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Assar ANDERSSON
Born on 22 April 1991 in Björlanda (Sweden)

HAIRED AND MULTI-ORIENTED GRAPH COMPLEXES WITH APPLICATIONS TO ALGEBRA AND GEOMETRY

Dissertation defence committee

Dr Sergei Merkulov, Dissertation Supervisor
Professor, Université du Luxembourg

Dr Norbert Poncin, Chairman
Professor, Université du Luxembourg

Dr Hugo Parlier, Vice Chairman
Professor, Université du Luxembourg

Dr Vladimir Dotsenko
Professor, Université de Strasbourg

Dr Bruno Vallette
Professor, Université Sorbonne Paris Nord

Abstract

In this thesis we study graph complexes and their applications to homotopy algebra, differential geometry, and the cohomology of the moduli space of algebraic curves.

The first main topic of this thesis is the study of a new higher dimensional incarnation of one of the most mysterious mathematical structures, the Grothendieck-Teichmüller group, using methods and ideas of multi-oriented props and graph complexes.

In Chapter 1, we fix our notation and conventions. We recall the definitions of operads and props following J.-L. Loday and B. Vallette [29], [47]. and D.V. Borisov and Y.I. Manin [8]. We also recall the basic tools of the deformation theory of props developed by B. Vallette and S. Merkulov in [38], and generalize them, rather straightforwardly, to the multi-oriented setting.

In Chapter 2, we study a family of multi-oriented properads of multi-oriented homotopy Lie bialgebras $\mathsf{holieB}_d^{k\uparrow l}$ parametrized by the valued parameter d and equipped with k directions, l of them being oriented. Multi-oriented properads were introduced by S. Merkulov in [35], where it is shown that they admit representations in vector spaces with branes, and provide us with a far reaching generalization of Drinfeld's notion of a *Manin triple* [14]. The multi-oriented properads $\mathsf{holieB}_d^{k\uparrow l}$ of homotopy Lie bialgebras, also introduced in [35], are from a combinatorial point of view a natural extension of the ordinary properad of Lie bialgebras. Their representations on vector spaces with branes are not yet fully understood. However, it is shown in [35] that representations of a two oriented operad $\mathsf{Lie}^{2\uparrow}$, obtained from a similar 'combinatorial lifting' of the Lie operad Lie, do really govern Manin triples as defined in [14].

In Section 2.2, we prove that the derivation complex of the k-oriented prop of l-oriented homotopy Lie bialgebras is quasi-isomorphic to the k-directed, l-oriented graph complex. This is a generalization of the result in [43], where the deformation complex of the (1-oriented) prop of homotopy Lie bialgebras is studied. Combining this with the results of T. Willwacher [51] and M. Živković [54], we get that the Grothendieck-Teichmüller group acts on the multi-oriented properads of Lie bialgebras.

In Section 2.3, we turn our attention to the (co)homology of the first non-trivial multi-oriented properad, the 2-oriented properad $\mathsf{holieB}_d^{2\uparrow 2}$. We prove that $\mathsf{holieB}_d^{2\uparrow 2}$ is indeed a minimal model of the 2-oriented properad of 2-oriented Lie bialgebras $\mathsf{LieB}_d^{2\uparrow 2}$. It is worth emphasizing that this proof does *not* follow the scenario of the famous constructions of the minimal resolutions of the ordinary *1-oriented*, props of Lie bialgebras [27], [47], [32], [34]. The main problem is that the key idea of using spectral sequences associated with path filtrations does not completely work in the multi-oriented case.

The second main topic of this thesis is the study of hairy graph complexes and their applications to the theory of cohomology groups of moduli spaces of genus g algebraic curves with $n \geq 1$ punctures. In Chapter 3, we show that the hairy graph complex $(\mathrm{HGC}_{d,d}, \delta)$, studied in e.g. [7], [22], can be understood as an associated graded complex of the oriented graph complex

$(\mathrm{OGC}_{d+1}, \delta)$, subject to a filtration on the number of target vertices, or equivalently source vertices, called the *fixed source graph complex.* The fixed source graph complex $(\mathrm{OGC}_1, \delta_0)$ maps into the ribbon graph complex RGC [42], which models the moduli space of Riemann surfaces with marked points [28]. The full differential δ on the oriented graph complex OGC_{d+1} corresponds to the deformed differential $\delta + \chi$ on the hairy graph complex $\mathrm{HGC}_{d,d}$, where χ adds a hair. This deformed complex $(\mathrm{HGC}_{d,d}, \delta + \chi)$ is already known to be quasi-isomorphic to the standard Kontsevich's graph complex GC_d [22]. This chapter is based on joint work with M. Živković [6].

The third main topic of this thesis is a new application of the remarkable theory of differential forms with logarithmic singularities developed in [3] for constructing a new universal transcendent formula for an exotic Lie ∞-automorphism of the Schouten-Nijenhuis Lie algebra of polyvector fields.

In Chapter 4, we develop a new (regularized) De Rham field theory based on a two parameter propagator with logarithmic singularities. We use this for constructing a new two parametric family of exotic Lie ∞-automorphisms of Schouten-Nijenhuis Lie algebra of polyvector fields on an arbitrary affine space

$$\mathcal{F}^{t,\lambda} : T_{poly}(\mathbb{R}^d) \rightsquigarrow T_{poly}(\mathbb{R}^d).$$

This universal formula involves all odd Riemann zeta values $\frac{1}{2\pi}\zeta(2n+1)$, $n \geq 1$. This is a new application of the regularized Stokes formula, introduced by A. Alekseev, C. A. Rossi, C. Torossian and T. Willwacher in [3], in order to prove a statement by M. Kontsevich [25] regarding the existence of a formality morphism

$$\mathcal{U}^{\log} : T_{poly}(\mathbb{R}^d) \rightsquigarrow D_{poly}(\mathbb{R}^d)$$

with weights obtained by integrating logarithmic forms.

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Contents

Introduction

This thesis belongs to a rapidly developing area of mathematics which studies graphs complexes, properads, as well as their deformation theory with applications to algebra and geometry. One of the main motivations of the work is to better understand one of the most mysterious structures in modern mathematics, namely the Grothendieck-Teichmüller group. It has recently emerged in very different parts of mathematics, such as deformation quantization of Lie bialgebras [15] [16], number theory [14], theory of braid groups [15], deformation quantization of Poisson structures [26], [50], [46], [5], Lie theory [4], Goldman-Turaev theory [1], [2], and many others. The Grothendieck-Teichmüller group has many different incarnations in dimension two [26], [51], and three [52], [43].

In our work we found higher dimensional incarnations of this mysterious group, using the new notion of multi-oriented graphs and properads, defined in [35] and [54].

In Chapter 2 we study a family of multi-oriented properads of multi-oriented homotopy Lie bialgebras $\mathsf{holieB}_d^{k\uparrow l}$ parametrized by the valued parameter d and equipped with k directions, l of them being oriented. One of the main results here is obtained in Section 2.2 where we show that the derivation complex of the multi-oriented genus completed properads $\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star}$ is quasi-isomorphic to the multi-oriented version of Kontsevich graph complex $\mathrm{GC}_{d+1}^{k\uparrow l}$ up to some rescaling classes. Combining this result with the results of T. Willwacher [51] and M. Živković [54], we conclude that the Grothendieck-Teichmüller group acts on the d oriented properad $\widehat{\mathsf{holieB}}_{d+1}^{d\uparrow d,\star}$ in a homotopy non-trivial way.

When considering (co)homology of graph complexes, subtle combinatorial differences in the definitions can have a great impact. For example, it is shown in [32] and [47] that holieB is a minimal model of LieB, i.e. the cohomology of the graph complex holieB is spanned by 3-regular graphs modulo some quadratic relations. If we instead consider with the wheeled closure $\mathsf{holieB}^{\circlearrowleft}$ of holieB, the main combinatorial tool in [32], a path filtration suggested by M. Kontsevich [27], as well as the distributive laws by B. Vallette [47], fails to work. As a consequence, the cohomology of $\mathsf{holieB}^{\circlearrowleft}$ is largely unknown. It was shown in [36] (along with the introduction of $\mathsf{holieB}^{\circlearrowleft}$), that $\mathsf{holieB}^{\circlearrowleft}$ is not a minimal resolution of $\mathsf{LieB}^{\circlearrowleft}$. It is therefore a highly non-trivial problem to compute the cohomology of the dg multi-oriented properads $\mathsf{holieB}^{k\uparrow l}$.

In Chapter 2, Section 2.3, we show that the 2-oriented properad $\mathsf{holieB}^{2\uparrow 2}$ is a minimal resolution of what we call the 2-oriented properad of Lie bialgebras $\mathsf{LieB}^{2\uparrow 2}$. The proof of this statement relies only partially on spectral sequences induced by path filtrations, as suggested in

[27]. As we shall see below, this line of thought does not lead us immediately to the solution of the problem, we are therefore forced to look for new arguments to finish the proof.

Another important theme of this thesis is the study of the deep relationship between the cohomology of the hairy graph complexes HGC and the cohomology of the moduli spaces $\mathcal{M}_{g,n}$ of of genus g Riemann surfaces with n marked points.

In [11] and [12], M. Chan, S. Galatius and S. Payne prove that there exists injections

$$H(\mathrm{GC}_0) \hookrightarrow \prod_{g\geq 1} H_c(\mathcal{M}_g, \mathbb{Q}), \tag{1}$$

and

$$H(\mathrm{GC}_0^{[n]}) \hookrightarrow \prod_{g\geq 1}\prod_{n\geq 0} H_c(\mathcal{M}_{g,n}, \mathbb{Q}), \tag{2}$$

where $H_c(\mathcal{M}_g, \mathbb{Q})$ and $H_c(\mathcal{M}_{g,n}, \mathbb{Q})$ is the compact support cohomology of the moduli space of Riemann surfaces of genus g without punctures and, respectively, with n marked points. The graph complex GC_0 is the (degree shifted) Kontsevich graph complex introduced in [24], and $\mathrm{GC}_0^{[n]}$ is a version of GC_0 where vertices may be 'marked' by elements in $[n]$.

In [42], S. Merkulov and T. Willwacher connected the Kontsevich graph complex and Penner's ribbon graph complex controlling the cohomology of moduli spaces in a different way. More precisely, they construct a map

$$F : (\mathrm{OGC}_1, d) \to (\mathrm{RGC}[1], \delta + \Delta_1), \tag{3}$$

where OGC_1 is the oriented version of Kontsevich graph complex and $(\mathrm{RGC}, \delta+\Delta_1)$ is a complex of ribbon graphs. Ribbon graphs are closely related to the moduli space of Riemann curves $\mathcal{M}_{g,n}$, see [44]. More specifically, it is shown in [28] that

$$H^k(\mathrm{RGC}, \delta) \cong \prod_{g,n\geq 1} \left(H_c^{k-n}(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes \mathrm{sgn}_n\right)^{\mathbb{S}_n} \oplus \begin{cases} \mathbb{Q} & \text{for } k = 1, 5, 9, \ldots \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

In this context, the differential $\delta + \Delta_1$ constructed in [42] is a deformation of the classical differential δ on RGC. It is conjectured that the explicit formula established in [42] is related to the inclusion (1) constructed in [11].

In chapter 3, we construct a quasi-isomorphism

$$\Phi : (\mathrm{OGC}_1, \delta_0) \to (\mathrm{HGC}_{0,0}, \delta) \cong \left(\mathrm{GC}_0^{[n]} \otimes \mathrm{sgn}_n\right)^{\mathbb{S}_n},$$

where δ_0 is a new differential on the oriented graph complex, constructed so that the same explicit formula constructed in (3) gives a morphism of dg vector spaces

$$F : (\mathrm{OGC}_1, \delta_0) \to (\mathrm{RGC}[1], \delta).$$

This suggests that the map (3) is also related to the map (2) constructed in [12].

The final chapter of our thesis is devoted to constructing an explicit exotic automorphism of polyvector fields.

In [26], M. Kontsevich proves his formality conjecture, giving an explicit construction of Lie infinity quasi-isomorphism

$$\mathcal{U} : T_{poly}(\mathbb{R}^d) \rightsquigarrow D_{poly}(\mathbb{R}^d)$$

from the Lie algebra of polyvector fields $T_{poly}(\mathbb{R}^d)$ to the dg Lie algebra of poly-differential operators on $\mathbb{R}^d$. This construction uses integral weights

$$\varpi_\Gamma := \int_{\mathrm{Conf}_{n,m}} \bigwedge_{(i,j)\in E(\Gamma)} \omega_{(i,j)},$$

where $\mathrm{Conf}_{n,m}$ is a compactified configuration space of points z_i, $i \in [n+m]$ on the closed upper half-plane $\mathbb{H}$ modulo real translations and scaling, with the m last points on the real line, and $\omega_{(i,j)}$ is the 1-form

$$\omega_{(i,j)} := \frac{1}{2\pi} d \arg \left(\frac{z_i - z_j}{\bar{z}_i - z_j} \right),$$

and Γ is a graph with $n + m$ vertices.

In [25], M. Kontsevich conjectured that the form $\omega_{(i,j)}$ can be replaced by the following,

$$\omega^{\log}_{(i,j)} := \frac{1}{2\pi i} d \log \left(\frac{z_i - z_j}{\bar{z}_i - z_j} \right),$$

thus creating another formality morphism $\mathcal{U}^{\log}$. However, as $\omega^{\log}_{(i,j)}$ has singularities as $\bar{z}_i - z_j \to 0$, it is not clear that the weights

$$\varpi^{\log}_\Gamma := \int_{\mathrm{Conf}_{n,m}} \bigwedge_{(i,j)\in E(\Gamma)} \omega^{\log}_{(i,j)}, \tag{5}$$

are well defined.

The integrals (5) were later proven to converge by A. Alekseev, C. A. Rossi, C. Torossian and T. Willwacher [3], using a new version of Stokes' formula that works for differential forms with some well behaved singularities on the boundary.

We found a new application of this new version of Stokes' formula by constructing a family of explicit exotic automorphism of polyvector fields, using a two parameter family of logarithmic propagators

$$\omega^{t,\lambda}_{(i,j)} := \frac{1-t}{2\pi i} d \log \left(\frac{z_i - z_j}{1 + \lambda |z_i - z_j|} \right) - \frac{t}{2\pi i} d \log \left(\frac{\bar{z}_i - \bar{z}_j}{1 + \lambda |z_i - z_j|} \right), \quad t, \lambda \in \mathbb{R},\ \lambda > 0, \tag{6}$$

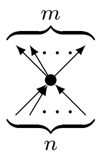
on a compactified configuration space $\mathfrak{C}_n$ of n points z_i on the complex plane $\mathbb{C}$ modulo translations.

Chapter 1

Graphs and props

Introduction

The notion of a *prop* (short for *product and permutation category*) was introduced by S. MacLane in [30] as a way of unifying the concept of (bi)algebraic structures. A prop can be viewed, in short, as a collection of operations with multiple inputs and outputs



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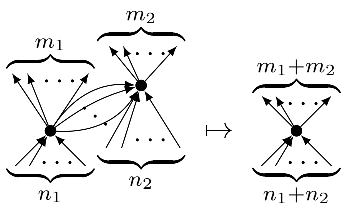
such that outputs can be composed to inputs



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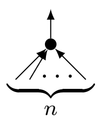
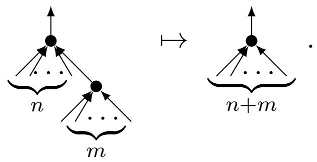
creating new operations. For different purposes, many variations of props exist in the literature. Perhaps the most used one is *operads*. The term operad and the corresponding formal definition were coined by P. May [33]. In brief, an operad can be viewed as a collection of operations with several inputs and a unique output



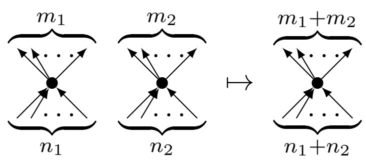
,

which can be composed



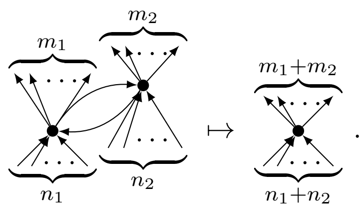
Operads give us an effective tool to treat many mathematical structures in a unifying way. There is, for example, an operad Ass that governs associative algebras in the sense that every associative algebra on a vector space V can be viewed as a map of operads from Ass to an operad End_V [29]. Similarly, there is an operad Com that governs commutative algebras, an operad Lie that governs Lie algebras, an operad Pois that governs Poisson algebras etc.

Other variations of props include *properads*, introduced by B. Vallette [47], which are very similar to the original definition of props, but with the subtle difference that horizontal compositions



are considered in props but not properads.

Wheeled props and properads, introduced in [31], is a variation of a prop that allows non-oriented compositions



Although not always perceived as such, associative algebras can be viewed as a type of props, where operations are only allowed to have one input and one output.

All variations of props presented above have some common features. Some ideas that work for one variation may also be applicable to other variations. See [29], [47], [38], [31]. Such a transfer of ideas and methods from one variation to another may, however, be far from straightforward.

A unifying definition of different variations of props was given by D.V. Borisov and Y.I. Manin [8], using a notion of *categories of abstract graphs.* In this section, we shall recall their definition of generalized props. Our main motivation for using the language of D.V. Borisov and Y.I. Manin is to transfer the definitions of a derivation complex and a deformation complex of a prop [38] to the setting of *multi-oriented props*, first considered in [35].

1.1 Categories of graphs and props

In this section, we will follow D.V. Borisov and Y.I. Manin [8] in order to define graphs and props.

1.1.1 Non-directed Graphs

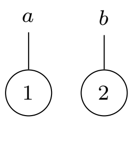
Definition 1.1.1. *A graph* $\Gamma = (F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma)$ *consists of:*

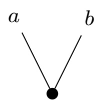
1. *A finite set* F_Γ*, called* flags*.*
2. *A finite set* V_Γ*, called* vertices*.*
3. *A map*
$$\partial_\Gamma : F_\Gamma \to V_\Gamma;$$
4. *An involution*
$$\iota_\Gamma : F_\Gamma \to F_\Gamma, \quad \iota^2 = \mathrm{id}\,.$$

The orbits of ι_Γ are called *edges* and will be denoted by $E(\Gamma)$. The orbits containing 2 elements are called *internal edges*, and will be denoted by $E_{int}(\Gamma)$, and the orbits containing a single element are called *external edges* or *legs*, and will be denoted by $L(\Gamma)$. A graph may be realized pictorially, by the following steps:

1. Draw a dot for each element in V_Γ.
2. For each 2-orbit (i, j) of ι_Γ draw a line between the vertices $\partial_\Gamma(i)$ and $\partial_\Gamma(j)$.
3. For each 1-orbit i of ι, draw a line connected in one end to $\partial_\Gamma(i)$.

For example:

$(\{a, b\}, \{1, 2\}, (a \mapsto 1, b \mapsto 2), \mathrm{id}) =$ 

$(\{a, b\}, \{\bullet\}, \{a, b\} \mapsto \{\bullet\}, \mathrm{id}) =$ 

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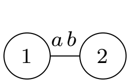
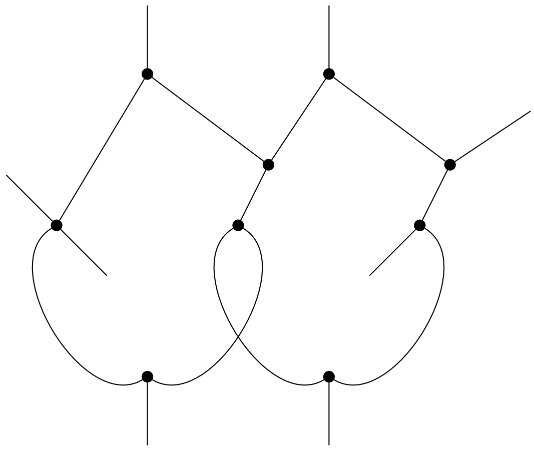
$(\{a, b\}, \{1, 2\}), (a \mapsto 1, b \mapsto 2), (ab)) =$ 

$(\{a, b\}, \{\bullet\}, \{a, b\} \to \{\bullet\}, (ab)) =$ 

.

Remark 1.1.2. The quadruple of data $(F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma)$ given in definition 1.1.1 is necessary in order to make precise definitions. Otherwise, we shall almost always think of graphs as pictures.

For bigger graphs than the ones above, e.g. this one



,

the data $(F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma)$ would (at least in the author's opinion) be not be of any use.

Definition 1.1.3. *A* morphism of graphs $h = (h^F, h_V, j_h) : \Gamma_1 \to \Gamma_2$ *consists of:*

1. *An injective map*
$$h^F : F_{\Gamma_2} \to F_{\Gamma_1}.$$
2. *A surjective map*
$$h_V : V_{\Gamma_1} \to V_{\Gamma_2}.$$
3. *An involution*
$$\iota_h : F_{\Gamma_1} \setminus h_F(\Gamma_2) \to F_{\Gamma_1} \setminus h_F(\Gamma_2), \quad \iota_h^2 = \mathrm{id}\,.$$

Such that

(a) *The image of h^F and its complement $F_{\Gamma_1} \setminus h^F(F_{\Gamma_2})$ are ι_{Γ_1}-invariant.*

(b) *The involution ι_h does not fix any element, and it agrees with ι_{Γ_1} for the non-fixed elements. Furthermore, we must have that*

$$\begin{array}{ccccc} F_{\Gamma_1} \setminus h^F(F_{\Gamma_2}) & \xrightarrow{\partial_{\Gamma_1}} & V_{\Gamma_1} & \xrightarrow{h_V} & V_{\Gamma_2} \\ \downarrow{\scriptstyle \iota_h} & & & \nearrow{\scriptstyle h_V} & \\ F_{\Gamma_1} \setminus h^F(F_{\Gamma_2}) & \xrightarrow{\partial_{\Gamma_1}} & V_{\Gamma_1} & & \end{array}$$

commutes.

(c) *The diagram*

$$\begin{array}{ccc} F_{\Gamma_2} & \xrightarrow{\partial_{\Gamma_2}} & V_{\Gamma_2} \\ \downarrow{\scriptstyle h^F} & & \uparrow{\scriptstyle h_V} \\ F_{\Gamma_1} & \xrightarrow{\partial_{\Gamma_1}} & V_{\Gamma_1} \end{array}$$

commutes.

(d) The bijection

$$(h^F)^{-1} : h^F(F_{\Gamma_2}) \to F_{\Gamma_2}$$

maps internal edges of Γ_1 *to internal edges of* Γ_2*.*

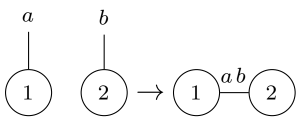
Two morphisms $h_1 = (h_1^F, h_V^1, j_h^1) : \Gamma_1 \to \Gamma_2$, $h_2 = (h_2^F, h_V^2, j_h^2) : \Gamma_2 \to \Gamma_3$, may be composed by

$$h_2 \circ h_1 = (h_1^F \circ h_2^F, h_V^2 \circ h_V^1, j_h^1 \sqcup j_h^2).$$

Note that j_h^1 and j_h^2 are involutions on disjoint sets. We will denote the category of graphs and morphisms as above by Gr.

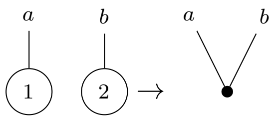
In order to get a better intuition of morphisms of graphs, let us consider the following 3 types of morphisms of graphs:

1. **Grafting:** A morphism $(h^F, h_V, \mathrm{id}) : \Gamma_1 \to \Gamma_2$, where both h^F and h_V are bijective, is called a *grafting morphism.* Note that a grafting morphism is not necessarily an isomorphism, as by (d), the map $(h^F)^{-1} : F_{\Gamma_1} \to F_{\Gamma_2}$ may map a pair of external legs to an internal edge, but not the other way around. For example

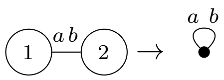
   $(\mathrm{id}, \mathrm{id}, \mathrm{id}) :$ 

   is a non-invertible grafting morphism.

2. **Vertex Merging** A morphism $(h^F, h_V, \mathrm{id}) : \Gamma_1 \to \Gamma_2$ where h_V is not bijective, while h^F if bijective and $\iota_{\Gamma_1} h^F = h^F \iota_{\Gamma_2}$ is called a *vertex merger.* A vertex merger preserves the number of internal edges and the number of external legs. For example

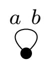
   $(\mathrm{id}, 1, 2 \mapsto \bullet, \mathrm{id}) :$  ,

   or

   $(\mathrm{id}, 1, 2 \mapsto \bullet, \mathrm{id}) :$  .

3. **Tadpole Deletion:** An edge starting and ending at the same vertex is called a *tadpole.* A morphism $(h^F, h_V, \iota_h) : \Gamma_1 \to \Gamma_2$ where h_V is bijective, and ι_{Γ_1} is non-fixed on the complement $F_{\Gamma_1} \setminus h^F(F_{\Gamma_2})$, is called a *tadpole deletion morphism.*

   Such a morphism h preserves the number of vertices while it forgets some internal edges. It follows from (c) that the forgotten edges must be tadpoles. Furthermore, it follows from (b) that ι_h must agree with ι_{Γ_1}. An example of a tadpole deletion morphism is

   $(\emptyset, \mathrm{id}, (ab)) :$  $\to \bullet$.

Proposition 1.1.4. *Any morphism $h \in Mor(Gr)$ can be written as a composition $h = t \circ m \circ k$, where k is a grafting morphism, m is a vertex merger and t is a tadpole deletion morphism.*

Proof. Pick an arbitrary morphism

$$(h^F, h_V, \iota_h) : (F_{\Gamma_1}, V_{\Gamma_1}, \partial_{\Gamma_1}, \iota_{\Gamma_1}) \to (F_{\Gamma_2}, V_{\Gamma_2}, \partial_{\Gamma_2}, \iota_{\Gamma_2}).$$

Consider the graph

$$\Gamma_1' := (F_{\Gamma_1}, V_{\Gamma_1}, \partial_{\Gamma_1}, \iota_{\Gamma_1}')$$

where ι_{Γ_1}' agrees with ι_{Γ_1} on the image $h^F(F_{\Gamma_2})$, and with ι_h on the compliment $F_{\Gamma_1} \setminus h^F(F_{\Gamma_2})$. Let k be the grafting morphism

$$k := (\mathrm{id}, \mathrm{id}, \emptyset) : \Gamma_1 \to \Gamma_1'.$$

Next, consider the graph

$$\Gamma_2' := (F_{\Gamma_1}, V_{\Gamma_2}, \partial_{\Gamma_2}', \iota_{\Gamma_1}')$$

where $\partial_{\Gamma_2}' : F_{\Gamma_1} \to V_{\Gamma_2}$ is given by $\partial_{\Gamma_2}' := \partial_{\Gamma_2} h_V$. Let m be the vertex merger

$$m := (\mathrm{id}, h_V, \emptyset) : \Gamma_1' \to \Gamma_2'.$$

Finally, there is a tadpole deletion morphism

$$t := (h^F, \mathrm{id}, \iota_h) : \Gamma_2' \to \Gamma_2.$$

It is clear that $h = t \circ m \circ k$. □

A morphism that is a composition of vertex mergers and tadpole deletions is called a *contraction morphism.*

A graph γ is called a *subgraph* of Γ if $F_\gamma \subset F_\Gamma$, $V_\gamma \subset V_\Gamma$, $\partial_\gamma = \partial_\Gamma|_{F_\gamma}$, and $\iota_\gamma = \iota_\Gamma|_{F_\gamma}$. We will write $\gamma \subset \Gamma$ if γ is a subgraph of Γ. For a subgraph $\gamma \subset \Gamma$ without external legs, let Γ/γ be the graph

$$\Gamma/\gamma := (F_\Gamma \setminus F_\gamma, V_\Gamma/V_\gamma, \partial_{\Gamma/\gamma}, \iota_\Gamma).$$

Let $con_\gamma : \Gamma \to \Gamma/\gamma$ be the morphism given by

$$con_\gamma^F : F_\Gamma \setminus F_\gamma \hookrightarrow F_\Gamma,$$

$$con_{\gamma,V} : V_\Gamma \to V_\Gamma/V_\gamma,$$

$$\iota^{con_\Gamma} = \iota_\Gamma|_{F_\gamma}.$$

We say that two subgraphs $\gamma_1, \gamma_2 \subset \Gamma$ are *disjoint* if their flag sets F_{γ_1} and F_{γ_2} are disjoint (note that with this convention, γ_1 and γ_2 may share vertices). For two disjoint subgraphs $\gamma_1, \gamma_2 \subset \Gamma$, we get that $\gamma_1 \subset \Gamma/\gamma_2$ and vice versa $\gamma_2 \subset \Gamma/\gamma_1$. We may therefore write

$$\Gamma/(\gamma_1, \gamma_2, \ldots, \gamma_n) := (\cdots(\Gamma/\gamma_1)/\gamma_2)/, \ldots)/\gamma_n)$$

for any disjoint subgraphs $\gamma_1, \ldots, \gamma_n \subset \Gamma$

Furthermore, It is clear that

$$(\Gamma/\gamma_1)/\gamma_2 = (\Gamma/\gamma_2)/\gamma_1,$$

and $con_{\gamma_1} con_{\gamma_2} = con_{\gamma_2} con_{\gamma_1}$. Hence, we have $\Gamma/(\gamma_1, \gamma_2, \ldots, \gamma_n) = \Gamma/(\gamma_{\sigma(1)}, \gamma_{\sigma(2}, \ldots, \gamma_{\sigma(n)})$ for any permutation $\sigma \in \mathbb{S}_n$.

The disjoint union $\sqcup$, by

$$\Gamma_1 \sqcup \Gamma_2 := (F_{\Gamma_1} \sqcup F_{\Gamma_2}, V_{\Gamma_1} \sqcup V_{\Gamma_2}, \partial_{\Gamma_1} \sqcup \partial_{\Gamma_2}, \iota_{\Gamma_1} \sqcup \iota_{\Gamma_2})$$

makes the category of graphs Gr into an symmetric monoidal category $(Gr, \sqcup)$. We say that a graph Γ in Gr is *connected* if there is no isomorphism

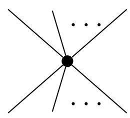
$$\Gamma_1 \sqcup \Gamma_2 \to \Gamma,$$

for graphs $\Gamma_1, \ \Gamma_2 \neq \emptyset$.

1.1.2 Properties of the category $(Gr, \sqcup)$

A graph that only contains one vertex, and no internal edges is called a *corolla*. That is a graph

$$c = (F_c, V_c, \partial_c, \iota_c) =$$



,

with $|V_c| = 1$ and $\iota_c = \mathrm{id}$.

For any graph Γ, there is a morphism con_Γ to the corolla $con(\Gamma) := (L(\Gamma), \{*\}, \partial, \mathrm{id})$, called the *full contraction morphism*, given by merging all the vertices followed by deleting all tadpoles. Furthermore, for every graph Γ, there is a canonical grafting morphism from a disjoint union of corollas to Γ,

$$\circ_\Gamma : \bigsqcup_{v \in V(\Gamma)} v (:= (\partial_\Gamma^{-1}(v), \{v\}), \partial_\Gamma, \mathrm{id})) \to \Gamma$$

called a *total grafting morphism*.

For a morphism $h : \Gamma_1 \to \Gamma_2$, there is a commutative diagram

$$\begin{array}{ccc} \bigsqcup_{v \in V_{\Gamma_2}} h^{-1}(v) & \xrightarrow{\sqcup h_v} & \bigsqcup_{v \in V_{\Gamma_2}} v \\ \downarrow k & & \downarrow \circ_{\Gamma_2} \\ \Gamma_1 & \xrightarrow{h} & \Gamma_2 \end{array}$$

called the *atomization* of h. Here $h^{-1}(v)$ is the graph $(\partial_{\Gamma_1}^{-1}(h_V^{-1}(v)), h_V^{-1}(v), \partial_{\Gamma_1}, \iota_{\Gamma_1})$ and $h_v : h_V^{-1}(v) \to v$ is the canonical restriction of h. The morphism k is a grafting morphism where k_V and k_F are identity maps.

Furthermore, for any morphism to a disjoint union of corollas

$$\sqcup h_v : \bigsqcup_{v \in \Gamma_2} \gamma_v \to \bigsqcup v,$$

and a total grafting morphism

$$\circ_{\Gamma_2} : \bigsqcup_{v \in V(\Gamma_2)} \to \Gamma_2,$$

there is a unique graph Γ_1 and a contraction morphism $h : \Gamma_1 \to \Gamma_2$ such that

$$\begin{array}{ccc} \bigsqcup_{v \in V_{\Gamma_2}} \Gamma_v & \xrightarrow{\sqcup h_v} & \bigsqcup_{v \in V_{\Gamma_2}} v \\ \downarrow k & & \downarrow \circ_{\Gamma_2} \\ \Gamma_1 & \xrightarrow{h} & \Gamma_2 \end{array}$$

is the atomization diagram of h.

1.1.3 Categories of abstract graphs and props

Definition 1.1.5. *A category of* Abstract Graphs *is a symmetric monoidal category* $(\mathcal{G}, \sqcup)$, *together with a functor*

$$\psi : (\mathcal{G}, \sqcup) \to (Gr, \sqcup)$$

satisfying the following:

1. *The functor ψ is faithful, i.e. for any two objects Γ_1, Γ_2 in $\mathcal{G}$, the map*

   $$\psi : Mor(\Gamma_1, \Gamma_2) \to Mor(\psi(\Gamma_1), \psi(\Gamma_2))$$

   is injective. We shall call a morphism $h \in \mathcal{M}or(\mathcal{G})$ a grafting morphism *or a* contraction morphism *if $\psi(h)$ is a grafting morphism or a contraction morphism respectively.*

2. *An object Γ in $\mathcal{G}$ is called* connected *if $\psi(\Gamma)$ is connected, and an object $c \in \mathcal{G}$ is called a $\mathcal{G}$-*corolla *if $\psi(c)$ is a corolla. For any connected object Γ in $\mathcal{G}$ there exists a $\mathcal{G}$-corolla $con(\Gamma)$ and a morphism $con_\Gamma : \Gamma \to con(\Gamma)$ such that*

   $$\psi(con_\Gamma) : \psi(\Gamma) \to \psi(con(\Gamma))$$

   is the full contraction morphism of $\psi(\Gamma)$.

3. *Any object Γ in $\mathcal{G}$ is the target of a morphism from a disjoint union of corollas*

   $$\circ_\Gamma : \bigsqcup c_i \to \Gamma,$$

   such that

   $$\psi(\circ_\Gamma) = \circ_{\psi(\Gamma)}.$$

4. *For each morphism $h : \Gamma_1 \to \Gamma_2$ in $\mathcal{G}$, there is a unique atomization diagram D such that $\psi(D)$ is the atomization diagram of $\psi(h)$.*

5. *For any fixed object* Γ_2 *in* $\mathcal{G}$*, and morphisms* $h_v : \gamma_v \to v$ *for each* $v \in V_\Gamma$*, there is a unique object* Γ_1*, and a contraction morphism* $h : \Gamma_1 \to \Gamma_2$ *such that*

$$\begin{array}{ccc} \bigsqcup_{v\in V_{\psi(\Gamma_2)}} \gamma_v & \xrightarrow{\sqcup h_v} & \bigsqcup_{v\in V_{\Gamma_2}} v \\ \downarrow k & & \downarrow \circ_{\Gamma_2} \\ \Gamma_1 & \xrightarrow{h} & \Gamma_2 \end{array}$$

is the atomization diagram of h*.*

Categories of abstract graphs shall almost always be thought of as graphs with extra decoration on edges or vertices, or subcategories of graphs. We will give several examples in Section 1.2.

