.17) that they are smooth vectors in the right-regular representation $(R, L^2(M_\lambda))$ of M_λ .

The idea is to combine this observation with an appropriate form of Frobenius' reciprocity (Theorem 3.2.24), due to Casselman, to construct $(\mathfrak{g}, K)$-invariant maps

$$T_{w_2} : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda, K_\lambda}(H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\ T_{w_2}(v)(k)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, \pi(k)v, w_2)$$

and

$$T_{w_4} : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda, K_\lambda}(H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\ T_{w_4}(v)(k)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, \pi(k)v, w_4).$$

Here, the subgroup $\overline{P}_\lambda$ is the parabolic subgroup opposite to P_λ. The notation $\mathrm{Ind}_{\overline{P}_\lambda, K_\lambda}(H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$ stands for the space of K-finite vectors in the representation induced from the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K \cap M_\lambda)$-module

$$H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$$

for an appropriately chosen admissible, unitary, sub-representation (σ, H_σ) of $(R, L^2(M_\lambda))$.

To apply the required form of Frobenius' reciprocity, we need to show that the maps

$$S_{w_2} : H_K \longrightarrow H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},\ S_{w_2}(v)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, v, w_2)$$

and

$$S_{w_4} : H_K \longrightarrow H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},\ S_{w_4}(v)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, v, w_4)$$

descend to $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant maps on $H_K/\mathfrak{n}_{\lambda_0} H_K$. Establishing this result is the technical heart of the article.

Assuming it, the integral

$$\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\rangle_{L^2(M_\lambda)}\, dk$$

is nothing but

$$\langle \dot{\mathrm{Ind}}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X)\Gamma_{\lambda,l}(m_\lambda, v_1, w_2), \Gamma_{\mu,m}(m_\lambda, v_3, w_4)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})},$$

where

$$\langle \cdot, \cdot \rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})}$$

is the inner product on $\mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$. We will see that this makes sense since the inducing data ensure unitarity. The sought equality will then follow from the skew-invariance of the inner product on a unitary representation with respect to the action of the Lie algebra.

To explain how the functions $\Gamma_{\lambda,l}(\cdot, v_1, v_2)$ and $\Gamma_{\mu,m}(\cdot, v_3, v_4)$ arise, we need to recall the main features of the asymptotic expansions of K-finite matrix coefficients of tempered representations. If $\phi_{v,w}$ is such a matrix coefficient, then its asymptotic expansion relative to the minimal parabolic subgroup P of G can be thought of as a sum indexed by a countable collection

$$\Lambda := \{(\lambda, l)\}_{\lambda \in \mathcal{E},\ l \in \mathbb{Z}^n_{\geq 0} : |l| \leq l_0}.$$

The set $\mathcal{E}$ is a collection of complex-valued real-linear functionals on $\mathrm{Lie}(A)$ depending on (π, H) and not on the particular choice of $v, w \in H_K$. It is the set of **exponents** of (π, H). The number n is the rank of G and l_0, too, depends on (π, H) only.

The term indexed by (λ, l) is multiplied by a complex coefficient $c_{\lambda,l}(v, w)$. The choice of $v, w \in H_K$ determines the pairs in $\mathcal{C}$ for which $c_{\lambda,l}(v, w) \neq 0$. If $\lambda \in \mathcal{E}$, there exists at least a pair of $v, w \in H_K$ such that, for some $l \in \mathbb{Z}^n_{\geq 0}$ with $|l| \leq l_0$, we have $c_{\lambda,l}(v, w) \neq 0$.

For any standard (for P) parabolic subgroup $P' = M'A'N'$ of G, the matrix coefficient $\phi_{v,w}$ admits a similar asymptotic expansion. It can be thought of as a sum indexed by a countable collection

$$\Lambda' := \{(\nu, q)\}_{\nu \in \mathcal{E}',\ q \in \mathbb{Z}^r_{\geq 0} : |q| \leq q_0}.$$

Here, $r \leq n$ is the dimension of A', the set $\mathcal{E}'$ consists of complex-valued real-linear functionals on $\mathrm{Lie}(A')$. On regions on which both the expansion relative to P and the expansion relative to P' are meaningful, by comparing the two it turns out that the element in $\mathcal{E}'$ are precisely the restrictions to $\mathrm{Lie}(A')$ of the elements in $\mathcal{E}$ and, making the appropriate identifications following from $A' \subset A$, each q is the projection to $\mathbb{Z}^r_{\geq 0}$ of an l appearing in the expansion relative to P.

While in the expansion relative to P the term indexed by (λ, l) is multiplied by the complex coefficient $c_{\lambda,l}(v, w)$, the term indexed by (ν, q) in the expansion relative to P' is mutiplied by a real-analytic function

$$c_{\nu,q}(\cdot, v, w) : M' \longrightarrow \mathbb{C}.$$

We need one more piece of information to explain how $\Gamma_{\lambda,l}(\cdot, v_1, v_2)$ and $\Gamma_{\mu,m}(\cdot, v_3, v_4)$ arise: the construction of $\mathbf{d}(\pi)$ in [23]. The idea is as follows. We can think of $\lambda \in \mathcal{E}$ as an n-tuple of complex numbers $(\lambda_1, \cdots, \lambda_n)$. It can be shown that there exist a finite sub-collection $\mathcal{E}_0 \subset \mathcal{E}$ such that, for every $\lambda \in \mathcal{E}$, there exists $\hat{\lambda} \in \mathcal{E}_0$ such that

$$\hat{\lambda} - \lambda \in \mathbb{Z}^n_{\geq 0}.$$

Moreover, any two distinct elements in $\mathcal{E}_0$ are integrally inequivalent: their difference does not belong to $\mathbb{Z}^n$. By a result of Casselman (Theorem 3.3.2), for every $\hat{\lambda} \in \mathcal{E}_0$ and for every $i \in \{1, \ldots, n\}$, we have

$$\mathrm{Re}\hat{\lambda}_i \leq 0$$

and it is clear that this holds for every $\lambda \in \mathcal{E}$.

For $(\lambda, l) \in \Lambda$, we introduce the set $I_\lambda := \{i \in \{1, \ldots, n\} | \mathrm{Re}\lambda_i < 0\}$, we define

$$\mathbf{d}_P(\lambda, l) := |I^c_\lambda| + \sum_{i \in I^c_\lambda} 2l_i \tag{1}$$

and we take the maximum, $\mathbf{d}_P$, as (λ, l) ranges over all the pairs with $\lambda \in \mathcal{E}_0$.

We can proceed analogously for every standard parabolic P' and obtain a non-negative integer $\mathbf{d}_{P'}$. The maximum over all P' is $\mathbf{d}(\pi)$.

Now, given $\lambda \in \mathcal{E}_0$, identifying I_λ with a subset of the simple roots determined by an order on the root system $(\mathfrak{g}, \mathfrak{a})$, we can construct a standard (for P) parabolic subgroup $P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0}$

associated to I_λ. We will show that if $(\lambda, l) \in \Lambda$ satisfies $\lambda \in \mathcal{E}_0$ and $\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi)$, then $\Gamma_{\lambda,l}(\cdot, v_1, v_2)$ is precisely the function $c^{P_\lambda}_{\nu,q}(\cdot, v_1, v_2)$ with $\nu := \lambda|_{\mathfrak{a}_{\lambda_0}}$, where $\mathfrak{a}_{\lambda_0} := \mathrm{Lie}(A_{\lambda_0})$, and q equal to the projection of l to $\mathbb{Z}^{I^c_\lambda}_{\geq 0}$.

Finally, we mentioned that in the integral

$$\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\rangle_{L^2(M_\lambda)}\, dk$$

the pairs (λ, l) and (μ, m) must be related in a precise way. First of all, $(\mu, m) \in \Lambda$ satisfies $\mu \in \mathcal{E}_0$ and $\mathbf{d}_P(\mu, m) = \mathbf{d}(\pi)$. In addition, we must have $I_\lambda = I_\mu$ (so that $P_\lambda = P_\mu$) and $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$. The last condition, together with the unitarity of the representation (σ, H_σ) introduced above, is precisely what ensures that $\mathrm{Ind}_{P_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$ is unitary.

Implementing the strategy sketched above requires gathering a number of intermediate results. Several are inspired from the chapter in [24] on Langland's classification of tempered representations. Here is a more detailed outline of the article.

Section 2: The first part includes a discussion of the $(\mathfrak{g}, K)$-module version of Schur's lemma (Theorem 3.2.13). In the second part, we recall the result of Harish-Chandra establishing that smooth, $Z(\mathfrak{g}_\mathbb{C})$-finite, K-finite, square-integrable functions on reductive groups are smooth vectors in the right-regular representation (Theorem 3.2.16). As a consequence, we prove that, on such a function, the action of $\mathfrak{g}$ through differentiation is the same as the action of the Lie algebra through the right-regular representation (Proposition 3.2.18). After stating the basic facts on parabolically induced representations that we need, we discuss Casselman's version of Frobenius's reciprocity (Theorem 3.2.24).

Section 3: In the first part, we recall the theory of asymptotic expansions of matrix coefficients of tempered representations both with respect to a minimal parabolic subgroup and with respect to standard parabolic subgroups. We then explain in detail how the functions $\Gamma_{\lambda,l}(\cdot, v_1, v_2)$, $\Gamma_{\mu,m}(\cdot, v_3, v_4)$ arise. We begin by introducing an equivalence relation on the data indexing the asymptotic expansion relative to P of the K-finite matrix coefficients of a tempered, irreducible, Hilbert representation (π, H). This equivalence relation is motivated by construction of $\mathbf{d}(\pi)$ in [23] and it is meant to exploit the criteria for the computation of asymptotic integrals in Appendix A in loc. cit. Imposing the conditions on (λ, l) and (μ, m) that we discussed above, we identify the functions $\Gamma_{\lambda,l}(\cdot, v_1, v_2)$ and $\Gamma_{\mu,m}(\cdot, v_3, v_4)$ with the coefficient functions in the asymptotic expansion relative to P_λ of ϕ_{v_1,v_2} and ϕ_{v_3,v_4} (Proposition 3.3.4). We then prove that they are smooth vectors in $(R, L^2(M_\lambda))$ (Proposition 3.3.6). Combining Proposition 3.3.6 with the technical Lemma 3.3.7 and Lemma 3.3.8, we are in position to construct unitary, admissible, finitely generated representations (σ_1, H_{σ_1}) and (σ_2, H_{σ_2}) whose direct sum is the unitary, admissible, finitely generated representation (σ, H_σ) introduced above (Proposition 3.3.10).

Section 4: Having gathered the results we need, we prove Proposition 3.1.4 (Proposition 3.4.2). We begin with a computational Lemma which shows that the second identity in Proposition 3.1.4 follows from the first (Lemma 3.4.1). The first part of the proof of Proposition 3.4.2 consists of an application of the considerations in Appendix A of [23] and a series of integral manipulations (justified in Lemma 3.4.3) aimed at showing that the integral

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1,v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg$$

can be computed in terms of a sum of integrals of the form

$$\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \overline{\Gamma}_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\rangle \, dk$$

with the pairs (λ, l) and (μ, m) both belonging to Λ with $\lambda, \mu \in \mathcal{E}_0$, $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}_P(\mu, m) = \mathbf{d}(\pi).$$

We explained how to construct the unitary, finitely generated, admissible representation (σ, H_σ) needed to apply Casselman's version of Frobenius' reciprocity. The discussion following Proposition 3.3.10 therefore gives $(\mathfrak{g}, K)$-equivariant maps

$$T_{w_2} : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\; T_{w_2}(v)(k)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, \pi(k)v, w_2)$$

and

$$T_{w_4} : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\; T_{w_4}(v)(k)(m_\lambda) := \Gamma_{\mu,m}(m_\lambda, \pi(k)v, w_4).$$

The condition $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and the fact that, by the definition of I_λ, the functional $\lambda|_{\mathfrak{a}_{\lambda_0}}$ is totally imaginary, shows that

$$\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \overline{\Gamma}_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\rangle \, dk$$

is equal to

$$\langle T_{w_2}(\dot{\pi}(X)v_1), T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})}.$$

The $(\mathfrak{g}, K)$-equivariance of T_{w_2} gives

$$\langle T_{w_2}(\dot{\pi}(X)v_1), T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})} = \langle \dot{\mathrm{Ind}}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X)T_{w_2}(v_1), T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})},$$

by Corollary 3.2.20 we have

$$\langle \dot{\mathrm{Ind}}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X)T_{w_2}(v_1), T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})} = -\langle T_{w_2}(v_1), \dot{\mathrm{Ind}}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X)T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}$$

and the $(\mathfrak{g}, K)$-equivariance of T_{w_4} gives

$$\langle T_{w_2}(v_1), \dot{\mathrm{Ind}}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X)T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})} = \langle T_{w_2}(v_1), T_{w_4}(\dot{\pi}(X)v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})},$$

thus completing the proof of Proposition 3.4.2.

Finally, we proceed as explained in the first part of the Introduction to prove Theorem 2.3.3 (Theorem 3.4.4).

3.2 Recollections on Representation Theory

Our presentation of the theory of $(\mathfrak{g}, K)$-modules follows [35]. To discuss its basic features, we need to gather some results on unitary representations of compact groups. We begin by recalling the basic notions in the study of representations of topological groups, which we always assume to be Hausdorff.

First, following [35], Section 1.1, let G denote a second-countable, locally compact group, equipped with a left Haar measure dg, and let V denote a complex topological vector space. We denote by $\mathrm{GL}(V)$ the group of invertible continuous endomorphisms of V. A **representation** of G on V is a strongly continuous homomorphism $\pi : G \longrightarrow \mathrm{GL}(V)$. Let (π, V) denote the datum of a representation of G. A subspace of V which is stable under the action of G through π is called an **invariant subspace**. A representation is said to be **irreducible** if the only closed invariant subspaces are the trivial subspace and V itself.

If $(H, \langle \cdot, \cdot \rangle)$ is a separable Hilbert space, a representation π of G on H is termed a **Hilbert representation**. If, in addition, G acts by unitary operators through π, the representation is said to be **unitary**.

Next, following [27], Section 10, we introduce the basic features of the theory of vector-valued integration.

Let $(X,\, dx)$ be a Radon measure space, let H be a Hilbert space and assume that

$$f : X \longrightarrow H$$

is measurable. The function f is **integrable** if it satisfies the following two conditions:

(1) For all $v \in H$,

$$\int_X |\langle f(x), v \rangle| \, dx < \infty.$$

(2) The map

$$v \mapsto \int_X |\langle f(x), v \rangle| \, dx$$

is a bounded conjugate-linear functional.

If $f : X \longrightarrow H$ is integrable, then, by the Riesz' representation theorem, there exists a unique element in H, denoted

$$\int_X f(x) \, dx,$$

such that, for all $v \in H$, we have

$$\left\langle \int_X f(x) \, dx, v \right\rangle = \int_X \langle f(x), v \rangle \, dx.$$

Proposition 3.2.1. Let $(X,\, dx)$ be as above. Let H, E be Hilbert spaces,
$f : X \longrightarrow H$ a measurable function and $T : H \longrightarrow E$ a bounded linear operator. Then the following holds:

(1) If

$$\int_X \|f(x)\|\, dx < \infty,$$

then $f : X \longrightarrow H$ is integrable.

(2) If $f : X \longrightarrow H$ is integrable, then so is $Tf : X \longrightarrow E$. Moreover,

$$T\left(\int_X f(x)\, dx\right) = \int_X Tf(x)\, dx.$$

Proof. See [27], Proposition 10.8 and Proposition 10.9. □

Now, let (π, H) be a unitary representation of G. Let $v \in H$ and $f : G \longrightarrow H$ be such that the map

$$g \mapsto f(g)\pi(g)v$$

is integrable. Let $\pi(f)v$ denote the unique element in H such that, for all $w \in H$, we have

$$\langle \pi(f)v, w\rangle = \int_G f(g)\langle \pi(g)v, w\rangle\, dg.$$

Proposition 3.2.2. Let (π, H) be as above. If $f \in L^1(G)$, then, for all $v \in H$, the map $g \mapsto f(g)\pi(g)v$ is integrable and the prescription

$$\pi(f) : H \longrightarrow H,\ v \mapsto \pi(f)v$$

defines a bounded linear operator.

Proof. See [27], Proposition 10.20. □

With the integral operators introduced in Proposition 3.2.2 at our disposal, we have all the tools needed to state the main results on the unitary representations of compact groups.

Let K be a compact group. Let $\widehat{K}$ denote the set of equivalence classes of irreducible unitary representations of K. If (π, H) is a unitary representation, for each $\gamma \in \widehat{K}$ let $H(\gamma)$ denote the closure of the sum of all the closed invariant subspaces of H in the equivalence class of γ. We refer to $H(\gamma)$ as the γ-isotypic component of H.