Definition 1.1.6. *Given an abstract category of graphs* $(\mathcal{G}, \sqcup)$*, and a symmetric monoidal category* $(\mathcal{C}, \otimes)$*, a* $\mathcal{GC}$*-prop is a strong monoidal functor*

$$\mathcal{P} : (\mathcal{G}, \sqcup) \to (\mathcal{C}, \otimes)$$

that maps grafting morphisms to isomorphisms.

Morphisms of $\mathcal{GC}$*-props are natural transformations of functors.*

Remark 1.1.7. Most commonly, the category $(\mathcal{C}, \otimes)$ will be the category of vector spaces $(\mathsf{vect}_\Bbbk, \otimes)$, or the category of differential graded (dg) vector spaces $(\mathsf{dg\text{-}vect}_\Bbbk, \otimes)$ over a field $\Bbbk$ (of characteristic 0). Such props will often be referred to as just props and dg props, respectively.

Definition 1.1.8. *A functor* $E : cor(\mathcal{G}) \to \mathcal{C}$ *from a category (groupoid) of corollas* $cor(\mathcal{G})$*, of a category of abstract graphs* $\mathcal{G}$*, to a category* $\mathcal{C}$ *is called a* $\mathcal{G}$-collection*.*

Morphisms of $\mathcal{G}$*-collections are natural transformations of functors.*

1.1.4 The free $\mathcal{GC}$-prop

Given an abstract category of graphs $\mathcal{G}$ and a disjoint union of corollas $\sqcup c$ in $\bigsqcup cor(\mathcal{G})$, let $\Rightarrow \sqcup c$ be the category with objects being morphisms $h : \Gamma \to \sqcup c$ in $Mor(\mathcal{G})$, and morphisms being transformations

$$h(: \Gamma \to c) \mapsto hf^{-1}(: \Gamma' \to c),$$

where f is an isomorphism $\Gamma' \to \Gamma$ in $\mathcal{G}$.

For a $\mathcal{G}$-collection E in a symmetric monoidal category $(\mathcal{C}, \otimes)$, let $\bar{E}$ be the functor

$$\begin{array}{rcl} \bar{E} : (\Rightarrow \sqcup c) & \to & \mathcal{C}, \\ h : (\Gamma \to \sqcup c) & \mapsto & \bigotimes_{v \in V_\Gamma} E(v). \end{array}$$

Next, for a graph Γ we define

$$\mathcal{F}(E)(\Gamma) := \mathrm{colim}_{\Rightarrow V(\Gamma)} \bar{E},$$

and for a morphism $g : \Gamma_1 \to \Gamma_2$, let $\mathcal{F}(E)(g)$ be the unique morphism that exists by the universal property of colimits, such that the following diagram commutes

$$\begin{array}{ccc} \bar{E}(h) & \xrightarrow{\cong} & \bar{E}((\bigsqcup_{v\in V_{\Gamma_2}} g_v)\circ h) \\ \downarrow{\scriptstyle \phi_h} & & \downarrow{\scriptstyle \phi_{\sqcup g \circ h}} \\ \mathcal{F}(E)(\Gamma_1) & \xrightarrow{\mathcal{F}(E)(g)} & \mathcal{F}(E)(\Gamma_2). \end{array}$$

Here, $\bigsqcup g_v : \bigsqcup_{V_{\Gamma_2}} g^{-1}(v) \to \bigsqcup_{V_{\Gamma_2}} v$ is the atomization of g.

Proposition 1.1.9. *The construction $\mathcal{F}(E)$ is a well defined functor of symmetric monoidal categories and a prop.*

Proof. As there are no morphisms from a connected graph to a disconnected graph in $\mathcal{G}$, we have an equivalence of categories

$$\left(\Rightarrow \bigsqcup_{c\in V_\Gamma} c\right) \cong \bigtimes_{c\in V_\Gamma} (\Rightarrow c)\,.$$

It follows that

$$\mathcal{F}(\Gamma) = \operatorname{colim}_{\Rightarrow V(\Gamma)} \bar{E} \cong \operatorname{colim}_{\times_{c\in V_\Gamma}\Rightarrow c} \bigotimes_{c\in V_\Gamma} \bar{E}_c \cong \bigotimes_{c\in V_\Gamma} \operatorname{colim}_{\Rightarrow c} \bar{E} = \bigotimes_{c\in V_\Gamma} \mathcal{F}(E)(c).$$

Hence, $\mathcal{F}(E)$ is a functor of symmetric monoidal categories.

Now we need to prove that $\mathcal{F}(E)$ maps grafting morphisms to isomorphisms. If $g : \Gamma_1 \to \Gamma_2$ is a grafting morphism, then its atomization $\sqcup_{g_v} : \bigsqcup_{v\in V_{\Gamma_2}} g^{-1}(v) \to \bigsqcup_{v\in V_{\Gamma_2}} v$ is an isomorphism. By the universal property of colimits, there exists a map $\mathcal{F}(E)(g)^{-1}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{E}((\bigsqcup_{v\in V_{\Gamma_2}} g_v)^{-1}\circ h) & \xleftarrow[\cong]{} & \bar{E}(h) \\ \downarrow{\scriptstyle \phi_{\circ \sqcup g^{-1}\circ h}} & & \downarrow{\scriptstyle \phi_h} \\ \mathcal{F}(E)(\Gamma_1) & \xleftarrow[\mathcal{F}(E)(g)^{-1}]{} & \mathcal{F}(E)(\Gamma_2). \end{array}$$

It is evident that $\mathcal{F}(E)(g)^{-1} \circ \mathcal{F}(E)(g) = \mathrm{id}$. ☐

Proposition 1.1.10. *The prop $\mathcal{F}(E)$ is the free prop over the $\mathcal{G}$-collection E in the sense that, for every prop $\mathcal{P}$ and $\mathcal{G}$-collection map $\phi : E \to \mathcal{P}$, there exists a unique map of props $\bar{\phi} : \mathcal{F}(E) \to \mathcal{P}$ such that the following diagram commutes*

$$\begin{array}{ccc} E & & \\ \downarrow & \searrow{\scriptstyle \phi} & \\ \mathcal{F}(E) & \xrightarrow{\bar{\phi}} & \mathcal{P} \end{array}\quad .$$

Proof. By the universal property of colimits, there exists a map $\bar{\phi}(c) : \mathcal{F}(E)(c) \to \mathcal{P}(c)$, for each corolla c, such that the diagram

$$\begin{array}{ccccc} \bar{E}(h) = \bigotimes_{v \in V_\Gamma} E(v) & \xrightarrow{\bigotimes \phi(v)} & \bigotimes_{v \in V_\Gamma} \mathcal{P}(v) & \xrightarrow{\mathcal{P}(h)} & \mathcal{P}(c) \\ \downarrow & & & \nearrow_{\bar{\phi}(c)} & \\ \mathcal{F}(E)(c) & & & & \end{array}$$

commutes for each contraction morphism $h : \Gamma \to c$. This gives our desired morphism of props. □

1.2 Examples of abstract categories of graphs and their associated props

1.2.1 Multi-directed and multi-oriented Graphs

Let $\mathcal{O}r_k$ be the set of maps $\mathfrak{s} : [k] \to \{in, out\}$. This set comes with the obvious involution $\iota : \mathcal{O}r^k \to \mathcal{O}r^k$, by flipping each value.

Definition 1.2.1. *A k-*directed graph $\Gamma = (F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma, Or_\Gamma, I_\Gamma)$ *consists of:*

1. *A graph* $\overline{\Gamma} := (F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma)$ *in* $\mathcal{G}$.
2. *A map*
$$Or_\Gamma : F_\Gamma \to \mathcal{O}r_k,$$
such that, for all internal edges (f, g)*, we must have*
$$Or_\Gamma(f) = \iota(Or_\Gamma(g)).$$

Definition 1.2.2. *A morphism* $h : \Gamma_1 \to \Gamma_2$ *between two k-directed graphs is a morphism of non-directed graphs*
$$(h^F, h_V, \iota_h) : \Gamma_1 \to \Gamma_2$$
such that $Or_\Gamma(f) = Or_\Gamma(h^F f)$.

Let the *category of k directed graphs* $\mathcal{G}^k$ be the category whose objects are k-directed graphs, and whose morphisms are morphisms of k-directed graphs.

For a graph Γ in $\mathcal{G}^k$, let a *path* in direction $i \in [k]$ be a string of flags in

$$(f_1, f^1), (f_2, f^2), \dots (f_k, f^k)$$

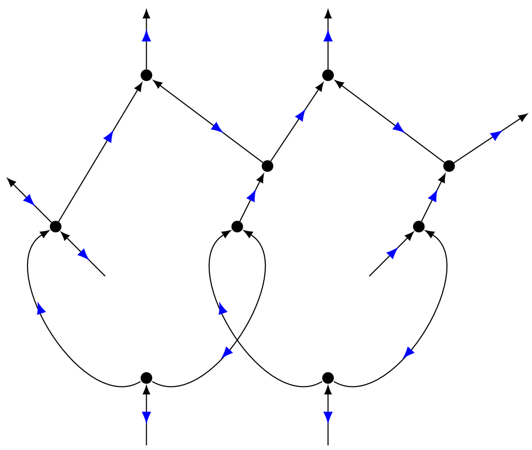
such that

$$\iota(f_r) = f^r, \quad \mathcal{O}r(f_r)(i) = out, \quad \mathcal{O}r(f^r)(i) = in, \text{ and } \partial_{\Gamma(f^r)} = \partial_{\Gamma(f_{r+1})}.$$

A path may also start or end at an external leg of Γ.

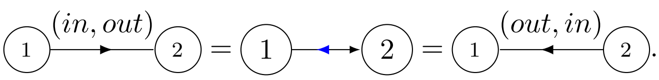
We will call a path starting and ending at the same edge a *wheel*. For $l \leq k$, let $\mathcal{G}^{k\uparrow l}$ be the category of *k-directed, l oriented graphs* consisting of graphs in $\mathcal{G}^k$ without wheels in the first l directions. Morphisms in $\mathcal{G}^{k\uparrow l}$ are morphisms of k-directed graphs $h : \Gamma_1 \to \Gamma_2$ that can be decomposed as $h = h' \circ k$, where h' is a contraction morphism and k is a grafting morphism such that $k\Gamma_1$ is oriented.

A multi-directed graph can be realized visually by a graph with arrows of different colors on each edge. For example



is a picture of a 2 directed (and oriented) graph, where the first direction is indicated by a black arrow in the end of each edge, and the second is indicated by a blue arrow in the middle of the edge.

Alternatively, we may choose to decorate each edge by an arrow, and an element $\mathfrak{s} \in Or_k$, where we say that direction i agrees with the arrow if $\mathfrak{s}(i) = in$, and disagrees with the arrow if $\mathfrak{s}(i) = out$. For example

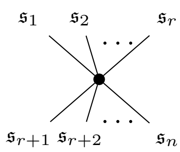


It may sometimes be useful to consider an ordering on $\mathcal{O}r_k$, for example

$$\mathfrak{s} > \mathfrak{t} \quad \text{if there exists } i \in [k] \text{ such that } \mathfrak{s}(i) = in, \text{ and } j \leq i \Rightarrow t(i) = out.$$

With this choice of ordering, $\overline{out} := (out, \ldots, out)$ is the minimal element and $\overline{in} := (in, \ldots, in)$ is the maximal element.

Each corolla in $cor(\mathcal{G}^{k\uparrow l})$ is isomorphic to a corolla

$c =$ 

,

which can be ordered so that $i \leq j$ implies that $\mathfrak{s}_i \leq \mathfrak{s}_j$. The automorphism group of such a corolla is given by $\bigtimes_{\mathfrak{s} \in Or_k} \mathbb{S}_{|Or_c^{-1}(\mathfrak{s})|}$.

It follows that a $\mathcal{G}^{k \uparrow l}\mathcal{C}$-collection

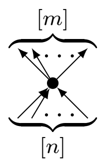
$$E : cor(\mathcal{G}^{k \uparrow l}) \to \mathcal{C}$$

is a collection of objects $\{E(\{n_{\mathfrak{s}}\})\}_{\mathfrak{s} \in \mathcal{O}r^k, n_{\mathfrak{s}} \geq 0}$ together with a $\bigtimes_{\mathfrak{s} \in Or_k} \mathbb{S}_{|Or_c^{-1}(\mathfrak{s})|}$ action on each $E(\{n_{\mathfrak{s}}\})$.

1.2.2 On one oriented graphs

Consider the category $\mathcal{G}^{1 \uparrow 1}$ consisting of graphs without wheels. A $\mathcal{G}^{1 \uparrow 1}\mathcal{C}$-prop is what is generally known as just a prop.

Each corolla in $cor(\mathcal{G}^{1 \uparrow 1})$ is isomorphic to a corolla



for some $n, m \geq 0$. The automorphism group of such a corolla is given by $\mathbb{S}_n \times \mathbb{S}_m$. Therefore, we get that $\mathcal{G}^{1 \uparrow 1}$-collection

$$E : cor(\mathcal{G}^1) \to \mathcal{C}$$

is equivalent to an $\mathbb{S}$-bimodule. That is a collection of objects

$$\{E(n, m)\}_{n,m}$$

together with an $\mathbb{S}_n \times \mathbb{S}_m$ action on each $E(n, m)$.

Let γ be a subgraph of an oriented graph Γ. It is clear that γ is also oriented. However, Γ/γ is not necessarily oriented. A trivial example of this is

Hence, a morphism $h : \Gamma_1 \to \Gamma_2$ can not necessarily be decomposed to $h = t \circ m \circ k$, where k is a grafting morphism, m is a vertex merger and t is a tadpole deletion, as in Proposition 1.1.4. Instead we have the following proposition.

Proposition 1.2.3. *The full contraction morphism $con_\Gamma : \Gamma \to c$ of an oriented graph Γ can be written as a chain of contraction morphisms*

$$\Gamma = \Gamma_1 \to \Gamma_2 \to \cdots \to \Gamma_n = c,$$

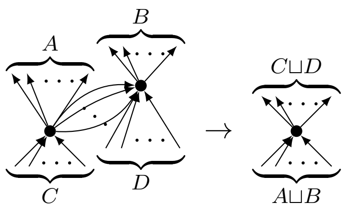
where $\Gamma_{i+1} = \Gamma_i/\gamma_i$ subgraphs $\gamma_i \subset \Gamma_1$ that only contain two vertices. Furthermore, if Γ is connected, one can choose the graphs γ_i so that they are all connected.

Proof. We need to show that for every oriented and connected graph Γ, there exists a connected subgraph $\gamma \subset \Gamma$ with 2 vertices, such that Γ/γ is oriented.

Let $\bar{\Gamma}$ be the graph obtained from Γ by removing all external legs. Next, let $S(\bar{\Gamma}) \subset V_\Gamma$ be the set of *sources* of $\bar{\Gamma}$, i.e. vertices without ingoing adjacent edges. As $\bar{\Gamma}$ is an oriented graph without external legs, the set $S(\bar{\Gamma})$ must be non-empty. Let $\bar{\Gamma} \setminus S(\bar{\Gamma})$ be the graph obtained from $\bar{\Gamma}$ by removing all sources and their adjacent edges. The graph $\bar{\Gamma} \setminus S(\bar{\Gamma})$ must be an oriented graph, hence there is a non empty set $S(\bar{\Gamma} \setminus S(\bar{\Gamma})) \subset V_\Gamma$ consisting of vertices that are sources in $\bar{\Gamma} \setminus S(\bar{\Gamma})$. Pick a vertex $v_1 \in S(\bar{\Gamma} \setminus S(\bar{\Gamma}))$. As v_1 is not a source in $\bar{\Gamma}$, v_1 must be connected to at least one vertex $v_2 \in S(\bar{\Gamma})$. Let $\gamma \subset \Gamma$ be the subgraph consisting of v_1 and v_2 and all edges connecting v_1 and v_2.

If Γ/γ is not oriented, Γ/γ must contain a path that contains the vertex $\{v_1, v_2\}$. However, as all incoming edges to $\{v_1, v_2\}$ come from vertices in $S(\bar{\Gamma})$, such a wheel would also have to contain a source vertex, which is impossible. Hence, Γ/γ must be oriented. □

By Proposition 1.2.3, a morphism $\mathcal{P}(con_\Gamma) : \mathcal{P}(\Gamma) \to \mathcal{P}(c)$ is uniquely determined by contraction morphisms $\mathcal{P}(con_\gamma) : \mathcal{P}(\gamma) \to \mathcal{P}(c)$, where γ is a graph with two vertices

$con_\gamma :$ 

.

This leads to the equivalence between this definition of props, which was first given in [8], and the monoidal definition of props given in e.g. [30] and [47].

For a vector space V, let EndB_V be the prop given by

$$\mathsf{EndB}_V(n, m) := \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}).$$

The group $\mathbb{S}_n \times \mathbb{S}_m$ acts on $\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$ by $(\sigma, \tau).f = \sigma f \tau^{-1}$, where $\sigma \in \mathbb{S}_n$ is interpreted as the isomorphism

$$\sigma : V_1 \otimes V_2 \otimes \ldots \otimes V_n \to V_{\sigma(1)} \otimes V_{\sigma(2)} \otimes \cdots \otimes V(\sigma_n)$$

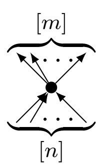
that comes with each symmetric monoidal category.

Remark 1.2.4. When we work with (differential) graded vector spaces, the Koszul sign rule applies. This means that

$$\sigma : A \otimes B \to B \otimes A$$

is given by $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$.

For an oriented graph Γ and a contraction morphism

$con_\Gamma : \Gamma \to$ 

,

the map

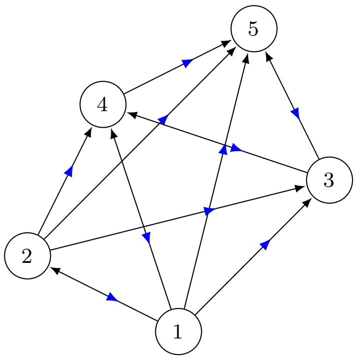
$$\mathsf{EndB}_V(con_\Gamma) : \bigotimes_{v\in V_\Gamma} \mathsf{EndB}_V(|in(v)|, |out(v)|) \to \mathsf{EndB}_V(n,m)$$

is given by composing the linear maps along the internal edges. Here, $in(v)$ and $(out(v))$ are, respectively, the sets of incoming and outgoing adjacent edges to v.

Definition 1.2.5. *A* (bi)algebra over a prop $\mathcal{P}$ *is a morphism of props* $\mu : \mathcal{P} \to \mathsf{EndB}_V$*.*

1.2.3 Multi-oriented props

A $\mathcal{G}^{k\uparrow l}\mathcal{C}$-prop is called a *k-directed, l-oriented prop*, and was first introduced in [35]. For multi-oriented graphs, there is no equivalent to Proposition 1.2.3. Consider for example the 2-oriented graph

$\Gamma =$  $\in \mathcal{G}^{2\uparrow 2}$,

where the first direction is directed $5 \succ 4 \succ 3 \succ 2 \succ 1$, and the second direction is directed $3 \succ 5 \succ 1 \succ 4 \succ 2$. One can see that there are no subgraphs $\gamma \subsetneq \Gamma$, $\gamma \neq \bullet$ such that Γ/γ is oriented in both directions. The existence of such graphs makes it impossible to give a monoidal definition of multi-oriented props.

A multi-oriented analogue to the endomorphism prop is given in [35].

1.2.4 Properads

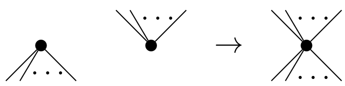
Using this framework to define props, the definition of a properad, as in [47], may be obtained in the following way.

In the category of graphs Gr, let us call a contraction morphism

$$h : \Gamma \to \Gamma/\gamma$$

properadic, if γ is a connected subgraph of Γ.

Consider the subcategory Pr of Gr that only allows morphisms $h : \Gamma_1 \to \Gamma_2$ if h can be written as a composition of grafting morphisms and properadic contraction morphisms. The main point is that vertex mergers



are not allowed.

For an abstract category of graphs $\mathcal{G}$, let $Pr\mathcal{G}$ be the category that contains all objects of $\mathcal{G}$ but only morphisms h such that $\psi(h) \in \mathcal{M}or(Pr)$.

Proposition 1.2.6. *The category Pr is an abstract category of graphs. Also, for any abstract category of graphs $\mathcal{G}$, the category $Pr\mathcal{G}$ is an abstract category of graphs.*

Proof. The first 3 conditions are obvious. Consider a atomization diagram

$$\begin{array}{ccc} \bigsqcup_{v\in V_{\psi(\Gamma_2)}} \gamma_v & \xrightarrow{\sqcup h_v} & \bigsqcup_{v\in V_{\Gamma_2}} v \\ \downarrow k & & \downarrow \circ_{\Gamma_2} \\ \Gamma_1 & \xrightarrow{h} & \Gamma_2 \end{array}.$$

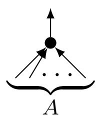
It is clear that if h is properadic, then so are all morphisms h_v. Conversely, if all maps h_v are properadic, then so is h. Hence, we also have the last two conditions. □

Definition 1.2.7. *We say that a $\mathcal{GC}$-*properad *is a $Pr\mathcal{GC}$-prop.*

It is often more convenient to study properads than props.

1.2.5 Rooted trees and operads

Let the category of *rooted trees*, $\mathcal{T}$, be the subcategory of $\mathcal{G}^{1\uparrow 1}$ consisting of graphs where every vertex has precisely 1 vertex has precisely one adjacent outgoing edge. Morphisms in $\mathcal{T}$ are morphisms of oriented graphs. The corollas $cor(\mathcal{T})$ are given by



$$, \quad |A| \geq 0.$$

Note that each connected component T in $\mathcal{T}$ must be of genus 0, i.e. $|V_T| - |E_T| - 1 = 0$, or T is a *tree*.

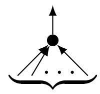
Proposition 1.2.8. *The category $\mathcal{T}$ is an abstract category of graphs.*

Proof. The conditions 1 and 3 are clear. For 2, note that each connected component T of $\mathcal{T}$ must have precisely one outgoing external leg. Hence, it contracts to a corolla in $cor(\mathcal{T})$. Next consider an atomization diagram

$$\begin{array}{ccc} \bigsqcup_{v\in V_{\psi(\Gamma_2)}} \gamma_v & \xrightarrow{\sqcup h_v} & \bigsqcup_{v\in V_{\Gamma_2}} v \\ \downarrow k & & \downarrow \circ_{\Gamma_2} \\ \Gamma_1 & \xrightarrow{h} & \Gamma_2 \end{array}.$$

It is clear that if Γ_1 and Γ_2 are rooted trees, then the subgraphs γ_v must be rooted trees, and the corollas $v \in V_{\Gamma_2}$ must be in $cor(\mathcal{T})$. Conversely, if each γ_v, and Γ_2 are rooted trees, then Γ_1 must also be a rooted tree. □

A $\mathcal{TC}$-prop is called an *operad*. As rooted trees are simpler objects than general graphs, there are other equivalent, perhaps simpler and more practical definitions of an operad (see for example [29]). As the automorphism group of a corolla with n ingoing legs and 1 outgoing leg



is given by the symmetric group $\mathbb{S}_n$, a $\mathcal{T}$-collection is equivalent to what is known as an $\mathbb{S}$-*module*. An $\mathbb{S}$-module a collection of objects $\{E(n)\}_{n\geq 0}$ in $\mathcal{C}$ together with an $\mathbb{S}_n$ action on each $E(n)$. This leads to the partial definition of an operad, given in for example [29].

Proposition 1.2.9 (e.g. J.-L. Loday, B. Vallette [29], Section 5.3.7)**.** *An operad $\mathcal{P} : \mathcal{T} \to \mathcal{C}$ is equivalent to an $\mathbb{S}$-module $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ together with partial composition maps*

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1), \quad 1 \leq i \leq n,$$

such that

$$(\mu \circ_i \nu) \circ_j \xi = \begin{cases} (\mu \circ_j \xi) \circ_{i+n_3-1} \nu) & j \leq i-1 \\ \mu \circ_i (\nu \circ_{j+i-1} \xi) & i \leq j \leq i+n_2-1 \\ (\mu \circ_{j-n_2+1} \xi) \circ_i \nu) & i+n_2 \leq j. \end{cases}$$

for all $\mu \in \mathcal{P}(n_1)$, $\nu \in \mathcal{P}(n_2)$, $\xi \in \mathcal{P}(n_3)$. Furthermore, these maps should satisfy the equivariance conditions

$$\mu \circ_i \sigma(\nu) = \sigma'(\mu \circ_i \nu)$$

for each $\sigma \in \mathbb{S}_m$, where $\sigma' \in \mathbb{S}_{n+n-1}$ is the permutation that acts as σ on the block $\{i, i+1, \dots, i+m-1\}$, and

$$\sigma(\mu) \circ_i \nu = \sigma''(\mu \circ_{\sigma(i)} \nu),$$

for each $\sigma \in \mathbb{S}_n$, where $\sigma'' \in \mathbb{S}_{n+n-1}$ is the permutation that acts like σ on $[n+m-1]/\{i, i+1, \dots, i+m-1\}$, while preserving the orders in $\{i, i+1, \dots, i+m-1\}$.

When considering regular operads, this is often a more practical definition.

For a vector space V, let End_V be the operad given by

$$\mathrm{End}_V(n) := \mathrm{Hom}(V^{\otimes n}, V) = \mathsf{EndB}_V(n, 1).$$

Definition 1.2.10. *An algebra over an operad $\mathcal{P}$ is a morphism of operads*

$$\mu : \mathcal{P} \to \mathrm{End}_A .$$

1.2.6 Symmetric graphs and props

For a corolla $c \in cor(Gr)$, let $Gr_{n,m}(c)$ be the set of graphs with vertices labeled by elements in $[n]$, internal edges labeled by elements in $[m]$ and external legs labeled as the legs of the corolla c. That is graphs

$$\Gamma = (F_\Gamma = ([m]_1 \sqcup [m]_2) \sqcup L_c, V_\Gamma = [n], \partial_\Gamma, \iota_m \sqcup \mathrm{id}_{L_c}),$$

where $\iota_m : [m]_1 \sqcup [m]_2 \to [m]_1 \sqcup [m]_2$ is the obvious involution $i_1 \rightleftarrows i_2$.

The edge labeled by i is then the orbit $(i_1, i_2) \subset [m]_1 \sqcup [m]_2$ of ι_m. We say that (i_1, i_2) is *intrinsically directed* from the vertex $\partial(i_1)$ to the vertex $\partial(i_2)$.

Let $\mathbb{P}_m \cong \mathbb{S}_m \ltimes \mathbb{S}_2^m$ be the group of permutations

$$\tau : [m]_1 \sqcup [m]_2 \to [m]_1 \sqcup [m]_2,$$

such that $\iota_m \tau = \tau \iota_m$. The group $\mathbb{S}^{\mathcal{G}}_{n,m} := \mathbb{S}_n \times \mathbb{P}_m$ acts on $Gr_{n,m}(c)$ by permuting vertex labels, permuting edge labels, and flipping the intrinsic direction of edges

$$(\sigma, \tau).\,(([m]_1 \sqcup [m]_2) \sqcup L_c, [n], \partial_\Gamma, \iota_m \sqcup \mathrm{id}_k) :=$$
$$\Big(([m]_1 \sqcup [m]_2) \sqcup L_c, [n], \sigma \circ \partial_\Gamma \tau^{-1} \sqcup \mathrm{id}_{L_c}, \iota_m \sqcup \mathrm{id}_{L_c}\Big).$$

It is clear that every graph $\Gamma \in Gr$ is isomorphic to some graph in $Gr_{n,m}(c)$, for some $n, m \geq 0$ and some corolla c. Furthermore, the $\mathbb{S}^{\mathcal{G}}_{n,m}$ action on $Gr_{n,m}(c)$ covers all internal isomorphisms between graphs in $Gr_{n,m}(c)$, i.e. any isomorphism $h : \Gamma \to \Gamma'$, such that

$$\begin{array}{ccc} \Gamma & & \\ \downarrow h & \searrow^{con_\Gamma} & \\ \Gamma' & \xrightarrow{con_{\Gamma'}} & c \end{array}$$

commutes is on the form

$$h = (h^F, h_V, \emptyset) = (\tau^{-1} \sqcup \mathrm{id}_{L_c}, \sigma, \emptyset),$$

for some $\tau \in \mathbb{P}_m$ and $\sigma \in \mathbb{S}_n$.

For an abstract category of graphs $\mathcal{G}$, and a corolla c in $cor(\mathcal{G})$, with set of external legs L_c, let $\mathcal{G}_{n,m}(c)$ be the set of graphs Γ in $\mathcal{G}$, such that:

1. $\psi(\Gamma) \in Gr_{n,m}(\psi(c))$,
2. there exists a contraction morphism $con_\Gamma : \Gamma \to c$.

Definition 1.2.11. *We say that an abstract category of graphs is* symmetric *if:*

1. *Any object Γ in $\mathcal{G}$ is isomorphic to some object in some $\mathcal{G}_{n,m}(c)$.*
2. *The group $\mathbb{S}^{\mathcal{G}}_{n,m}$ acts on $\mathcal{G}_{n,m}(c)$ in such a way so that*

$$\psi((\sigma, \tau).\Gamma) = (\sigma, \tau).\psi(\Gamma).$$

All our examples of abstract categories of graphs so far are symmetric. For example, $\mathcal{G}^{k\uparrow l}_{n,m}(c)$ is the set of k-directed l-oriented graphs

$$\Gamma = ([m]_1 \sqcup [m]_2 \sqcup L_c, [n], \partial_\Gamma, \iota_\Gamma, Or_\Gamma).$$

The group $\mathbb{S}_n \times \mathbb{P}_m$ acts on $\mathcal{G}^{k\uparrow l}_{n,m}(c)$ by

$$(\sigma,\tau).([m]_1 \sqcup [m]_2 \sqcup L_c, [n], \partial_\Gamma, \iota_\Gamma, Or_\Gamma) :=$$
$$([m]_1 \sqcup [m]_2 \sqcup L_c, [n], \sigma\partial_\Gamma(\tau^{-1} \sqcup \mathrm{id}_{L_c}), \iota_\Gamma, Or_\Gamma(\tau^{-1} \sqcup \mathrm{id}_{L_c})).$$

For the properadic categories of abstract graphs $Pr\mathcal{G}^{k\uparrow l}(c)$, we have that $Pr\mathcal{G}^{k\uparrow l}_{n,m}(c)$ consists of connected graphs in $\mathcal{G}^{k\uparrow l}_{n,m}(c)$.

For a field $\mathbb{k}$ of characteristic 0, let $(\mathsf{vect}_\mathbb{k}, \otimes)$ be the category of vector spaces over the field $\mathbb{k}$.

Proposition 1.2.12. *Let $\mathcal{G}$ be a symmetric abstract category of graphs, and let $E : cor(\mathcal{G}) \to \mathsf{vect}_\mathbb{k}$ be a $\mathcal{G}$-collection. Then, the free prop over E*

$$\mathcal{F}(E) : (\mathcal{G}, \sqcup) \to (\mathsf{vect}_\mathbb{k}, \otimes)$$

is given by

$$\mathcal{F}(E)(c) \cong \bigoplus_{n,m\geq 0} \left(\bigoplus_{\Gamma\in\mathcal{G}_{n,m}(c)} \bigotimes_{v\in V_\Gamma} E(v) \right)^{\mathbb{S}^{\mathcal{G}}_{n,m}}.$$

Proof. Let $\Rightarrow_{n,m} c$ be the category whose objects are morphisms $\Gamma \to c$, $\Gamma \in \mathcal{G}^{k\uparrow l}_{n,m}(c)$, and whose morphisms are transformations

$$h : \Gamma \to c \mapsto f^{-1}h : \Gamma' \to c,$$

where $f \in \mathbb{S}^{\mathcal{G}}_{n,m}$.

As each object in $\mathcal{G}$ is isomorphic to some object in some $\mathcal{G}^{k\uparrow l}_{n,m}(c)$, we have that

$$\mathcal{F}(E)(c) := \mathrm{colim}_{\Rightarrow c} \bigotimes_{v\in V_\Gamma} E(v) \cong \mathrm{colim}_{\times_{n,m\geq 0} \Rightarrow_{n,m} c} \bigotimes_{v\in V_\Gamma} E(v) = \left(\bigoplus_{\Gamma\in\mathcal{G}^{k\uparrow l}_{n,m}(c)} \bigotimes_{v\in V_\Gamma} E(v) \right)^{\mathbb{S}^{\mathcal{G}}_{n,m}}.$$

□

For a graph $\Gamma \in \mathcal{G}_{n,m}(c)$, and graphs $\gamma_v \in \mathcal{G}_{n_v,m_v}(v)$ for all vertices $v \in V(\Gamma)$, let

$$\Gamma(\gamma_1, \ldots, \gamma_n) \in \mathcal{G}_{N,M}(c),$$

be the graph with vertex set

$$V_{\Gamma(\gamma_1,\ldots,\gamma_n)} = \bigsqcup_{v\in V_\Gamma} [n_v] = [N],$$

flag set

$$\left([m] \sqcup \bigsqcup [m_v]\right)_1 \sqcup \left([m] \sqcup \bigsqcup [m_v]\right)_2 \sqcup L_c = [M]_1 \sqcup [M]_1 \sqcup L_c,$$

connection map

$$\partial_{\Gamma(\gamma_1,\ldots,\gamma_n)} = \partial_\Gamma \sqcup \bigsqcup_{v \in V_\Gamma} \partial_{\gamma_v},$$

and orientation map

$$Or_{\Gamma(\gamma_1,\ldots,\gamma_n)} = Or_\Gamma \sqcup \bigsqcup_{v \in V_\Gamma} Or_{\gamma_v}.$$

The contraction map

$$\mathcal{F}(E)(con_{\gamma_1,\ldots,\gamma_n}) : \underbrace{\bigotimes_{v \in V(\Gamma)} \left(\bigoplus_{n_v, m_v \geq 0} \left(\bigoplus_{\gamma_v \in \mathcal{G}^{k\uparrow l}_{n_v,m_m}(v)} \bigotimes_{w \in V_{\gamma_v}} E(w) \right)^{\mathbb{S}^{\mathcal{G}}_{n_v,m_v}} \right)}_{\mathcal{F}(E)(\Gamma)} \to$$

$$\to \underbrace{\bigoplus_{N,M \geq 0} \left(\bigoplus_{\Gamma \in \mathcal{G}^{k\uparrow l}_{N,M}(c)} \bigotimes_{w \in V_\Gamma} E(w) \right)^{\mathbb{S}^{\mathcal{G}}_{N,M}}}_{\mathcal{F}(E)(c)}.$$

is then induced by the canonical map

$$\bigotimes_{v \in V(\Gamma)} \left(\bigoplus_{n_v, m_v \geq 0} \bigoplus_{\gamma_v \in \mathcal{G}^{k\uparrow l}_{n_v,m_m}(v)} \bigotimes_{w \in V_{\gamma_v}} E(w) \right) \to \bigoplus_{n_v, m_v \geq 0} \bigoplus_{\gamma_v \in \mathcal{G}^{k\uparrow l}_{n_v,m_m}(v)} \bigotimes_{w \in V_{\Gamma(\gamma_1,\ldots,\gamma_n)}} E(w) \hookrightarrow$$

$$\hookrightarrow \bigoplus_{N,M \geq 0} \bigoplus_{\Gamma \in \mathcal{G}^{k\uparrow l}_{N,M}(c)} \bigotimes_{w \in V_\Gamma} E(w).$$

Note that $\bigtimes_{v \in V_\gamma} \mathbb{S}_{n_v,m_v}$ is a subgroup of $\mathbb{S}^{\mathcal{G}}_{N,M}$.

1.2.7 Ideals of props

Definition 1.2.13. *For a prop*

$$\mathcal{P} : \mathcal{G} \to \mathsf{vect}_{\Bbbk},$$

we say that an ideal $I \subset \mathcal{P}$ *is a* $\mathcal{G}$*-collection* I *such that for any corolla* c *in* $cor(\mathcal{G})$*, graphs* $\Gamma_1 \sqcup \Gamma_2$ *in* $\mathcal{G}$*, and morphisms* $h : \Gamma_1 \sqcup \Gamma_2 \to c$*; the map*

$$\mathcal{P}(h) : \mathcal{P}(\Gamma_1) \otimes I(\Gamma_2) \to \mathcal{P}(c)$$

has its image in $I(c)$*.*

For a prop $\mathcal{P}$ in the category of vector spaces and an ideal $I \subset \mathcal{P}$, we may define a prop

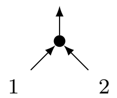
$$\mathcal{P}/I(\Gamma) := \mathcal{P}(\Gamma)/I(\Gamma). \tag{1.1}$$

For a subspace $R \subset \prod_{c \in cor(\mathcal{G})} \mathcal{P}(c)$, we let $\langle R \rangle$ denote the ideal generated by R, i.e. the minimal ideal containing R

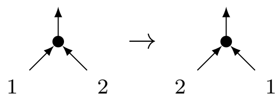
$$(R)(c) := \{\mu \in \mathcal{P} : \mu \in \operatorname{im}(\mathcal{P}(h)|_{\mathcal{P}(\Gamma_1) \otimes R(\Gamma_2)}) \text{ for some } h : \Gamma_1 \sqcup \Gamma_2 \to c\}. \tag{1.2}$$

We may use (1.1) and (1.2) to define examples of props.

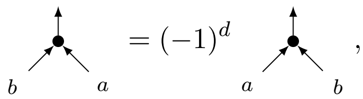
For example, let E_d be the $\mathcal{T}$-collection that maps the 3-valent corolla to a 1-dimensional vector space of degree $1-d$

E_d () $:= \Bbbk[1-d]$,

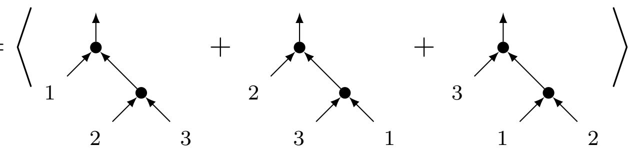
while the unique non-trivial automorphism of the 3-valent corolla

$\sigma :$ 

maps to the sign $(-1)^d$. Then, $\mathcal{F}(E)(n)$ is a graded vector space spanned by trees, generated by the corollas

 (1.3)

Let $J_d \subset \prod_{c \in cor(\mathcal{T})} \mathcal{F}(E_d)(c)$ be the ideal

$J_d =$  , (1.4)

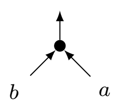
We may then define an operad

$$\mathsf{Lie}_d := \mathcal{F}(E_d)/J_d.$$

Proposition 1.2.14. *Algebras over the operad Lie_d are precisely ($(1-d)$-shifted) Lie algebras.*

Proof. It is evident that for any dg Lie algebra $(V, [-,-], \delta)$, there is a map

$$\mu : \mathsf{Lie}_d \to \operatorname{End}_{(V,\delta)}$$

 $\mapsto [-,-]$.

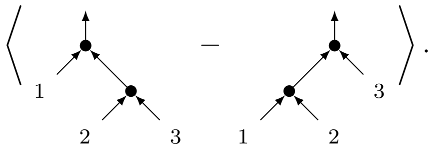
Conversely, each such map gives a dg Lie algebra. □

Proposition 1.2.14 gives the motivation to say that Lie_d is the operad that *governs* Lie algebras. We may define operads governing different types of algebras in a similar fashion. For example, let E_{Ass} be the $\mathbb{S}$-module given by

$$E_{\mathsf{Ass}}(n) := \begin{cases} \mathbb{k}(\mathbb{S}_2) & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

the operad that governs associative algebras is then given by

$$\mathsf{Ass} := \mathcal{F}(E_{\mathsf{Ass}}(n))/$$



1.2.8 Coprops

Using this framework of D.V. Borisov , Y.I. Manin [8] to define props, we may define coprops in the following way:

Definition 1.2.15. *A $\mathcal{GC}$-*coprop *is a contravariant functor*

$$\mathcal{P}^c : (\mathcal{G}, \sqcup) \to (\mathcal{C}, \otimes)$$

that maps grafting morphisms to isomorphisms.

Example 1.2.16. For a dg prop $\mathcal{P}$, where $\mathcal{P}(\Gamma)$ is finite dimensional for each graph Γ, its dual $\mathcal{P}^*$ given by

$$\mathcal{P}^*(\Gamma) := \hom(\mathcal{P}(\Gamma), \mathbb{k})$$

and

$$\mathcal{P}^*(h : \Gamma_1 \to \Gamma_2) := \mathcal{P}(h)^* : \hom(\mathcal{P}(\Gamma_2), \mathbb{k}) \to \hom(\mathcal{P}(\Gamma_1), \mathbb{k})$$

is a coprop.

1.3 Deformation theory of props

In this section we shall recall some concepts from [38] in the slightly more generalized setting of dg props over a symmetric category of graphs $\mathcal{G}$.

1.3.1 The space of derivations $\mathrm{Der}(\mathcal{P})$

In this section, we will define the space of derivations of a dg prop.

Definition 1.3.1. *For a dg prop $\mathcal{P}$ let $\hom^k_{ndg}(\mathcal{P})$ be the set of collections of linear maps $f_\Gamma : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ of degree k, not necessarily commuting with the differential, such that for each two graphs Γ_1, Γ_2 and morphism $h : \Gamma_1 \to \Gamma_2$, the diagram*

$$\begin{CD} \mathcal{P}(\Gamma_1) @>{f_{\Gamma_1}}>> \mathcal{P}(\Gamma_1) \\ @V{\mathcal{P}(h)}VV @VV{\mathcal{P}(h)}V \\ \mathcal{P}(\Gamma_2) @>{f_{\Gamma_2}}>> \mathcal{P}(\Gamma_2) \end{CD}$$

commutes.

Definition 1.3.2. *An element $d \in \hom^k_{ndg}(\mathcal{P})$ is called a* derivation *of $\mathcal{P}$ if the diagram*

$$\begin{CD} \mathcal{P}(\Gamma_1 \sqcup \Gamma_2) @>{d_{\Gamma_1 \sqcup \Gamma_2}}>> \mathcal{P}(\Gamma_1 \sqcup \Gamma_2) \\ @V{\cong}VV @VV{\cong}V \\ \mathcal{P}(\Gamma_1) \otimes \mathcal{P}(\Gamma_2) @>{\mathrm{id} \otimes d_{\Gamma_2} + d_{\Gamma_1} \otimes \mathrm{id}}>> \mathcal{P}(\Gamma_1) \otimes \mathcal{P}(\Gamma_2) \end{CD}$$

commutes for each Γ_1, Γ_2. Here the vertical maps are the coherence maps of the strong monoidal functor $\mathcal{P}$. We denote the set of derivations of a dg prop $\mathcal{P}$ by $\mathrm{Der}(\mathcal{P}) = \prod_{k \in \mathbb{Z}} \mathrm{Der}^k(\mathcal{P})$.

Note that for two derivations $d_1, d_2 \in \mathrm{Der}(\mathcal{P})^k$ and $x, y \in \mathbb{k}$, we have that $xd_1 + yd_2$ is also a derivation of degree k. Hence, $\mathrm{Der}(\mathcal{P})$ is a graded $\mathbb{k}$-vector space.

Proposition 1.3.3. *The space of derivations $\mathrm{Der}(\mathcal{P})$ forms a dg Lie algebra $(\mathrm{Der}(\mathcal{P},), [-, -], \delta)$ with the Lie bracket being*

$$[d_1, d_2] := d_1 \circ d_2 - (-1)^{|d_1||d_2|} d_2 \circ d_1,$$

and the differential being

$$\delta(d) = \delta_{\mathcal{P}} \circ d - (-1)^{|d|} d \circ \delta_{\mathcal{P}}.$$

Here $\delta_{\mathcal{P}}$ denotes the differential on each the dg vector space $\mathcal{P}(c)$, $c \in cor(\mathcal{G})$

Proof. It is clear that $[-, -]$ satisfies the Jacobi identity. However, we need to show that $[f, g]$ is a derivation.

We have

$$\begin{gathered} (\mathrm{id} \otimes f + f \otimes \mathrm{id}) \circ ((\mathrm{id} \otimes g) + g \otimes \mathrm{id}) = \\ = \mathrm{id} \otimes f \circ g + f \otimes g + (-1)^{|f||g|} g \otimes f + f \circ g \otimes \mathrm{id} \end{gathered}$$

and

$$\begin{gathered} (\mathrm{id} \otimes g + g \otimes \mathrm{id}) \circ (\mathrm{id} \otimes f + f \otimes \mathrm{id}) = \\ = \mathrm{id} \otimes g \circ f + g \otimes f + (-1)^{|g||f|} f \otimes g + g \circ f \otimes \mathrm{id}\,. \end{gathered}$$

This gives

$$\begin{gathered}[f\otimes \mathrm{id}+\mathrm{id}\otimes f, g\otimes \mathrm{id}+\mathrm{id}\otimes g]= \\ =(\mathrm{id}\otimes f+f\otimes \mathrm{id})\circ((\mathrm{id}\otimes g)+g\otimes \mathrm{id})-(-1)^{|f||g|}(\mathrm{id}\otimes g+g\otimes \mathrm{id})\circ(\mathrm{id}\otimes f+f\otimes \mathrm{id})= \\ =\mathrm{id}\otimes(f\circ g-(-1^{|f||g|})g\circ f)+(f\circ g-(-1^{|f||g|})g\circ f)\otimes \mathrm{id}= \\ =\mathrm{id}\otimes[f,g]+[f,g]\otimes \mathrm{id},\end{gathered}$$

which is what we want. Note that all terms $\pm(f\otimes g)$ and $\pm(g\otimes f)$ cancel out.

It follows that the diagram

$$\begin{array}{ccc}\mathcal{P}(\Gamma_1\sqcup\Gamma_2) & \xrightarrow{[f,g]_{\Gamma_1\sqcup\Gamma_2}} & \mathcal{P}(\Gamma_1\sqcup\Gamma_2) \\ \downarrow\cong & & \downarrow\cong \\ \mathcal{P}(\Gamma_1)\otimes\mathcal{P}(\Gamma_2) & \xrightarrow{\mathrm{id}\otimes[f,g]_{\Gamma_2}+[f,g]_{\Gamma_1}\otimes\mathrm{id}} & \mathcal{P}(\Gamma_1)\otimes\mathcal{P}(\Gamma_2)\end{array}$$

commutes for all derivations f, g, i.e. $[f,g]$ is a derivation.

The differential is compatible with the Lie bracket, as it is given by

$$\delta=[\delta_{\mathcal{P}},-].$$

Furthermore, $\delta^2=0$ as $[\delta_{\mathcal{P}},\delta_{\mathcal{P}}]=0$. □

Proposition 1.3.4. *There is a one-to-one correspondence between maps of $\mathcal{G}$-collections $d : E\to\mathcal{F}(E)$ and derivations $d\in \mathrm{Der}(\mathcal{F}(E))$, such that*

$$\begin{array}{ccc}E & & \\ \downarrow i & \searrow d & \\ \mathcal{F}(E) & \xrightarrow{d} & \mathcal{F}(E)\end{array}$$

commutes.

Proof. For any map of $\mathcal{G}$-collections $d : E\to\mathcal{F}(E)$, we get a derivation

$$d:\mathcal{F}(E)(c)\cong\bigoplus_{n,m\geq 0}\left(\bigoplus_{\Gamma\in\mathcal{G}_{n,m}(c)}\bigotimes_{v\in V_\Gamma}E(v)\right)^{\mathbb{S}^{\mathcal{G}}_{n,m}}\to$$

$$\to\bigoplus_{n,m\geq 0}\left(\bigoplus_{\Gamma\in\mathcal{G}_{n,m}(c)}\bigotimes_{v\in V_\Gamma}\mathcal{F}(E)(v)\right)^{\mathbb{S}^{\mathcal{G}}_{n,m}}\to\mathcal{F}(E)(c)$$

by

$$(\mu_1\otimes\ldots\otimes\mu_n)\mapsto\sum_{\alpha=1}^{n} i\mu_1\otimes\ldots\otimes i\mu_{\alpha-1}\otimes d\mu_\alpha\otimes i\mu_{\alpha+1}\otimes i\mu_n.$$

This is the only derivation that restricts to $d : E\to\mathcal{F}(E)$, as any other such derivation

$$d' : \mathcal{F}(E)\to\mathcal{F}(E)$$

would give a derivation $d'-d$ that restricts to $0 : E\to\mathcal{F}(E)$. It is clear that such a derivation must be 0. □

1.3.2 Minimal models

For a dg prop $\mathcal{P}$, let $H(\mathcal{P})$ be the dg prop given by

$$H(\mathcal{P})(\Gamma) := H(\mathcal{P}(\Gamma)).$$

A dg prop $\mathcal{P}_\infty$ is called *quasi-free* if

$$\mathcal{P}_\infty(c) := (\mathcal{F}(E)(c), \delta),$$

where $(\mathcal{F}(E), 0)$ is the free prop over some $\mathcal{G}$-collection E, and δ is a differential on the space $\mathcal{F}(E)(c)$.

Definition 1.3.5. *A* model *of a dg prop $\mathcal{P}$ is a quasi-free dg prop $\mathcal{P}_\infty$ with a quasi-isomorphism*

$$\epsilon : \mathcal{P}_\infty \to \mathcal{P}.$$

As any derivation on a free prop $\mathcal{F}(E)$, the differential δ of a quasi-free dg prop is uniquely determined by its restriction

$$\delta : E \to \mathcal{F}(E).$$

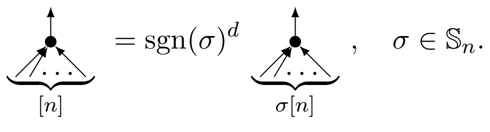
Let $\mathcal{F}^{(n)}(E) \subset \mathcal{F}(E)$ be the subspace spanned by decoreted graphs containing n vertices. Let $\delta^{(n)}$ be the part of $\delta (= \sum_{n\geq 1} \delta^{(n)})$ whose image lies in $\mathcal{F}^{(n)}(E)$. We say that δ is *decomposable*, if $\delta^{(1)} = 0$.

Definition 1.3.6. *A model $(\mathcal{F}(E), \delta)$ of a prop $\mathcal{P}$ is called a* minimal model *if the differential δ is decomposable.*

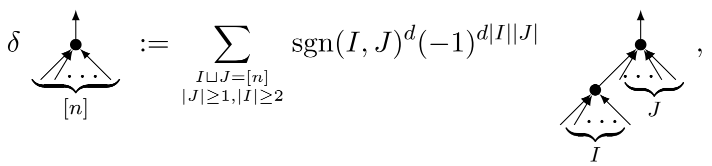
For example, the operad Lie_d admits a minimal model holie_d, first considered by V. Hinich, V. Schechtman [20]. It is given by $\mathsf{hoLieB}_d(c) := (\mathcal{F}(\mathcal{E}_d), \delta)$, where $\mathcal{E}_d$ is the $\mathbb{S}$-module given by

$$\mathcal{E}_d\,(n) := \begin{cases} \mathbb{k}[(1-n)d+1] & n \geq 2 \\ 0 & n \leq 1, \end{cases},$$

and $\mathbb{S}_n$ acts on $\mathcal{E}_d(n)$ by the sign



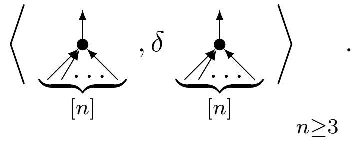
The differential δ is induced by



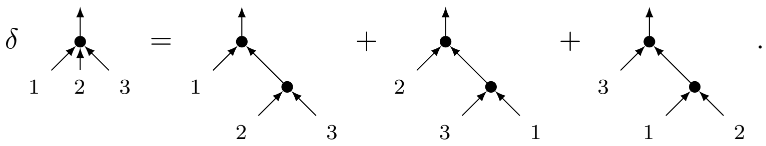
where $\mathrm{sgn}(I,J)$ is the sign of the unshuffle permutation

$$\sigma_{I,J} : [n] \mapsto I \sqcup J.$$

Let $\mathcal{R}_d$ be the ideal generated by



Note that



Hence, we have that $\mathsf{holie}_d/\mathcal{R}_d \cong \mathsf{Lie}_d$.

Theorem 1.3.7 (V. Hinich, V. Schechtman, [20])**.** *The operad holie_d is a minimal model of Lie_d.*

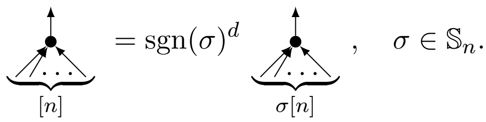
We shall also consider a quasi-free operad, called the *extended homotopy Lie operad*

$$\mathsf{holie}_d^+(c) := (\mathcal{F}(\mathcal{E}_d^+)(c), \delta),$$

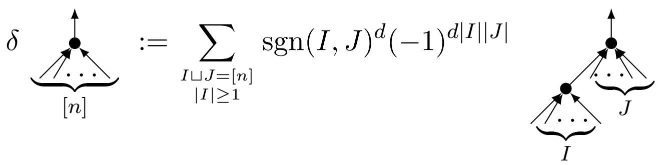
where $\mathcal{E}_d^+$ is the $\mathbb{S}$-module given by

$$\mathcal{E}_d^+(n) := \begin{cases} \mathbb{k}[(1-n)d+1] & n \geq 1 \\ 0 & n = 0, \end{cases},$$

and $\mathbb{S}_n$ again acts on $\mathcal{E}_d^+(n)$ by the sign



The differential δ is induced by



The difference between holie_d and holie_d^+ being that holie_d^+ has an additional generator

of degree 1, and the differential creates such corollas. This difference makes holie_d^+ acyclic.

Representations

$$\mu : \mathsf{holie}_d^+ \to \mathrm{End}_{(V,\delta)}$$

are $(1-d)$-shifted homotopy Lie algebras on the dg vector space $(V, \delta + \mu($  $))$.

1.3.3 The deformation complex

For maps of $\mathcal{G}$-collections $f_1, f_2, \ldots, f_n : E_1 \to E_2$ and a graph $\Gamma \in \mathcal{G}_{n,m}$, consider the map

$$\begin{array}{rcl} [f_1, \ldots, f_n]_\Gamma : \bigotimes_{v \in V_\Gamma} E_1(v) & \to & \bigotimes_{v \in V_\Gamma} E_2(v) \\ \mu_1 \otimes \ldots \otimes \mu_n & \mapsto & \sum_{\sigma \in \mathbb{S}_n} \pm f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}(\mu_1 \otimes \ldots \otimes \mu_n), \end{array}$$

where the signs $\pm$ are the Koszul signs of the permutation $f_1 \otimes \ldots \otimes f_n \mapsto f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}$. For an isomorphism of graphs $\tau : \Gamma_1 \to \Gamma_2$, it is clear that the following diagram commutes

$$\begin{array}{ccc} E_1(\Gamma_1) & \xrightarrow{[f_1,\ldots,f_n]_{\Gamma_1}} & E_2(\Gamma_1) \\ \downarrow E(\tau) & & E(\tau) \downarrow \\ E_1(\Gamma_2) & \xrightarrow{[f_1,\ldots,f_n]_{\Gamma_2}} & E_2(\Gamma_2). \end{array}$$

Hence, the collection of maps $\{[f_1, \ldots, f_n]_\Gamma\}_{\Gamma \in \mathcal{G}_{n,m}}$ gives a map

$$\underline{[f_1, \ldots, f_n]} : \mathcal{F}^{(n)}(E_1) \to \mathcal{F}^{(n)}(E_2).$$

For a quasi-free $\mathcal{G}$-dg prop $\mathcal{P}_\infty := (\mathcal{F}(E), \delta_{\mathcal{P}_\infty})$ and a $\mathcal{G}$-dg prop $(\mathcal{Q}, \delta_{\mathcal{Q}})$, we let the *deformation complex* $\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{0} \mathcal{Q})$, be the dg vector space spanned by (degree shifted) $\mathcal{G}$-collection maps

$$\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{0} \mathcal{Q}) := \mathrm{Hom}_{\mathbb{S}^{\mathcal{G}}}(E, \mathcal{Q})[-1],$$

with the differential $\delta(f) := \delta_Q \circ f + [f]$, where $[f]$ is the following map of $\mathcal{G}$-collections

$$(-1)^{|f|+1}[f] : E \xrightarrow{\delta^{(1)}_{\mathcal{P}_\infty}} \mathcal{F}^{(1)}(E) \xrightarrow{\underline{[f]}} \mathcal{F}^{(n)}(\mathcal{Q}) \xrightarrow{con} \mathcal{Q}\,.$$

This is equipped with a holie_1 structure, given by letting $[f_1, \ldots, f_n]$ be the following map of $\mathcal{G}$-collections

$$(-1)^{|f_1|+\ldots+|f_n|+1}[f_1, \ldots, f_n] : E \xrightarrow{\delta^{(n)}_{\mathcal{P}_\infty}} \mathcal{F}^{(n)}(E) \xrightarrow{\underline{[f_1,\ldots,f_n]}} \mathcal{F}^{(n)}(\mathcal{Q}) \xrightarrow{con} \mathcal{Q}\,.$$

Proposition 1.3.8. *The deformation complex* $\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{0} \mathcal{Q})$ *is indeed an homotopy Lie algebra.*

Proof. We will show that $\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{0} \mathcal{Q})[1]$ is a shifted homotopy Lie algebra, i.e.

$$\sum_{I\sqcup J=[n]} \pm[[f_{i_1},\dots,f_{i_{|I|}}],f_{j_1},\dots,f_{j_{|J|}}] \pm \delta_{\mathcal{Q}}[f_1,\dots,f_n] + \sum_{i=1}^{n} \pm[\delta_{\mathcal{Q}}(f_i),f_1,\dots,f_{i-1},f_{i+1},\dots,f_n] = 0,$$

and the signs $\pm$ arise from the Koszul sign rule of the permutations

$$(f_1,\dots f_n) \mapsto f_{i_1},\dots,f_{i_{|I|}},f_{j_1},\dots,f_{j_{|J|}} \text{ and } f_i,f_1,\dots,f_{i-1},f_{i+1},\dots,f_n.$$

As $\delta_{\mathcal{Q}}$ is a derivation, we have that

$$\delta_{\mathcal{Q}}[f_1,\dots,f_n] + \sum_{i=1}^{n} \pm[\delta_{\mathcal{Q}}(f_i),f_1,\dots,f_{i-1},f_{i+1},\dots,f_n] = 0.$$

The other terms

$$\sum_{I\sqcup J=[n]} \pm[[f_{i_1},\dots,f_{i_{|I|}}],f_{j_1},\dots,f_{j_{|J|}}]$$

is the map

$$E \xrightarrow{(\delta^2)^{(n)}} \mathcal{F}^{(n)}(E) \xrightarrow{\sum_{I\sqcup J=[n]} \pm[[f_{i_1},\dots,f_{i_{|I|}}],f_{j_1},\dots,f_{j_{|J|}}]} \mathcal{F}^{(n)}(\mathcal{Q}) \xrightarrow{con} \mathcal{Q},$$

where $(\delta^2)^{(n)}$ is the weight n part of δ^2. As $\delta^2 = 0$, we must also have that

$$\sum_{I\sqcup J=[n]} \pm[[f_{i_1},\dots,f_{i_{|I|}}],f_{j_1},\dots,f_{j_{|J|}}] = 0.$$

It follows that these brackets induce a holie_0 structure on $\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{0} \mathcal{Q})[1]$ and, therefore, a holie_1 structure on $\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{0} \mathcal{Q})$. ☐

Proposition 1.3.9. *A* $\mathcal{G}$*-collection map* $E \to \mathcal{Q}$ *induces a morphism of dg props* $\bar{f} : \mathcal{P}_\infty \to \mathcal{Q}$ *if and only if* f *is a Maurer Cartan element* $\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{0} \mathcal{Q})$*, i.e. a* $\mathcal{G}$*-collection map* $f : E \to \mathcal{Q}$ *of degree* 0*, such that*

$$\delta_{\mathcal{Q}} \circ f - \sum_{k\geq 1} \frac{1}{k!}[\underbrace{f,\dots,f}_{k}] = 0.$$

Proof. A $\mathcal{G}$-collection map $f : E \to \mathcal{Q}$ of degree 0, gives a map of non-dg props

$$\bar{f} : \mathcal{F}(E) \to \mathcal{Q}.$$

One can see that

$$\bar{f}\delta_{\mathcal{P}_\infty} = \sum_{k\geq 1} \frac{1}{k!}[\underbrace{f,\dots,f}_{k}].$$

Hence, we have that $\bar{f}\delta_{\mathcal{P}_\infty} = \delta_{\mathcal{Q}}\bar{f}$ if and only if f is an MC-element. ☐

By proposition 1.3.9, and the fact that each MC element gives a twisted differential [39], we have that each morphism $f : \mathcal{P}_\infty \to \mathcal{Q}$ gives a twisted differential on $\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{0} \mathcal{Q})$ given by

$$\delta_f(g) := \sum_{k \geq 1} \frac{1}{(k+1)!}[g, \underbrace{f, \ldots, f}_{k}].$$

We say that

$$\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{f} \mathcal{Q}) := (\mathrm{Hom}_{\mathbb{S}_\mathcal{G}}(E, \mathcal{Q})[-1], \delta + \delta_f)$$

is the deformation complex of the map $f : \mathcal{P}_\infty \to \mathcal{Q}$.

Note that

$$\mathsf{Def}(\mathcal{P}_\infty \xrightarrow{\mathrm{id}} \mathcal{P}_\infty) \cong \mathrm{Der}(\mathcal{P}_\infty)[-1]$$

as dg vector spaces. However, their (homotopy) Lie structures are different.

1.4 Multi-directed and oriented versions of Kontsevich graph complex

In this section we will give definitions of the multi-directed and multi-oriented versions of Kontsevich graph complex.

1.4.1 A vector space of graphs

Let $\mathrm{gra}^{k\uparrow l}_{n,m} := \mathcal{G}^{k\uparrow l}_{n,m}(\bullet)$, where $\bullet$ is the corolla without any legs. The set $\mathrm{gra}^{k\uparrow l}_{n,m}$ consists of graphs Γ in $\mathcal{G}^{k\uparrow l}$ with vertex set $V_\Gamma = [n]$, set of flags $F_\Gamma = [m]_1 \sqcup [m]_2$ and involution $\iota_\Gamma : [m]_1 \sqcup [m]_2 \to [m]_1 \sqcup [m]_2$ by $i_1 \mapsto i_2$, $i_2 \mapsto i_1$.

We may say that an element (graph) $\Gamma \in \mathrm{gra}^{k\uparrow l}_{n,m}$ consists of two maps

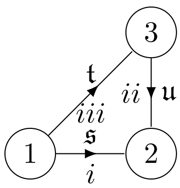
$$\Gamma = (\partial_\Gamma : [m]_1 \sqcup [m]_2 \to [n], Or_\Gamma : [m]_1 \sqcup [m]_2 \to \mathcal{O}r_k)$$

with $Or_\Gamma(i_1) = (Or_\Gamma(i_2))^{opp}$, and no closed paths in the first l directions.

We may also realize a graph $\Gamma \in \mathrm{gra}^{k\uparrow l}_{n,m}$ pictorially by:

1. Drawing n labeled vertices, and a line labeled by i between $\partial_\Gamma(i_1)$ to $\partial_\Gamma(i_2)$ for each $i \in [m]$.
2. Drawing an arrow on each line indicating the intrinsic orientation of the edge, and adding a proper labeling for the other orientations.

For example:



$= ([3]_1 \sqcup [3]_2, [3]_{(v)}, \partial, \iota_3, Or),$

where

$$\partial(1_1) = 1, \quad \partial(1_2) = 2, \quad \partial(2_1) = 3, \quad \partial(2_2) = 2, \quad \partial(3_1) = 1, \quad \partial(3_2) = 3,$$

we denote the edge (i_1, i_2) by a roman numeral, and

$$(Or(1_1) = \mathfrak{s}, \quad Or(2_1) = \mathfrak{u}, \; Or(3_1) = \mathfrak{t}).$$

The group $\mathbb{P}_m = \mathbb{S}_m \ltimes \mathbb{S}_2^m$ acts naturally on $\mathrm{gra}_{n,m}^{k\uparrow l}$ by permuting the edge labels and flipping the intrinsic direction

$$\begin{aligned} \sigma : \mathrm{gra}_{m,n}^{k\uparrow l} &\to \mathrm{gra}_{m,n}^{k\uparrow l} \\ \Gamma = ([m] \sqcup [m], [n], \partial_\Gamma, \iota, Or_\Gamma) &\mapsto \sigma\Gamma = ([m] \sqcup [m], [n], \partial_\Gamma \circ \sigma^{-1}, \iota, Or_\Gamma \circ \sigma^{-1}). \end{aligned}$$

Let $\mathrm{Gra}_d^{k\uparrow l}(n)$ be the vector space

$$\mathrm{Gra}_d^{k\uparrow l}(n) := \begin{cases} \prod_{m\geq 0} \left(\langle \mathrm{gra}_{n,m}^{k\uparrow l} \rangle \otimes_{\mathbb{P}_m} \mathrm{sgn}_m \right) [m(1-d)] & \text{d even,} \\ \prod_{m\geq 0} \left(\langle \mathrm{gra}_{n,m}^{k\uparrow l} \rangle \otimes_{\mathbb{P}_m} \mathrm{sgn}_2^{\otimes m} \right) [m(1-d)] & \text{d odd.} \end{cases}$$

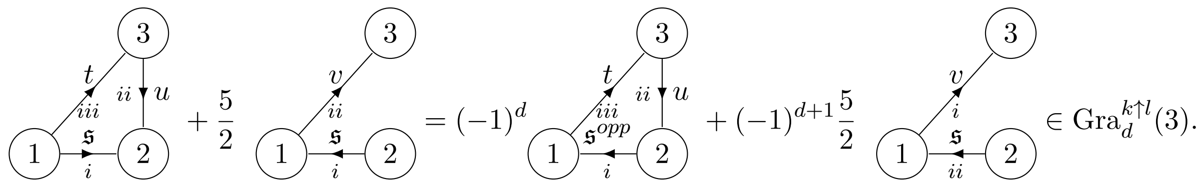
The elements in $\mathrm{Gra}_d^{k\uparrow l}(n)$ are linear combinations of graphs in $\mathrm{gra}_{n,-}^{k\uparrow l}$ modulo

$$\Gamma = \mathrm{sgn}_d^{\mathbb{P}_m}(\sigma)\sigma\Gamma,$$

for any $\sigma \in \mathbb{P}_m$, where

$$\mathrm{sgn}_d^{\mathbb{P}_m}(\sigma, \sigma_2) := \begin{cases} \mathrm{sgn}_m(\sigma_1) & d \text{ even} \\ \mathrm{sgn}_2^{\otimes n}(\sigma_2) & d \text{ odd.} \end{cases}$$

For example



1.4.2 An operad of graphs

For two graphs $\Gamma_1 \in \mathrm{gra}_{n_1,m_1}^{k\uparrow l}$, $\Gamma_2 \in \mathrm{gra}_{n_2,m_2}^{k\uparrow l}$, let

$$\mathrm{gra}_{\Gamma_1 \circ_i \Gamma_2} \subset \mathrm{gra}_{n_1+n_2-1, m_1+m_2}$$

be the set of graphs Γ, such that:

1. The subgraph Γ_2' of Γ, with vertices $i, i+1, \ldots, i+n_2-1$ and edges $m_1+1, \ldots, m_1+m_2$ is isomorphic to Γ_2, by a map that preserves the order of the vertices and the edges.

2. There is an isomorphism $\Gamma/\Gamma_2' \to \Gamma_1$ by collapsing the vertex set of Γ/Γ_2' to $\{1,\dots\{i,i+1,\dots,i+n_2-1\},i+n_2,\dots n_1+n_2-1\} \to [n_1]$.

More precisely, for two graphs

$$\Gamma_1 = (F_{\Gamma_1},[n_1],\partial_1,\iota_1,Or_1),$$

$$\Gamma_2 = (F_{\Gamma_2},[n_2],\partial_2,\iota_2,Or_2)$$

let $\mathrm{gra}_{\Gamma_1\circ_i\Gamma_2}$ be the set consisting of graphs

$$\Gamma = (F_{\Gamma_1}\sqcup F_{\Gamma_2},[n_1+n_2-1],\partial_\Gamma,\iota_1\sqcup\iota_2,Or_1\sqcup Or_2),$$

such that

$$\partial_\Gamma(f) \begin{cases} =\partial_1(f) & \text{if } f\in\partial_1^{-1}(\{1,\dots i-1\}) \\ \in\{i,i+1,\dots,i+n_2-1\} & \text{if } f\in\partial_1^{-1}(i) \\ =\partial_1(f)+n_2-1 & \text{if } f\in\partial_1^{-1}(\{i+1,\dots n_1\}) \\ =\partial_2(f)+i-1 & \text{if } f\in F_{\Gamma_2}. \end{cases}$$

We define the composition maps by

$$\begin{aligned} \circ_i : \mathrm{Gra}_d^{k\uparrow l}(n_1)\otimes\mathrm{Gra}_d^{k\uparrow l}(n_2) &\to \mathrm{Gra}_d^{k\uparrow l}(n_1+n_2-1) \\ \Gamma_1\otimes\Gamma_2 &\mapsto \Gamma_1\circ_i\Gamma_2 := \sum_{\Gamma\in\mathrm{gra}_{\Gamma_1\circ_i\Gamma_2}}\Gamma \end{aligned}$$

Pictorially, the element $\Gamma_1\circ_i\Gamma_2$ is obtained from Γ_1 and Γ_2 by removing the i'th vertex from Γ_1 and summing over all ways to reconnect the edges that were connected to vertex i to Γ_2.