Proposition 3.2.3. Let K be a compact group. Let (π, H) be an irreducible unitary representation of K. Then H is finite-dimensional.

Proof. See [35], Proposition 1.4.2. □

Given Proposition 3.2.3, we can associate, to each $\gamma \in \widehat{K}$, the function

$$\chi_\gamma : K \longrightarrow \mathbb{C},\ \chi_\gamma(g) := \mathrm{tr}\gamma(g),$$

the **character** of γ.

Recall that if $\{(\pi_i, H_i) | i \in I\}$ is a countable family of unitary representations of a topological group G, we can construct a new unitary representation of G, the **direct sum**, on the Hilbert space completion of the algebraic direct sum of the H_i's. We refer the reader to [35], Section 1.4.1, for the details of this construction. We let

$$\bigoplus_{i\in I} H_i$$

denote the direct sum of the family $\{(\pi_i, H_i)|i \in I\}$, dropping explicit reference to the π_i's.

Proposition 3.2.4. Let K be a compact group. Let (π, H) be a unitary representation of K. Then (π, H) is the direct sum representation of its K-isotypic components; that is,

$$H = \bigoplus_{\gamma\in\widehat{K}} H(\gamma).$$

Moreover, let α_γ denote the function

$$\alpha_\gamma(k) := \dim(\gamma)\overline{\chi_\gamma}(k).$$

Then the following holds:

$$H(\gamma) = \pi(\alpha_\gamma)H.$$

Proof. See [35], Lemma 1.4.7. □

Proposition 3.2.5. Let K be a compact group. If (π, H) is a Hilbert space representation of K, then there exists an inner product on H that induces the original topology on H and for which K acts unitarily through π.

Proof. See [35], Lemma 1.4.8. □

We are finally ready to introduce $(\mathfrak{g}, K)$-modules.

Definition 3.2.6. Let G be a connected, semisimple Lie group with finite centre. Let $\mathfrak{g}$ denote its Lie algebra. Let K be a maximal compact subgroup of G, which we fix from now on, with Lie algebra $\mathfrak{k}$. A vector space V, equipped with the structure of $\mathfrak{g}$-module and K-module, is called a $(\mathfrak{g}, K)$-**module** if the following conditions hold:

(1) For all $v \in V$, for all $X \in \mathfrak{g}$, for all $k \in K$,

$$kXv = \mathrm{Ad}(k)Xkv$$

(2) For all $v \in V$, the span of the set

$$Kv := \{kv|k \in K\}$$

is a finite-dimensional subspace of V, on which the action of K is continuous.

(3) For all $v \in V$, for all $Y \in \mathfrak{k}$,

$$\frac{d}{dt}\exp(tY)v|_{t=0} = Yv.$$

We remark that (3) implicitly uses the smoothness of the action of K on the span of Kv. This follows from the fact that a continuous group homomorphism between Lie groups is automatically smooth.

Let V and W be $(\mathfrak{g}, K)$-modules and let $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ denote the space of $\mathfrak{g}$-morphisms that are also K-equivariant. Then V and W are said to be **equivalent** if $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ contains an invertible element.

A $(\mathfrak{g}, K)$-module V is called **irreducible** if the only subspaces that are invariant under the actions of $\mathfrak{g}$ and K are the trivial subspace and V itself. In this case, we have the following theorem:

Theorem 3.2.7. Let V be an irreducible $(\mathfrak{g}, K)$-module. Then the space $\mathrm{Hom}_{\mathfrak{g},K}(V, V)$ is 1-dimensional.

Proof. This is the result actually proved in [35], Lemma 3.3.2, although the statement there says $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$, for an unspecified W. We believe it is a typo. □

Let V be a $(\mathfrak{g}, K)$-module. Since, given each $v \in V$, the span of Kv, say W_v, is a finite-dimensional continuous representation of K, we can use Proposition 3.2.5 and then apply Proposition 3.2.4, thus decomposing W_v into a finite sum of finite-dimensional K-invariant subspaces of V. For $\gamma \in \widehat{K}$, we let $V(\gamma)$ denote the sum of all the K-invariant finite dimensional subspaces in the equivalence class of γ. Then the discussion above implies that

$$V = \bigoplus_{\gamma \in \widehat{K}} V(\gamma)$$

as a K-module, with the direct sum indicating the algebraic direct sum. A $(\mathfrak{g}, K)$-module V is called **admissible** if, for all $\gamma \in \widehat{K}$, $V(\gamma)$ is finite-dimensional.

Given a unitary representation (π, H), there exists a $(\mathfrak{g}, K)$-module naturally associated to it. To define it, recall that a vector $v \in H$ is called **smooth** if the map

$$g \mapsto \pi(g)v$$

is smooth. Let H^∞ denote the subspace of smooth vectors of H. It is a standard fact that the prescription

$$\dot{\pi}(X) := \frac{d}{dt}\pi(\exp(tX))v|_{t=0},$$

for $v \in H^\infty$ and $X \in \mathfrak{g}$, defines an action of $\mathfrak{g}$ on H^∞. Recall that a vector $v \in H$ is K-**finite** if the span of the set

$$\pi(K)v := \{\pi(k)v | k \in K\}$$

is finite-dimensional. Let H_K denote the subspace of K-finite vectors of H. By [35], Lemma 3.3.5, with the action of $\mathfrak{g}$ so defined and with the action of K through π, the space $H_K \cap H^\infty$ is a $(\mathfrak{g}, K)$-module. The representation (π, H) is said to be **admissible** if $H_K \cap H^\infty$ is admissible as a $(\mathfrak{g}, K)$-module and (π, H) is called **infinitesimally irreducible** if $H_K \cap H^\infty$ is irreducible as a $(\mathfrak{g}, K)$-module. It is in general not true that a K-finite vector is smooth. However, if (π, H) is admissible, we have the following result:

Theorem 3.2.8. Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be an admissible representation of G. Then every K-finite vector is smooth.

Proof. See the proof [35], Theorem 3.4.10. □

In light of the following fundamental result of Harish-Chandra, Theorem 3.2.8 will play an important role in this article.

Theorem 3.2.9. Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be an irreducible, Hilbert representation of G. Then (π, H) is admissible.

Proof. See [26], Theorem 7.204. □

In the following, given a unitary representation (π, H), we will write H_K for the $(\mathfrak{g}, K)$-module $H_K \cap H^\infty$ even if (π, H) is not admissible. We believe it will not cause any confusion.
We are now in position to prove the version of Schur's lemma for sesquilinear forms that we will use in Section 3. It is given as Corollary 3.2.13 below. First, we need:

Theorem 3.2.10. Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be an admissible Hilbert representation of G. Then (π, H) is irreducible if and only if it is infinitesimally irreducible.

Proof. See [35], Theorem 3.4.11. □

Definition 3.2.11. Let V and W be $(\mathfrak{g}, K)$-modules. A sesquilinear form

$$B(\cdot, \cdot) : V \times W \longrightarrow \mathbb{C}$$

is $(\mathfrak{g}, K)$-invariant if it satisfies the following two conditions:

(i) For all $k_1, k_2 \in K$ and all $v, w \in V$ we have

$$B(k_1 v, k_2 w) = B(v, w).$$

(ii) For all $X \in \mathfrak{g}$ and all $v, w \in V$ we have

$$B(Xv, w) = -B(v, Xw).$$

Theorem 3.2.12. Let G be a connected, semisimple Lie group with finite centre. Let V be an admissible $(\mathfrak{g}, K)$-module. Suppose that there exist a $(\mathfrak{g}, K)$-module W and a non-degenerate $(\mathfrak{g}, K)$-invariant sesquilinear form

$$B(\cdot, \cdot) : V \times W \longrightarrow \mathbb{C}.$$

Then W is $(\mathfrak{g}, K)$-isomorphic to $\overline{V}$.

Proof. This is [35], Lemma 4.5.1, except for the fact that our form is sesquilinear. To account for it, we modify the definition of the map T in the reference by setting, for a given $w \in W$, $T(w)(v) = B(w, v)$ for all $v \in V$. This defines a map from W to $\overline{V}$ obtained by sending w to $T(w)$ which, by the argument in the reference, is a $(\mathfrak{g}, K)$-isomorphism. □

The next corollary is proved by adapting to our case the argument in [15], Proposition 8.5.12 and using the beginning of the proof of [24], Proposition 9.1.

Corollary 3.2.13. Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be an irreducible, admissible, Hilbert representation of G. Then, up to a constant, there exists at most one non-zero $(\mathfrak{g}, K)$-invariant sesquilinear form on H_K. In particular, if (π, H) is irreducible unitary, then every such form is a constant multiple of $\langle \cdot, \cdot \rangle$.

Proof. The irreducibility of H implies that of H_K, by Theorem 3.2.10 and by Theorem 3.2.8. Let $B(\cdot, \cdot)$ be a $(\mathfrak{g}, K)$-invariant sesquilinear form. Consider the linear subspace V_0 of H_K defined as

$$V_0 := \{v \in H_K | B(v, w) = 0 \text{ for all } w \in H_K\}.$$

Since $B(\cdot, \cdot)$ is non-zero, V_0 is a proper subspace of H_K. Since $B(\cdot, \cdot)$ is moreover $(\mathfrak{g}, K)$-invariant, it follows that V_0 is a $(\mathfrak{g}, K)$-invariant subspace of H_K, hence, by the irreducibility of H_K, it must be zero. Analogous considerations for the subspace

$$V^0 := \{w \in H_K | B(v, w) = 0 \text{ for all } v \in H_K\}$$

imply that $B(\cdot, \cdot)$ is non-degenerate. By Theorem 3.2.12 the map $v \mapsto T(v)$, $T(v)(\cdot) := B(v, \cdot)$, is a $(\mathfrak{g}, K)$-isomorphism. Since H_K is irreducible, the space $\mathrm{Hom}_{\mathfrak{g},K}(H_K, H_K)$ is 1-dimensional by Theorem 3.2.7. Now, let $B'(\cdot, \cdot)$ be another such form, with associated isomorphism T'. Then $T(T')^{-1} = cI$, for some $c \in \mathbb{C}$. For the last statement, the unitarity of (π, H) implies that $\langle \cdot, \cdot \rangle$ is a $(\mathfrak{g}, K)$-invariant non-degenerate sesquilinear form and Theorem 3.2.9 with the discussion above, implies the result.

□

Since we are assuming that G is connected, proving $(\mathfrak{g}, K)$-invariance reduces to proving $\mathfrak{g}$-invariance. Indeed, by Theorem 2.2, p. 256, in [21], any maximal compact subgroup K of G is connected. Therefore, by Corollary 4.48 in [25], the exponential map

$$\exp : \mathfrak{k} \longrightarrow K$$

is surjective.

Proposition 3.2.14. Let G be a connected, semisimple Lie group with finite centre. Let V be a $(\mathfrak{g}, K)$-module, let

$$B(\cdot, \cdot) : V \times V \longrightarrow \mathbb{C}$$

be a $\mathfrak{g}$-invariant sesquilinear form. Then $B(\cdot, \cdot)$ is K-invariant.

Proof. Given any pair of vectors $v, w \in V$, we can find a finite-dimensional subspace of V, say W, which contains both and on which K acts continuously through a representation π. The restriction of the bilinear form $B(\cdot, \cdot)$ to W is continuous. To prove that $B(\pi(k)v, \pi(k)w) = B(v, w)$ for all $k \in K$, it suffices to prove that $B(\pi(k)v, w) = B(v, \pi(k^{-1})w)$ for all $k \in K$. Given $k \in K$, let $X \in \mathfrak{k}$ be such that $k = \exp(X)$. We begin by writing

$$B(\pi(k)v, w) = B(\pi(\exp X)v, w).$$

Since $\pi(\exp X) = \exp\dot{\pi}(X)v$, we obtain

$$B(\pi(\exp X)v, w) = B(\exp\dot{\pi}(X)v, w).$$

The continuity of $B(\cdot,\cdot)$ on W gives

$$B(\exp\dot{\pi}(X)v, w) = \exp B(\dot{\pi}(X)v, w).$$

By the $\mathfrak{g}$-invariance of $B(\cdot,\cdot)$, we have

$$\exp B(\dot{\pi}(X)v, w) = \exp B(v, \dot{\pi}(-X)w)$$

and, finally,

$$\exp B(v, \dot{\pi}(-X)w) = B(v, \pi(\exp(-X))w).$$

□

Let us recall that any locally compact Hausdorff group G acts on the Hilbert space $L^2(G)$ by the prescription

$$R(g)f(x) := f(xg).$$

The representation so obtained is unitary and if G is a Lie group the notion of smooth vectors in $L^2(G)$ makes sense. In the next section, we will need a criterion to establish that certain functions are smooth vectors in $L^2(G)$. We will make use of the following notion:

Definition 3.2.15. Let G be a Lie group and let (π, H) be a Hilbert representation of G. The **Gårding subspace** of H is the vector subspace of H spanned by the set

$$\{\pi(f)v | v \in H,\ f \in C_c^\infty(G)\}.$$

Proposition 3.2.16. Let G be a Lie group with finitely many connected components, let (π, H) be a Hilbert representation of G. Then every vector in the Gårding subspace of H is a smooth vector in H.

Proof. See [35], Lemma 1.6.1.

□

Recall that $f \in C^\infty(G)$ is called **$Z(\mathfrak{g}_\mathbb{C})$-finite** if it is annihilated by an ideal of $Z(\mathfrak{g}_\mathbb{C})$ of finite codimension. The criterion we need is the following result of Harish-Chandra:

Theorem 3.2.17. Let G be a group in the class $\mathcal{H}$ as in [33], p. 192. Let $f \in C^\infty(G)$ be K-finite and $Z(\mathfrak{g}_\mathbb{C})$-finite. Then there exists a function $h \in C_c^\infty(G)$ which satisfies $h(kgk^{-1}) = h(g)$ for all $k \in K$ and for all $g \in G$ and such that $f * h = h$. If $f \in C^\infty(G)$, in addition, is square-integrable, then f is a smooth vector in $L^2(G)$.

Proof. The first statement is [33], Proposition 14, p. 352. The second conclusion follows from the observation found at the beginning of the proof of Corollary 8.42 in [24] that f is in the Gårding subspace of $L^2(G)$ and it is therefore smooth by Proposition 3.2.16. That f is indeed in the Gårding subspace of $L^2(G)$ follows from the equality

$$R(\tilde{h})f = f * h,$$

where $\tilde{h}(x) = h(x^{-1})$ and from the first statement.

□

Proposition 3.2.18. Let G be a group in the class $\mathcal{H}$. Let $f \in C^\infty(G)$ be K-finite, $Z(\mathfrak{g}_\mathbb{C})$-finite and square-integrable. Then, for every $X \in \mathfrak{g}$, we have

$$Xf = \dot{R}(X)f$$

where $Xf : G \longrightarrow \mathbb{C}$ is defined as

$$Xf(g) := \frac{d}{dt}\left[f(g\exp(tX))\right]|_{t=0} \tag{2}$$

Proof. By Theorem 3.2.17, there exists $h \in C_c^\infty(G)$ such that

$$f = f * h.$$

From the equalities

$$Xf = X(f * h) = f * Xh$$

and

$$f * Xh = \dot{R}(\widetilde{Xh})f,$$

we obtain

$$Xf = \dot{R}(\widetilde{Xh})f.$$

Since

$$\dot{R}(\widetilde{Xh})f = \dot{R}(X)R(\tilde{h})f$$

and

$$R(\tilde{h})f = f * h = f,$$

we conclude

$$Xf = \dot{R}(X)f.$$

□

We will apply Proposition 3.2.18 to the group M in the Langlands decomposition of a parabolic subgroup $P = MAN$ of a connected semisimple Lie group with finite centre. A group M of this form will not be connected, semisimple in general. However, it belongs to the class $\mathcal{H}$ by [17], Lemma 9, p. 108.

We briefly recall the construction of parabolically induced representations. We refer the reader to [26], Chapter XI, for a more thorough account.

Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. The group $K_M := K \cap M$ is a maximal compact subgroup of M. Let λ be a complex-valued real-linear functional on $\mathfrak{a}$ and let (σ, H_σ) be a Hilbert representation of M. We define an action of G on the space of functions

$$\{f \in C(K, H_\sigma)|\ f(mk) = \sigma(m)f(k) \text{ for all } m \in K_M \text{ and all } k \in K\}$$

by declaring

$$\mathrm{Ind}_P(\sigma, \lambda, g)f(k) := e^{(\lambda+\rho)(\mathbf{h}(kg))}\sigma(\mathbf{m}(kg))f(\mathbf{k}(kg)),$$

where, if $g = kman$ for some $k \in K$, $m \in M$, $a \in A$, $n \in N$, we set $\mathbf{k}(g) := k$, $\mathbf{m}(g) := m$, $\mathbf{h}(g) := \log(a)$, $\mathbf{n}(g) := n$. The symbol ρ denotes half of the sum of the positive restricted roots determined by $\mathfrak{a}$ counted with multiplicities. On this space of functions, we introduce the norm

$$\|f\|_{\mathrm{Ind}_P(\sigma,\lambda)} := (\int_K \|f(k)\|_\sigma^2 \, dk)^{\frac{1}{2}}$$

and, upon completing, we obtain a Hilbert representation of G which we denote $\mathrm{Ind}_P(\sigma, \lambda)$. We will denote $\mathrm{Ind}_{P,K_M}(\sigma, \lambda)$ the space of K_M-finite vectors in $\mathrm{Ind}_P(\sigma, \lambda)$.