Proposition 1.4.1. *The maps* $\circ_i : \mathrm{Gra}(n_1)\otimes\mathrm{Gra}(n_2)\to\mathrm{Gra}(n_1+n_2-1)$ *are well defined and they are indeed operadic composition maps.*

Proof. For $\Gamma_1\in\mathrm{gra}^{k\uparrow l}_{n_1,m_1},\Gamma_2\in\mathrm{gra}^{k\uparrow l}_{n_2,m_2}$, and $\sigma\in\mathbb{P}_{m_1}$, $\tau\in\mathbb{P}_{m_2}$, we have

$$\sigma\Gamma_1\circ_i\tau\Gamma_2 = \sum_{\Gamma'\in\mathrm{gra}_{\sigma\Gamma_1\circ_i\tau\Gamma_2}}\Gamma' = \sum_{\Gamma\in\mathrm{gra}_{\Gamma_1\circ_i\Gamma_2}}(\sigma\times\tau)\Gamma$$

Hence, the maps are well defined on

$$\bigoplus_{m,n}\Bbbk\langle\mathrm{gra}^{k\uparrow l}_{n,m}[m(d-1)]\rangle/(\Gamma+\mathrm{sgn}_d(\sigma)\sigma\Gamma)_{\sigma\in\mathbb{P}_m,\Gamma\in\mathrm{gra}^{k\uparrow l}_{-,m}} \cong \bigoplus\mathrm{Gra}_d^{k\uparrow l}(n).$$

To show that the maps $\circ_i$ define operadic composition maps, we need to show that

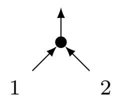
$$(\Gamma_1\circ_i\Gamma_2)\circ_j\Gamma_3 = \begin{cases} (-1)^{|\Gamma_2||\Gamma_3|}(\Gamma_1\circ_j\Gamma_3)\circ_{i+n_3-1}\Gamma_2) & j<i \\ \Gamma_1\circ_i(\Gamma_2\circ_{j-i+1}\Gamma_3) & i\le j\le j+i-1 \\ (-1)^{|\Gamma_2||\Gamma_3|}(\Gamma_1\circ_{j-n_2+1}\Gamma_3)\circ_i\Gamma_2) & i+n_2\le j\le n_1+n_2-1. \end{cases}$$

In each case, one can see that the graphs on the left hand side differ from the graphs on the right hand side by a permutation $\sigma\in\mathbb{P}_m$, which gives the desired sign. □

1.4.3 The graph complex

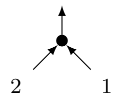
Proposition 1.4.2. *There is a map of operads*

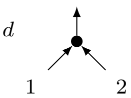
$$s : \mathsf{Lie}_d \;\to\; \mathrm{Gra}_d^{k\uparrow l}$$



$$\mapsto \quad \sum_{\mathfrak{s}\in\mathcal{O}r^k} \textcircled{1} \xrightarrow{\mathfrak{s}} \textcircled{2}.$$

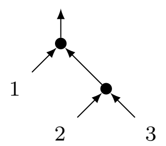
Proof. We have

$$s\Big($$  $$\Big) = \sum_{\mathfrak{s}\in\mathcal{O}r^k} \textcircled{2} \xrightarrow{\mathfrak{s}} \textcircled{1} = \sum_{\mathfrak{s}\in\mathcal{O}r^k} (-1)^d \textcircled{2} \xleftarrow{\mathfrak{s}^{opp}} \textcircled{1} =$$

$$= \sum_{\mathfrak{s}\in\mathcal{O}r^k} (-1)^d \textcircled{2} \xleftarrow{\mathfrak{s}} \textcircled{1} = s\Big((-1)^d$$  $$\Big).$$

Hence, s gives a well defined map of operads $\mathcal{F}(E_d) \to \mathrm{Gra}_d^{k\uparrow l}$, where E_d is the generating $\mathbb{S}$-module of Lie_d. It remains to show that $s(J_d) = 0$.

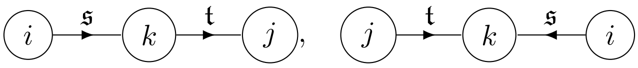
We have

$$s\Big($$  $$\Big) = \sum_{\mathfrak{s}_1,\mathfrak{s}_2\in\mathcal{O}r^k} \Big(\textcircled{1} \xrightarrow{\mathfrak{s}_1} \textcircled{2} \xrightarrow{\mathfrak{s}_2} \textcircled{3} + \textcircled{1} \xrightarrow{\mathfrak{s}_1} \textcircled{3} \xleftarrow{\mathfrak{s}_2} \textcircled{2} \Big).$$

It follows that

$$\begin{aligned} s(J_d) = \\ \sum_{\mathfrak{s}_1,\mathfrak{s}_2\in\mathcal{O}r^k} \Big(& \textcircled{1} \xrightarrow{\mathfrak{s}_1} \textcircled{2} \xrightarrow{\mathfrak{s}_2} \textcircled{3} + \textcircled{1} \xrightarrow{\mathfrak{s}_1} \textcircled{3} \xleftarrow{\mathfrak{s}_2} \textcircled{2} + \\ & + \textcircled{2} \xrightarrow{\mathfrak{s}_1} \textcircled{3} \xrightarrow{\mathfrak{s}_2} \textcircled{1} + \textcircled{2} \xrightarrow{\mathfrak{s}_1} \textcircled{1} \xleftarrow{\mathfrak{s}_2} \textcircled{3} + \\ & + \textcircled{3} \xrightarrow{\mathfrak{s}_1} \textcircled{1} \xrightarrow{\mathfrak{s}_2} \textcircled{2} + \textcircled{3} \xrightarrow{\mathfrak{s}_1} \textcircled{2} \xleftarrow{\mathfrak{s}_2} \textcircled{1} \Big) = \\ \sum_{\mathfrak{s}_1,\mathfrak{s}_2\in\mathcal{O}r^k} \Big(& \textcircled{1} \xrightarrow{\mathfrak{s}_1} \textcircled{2} \xrightarrow{\mathfrak{s}_2} \textcircled{3} + \textcircled{3} \xrightarrow{\mathfrak{s}_1} \textcircled{2} \xleftarrow{\mathfrak{s}_2} \textcircled{1} + \\ & + \textcircled{2} \xrightarrow{\mathfrak{s}_1} \textcircled{3} \xrightarrow{\mathfrak{s}_2} \textcircled{1} + \textcircled{1} \xrightarrow{\mathfrak{s}_1} \textcircled{3} \xleftarrow{\mathfrak{s}_2} \textcircled{2} + \\ & + \textcircled{3} \xrightarrow{\mathfrak{s}_1} \textcircled{1} \xrightarrow{\mathfrak{s}_2} \textcircled{2} + \textcircled{2} \xrightarrow{\mathfrak{s}_1} \textcircled{1} \xleftarrow{\mathfrak{s}_2} \textcircled{3} \Big). \end{aligned}$$

Each row in the last expression vanishes as the graphs



differ by one swap of edge labels and one flip of the intrinsic orientation. □

Let $\bar{s} : \mathsf{holie}_d^+ \to \mathrm{Gra}_d^{k\uparrow l}$ be the map

$$\bar{s} : \mathsf{holie}_d^+ \xrightarrow{\epsilon} \mathsf{Lie}_d \xrightarrow{s} \mathrm{Gra}_d^{k\uparrow l}.$$

We define the k-directed, l-oriented version of Kontsevich full graph complex $\mathrm{fGC}_d^{k\uparrow l}$ to be the deformation complex

$$\mathrm{fGC}_d^{k\uparrow l} := \mathsf{Def}(\mathsf{holie}_d^+ \xrightarrow{\bar{s}} \mathrm{Gra}_d^{k\uparrow l}).$$

Formally, an element Γ is a map of $\mathbb{S}$-modules

$$\Gamma : \prod_{n\geq 1} \mathcal{E}_d^+ \to \mathrm{Gra}_d^{k\uparrow l}(n),$$

where $\mathcal{E}_d^+$ is the generating $\mathbb{S}$-module of the quasi-free operad holie_d^+.

Consider the map of graded vector spaces

$$p : \prod_{n\geq 1} \mathrm{Gra}^{k\uparrow l}(n)[(n-1)d + (1-d)m] \to \mathrm{fGC}_d^{k\uparrow l},$$

given by

$$p(\Gamma) := \sum_{\sigma\in\mathbb{S}_n} \mathrm{sgn}(\sigma)^{d+1}\sigma\Gamma$$

The map p is clearly surjective, and its kernel is given by

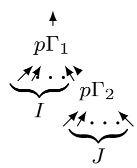
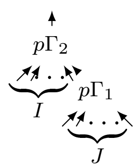
$$\ker(p) = \prod_{n,m} \Bbbk(\{\Gamma - \mathrm{sgn}_d^{\mathbb{S}_\mathcal{G}}(\sigma)\Gamma \ : \ \Gamma \in \mathrm{gra}_{n,m}, \sigma \in \mathbb{S}_\mathcal{G}\})[(n-1)d + (1-d)m],$$

where

$$\mathrm{sgn}_d(\sigma_v, \sigma_{e_1}, \sigma_{e_2}) = \mathrm{sgn}(\sigma_v)^d \,\mathrm{sgn}(\sigma_{e_1})^{d+1}\,\mathrm{sgn}(\sigma_{e_2})^d, \quad (\sigma_v, \sigma_{e_1}, \sigma_{e_2}) \in \mathbb{S}_n \times \mathbb{S}_m \ltimes \mathbb{S}_2^{\times m} = \mathbb{S}_\mathcal{G}.$$

If d is even, $\mathrm{fGC}_d^{k\uparrow l}$ is spanned by graphs in $\mathrm{gra}_{n,m}^{k\uparrow l}$ with symmetrized vertices and skew symmetrized edges. If d is odd, $\mathrm{fGC}_d^{k\uparrow l}$ is spanned by graphs in $\mathrm{gra}_{n,m}^{k\uparrow l}$ with skew symmetrized vertices and symmetrized edges.

For two graphs $\Gamma_1 \in \mathrm{gra}_{n_1,m_1}^{k\uparrow l}$, $\Gamma_2 \in \mathrm{gra}_{n_2,m_2}^{k\uparrow l}$, the Lie bracket in $\mathrm{fGC}^{k\uparrow l}$ is given by

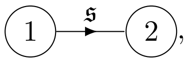
$$[p\Gamma_1, p\Gamma_2] = \sum_{I\sqcup J=[n_1+n_2-1]} \pm$$  $\pm$  $=$

$$= p\left(\sum_{i=1}^{n_1}\sum_{\Gamma\in \mathrm{gra}_{\Gamma_1\circ_i\Gamma_2}}\Gamma - \sum_{i=1}^{n_2}\sum_{\Gamma\in \mathrm{gra}_{\Gamma_2\circ_i\Gamma_1}}\Gamma\right).$$

Hence, we may write

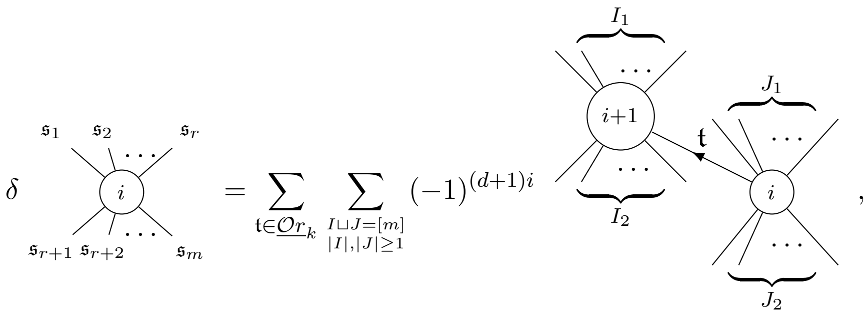
$$[\Gamma_1,\Gamma_2] := \sum_{i=1}^{n_1}\sum_{\Gamma\in \mathrm{gra}_{\Gamma_1\circ_i\Gamma_2}}\Gamma - \sum_{i=1}^{n_2}\sum_{\Gamma\in \mathrm{gra}_{\Gamma_2\circ_i\Gamma_1}}\Gamma$$

for two graphs $\Gamma_1 \in \mathrm{gra}^{k\uparrow l}_{n_1,m_1}$, $\Gamma_2 \in \mathrm{gra}^{k\uparrow l}_{n_2,m_2}$, representing elements in fGC_d. The map $\bar{s}[1] : \mathcal{E}^+_d[1] \to \mathrm{Gra}_d$ is given by

$$p\sum_{\mathfrak{s}\in\underline{\mathcal{O}r^k}}$$


where $\underline{\mathcal{O}r^k} \subset \mathcal{O}r^k$ is the subset such that $\mathcal{O}r^k = \underline{\mathcal{O}r^k} \sqcup (\underline{\mathcal{O}r^k})^{opp}$, e.g. The first direction agrees with the intrinsic direction for every $\mathfrak{s} \in \underline{\mathcal{O}r^k}$.

The differential δ on $\mathrm{fGC}^{k\uparrow l}$ acts on a graph $\delta p(\Gamma) = p\left(\sum_{i=V_\Gamma}\Gamma\circ_i \bullet\!\!-\!\!\bullet - (\bullet\!\!-\!\!\bullet\circ\Gamma)\right)$. Note that the terms $\bullet\!\!-\!\!\bullet\circ\Gamma$ cancel with the terms in $\Gamma\circ_i\bullet\!\!-\!\!\bullet$ where all the edges connected to i gets reconnected to one of the vertices in $\bullet\!\!-\!\!\bullet$. Hence, δ acts by splitting each vertex in a graph Γ, without creating univalent vertices



where $I_1 = I\cap[r]$, $I_2 = I\cap[r+1,m]$, $J_1 = J\cap[r]$, $J_2 = J\cap[r+1,m]$, and the new edge gets the last edge label.

1.4.4 Trimming of $\mathrm{fGC}^{k\uparrow l}$

In this section, we will define the *Kontsevich k-directed l-oriented graph complex* $\mathrm{GC}^{k\uparrow l}_d \subset \mathrm{fGC}^{k\uparrow l}_d$. It is spanned by fewer graphs than the full graph complex $\mathrm{fGC}^{k\uparrow l}$, but it contains the same information in the cohomology.

Let

$$\mathrm{fGC}^{k\uparrow l,conn}_d \subset \mathrm{fGC}^{k\uparrow l}_d$$

be the sub dg Lie algebra spanned by connected graphs. It is clear that

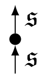
$$\mathrm{fGC}^{k\uparrow l} \cong \widehat{S}(\mathrm{fGC}^{k\uparrow l,conn}),$$

where $\widehat{S}$ is the completed symmetric tensor product.

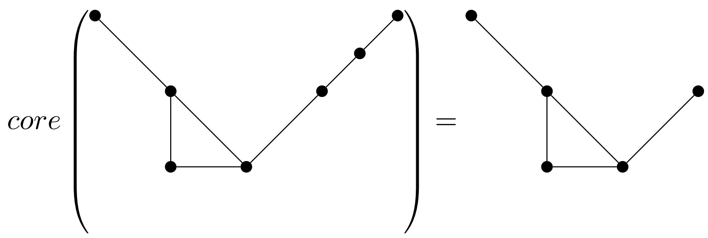
Next, we define the *Kontsevich k-directed l-oriented graph complex* $\mathrm{GC}_d^{k\uparrow} \subset \mathrm{fGC}_d^{k\uparrow l,conn}$ to be the sub dg Lie algebra spanned by graphs, where every vertex is at least 2-valent.

Proposition 1.4.3. *The inclusion* $i : \mathrm{GC}_d^{k\uparrow l} \hookrightarrow \mathrm{fGC}_d^{k\uparrow l,conn}$ *is a quasi-isomorphism.*

Proof. Let an antenna of a graph Γ be a subgraph A, consisting of a univalent vertex v together with a maximal string of passing vertices.

Remark 1.4.4. A *passing vertex* of a multi-oriented graph is a two valent corolla  that is not a source or target in any direction.

We say that an antenna is of length i if it contains i vertices. Let $core(\Gamma)$ be the graph obtained from Γ by contracting all antennas to length 1. For example:



The differential δ cannot decrease the number of vertices in $core(\Gamma)$. Hence, we may take a filtration on $\mathrm{fGC}^{k\uparrow l}$ by the number of vertices in the core. That is

$$F_p(\mathrm{fGC}^{k\uparrow l,conn}) := \{\Gamma \in \mathrm{fGC}^{k\uparrow l,conn} : |V_{core(\Gamma)}| \geq p, \text{ for all single term graphs } \Gamma\}.$$

The associated graded complex $gr(\mathrm{fGC}^{k\uparrow l,conn}, \delta) \cong (\mathrm{fGC}^{k\uparrow l,conn}, \delta_0)$ splits

$$(\mathrm{fGC}^{k\uparrow l,conn}, \delta_0) \cong \bigoplus_{\gamma} (C_\gamma, \delta_0),$$

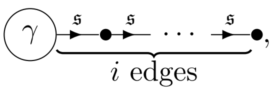
with a subcomplex (C_γ, δ_0), for each graph γ with only length 1 antennas, spanned by graphs Γ with $core(\Gamma) = \gamma$.

Each (C_γ, δ_0) is isomorphic to a tensor product

$$(C_\gamma, \delta_0) \cong \left(\bigotimes_{i \in V_{uni}(\gamma)} (A_i, \delta_0) \right)^{\mathbb{S}_\gamma},$$

where $V_{uni}(\gamma)$ is the set of univalent vertices of γ and where (A_i, δ_0) is the dg vector space that models the antenna associated with i, and $\mathbb{S}_\gamma$ is the automorphism group of γ.

Each A_i is spanned by elements

$a_i :=$ , $\quad i \geq 1$

and

$$\delta a_i = \begin{cases} a_{i+1} & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

It is clear that each (A_i, δ_0) is acyclic. Hence, (C_γ, δ_0) is acyclic unless γ does not have any univalent vertices, in which case $\gamma \in \mathrm{GC}^{k\uparrow l}$, and $C_\gamma = (\Bbbk)^{\mathbb{S}_\gamma}$.

It follows from the mapping lemma for spectral sequences [49] that the inclusion

$$i : \mathrm{GC}_d^{k\uparrow l} \hookrightarrow \mathrm{fGC}_d^{k\uparrow l, conn}$$

is a quasi-isomorphism. □

1.4.5 About the cohomology $H(\mathrm{GC}_d^{k\uparrow l}, \delta)$

In this section, we will present a short survey of known results about the cohomology $H(\mathrm{GC}_d^{k\uparrow l})$.

Theorem 1.4.5 (M. Živković, [54])**.** *For all $k \geq l$, we have*

$$H(\mathrm{GC}_d^{k\uparrow l}) \cong H(\mathrm{GC}_d^{(k+1)\uparrow l}).$$

For all $k > l$, we have

$$H(\mathrm{GC}_d^{k\uparrow l}) \cong H(\mathrm{GC}_{d+1}^{k\uparrow l+1}).$$

Remark 1.4.6. The first statement

$$H(\mathrm{GC}_d^{k\uparrow l}) \cong H(\mathrm{GC}_d^{(k+1)\uparrow l})$$

is rather simple to prove. One can see that there is a map

$$f : \mathrm{GC}_d^{k\uparrow l} \to \mathrm{GC}_d^{(k+1)\uparrow l}$$

by letting $f(\Gamma)$ be the sum of all ways to add an extra direction on each edge. One can then show that f is a quasi-isomorphism.

The second statement

$$H(\mathrm{GC}_d^{k\uparrow l}) \cong H(\mathrm{GC}_{d+1}^{k\uparrow l+1})$$

is less obvious. The special case $H(\mathrm{GC}_d^{1\uparrow 0}) \cong H(\mathrm{GC}_{d+1}^{1\uparrow 1})$ was first proven by T. Willwacher in [52], without giving an explicit map. In [54], Živković construct an explicit quasi-isomorphisms

$$h : (\mathrm{GC}_{d+1}^{(k+1)\uparrow(l+1)}, \delta) \to (\mathrm{GC}_d^{k\uparrow l}, \delta).$$

Perhaps the most astonishing result about the $H(\mathrm{GC}_2, \delta)$ is the following theorem by T. Willwacher.

Theorem 1.4.7 (T. Willwacher, [51])**.** *We have*

$$H^0(\mathrm{GC}_2^{0\uparrow 0}) \cong \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ is the Grothendieck-Teichmüller Lie algebra.

The Deligne-Drinfeld-Ihara conjecture states that the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ is conjectured to be isomorphic to the completed free Lie algebra with generators

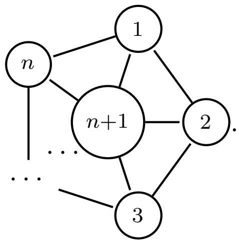
$$\sigma_3, \sigma_5, \ldots, \sigma_{2n+1}, \ldots, \quad n \geq 1.$$

(It is shown by F. Brown [9] that this is a Lie subalgebra of $\mathfrak{grt}_1$.)

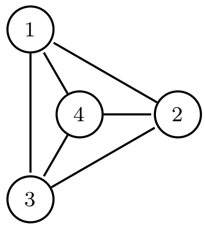
It is shown in [51], that each (conjectural) generator σ_{2n+1} corresponds to a cohomology class in GC_2 represented by

$$W_{2n+1} + (\text{other terms}),$$

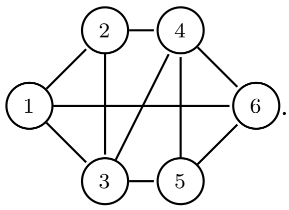
where W_n is the graph

$W_n :=$ 

The class associated to W_3, is simply given by

$W_3 =$ 

The next class, associated to W_5, is given by

$W_5 + \frac{5}{2}$ 

A representative for the cohomology class associated to W_7 is computed by R. Buring, A. Kiselev and N. Rutten in [10], and it is the sum of 46 different graphs. To the author's knowledge, no one has computed the class associated to W_9.

The differential on GC_d preserves the loop order (number of vertices minus number of edges), hence GC splits, with a sub-dg vector space $GC_d^{(p)}$ for each loop order p. The most extensive list (to the author's knowledge) of dimensions of $H(\mathrm{GC}_{even}^{(p)})$ and $H(\mathrm{GC}_{odd}^{(p)})$ is given in [21].

Chapter 2

Multi-directed and oriented Lie bialgebras

Introduction

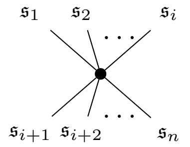
The (1-oriented) properad of homotopy Lie-bialgebras $\mathsf{holieB}_{c,d}$ (with two integer parameters c, d determining the degree shifts of the cobrackets and brackets respectively) is a well known object, defined and studied in e.g. [32], [43], [47]. The multi-oriented versions of $\mathsf{holieB}_{c,d}$, denoted $\mathsf{holieB}_d^{k\uparrow l}$, were defined by S. Merkulov in [35]. In this chapter, we will prove the claim that the Grothendieck-Teichmüller group acts faithfully on each $\mathsf{holieB}_d^{k\uparrow l}$. More specifically, we will show that $\mathrm{GC}_{d+1}^{k\uparrow l}$ acts faithfully on the completed properad $\widehat{\mathsf{holieB}}_d^{k\uparrow l}$ by derivations. This is a generalization of results in [43] and [5], where the 1 oriented and wheeled cases are considered respectively. Combining this result with Theorem 1.4.5 (M. Živković [54]) and Theorem 1.4.7 (T. Willwacher [51]), we get a d dimensional incarnation of the Grothendieck-Teichmüller group for any $d \geq 2$.

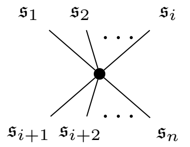
We will also show that the 2-oriented properad $\mathsf{holieB}_d^{2\uparrow 2}$ is a minimal model of the 2-oriented properad $\mathsf{LieB}_d^{2\uparrow 2}$. This proof partly relies on path filtrations, as in [32], originally suggested by M. Kontsevich [27]. However, as we will see that path filtrations do not take us all the way, this proof uses a new type of filtrations on graph complexes.

2.1 The definition of $\mathsf{holieB}^{k\uparrow l}$

2.1.1 The generating $\mathcal{G}^k$-collection

The multi-directed homotopy Lie-bialgebra props are quasi-free, meaning that they admit the structure

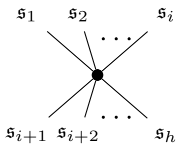
$\mathsf{holieB}_d^{k\uparrow l}$ (

) = ($\mathcal{F}^{k\uparrow l}(\mathcal{E}_d^k)$ (

) $, \delta$) ,

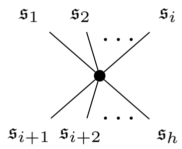
where $\mathcal{E}_d^k$ is a non-differential graded $\mathcal{G}^k$-collection, and δ is a degree 1 derivation on $\mathcal{F}^{k\uparrow l}(\mathcal{E}_d^k)$ that squares to 0. We will start by describing the generating $\mathcal{G}^{k\uparrow l}$-collection $\mathcal{E}^{k\uparrow l}$.

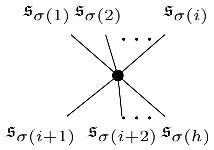
For an integer d, and a map $D : \mathcal{O}r^k \to \mathbb{Z}$ such that $D(\mathfrak{s}) + D(\mathfrak{s}^{opp}) = -d$, let $\mathcal{E}_d^k$ be the $\mathcal{G}^k$-collection given by

$\mathcal{E}_d^k$ 

$:= \mathbb{k}\left[1 + d + \sum_{j=1}^{h} D(\mathfrak{s}_j)\right];$

and for an isomorphism

$\sigma :$ 

$\to$ 

$, \quad \sigma \in \mathbb{S}_h$

we set

$$\mathcal{E}_d^k(\sigma) := \mathrm{sgn}_D(\sigma).$$

Here $\mathrm{sgn}_D(\sigma)$ is the graded sign of the permutation, uniquely determined by the rule $\mathrm{sgn}_D(\sigma\tau) = \mathrm{sgn}_{d+1}(\sigma)\,\mathrm{sgn}_D(\tau)$, and

$$\mathrm{sgn}_D(j(j+1)) = \begin{cases} -1 & \text{if both } D(\mathfrak{s}_j) \text{ and } D(\mathfrak{s}_{j+1}) \text{ are odd} \\ 1 & \text{otherwise,} \end{cases}$$

for all transpositions $(j(j+1))$.

Remark 2.1.1. We omit the D from the notation in $\mathcal{E}_d^k$, as we are almost always only interested in d. Or more specifically the parity of d. However, each $\mathcal{E}_d^k$ is understood to be equipped with a map $D : \mathcal{O}r^k \to \mathbb{Z}$, such that $D(\mathfrak{s}) + D(\mathfrak{s}^{opp}) = -d$. In the 1-directed case, we will, however, write $\mathcal{E}_{c,d}^{1\uparrow 1,0}$ and $\mathsf{holieB}_{c,d}^{1\uparrow 1,0}$. With that, we mean $D(in) = -c$ and $D(out) = -d$.

Let $\mathcal{F}^{k\uparrow l}(\mathcal{E}_d^k)$ be the free k-directed, l-oriented prop over the $\mathcal{G}^k$-collection $\mathcal{E}_d^k$. For a corolla c, we get

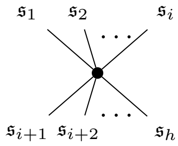
$$\mathcal{F}(\mathcal{E}_d^k)(c) \cong \bigoplus_{m,n\geq 0} \left(\bigoplus_{\Gamma \in \mathcal{G}_{m,n}^{k\uparrow l}(c)} \bigotimes_{v \in V(\Gamma)} \mathcal{E}_d^k(v) \right)^{\mathbb{S}_{n,m}^{\mathcal{G}}} \cong$$

$$\bigoplus_{m,n\geq 0} \Bbbk \left\langle \mathcal{G}^{k\uparrow l}_{n,m}(c) \left[n(d+1) - md + \sum_{i\in L(c)} D(Or(i)) \right] \right\rangle / (\Gamma - \mathrm{sgn}_{d+1}(\sigma)\sigma\Gamma)_{\Gamma\in\mathcal{G}^{k\uparrow l}_{n,m}(c), \sigma\in(\mathbb{S}^{\mathcal{G}}_{n,m})}.$$

In words, $\mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d)(c)$ is spanned by isomorphism classes of graphs with external legs labeled as the legs of c. Note that $\mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d)(\bullet) \cong \mathrm{fGC}^{k\uparrow l}_{d-1}[d+1]$ as graded vector spaces.

2.1.2 The differential

For a corolla

$$([h], \{\bullet\}, i \mapsto \bullet, \mathrm{id}, i \mapsto \mathfrak{s}_i) =$$  ,

consider the map

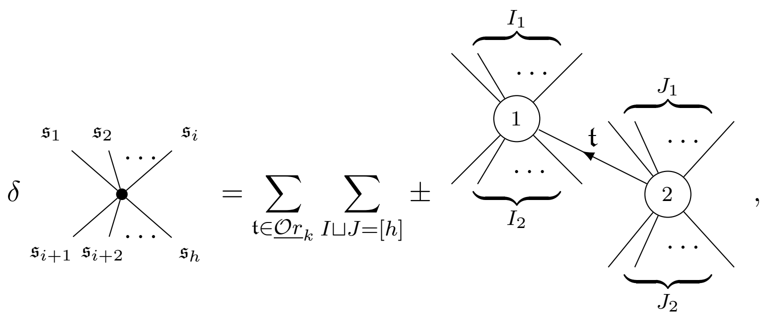
$$\delta : \mathcal{E}^k_d \to \mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d)$$

given by

$$\delta(([h],[1], i\mapsto 1, \mathrm{id}, i \mapsto \mathfrak{s}_i)) := \sum_{\mathfrak{t}\in\underline{\mathcal{O}r^k}} \sum_{I\sqcup J=[h]} ([h]\sqcup[1]_1\sqcup[1]_2, [2], \partial^I_J, \mathrm{id}, (1_1 \mapsto \mathfrak{t} \sqcup 1_2 \mapsto \mathfrak{t}^{opp}) \sqcup (i\mapsto \mathfrak{s}_i)), \quad (2.1)$$

where $\partial^I_J(1_1) = 1$, $\partial^I_J(1_2) = 2$, and $\partial^I_J|_I = 1$, $\partial^I_J|_J = 2$.

In pictures, this is



,

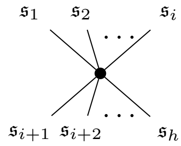
where $I_1 = I\cap[i]$, $I_2 = I\cap[i+1,h]$, $J_1 = I\cap[i]$, $I_1 = I\cap[i+1,h]$.

Remark 2.1.2. In the pictures, we consider the signs to be ambiguous as the order of the flags are not properly specified. Hence, we write the signs $\pm$. The proper signs are indicated in (2.1).

The map $\delta : \mathcal{E}^k_d \to \mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d)$ induces a derivation

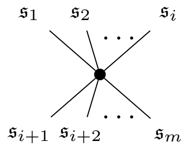
$$\delta : \mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d) \to \mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d).$$

Proposition 2.1.3. *The derivation $\delta : \mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d) \to \mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d)$ is of homological degree 1, and $\delta^2 = 0$.*

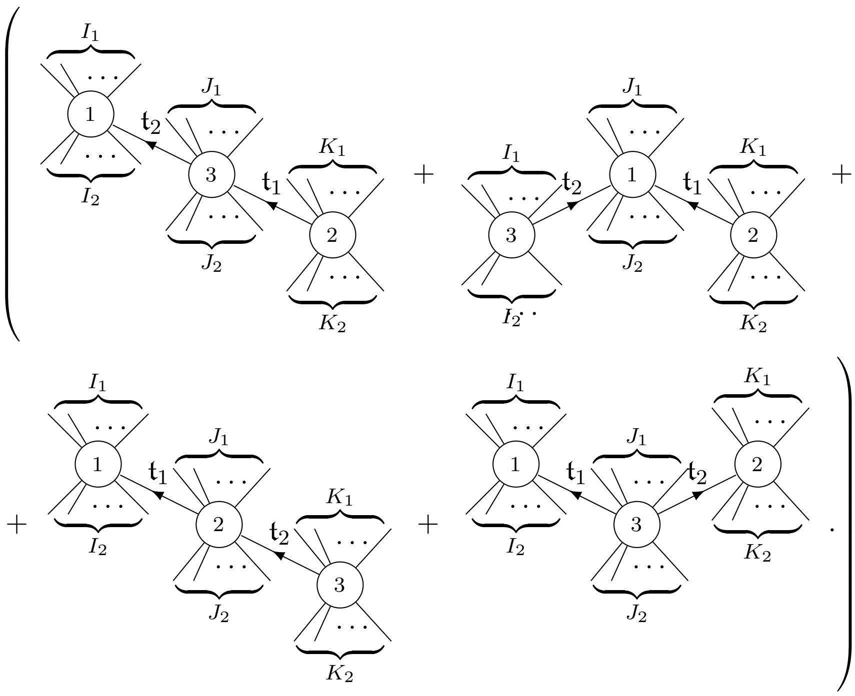
Proof. As δ adds 1 internal edge and 1 vertex, its homological degree is $(d+1)-d=1$. To see that δ squares to 0, it is sufficient to show that $\delta^2(c)=0$, for all corollas $c =$ 

.

We get

$$\sum_{\mathfrak{t}_1,\mathfrak{t}_2\in\underline{\mathcal{O}r}^k} \delta^2$$ 

$=$

$$\sum_{\mathfrak{t}_1,\mathfrak{t}_2\in\underline{\mathcal{O}r}^k} \sum_{I\sqcup J\sqcup K=[h]}$$ 

.

Remark 2.1.4. Here, we say that each picture represents a graph

$$(\underbrace{[h]\sqcup[2]_1\sqcup[2]_2}_{F_\Gamma}, \underbrace{[3]}_{V_\Gamma}, \partial, \iota, \mathcal{O}r).$$

As the numbering of the vertices and edges are indicated, the sign is not ambiguous.

Each term above has an isomorphic 'partner term', and the isomorphism is given by 1 transposition of vertex labels, and the transposition of the internal edge labels. This permutation is odd both when d is even and when d is odd. Hence, each term is canceled by another term, and $\delta^2=0$. ☐

2.1.3 The properads $\mathsf{holieB}^{\star}$, holieB^{+} and holieB

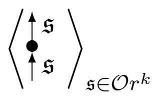
We will consider three different 'completion levels' of the properad of k-directed l-oriented homotopy Lie bialgebras. First, The properad of *full k-directed l-oriented homotopy Lie bialgebras*

$$\mathsf{holieB}_d^{k\uparrow l,\star} := (\mathcal{F}^{k\uparrow l}(\mathcal{E}_d^k), \delta).$$

Second, let ST be the ideal of $\mathsf{holieB}^{k\uparrow l,\star}$ generated by corollas that are sources or targets in some direction. The properad of *extended k-directed l-oriented homotopy Lie bialgebras* $\mathsf{holieB}^{k\uparrow l,+}$ is given by

$$\mathsf{holieB}_d^{k\uparrow l,+} := \mathsf{holieB}^{k\uparrow l,\star}/ST.$$

Next, let I be the ideal of $\mathsf{holieB}^{k\uparrow l,+}$ generated by the remaining 2-valent corollas

$I :=$ 

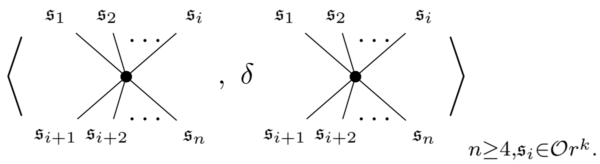
The properad of *(regular) k-directed l-oriented homotopy Lie bialgebras* $\mathsf{holieB}_d^{k\uparrow l}$ is given by

$$\mathsf{holieB}_d^{k\uparrow l} := \mathsf{holieB}_d^{k\uparrow l,+}/I.$$

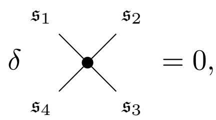
The multi-oriented properads of (multi-oriented) Lie bialgebras $\mathsf{LieB}_d^{k\uparrow l}$ are obtained as a quotient

$$\mathsf{LieB}_d^{k\uparrow l} := \mathsf{holieB}_d^{k\uparrow l}/K^k,$$

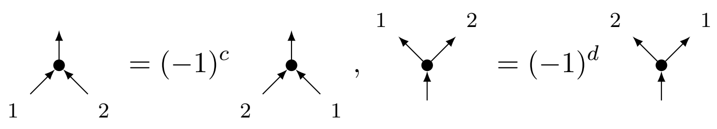
where

$K^k :=$ 

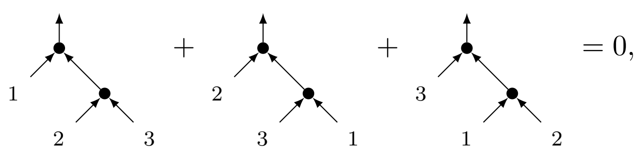
Each $\mathsf{LieB}_d^{k\uparrow l}$ is generated by 3-valent corollas, modulo quadratic relations given by

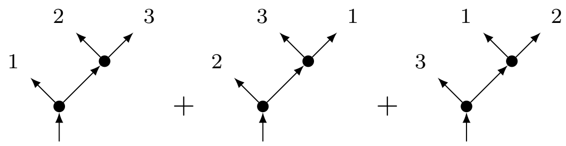


for all $\mathfrak{s}_1,\ \mathfrak{s}_2,\ \mathfrak{s}_3,\ \mathfrak{s}_4 \in \mathcal{O}r^k$, such that the corolla above is not a source or a target. For example, in the one directed case, we get that $\mathsf{LieB}_{c,d}^{1\uparrow 1}$ and $\mathsf{LieB}_{c,d}^{1\uparrow 0}$ are generated by corollas

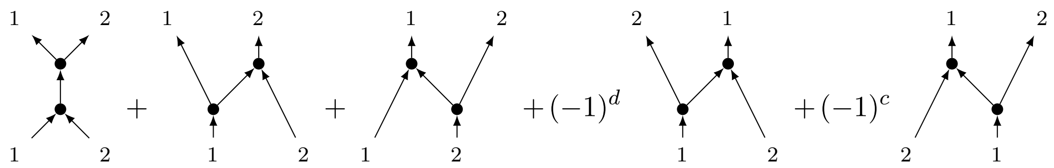
 (2.2)

modulo the relations

 (2.3)



$$= 0, \tag{2.4}$$



$$= 0. \tag{2.5}$$

The full list of relations for the 2-directed properad $\mathsf{LieB}^{2\uparrow 2,1,0}$ will be given in Section 2.3.3.

2.1.4 Completed variants

For a graph Γ, we say that its *loop order*, is given by the number of internal edges minus the number of vertices. For an $\mathcal{G}$-collection E in a symmetric category of graphs $\mathcal{G}$, let

$$G_p(c) := \bigoplus_{m-n\geq p} \left(\bigoplus_{\Gamma \in \mathcal{G}^{conn}_{n,m}(c)} \bigotimes E(v) \right)^{\mathbb{S}^{\mathcal{G}}_{n,m}} \subset \mathcal{F}(E)(c)$$

be the ideal of the free properad generated by all decorated graphs with loop order greater or equal to p. We define the *genus completed free properad* to be the projective limit

$$\widehat{\mathcal{F}(E)} := \lim_{\leftarrow p} \mathcal{F}(E)/G_p.$$

Elements in the vector space $\widehat{\mathcal{F}(E)}(c)$ are possibly infinite sums of isomorphism classes of graphs, with vertices decorated by elements in E. Note, however, that arbitrary infinite sums are not allowed. More precisely, for every fixed p, there can only be a finite number of graphs with loop order p.

We shall consider the derivation complex $\mathrm{Der}(\widehat{\mathcal{F}(E)})$ to be the space of continuous derivations, i.e. derivations

$$d : \widehat{\mathcal{F}(E)} \to \widehat{\mathcal{F}(E)},$$

such that

$$d\left(\sum_{i=1}^{\infty} \mu_i\right) = \sum_{i=1}^{\infty} d\left(\mu_i\right).$$

Practically, such derivations are uniquely determined by a map of $\mathcal{G}$-collections

$$d : E \to \widehat{\mathcal{F}(E)}.$$

The differential on $\mathsf{holieB}_d^{k\uparrow l}$ does not change the loop order. Hence, we may define

$$\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star}, \quad \widehat{\mathsf{holieB}}_d^{k\uparrow l,+} \text{ and } \widehat{\mathsf{holieB}}_d^{k\uparrow l}$$

to be the genus completed versions of $\mathsf{holieB}_d^{k\uparrow l,\star}$, $\mathsf{holieB}_d^{k\uparrow l,+}$ and $\mathsf{holieB}_d^{k\uparrow l}$ respectively. Note that for each fixed degree, the vector space

$$\bigoplus_{n-m=p} \left(\bigoplus_{\Gamma\in\mathcal{G}^{k\uparrow l}_{n,m}} \bigotimes_{v\in V(\Gamma)} \mathcal{E}^{k\uparrow l}(v) \right)^{\mathbb{S}^{\mathcal{G}}_{n,m}}$$

is finite dimensional. Hence, elements in the genus completed properads $\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star}$, $\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}$ and $\widehat{\mathsf{holieB}}_d^{k\uparrow l}$ are infinite sums of graphs with no finiteness conditions.

2.2 Derivations of $\widehat{\mathsf{holieB}}^{k\uparrow l,\star}$

Let us consider the dg Lie algebra of derivations $\mathrm{Der}(\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star})$. As a vector space it is equivalent to the space of maps of $\mathcal{G}^k$-collections

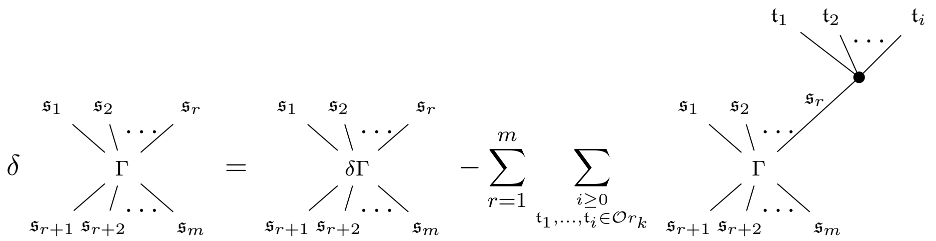
$$\Gamma : \mathcal{E}^k_d \to \widehat{\mathcal{F}^{k\uparrow l}(\mathcal{E}^k_d)}.$$

We get that

$$\mathrm{Der}(\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star}) \cong \prod_{\substack{h\geq 0\\ c\in cor^k(h)}} (\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star}(c)[-|c|])_{Aut(c)} \cong \prod_{h\geq 0}\prod_{n,m} \mathbb{k}(\mathcal{G}^{k\uparrow l}_{n,m,h})^{\mathbb{S}^{\mathcal{G}}_{n,m,h}}[n(d+1)-md],$$

where $cor^k(h)$ is the set of k-directed corollas with legs labeled by h, is a dg vector space spanned by isomorphism classes of k-directed l-oriented graphs with external legs. Here, $\mathcal{G}^{k\uparrow}_{n,m,h}$ is the set of graphs with n labeled vertices, m labeled edges and h labeled external legs. The group $\mathbb{S}^{\mathcal{G}}_{n,m,h} := \mathbb{S}^{\mathcal{G}}_{n,m} \times \mathbb{S}_h$, where $\mathbb{S}_h$ acts by permuting the external legs.

The differential δ acts on such a graph by vertex splitting, and attaching corollas to the external legs



In terms of graphs, the Lie bracket is then given by

$$[\Gamma_1,\Gamma_2] = \sum_{v\in V_{\Gamma_1}} \Gamma_1 \circ_v \Gamma_2 - \sum_{v\in V_{\Gamma_2}} \Gamma_2 \circ_v \Gamma_1.$$

Here, $\Gamma_1 \circ_v \Gamma_2$ is the sum of all graphs Γ such that $\Gamma_2 \subset \Gamma$, and $\Gamma/\bar{\Gamma}_2 = \Gamma_1$, where $\bar{\Gamma}_2$ is the graph obtained from Γ_2 by removing the external legs.

We define a map

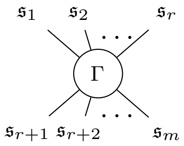
$$F : \mathrm{fGC}_{d+1}^{k\uparrow l, conn} \to \mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star})$$

by sending each graph Γ to the sum over all ways to attach external legs to Γ. More precisely, let $\Gamma = (E_\Gamma, V_\Gamma, \partial_\Gamma, Or_\Gamma)$ be s single term graph of $\mathrm{fGC}^{k\uparrow l}$. Then,

$$F(E_\Gamma, V_\Gamma, \partial, Or) := \sum (E_\Gamma \sqcup [h], V_\Gamma, \partial \sqcup \partial_h, Or_\Gamma \sqcup Or_h),$$

where the sum runs over all $h \geq 0$, $\partial_h : [h] \to V_\Gamma$ and $Or_h : [h] \to Or_k$. In pictures, we will write

$$F(\Gamma) = \sum_{\substack{m\geq 0 \\ \mathfrak{s}_1,\dots,\mathfrak{s}_m}}$$



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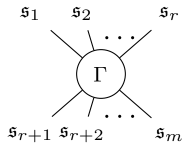
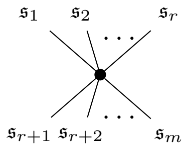
Here,



is understood to be the sum of all graphs whose internal part is equal to Γ that contracts to



.

Proposition 2.2.1. *The map*

$$F : \mathrm{fGC}_{d+1}^{k\uparrow l, conn} \to \mathrm{Der}(\mathit{holieB}_d^{k\uparrow l,\star})$$

is a map of dg Lie algebras.

Proof. One can see that, for a vertex $v \in V_{\Gamma_1}$, we get

$$F(\Gamma_1) \circ_v F(\Gamma_2) = F(\Gamma_1 \circ_v \Gamma_2),$$

where the $\circ_v$ on the left hand side is the linear extension of the operation defined above, and the $\circ_v$ on the left hand side is the operadic composition map of $\mathrm{Gra}^{k\uparrow l}$. It follows that

$$[F(\Gamma_1), F(\Gamma_2)] = F([\Gamma_1, \Gamma_2]).$$

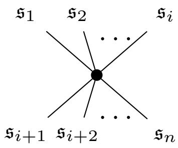
Furthermore, the derivation $F(\bullet\!\!-\!\!\bullet)$ is precisely the differential on $\widehat{\mathsf{hoLieB}}_d^{k\uparrow l}$. Hence, the differential on $\mathrm{Der}(\mathsf{hoLieB}_d^{k\uparrow l})$ is given by

$$\Gamma \mapsto [\Gamma, F(\bullet\!\!-\!\!\bullet)].$$

It follows that F is also a map of dg vector spaces. □

2.2.1 Rescaling classes

For $\mathfrak{t} \in \mathcal{O}r_k$, consider the element $r_\mathfrak{t} \in \mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star})$ by

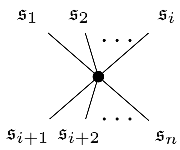
$$r_\mathfrak{t} := \sum_{\substack{n\geq 1 \\ \mathfrak{s}_1\leq\ldots\leq\mathfrak{s}_n\in\mathcal{O}r_k}} (\#\{i\in[k] : \mathfrak{s}_i = \mathfrak{t}\} - 1)$$  .

We get that

$$\delta(r_\mathfrak{t} - r_{\mathfrak{t}^{opp}}) = 0,$$

and as there are no graphs without vertices, there cannot be an element $R_\mathfrak{t}$ such that $\delta R_\mathfrak{t} = (r_\mathfrak{t} - r_{\mathfrak{t}^{opp}})$. Hence, each $r_\mathfrak{t} - r_{\mathfrak{t}^{opp}}$ represents a cohomology class in $\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star})$.

Similarly, we get that

$$r_\emptyset := \sum_{\substack{i\geq 1 \\ \mathfrak{s}_1,\ldots,\mathfrak{s}_n\in\mathcal{O}r_k}} (n-2) \tag{2.6}$$ 

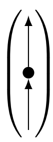
represents a cohomology class in $\mathrm{Der}(\widehat{\mathsf{hoLieB}}_{c,d}^{k\uparrow l,\star})$. We will call the classes $[r_\mathfrak{t} - r_{\mathfrak{t}^{opp}}], [r_\emptyset] \in \mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star})$ the *rescaling classes*, as they only act by rescaling corollas.

Remark 2.2.2. In [43] and [5], the special graph $\uparrow$, which is an edge not connected to any vertex, is considered to be an element in $\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{1\uparrow 1})$, with

$$\delta\uparrow = r_\uparrow - r_\downarrow.$$

The special graph $\uparrow$ may be viewed as the derivation generated by

 $\mapsto \mathrm{id}$.

With our definition, id is not a member of $\mathsf{hoLieB}_{c,d}^{1\uparrow 1}$  . Hence, we do not consider the special graph $\uparrow$ to be a member of $\mathrm{Der}(\widehat{\mathsf{hoLieB}}_{c,d}^{1\uparrow 1})$.

2.2.2 Cohomology of the full derivation complex

Theorem 2.2.3. *The map*

$$F : \mathrm{GC}_d^{k\uparrow l} \to \mathrm{Der}(\widehat{\mathcal{H}oLieB}_d^{k\uparrow l,\star})$$

is a quasi-isomorphism up to the rescaling classes $r_{\mathfrak{t}} - r_{\mathfrak{t}^{opp}}$, $r_{\emptyset}$.

The 1-directed case of this theorem is considered in [5].

Proof. Take a filtration on $\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star})$ by the number of external legs plus the number of internal edges

$$F_p(\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l\star})) := \prod_{n\geq 1, m+h\geq p} \mathbb{k}(\mathcal{G}_{n,m,h})^{\mathbb{S}^{\mathcal{G}}_{n,m,h}}.$$

Also take a filtration on $\mathrm{GC}_d^{k\uparrow l}$ by the number of internal edges

$$F_p\mathrm{GC}_d^{k\uparrow l} := \prod_{n\geq 1, m\geq p} \mathbb{k}(\mathcal{G}_{n,m})^{\mathbb{S}^{\mathcal{G}_{n,m,0}}}.$$

Note that the map F maps $F_p\mathrm{GC}_d^{k\uparrow l}$ into $F_p\,\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l})$.

On the associated graded complex

$$gr(\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star}), \delta) := \prod_p (F_p(\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star}), \delta)/F_{p+1}(\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star}), \delta) \cong$$

$$\cong \prod_p \left(\prod_{n\geq 1, m+h=p} \mathcal{G}_{n,m,h}, \delta_0 \right)^{\mathbb{S}^{\mathcal{G}}_{n,m,h}},$$

the differential δ_0 acts by attaching a univalent vertex to each external leg. One can see that all other terms of δ add an internal edge or external legs.

For a graph $\Gamma \in \mathcal{G}_{n,m,h}$, let its *non-univalent part* $\nu(\Gamma)$ be the graph obtained from Γ by removing all univalent vertices v, transforming the internal edge that was connected to v to an external hair. The differential δ_0 preserves the non-univalent part of each graph. Hence, the associated graded complex splits

$$(\mathrm{Der}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,\star}), \delta_0) \cong \prod_{n,m,h\geq 0} \bigoplus_{\gamma\in iso(\nu(\mathcal{G}_{n,m,h}))} (C_\gamma, \delta_0),$$

where the sum runs over all isomorphism classes of graphs without univalent vertices γ, including special graphs

$$\gamma = \xrightarrow{\mathfrak{s}}$$

without vertices, and (C_γ, δ_0) is the sub-dg vector space spanned by graphs Γ such that $\nu(\Gamma) \cong \gamma$.

If $\gamma = \xrightarrow{\mathfrak{s}}$, we get that $\mathbb{C}_\gamma$ is spanned by elements

$$\bullet\!\xrightarrow{\mathfrak{s}}\ ,\ \xrightarrow{\mathfrak{s}}\!\bullet,\ \bullet\!\xrightarrow{\mathfrak{s}}\!\bullet,$$

with

$$\delta_0 \bullet\!\xrightarrow{\mathfrak{s}} = \delta_0 \xrightarrow{\mathfrak{s}}\!\bullet = \bullet\!\xrightarrow{\mathfrak{s}}\!\bullet.$$

In this case, $\mathbb{C}_\gamma$ contains one cohomology class represented by

$$\bullet\!\xrightarrow{\mathfrak{s}} - \xrightarrow{\mathfrak{s}}\!\bullet.$$

This class corresponds to the rescaling class $r_{\mathfrak{s}} - r_{\mathfrak{s}^{opp}}$.

For a fixed graph $\gamma =\in \mathcal{G}_{n,m,h}$ without univalent vertices, we have

$$C_\gamma \cong \left(\bigotimes_{e\in(L_\Gamma)} D_l \right)^{\mathbb{S}^{\mathcal{G}}_{n,m,h}},$$

where D_l is the dg vector space spanned by configurations

$$\gamma \xrightarrow{Or_\gamma(l)}$$

and

$$\gamma \xrightarrow{Or_\gamma(l)}\!\bullet$$

with

$$\delta_0\ \gamma \xrightarrow{Or_\gamma(l)} \ = \gamma \xrightarrow{Or_\gamma(l)}\!\bullet.$$

It is clear that each D_l is acyclic. It follows that C_γ is acyclic for all graphs with a non-empty set of external legs. If γ does not contain any external legs, then $C_\gamma \cong (\mathbb{k}[|\gamma|])^{Aut_\gamma}$.

Now, on the second page of the spectral sequence, we have

$$H(gr\,\mathrm{Der}(\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star}), \delta_0) \cong \left(\prod_{n\geq 1, m=p} \nu(\mathcal{G}_{n,m,0}) \right)^{\mathbb{S}^{\mathcal{G}_{n,m}}} \cong (\mathrm{GC}^{k\uparrow l}) \oplus \mathbb{k}(\text{rescaling classes}).$$

Note that the rescaling class $r_\emptyset$ corresponds to the class in $C_\bullet$. It is now clear that the map

$$F \oplus i : \mathrm{GC}_{d+1}^{k\uparrow l} \oplus \mathbb{k}(\text{rescaling classes}) \to \mathsf{holieB}_d^{k\uparrow l}$$

induces an isomorphism

$$F \oplus i : (\mathrm{GC}_{d+1}^{k\uparrow l}, 0) \oplus \mathbb{k}(\text{rescaling classes}) \to H(gr\,\mathrm{Der}(\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star}), \delta_0).$$

By the mapping lemma for spectral sequences, we have that F is a quasi-isomorphism $\square$

2.2.3 Another derivation complex $\mathrm{Der}^\star(\widehat{\mathbf{holieB}}_d^{k\uparrow l,+})$

Consider the derivation complex $\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}) := \mathsf{Def}(\widehat{\mathsf{holieB}}_d^{k\uparrow l,\star} \to \widehat{\mathsf{holieB}}_d^{k\uparrow l,+})[1]$ consisting of derivations

$$\Gamma : \widehat{\mathsf{holieB}}^{k\uparrow l,\star} \to \widehat{\mathsf{holieB}}_d^{k\uparrow l,+}.$$

As a graph complex, $\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+})$ consists of (possibly infinite) sums of graphs without source or target vertices. As a vector space, we may write

$$\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}) \cong \prod_{m,n,h} \mathbb{k}(\mathcal{G}_{n,m,h}^{k\uparrow l,\not{st}}[d(n+1)-dm])^{\mathbb{S}^{\mathcal{G}}_{n,m,h}},$$

where $\mathcal{G}_{n,m,h}^{k\uparrow l,\not{st}}$ is the set of graphs without source or target vertices.

Remark 2.2.4. In the fully oriented case, we have

$$\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow k,+}) \cong \mathrm{Der}(\widehat{\mathsf{holieB}}_d^{k\uparrow k,+}).$$

Theorem 2.2.5. *The cohomology of the derivation complex is given by*

$$H(\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+})) \cong \mathrm{GC}^{k\uparrow l} \oplus H(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}(\bullet))[-(d+1)] \oplus \mathbb{k}(\textit{rescaling classes}).$$

Remark 2.2.6. If at least 1 direction is oriented, i.e. $l \geq 1$, we have that $\widehat{\mathsf{holieB}}^{k\uparrow l,+}(\bullet) = 0$. Hence, we have $H(\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+})) \cong \mathrm{GC}^{k\uparrow l} \oplus$ (rescaling classes) for $l \geq 1$.

This theorem looks very similar to Theorem 2.2.3. However, as there are no 1-valent corollas in the graph complex $\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+})$, we will have to use different spectral sequences to prove it. First note that the dg vector space $(\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}), \delta)$ splits

$$\mathrm{Der}^\star(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}), \delta) \cong (\mathrm{Der}^{\star\setminus\bullet}(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}), \delta) \oplus (\mathrm{Der}^\bullet(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}), \delta),$$

where $(\mathrm{Der}^\bullet(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}), \delta)$ is the dg vector space of derivations

$$\bullet \to \widehat{\mathsf{holieB}}_d^{k\uparrow l,+}.$$

It is clear that $(\mathrm{Der}^\bullet(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}), \delta)$ is isomorphic to $\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}(\bullet)[-(d+1)]$. The remaining part of the dg vector space $(\mathrm{Der}^{\star\setminus\bullet}(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}), \delta)$ is spanned by multi-directed graphs with at least 1 external leg

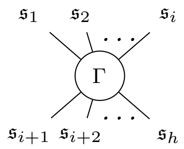
$$(\mathrm{Der}^{\star\setminus\bullet}(\widehat{\mathsf{holieB}}_d^{k\uparrow l,+}), \delta) \cong \prod_{n\geq 1, m\geq 0, h\geq 1} \mathbb{k}(\mathcal{G}_{n,m,h}^{k\uparrow l,\not{st}}[d(n+1)-dm])^{\mathbb{S}^{\mathcal{G}}_{n,m,h}}.$$

Theorem 2.2.5 will follow from the following proposition.

Proposition 2.2.7. *The map*

$$F^+ : \mathrm{GC}^{k\uparrow l} \;\to\; \mathrm{Der}^{\star\backslash\bullet}(\widehat{\mathit{hoLieB}}_d^{k\uparrow l})$$

$$\Gamma \;\mapsto\; \sum_{h\geq 1}$$



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is a quasi-isomorphism up to the re-scaling classes $r_{\mathfrak{t}}$, $r_\emptyset$.

The 1 oriented case of Proposition 2.2.7 is proven in [43].

2.2.4 Outgoing antennas

In this section, we will construct a multi-directed version $\widehat{\mathrm{GC}}_d^{k\uparrow l}$ of the graph complex $\widehat{\mathrm{GC}}$ introduced in [52]. The purpose of this is that the map

$$F^+ : \mathrm{fGC}^{k\uparrow l} \to \mathrm{Der}^{\star\backslash\bullet}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,+})$$

factors through $\widehat{\mathrm{GC}}_d^{k\uparrow l}$. We will then separately show that the maps

$$F_1^+ : \mathrm{fGC}^{k\uparrow l} \to \widehat{\mathrm{GC}}_d^{k\uparrow l}$$

and

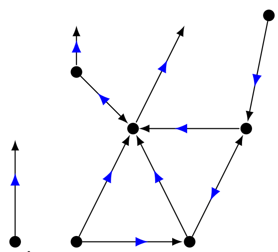
$$F_2^+ \widehat{\mathrm{GC}}_d^{k\uparrow l} \to \mathrm{Der}^{\star}(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,+}),$$

such that $F^+ = F_2^+ \circ F_1^+$, are quasi-isomorphisms up to the rescaling classes.

Let $\widehat{\mathcal{G}}_{n,m,h}^{k\uparrow l} \subset \mathcal{G}_{n,m,h}^{k\uparrow l}$ be the set of connected graphs Γ with n vertices ($V_\Gamma = [n]$), m internal edges, and h external legs ($F_\Gamma = ([m] \sqcup [m] \sqcup [h])$) such that:

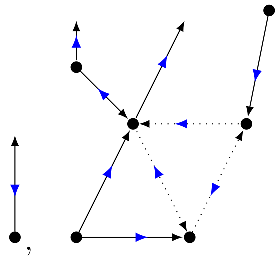
1. No external leg is coherently directed inwards, i.e. for each external leg $e \in (L(\Gamma))$, we must have that
$$(\mathcal{O}r_\Gamma(e))^{-1}(out) \neq \emptyset.$$
2. Each vertex v has at least one outgoing adjacent edge in each direction, i.e. for each vertex $v \in V_\Gamma$, we have that
$$(\mathcal{O}r_\Gamma(\partial_\Gamma^{-1}(v)))^{-1}(out) \neq \emptyset.$$

For example



$$\in \widehat{\mathcal{G}}^{2\uparrow 2},$$

while

 $\notin \widehat{\mathcal{G}}^{2\uparrow 2}$,

as the first graph has a vertex without an outgoing edge in the blue direction, and the second graph has a wheel in the black direction (marked by dotted edges).

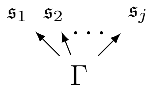
Now, let $\widehat{\mathrm{GC}}_d^{k\uparrow l}$ be the graded vector space

$$\widehat{\mathrm{GC}}_d^{k\uparrow l} := \prod_{n,m,h\geq 0} \mathbb{k}(\widehat{\mathcal{G}}_{n,m,h}^{k\uparrow l}[(n-1)d+(1-d)m])^{\mathbb{S}_{n,m,h}}.$$

We have a map of vector spaces

$$\begin{array}{rcl} F_1 : \mathrm{fGC}_{d+1}^{k\uparrow l} & \to & \widehat{\mathrm{GC}}_{d+1}^{k\uparrow l} \\ (F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma, Or_\Gamma) & \mapsto & \displaystyle\sum_{j\geq 0} \sum_{Or_j:[j]\to \mathcal{O}r^k} \sum_{\partial_j:[j]\to V_\Gamma} (F_\Gamma \sqcup [j], V_\Gamma, \partial_\Gamma \sqcup \partial_j, \iota_\Gamma \sqcup \mathrm{id}_j, Or_\Gamma \sqcup Or_j), \end{array}$$

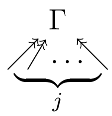
where $Or_j(i) \neq \overline{in}$ for all $i \in [j]$, and we identify terms containing a target vertex with 0. Pictorially, F_1 is given by

$\Gamma \mapsto \sum_{j\geq 0}$  .

We also have a map

$$\begin{array}{rcl} F_2^+ : \widehat{\mathrm{GC}}_{d+1}^{k\uparrow l} & \to & \mathrm{Der}^\star(\widehat{\mathsf{hoLieB}}_d^{k\uparrow l,+}) \\ (F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma, Or_\Gamma) & \mapsto & \displaystyle\sum_{j\geq 0} \sum_{p_j:[j]\to V_\Gamma} (F_\Gamma \sqcup [j], V_\Gamma, P_\Gamma \sqcup \partial_j, \iota_\Gamma \sqcup \mathrm{id}_j, \mathcal{O}r_\Gamma \sqcup in_j), \end{array}$$

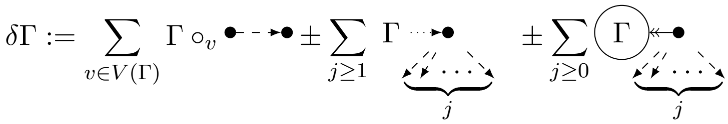
pictorially given by

$\Gamma \quad \mapsto \quad \sum_{j\geq 0}$  .

Here, a double headed arrow represents a coherently directed edge. We may equip $\widehat{\mathrm{GC}}_d^{k\uparrow l}$ with the differential δ, such that

$$\begin{array}{ccc} \mathrm{im}(F_2) & \xrightarrow{\delta} & \mathrm{im}(F_2) \\ {\scriptstyle F_2}\uparrow & & \downarrow{\scriptstyle F_2^{-1}} \\ \widehat{\mathrm{GC}}_d^{k\uparrow l} & \xrightarrow{\delta} & \widehat{\mathrm{GC}}_d^{k\uparrow l} \end{array}$$

commutes. Pictorially, the differential δ is given by

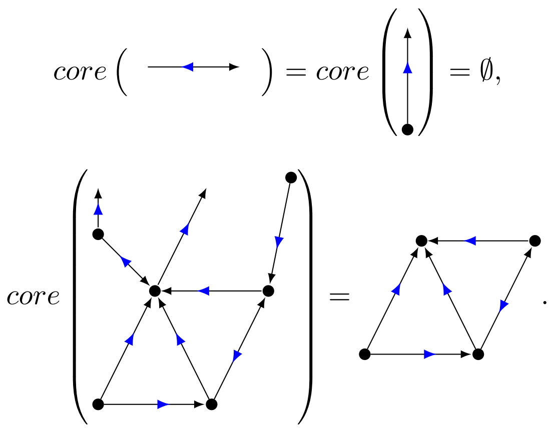

(2.7)

Here, the first term is vertex splitting, the second term is attaching corollas as in $\mathrm{Der}(\mathsf{holieB})$, and the third term is attaching a coherently incoming edge to the graph with all possible combinations of legs attached to it.

Proposition 2.2.8. *The map* $F_1 : \mathrm{GC}^{k\uparrow l} \to \widehat{\mathrm{GC}}^{k\uparrow l}$ *above is a quasi-isomorphism up to the rescaling classes* $r_\emptyset$*, and* $r_{\mathfrak{s}} - r_{\mathfrak{s}^{opp}}$*, for* $\mathfrak{s} \in \mathcal{O}r^k$*,* $\mathfrak{s} \neq \overline{in}, \overline{out}$*.*

If $k = 0$, then this is Proposition 3 in [52]. The proof is analogous.

Proof. We define an *antenna* of a graph $\Gamma \in \widehat{\mathcal{G}}^{k\uparrow l}_{n,m,h}$ to be a subgraph containing either an external leg or a univalent vertex together with a maximal string of bivalent vertices. We obtain the graph $core(\Gamma)$ of a graph Γ by removing all antennas. For example



The differential δ can only preserve or increase the number of vertices in $core(\Gamma)$. Hence, we may define a descending filtration by the number of vertices in the core

$$F_p\widehat{\mathrm{GC}}^{k\uparrow l}_d := \begin{cases} \prod_{n,m,h\geq 0} \Bbbk(\{\Gamma \in \widehat{\mathcal{G}}^{k\uparrow l}_{n,m,h} : |V_{core(\Gamma)}| \geq p\}[(n-1)d + (1-d)m])^{\mathbb{S}^{\mathcal{G}}_{n,m,h}} & p \geq 0 \\ 0 & p \leq -1 \end{cases}$$

The differential δ_0 on the associated graded complex

$$gr(\widehat{\mathrm{GC}}^{k\uparrow l}_d) := \prod_{p\geq 0} F_p\widehat{\mathrm{GC}}^{k\uparrow l}_d / F_{p+1}\widehat{\mathrm{GC}}^{k\uparrow l}_d$$

only sees the terms that preserve the number of vertices in the core. One can see that δ_0 also preserves the structure of the core.

Hence, the associated graded complex splits

$$(gr(\widehat{\mathrm{GC}}_d^{k\uparrow l}), \delta_0) \cong \prod_{\gamma \in core(\widehat{\mathcal{G}^{k\uparrow l}})} (C_\gamma, \delta_0),$$

where the product runs over all isomorphism classes of possible core graphs γ and (C_γ, δ_0) is the associated sub-dg vector space spanned by graphs Γ with $core(\Gamma) = \gamma$.

If the core γ is not empty, we get that

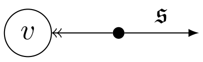
$$(C_\gamma, \delta_0) \cong \left(\bigotimes_{v \in V_\gamma} (D_v, \delta_0) \right)^{\mathbb{S}_\gamma},$$

where v runs over all vertices in the core, $\mathbb{S}_\gamma$ is the automorphism group of γ, and (D_v, δ_0) is the dg vector space that models configurations of antennas connected to v.

If v is at least 2-valent and not a target vertex, then

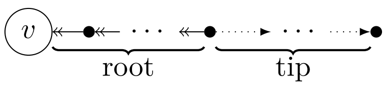
$$D_v \cong (\widehat{S}(A), d_1 + d_2),$$

where A is the dg vector space that models a single antenna attached to v, $\widehat{S}(A)$ denotes the completed symmetric algebra over A, d_1 is the differential induced on $\widehat{S}(A)$ by the differential on A, and d_2 is multiplication by the element

$$\sum_{\mathfrak{s} \in \mathcal{O}r_k \setminus \{\overline{in}\}}$$ 

.

For an antenna $a \in A$, we define the *root* of the antenna to be a maximal string of coherently ingoing edges, staring at the root-vertex of the antenna. We call the non-root part of the antenna the *tip* of the antenna.

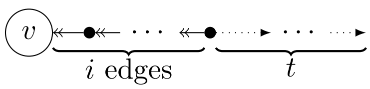


We define a descending filtration on (A, d_1) by the number of edges in the tip of the antenna. The differential $d_{1,0}$ on the associated graded complex only sees the terms that leave the tip intact. Hence, it splits

$$(grA, d_{1,0}) \cong \prod_t (R_t, d_{1,0}),$$

where R_t is the dg vector space that models the root of an antenna with a tip t.

If the tip is not empty, R_t is spanned by elements

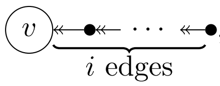
$r_i =$ 

where $i \geq 0$. One can see that the differential is given by

$$d_{1,0} r_i = \begin{cases} \pm r_{i+1} & i \text{ even} \\ 0 & i \text{ odd}, \end{cases}$$

and that the dg vector space R_t is acyclic.

If the tip is empty ($t = 0$), we get that R_0 is spanned by elements

$r'_i =$ 

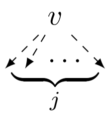
,

where $i \geq 1$. Here, we get that

$$d_{1,0} r'_i = \begin{cases} 0 & i \text{ even} \\ \pm r_{i+1} & i \text{ odd}, \end{cases},$$

so $(R_0, d_1, 0)$ is also acyclic. Note that splitting of the univalent vertex is canceled by the third term in (2.7) with $j = 0$. It follows that (A, d_1) is acyclic.

Using that (A, d_1) is acyclic, and that the differential increases the number of antennas attached to v, one can see that each cohomology class of $\widehat{S}(A)$ must contain the empty configuration. Furthermore, $\widehat{S}(A)$ contains exactly one such cohomology class represented by

$\sum_{j \geq 0}$ 

If v is at least two-valent, but a target in the core, we get that

$$C_v \cong \widehat{S}(A)/B_v,$$

where $B_v \subset \widehat{S}(A)$ is the sub-dg vector space spanned by antenna configurations that does not remove the target status of v.

Once again, we can see that any cohomology class of B_v must contain the empty configuration. However, there are no cocycles of B_v containing the empty configuration. Hence, $H(B_v) = 0$, and we have

$$H(\widehat{S}(A)/B_v) \cong H(\widehat{S}(A)).$$

If v is one-valent in the core, the dg vector space C_v needs at least two antennas. Hence,

$$C_v \cong \widehat{S}^{\geq 2}(A)/(B_v)^{\geq 2},$$

which is acyclic as both (A, d_1) and $(B_v^{\geq 2}, d_1)$ are acyclic.

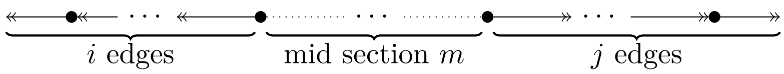
If v is 0 valent in the core (The core is just a single vertex v), we must have at least 3 antennas attached to v. Hence,

$$C_v \cong \widehat{S}^{\geq 3}(A)/B_v^{\geq 3},$$

which is also acyclic.

We now have showed that $H(gr(\widehat{\mathrm{GC}}_d^{k\uparrow l}, \delta))$ contains one cohomology class for every core graph Γ, with vertices being at least bivalent. Those are precisely the graphs in $\mathrm{GC}_d^{k\uparrow l}$. Furthermore, we have that each such class is represented by $F_1^+(\Gamma)$.

Finally, we will treat the case when the core is empty. An element in $gr_0\widehat{\mathrm{GC}}$ must have the form

$F^m_{i,j} =$ 

,

where the *mid-section* m is a string of non-coherently directed edges, and $i, j \geq 0$ counts the number of coherently directed edges on the sides of the m. Put a descending filtration on $gr_0\widehat{\mathrm{GC}}^{k\uparrow l}$ by the number of edges in the mid section. The associated graded complex splits again, with a sub-dg vector space A^m for each mid-section m.