Proposition 3.2.19. Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. Let λ be a complex-valued, real-linear, totally imaginary functional on $\mathfrak{a}$ and let (σ, H_σ) be a unitary representation of M. Then $\mathrm{Ind}_P(\sigma, \lambda)$ is a unitary representation of G.

Proof. See [26], Corollary 11.39.

□

Corollary 3.2.20. Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. Let λ be a complex-valued, real-linear, totally imaginary functional on $\mathfrak{a}$ and let (σ, H_σ) be a unitary representation of M. Then, for every $f_1, f_2 \in \mathrm{Ind}_{P,K_M}(\sigma, \lambda)$ and for every $X \in \mathfrak{g}$, we have

$$\langle \dot{\mathrm{Ind}}_P(\sigma, \lambda)(X)f_1, f_2\rangle_{\mathrm{Ind}_P(\sigma,\lambda)} = -\langle f_1, \dot{\mathrm{Ind}}_P(\sigma, \lambda)(X)f_2\rangle_{\mathrm{Ind}_P(\sigma,\lambda)}.$$

Proof. This is a consequence of Proposition 3.2.19 and the skew-invariance of the inner product on a unitary representation with respect to the action of the Lie algebra on the space of smooth vectors ([37], p. 266).

□

Next, we recall a form of Frobenius' reciprocity originally observed by Casselman. We first need some preparation.

First of all, we record the following.

Lemma 3.2.21. Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. If V is a $(\mathfrak{g}, K)$-module, then $V/\mathfrak{n}V$ can be equipped with the structure of an $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module in such a way that the quotient map

$$q : V \longrightarrow V/\mathfrak{n}V$$

is $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-equivariant.

Proof. It suffices to show that if $v \in V$ is of the form

$$v = Xw$$

for some $w \in V$ and $X \in \mathfrak{n}$, then, for all $\xi \in K_M$, we have

$$\xi v \in \mathfrak{n}V,$$

and, for all $Y \in \mathfrak{m} \oplus \mathfrak{a}$, we have

$$Yv \in \mathfrak{n}V.$$

Let $\xi \in K_m$. We have

$$\xi v = \xi X w = \mathrm{Ad}(\xi) X \xi w$$

and, since K_M, being contained in M, normalises $\mathfrak{n}$ by [25], Proposition 7.83, it follows that $\mathrm{Ad}(\xi)X \in \mathfrak{n}$.

Let $Y \in \mathfrak{m} \oplus \mathfrak{a}$. We have

$$Yv = YXw = [Y, X]\, w + XYw.$$

The second term in the RHS belongs to $\mathfrak{n}V$ because $X \in \mathfrak{n}$ and the first belongs to $\mathfrak{n}V$ because $\mathfrak{n}$ is an ideal in $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ by [25], Proposition 7.78. □

Let us recall that a $(\mathfrak{g}, K)$-module is **finitely generated** if it is a finitely generated $U(\mathfrak{g}_{\mathbb{C}})$-module. We say that a Hilbert representation (π, H) of G is finitely generated if H_K is finitely generated. We record the following result of Casselman.

Theorem 3.2.22. Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. Let V be an admissible, finitely generated $(\mathfrak{g}, K)$-module. Then $V/\mathfrak{n}V$ is an admissible, finitely generated $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module.

Proof. See [35], Lemma 4.3.1. □

Corollary 3.2.23. Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. If V is an irreducible $(\mathfrak{g}, K)$-module admitting an infinitesimal character, then $V/\mathfrak{n}V$ is an admissible, finitely generated $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module.

Proof. By [21], Theorem 2.2, p. 256, K is connected. By [26], Theorem 7.204, V is admissible. Combining [26], Example 1, p. 442, and [26], Corollary 7.207, it follows that V is finitely generated. The result now follows from Theorem 3.2.22.

□

Let $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of P, M, A and N, respectively.

Let (σ, H_σ) be an admissible and finitely generated Hilbert representation of M which is unitary when restricted to K_M. Let λ be a complex-valued real-linear functional on $\mathfrak{a}$. Consider the $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module H^λ_{σ,K_M} defined as

$$H^\lambda_{\sigma,K_M} := H_{\sigma,K_M} \otimes \mathbb{C}_{\lambda+\rho}$$

where the pair $(\mathfrak{m}, K_M)$ acts on H_{σ,K_M} and $\mathfrak{a}$ acts on $\mathbb{C}_{\lambda+\rho}$ via the functional $\lambda + \rho$.

If V is a $(\mathfrak{g}, K)$-module and $T \in \mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{P,K_M}(\sigma, \lambda))$, then we can define an element $\hat{T} \in \mathrm{Hom}_{\mathfrak{m}\oplus\mathfrak{a},K_M}(V/\mathfrak{n}V, H^\lambda_{\sigma,K_M})$ by setting

$$\hat{T}(v) := T(v)(1).$$

Theorem 3.2.24. Let G be a connected, semisimple Lie group with finite centre. Let V be a $(\mathfrak{g}, K)$-module. Let (σ, H_σ) be an admissible and finitely generated Hilbert representation of M which is unitary when restricted to K_M and let λ be a complex-valued real-linear functional on $\mathfrak{a}$. Consider the $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module H^λ_{σ,K_M}. Then the map

$$\mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{P,K_M}(\sigma, \lambda)) \longrightarrow \mathrm{Hom}_{\mathfrak{m}\oplus\mathfrak{a},K_M}(V/\mathfrak{n}V, H^\lambda_{\sigma,K_M}),\ T \mapsto \hat{T}$$

is a bijection.

Proof. See [35], Lemma 5.2.3 and the discussion preceding it. See also Lemma 3.8.2 for the specialisation to the case of a minimal parabolic. Alternatively [20], Theorem 4.9.

□

The inverse of the map $T \mapsto \hat{T}$ is constructed as follows. Let $S \in \mathrm{Hom}_{\mathfrak{m}\oplus\mathfrak{a},K_M}(V/\mathfrak{n}V, H^\lambda_{\sigma,K_M})$. Then we obtain an element $\tilde{S} \in \mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{P,K_M}(\sigma, \lambda))$ by setting

$$\tilde{S}(v)(k) := S(q(kv)),$$

where $q : V \longrightarrow V/\mathfrak{n}V$ denotes the quotient map. Then the inverse of $T \mapsto \hat{T}$ is given by the map

$$\mathrm{Hom}_{\mathfrak{m}\oplus\mathfrak{a},K_M}(V/\mathfrak{n}V, H^\lambda_{\sigma,K_M}) \longrightarrow \mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{P,K_M}(\sigma, \lambda)),\ S \mapsto \tilde{S}.$$

3.3 Asymptotic behaviour of representations

We begin by collecting the fundamental facts concerning asymptotic expansions of matrix coefficients of tempered representations. We refer the reader to [24], Chapter VIII, for a more thorough exposition of the topic.

Let G be a connected, semisimple Lie group with finite centre, let K be a fixed maximal compact subgroup of G corresponding to a Cartan decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $P = MAN$ denote the minimal parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Given a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, we call A the corresponding subgroup of P and M the centraliser of A in K. We fix a system Δ of simple roots of the root system $(\mathfrak{g}, \mathfrak{a})$, we use Δ^+ to denote the corresponding set of positive roots.

Let $\mathfrak{a}^+$ denote the set $\{H \in \mathfrak{a} | \alpha(H) > 0 \text{ for all } \alpha \in \Delta\}$. Then the subset of regular elements G^{reg} of G admits a decomposition as $G^{\text{reg}} = K\exp(\mathfrak{a}^+)K$ and G itself admits a decomposition $G = K\overline{\exp(\mathfrak{a}^+)}K$.

We write $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and we identify it with the ordered set $\{1, \dots, n\}$ in the obvious way. We adopt the following notation to simplify the appearance of the expansions we are going to work with.

For $H \in \mathfrak{a}$ and $l \in \mathbb{Z}^n_{\geq 0}$, we set $\alpha(H)^l := \prod_{i=1}^n \alpha_i(H)^{l_i}$.

If λ is a real-linear complex-valued functional on $\mathfrak{a}$, since, for every $H \in \mathfrak{a}$, we have

$$\lambda(H) = \sum_{i=1}^n \lambda_i \alpha_i(H)$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, we will often identify λ with the n-tuple $(\lambda_1, \dots, \lambda_n)$.

The next result is concerned with the expansion of K-finite matrix coefficients relative to P.

Theorem 3.3.1. Let G be a connected, semisimple Lie group with finite centre and let (π, H) be an irreducible, Hilbert representation of G. Then there exist a non-negative integer l_0 and a finite set of real-linear complex-valued functionals on $\mathfrak{a}$, denoted $\mathcal{E}_0$, such that, for every $v, w \in H_K$, the restriction to $\exp(\mathfrak{a}^+)$ of the matrix coefficient

$$\phi_{v,w}(g) = \langle \pi(g)v, w \rangle$$

admits a uniformly and absolutely convergent expansion as

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{\lambda \in \mathcal{E}_0} \sum_{l \in \mathbb{Z}^n_{\geq 0} : |l| \leq l_0} \sum_{k \in \mathbb{Z}^n_{\geq 0}} \alpha(H)^l e^{(\lambda - k)(H)} \langle c_{\lambda - k, l}(v), w \rangle,$$

where each $c_{\lambda-k,l} : H_K \longrightarrow H_K$ is a complex-linear map and $\rho_{\mathfrak{p}}$ denotes half of the sum of the elements in Δ^+ counted with multiplicities.

Proof. By [26], Theorem 7.204, (π, H) is admissible and therefore has an infinitesimal character. By [24], Theorem 8.32, we have the stated expansion for any τ-spherical function (in the sense of [24], p.215) F on G of the form

$$F(g) = E_2 \pi(g) E_1,$$

where τ_1 and τ_2 are sub-representations of

$$\pi|_K \cong \bigoplus_{\gamma \in \tilde{K}} n_\gamma \gamma$$

of the form

$$\tau_1 := \bigoplus_{\gamma \in \Theta_1} n_\gamma \gamma \text{ and } \tau_2 := \bigoplus_{\gamma \in \Theta_2} n_\gamma \gamma$$

for finite collections $\Theta_1, \Theta_2 \in \hat{K}$, and E_1, E_2 are the orthogonal projections to τ_1, τ_2, respectively. In this expansion, the set $\mathcal{E}_0$, the maps $c_{\lambda-k,l}$ and the number l_0 depend on $\tau = (\tau_1, \tau_2)$ and we can expand $\phi_{v,w}$ provided that $v \in \tau_1$ and $w \in \tau_2$. To obtain an expansion valid for every $v, w \in H_K$ and with F, l_0 and the $c_{\lambda-k,l}$ independent of τ, we appeal to [8], Theorem in 8.8, which we can apply since (π, H) is finitely generated by [26], Corollary 7.207.

□

We recall that if ν, ν' are real-linear complex-valued functionals on $\mathfrak{a}$ such that $\nu - \nu'$ is an integral linear combination of the simple roots, then we say that ν and ν' are **integrally equivalent**.

The set $\mathcal{E}_0$ has the property that if $\lambda, \lambda' \in \mathcal{E}_0$ with $\lambda \neq \lambda'$, then λ and λ' are not integrally equivalent.

If ν and ν' are integrally equivalent and $\nu - \nu'$ is a non-negative integral combination of the simple roots, we write $\nu \geq \nu'$, thus introducing an order relation among integrally equivalent functionals on $\mathfrak{a}$.

If $k \in \mathbb{Z}^n_{\geq 0}$ is such that the term

$$\alpha(H)^l e^{(\lambda-k)(H)} \langle c_{\lambda-k,l}(v), w \rangle$$

is non-zero for some $\lambda \in \mathcal{E}_0$ and for some $v, w \in H_K$, then we say that $\nu := (\lambda - k)$ is an **exponent**. The exponents which are maximal with respect to the order relation introduced above are called **leading exponents**: $\mathcal{E}_0$ is precisely the set of leading exponents.

The following result is used crucially in [23] and in the following.

Theorem 3.3.2. Let (π, H) be an irreducible, tempered, Hilbert representation of G. Then every $\lambda \in \mathcal{E}_0$ satisfies

$$\mathrm{Re}\lambda_i \leq 0$$

for every $i \in \{1, \ldots, n\}$.

Proof. See [24], Theorem 8.53. Strictly speaking, in loc. cit. the theorem is formulated under some restrictions on G, but it is a convenient reference since we are adopting the same normalisation of the exponents. See [5], Proposition 3.7, p. 83, or [8], Corollary 8.12 for proofs for more general groups.

□

We now turn to asymptotic expansions of matrix coefficients of (π, H) relative to standard (for P) parabolic subgroups of G. We follow [24], Chapter VIII, Section 12.

Given a subset $I \subset \{1, ..., n\}$, and recalling that we identified Δ with $\{1, \ldots, n\}$, we can associate to it a parabolic subgroup

$$P_I = M_I A_{I^c} N_{I^c}$$

of G containing P in such a way that $\mathfrak{g}_{-\alpha} \subset \mathfrak{m}_I$ if and only if $\alpha \in I$ (with $\mathfrak{m}_I$ denoting the Lie algebra of M_I). For the details, we refer the reader to [25], Chapter VII and [24], Proposition 5.23.

First, we introduce the basis $\{H_1, \ldots, H_n\}$ of $\mathfrak{a}$ dual to Δ. We define the Lie algebra $\mathfrak{a}_I$ as

$$\mathfrak{a}_I := \sum_{i \in I} \mathbb{R} H_i$$

and the group A_I as

$$A_I := \exp(\sum_{i \in I} \mathbb{R} \alpha_i).$$

We can then write

$$\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}_{I^c}$$

and

$$A = A_I A_{I^c}.$$

The groups N_I and N_{I^c} correspond to the Lie algebras

$$\mathfrak{n}_I := \sum_{\beta \in \Delta^+ : \beta|_{\mathfrak{a}_{I^c}} = 0} \mathfrak{g}_\beta \text{ and } \mathfrak{n}_{I^c} := \sum_{\beta \in \Delta^+ : \beta|_{\mathfrak{a}_{I^c}} \neq 0} \mathfrak{g}_\beta.$$

We have

$$\rho = \rho_I + \rho_{I^c}$$

with

$$\rho_I := \frac{1}{2} \sum_{\beta \in \Delta^+ : \beta|_{\mathfrak{a}_{I^c}} = 0} (\dim \mathfrak{g}_\beta) \beta$$

and analogously for ρ_{I^c}. Denoting $M_{0,I}$ the group corresponding to the Lie algebra

$$\mathfrak{m}_I = \mathfrak{m} \oplus \mathfrak{a}_I \oplus \mathfrak{n}_I \oplus \overline{\mathfrak{n}}_I,$$

the group M_I is then given by

$$M_I := Z_K(\mathfrak{a}_{I^c}) M_{0,I}.$$

Finally, $K_I := K \cap M_I$ is a maximal compact subgroup of M_I and $M A_I N_I$ is a minimal parabolic subgroup of M_I.

Theorem 3.3.3. Let G be a connected, semisimple Lie group with finite centre and let (π, H) be an irreducible, Hilbert representation of G. Let C be a compact subset of M_I satisfying $K_I C K_I = C$. Then there exists a positive real number R depending on C such that, for every $m \in C$ and for every $a = \exp H \in A_{I^c}$ which satisfies $\alpha_i(H) > \log R$ for every $i \in I^c$, we have

$$\phi_{v,w}(m\exp H) = e^{-\rho_{I^c}(H)} \sum_{\nu \in \mathcal{E}_I} \sum_{q \in \mathbb{Z}_{\geq 0}^{I^c} : |q| \leq q_0} \alpha(H)^q e^{\nu(H)} c_{\nu,q}^{P_I}(m, v, w)$$

for every $v, w \in H_K$. Here, $\mathcal{E}_I$ is a countable set of real-linear complex-valued functionals on $\mathfrak{a}_{I^c}$, each $c_{\nu,q}^{P_I}$ extends to a real analytic function on M_I and satisfies

$$c_{\nu,q}^{P_I}(\xi_2 m \xi_1, v, w) = c_{\nu,q}^{P_I}(m, \pi(\xi_1)v, \pi(\xi_2^{-1})w)$$

for every $\xi_1, \xi_2 \in K_I$. Moreover, for every $m \in M_I$ and $w \in H_K$, the map

$$H_K \longrightarrow \mathbb{C}, \ v \mapsto c_{\nu,q}^{P_I}(m, v, w)$$

is complex-linear and, for every $m \in M_I$ and $v \in H_K$, the map

$$H_K \longrightarrow \mathbb{C}, \ w \mapsto c_{\nu,q}^{P_I}(m, v, w)$$

is conjugate-linear.