If m contains at least two edges, then $F^m_{0,0}$ is allowed. Also note that the mid section cannot be symmetric, hence $F^m_{i,j} \neq \pm F^m_{j,i}$. We get that $A^m \cong R \otimes R$, where R is the dg vector space with one generator r_i in each degree $i \geq 0$, and

$$dr_i = \begin{cases} \pm r_{i+1} & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

It is clear that A^m is acyclic.

If m consist of a single edge $m =$ , the graph $F^m_{0,0}$ is not allowed. Hence,

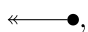
d_1"$F^m_{0,0}$" $=$  ,

represents a cohomology class. This cohomology class corresponds to $(r_{\mathfrak{t}} - r_{\mathfrak{t}^{opp}})$.

If the mid-section is just a vertex $m = \bullet$, then we have a symmetry relation $F^\bullet_{i,j} = \pm F^\bullet_{j,i}$, and also $F^\bullet_{0,0}$ is not allowed. Hence,

$$A^\bullet \cong R^{deg\geq 1} \wedge R^{deg \geq 1},$$

which contains a single cohomology class represented by

$F^\bullet_{0,1} = F^\bullet_{1,0} =$ ,

which is the term of $r_\emptyset - r_{\overline{in}} + r_{\overline{out}}$ with an empty core. □

2.2.5 Ingoing trees

Proposition 2.2.9. *The inclusion* $F_2 : \widehat{\mathrm{GC}}^{k\uparrow l}_{d+1} \to \mathrm{Der}(\widehat{\mathsf{hoLieB}}^{k\uparrow l}_d)$ *is a quasi-isomorphism, up to* 1 *rescaling class.*

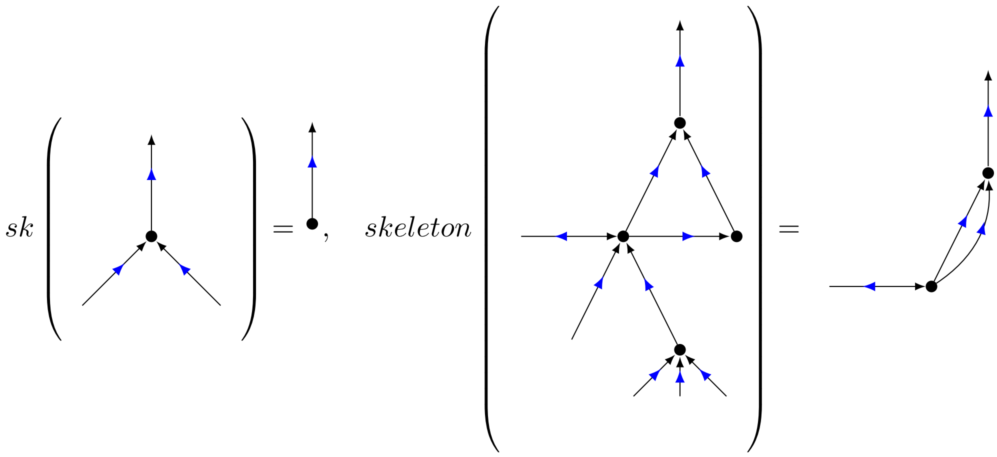
The proof of this is analogous to Proposition 4.1.2 in [43].

Proof. For a graph $\mathrm{Der}(\mathsf{holieB}^{k\uparrow l})$, we define an *inwards directed external tree* to be a tree that is connected to the graph in one point, and is oriented inwards in all directions.

For a graph in $\Gamma \in \mathrm{Der}(\mathsf{holieB}^{k\uparrow l,+})$, we define its *skeleton*, $sk(\Gamma)$ to be the graph obtained from Γ by:

1. Removing all inwards external trees from Γ, to obtain a graph γ.
2. Removing all passing vertices from γ, and connect the two edges.

For example



Similarly, for a graph in $\widehat{\mathrm{GC}}_d^{k\uparrow l}$, we obtain its skeleton by removing passing vertices.

We get descending filtration on $\mathrm{Der}(\mathsf{holieB}^{k\uparrow l})$ and $\widehat{\mathrm{GC}}_d^{k\uparrow l}$, by the number of vertices in the skeleton. The map F_2 maps a graph with skeleton γ to a sum of graph with the same skeleton γ. Hence, it induces a map of spectral sequences.

It is a standard argument to show that $H(gr\widehat{\mathrm{GC}}_d^{k\uparrow l})$ contains one cohomology class for each skeleton $\tilde{\gamma}$.

On the other hand, one can see that the differential δ_0 on the associated graded complex $gr\,\mathrm{Der}(\mathsf{holieB}_d^{k\uparrow l})$ only sees the terms that leave the skeleton intact. Hence $(\mathrm{Der}(\mathsf{holieB}_d^{k\uparrow l}), \delta_0)$ splits as a product with a sub-dg vector space $(C_{\tilde{\gamma}}, \delta_0)$ for each skeleton $\tilde{\gamma}$. We need to show that each $H(C_{\tilde{\gamma}}, \delta_0)$ is one dimensional, with its cohomology class being represented by $F_2(\tilde{\gamma})$.

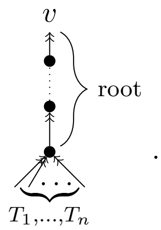
Take another filtration on each $C_{\tilde{\gamma}}$ by the number of removed passing vertices. The associated graded complex $grC_{\tilde{\gamma}}$ splits into a direct sum $\bigoplus(D_\gamma, \delta_{0,0})$, with a sub-dg vector space $(D_\gamma, \delta_{0,0})$ for each graph γ obtained from $\tilde{\gamma}$ by adding passing vertices to edges. Furthermore, the differential $\delta_{0,0}$ on D_γ only acts on the input forest to each vertex $v \in V(\gamma)$. Hence D_γ splits into a tensor product

$$(D_\gamma, \delta_{0,0}) \cong \left(\bigotimes_{v \in V(\gamma)} (E_v, \delta_{0,0}) \right)^{\mathbb{S}_\gamma},$$

where E_v is the dg vector space of input forest configurations attached to v.

If v is not a source in any direction, then all forest configurations are legal, including the empty one.

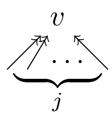
For an input forest configuration T, consider its *root* to be the maximal string of vertices that are ancestors to all other vertices



.

Similarly to the antenna complex in the proof of Proposition 2.2.8, we can take a filtration on the number of edges in the non-root part. It is easy to see that the associated graded complex is acyclic almost everywhere, except for the 2 classes represented by the empty configuration, and the configuration of a single input leg. Hence, $H(E_v)$ is at most 2-dimensional. Furthermore, one can see that $H(E_v)$ contains one cohomology class represented by the empty configuration, and one cohomology class represented by

$$\sum_{j\geq 0}$$



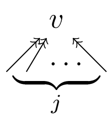
If v is a source in some direction, then the empty configuration is not included in E_v, and hence $H(E_v)$ is one-dimensional. Thus, the cohomology of the associated graded complex $grC_{\tilde{\gamma}}$ is spanned by graphs γ obtained from $\tilde{\gamma}$ by:

1. Adding vertices on edges.

2. For passing vertices and sources, we sum over all ways to attach coherently ingoing legs to the vertex.

3. For other vertices v, there is a choice of either not attaching anything to v, or attaching

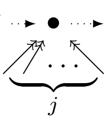
$$\sum_{j\geq 0}$$



to v.

Now, let us look at the second page of the spectral sequence $(H(C_{\tilde{\gamma}}, \delta_{0,0}), \delta_{0,1})$. The differential $\delta_{0,1}$ creates new passing vertices by splitting vertices in $\tilde{\gamma}$, splitting passing vertices, and adding corollas

$$\sum_{j\geq 0} \gamma$$

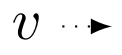


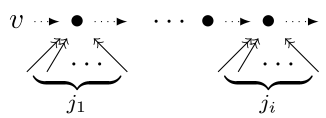
to the external legs of $\tilde{\gamma}$.

Put a descending filtration on $(H(C_{\tilde{\gamma}}, \delta_{0,0}), \delta_{0,1})$ by the number of vertices without coherently ingoing legs. The associated graded complex splits into a tensor product

$$gr(H(C_{\tilde{\gamma}}), \delta_{0,1}) \cong \bigotimes_{e \in E(\tilde{\gamma})} (A_e, \delta_{0,1,0}),$$

where $(A_e, \delta_{0,1,0})$ is the dg vector space that models passing vertices with attached input trees between the endpoints of e.

If $e =$  is an external leg, then A_e is spanned by elements

$$F_i = \sum_{j_1,\ldots,j_i \geq 1}$$ 

where $i \geq 0$ counts the number of passing vertices. We get that

$$\delta_{0,1,0} F_i = \begin{cases} \epsilon_i F_{i+1} & \text{if the root is bald} \\ \epsilon_{i+1} F_{i+1} & \text{if the root is hairy,} \end{cases}$$

and we can see that A_e is acyclic if the root is bald, and that it contains a single cohomology class represented by F_0 if the root is hairy.

If e is an internal edge, then

$$\delta_{0,1,0} F_i = \begin{cases} \epsilon_{i+1} F_{i+1} & \text{if both the endpoints are hairy} \\ \epsilon_i F_{i+1} & \text{if one of the endpoints is hairy and the other is bald} \\ \epsilon_{i+1} F_{i+1} & \text{if both the endpoints are bald.} \end{cases}$$

Thus, $(A_e, \delta_{0,1,0})$ is acyclic if one of the endpoints is bald and the other is hairy. Since we must have at least 1 hairy vertex, we get that all vertices must be hairy for a graph to not vanish in the cohomology.

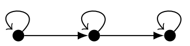
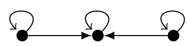
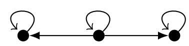
It follows that each $H(C_{\tilde{\gamma}}, \delta_0)$ is one dimensional, with its cohomology class being represented by $F_2(\tilde{\gamma})$. The only class not covered by the image F_2 is the class in $C_\emptyset$, which is the minimal skeleton part of the rescaling class $r_\emptyset + r_{\overline{in}} - r_{\overline{out}}$. Hence, the map

$$F_2 \oplus i : \widehat{\mathrm{GC}}_{d+1} \oplus \mathbb{k}(r_\emptyset + r_{\overline{in}} - r_{\overline{out}}) \to H(\widehat{\mathsf{holieB}}_d^{k\uparrow l}, \delta_0)$$

is a quasi-isomorphism. □

2.3 Cohomology of $\mathsf{holieB}_d^{2\uparrow 2}$

It is shown in [32] and [47] that $\mathsf{holieB}_d^{1\uparrow 1}$ is a minimal model of $\mathsf{LieB}_d^{1\uparrow 1}$. In [36] it is shown that $\mathsf{holieB}_d^{1\uparrow 0}$ contains a cohomology class

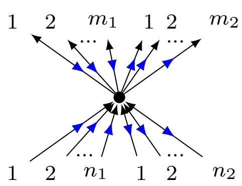
 $-$  $+$  $\in \mathsf{holieB}_{odd}^{1\uparrow 0}$.

Hence, the wheeled properad $\mathsf{holieB}^{1\uparrow 0}_{odd}$ is not a minimal model of $\mathsf{LieB}^{1\uparrow 0}_{odd}$.

In this chapter we shall prove the following theorem:

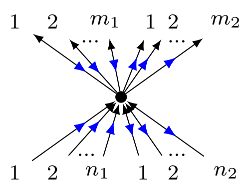
Theorem 2.3.1. *The projection* $(\mathsf{holieB}^{2\uparrow 2}_{d}, \delta) \to (\mathsf{LieB}^{2\uparrow 2}_{d}, 0)$ *is a quasi-isomorphism, i.e.* $(\mathsf{holieB}^{2\uparrow 2}_{d}, \delta)$ *is a minimal model of* $(\mathsf{LieB}^{2\uparrow 2}_{d}, 0)$.

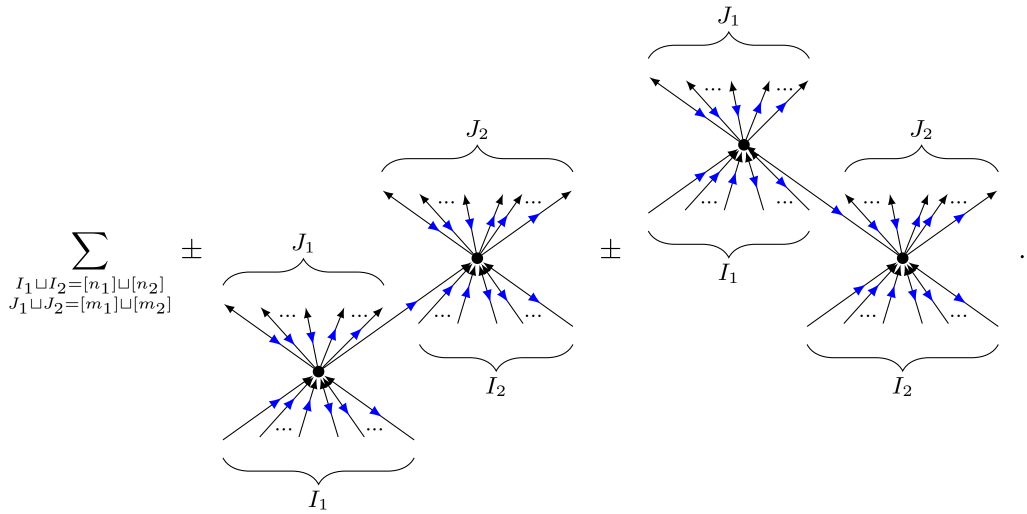
The two oriented properad $\mathsf{holieB}^{2\uparrow 2}_{d}$ is generated by (skew)symmetric corollas



$$, \quad \begin{cases} n_1 + n_2 + m_1 + m_2 \geq 3 \\ n_1 + n_2 \geq 1 \\ m_1 + m_2 \geq 1 \\ n_1 + m_1 \geq 1 \\ n_2 + m_2 \geq 1. \end{cases}$$

The differential δ acts on each generating corolla by vertex splitting

δ  $:=$

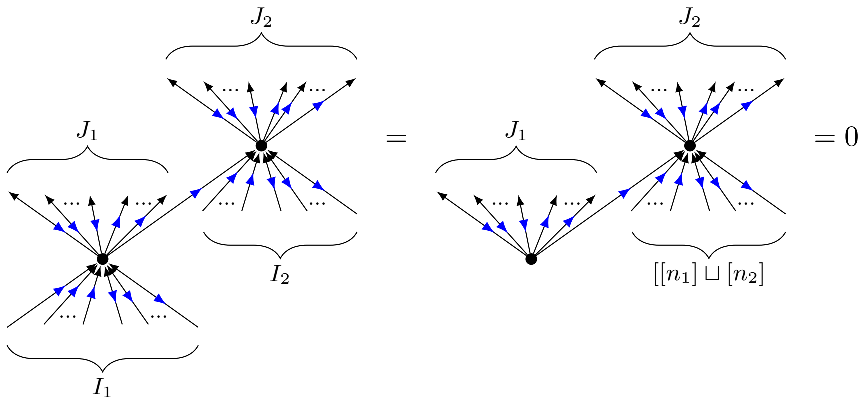


(2.8)

Remark 2.3.2. When considering 2-oriented graphs, we can draw the pictures so that the first direction is given by black arrows at the end of each edge, that flow from down to up; and the second direction is given by blue arrows at the middle of each edge, that flow from left to right.

Remark 2.3.3. In the formula above, we identify corollas that are targets or sources with 0.

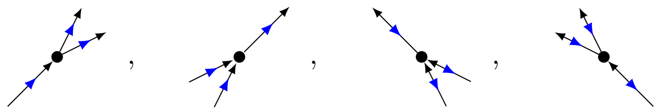
For example, if $I_1 = \emptyset$, then



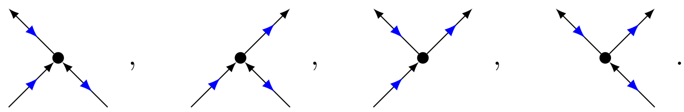
The 2-oriented properad of 2-oriented Lie-bialgebras $\mathsf{LieB}_{c,d}^{2\uparrow 2}$ given by the quotient

$$\mathsf{LieB}_d^{2\uparrow 2} := \mathsf{holieB}_d^{2\uparrow 2}/K,$$

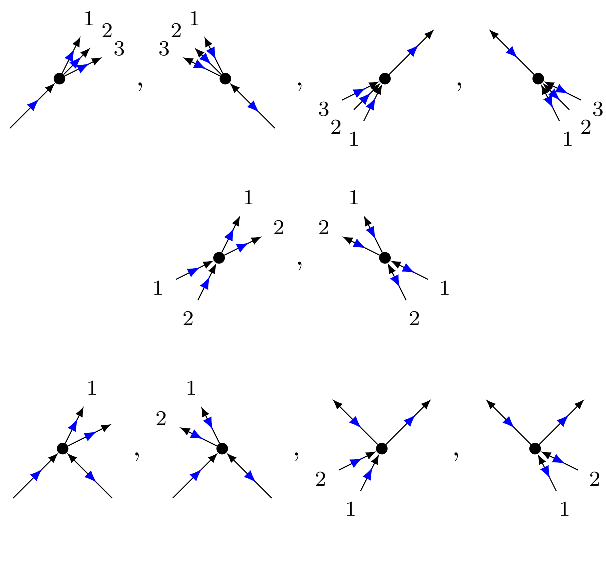
is generated by 8 different 3-valent corollas, 4 operadic corollas

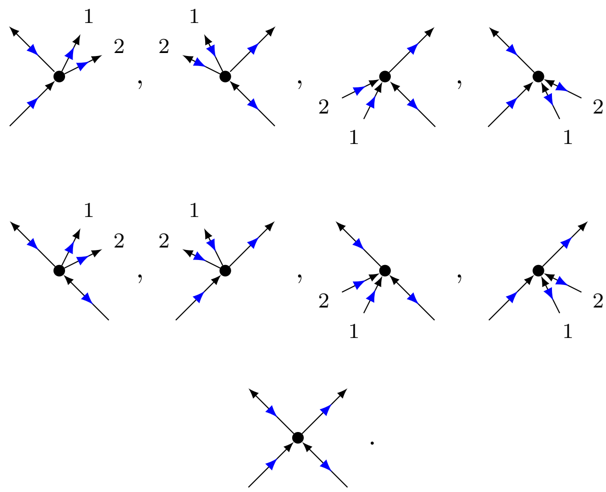


and 4 other corollas



Modulo 19 quadratic relations, induced by the differential of the 4-valent corollas in holieB_d^2;





Full pictures of all relation are presented in Section 2.3.3.

Following the methods used in [32] to show that holieB is quasi-isomorphic to LieB, we will establish a spectral sequence on $\mathsf{holieB}_d^{2\uparrow 2}$, by the number of directed paths through a graph.

For a graph $\Gamma \in \mathsf{holieB}_d^{2\uparrow 2}$, define $P_i(\Gamma)$ to be the set of paths following direction i from an i-input leg to an i-output leg. The differential can only decrease the number of paths through a graph, hence we obtain filtrations on $\mathsf{holieB}_d^{2\uparrow 2}$ by only considering graphs with a limited number of directed paths.

We will start by taking a filtration on the number of paths following the first (black) direction

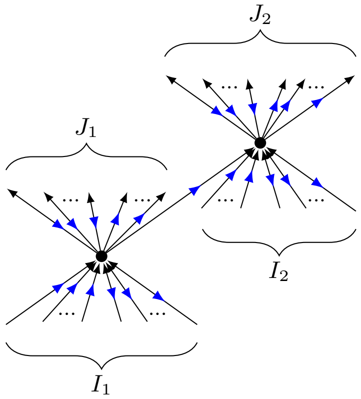
$$F_p\mathsf{holieB}_d^{2\uparrow 2} := \{\Gamma \in \mathsf{holieB}_d^{2\uparrow 2} || P_1(\Gamma)| \leq p\},$$

and we will show that the cohomology of the associated graded complex

$$gr(\mathsf{holieB}_d^{2\uparrow 2}, \delta) := \bigoplus_p (F_p\mathsf{holieB}_d^{2\uparrow 2}/F_{p-1}\mathsf{holieB}_d^{2\uparrow 2}, \delta) = (\mathsf{holieB}_d^{2\uparrow 2}, \delta_0)$$

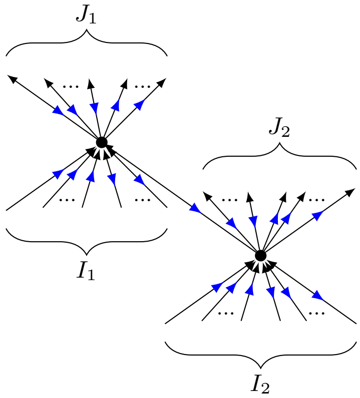
is generated by 3-regular graphs. As $\dim H(\mathsf{holieB}_d^{2\uparrow 2}, \delta_0) \leq \dim H(\mathsf{holieB}_d^{2\uparrow 2}, \delta)$, it follows that $(\mathsf{holieB}_d^{2\uparrow 2}, \delta)$ is generated by 3-regular graphs.

The differential δ_0 sees only the terms in (2.8) where all paths in the black direction are preserved. These are terms



with $J_1 = \emptyset$ or $I_2 = \emptyset$,

and



with $J_2 = \emptyset$ or $I_1 = \emptyset$.

Remark 2.3.4. If we naively try to follow the proof of the 1 oriented case in [32], we would say that $(gr\mathsf{holieB}_d^{2\uparrow 2}, \delta_0)$ splits

$$(gr\mathsf{holieB}_d^{2\uparrow 2}, \delta_0) \cong \bigoplus_{\gamma} (C_\gamma, \delta_0),$$

where γ is a "reduced graph" preserved by the differential δ_0, and (C_γ, δ_0) is the sub-dg vector space spanned by graphs that reduces to γ.

Next, we would be tempted to say that each C_γ is a tensor product

$$C_\gamma \cong \left(\bigotimes_{v \in V(\gamma)} (D_v, \delta_0) \right)^{\mathbb{S}_\gamma},$$

where D_v is the dg vector space spanned by graphs that reduces to v. Finally, we would conclude that each $H(D_v, \delta_0)$ is spanned by 3-valent graphs.

However, as we will find that the reduced graph γ may contain wheels in the second (blue) direction, we can only say that

$$C_\gamma \subset \left(\bigotimes_{v \in V(\gamma)} (D_v, \delta_0) \right)^{\mathbb{S}_\gamma}.$$

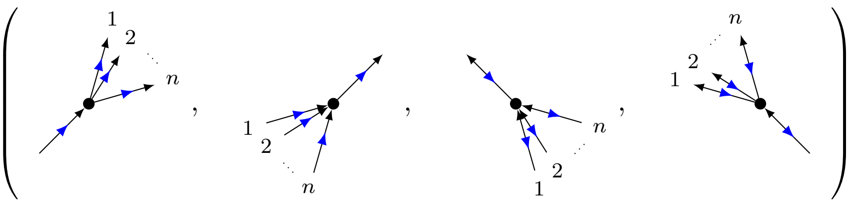
Hence, it is not clear that the cohomology is completely determined by $H(D_v, \delta_0)$. Before we tackle this problem, let us, in the next section, reduce the number of corollas we have to consider.

2.3.1 Auxiliary properads $\mathcal{P}^{2\uparrow 2}$ and $ho\mathcal{P}^{2\uparrow 2}$

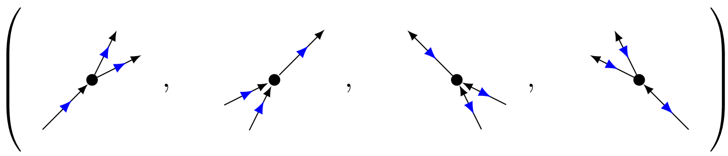
We will define two 2-oriented dg properads $(\mathsf{ho}\mathcal{P}^{2\uparrow 2}, \delta_0)$, $(\mathcal{P}^{2\uparrow 2}, \delta_0)$ such that:

1. We have quasi-isomorphisms of dg vector spaces
$$\mathsf{ho}\mathcal{P}^{2\uparrow 2} \to gr(\mathsf{holieB}_d^{2\uparrow 2}) \text{ and } \mathsf{ho}\mathcal{P}^{2\uparrow 2} \to \mathcal{P}^{2\uparrow 2}.$$
2. The properad $\mathcal{P}^{2\uparrow 2}$ contains fewer corollas, and it is, therefore, easier to prove that $H(\mathcal{P}^{2\uparrow 2})$ is spanned by 3-valent corollas.

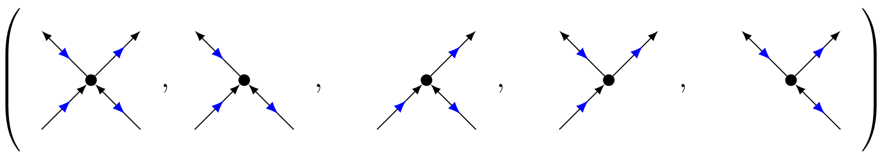
Let
$$\mathsf{ho}\mathcal{P}^{2\uparrow 2} := \mathcal{F}ree(\mathcal{O}\mathrm{p}, \mathcal{S}\mathrm{tar})/\mathcal{R}$$
and
$$\mathcal{P}^{2\uparrow 2} := \mathcal{F}^{2\uparrow 2}(\mathcal{O}\mathrm{p}(2), \mathcal{S}\mathrm{tar})/(\mathcal{R}, \mathcal{J}),$$
where $\mathcal{O}$p is the set of *operadic corollas*

$Op :=$


$_{n\geq 2}$,

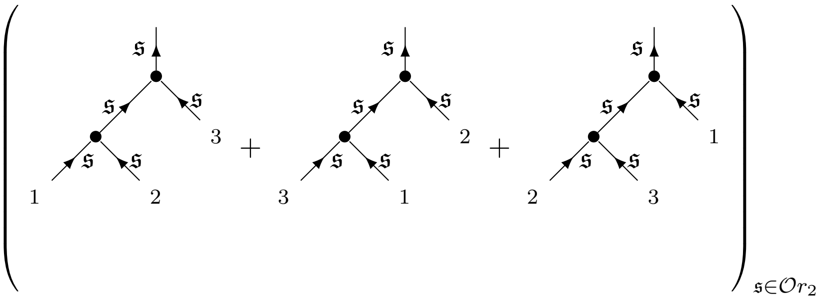
$Op(2) :=$

,

where one leg has the opposite multi-direction to all other legs; and $\mathcal{S}$tar is the set of *star corollas*


,

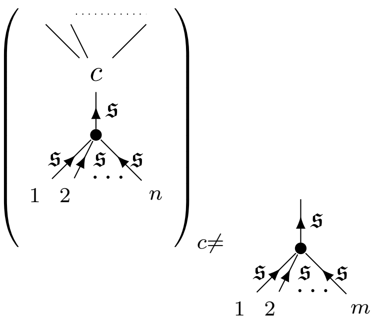
where each leg has a unique multi-direction.

The relations are given by the Jacobi identity

$\mathcal{J} =$ 

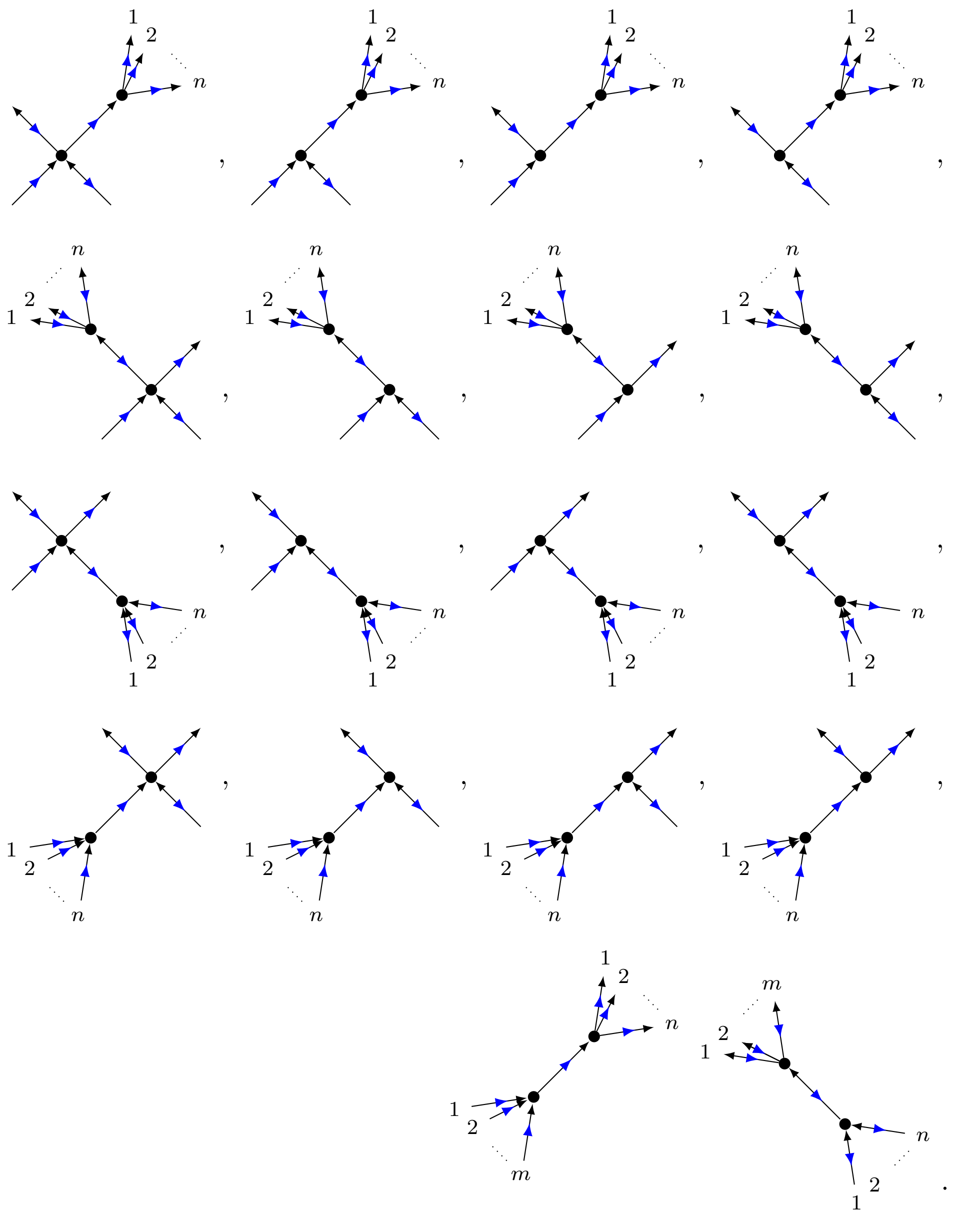
, (2.9)

and $\mathcal{R}$ is an ideal of 1 term relations, were an operadic corolla is connected to a star corolla, or the opposite directed operadic corolla, with its unique leg

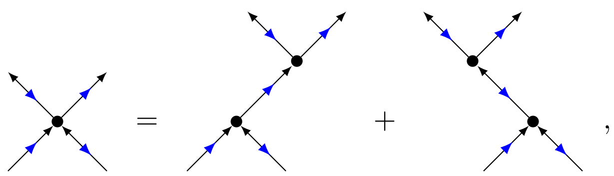
$\mathcal{R} :=$ 

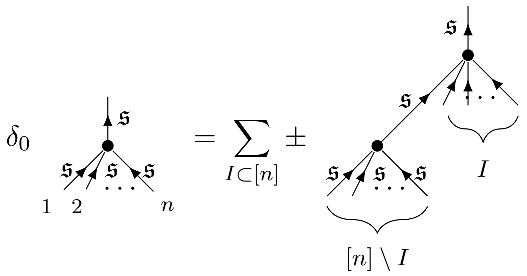
. (2.10)

In full, $\mathcal{R}$ is generated by:



The differential δ_0 acts by vertex splitting while preserving all paths in the black direction

δ_0  (2.11) and the holie differential on operadic corollas

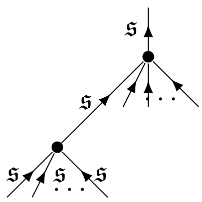


(2.12)

Remark 2.3.5. It is easy to verify that the differential δ_0 preserves the ideals $\mathcal{R}$ and $\mathcal{J}$.

As a dg vector space, $\mathsf{ho}\mathcal{P}^{2\uparrow 2}$ is a subspace of $\mathsf{holieB}_d^{2\uparrow 2}$, spanned by graphs such that:

1. Each corolla is either an operadic corolla or a star corolla;
2. For each operadic corolla, the unique leg is either an external leg, or it is connected to another operadic corolla of the same type



.

There is a canonical inclusion of dg vector spaces

$$i : ho\mathcal{P}^{2\uparrow 2} \hookrightarrow gr(\mathsf{holieB}_d^{2\uparrow 2}), \tag{2.13}$$

as well as a projection

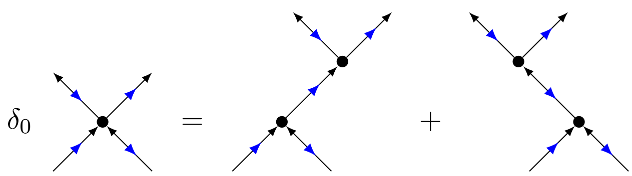
$$p : \mathsf{ho}\mathcal{P}^{2\uparrow 2} \to \mathcal{P}^{2\uparrow 2}. \tag{2.14}$$

Proposition 2.3.6. *The maps i and p above are quasi-isomorphisms, i.e.*

$$gr(\mathsf{holieB}_d^{2\uparrow 2}) \cong H(ho\mathcal{P}^{2\uparrow 2}) \cong H(\mathcal{P}^{2\uparrow 2}).$$

Proof. First, take a filtration on $gr(\mathsf{holieB}_d^{2\uparrow 2})$, $\mathsf{ho}\mathcal{P}^{2\uparrow 2}$, and $\mathcal{P}^{2\uparrow 2}$ by the number of paths in the second (blue) direction.

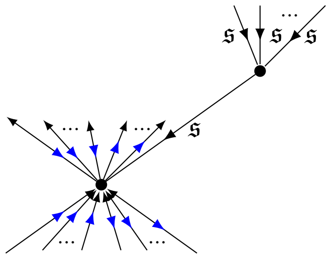
As neither of the terms in



preserve the paths in the blue direction, we have that the differential on the associated graded complex $gr\mathcal{P}^{2\uparrow 2}$ vanishes.

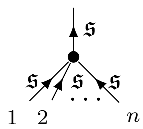
On $gr\mathsf{ho}\mathcal{P}^{2\uparrow 2}$, the differential $\delta_{0,0}$ vanishes on star corollas, and acts normally on the operadic corollas.

On the (double) associated graded complex $grgr\mathsf{holieB}_d^{2\uparrow 2}$, the differential $\delta_{0,0}$ only sees terms on the form



, (2.15)

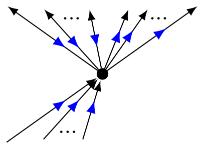
where one (or both) of the corollas is an operadic corolla

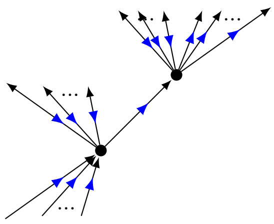


$\in \mathcal{O}\mathrm{p}$,

pointing towards the other corolla.

Remark 2.3.7. We always say that an operadic-corolla is pointing towards its unique leg, even if the unique leg is directed the opposite direction.