Proof. For a τ-spherical function F as in the proof of Theorem 3.3.1, the result follows from [24], Theorem 8.45. To obtain an expansion independent of τ, it suffices to prove that each $F_{\nu-\rho_{I_\lambda^c}}$ is independent of τ.

Let $m \in M_I$ and write $m = \xi_2 a_I \xi_2$ for some $a_I \in \overline{\exp(\mathfrak{a}_I^+)}$ and some $\xi_1, \xi_2 \in K_I$. Since

$$F_{\nu-\rho_{I^c}}(ma, v, w) = F_{\nu-\rho_{I^c}}(a_I a, \pi(\xi_1)v, \pi(\xi_2^{-1})w),$$

re-labeling things, it suffices to prove that $F_{\nu-\rho_{I^c}}(\cdot, v, w)$ is independent of τ as a function on $\overline{\exp(\mathfrak{a}_I^+)}A_{I^c}$. By [24], Corollary 8.46, the functional $\nu \in \mathcal{E}_\mathcal{I}$ is the restriction of an element in the set of exponents $\mathcal{E}$ in the expansion relative to P and this set is independent of τ by [8], Theorem 8.8. Therefore, it remains to prove that each $c_{\nu,q}^{P_\lambda}$ is independent of τ. Since $c_{\nu,q}^{P_\lambda}$ is analytic on M_I, it suffices to prove that $c_{\nu,q}^{P_\lambda}(\cdot, v, w)$ as a function on $\exp(\mathfrak{a}_I^+)$ is independent of τ. Given $a_I \in \exp(\mathfrak{a}_I^+)$, we can find a compact subset C of M_I containing and a_I such that $K_I C K_I = C$, and a positive R depending on C, such that for every $H \in \mathfrak{a}_{I^c}$ satisfying $\alpha_i(H) > \log R$ for every $i \in I^c$, the expansion of $\phi_{v,w}(a_I a)$ relative to P and the expansion relative to P_I are both valid. Comparing them as in [24], p. 251, it follows that expansion relative to P_I is completely determined by the expansion relative to P and the latter is independent of τ by Theorem 3.3.1. ☐

For every $\nu \in \mathcal{E}_I$, the term

$$\alpha(H)^q e^{(\nu-\rho_{I^c})(H)} c_{\nu,q}^{P_I}(m, v, w)$$

is non-zero for some $v, w \in H_K$ and some $m \in M$. The set $\mathcal{E}_I$ is the set of **exponents relative to P_I.**

We are ready to define the functions of the form $\Gamma_{\lambda,l}$ discussed in the Introduction. The first step consists in associating a standard (for P) parabolic subgroup of G to each $\lambda \in \mathcal{E}_0$.

Let (π, H) be an irreducible, tempered, Hilbert representation of G and let $\lambda \in \mathcal{E}_0$. We set $I_\lambda := \{i \in \{1, \ldots, n\} | \mathrm{Re}\lambda_i < 0\}$ which we identify with the subset Δ_λ of Δ defined as

$$\Delta_\lambda := \{\alpha_i \in \Delta | i \in I_\lambda\}.$$

The construction of standard parabolic subgroups from the datum of a subset of Δ assigns to I_λ the standard parabolic subgroup P_λ defined as

$$P_\lambda := P_{I_\lambda}.$$

It admits a decomposition

$$P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0},$$

where

$$A_{\lambda_0} := A_{I_\lambda^c}.$$

The subgroup M admits a decomposition

$$M_\lambda = K_\lambda \overline{\exp(\mathfrak{a}_\lambda^+)} K_\lambda,$$

where

$$\mathfrak{a}_\lambda := \mathfrak{a}_{I_\lambda}$$

and

$$K_\lambda := K \cap M_\lambda.$$

The group A decomposes as $A = A_\lambda A_{\lambda_0}$. We write $\mathfrak{a}_\lambda$ and $\mathfrak{a}_{\lambda_0}$ for $\mathfrak{a}_{I_\lambda}$ and $\mathfrak{a}_{I_\lambda^c}$, respectively. Similarly, we write ρ_λ and ρ_{λ_0} for ρ_{I_λ} and $\rho_{I_\lambda^c}$, respectively.

We are going to introduce an equivalence relation on the data indexing the expansion of $\phi_{v,w}$ relative to P. The definition is motivated by the construction of $\mathbf{d}(\pi)$ in [23]. Let $v, w \in H_K$. We have

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{\lambda \in \mathcal{E}_0} \sum_{l \in \mathbb{Z}^n_{\geq 0} : |l| \leq l_0} \alpha(H)^l e^{\lambda(H)} \Phi^{v,w}_{\lambda,l}(H)$$

where

$$\Phi^{v,w}_{\lambda,l}(H) := \sum_{k \in \mathbb{Z}^n_{\geq 0}} e^{-k(H)} \langle c_{\lambda-k,l}(v_1), v_2 \rangle.$$

The terms in this expansion are indexed by the finite set

$$\mathcal{C} := \{(\lambda, l)\}_{\lambda \in \mathcal{E}_0,\ l \in \mathbb{Z}^n_{\geq 0} : |l| \leq l_0}.$$

We introduce a relation on $\mathcal{C}$ by declaring that $(\lambda, l) \sim (\mu, m)$ if $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and $\mathrm{res}_{I_\lambda^c} l = \mathrm{res}_{I_\mu^c} m$.

It is clear that $\sim$ is an equivalence relation. We denote $[\lambda, l]$ the equivalence class containing (λ, l).

We can therefore re-group the expansion of $\phi_{v,w}$ as follows:

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{[\lambda,l]\in\mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda|_{\mathfrak{a}_{\lambda_0}}(H_{\lambda_0})} \sum_{(\lambda',l')\in[\lambda,l]} \alpha(H_\lambda)^{l'_\lambda} e^{\lambda'|_{\mathfrak{a}_\lambda}(H_\lambda)} \Phi^{v,w}_{\lambda',l'}(H),$$

where

$$l_{\lambda_0} := \mathrm{res}_{I^c_\lambda} l,\ \alpha(H_{\lambda_0})^{l_{\lambda_0}} := \prod_{i\in I^c_\lambda} \alpha_i(H_{\lambda_0})^{l_i},\ l'_\lambda := \mathrm{res}_{I_\lambda} l',\ \alpha(H_\lambda)^{l'_\lambda} := \prod_{i\in I_\lambda} \alpha(H_\lambda)^{l'_i}$$

and $H = H_{\lambda_0} + H_\lambda$ corresponds to the decomposition

$$\mathfrak{a}^+ = \mathfrak{a}^+_{\lambda_0} \oplus \mathfrak{a}^+_\lambda.$$

We are also implicitly using the fact that $\alpha(H)^l = \alpha(H_\lambda)^{l_\lambda}\alpha(H_{\lambda_0})^{l_{\lambda_0}}$ which follows from writing H with respect to the basis dual to Δ.

Let us assume that $[\lambda, l] \in \mathcal{C}/\sim$ satisfies

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi),$$

where $\mathbf{d}_P(\lambda, l)$ is defined by (1).

For $H_\lambda \in \mathfrak{a}^+_\lambda$, we set

$$\Gamma_{\lambda,l}(\exp H_\lambda, v, w) := e^{-\rho(H)} \sum_{(\lambda',l')\in[\lambda,l]} \alpha(H_\lambda)^{l'_\lambda} e^{\lambda'|_{\mathfrak{a}_{\lambda_0}}(H_\lambda)} \Phi^{v,w}_{\lambda',l'}(H_\lambda). \tag{3}$$

Before establishing the properties of $\Gamma_{\lambda,l}$, let us pause to explain the motivation behind the condition on the equivalence class $[\lambda, l]$. The discussion that follows will be used only in Section 4. The reader who prefers to do so can skip to Proposition 3.3.4 without any loss of continuity.

Let $v_1, v_2, v_3, v_4 \in H_K$. We will be considering integrals of the form

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{v_1,v_2}(\exp H)\overline{\phi}_{v_3,v_4}(\exp H) \prod_{\beta\in\Delta^+} (e^{\beta(H)} - e^{-\beta(H)})^{\dim\mathfrak{g}_\beta}\, dH,$$

where

$$\mathfrak{a}^+_{<r} := \mathfrak{a}^+ \cap \{H \in \mathfrak{a} | \beta(H) < r \text{ for all } \beta \in \Delta^+\}. \tag{4}$$

Treating these is the content of Appendix A in [23]. We remark that our region of integration is defined as to exclude the subset of $\overline{\mathfrak{a}^+}$ where at least one of the simple roots vanishes. It is a set of measure zero.

We want to interpret Lemma A.5 in [23] in group-theoretic terms.

Let us consider the matrix coefficients ϕ_{v_1,v_2} and ϕ_{v_3,v_4}. On $A^+ := \exp(\mathfrak{a}^+)$, they can be expanded as

$$\phi_{v_1,v_2}(\exp H) = e^{-\rho(H)} \sum_{[\lambda,l]\in\mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda|_{\mathfrak{a}_{\lambda_0}}(H_{\lambda_0})} \sum_{(\lambda',l')\in[\lambda,l]} \Psi^{v_1,v_2}_{\lambda',l'}(H)$$

and

$$\phi_{v_3,v_4}(\exp H) = e^{-\rho(H)} \sum_{[\mu,m]\in\mathcal{C}/\sim} \alpha(H_{\mu_0})^{m_{\mu_0}} e^{\mu|_{\mathfrak{a}_{\mu_0}}(H_{\mu_0})} \sum_{(\mu',m')\in[\mu,m]} \Psi^{v_3,v_4}_{\mu',m'}(H).$$

where, for $(\lambda',l')\in[\lambda,l]$, we set

$$\Psi^{v_1,v_2}_{\lambda',l'}(H) := \alpha(H_\lambda)^{l'_\lambda} e^{\lambda'|_{\mathfrak{a}_\lambda}(H_\lambda)} \Phi^{v_1,v_2}_{\lambda',l'}(H)$$

and similarly for $(\mu',m')\in[\mu,m]$.

Let $[\lambda,l]\in\mathcal{C}/\sim$ and $[\mu,m]\in\mathcal{C}/\sim$ be such that $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and

$$\mathbf{d}(\pi) = |I^c_\lambda| + \sum_{i\in I^c_\lambda}(l_i+m_i).$$

In view of the first condition, the third is equivalent to the requirement

$$\mathbf{d}_P(\lambda,l) = \mathbf{d}(\pi) \text{ and } \mathbf{d}_P(\mu,m) = \mathbf{d}(\pi).$$

Consider the summand

$$e^{-2\rho(H)}\alpha(H)^{l'+m'}e^{(\lambda'+\overline{\mu'})(H)}\Phi^{v_1,v_2}_{\lambda',l'}\overline{\Phi}^{v_3,v_4}_{\mu',m'}(H)$$

in the expansion of the product $\phi_{v_1,v_2}\overline{\phi}_{v_3,v_4}$ on A^+.

Taking into account the factor $e^{-2\rho(H)}$ and the fact that the term

$$\Omega(H) := \prod_{\beta\in\Delta^+}(e^{\beta(H)} - e^{-\beta(H)})^{\dim \mathfrak{g}_\beta} \tag{5}$$

is incorporated in the function ϕ in Lemma A.5 in [23] (compare section 4.7 in loc. cit.), this lemma shows that, as $r\to\infty$, the integral

$$\frac{1}{r^{\mathbf{d}(\pi)}}\int_{A^+_{<r}} e^{-2\rho(H)}\alpha(H)^{l'+m'}e^{(\lambda'+\overline{\mu'})(H)}\Phi^{v_1,v_2}_{\lambda',l'}\overline{\Phi}^{v_3,v_4}_{\mu',m'}(H)\Omega(H)\,dH$$

tends to

$$C(\lambda,l,m)\int_{A^+_\lambda} e^{-2\rho_\lambda(H_\lambda)}\left[\Psi^{v_1,v_2}_{\lambda',l'}\overline{\Psi}^{v_3,v_4}_{\mu',m'}\right]|_{\mathfrak{a}_\lambda}(H_\lambda)\Omega_\lambda(H_\lambda)\,dH_\lambda$$

where

$$\Omega_\lambda(H_\lambda) := \prod_{\beta\in\Delta^+_\lambda}(e^{\beta(H_\lambda)} - e^{-\beta(H_\lambda)})^{\dim \mathfrak{g}_\beta}, \tag{6}$$

with

$$\Delta_\lambda^+ := \{\beta \in \Delta^+ | \beta|_{\mathfrak{a}_{\lambda_0}} = 0\}$$

and the quantity $C(\lambda, l, m)$ is given by

$$C(\lambda, l, m) := \int_{\{H \in A_{\lambda_0} | \mathrm{ext}^{I_\lambda^c}(H) \in A_{<1}^+\}} \alpha(H_{\lambda_0})^{l_{\lambda_0} + m_{\mu_0}} \, dH_{\lambda_0} \tag{7}$$

Now, summing over all $(\lambda', l') \in [\lambda, l]$ and over all $(\mu', m') \in [\mu, m]$, we obtain that the integral over $A_{<r}^+$ of

$$e^{-2\rho(H)} \sum_{(\lambda', l') \in [\lambda, l]} \sum_{(\mu', m') \in [\mu, m]} \alpha(H)^{l' + m'} e^{(\lambda' + \overline{\mu'})(H)} \Phi_{\lambda', l'}^{v_1, v_2} \overline{\Phi}_{\mu', m}^{v_3, v_4}(H) \Omega(H),$$

upon multiplying by $\frac{1}{r^{\mathbf{d}(\pi)}}$ and letting $r \to \infty$, equals

$$C(\lambda, l, m) \int_{A_\lambda^+} e^{-2\rho_\lambda(H_\lambda)} \sum_{(\lambda', l') \in [\lambda, l]} \sum_{(\mu', m') \in [\mu, m]} \left[\Psi_{\lambda', l'}^{v_1, v_2} \overline{\Psi}_{\mu', m'}^{v_3, v_4} \right] |_{\mathfrak{a}_\lambda}(H_\lambda) \Omega_\lambda(H_\lambda) \, dH_\lambda.$$

Finally, since

$$\Phi_{\lambda', l'}^{v_1, v_2}|_{\mathfrak{a}_\lambda}(H_\lambda) = \sum_{k \in \mathbb{Z}_{\geq 0}^{I_\lambda}} e^{-k(H_\lambda)} \langle c_{\lambda' - k, l'}(v_1), v_2 \rangle,$$

and similarly for $\Phi_{\mu', m,}^{v_3, v_4}$, the integral above equals

$$C(\lambda, l, m) \int_{A_\lambda^+} \Gamma_{\lambda, l}(\exp H_\lambda, v, w) \overline{\Gamma}_{\mu, m}(\exp H_\lambda, v, w) \Omega_\lambda(H_\lambda) \, dH_\lambda.$$

If $[\lambda, l], [\mu, m] \in \mathcal{C}/\sim$ fail to satisfy any of the three conditions $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_\lambda}$ and

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi) = \mathbf{d}_P(\mu, m),$$

then, for every $(\lambda', l') \in [\lambda, l]$ and for every $(\mu', m') \in [\mu, m]$, by the considerations in the proof of Claim A.6 and Lemma A.5 in [23], the integral

$$\frac{1}{r^{\mathbf{d}(\pi)}} \int_{A_{<r}^+} e^{-2\rho(H)} \alpha(H)^{l' + m'} e^{(\lambda' + \overline{\mu'})(H)} \Phi_{\lambda', l'}^{v_1, v_2} \overline{\Phi}_{\mu', m'}^{v_3, v_4}(H) \Omega(H) \, dH$$

vanishes as $r \to \infty$.

Therefore, the equivalence classes $[\lambda, l] \in \mathcal{C}/\sim$ for which $\Gamma_{\lambda, l}$ is defined are precisely the ones that may contribute a non-zero term to the expression

$$\lim_{r \to \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{A_{<r}^+} \phi_{v_1, v_2}(\exp H) \overline{\phi}_{v_3, v_4}(\exp H) \Omega(H) \, dH.$$

Throughout the rest of this section, we fix an irreducible, tempered, Hilbert representation of a connected, semisimple Lie group G with finite centre.