If the corolla only contains 3 types of legs, e.g.  , then δ_0 also sees terms on the form



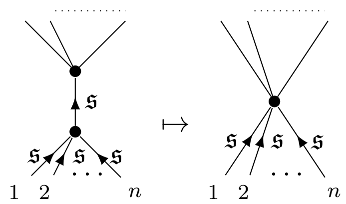
, (2.16)

where the internal edge is a unique input (or output) in both directions but to the opposite corollas (in the example above, the internal edge is the unique output, in the blue direction, to the lower left corolla; and the unique input, in the black direction, to the upper right corolla). Let us write

$$\delta_{0,0}\Gamma = \sum \Gamma_\alpha + \sum \Gamma_\beta, \quad (2.17)$$

where Γ_α are graphs obtained from Γ by splitting a vertex such as (2.15), and Γ_β are graphs obtained from Γ by splitting a vertex such as (2.16).

Next, for a graph $\Gamma \in (grgr)\mathsf{holieB}^2_d$, $gr\mathsf{ho}\mathcal{P}^{2\uparrow 2}$, or $\mathcal{P}^{2\uparrow 2}$ we define the *reduced graph* $R(\Gamma)$, to be the graph obtained from Γ by contracting the unique edge in all operadic corollas



, (2.18)

(if it is internal).

Note that each term $\sum \Gamma_\alpha$ in (2.17) has the same reduced graph as Γ, while the terms in $\sum \Gamma_\beta$ have more vertices in the reduced graph than Γ, i.e.

$$R(\Gamma_\alpha) = R(\Gamma) \qquad |V(R(\Gamma_\beta))| > |V(R(\Gamma))|.$$

Hence, we may take a filtration on the $(grgr\mathsf{holieB}^{2\uparrow 2}_d, \delta_{0,0})$, $(gr\mathsf{ho}\mathcal{P}^{2\uparrow 2}, \delta_{0,0})$, and $(gr\mathcal{P}^{2\uparrow 2}, \delta_{0,0})$ by the number of vertices in the reduced graph

$$G_p grgr\mathsf{holieB}^{2\uparrow 2}_d := \{\Gamma \in gr\mathsf{holieB}^{2\uparrow 2}_d || R(V(\Gamma))| \geq p\}.$$

The differential $\delta_{0,0,0}$ on the associated graded complex $grgrgr\mathsf{holieB}^{2\uparrow 2}_d$ only sees the terms

$$\delta_{0,0,0} = \sum \Gamma_\alpha$$

that completely preserve the reduced graph. Hence, $(\mathsf{holieB}^{2\uparrow 2}_d, \delta_{0,0,0})$ splits

$$(\mathsf{holieB}^{2\uparrow 2}_d, \delta_{0,0,0}) \cong \bigoplus_\gamma (C_\gamma, \delta_{0,0,0}),$$

where the sum runs over all possible reduced graphs γ, and C_γ is the sub-dg vector space spanned by graphs Γ with $R(\Gamma) = \gamma$.

Next, each C_γ splits into a tensor product,

$$C_\gamma \cong \left(\bigotimes_{v \in V(\gamma)} (D_v, \delta_{0,0,0}) \right)^{\mathbb{S}_\gamma},$$

where D_v is the dg vector space spanned by graphs G such that $R(G) = v$, and $\mathbb{S}_\gamma$ is the symmetry group that acts on γ. We can see that D_v consists of genus 0 graphs with at most one vertex x not being an operadic corolla, connected to trees of operadic corollas directed towards x.

Similarly, $(\mathsf{ho}\mathcal{P}^{2\uparrow 2}, \delta_{0,0,0})$ and $(\mathcal{P}^{2\uparrow 2}, \delta_{0,0,0}(=0))$ split, into

$$(\mathsf{ho}\mathcal{P}^{2\uparrow 2}, \delta_{0,0,0}) \cong \bigoplus_\gamma (C'_\gamma, \delta_{0,0,0})$$

and

$$(\mathcal{P}^{2\uparrow 2}, 0) \cong \bigoplus_{\gamma} (C''_\gamma, 0),$$

with

$$(C'_\gamma, \delta_{0,0,0}) \cong \left(\bigotimes_{v \in V(\gamma)} (D'_v, \delta_{0,0,0}) \right)^{\mathbb{S}_\gamma},$$

$$(C''_\gamma, 0) \cong \left(\bigotimes_{v \in V(\gamma)} (D''_v, 0) \right)^{\mathbb{S}_\gamma}.$$

We need to show that, for each 2-directed (skew) symmetric corolla v, the maps

$$i : \mathsf{ho}\mathcal{P}^{2\uparrow 2} \hookrightarrow gr\mathsf{holieB}_d^{2\uparrow 2},$$

and

$$p : \mathsf{ho}\mathcal{P}^{2\uparrow 2} \to \mathcal{P}^{2\uparrow 2}$$

induces quasi-isomorphisms

$$(D_v, \delta_{0,0,0}) \to (D'_v, \delta_{0,0,0}) \leftarrow (D''_v, 0).$$

1. If v is an operadic corolla with $n+1$ legs, then we can see that

   $$(D_v, \delta_{0,0,0}) = (D'_v, \delta_{0,0,0}) \cong (\mathsf{hoLie}(n), \delta),$$

   and

   $$(D'_v, 0) \cong (\mathsf{Lie}(n), 0).$$

   It follows from Theorem 1.3.7, that both maps are quasi-isomorphisms.

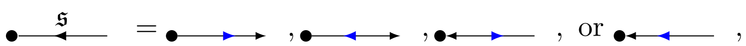
2. If v is a star corolla, then there is only one graph that reduces to v, namely v itself. Hence, D_v, D'_v and D''_v are all 1-dimensional.

3. Otherwise, if v is neither a star corolla nor an operadic corolla, we have

   $$D'_v = D''_v = 0.$$

   We need to show that D_v is acyclic for such corollas.

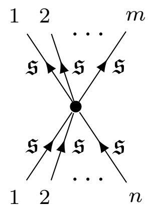
Pick a multi-direction



of which v has at least 2 legs. Define a weight grading on D_v, by

$$w(x) = \begin{cases} 1 & \text{if the non-operadic corolla only has 1 leg of direction } \mathfrak{s} \\ 0 & \text{otherwise.} \end{cases}$$

for every generating graph in D_v. Alternatively, if v is a *properadic corolla*

$v =$ ,

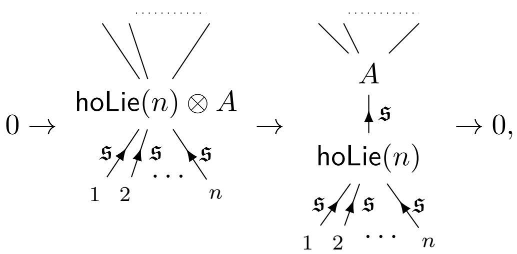
we say that

$$w(x) = \begin{cases} 1 & \text{if all corollas are operadic} \\ 0 & \text{otherwise.} \end{cases}$$

We can make a filtration by

$$F_p D_v := \{x \in D_v : deg(x) - w(x) \geq p\},$$

where $deg(x)$ is the cohomological degree. Then, the associated graded complex consists of short exact sequences



which are acyclic. Here, A is the vector space of spanned by trees of operadic corollas of the other multi-directions.

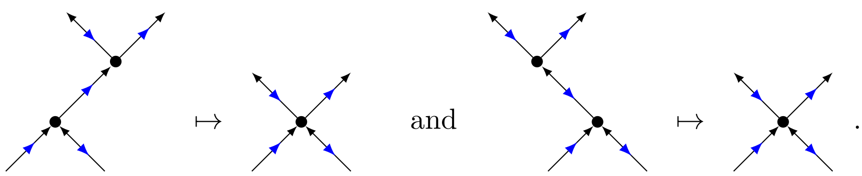
It follows that the maps in (2.13) and (2.14) are quasi-isomorphisms. □

2.3.2 Cohomology of $\mathcal{P}^{2\uparrow 2}$.

In this section, we will prove the following proposition, which will finish the proof of Theorem 2.3.1.

Proposition 2.3.8. *All cohomology classes in* $H(\mathcal{P}^{2\uparrow 2}, \delta_0)$ *are represented by strictly* 3*-valent graphs.*

For a graph $\Gamma \in \mathcal{P}^{2\uparrow 2}$, we define another reduced graph $R'(\Gamma)$, obtained by contracting subgraphs



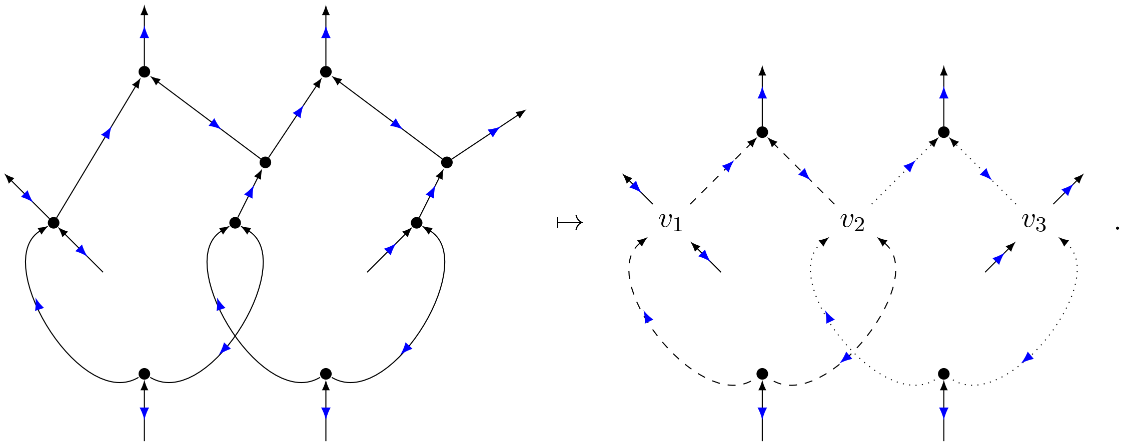


Figure 2.1: The graph to the left reduces to the graph to the right. We labeled the four valent star corollas by v_1, v_2 and v_3, and we marked the wheels by dotted and dashed edges.

One can see that the differential δ_0 preserves $R'(\Gamma)$. Hence, $(\mathcal{P}^{2\uparrow 2}, \delta_0)$ splits

$$\mathcal{P}^{2\uparrow 2} \cong \bigoplus_{\gamma} (\mathcal{C}_\gamma, \delta_0),$$

where the sum runs over all reduced graphs γ, and $(\mathcal{C}_\gamma, \delta_0)$ is the dg vector space spanned by graphs with $R'(\Gamma) = \gamma$.

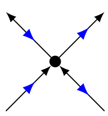
However, as $R'(\Gamma)$ may contain cycles in the second (blue) direction (see Figure 2.1), we do not get that each $(\mathcal{C}_\gamma, \delta_0)$ is a tensor product of simple dg vector spaces.

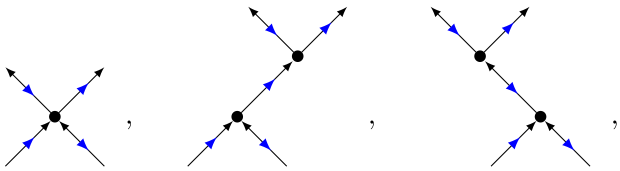
For this purpose, consider the 2-directed 1-oriented properad $(\mathcal{P}^{2\uparrow 1}, \delta_0) \supset (\mathcal{P}^{2\uparrow 2}, \delta_0)$, generated by the same corollas and relations as $\mathcal{P}^{2\uparrow 2}$, except that we allow wheels in the blue direction. We have that

$$(\mathcal{P}^{2\uparrow 1}, \delta_0) \cong \bigoplus_{\gamma} (\mathcal{C}^{\circlearrowleft}_\gamma, \delta_0),$$

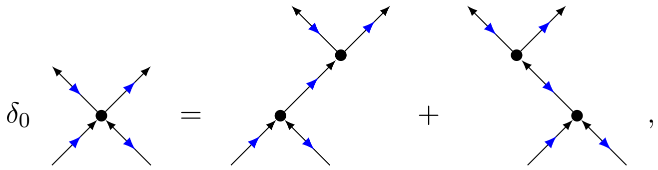
where $\mathcal{C}^{\circlearrowleft}_\gamma \supset \mathcal{C}_\gamma$ is the dg vector space spanned by graphs $\Gamma \in \mathcal{P}^{2\uparrow 1}$ with $R'(\Gamma) = \gamma$. Here, we do have

$$\mathcal{C}^{\circlearrowleft}_\gamma \cong \left(\bigotimes_{v \in S(\gamma)} (D_v, \delta_0) \right)^{\mathbb{S}_\gamma},$$

where $S(\gamma)$ is the set of  corollas in γ, and D_v is a 3-dimensional dg vector space spanned by substitutions of



to the vertex $v \in S(\gamma)$, with



and $\mathbb{S}_\gamma$ is the symmetry group of γ.

As $H(D_v)$ is one-dimensional, it is easy to see the following lemma:

Lemma 2.3.9. *For each reduced graph γ, we have that $H(C_\gamma^{\circlearrowleft})$ is one-dimensional, and the cohomology class is represented by a 3-regular graph.*

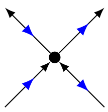
However, as $\mathcal{C}_\gamma^{\circlearrowleft}/\mathcal{C}_\gamma$ is not a sub-dg vector space, i.e. $(\mathcal{C}_\gamma^{\circlearrowleft}, \delta_0)$ does not split

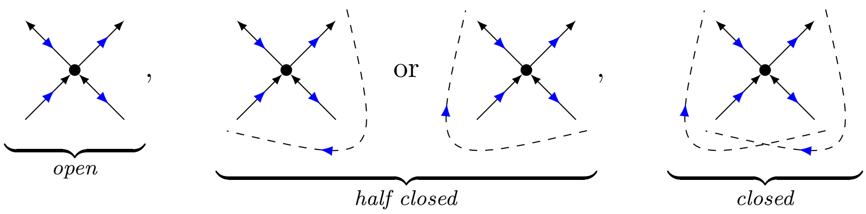
$$(\mathcal{C}_\gamma^{\circlearrowleft}, \delta_0) \ncong (\mathcal{C}_\gamma, \delta_0) \oplus (\mathcal{C}_\gamma^{\circlearrowleft}/\mathcal{C}_\gamma, \delta_0),$$

the following lemma, which finishes the proof of Theorem 2.3.1, is not as clear.

Lemma 2.3.10. *For each reduced graph γ, we have that $H(\mathcal{C}_\gamma, \delta_0) \subset H(\mathcal{C}_\gamma^{\circlearrowleft}, \delta_0)$. More precisely, $H(\mathcal{C}_\gamma, \delta_0)$ is either one-dimensional with a cohomology class represented by any strictly 3-valent graph in $\mathcal{C}_\gamma$, or $\mathcal{C}_\gamma$ is empty.*

If $S(\gamma)$ is empty, then it is clear that $\mathcal{C}_\gamma$ is one-dimensional. Assume that $\mathcal{C}_{\gamma'}$ is one-dimensional for each γ' with $|S(\gamma')| < n$, and let γ be a reduced graph with $|S(\gamma)| = n$.

Each  $\in S(\gamma)$ may be "tangled up" in the wheels of γ in one of three possible ways, pictured and named as



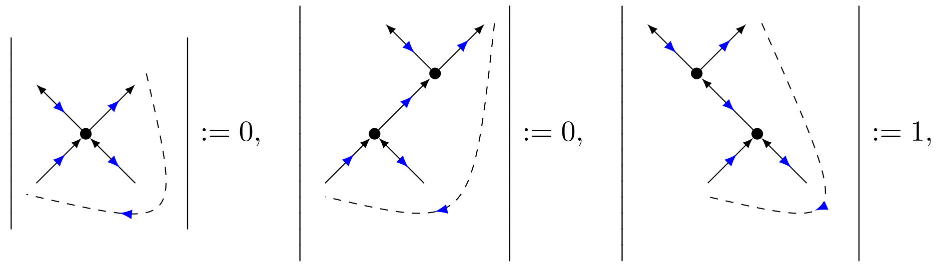
(2.19)

We denote the set of open corollas by $F(\gamma)$, the set of half closed corollas by $hB(\gamma)$, and the set of closed corollas by $B(\gamma)$.

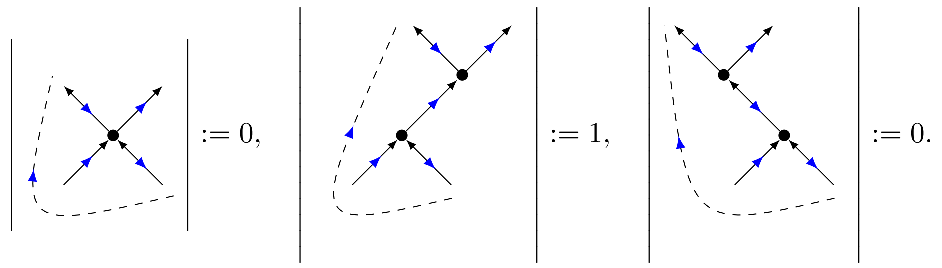
Remark 2.3.11. Note that, if γ contains a wheel, then γ must contain at least one half closed corolla. Otherwise, $\mathcal{C}_\gamma$ is empty.

We will put a weight grading on each D_v, by setting the weight of a substitution to 1 if it "breaks a wheel" in γ, and 0 if it does not. That is:

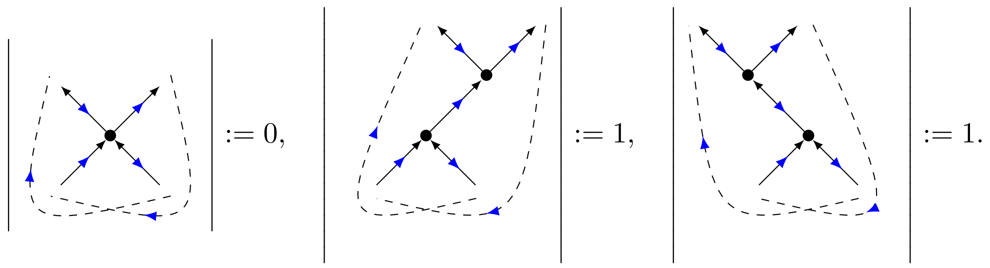
1. If v is open, then all substitutions carry degree 0.
2. If v is half closed, then



or in the opposite case



3. If v is closed, then



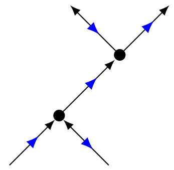
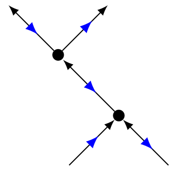
The differential can only increase the weight grading (by 1 or 0). Hence, this weight grading induces a spectral sequence on $\mathcal{C}_\gamma$ and on $\mathcal{C}_\gamma^\circlearrowleft$. On the associated graded complexes, which we will denote $(\mathcal{C}_\gamma^\circlearrowleft, d_0)$ and $(\mathcal{C}_\gamma, d_0)$, we do have that

$$(\mathcal{C}_\gamma^\circlearrowleft, d_0) \cong (\mathcal{C}_\gamma, d_0) \oplus (\mathcal{C}_\gamma^\circlearrowleft/\mathcal{C}_\gamma, d_0),$$

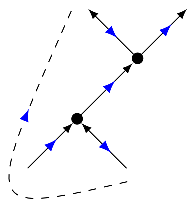
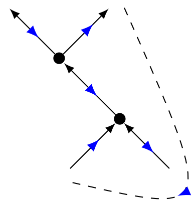
Hence,

$$(\mathcal{C}_\gamma^\circlearrowleft/\mathcal{C}_\gamma, d_0) \oplus (\mathcal{C}_\gamma, d_0) \cong (\mathcal{C}_\gamma^\circlearrowleft, d_0) \cong \bigotimes_{v\in F(\gamma)} (D_v, d_0(=\delta_0)) \bigotimes_{v\in hB(\gamma)} (D_v, d_0) \bigotimes_{v\in B(\gamma)} (D_v, d_0(=0)).$$

For the open star corollas $v \in F(\gamma)$, the cohomology of $(D_v, d_0) \cong (D_v, \delta_0)$ is one-dimensional, with a cohomology class represented by either of the extended substitutions;

$H(D_v, d_0) = \Big\langle \Big[$  $\Big] = \Big[$  $\Big] \Big\rangle.$ (2.20)

For the half closed star corollas $v \in hB(\gamma)$, the cohomology of (D_v, d_0) is again one-dimensional, however, the cohomology class is only represented by the substitution that breaks the wheels passing through v;

$H(D_v, d_0) = \Big\langle \Big[$  $\Big] \Big\rangle \Big($ or $H(D_v, d_0) = \Big\langle \Big[$  $\Big] \Big\rangle \Big).$ (2.21)

Now, consider the second page of both spectral sequences, say $(E_2^{\circlearrowleft}, d_1)$ and (E_2, d_1),

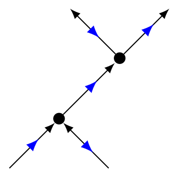
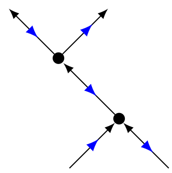
$$E_2^{\circlearrowleft} \cong H(\mathcal{C}_\gamma^{\circlearrowleft}, d_0),$$

and

$$E_2 \cong H(\mathcal{C}_\gamma^{\circlearrowleft}, d_0).$$

By (2.20) and (2.21), we get that $E_2^{\circlearrowleft}$ consists of graphs in $\Gamma \in \mathcal{C}^{\circlearrowleft}$, while E_2 consists of graphs in $\mathcal{C}_\gamma$, such that:

1. The half closed star corollas in γ are extended the way that breaks the wheels passing through them.

2. On the open corollas, we identify  with $-$  .

It follows that

$$(E_2, d_1) \subset (E_2^{\circlearrowleft}, d_1) \cong \bigotimes_{v \in B(\gamma)} (D_v, \delta_0).$$

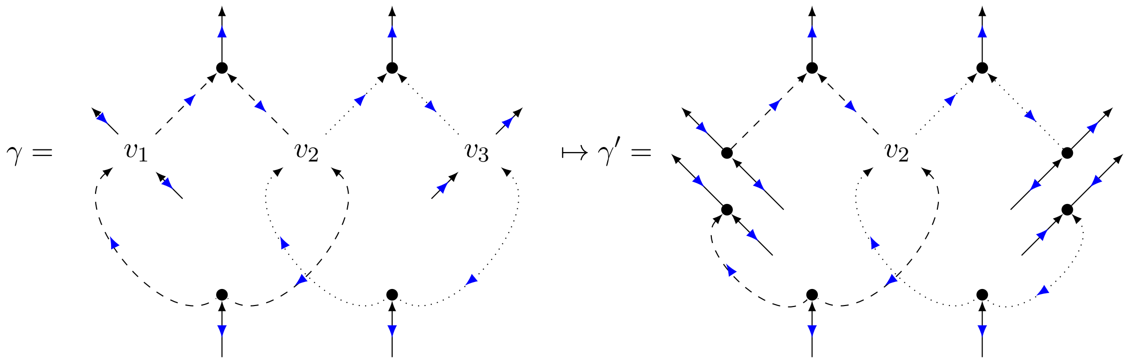
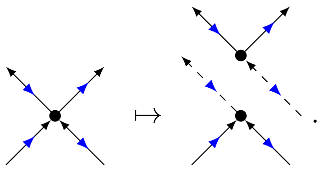


Figure 2.2: One can see that the vertices v_1 and v_3 are "half closed", while v_2 is "closed" in γ. In γ', the vertex v_2 is "open".

Lemma 2.3.12. *The second page of this spectral sequence, (E_2, d_1), is identical to a dg vector space $(\mathcal{C}_{\gamma'}, \delta_0)$ where γ' is a reduced graph obtained from γ by substituting the open and the half closed corollas*



Here, the dashed arrows are external legs. See Figure 2.2.

Proof. One can see that there is a canonical isomorphism

$$i : (E_2^{\circlearrowleft}, d_1) = \bigotimes_{v \in B(\gamma)} (D_v, \delta_0) \to \bigotimes_{v \in S(\gamma')} (D_v, \delta_0) = (\mathcal{C}_{\gamma'}^{\circlearrowleft}, \delta_0).$$

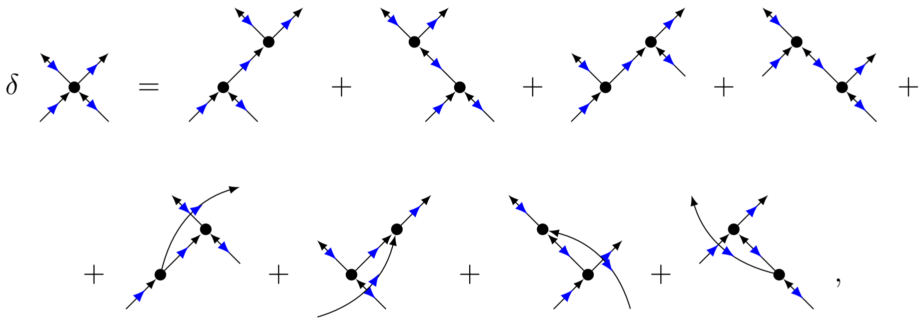
Furthermore, as the graphs in $E_2^{\circlearrowleft}$ does not have wheels passing through the half closed corollas, this map sends oriented graphs precisely to oriented graphs. Hence, it restricts to an isomorphism

$$(E_2, d_1) \to (\mathcal{C}_{\gamma'}, \delta_0).$$

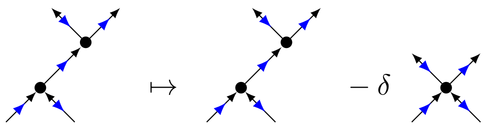
□

As γ' contains fewer 4-valent vertices than γ, our induction assumption gives us that $H(C_\gamma) \cong H(E_2) \cong H(C_{\gamma'})$ is spanned by 3-valent corollas. This finishes the proof of Theorem 2.3.1.

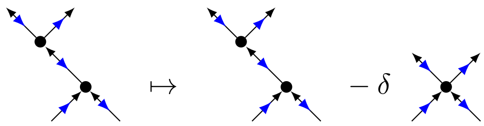
Remark 2.3.13. If we instead try to follow B. Vallette's proof [47] of $H(\mathsf{hoLieB}) \cong \mathsf{LieB}$, we would find that the relation

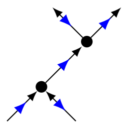
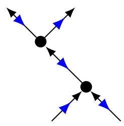


in $\mathsf{LieB}_d^{2\uparrow 2}$, does not have a simple replacement rule. This could possibly be fixed by a 'replacement algorithm', which determines whether to replace



or

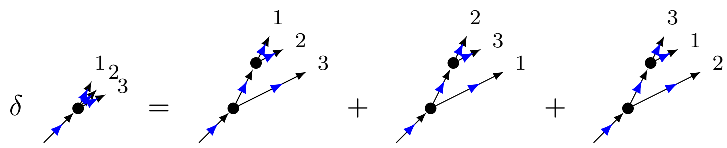


depending on whether the  or  belongs to an open, closed or half closed corolla of a reduced graph $R'(\Gamma)$.

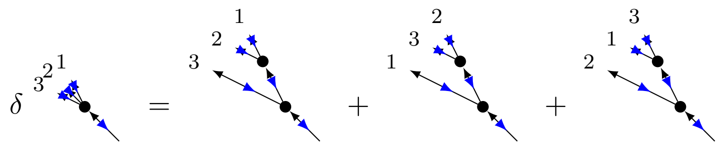
2.3.3 Relations for 2-directed Lie bialgebras

In this section, we will give full formulas for the 19 relations in $\mathsf{LieB}_0^{2\uparrow 2}$, $\mathsf{LieB}_0^{2\uparrow 1}$ and $\mathsf{LieB}_0^{2\uparrow 0}$. They are as follows:

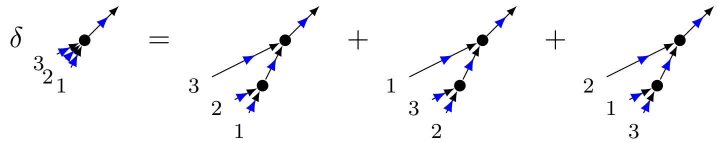
1.



2.



3.



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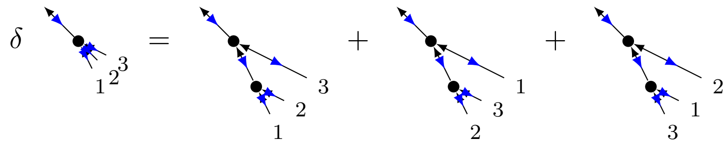
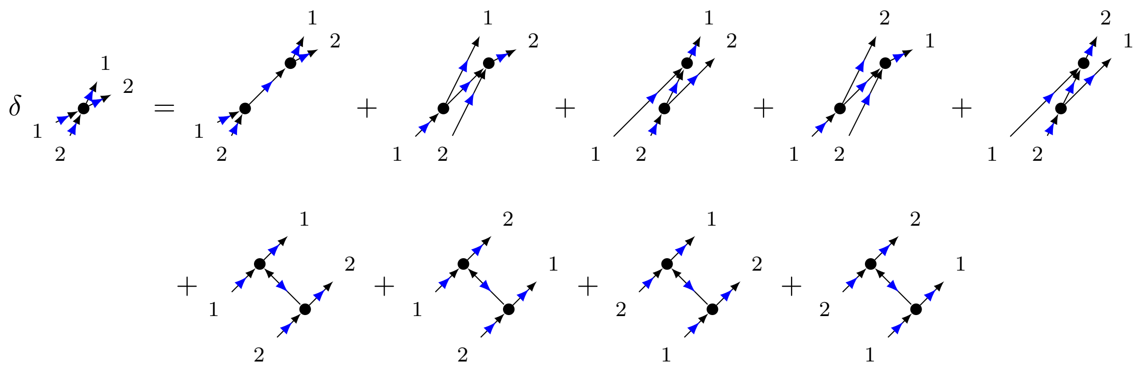
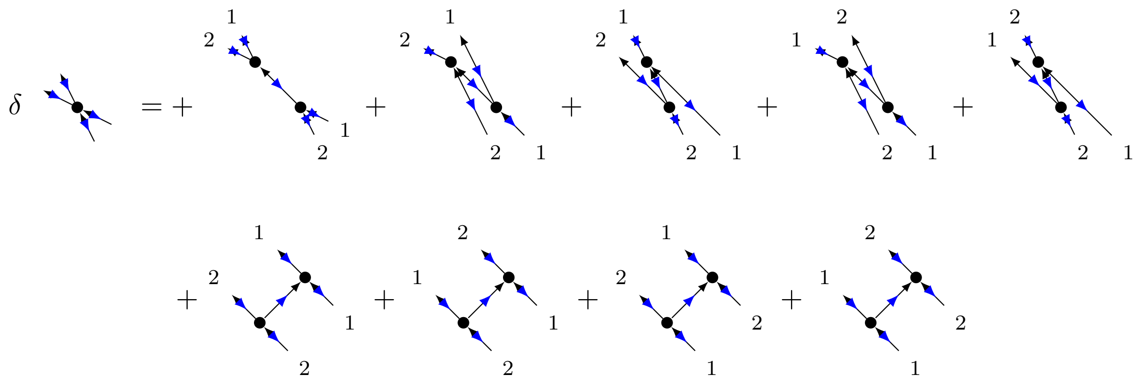


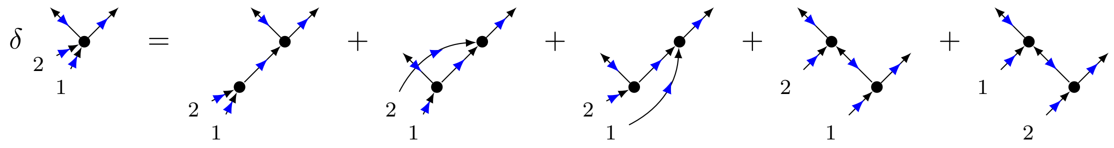
5.



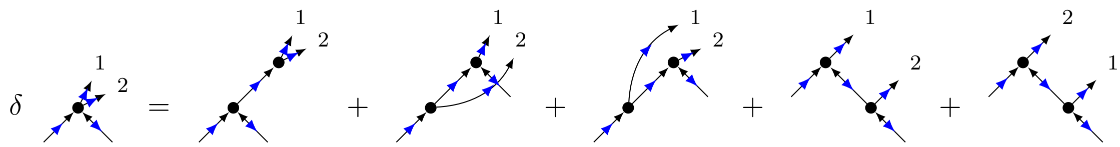
6.



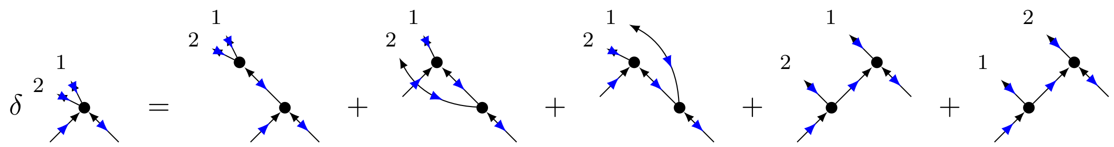
7.



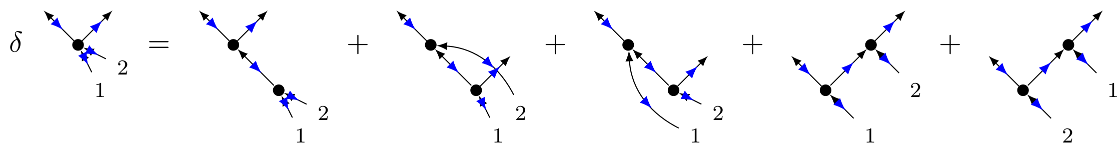
8.



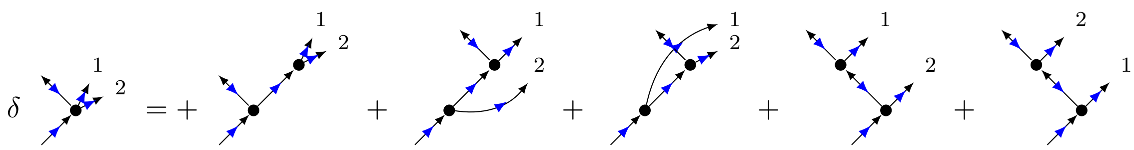
9.



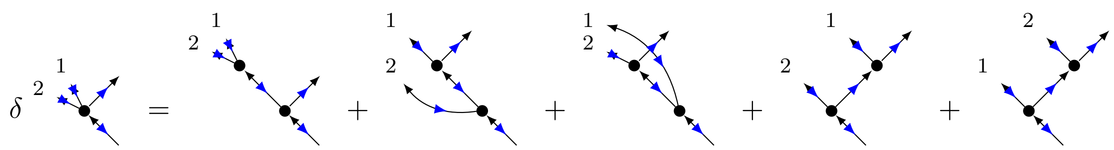
10.



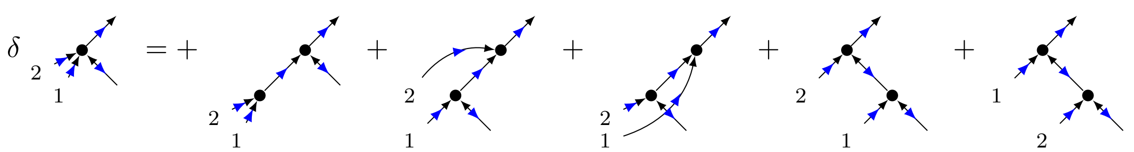
11.