To study the properties of $\Gamma_{\lambda, l}$, we begin by showing that it is equal to a function of the form $c_{\nu, q}^{P_\lambda}$. More precisely, we have:

Proposition 3.3.4. Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l),$$

Set $\nu := \lambda|_{\mathfrak{a}_{\lambda_0}}$ and $q := l_{\lambda_0}$. Then, for every $H_\lambda \in \mathfrak{a}_\lambda^+$, we have

$$\Gamma_{\lambda,l}(\exp H_\lambda, v, w) = c_{\nu,q}^{P_\lambda}(\exp H_\lambda, v, w).$$

Proof. For every $H_\lambda \in \mathfrak{a}_\lambda^+$, we can find a compact subset C of M_λ such that $K_\lambda C K_\lambda = C$ and which contains H_λ, and a positive real $R > 0$ such that if $H_{\lambda_0} \in \mathfrak{a}_{\lambda_0}^+$ satisfies $\alpha_i(H_{\lambda_0}) > \log R$ for every $i \in I_\lambda^c$, then the expansion of $\phi_{v,w}$ with respect to P and the expansion with respect to P_λ are both valid at $H = H_\lambda + H_{\lambda_0}$. Comparing them as in [24], p. 251, we see that

$$c_{\nu,q}^{P_\lambda}(\exp H_\lambda, v, w) = \sum_{\lambda' \in \mathcal{E}_0 : \lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu} \quad \sum_{l' : |l'| \leq l_0 \text{ and } l'_{\lambda_0} = q} e^{-\rho_\lambda(H_\lambda)} \Psi_{\lambda',l'}^{v,w}(H_\lambda).$$

Since, by definition of $\Gamma_{\lambda,l}(\cdot, v, w)$, we have

$$\Gamma_{\lambda,l}(\exp H_\lambda, v, w) = e^{-\rho(H_\lambda)} \sum_{(\lambda',l') \in [\lambda,l]} \Psi_{\lambda',l'}^{v,w}(H_\lambda),$$

recalling the definition of the equivalence relation that we imposed on $\mathcal{C}$, we only need to show that the set

$$\{\lambda' \in \mathcal{E}_0 | \lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu\}$$

is equal to the set

$$\{\lambda \in \mathcal{E}_0 | I_{\lambda'} = I_\lambda \text{ and } \lambda'|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}}\}.$$

Because of the assumption on $[\lambda, l]$, for every $\lambda' \in \mathcal{E}_0$ such that $\lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu$, we have $\mathrm{Re}\lambda'_j \neq 0$ for every $j \in I_\lambda$. Indeed, if there existed a $j \in I_\lambda$ for which $\mathrm{Re}\lambda'_j = 0$, we would have

$$|I_{\lambda'}^c| \geq 1 + |I_\lambda^c|$$

and, since $l'_{\lambda_0} = l_{\lambda_0}$, this would imply

$$\mathbf{d}_P(\lambda', l') > |I_\lambda^c| + \sum_{i \in I_{\lambda'}^c} 2l'_i \geq \mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi),$$

contradicting the maximality of $\mathbf{d}(\pi)$. Since, by Theorem 3.3.2, we have $\mathrm{Re}\lambda'_i \leq 0$ for every $i \in \{1, \ldots, n\}$, this concludes the proof. □

Theorem 8.45 in [24] and the discussion at the beginning of p. 251 in loc. cit. now show that $\Gamma_{\lambda,l}(\cdot, v, w)$, being equal to $c_{\nu,q}^{P_\lambda}$, extends to an analytic function on M_λ, which we denote again $\Gamma_{\lambda,l}(\cdot, v, w)$. If we decompose M_λ as

$$M_\lambda = K_\lambda \overline{\exp(\mathfrak{a}_\lambda^+)} K_\lambda,$$

and if we write $m \in M_\lambda$ as $m = \xi_2 \exp H_\lambda \xi_1$ for some $\xi_1, \xi_2 \in K_\lambda$ and some $H_\lambda \in \overline{\mathfrak{a}_\lambda^+}$, then we have

$$\Gamma_{\lambda,l}(m, v, w) = \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi_1)v, \pi(\xi_2)^{-1}w)$$

because $c^{P_\lambda}_{\nu,q}(\cdot, v, w)$ exhibits the same behaviour.

We want to prove that $\Gamma_{\lambda,l}(\cdot, v, w)$ belongs to $L^2(M_\lambda)$ and it is $Z(\mathfrak{m}_{\lambda\mathbb{C}})$-finite. An application of Theorem 3.2.17 will imply that $\Gamma_{\lambda,l}(\cdot, v, w)$ is a smooth vector in $L^2(M_\lambda)$. Similar ideas appear in [24], Chapter VIII, and in [29].

We recall that there exists an injective algebra homomorphism

$$\mu_{P_\lambda} : Z(\mathfrak{g}_\mathbb{C}) \longrightarrow Z((\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0})_\mathbb{C}) \cong Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$$

which turns $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ into a free module of finite rank over $\mu_{P_\lambda}(Z(\mathfrak{g}_\mathbb{C}))$ by ([18], Lemma 21).

Proposition 3.3.5. Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then $\Gamma_{\lambda,l}(\cdot, v, w)$ belongs to $L^2(M_\lambda)$.

Proof. We argue as in the proof of Lemma 4.10 in [29]. By the proof of Proposition 3.3.4, we have $\mathrm{Re}\lambda'_i < 0$ for every λ' appearing in the expansion of $\Gamma_{\lambda,l}(\cdot, v, w)$ on $\exp(\mathfrak{a}^+_\lambda)$ and for every $i \in I_\lambda$. Since $\Gamma_{\lambda,l}(\cdot, v, w)$ is analytic on $\overline{\exp(\mathfrak{a}^+_\lambda)}$, we can apply Theorem 4 in [19] and then argue as in Theorem 7.5 of [8] to establish the desired square-integrability on $\overline{\exp(\mathfrak{a}^+_\lambda)}$. The square-integrability on M_λ follows from combining the decomposition of M_λ as $M_\lambda = K_\lambda\overline{\exp(\mathfrak{a}^+_\lambda)}K_\lambda$, the corresponding integral formula and the fact that if $m = \xi_2 \exp H_\lambda \xi_2$, for some $H_\lambda \in \overline{\mathfrak{a}^+_\lambda}$ and some $\xi_1, \xi_2 \in K_\lambda$, then

$$\Gamma_{\lambda,l}(m, v, w) = \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi_1)v, \pi(\xi_2)^{-1}w).$$

□

Proposition 3.3.6. Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then, for every $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$ and for every $m \in M_\lambda$, we have

$$X\Gamma_{\lambda,l}(m, v, w) = \Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w).$$

Moreover, the function $\Gamma_{\lambda,l}(\cdot, v, w)$ is a smooth vector in the representation $(R, L^2(M_\lambda))$ of M_λ.

Proof. For a given $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$ and every $g \in G$, we have

$$X\phi_{v,w}(g) = \phi_{\dot{\pi}(X)v,w}(g).$$

Therefore, the restriction of $X\phi_{v,w}(\cdot)$ to $M_\lambda A_{\lambda_0}$ satisfies

$$X\phi_{v,w}(ma) = \phi_{\dot{\pi}(X)v,w}(ma).$$

Given $m \in M_\lambda$ we can find a compact subset C of M_λ containing m such that $K_\lambda C K_\lambda = C$ and a positive R depending on C such that if $a = \exp H \in \exp(\mathfrak{a}^+_{\lambda_0})$ satisfies $\alpha_i(H) > \log R$ for every $i \in I^c_\lambda$, then $\phi_{\dot{\pi}(X)v,w}(ma)$ may be expanded with respect to P_λ.

Since $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$, the restriction of $X\phi_{v,w}(\cdot)$ to $M_\lambda A_{\lambda_0}$ can also be computed as the action of the differential operator X on the restriction of $\phi_{v,w}(\cdot)$ to $M_\lambda A_{\lambda_0}$.
For $m \in M_\lambda$ and $a \in \exp(\mathfrak{a}_{\lambda_0}^+)$ as above, we expand the function so obtained with respect to P_λ and, as in the proof of (4.8) in [29], because of the convergence of the series, we can apply the differential operator term by term. By comparing the resulting expansion with the expansion of $\phi_{\dot{\pi}(X)v,w}(ma)$, and invoking Corollary B.26 of [24], we obtain

$$Xc_{\nu,q}^{P_\lambda}(m, v, w) = c_{\nu,q}^{P_\lambda}(m, \dot{\pi}(X)v, w)$$

for every $\nu \in \mathcal{E}_I$ and every $q \in \mathbb{Z}_{\geq 0}^{I_\lambda^c}$. The first statement now follows from choosing ν and q as in Proposition 3.3.4.

For the last statement, we need to show that $\Gamma_{\lambda,l}(\cdot, v, w)$ is annihilated by an ideal of finite codimension in $Z(\mathfrak{m}_{\lambda\mathbb{C}})$; the result will then follow from Theorem 3.2.17. Let J be the kernel of the infinitesimal character of (π, H). Then J is an ideal of finite codimension in $Z(\mathfrak{g}_\mathbb{C})$. As observed in [17], p.182, the inverse image $J_{\mathfrak{m}_\lambda}$ along the inclusion

$$Z(\mathfrak{m}_{\lambda\mathbb{C}}) \longrightarrow Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}}),\ X \mapsto X \otimes 1$$

of the ideal generated by $\mu_{P_\lambda}(J)$ in $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ is an ideal of finite codimension in $Z(\mathfrak{m}_{\lambda\mathbb{C}})$. This follows from the fact that the ideal generated by $\mu_{P_\lambda}(J)$ is of finite codimension in $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$, since $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ is a free module of finite type over $\mu_{P_\lambda}(Z(\mathfrak{g}_\mathbb{C}))$ by [18], Lemma 21. Denoting $\mu_{P_\lambda}(J)^e$ the ideal generated by $\mu_{P_\lambda}(J)$, we see that $J_{\mathfrak{m}_\lambda}$ is precisely the kernel of the homomorphism

$$Z(\mathfrak{m}_{\lambda\mathbb{C}}) \longrightarrow (Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}}))/\mu_{P_\lambda}(J)^e,\ X \mapsto (X \otimes 1) + \mu_{P_\lambda}(J)^e.$$

This exhibits $J_{\mathfrak{m}_\lambda}$ as an ideal of finite codimension in $Z(\mathfrak{m}_{\lambda\mathbb{C}})$. Now, if $X \in J_{\mathfrak{m}_\lambda}$, then $X \otimes 1$ belongs to $\mu_{P_\lambda}(J)^e$. Hence $X \otimes 1$ can be written as

$$X \otimes 1 = \sum_{i=1}^{r} Y_i \mu_{P_\lambda}(Z_i)$$

with $Y_i \in Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ and $Z_i \in J$. For every $i \in \{1, \ldots, r\}$, by (8.68) in [24], p. 251, the differential operator $\mu_{P_\lambda}(Z_i)$ annihilates the function

$$F_{\nu-\rho_{\lambda_0}}(ma, v, w) := \sum_{q:|q|\leq q_0} c_{\nu,q}^{P_\lambda}(m, v, w)\alpha(H)^q e^{(\nu-\rho_{\lambda_0})(H)}.$$

Therefore, $X \otimes 1$ annihilates it, as well. On the other hand, by the first of the proof, we have

$$(X \otimes 1)F_{\nu-\rho_{\lambda_0}}(ma, v, w) = \sum_{q:|q|\leq q_0} c_{\nu,q}^{P_\lambda}(m, \dot{\pi}(X)v, w)\alpha(H)^q e^{(\nu-\rho_{\lambda_0})(H)}.$$

Since the LHS vanishes identically on $M_\lambda A_{\lambda_0}$, it follows that

$$c_{\nu,q}^{P_\lambda}(m, \dot{\pi}(X)v, w) = 0$$

for every $m \in M_\lambda$. Choosing ν and q as in Proposition 3.3.4, we find that $\Gamma_{\lambda,l}(\cdot, v, w)$ is annihilated by $J_{\mathfrak{m}_\lambda}$.

□

Let $w \in H_K$. The next two technical lemmata, together with Proposition 3.3.6, will be used to prove the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariance of the map

$$S_w : H_K \longrightarrow L^2(M_\lambda) \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},\ S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w).$$

We are not claiming that for every $w \in H_K$ this map is non-zero: the only thing we need to know is that, whenever $w \in H_K$ is such that S_w is not identically zero, then S_w is $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant. Having established this, we show that the existence of an admissible, finitely generated, unitary representation (σ, H_σ) of M_λ which will allow us to apply Theorem 3.2.24 in the way we explained in the Introduction. This is the content of the last two results of this section.

Lemma 3.3.7. Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then, for every $X \in \mathfrak{a}_{\lambda_0}$ and every $m \in M_\lambda$, we have

$$\Gamma_{\lambda,l}(m, \dot\pi(X)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda,l}(m, v, w).$$

Proof. We write $m \in M_\lambda$ as $m = \xi_2 a_\lambda \xi_2$ for some $\xi_1, \xi_2 \in K_\lambda$ and some $a_\lambda \in \overline{\exp(\mathfrak{a}_\lambda^+)}$. Then we have

$$\Gamma_{\lambda,l}(m, \dot\pi(X)v, w) = \Gamma_{\lambda,l}(a_\lambda, \pi(\xi_1)\dot\pi(X)v, \pi(\xi_2^{-1})w).$$

Recalling that

$$\pi(\xi_1)\dot\pi(X)v = \dot\pi(\mathrm{Ad}(\xi_1)X)\pi(\xi_1)v,$$

since M_λ centralises $\mathfrak{a}_{\lambda_0}$ ([25], Proposition 7.82) and K_λ is contained in M_λ, we have

$$\Gamma_{\lambda,l}(a_\lambda, \pi(\xi_1)\dot\pi(X)v, \pi(\xi_2^{-1})w) = \Gamma_{\lambda,l}(a_\lambda, \dot\pi(X)\pi(\xi_1)v, \pi(\xi_2^{-1})w).$$

Therefore, re-labeling things, it suffices to prove that for every $X \in \mathfrak{a}_{\lambda_0}$ and for every $a_\lambda \in \overline{\exp(\mathfrak{a}_\lambda^+)}$, we have

$$\Gamma_{\lambda,l}(a_\lambda, \dot\pi(X)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda,l}(a_\lambda, v, w).$$

Moreover, since $\Gamma_{\lambda,l}(\cdot, v, w)$ is analytic, it suffices to prove the identity for every $a_\lambda \in \exp(\mathfrak{a}_\lambda^+)$.

Let $a_\lambda = \exp H_\lambda$, $H_\lambda \in \mathfrak{a}_\lambda^+$. Then there exist a compact subset C of M_λ containing a_λ and such that $K_\lambda C K_\lambda = C$, and a positive R depending on C such that, for all $H_{\lambda_0} \in \mathfrak{a}_{\lambda_0}^+$ satisfying $\alpha_i(H_{\lambda_0}) > \log R$ for every $i \in I_\lambda^c$, the expansion of $\phi_{\dot\pi(X)v,w}(a_\lambda \exp H_{\lambda_0})$ relative to P and the expansion of $\phi_{\dot\pi(X)v,w}(a_\lambda \exp H_{\lambda_0})$ relative to P_λ are both valid.

Setting $H := H_\lambda + H_{\lambda_0}$ for H_{λ_0} as above, the first expansion gives

$$\phi_{\dot\pi(X)v,w}(H) = \sum_{\tilde\lambda \in \mathcal{E}} \sum_{\tilde l \in \mathbb{Z}^n_{\geq 0} : |\tilde l| \leq l_0} \alpha(H)^{\tilde l} e^{(\tilde\lambda - \rho)(H)} \langle c_{\tilde\lambda, \tilde l}(\dot\pi(X)v), w\rangle$$

By linearity we can assume that $X = H_i$ for some $i \in I_\lambda^c$, where H_i, we recall, is the element in $\mathfrak{a}_{\lambda_0}$ dual to to the simple root α_i.

Differentiating term by term and taking into account the computation

$$H_i\left[\alpha(H)^{\tilde{l}}e^{(\tilde{\lambda}-\rho)(H)}\right] = \tilde{l}_i\alpha(H)^{\tilde{l}-e_i}e^{(\tilde{\lambda}-\rho)(H)} + (\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} - \rho)(H_i)\alpha(H)^{\tilde{l}}e^{(\tilde{\lambda}-\rho)(H)},$$

where e_i is the element in $\mathbb{Z}^n_{\geq 0}$ having 1 as its i-th co-ordinate and 0 as every other co-ordinate, we observe that the only terms in the expansion

$$\phi_{v,w}(H) = \sum_{\tilde{\lambda}\in\mathcal{E}} \sum_{\tilde{l}\in\mathbb{Z}^n_{\geq 0}:|\tilde{l}|\leq l_0} \alpha(H)^{\tilde{l}}e^{(\tilde{\lambda}-\rho)(H)}\langle c_{\tilde{\lambda},\tilde{l}}(v), w\rangle$$

that after differentiation by $H_i \in \mathfrak{a}_{\lambda_0}$ can contribute a term of the form

$$c\alpha(H)^{\tilde{l}}e^{(\tilde{\lambda}-\rho)(H)}\langle c_{\tilde{\lambda},\tilde{l}}(v), w\rangle,$$

with $c \in \mathbb{C}$, to the expansion of $\phi_{\dot{\pi}(X)v,w}(H)$, are precisely

$$\alpha(H)^{\tilde{l}}e^{(\tilde{\lambda}-\rho)(H)}\langle c_{\tilde{\lambda},\tilde{l}}(v), w\rangle \text{ and } \alpha(H)^{\tilde{l}+e_i}e^{(\tilde{\lambda}-\rho)(H)}\langle c_{\tilde{\lambda},\tilde{l}}(v), w\rangle.$$

Assume $\lambda \in \mathcal{E}_0$ and $l \in \mathbb{Z}^n_{\geq 0}$ with $|l| \leq l_0$ satisfy

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi).$$

Then, if the term

$$\alpha(H)^{l+e_i}e^{(\lambda-\rho)(H)}\langle c_{\lambda,l}(v), w\rangle$$

appeared in the expansion of $\phi_{v,w}(H)$, we would have

$$\mathbf{d}_P(\lambda, l+e_i) > |I^c_\lambda| + \sum_{i\in I^c_\lambda} 2l_i = \mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi),$$

contradicting the maximality of $\mathbf{d}(\pi)$. This reasoning shows that in the expansion

$$\phi_{\dot{\pi}(H_i)v,w}(a_\lambda \exp H_{\lambda_0}) = \sum_{\nu\in\mathcal{E}_I} \sum_{q\in\mathbb{Z}^{I^c_\lambda}_{\geq 0}:|q|\leq q_0} \alpha(H_{\lambda_0})^q e^{(\nu-\rho_{\lambda_0})(H_{\lambda_0})} c^{P_\lambda}_{\nu,q}(a_\lambda, \dot{\pi}(H_i)v, w)$$

relative to P_λ, the term indexed by (ν, q) with $\nu = \lambda|_{\mathfrak{a}_{\lambda_0}}$ and $q = l_{\lambda_0}$ satisfies

$$c^{P_\lambda}_{\nu,q}(a_\lambda, \dot{\pi}(H_i)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i)c^{P_\lambda}_{\nu,q}(a_\lambda, v, w).$$

Indeed, the comparison in [24], p. 251, shows that

$$\alpha(H_{\lambda_0})^q e^{(\nu-\rho_{\lambda_0})(H_{\lambda_0})} c^{P_\lambda}_{\nu,q}(a_\lambda, \dot{\pi}(H_i)v, w)$$

is the sum of all the terms in the expansion of $\phi_{\dot{\pi}(H_i)v,w}(H)$ relative to P which are indexed by couples $(\tilde{\lambda}, \tilde{l})$ satisfying

$$\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}} \text{ and } \tilde{l}_{\lambda_0} = l_{\lambda_0}$$

and, as we saw, these are the terms of the form

$$(\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i)\alpha(H)^{\tilde{l}}e^{(\tilde{\lambda}-\rho)(H)}\langle c_{\tilde{\lambda},\tilde{l}}(v), w\rangle.$$

Finally, since

$$\Gamma_{\lambda,l}(a_\lambda, v, w) = c^{P_\lambda}_{\nu,q}(a_\lambda, v, w)$$

by Proposition 3.3.4, we obtain

$$\Gamma_{\lambda,l}(a_\lambda, v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i)\Gamma_{\lambda,l}(a_\lambda, \dot{\pi}(H_i)v, w).$$

□

Lemma 3.3.8. Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then, for every $X \in \mathfrak{n}_{\lambda_0}$ and every $m \in M_\lambda$, we have

$$\Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w) = 0.$$

Proof. We write $m \in M_\lambda$ as $m = \xi_2 a_\lambda \xi_2$ for some $\xi_1, \xi_2 \in K_\lambda$ and some $a_\lambda \in \overline{\exp(\mathfrak{a}_\lambda^+)}$. Then we have

$$\Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w) = \Gamma_{\lambda,l}(a_\lambda, \pi(\xi_1)\dot{\pi}(X)v, \pi(\xi_2^{-1})w).$$

Recalling that

$$\pi(\xi_1)\dot{\pi}(X)v = \dot{\pi}(\mathrm{Ad}(\xi_1)X)\pi(\xi_1)v,$$

since M_λ normalises $\mathfrak{n}_{\lambda_0}$ ([25], Proposition 7.83) and K_λ is contained in M_λ, we have

$$\Gamma_{\lambda,l}(a_\lambda, \pi(\xi_1)\dot{\pi}(X)v, \pi(\xi_2^{-1})w) = \Gamma_{\lambda,l}(a_\lambda, \dot{\pi}(X')\pi(\xi_1)v, \pi(\xi_2^{-1})w)$$

for some $X' \in \mathfrak{n}_{\lambda_0}$. Therefore, re-labeling things, it suffices to prove that for every $X \in \mathfrak{n}_{\lambda_0}$ and for every $a_\lambda \in \overline{\exp(\mathfrak{a}_\lambda^+)}$, we have

$$\Gamma_{\lambda,l}(a_\lambda, \dot{\pi}(X)v, w) = 0.$$

Moreover, since $\Gamma_{\lambda,l}(\cdot, v, w)$ is analytic, it suffices to prove the identity for every $a_\lambda \in \exp(\mathfrak{a}_\lambda^+)$.

Let $a_\lambda = \exp H_\lambda$, $H_\lambda \in \mathfrak{a}_\lambda^+$. Then there exist a compact subset C of M_λ containing H_λ and such that $K_\lambda C K_\lambda = C$, and a positive R depending on C such that, for all $H_{\lambda_0} \in \mathfrak{a}_{\lambda_0}^+$ satisfying $\alpha_i(H_{\lambda_0}) > \log R$ for every $i \in I_\lambda^c$, the expansion of $\phi_{\dot{\pi}(X)v,w}(a_\lambda \exp H_{\lambda_0})$ relative to P and the expansion of $\phi_{\dot{\pi}(X)v,w}(a_\lambda \exp H_{\lambda_0})$ relative to P_λ are both valid.

Setting $H := H_\lambda + H_{\lambda_0}$ for H_{λ_0} as above, the first expansion gives

$$\phi_{\dot{\pi}(X)v,w}(H) = \sum_{\tilde{\lambda} \in \mathcal{E}} \sum_{\tilde{l} \in \mathbb{Z}_{\geq 0}^n : |\tilde{l}| \leq l_0} \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda},\tilde{l}}(\dot{\pi}(X)v), w\rangle$$

and the second gives

$$\phi_{\dot{\pi}(X)v,w}(a_\lambda \exp H_{\lambda_0}) = \sum_{\nu \in \mathcal{E}_I} \sum_{q \in \mathbb{Z}_{\geq 0}^{I_\lambda^c} : |q| \leq q_0} \alpha(H_{\lambda_0})^q e^{(\nu - \rho_{\lambda_0})(H_{\lambda_0})} c_{\nu,q}^{P_\lambda}(a_\lambda, \dot{\pi}(X)v, w)$$

By [24], Corollary 8.46, each $\nu - \rho_{\lambda_0}$ in the second expansion is of the form $\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}$ for some exponent $\tilde{\lambda}$ in the first expansion. Therefore, it suffices to prove that if $\lambda \in \mathcal{E}_0$ and $l \in \mathbb{Z}_{\geq 0}^n$ with $|l| \leq l_0$ satisfy

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi),$$

then no term with exponent $\tilde{\lambda} - \rho$ for which $\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}}$ appears in the first expansion. Indeed, if we can show this, since by the comparison in [24], p. 251, the term

$$\alpha(H_{\lambda_0})^q e^{(\nu - \rho_{\lambda_0})(H_{\lambda_0})} c^{P_\lambda}_{\nu,q}(a_\lambda, \dot{\pi}(X)v, w),$$

for $\nu = \lambda|_{\mathfrak{a}_{\lambda_0}}$ and $q_{\lambda_0} = l_{\lambda_0}$ is the sum of all the terms in the expansion of $\phi_{\dot{\pi}(X)v,w}(H)$ relative to P which are indexed by couples $(\tilde{\lambda}, \tilde{l})$ satisfying

$$\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}} \text{ and } \tilde{l}_{\lambda_0} = l_{\lambda_0},$$

it would follow that

$$c^{P_\lambda}_{\nu,q}(a_\lambda, \dot{\pi}(X)v, w) = 0$$

and therefore

$$\Gamma_{\lambda,l}(a_\lambda, \dot{\pi}(X)v, w) = 0.$$

By linearity we can assume that $X \in \mathfrak{g}_{-\alpha_i}$ for some $i \in I^c_\lambda$ ([24], Proposition 5.23).

Computing as in [8], Lemma 8.16, we have

$$\phi_{\dot{\pi}(X)v,w}(a) = \langle \dot{\pi}(\mathrm{Ad}(a)X)\pi(a)v, w\rangle = -e^{-\alpha_i(H)}\phi_{v,\dot{\pi}(X)w}(a).$$

Hence every exponent in the expansion of $\phi_{\dot{\pi}(X)v,w}(a)$ relative to P is of the form $\tilde{\lambda} = \lambda' - e_i$ for some $\lambda' \in \mathcal{E}$. Now, if there existed $\lambda' \in \mathcal{E}$ with

$$(\lambda' - e_i)|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}},$$

we would have

$$\mathrm{Re}(\lambda' - e_i)_i = \mathrm{Re}\lambda_i = 0$$

since $i \in I^c_\lambda$. This means that $\mathrm{Re}\lambda'_i > 0$, a contradiction. Indeed, since (π, H) is tempered, the real part of every co-ordinate of each leading exponent is at most zero by Theorem 3.3.2 and it follows that the same property holds for every element in $\mathcal{E}$. This concludes the proof.

□

Lemma 3.3.9. Let $w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then there exists a collection $\{(\theta, H_\theta)\}_{\theta\in\Theta}$ of orthogonal irreducible sub-representations of $L^2(M_\lambda)$, the direct sum of which we denote (σ, H_σ), such that the image of the $(\mathfrak{m}_\lambda, K_\lambda)$-equivariant map

$$S_w : H_K \longrightarrow L^2(M_\lambda)_{K_\lambda},\ S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w)$$

is the $(\mathfrak{m}_\lambda, K_\lambda)$-module H_{σ,K_λ}.

Proof. The map S_w is well-defined by Proposition 3.3.6. For every $\xi \in K_\lambda$ and every $m \in M_\lambda$, we have

$$S_w(\pi(\xi)v)(m) = \Gamma_{\lambda,l}(m, \pi(\xi)v, w) = \Gamma_{\lambda,l}(m\xi, v, w) = R(\xi)S_w(v)(m).$$

By Proposition 3.3.6, for all $X \in \mathfrak{m}_\lambda$ and for all $m \in M_\lambda$, we have

$$S_w(\dot{\pi}(X)v)(m) = X\Gamma_{\lambda,l}(m, v, w)$$

and, by Proposition 3.2.18, we have

$$X\Gamma_{\lambda,l}(m, v, w) = \dot{R}(X)\Gamma_{\lambda,l}(m, v, w).$$

Therefore

$$S_w(\dot{\pi}(X)v)(m) = \dot{R}(X)S_w(v)(m)$$

and this concludes the proof that S_w is $(\mathfrak{m}_\lambda, K_\lambda)$-equivariant.

In the proof of Proposition 3.3.6, we showed that, for each $v \in H_K$, the function $\Gamma_{\lambda,l}(\cdot, v, w)$ is a $Z(\mathfrak{m}_\lambda)$-finite function in $L^2(M_\lambda)$. By [24], Corollary 8.42, there exist finitely many orthogonal irreducible sub-representations of $(R, L^2(M_\lambda))$ such that $\Gamma_{\lambda,l}(\cdot, v, w)$ is contained in their direct sum. It follows that there exists a (not necessarily finite) collection $\{(\theta, H_\theta)\}_{\theta\in\Theta}$ of orthogonal irreducible sub-representations of $(R, L^2(M_\lambda))$ such that $S_w(H_K)$ is contained in their direct sum. Let (σ, H_σ) denote the direct sum of the sub-representations in this collection.

We need to show that the image of S_w is precisely H_{σ,K_λ}. We begin by observing that, for any given $v \in H_K$, we have $\Gamma_{\lambda,l}(\cdot, v, w) \in H_{\sigma,K_\lambda}$. Indeed, the K_λ-finiteness of v implies the existence of finitely many $v_1, \ldots, v_r \in H_K$ such that

$$R(K_\lambda)\Gamma_{\lambda,l}(\cdot, v, w) \in \operatorname{span}\{\Gamma_{\lambda,l}(\cdot, v_i, w) | i \in \{1, \ldots, r\}\}.$$

Hence, $\Gamma_{\lambda,l}(\cdot, v, w)$ is K_λ-finite and, since it is a smooth vector in $(R, L^2(M_\lambda))$ by Proposition 3.3.6, it belongs to $H_\sigma \cap L^2(M_\lambda)_{K_\lambda} = H_{\sigma,K_\lambda}$ and it follows that $S_w(H_K) \subset H_{\sigma,K_\lambda}$. For the reverse inclusion, the irreducibility of each (θ, H_θ) implies that $S_w(H_K) \cap H_{\theta,K_\lambda} = H_{\theta,K_\lambda}$. Therefore H_{σ,K_λ} is contained in the image of S_w, completing the proof.

□

Proposition 3.3.10. Let $w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then there exists an admissible, finitely generated, unitary representation (σ, H_σ) of M_λ such that the map

$$S_w : H_K \longrightarrow H_{\sigma,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \; S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w)$$

is $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant.

Proof. By Lemma 3.3.9, there exists a unitary representation (σ, H_σ) of M_λ such that $S_w(H_K) = H_{\sigma,K_\lambda}$. By Lemma 3.3.7, for all $X \in \mathfrak{a}_{\lambda_0}$ and for all $m \in M_\lambda$, we have

$$S_w(\dot{\pi}(X)v)(m) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda,l}(m, v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)S_w(v).$$

By Lemma 3.3.8, for all $X \in \mathfrak{n}_{\lambda_0}$ and for all $m \in M_\lambda$, we have

$$S_w(\dot{\pi}(X)v)(m) = \Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w) = 0.$$

We thus obtained an $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant map

$$S_w : H_K \longrightarrow H_{\sigma,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},\ S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w)$$

which factors through the quotient map

$$q : H_K \longrightarrow H_K/\mathfrak{n}_{\lambda_0} H_K$$

which is $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant by Lemma 3.2.21.
Since H_K, being irreducible (and hence admissible by [26], Theorem 7.204), has an infinitesimal character, by Corollary 3.2.23 the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-module $H_K/\mathfrak{n}_{\lambda_0} H_K$ is admissible and finitely generated. It follows that

$$S_w(H_K) = H_{\sigma,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$$

is an admissible and finitely generated $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-module. The fact that $\mathfrak{a}_{\lambda_0}$ acts by scalars, implies that H_{σ,K_λ} itself is finitely generated (as $U(\mathfrak{m}_{\lambda\mathbb{C}})$-module) and admissible.

□

Proposition 3.3.10, in combination with Theorem 3.2.24 and the discussion following it, implies that the map

$$T_w : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda, K_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\ T_w(v)(k)(m) := \Gamma_{\lambda,l}(m, \pi(k)v, w)$$

is $(\mathfrak{g}, K)$-equivariant. Here, $\overline{P}_\lambda$ denotes the parabolic subgroup opposite to P_λ and we recall that the half-sum of positive roots determined by $\overline{P}_\lambda$ is precisely $-\rho_{\lambda_0}$. The observation above follows from the fact that $T_w = \tilde{S}_w$ in the notation of the discussion following Theorem 3.2.24.

3.4 Asymptotic Orthogonality

For a tempered, irreducible, Hilbert representation (π, H) of G, for $v, w \in H$, let

$$\phi_{v,w}(g) := \langle \pi(g)v, w \rangle$$

denote the associated matrix coefficient. By (2) of Theorem 1.2, there exists $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$ such that

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} |\phi_{v,w}(g)|^2 \, dg < \infty$$

for all $v, w \in H_K$.

As in Section 4.1 of [23], by the polarisation identity and by (2) of Theorem 1.2, the prescription

$$D(v_1, v_2, v_3, v_4) := \lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g) \overline{\phi}_{v_3,v_4}(g) \, dg$$

is a well-defined form on H_K that is linear in the first and fourth variable, conjugate-linear in the second and the third.

We explained in the Introduction that the crucial point is the proof of Proposition 3.1.4. We begin with the following reduction.

Lemma 3.4.1. Let G be a connected, semisimple Lie group with finite centre and let (π, H) be a tempered, irreducible, Hilbert representation of G. If for all $X \in \mathfrak{g}$ and for all $v_1, v_2, v_3, v_4 \in H_K$ we have

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1,v_2}(g) \overline{\phi}_{v_3,v_4}(g) \, dg = -\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g) \overline{\phi}_{\dot{\pi}(X)v_3,v_4}(g) \, dg$$

then the equality

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,\dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3,v_4}(g) \, dg = -\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g) \overline{\phi}_{v_3,\dot{\pi}v_4}(g) \, dg$$

holds for every $X \in \mathfrak{g}$ and for every $v_1, v_2, v_3, v_4 \in H_K$.

Proof. We write

$$\phi_{v_1,\dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3,v_4}(g) = \langle v_1, \pi(g^{-1})\dot{\pi}(X)v_2 \rangle \overline{\langle v_3, \pi(g^{-1})v_4 \rangle}$$

and since $\langle \cdot, \cdot \rangle$ is Hermitian we have

$$\langle v_1, \pi(g^{-1})\dot{\pi}(X)v_2 \rangle \overline{\langle v_3, \pi(g^{-1})v_4 \rangle} = \phi_{v_4,v_3}(g^{-1}) \overline{\phi}_{\dot{\pi}(X)v_2,v_1}(g^{-1}).$$

Now, since $G_{<r}$ is invariant under $\iota(g) = g^{-1}$ and G is unimodular, we have

$$\int_{G_{<r}} \phi_{v_4,v_3}(g^{-1}) \overline{\phi}_{\dot{\pi}(X)v_2,v_1}(g^{-1}) \, dg = \int_{G_{<r}} \phi_{v_4,v_3}(g) \overline{\phi}_{\dot{\pi}(X)v_2,v_1}(g) \, dg$$

and therefore

$$\int_{G_{<r}} \phi_{v_1,\dot{\pi}(X)v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg = \int_{G_{<r}} \phi_{v_4,v_3}(g)\overline{\phi}_{\dot{\pi}(X)v_2,v_1}(g)\, dg.$$

Applying complex conjugation, we obtain

$$\overline{\int_{G_{<r}} \phi_{v_1,\dot{\pi}(X)v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg} = \int_{G_{<r}} \phi_{\dot{\pi}(X)v_2,v_1}(g)\overline{\phi}_{v_4,v_3}(g)\, dg.$$

Assuming the validity of the first identity in the statement, we can write

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_2,v_1}(g)\overline{\phi}_{v_4,v_3}(g)\, dg = -\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_2,v_1}(g)\overline{\phi}_{\dot{\pi}(X)v_4,v_3}(g)\, dg.$$

Now, since

$$\int_{G_{<r}} \phi_{\dot{\pi}(X)v_2,v_1}(g)\overline{\phi}_{v_4,v_3}(g)\, dg = \overline{\int_{G_{<r}} \phi_{v_1,\dot{\pi}(X)v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg},$$

it follows that

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,\dot{\pi}(X)v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg = -\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \overline{\int_{G_{<r}} \phi_{v_2,v_1}(g)\overline{\phi}_{\dot{\pi}(X)v_4,v_3}(g)\, dg}.$$

Observing that

$$\overline{\int_{G_{<r}} \phi_{v_2,v_1}(g)\overline{\phi}_{\dot{\pi}(X)v_4,v_3}(g)\, dg} = \int_{G_{<r}} \phi_{\dot{\pi}(X)v_4,v_3}(g)\overline{\phi}_{v_2,v_1}(g)\, dg$$

and that, using the invariance of $G_{<r}$ under $\iota(g) = g^{-1}$ and the unimodularity of G, we have

$$\int_{G_{<r}} \phi_{\dot{\pi}(X)v_4,v_3}(g)\overline{\phi}_{v_2,v_1}(g)\, dg = \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi}_{v_3,\dot{\pi}(X)v_4}(g)\, dg,$$

we finally obtain

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,\dot{\pi}(X)v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg = -\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi}_{v_3,\dot{\pi}(X)v_4}(g)\, dg.$$

□

Proposition 3.4.2. Let G be a connected, semisimple Lie group with finite centre and let (π, H) be a tempered, irreducible, Hilbert representation of G. Then, for all $X \in \mathfrak{g}$ and for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1,v_2}(g)\overline{\phi}_{v_3,v_4}(g)\, dg = -\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi}_{\dot{\pi}(X)v_3,v_4}(g)\, dg.$$

Proof. We can assume $\mathbf{d}(\pi) \neq 0$, for otherwise (π, H) is a discrete series and the result is a consequence of the fact that $(R, L^2(G))$ is unitary.

The integral formula for the Cartan decomposition, taking into account the fact that, except for a set of measure zero, every $g \in G_{<r}$ can be written as $g = k_2 \exp H k_1$, for some $k_1, k_2 \in K$ and some $H \in \mathfrak{a}^+_{<r}$, with $\mathfrak{a}^+_{<r}$ as in (4), gives

$$\int_{G_{<r}} \phi_{\dot\pi(X)v_1,v_2}(g)\overline{\phi}_{v_3,v_4}(g)\,dg = \int_K \int_{\mathfrak{a}^+_{<r}} \int_K \phi_{\dot\pi(X)v_1,v_2}(k_2 \exp H k_1)\overline{\phi}_{v_3,v_4}(k_2 \exp H k_1)\Omega(H)\,dk_1\,dH\,dk_2$$

with $\Omega(H)$ defined in (5).

Arguing as in [23], p. 258, we can interchange the two innermost integrals in the RHS and, upon multiplying both sides by $\frac{1}{r^{\mathbf{d}(\pi)}}$ and taking the limit as $r \to \infty$, the RHS can be computed as the integral over $K \times K$ of

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{\dot\pi(X)v_1,v_2}(k_2 \exp H k_1)\overline{\phi}_{v_3,v_4}(k_2 \exp H k_1)\Omega(H)\,dH.$$

We expand ϕ_{v_1,v_2} and ϕ_{v_3,v_4} as

$$\phi_{v_1,v_2}(k_2 \exp H k_1) = e^{-\rho_{\mathfrak{p}}(H)} \sum_{[\lambda,l]\in\mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda|_{\mathfrak{a}_{\lambda_0}}(H_{\lambda_0})} \sum_{(\lambda',l')\in[\lambda,l]} \Psi^{\pi(k_1)v_1,\pi(k_2^{-1})v_2}_{\lambda',l'}(H)$$

and

$$\phi_{v_3,v_4}(k_2 \exp H k_1) = e^{-\rho_{\mathfrak{p}}(H)} \sum_{[\mu,m]\in\mathcal{C}/\sim} \alpha(H_{\mu_0})^{m_{\mu_0}} e^{\mu|_{\mathfrak{a}_{\mu_0}}(H_{\mu_0})} \sum_{(\mu',m')\in[\mu,m]} \Psi^{\pi(k_1)v_1,\pi(k_2^{-1})v_2}_{\mu',m'}(H).$$

By [23] Lemma A.5 and Claim A.6, the only non-zero contributions to

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{\dot\pi(X)v_1,v_2}(k_2 \exp H k_1)\overline{\phi}_{v_3,v_4}(k_2 \exp H k_1)\Omega(H)\,dH$$

may come from those $[\lambda, l] \in \mathcal{C}/\sim$ and those $[\mu, m] \in \mathcal{C}/\sim$ for which $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and

$$\mathbf{d}(\pi) = |I^c_\lambda| + \sum_{i\in I^c_\lambda}(l_i + m_i).$$

In view of the first condition, the third is equivalent to requiring that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l) = \mathbf{d}_P(\mu, m),$$

where $\mathbf{d}_P(\lambda, l)$ and $\mathbf{d}_P(\mu, m)$ are defined by (1).

By the discussion in Section 3 and by Proposition 3.3.4, the expression

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{\dot\pi(X)v_1,v_2}(k_2 \exp H k_1)\overline{\phi}_{v_3,v_4}(k_2 \exp H k_1)\Omega(H)\,dH$$

is equal to a finite sum terms of the form

$$C(\lambda, l, m) \int_{\mathfrak{a}^+_\lambda} \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)\dot\pi(X)v_1, \pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4)\Omega_\lambda(H_\lambda)\,dH_\lambda,$$

with $C(\lambda, l, m)$ as in (7), the functions $\Gamma_{\lambda,l}$ and $\Gamma_{\mu,m}$ defined as in (3) and $\Omega_\lambda(H_\lambda)$ defined as in (6).

Taking into account the integration over $K \times K$, we proved that

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_K \int_{\mathfrak{a}^+_{<r}} \int_K \phi_{\dot{\pi}(X)v_1,v_2}(k_2 \exp H k_1)\overline{\phi}_{v_3,v_4}(k_2 \exp H k_1)\Omega(H)\, dk_1\, dH\, dk_2$$

is equal to a finite sum of terms of the form

$$C(\lambda, l, m) \int_K \int_K \int_{\mathfrak{a}^+_\lambda} \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4)\Omega_\lambda(H_\lambda)\, dH_\lambda\, dk_1\, dk_2.$$

We remark that if $[\lambda, l], [\mu, m] \in \mathcal{C}/\sim$ are such that $I_\lambda = I_\mu$ is empty, then $\Gamma_{\lambda,l}$ and $\Gamma_{\mu,m}$ have no dependence on H_λ: this corresponds to the fact that they arise from the expansion of the matrix coefficients relative to the fixed minimal parabolic subgroup of G. For the moment we consider those terms for which I_λ is not empty. At the end of the proof, we will indicate the modifications required to handle the remaining cases.

By (1) of Lemma 3.4.3 and applying the Fubini-Tonelli theorem, we can interchange the two innermost integral and we therefore need to prove that

$$\int_K \int_{\mathfrak{a}^+_\lambda} \int_K \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4)\Omega_\lambda(H_\lambda)\, dk_1\, dH_\lambda\, dk_2$$

is equal to

$$-\int_K \int_{\mathfrak{a}^+_\lambda} \int_K \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)v_1, \pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_3, \pi(k_2^{-1})v_4)\Omega_\lambda(H_\lambda)\, dk_1\, dH_\lambda\, dk_2$$

Set

$$\mathcal{I}(\exp H_\lambda, k_1, k_2^{-1}) := \Gamma_{\lambda,l}(\exp_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4).$$

We apply the quotient integral formula ([13], Theorem 2.51) to write the integral

$$\int_K \int_{\mathfrak{a}^+_\lambda} \int_K \mathcal{I}(\exp H_\lambda, k_1, k_2^{-1})\Omega(H_\lambda)\, dk_1\, dH_\lambda\, dk_2$$

as

$$\int_K \int_{\mathfrak{a}^+_\lambda} \int_{K_\lambda\backslash K} \int_{K_\lambda} \mathcal{I}(\exp H_\lambda, \xi_1 k_1, k_2^{-1})\Omega_\lambda(H_\lambda)\, d\xi_1\, d\dot{k}_1\, dH_\lambda\, dk_2$$

and again to write it as

$$\int_{K/K_\lambda} \int_{K_\lambda} \int_{\mathfrak{a}^+_\lambda} \int_{K_\lambda\backslash K} \int_{K_\lambda} \mathcal{I}(\exp H_\lambda, \xi_1 k_1, \xi_2^{-1} k_2^{-1})\Omega_\lambda(H_\lambda)\, d\xi_1\, d\dot{k}_1\, dH_\lambda\, d\xi_2\, d\dot{k}_2.$$

By (3) of Lemma 3.4.3, we can appeal to the Fubini-Tonelli theorem to interchange the two innermost integrals and to obtain

$$\int_{K/K_\lambda}\int_{K_\lambda}\int_{\mathfrak{a}_\lambda^+}\int_{K_\lambda}\int_{K_\lambda\backslash K}\mathcal{I}(\exp H_\lambda,\xi_1k_1,\xi_2^{-1}k_2^{-1})\Omega_\lambda(H_\lambda)\,d\dot{k}_1\,d\xi_1\,dH_\lambda\,d\xi_2\,d\dot{k}_2.$$

Now, combining the fact that $M_\lambda^{\mathrm{reg}}=K_\lambda\exp(\mathfrak{a}_\lambda^+)K_\lambda$, the relevant integral formula and the fact that the complement of M^{reg} has measure zero in M, it follows that the integral

$$\int_{K_\lambda}\int_{\mathfrak{a}_\lambda^+}\int_{K_\lambda}\int_{K_\lambda\backslash K}\mathcal{I}(\exp H_\lambda,\xi_1k_1,\xi_2^{-1}k_2^{-1})\Omega_\lambda(H_\lambda)\,d\dot{k}_1\,d\xi_1\,dH_\lambda\,d\xi_2$$

is equal to

$$\int_{M_\lambda}\int_{K_\lambda\backslash K}\Gamma_{\lambda,l}(m_\lambda,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(m_\lambda,\pi(k_1)v_3,\pi(k_2^{-1})v_4)\,d\dot{k}_1\,dm_\lambda.$$

For $k_1\in K$, we define

$$f(k_1):=\langle\Gamma_{\lambda,l}(m_\lambda,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2),\Gamma_{\mu,m}(m_\lambda,\pi(k_1)v_3,\pi(k_2^{-1})v_4)\rangle_{L^2(M_\lambda)}.$$

The function f is invariant under left-multiplication by K_λ. Indeed, if
$\xi\in K_\lambda$, then

$$\Gamma_{\lambda,l}(m_\lambda,\pi(\xi k_1)\dot{\pi}(X)v_1,\pi(k_2)v_2)=\Gamma_{\lambda,l}(m_\lambda\xi,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2)v_2)$$

and similarly for the $\Gamma_{\mu,m}$-term. Since the right-regular representation of M_λ is unitary, we have

$$\langle\Gamma_{\lambda,l}(m_\lambda\xi,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2),\Gamma_{\mu,m}(m_\lambda\xi,\pi(k_1)v_3,\pi(k_2^{-1})v_4)\rangle_{L^2(M_\lambda)}=f(k_1).$$

An application of the quotient integral formula ([13], Theorem 2.51) gives

$$\int_K f(k_1)\,dk_1=\int_{K_\lambda\backslash K}\int_{K_\lambda}f(\xi k_1)\,d\xi\,d\dot{k}_1=\mathrm{vol}(K_\lambda)\int_{K_\lambda\backslash K}f(k_1)\,d\dot{k}_1.$$

By (2) in Lemma 3.4.3 and appealing again to the Fubini-Tonelli theorem, we interchange the integrals over M_λ and $K_\lambda\backslash K$ to obtain that

$$\int_{K/K_\lambda}\int_{M_\lambda}\int_{K_\lambda\backslash K}\Gamma_{\lambda,l}(m_\lambda,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(m_\lambda,\pi(k_1)v_3,\pi(k_2^{-1})v_4)\,d\dot{k}_1dm_\lambda\,d\dot{k}_2$$

equals

$$\frac{1}{\mathrm{vol}(K_\lambda)}\int_{K/K_\lambda}\int_K f(k_1)\,dk_1\,d\dot{k}_2$$

which, in turn, equals

$$\frac{1}{\mathrm{vol}(K_\lambda)}\int_{K/K_\lambda}\int_K\langle\Gamma_{\lambda,l}(m_\lambda,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2),\Gamma_{\mu,m}(m_\lambda,\pi(k_1)v_3,\pi(k_2^{-1})v_4)\rangle_{L^2(M_\lambda)}\,dk_1\,d\dot{k}_2.$$

For fixed $k_2\in K$, set $w_2:=\pi(k_2^{-1})v_2$ and $w_4:=\pi(k_2^{-1})v_4$. We reduced the problem to proving that

$$\int_K\langle\Gamma_{\lambda,l}(m_\lambda,\pi(k_1)\dot{\pi}(X)v_1,w_2),\Gamma_{\mu,m}(m_\lambda,\pi(k_1)v_3,w_4)\rangle_{L^2(M_\lambda)}\,dk_1$$

equals

$$-\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_3, w_4)\rangle_{L^2(M_\lambda)}\, dk_1.$$

By Proposition 3.3.10, there exist an admissible, finitely generated, unitary representation (σ_1, H_{σ_1}) of M_λ such that the image of the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant map

$$S_{w_2} : H_K \longrightarrow L^2(M_\lambda)_{K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},\ S_{w_2}(v)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, v, w_2)$$

is precisely $H_{\sigma_1,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$ and an admissible, finitely generated, unitary representation (σ_2, H_{σ_2}) such that the image of the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant map

$$S_{w_4} : H_K \longrightarrow L^2(M_\lambda)_{K_\lambda} \otimes \mathbb{C}_{\mu|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},\ S_{w_4}(v)(m_\lambda) := \Gamma_{\mu,m}(m_\lambda, v, w_4)$$

is precisely $H_{\sigma_2,K_\lambda} \otimes \mathbb{C}_{\mu|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$. Let (σ, H_σ) denote the direct sum of (σ_1, H_{σ_1}) and (σ_2, H_{σ_2}). It is an an admissible, finitely generated, unitary representation which restricts to a unitary representation of K_λ. Since $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$, by the same computations as in Lemma 3.3.9 and Proposition 3.3.10 we obtain $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant maps

$$S_{w_2} : H_K \longrightarrow H_{\sigma,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},\ S_{w_2}(v)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, v, w_2)$$

and

$$S_{w_4} : H_K \longrightarrow H_{\sigma,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},\ S_{w_4}(v)(m_\lambda) := \Gamma_{\mu,m}(m_\lambda, v, w_4)$$

factoring through the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant quotient map

$$q : H_K \longrightarrow H_K/\mathfrak{n}_{\lambda_0} H_K.$$

From the discussion following Proposition 3.3.10, we obtain $(\mathfrak{g}, K)$-equivariant maps

$$T_{w_2} : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda, K_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\ T_{w_2}(v)(k_1)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)v, w_2)$$

and

$$T_{w_4} : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda, K_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\ T_{w_4}(v)(k_1)(m_\lambda) := \Gamma_{\mu,m}(m_\lambda, \pi(k_1)v, w_4).$$

By definition of the inner product on $\mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$, we see that proving the sought identity is equivalent to proving that

$$\langle T_{w_2}(\dot{\pi}(X)v_1), T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})} = -\langle T_{w_2}(v_1), T_{w_4}(\dot{\pi}(X)v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}.$$

By the $(\mathfrak{g}, K)$-equivariance of T_{w_2}, we have

$$\langle T_{w_2}(\dot{\pi}(X)v_1), T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})} = \langle \dot{\mathrm{Ind}}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X) T_{w_2}(v_1), T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}$$

and, since $\lambda|_{\mathfrak{a}_{\lambda_0}}$ is totally imaginary, from Corollary 3.2.20 we deduce

$$\langle \dot{\mathrm{Ind}}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X) T_{w_2}(v_1), T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})} = -\langle T_{w_2}(v_1), \dot{\mathrm{Ind}}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X) T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}.$$

The result follows from the $(\mathfrak{g}, K)$-equivariance of T_{w_4}.

Finally, we consider the terms for which $I_\lambda = I_\mu$ is empty, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l) = \mathbf{d}_P(\mu, m).$$

In this case we have to prove that

$$\int_K \int_K \Gamma_{\lambda,l}(\pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(\pi(k_1)v_3, \pi(k_2^{-1})v_4)\, dk_1\, dk_2$$

is equal to

$$-\int_K \int_K \Gamma_{\lambda,l}(\pi(k_1)v_1, \pi(k_2^{-1})v_2)\overline{\Gamma}_{\mu,m}(\pi(k_1)\dot{\pi}(X)v_3, \pi(k_2^{-1})v_4)\, dk_1\, dk_2.$$

So it suffices to prove that

$$\int_K \Gamma_{\lambda,l}(\pi(k_1)\dot{\pi}(X)v_1, w_2)\overline{\Gamma}_{\mu,m}(\pi(k_1)v_3, w_4)\, dk_1$$

is equal to

$$-\int_K \Gamma_{\lambda,l}(\pi(k_1)v_1, w_2)\overline{\Gamma}_{\mu,m}(\pi(k_1)\dot{\pi}(X)v_3, w_4)\, dk_1.$$

We begin by showing that $S_w(v)(m_\lambda) := \Gamma_{\lambda,l}(\pi(m_\lambda)v, w)$ belongs to $L^2(M_\lambda)$, with M_λ now denoting the centraliser of A_{λ_0} in K and $v, w \in H_K$. Because M_λ is compact, it suffices to prove that $S_{w_2}(v)$ is continuous as a function of m_λ and this follows from the assumption that v is K-finite and hence M_λ-finite. We thus obtained a map

$$S_w : H_K \longrightarrow L^2(M_\lambda),\ \ v \mapsto S_w(v).$$

The proofs of Lemma 3.3.7 and Lemma 3.3.8 specialise to the case at hand, moreover S_w is clearly M_λ-equivariant. Thus we obtained an $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, M_\lambda)$-equivariant map

$$S_w : H_K \longrightarrow L^2(M_\lambda) \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$$

which factors through $H_K/\mathfrak{n}_{\lambda_0}H_K$. To construct the appropriate representation (σ, H_σ) of M_λ for the case at hand, we can use the M_λ-finiteness of $S_w(v)$ to deduce the existence of a finite collection of irreducible orthogonal subrepresentations of $L^2(M_\lambda)$ containing $S_w(v)$. This implies the existence of a collection $\{(\theta, H_\theta)\}_{\theta\in\Theta}$ as in the discussion before Proposition 3.3.10. Reasoning as above, we thus obtain $(\mathfrak{g}, K)$-equivariant maps

$$T_{w_2} : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\ \ T_{w_2}(v)(k)(m_\lambda) := \Gamma_{\lambda,l}(\pi(m_\lambda k)v, w_2)$$

and

$$T_{w_4} : H_K \longrightarrow \mathrm{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}),\ \ T_{w_4}(v)(k)(m_\lambda) := \Gamma_{\mu,m}(\pi(m_\lambda k)v, w_4).$$

To complete the proof it suffices to write

$$\int_K \Gamma_{\lambda,l}(\pi(k_1)\dot{\pi}(X)v_1, w_2)\overline{\Gamma}_{\mu,m}(\pi(k_1)v_3, w_4)\, dk_1$$

as

$$\int_{M_\lambda\backslash K}\int_{M_\lambda}\Gamma_{\lambda,l}(\pi(m_\lambda k_1)\dot{\pi}(X)v_1,w_2)\overline{\Gamma}_{\mu,m}(\pi(m_\lambda k_1)v_3,w_4)\,dm_\lambda\,d\dot{k}_1$$

and to observe that, by [34], 8.4.1, the last expression is precisely

$$\langle T_{w_2}(\dot{\pi}(X)v_1),T_{w_4}(v_3)\rangle_{\mathrm{Ind}_{\overline{P}_\lambda}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}.$$

□

Lemma 3.4.3. Let $v_1,w_2,v_3,w_4\in H_K$. Let $[\lambda,l],[\mu,m]\in\mathcal{C}/\sim$ be such that $I_\lambda=I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}}=\mu|_{\mathfrak{a}_{\lambda_0}}$ and $\mathbf{d}(\pi)=|I_\lambda^c|+\sum_{i\in I_\lambda^c}(l_i+m_i)$. Then the following holds:

(1)
$$\int_K\int_{\mathfrak{a}_\lambda^+}|\Gamma_{\lambda,l}(\exp H_\lambda,\pi(k_1)v_1,w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,\pi(k_1)v_3,w_4)|\,dH_\lambda\,dk_1<\infty$$

(2)
$$\int_{K_\lambda\backslash K}\int_{M_\lambda}|\Gamma_{\lambda,l}(m_\lambda,\pi(k)v_1,w_2)\overline{\Gamma}_{\mu,m}(m_\lambda,\pi(k)v_3,w_4)|\,dm_\lambda\,d\dot{k}<\infty$$

(3) For any fixed $H_\lambda\in\mathfrak{a}_\lambda^+$, we have

$$\int_{K_\lambda\backslash K}\int_{K_\lambda}|\Gamma_{\lambda,l}(\exp H_\lambda,\pi(\xi k)v_1,w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,\pi(\xi k)v_3,w_4)|\,d\xi\,d\dot{k}<\infty.$$

Proof. For (1), we begin by observing that for fixed $k\in K$, the functions $\Gamma_{\lambda,l}(\exp H_\lambda,\pi(k)v_1,v_2)$ and $\Gamma_{\mu,m}(\exp H_\lambda,\pi(k)v_3,v_4)$ are square-integrable on $\mathfrak{a}_\lambda^+$ by Proposition 3.3.5. Therefore, we have

$$\int_{\mathfrak{a}_\lambda^+}|\Gamma_{\lambda,l}(\exp H_\lambda,\pi(k)v_1,w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,\pi(k)v_3,w_4)|\,dm_\lambda<\infty.$$

Hence, we can define the function

$$h:K\longrightarrow\mathbb{R}_{\geq 0},\;\;h(k)=\int_{\mathfrak{a}_\lambda^+}|\Gamma_{\lambda,l}(\exp H_\lambda,\pi(k)v_1,w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,\pi(k)v_3,w_4)|\,dH_\lambda$$

and the result will follow if we establish the continuity of h. The K-finiteness of v_1 and v_3 implies the existence of finitely many K-finite vectors $v_1^{(1)},\dots,v_1^{(p)}$ and finitely many K-finite vectors $v_3^{(1)},\dots,v_3^{(q)}$ such that

$$\pi(k)v_1=\sum_{i=1}^{p}a_i(k)v_1^{(i)},\text{ and }\pi(k)v_3=\sum_{j=1}^{q}b_j(k)v_3^{(j)}$$

for continuous complex-valued functions a_i and b_j. Let $k_0\in K$. Then

$$|h(k)-h(k_0)|$$

is majorised by the integral over $\mathfrak{a}_\lambda^+$ of

$$||\Gamma_{\lambda,l}(\exp H_\lambda,\pi(k)v_1,w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,\pi(k)v_3,w_4)|-|\Gamma_{\lambda,l}(\exp H_\lambda,\pi(k_0)v_1,w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,\pi(k_0)v_3,w_4)||.$$

By reverse triangle inequality, the integrand is majorised by

$$|\Gamma_{\lambda,l}(\exp_\lambda,\pi(k)v_1,w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,\pi(k)v_3,w_4) - \Gamma_{\lambda,l}(\exp H_\lambda,\pi(k_0)v_1,w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,\pi(k_0)v_3,w_4)|$$

which, in turn, is less than or equal to

$$\sum_{i=1}^{p}\sum_{j=1}^{q}|a_i(k)\overline{b}_j(k)-a_i(k_0)\overline{b}_j(k_0)||\Gamma_{\lambda,l}(\exp H_\lambda,v_1^{(i)},w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,v_3^{(j)},w_4)|.$$

We obtained

$$|h(k)-h(k_0)| \leq \sum_{i=1}^{p}\sum_{j=1}^{q}|a_i(k)\overline{b}_j(k)-a_i(k_0)\overline{b}_j(k_0)|\int_{\mathfrak{a}_\lambda^+}|\Gamma_{\lambda,l}(\exp H_\lambda,v_1^{(i)},w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda,v_3^{(j)},w_4)|\,dH_\lambda$$

and the continuity follows from the continuity of the a_i's and b_j's.

For (2), we begin by observing that for fixed $k \in K$, the functions $\Gamma_{\lambda,l}(m_\lambda,\pi(k)v_1,w_2)$ and $\Gamma_{\mu,m}(m_\lambda,\pi(k)v_3,w_4)$ are square-integrable on M_λ by Proposition 3.3.5. Therefore, we have

$$\int_{M_\lambda}|\Gamma_{\lambda,l}(m_\lambda,\pi(k)v_1,w_2)\overline{\Gamma}_{\mu,m}(m_\lambda,\pi(k)v_3,w_4)|\,dm_\lambda < \infty.$$

Hence, we can define the function

$$h : K \longrightarrow \mathbb{R}_{\geq 0},\ h(k) = \int_{M_\lambda}|\Gamma_{\lambda,l}(m_\lambda,\pi(k)v_1,w_2)\overline{\Gamma}_{\mu,m}(m_\lambda,\pi(k)v_3,w_4)|\,dm_\lambda.$$

Arguing as for (1), we obtain that h is continuous.

By the right-invariance of the Haar measure on M_λ and since

$$\Gamma_{\lambda,l}(m_\lambda,\pi(\xi k)v_1,w_2) = \Gamma_{\lambda,l}(m_\lambda\xi,\pi(k)v_1,w_2)$$

for every $\xi \in K_\lambda$ (and similarly for the $\Gamma_{\mu,m}$-term), the function h is invariant under multiplication on the left by elements in K_λ and it therefore descends to a continuous function on $K_\lambda\backslash K$, concluding the proof of the first statement.

For (3), given a fixed $H_\lambda \in \mathfrak{a}_\lambda^+$ the function

$$K_\lambda \longrightarrow \mathbb{C},\ \xi \mapsto \Gamma_{\lambda,l}(\exp H_\lambda,\pi(\xi k)v_1,w_2)$$

is continuous. Indeed, let $\xi_0 \in K_\lambda$. Since $\pi(k)v$ is K-finite, it is in particular K_λ-finite. Hence, there exist finitely many K_λ-finite vectors $v_1,\ldots,v_r$ such that

$$\pi(\xi)\pi(k)v = \sum_{i=1}^{r}c_i(\xi)v_i,$$

where each c_i is a complex-valued continuous function on K_λ. Therefore, the quantity

$$|\Gamma_{\lambda,l}(\exp H_\lambda,\pi(\xi k)v_1,w_2) - \Gamma_{\lambda,l}(\exp H_\lambda,\pi(\xi_0 k)v_1,w_2)|$$

is bounded by

$$\sum_{i=1}^{r} |c_i(\xi) - c_i(\xi_0)||\Gamma_{\lambda,l}(\exp H_\lambda, v_i, w_2)|$$

and the claim follows from the continuity of the c_i's.

The same argument shows that, for fixed $H_\lambda \in \mathfrak{a}_\lambda^+$, the function

$$K_\lambda \longrightarrow \mathbb{C},\ \xi \mapsto \Gamma_{\mu,m}(\exp H_\lambda, \pi(\xi)v_3, w_4)$$

is continuous and it follows that

$$\int_{K_\lambda} |\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(\xi k)v_3, w_4)|\, d\xi < \infty.$$

Hence, we can define the function

$$f : K \to \mathbb{R}_{\geq 0},\ f(k) = \int_{K_\lambda} |\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2)\overline{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(\xi k)v_3, w_4)|\, d\xi$$

and argue as in the proof of (2).

□

We can now complete the strategy outlined in the Introduction. For fixed $v_2, v_4 \in H_K$, we define

$$A_{v_2,v_4}(\cdot,\cdot) := D(\cdot, v_2, \cdot, v_4),$$

which is linear in the first variable and conjugate linear in the second. For fixed $v_1, v_3 \in H_K$, we define

$$B_{v_1,v_3}(\cdot,\cdot) := D(v_1, \cdot, v_3, \cdot),$$

which is conjugate-linear in the first variable and linear in the second.

Theorem 3.4.4. Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be a tempered, irreducible, Hilbert representation of G. Then there exists $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\lim_{r\to\infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

Proof. Fix $v_2, v_4 \in H_K$. By Proposition 3.4.2 we can apply Corollary 3.2.13 to the form A_{v_2,v_4}. Hence there exists $c_{v_2,v_4} \in \mathbb{C}$ such that for all $v_1, v_3 \in H_K$ we have

$$A_{v_2,v_4}(v_1, v_3) = c_{v_2,v_4}\langle v_1, v_3\rangle.$$

Similarly, fixing $v_1, v_3 \in H_K$, by Proposition 3.4.2 and Lemma 3.4.1 there exists a $d_{v_1,v_3} \in \mathbb{C}$ such that

$$\overline{B}_{v_3,v_1}(v_4, v_2) = d_{v_1,v_3}\langle v_4, v_2\rangle,$$

since the left-hand side is conjugate-linear in the first variable. Hence, since

$$\overline{B}_{v_3,v_1}(v_4, v_2) = B_{v_1,v_3}(v_2, v_4),$$

we obtain

$$B_{v_1,v_3}(v_2, v_4) = d_{v_1,v_3}\overline{\langle v_2, v_4\rangle}$$

By definition, we have

$$D(v_1, v_2, v_3, v_4) = A_{v_2,v_4}(v_1, v_3) = B_{v_1,v_3}(v_2, v_4),$$

so, for a vector $v_0 \in H_K$ of norm 1, using (2) of Theorem 1.2, we obtain a real number $C(v_0, v_0) > 0$ such that

$$D(v_0, v_0, v_0, v_0) = C(v_0, v_0) = c_{v_0,v_0} = d_{v_0,v_0}.$$

Computing $D(v_1, v_0, v_3, v_0)$, we have

$$d_{v_1,v_3} = c_{v_0,v_0}\langle v_1, v_3\rangle.$$

Therefore, we obtained

$$D(v_1, v_2, v_3, v_4) = c_{v_0,v_0}\langle v_1, v_3\rangle\overline{\langle v_2, v_4\rangle},$$

showing that $\mathbf{f}(\pi) := \frac{1}{C(v_0,v_0)}$ does not depend on the choice of v_0, as required.

□

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