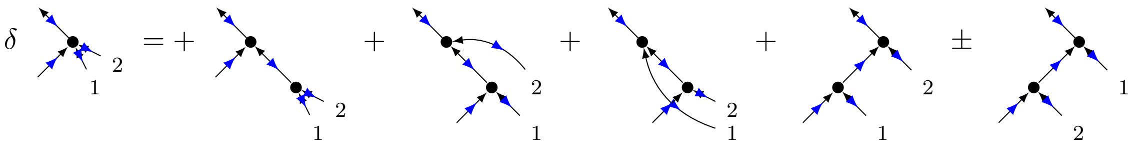
12.



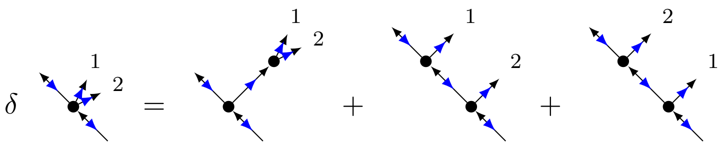
13.



14.



15.



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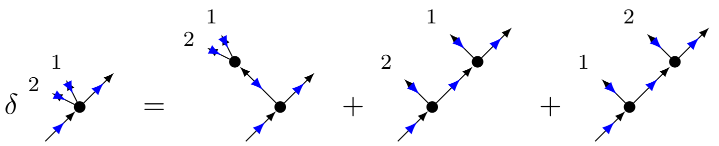
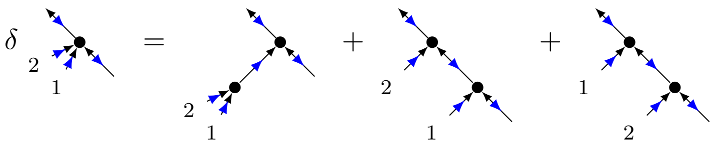
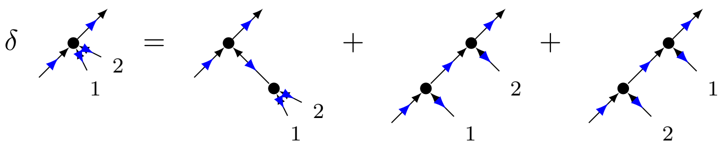


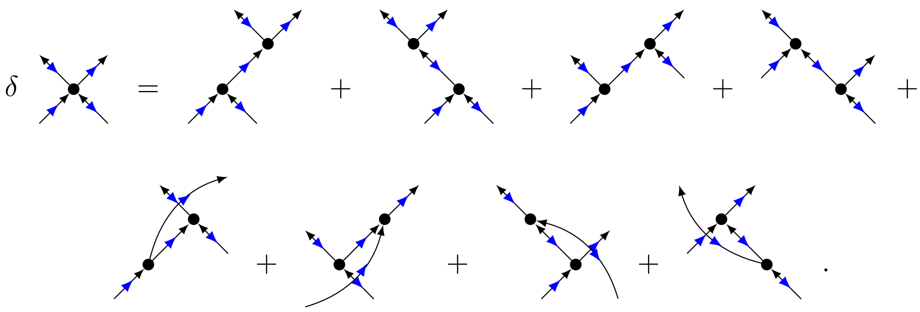
17.



18.



19.



The relations of $\mathsf{LieB}_d^{2\uparrow 2}$, for $d \neq 0$ are as above, but with additional signs.

Chapter 3

A fixed source graph complex

Introduction

The main motivation for the work in this chapter is the explicit morphism of dg vector spaces

$$F : (\mathrm{OGC}_1, \delta) \to (\mathrm{RGC}[1], \delta + \Delta_1)$$

constructed by S. Merkulov and T. Willwacher in [42]. Here, (OGC_1, δ) is the oriented version of Kontsevich's graph complex, and $(\mathrm{RGC}[1], \delta + \Delta_1)$ is a dg vector space spanned by ribbon graphs. Ribbon graphs (sometimes called fat graphs) model Riemann surfaces with marked points [44]. For our ribbon graph complex RGC, we get

$$H^k(\mathrm{RGC}, \delta) \cong \prod_{g,n} \left(H_c^{k-n}(\mathcal{M}_{g,n}, \mathbb{Q}) \otimes \mathrm{sgn}_n \right)^{\mathbb{S}_n} \oplus \begin{cases} \mathbb{Q} & \text{for } k = 1, 5, 9, \dots \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where $H_c(\mathcal{M}_{g,n}, \mathbb{Q})$ is the compact support cohomology of the moduli space of Riemann surfaces of genus g with n marked points (see [28] for more details). In this context, the differential $\delta + \Delta_1$ constructed in [42] is a deformation of the classical differential δ on RGC. A simple observation gives that the same explicit formula F is also a map of dg vector spaces

$$F : (\mathrm{OGC}_1, \delta_0) \to (\mathrm{RGC}[1], \delta), \tag{3.2}$$

where, this time, it is the differential δ on OGC_1 that is not standard. The differential δ_0 splits vertices of graphs in OGC in a way that preserves the number of target vertices. In this chapter, we show that there is a quasi-isomorphism from the oriented graph complex with this new differential to the better known hairy graph complex $\mathrm{HGC}_{n,n}$, studied in e.g. [7], [22].

Theorem 3.0.1. *There is a map of graded vector spaces*

$$\Phi : \mathrm{OGC}_{n+1} \to \mathrm{HGC}_{n,n},$$

such that the associated morphisms of dg vector spaces

$$\Phi : (\mathrm{OGC}_{n+1}, \delta_0) \to (\mathrm{HGC}_{n,n}, \delta),$$

and

$$\Phi : (\mathrm{OGC}_{n+1}, d) \to (\mathrm{HGC}_{n,n}, \delta + \chi)$$

are quasi-isomorphisms. Here, χ is the extra differential on $\mathrm{HGC}_{d,d}$ that adds a hair, considered in [22].

As a corollary, we get a relationship between the ribbon graph complex and the hairy graph complex.

Corollary 3.0.2. *We have an explicit zig-zag of morphisms*

$$(\mathrm{HGC}_{0,0}, \delta) \leftarrow (\mathsf{O}\mathrm{GC}_1, \delta_0) \to (\mathrm{RGC}[1], \delta),$$

where the left map is a quasi-isomorphism.

A recent result by M. Chan, S. Galatius and S. Payne [12] states that there exists an embedding

$$H^k(\mathrm{GC}_0^{[n]}, d) \to \prod_g H_c^k(\mathcal{M}_{g,n}, \mathbb{Q}).$$

Here, $\mathrm{GC}^{[n]}$ is a dg vector space spanned by hairy graphs where each hair is labeled by an integer in $[n]$. The map $\Phi : (\mathrm{OGC}_1, \delta_0) \to (\mathrm{RGC}[1], \delta)$ is believed to be related to this embedding.

This chapter is based on joint work with M. Živković [6].

3.1 The hairy graph complex

The hairy graph complex $(\mathrm{HGC}_{d,c}, \delta)$ is a version of Kontsevich Graph complex, where graphs are allowed to have external legs, which we in this context call *hairs*.

Let $\mathrm{H}_k\mathcal{G}_{n,m}$ be the set of connected graphs without univalent vertices, with vertices labeled by $[n]$, internal edges labeled by $[m]$, and hairs labeled by $[k]$. That is connected graphs $\Gamma = (F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma)$ in Gr, with flag set $F_\Gamma = [m]_1 \sqcup [m]_2 \sqcup [k]$, vertex set $V_\Gamma = [n]$, involution $\iota_\Gamma = \iota_m \sqcup \mathrm{id}_{[k]}$, and $|\partial_\Gamma^{-1}(v)| \geq 2$ for all vertices $v \in V_\Gamma$. The group $\mathbb{S}^{\mathrm{H}\mathcal{G}}_{n,m,k} := \mathbb{S}_n \times \mathbb{S}_m \times \mathbb{S}_2^m \times \mathbb{S}_k$ acts on $\mathrm{H}_k\mathcal{G}_{n,m}$ by permuting vertex labels, permuting edge labels, flipping the intrinsic direction of internal edges, and permuting hair labels. That is

$$(\sigma, \tau_1, \tau_2, \upsilon)(F_\Gamma, V_\Gamma, \partial_\Gamma, \iota_\Gamma) = (F_\Gamma, V_\Gamma, \sigma \circ \partial_\Gamma \circ ((\tau, \tau_2)^{-1} \sqcup \upsilon^{-1}), \iota_\Gamma), \quad \sigma \in \mathbb{S}_n,$$

for all $(\tau_1, \tau_2) \in \mathbb{S}_m \times \mathbb{S}_2^{\times m}, \upsilon \in \mathbb{S}_k$.

Let

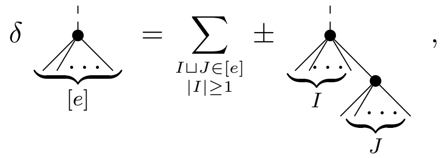
$$\mathrm{Hgra}_d(n, m, k) := \mathbb{k}(\mathrm{H}_k\mathcal{G}_{n,m})[d(n-1) + (1-d)m - k + 1],$$

and let

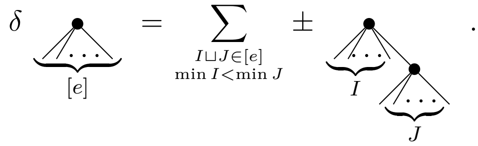
$$\mathrm{HGC}_d := \begin{cases} \prod_{n\geq 1, m\geq 0, k\geq 1} \mathrm{Hgra}_d(n,m,k) \otimes_{\mathbb{S}^{\mathrm{H}\mathcal{G}}_{n,m,k}} (\mathrm{sgn}_n \otimes \mathbb{k} \otimes \mathrm{sgn}_2^{\otimes m} \otimes \mathrm{sgn}_k) & d \text{ odd}, \\ \prod_{n\geq 1, m\geq 0, k\geq 1} \mathrm{Hgra}_d(n,m,k) \otimes_{\mathbb{S}^{\mathrm{H}\mathcal{G}}_{n,m,k}} (\mathbb{k} \otimes \mathrm{sgn}_m \otimes \mathbb{k} \otimes \mathrm{sgn}_k) & d \text{ even}. \end{cases}$$

Remark 3.1.1. Here, we define the hairy graph complex HGC_d so that hairs have degree 1. It is also possible (and required for some applications, for example in [7]) to define the graph complex so that the hairs have other degrees. In our hairy graph complex HGC_d, we do not allow graphs without hairs, however, in some applications it may be more suitable to also include hairless graphs.

The differential δ is given by vertex splitting as in GC, minus the terms created where a hairy vertex only has one adjacent internal edge. This means that δ acts on each vertex with e adjacent internal edges, and one adjacent hair, by

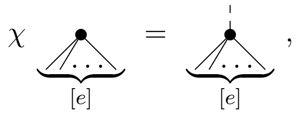


where the dashed line represents the hair. On non-hairy vertices, δ acts by



The sign is determined by giving the new edge the last edge label, while the lower of the two vertices inherits the label of the split vertex, the upper vertex gets the last vertex label, and the new edge is intrinsically oriented from the lower vertex to the upper vertex.

It was shown in [22] that HGC_d admits an extra differential χ given by attaching a hair to each vertex



where the sign is determined by giving the new hair the last hair label.

Proposition 3.1.2 (A. Khoroshkin, T. Willwacher, M. Živković [22]). *If we require graphs in* HGC_d *to have at least one hair, we get*

$$H(\mathrm{HGC}_d, \delta + \chi) \cong H(\mathrm{GC}_d, \delta).$$

3.1.1 A fixed source graph complex

To clear up some space for superscripts, let us denote the set of 1-oriented graphs and the 1-oriented graph complex by $\mathsf{O}\mathcal{G}$ and $\mathsf{O}\mathrm{GC}_d$ respectively That is

$$\mathsf{O}\mathcal{G}_{n,m} := \mathcal{G}^{1\uparrow 1}_{n,m},$$

and

$$\mathsf{OGC}_d := \mathrm{GC}_d^{1\uparrow 1}.$$

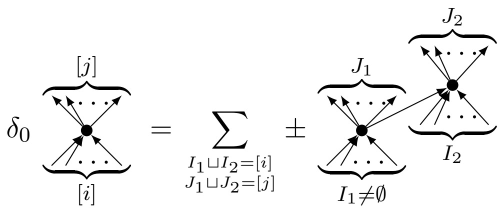
For an oriented graph Γ, let $w_{source}(\Gamma)$ be the number of source vertices in Γ. Note that splitting a vertex of Γ cannot decrease the number of source vertices in Γ. Hence, we obtain a filtration

$$F_p(\mathsf{OGC}, \delta) := (\{\Gamma \in \mathsf{OGC} : \ w_{source}(\Gamma) \geq p\}, \delta).$$

The associated graded complex

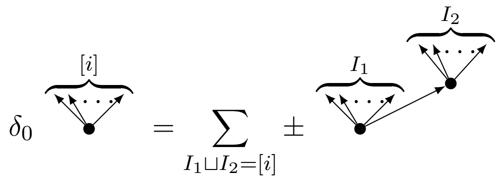
$$(\mathsf{OGC}, \delta_0) := \left(\prod_{p \geq 1} F_p \mathsf{OGC} / F_{p+1} \mathsf{OGC}, \delta \right)$$

admits a differential δ_0 given by vertex splitting without creating source vertices. That is



$, \quad J_1 \sqcup I_1 \neq \emptyset, \quad J_2 \sqcup I_2 \neq \emptyset,$

for non-source vertices, and



$, \quad I_1, I_2 \neq \emptyset,$

for source vertices. We will call the graph complex $(\mathsf{OGC}_d, \delta_0)$ the *fixed source oriented graph complex*.

3.1.2 A map $\mathrm{OGC}_{d+1} \to \mathrm{HGC}_d$

In this section, we will create a map of graded vector spaces

$$\Phi : \mathsf{OGC}_{d+1} \to \mathrm{HGC}_d.$$

Let $\mathsf{O}_{k,t}\mathcal{G}_{n,m} \subset \mathsf{O}\mathcal{G}_{n,m}$ be the set of oriented graphs with k source vertices and t bi valent target vertices. For a graph $\Gamma \in \mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$, we define the *hairy skeleton* $hs(\Gamma)$ to be the isomorphism class of hairy graphs with n vertices, m internal edges and k hairs, obtained from Γ by:

1. Putting a hair on each source vertex.
2. Contracting each occurrence of bivalent target vertices $\rightarrow\bullet\leftarrow$ into a single non-oriented edge.

3. Forgetting the orientation of all other edges.

For a graph $\Gamma \in \mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$, the map Φ is roughly defined as

$$\Phi(\Gamma) := \begin{cases} hs(\Gamma) & \text{if } t = m - n + k \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Unfortunately, (3.3) does not provide enough information to say that it is a well defined map of graded vector spaces. In order to make a well defined map, let us compare the graded vector spaces

$$\mathsf{OGC}_{d+1} \cong \prod_{n,m} \Bbbk(\mathsf{O}\mathcal{G}_{n,m})[(d+1)(n-1) - dm]/(\Gamma - \operatorname{sgn}_{d+1}(\sigma)\sigma\Gamma)_{\Gamma \in \mathsf{O}\mathcal{G}_{n,m}),\sigma \in \mathbb{S}^{\mathcal{G}}_{n,m}},$$

where $\operatorname{sgn}_{d+1}(\sigma_1, \sigma_2, \sigma_3) = (\operatorname{sgn}(\sigma_1)^{d+1} \operatorname{sgn}(\sigma_2)^d \operatorname{sgn}(\sigma_3)^{d+1})$, for

$$(\sigma_v, \sigma_e, \tau_e) \in \mathbb{S}_n \times \mathbb{S}_m \times \mathbb{S}_2^{\times m} = \mathbb{S}^{\mathcal{G}}_{n,m};$$

and +*-

$$\mathrm{HGC}_d \cong \prod_{n,m,k} \Bbbk(\mathrm{H}_k\mathcal{G}_{n,m})[d(n-1) + (1-d)m + k]/(\Gamma - \operatorname{sgn}_d(\tau)\tau\Gamma)_{\Gamma \in \mathrm{H}_k\mathcal{G}_{n,m}),\tau \in \mathbb{S}^{\mathrm{H}\mathcal{G}}_{n,m,k}},$$

where $\operatorname{sgn}_d(\tau_1, \tau_2, \tau_3, \tau_4) = \operatorname{sgn}(\tau_1)^d \operatorname{sgn}(\tau_2)^{d+1} \operatorname{sgn}(\tau_3)^d \operatorname{sgn}(\tau_4)$, for $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathbb{S}_n \times \mathbb{S}_m \times \mathbb{S}_2^{\times m} \times \mathbb{S}_k = \mathbb{S}^{\mathrm{H}\mathcal{G}}_{n,m,k}$.

Let us first check that Φ is of degree 0. Set $t = m - n + k$, we have that a graph $\Gamma \in \mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t} = \mathsf{O}_{k,m-n+k}\mathcal{G}_{m+k,2m-n+k}$ is of degree

$$\deg(\Gamma) = (d+1)(m+k-1) - d(2m-n+k) = d(n-1) + (1-d)m + k = \deg(\Phi\Gamma),$$

which is what we need.

To properly define Φ, we will complete the following steps:

1. For $t = m - n + k$, we define a subset $\widetilde{\mathsf{O}_{k,t}\mathcal{G}}_{n+t,m+t} \subset \mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$, called the *model enumerated graphs*, such that each graph $\Gamma \in \mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$ is isomorphic to a graph $\Gamma' \in \widetilde{\mathsf{O}_{k,t}\mathcal{G}}_{n+t,m+t}$. It can then be concluded that there is an isomorphism of vector spaces

$$\mathsf{O}_{k,t}\mathrm{GC}_{d+1} \to \prod_{n-m=t-k} \widetilde{\mathsf{O}_{k,t}\mathcal{G}}_{n+t,m+t}[d(n-1) + (1-d)m + k]/\sim,$$

where $\sim$ is the relation $\Gamma = \operatorname{sgn}_{d+1}(\sigma)\sigma\Gamma$ for all $\Gamma \in \widetilde{\mathsf{O}_{k,t}\mathcal{G}}_{n+t,m+t}$ and $\sigma \in \mathbb{S}^{\widetilde{\mathcal{G}}}$, where $\mathsf{O}_{k,t}\mathrm{GC}_{d+1} \subset \mathsf{OGC}_{d+1}$ is the sub vector space spanned by graphs in $\mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$, and $\mathbb{S}^{\widetilde{\mathcal{G}}}$ is the set of isomorphisms between graphs in $\widetilde{\mathsf{O}_{k,t}\mathcal{G}}_{n+t,m+t}$.

2. We associate each graph $\Gamma \in \widetilde{\mathsf{O}_{k,t}\mathcal{G}}_{n+t,m+t}$ with a graph $\widetilde{hs}(\Gamma) \in \mathrm{H}_k\mathcal{G}_{n,m}$, of the isomorphism class $hs(\Gamma)$.

3. For any isomorphism $\sigma(: \Gamma_1 \to \Gamma_2) \in \mathbb{S}^{\widetilde{\mathcal{G}}}$, let τ be the isomorphism of hairy graphs $\tau : \widetilde{hs}(\Gamma_1) \to \widetilde{hs}(\Gamma_2)$. We will show that $\mathrm{sgn}_{d+1}(\sigma) = \mathrm{sgn}_d(\tau)$. As a consequence we get a map of graded vector spaces

$$\prod_{n,m} \widetilde{\mathsf{O}\mathcal{G}}_{n,m}[d(n-1) + (1-d)m + k]/ \sim \to \mathrm{HGC}_d.$$

We can then give a proper definition of $\Phi : \mathrm{OGC}_{d+1} \to \mathrm{HGC}_d$, by

$$\Phi(\Gamma) := \begin{cases} \mathrm{sgn}_{d+1}(\sigma_\Gamma)\widetilde{hs}(\sigma_\Gamma \Gamma) & \text{if } \Gamma \in \mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}, \text{ for } t = m - n + k \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

where σ_Γ is an isomorphism $\Gamma \to \Gamma'$ for some arbitrary model enumerated graph Γ'.

3.1.3 Counting sources

Lemma 3.1.3. *If $t = m - n + k$, then each vertex of a graph $\Gamma \in \mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$ is either a source vertex, a bivalent target vertex, or it has a unique incoming edge.*

Proof. Consider the graph γ obtained from Γ by removing each bivalent target vertex and its adjacent edges. The graph γ then contains n vertices and $m - (m - n + k) = n - k$ edges. Such a graph γ must contain at least k separate connected components. Since γ is oriented, each connected component must contain at least 1 source vertex. As γ contains precisely k source vertices, we must have that γ contains precisely k connected components $\tau_1, \ldots, \tau_k$ with 1 source vertex each. Furthermore, γ contains precisely k components if and only if each component τ_i is of genus 0 i.e. each τ_i is a tree. Finally, we note that each tree τ_i contains precisely 1 source vertex if and only if all edges in τ_i are directed outwards from the single source vertex. □

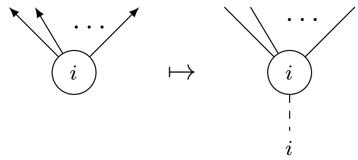
Let $t = m - n + k$, we say that a graph $\Gamma \in \mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$ with k source vertices is of *model enumeration* if:

1. The source vertices occupy vertex labels $1, \ldots, k$;
2. The bivalent target vertices occupy the last vertex labels, $n + 1, \ldots, n + t$;
3. The two adjacent edges to a bivalent target vertex $n + i$, are labeled $m - t + i$ and $m + i$.
4. The unique incoming edge to each vertex $i \in \{k + 1, n\}$ is labeled by i.
5. The intrinsic orientation agrees with the orientation for all edges.

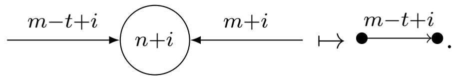
Let $\widetilde{\mathsf{O}_{k,t}\mathcal{G}}_{n+t,m+t}$ be the set of model enumerated graphs. It is clear that any graph in $\mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$ is isomorphic to a (several) model enumerated graph(s). Hence, step 1 is completed.

For each graph $\Gamma \in \widetilde{\mathsf{O}_{k,t}\mathcal{G}}_{n+t,m+t}$, let $\widetilde{hs}(\Gamma) \in \mathrm{H}_k\mathcal{G}_{n,m}$ be the graph obtained from Γ by:

1. Forgetting the orientation of the edges. (As the intrinsic orientation agrees with the orientation for each edge of a graph in $\mathsf{O}\widetilde{\mathcal{G}}$, the orientation is not really forgotten).
2. Attaching a hair labeled i to each vertex $i = 1, \dots, k$



3. Contracting the edges $m+1, \dots, m+t$



This completes step 2. The following Lemma will complete step 3.

Lemma 3.1.4. *Let Γ_1, Γ_2 be two model enumerated graphs in $\mathsf{O}_{k,t}\mathcal{G}_{n+t,m+t}$, let $\sigma \in \mathbb{S}^{\mathcal{G}}_{n+t,m+t}$ be a permutation that induces an isomorphism*

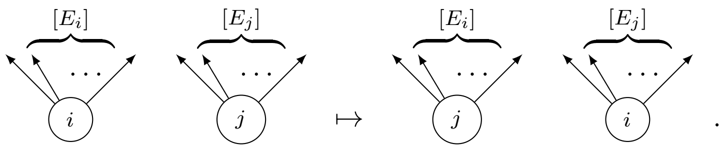
$$\sigma : \Gamma_1 \to \Gamma_2,$$

and let $\tau \in \mathbb{S}^{\mathrm{H}\mathcal{G}}$ be the permutation that induces an isomorphism

$$\tau : \widetilde{hs}(\Gamma_1) \to \widetilde{hs}(\Gamma_2).$$

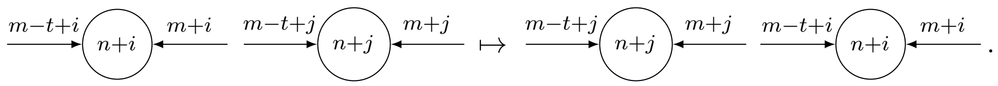
Then $\mathrm{sgn}_{d+1}(\sigma) = \mathrm{sgn}_d(\tau)$.

Proof. We may write $\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_4$, where the permutation $\sigma_1 \in \mathbb{S}_k$ permutes the source vertices, e.g.



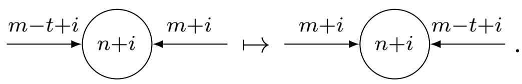
The graded sign of such a permutation is given by $\mathrm{sgn}_{d+1}(\sigma_1) = \mathrm{sgn}(\sigma_1)^{d+1}$.

Next, $\sigma_2 \in \mathbb{S}_t$ permutes the labels of bivalent target vetrices along with their adjacent edges. e.g.



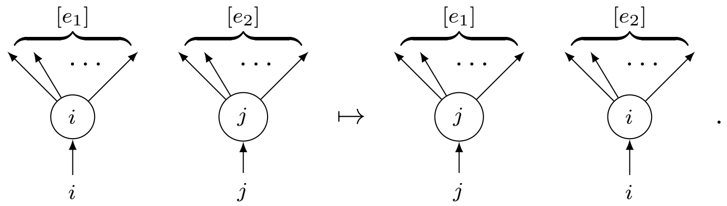
The graded sign of such a permutation is given by $\mathrm{sgn}_{d+1}(\sigma_2) = \mathrm{sgn}(\sigma_2)^{d+1}(\mathrm{sgn}(\sigma_2)^d)^2 = \mathrm{sgn}(\sigma_2)^{d+1}$.

Next, $\sigma_3 \in \mathbb{S}_2^{\times t}$ swaps the labels of two edges adjacent to a bivalent target vertex. e.g.



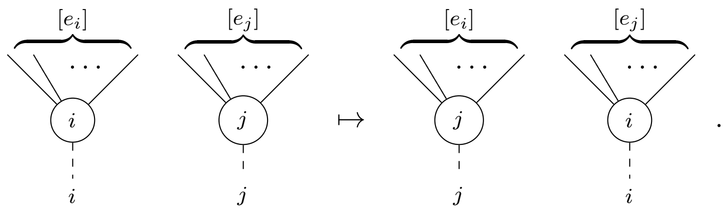
The graded sign of such a permutation is given by $\mathrm{sgn}_{d+1} = \mathrm{sgn}(\sigma_3)^d$.

Finally, $\sigma_4 \in \mathbb{S}_{n-k}$ permutes the labels of the remaining vertices, along with the labels of their adjacent ingoing edges



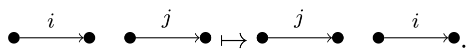
The graded sign of such a permutation is given by $\mathrm{sgn}(\sigma_4) = \mathrm{sgn}(\sigma_4)^{d+1}\,\mathrm{sgn}(\sigma_4)^d = \mathrm{sgn}(\sigma_4)$.

The associated isomorphisms $\tau = \tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_4 : \widetilde{hs}(\Gamma_1) \to \widetilde{hs}(\Gamma_2)$ are given respectively by a permutation $\tau_1 \in \mathbb{S}_k$ that permutes the labels of the hairy vertices along with the hair labels. e.g.



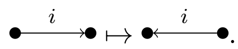
We get $\mathrm{sgn}_d(\tau_1) = \mathrm{sgn}(\tau)^{d+1} = \mathrm{sgn}_{d+1}(\sigma_1)$.

The permutation $\tau_2 \in \mathbb{S}_t$ permutes the labels of the edges $m-t+1, \ldots, m$



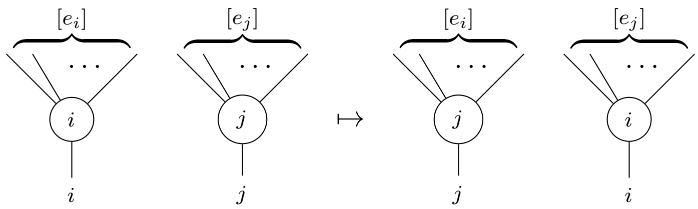
We get $\mathrm{sgn}_d(\tau_2) = \mathrm{sgn}(\tau_2)^{d+1} = \mathrm{sgn}_{d+1}(\sigma_2)$.

The permutation $\tau_3 \in \mathbb{S}_2^{\times t}$ swaps the intrinsic orientation of the edges $m-t+1, \ldots, m$,



We get $\mathrm{sgn}_d(\tau_3) = \mathrm{sgn}(\tau_3)^d = \mathrm{sgn}_{d+1}(\sigma_3)$.

Finally, $\tau_4 \in \mathbb{S}_{n-k}$ permutes the vertices $k+1, \ldots, n$ along with the edges $k+1, \ldots, n$, e.g.



We get $\mathrm{sgn}_d(\tau_4) = \mathrm{sgn}(\tau_4) = \mathrm{sgn}_{d+1}(\sigma_4)$. □

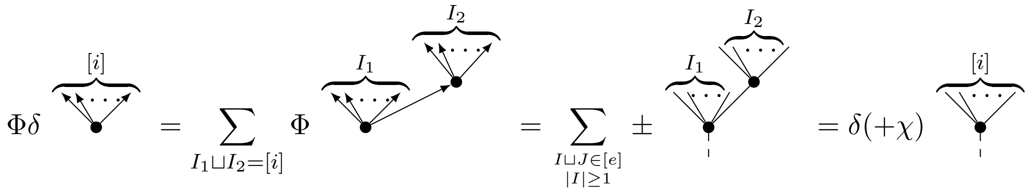
Proposition 3.1.5. *The map* Φ *is a map of dg vector spaces*

$$\Phi : (\mathsf{OGC}_{d+1}, \delta) \to (\mathrm{HGC}_d, \delta + \chi)$$

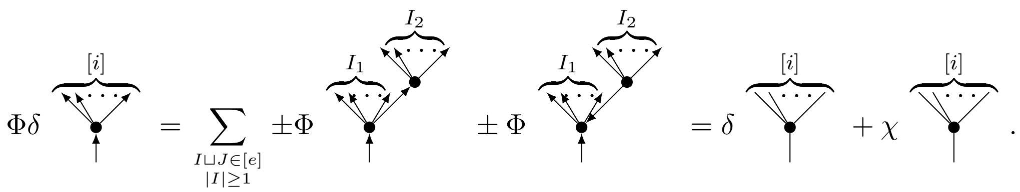
and

$$\Phi : (\mathsf{OGC}_{d+1}, \delta_0) \to (\mathrm{HGC}_d, \delta).$$

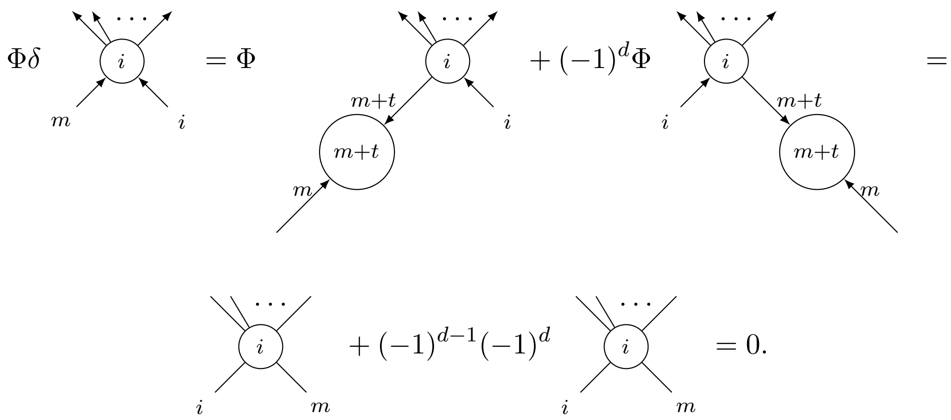
Proof. For source vertices, we have



For vertices with 1 ingoing edge, we have

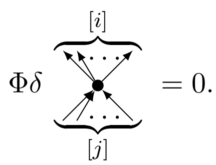


For a vertex with 2 ingoing legs, we get



Here, the $(-1)^d$ sign comes from a swap of edge labels in the oriented graph complex OGC_{d+1}, and the $(-1)^{d-1}$ sign comes from a swap of edge labels in HGC_d.

Finally, for a vertex with three or more ingoing legs, it is clear that



It follows that $\Phi\delta = (\delta + \chi)\Phi$. As Φ maps a graph with k sources to a hairy graph with precisely k hairs, it is clear that we also have $\Phi\delta_0 = \delta\Phi$. $\square$

3.1.4 The map Φ is a quasi-isomorphism

Theorem 3.1.6. *The maps*

$$\Phi : (\mathsf{O}\mathrm{GC}_{d+1}, \delta) \to (\mathrm{HGC}_d, \delta + \chi)$$

and

$$\Phi : (\mathsf{O}\mathrm{GC}_{d+1}, \delta_0) \to (\mathrm{HGC}_d, \delta)$$

are quasi-isomorphisms.

Remark 3.1.7. This is an alternative proof to the proof given in [6], where the dual complexes are considered.

Proof. It is enough to show that

$$\Phi : (\mathsf{O}\mathrm{GC}_{d+1}, \delta_0) \to (\mathrm{HGC}_d, \delta)$$

is a quasi-isomorphism, as it is the first page of a spectral sequence, it follows by the mapping lemma for spectral sequences [49], that the first map is also an isomorphism. For a graph $\Gamma \in \mathsf{O}\mathrm{GC}_d$, count the weight of a vertex $v \in V(\Gamma)$ by

$$w(v) := \begin{cases} \text{valence}(v) & \text{if } v \text{ is a source} \\ \text{valence}(v) - 1 & \text{if } v \text{ is not a source.} \end{cases},$$

and take a filtration on the number of vertices with $w(v) \geq 1$. The differential $\delta_{0,0}$ on the associated graded complex $(\mathsf{O}\mathrm{GC}_d, \delta_{0,0}) := gr(\mathsf{O}\mathrm{GC}, \delta_0)$ only creates bivalent target vertices. Moreover, it fixes the hairy skeleton. Hence the associated graded complex splits

$$(\mathsf{O}\mathrm{GC}_d, \delta_{0,0}) \cong \bigoplus_{\gamma} (C_\gamma, \delta_{0,0}),$$

where the sum runs over all hairy graphs γ, and C_γ is the sub-dg vector space spanned by graphs Γ with hairy skeleton $hs(\Gamma) = \gamma$.

Each C_γ is a sub-dg vector space of a quotient of a tensor product

$$(C_\gamma, \delta_{0,0}) \subset \left(\left(\bigotimes_{e \in E_{internal}(\gamma)} (D_e, \delta_{0,0}) \right) / B_\gamma \right)^{\mathbb{S}_\gamma},$$

where $D_{(v,w)}$ is the dg vector space spanned by substitutions of either

$$v \to w, v \leftarrow w \text{ or } v \to\bullet\leftarrow w;$$

to the edge e between two vertices v and w. The sub-dg vector space $B_\gamma \subset \otimes D_e$ consists of graphs where at least one non-hairy vertex $v \in V(\gamma)$ is a source vertex, and $\mathbb{S}_\gamma$ is the automorphism group of γ. The graphs not included in C_γ are graphs with either a closed path, or where a hairy vertex is not an internal source.

Let

$$(\overline{C}_\gamma, \delta_{0,0}) \subset \left(\bigotimes_{(v,w)\in E_{internal}(\gamma)} (D_{(v,w)}, \delta_{0,0}) \right) / B_\gamma$$

be the dg vector space such that $\left(\overline{C}_\gamma\right)^{\mathbb{S}_\gamma} \cong C_\gamma$. We need to show that each $(\overline{C}_\gamma, \delta_{0,0})$ is one-dimensional, with a cohomology class represented by any graph Γ such that $\Phi(\Gamma) = \gamma$.

We note that if all vertices in γ are hairy, then $\overline{C}_\gamma$ is one-dimensional, with the only legal graph being

$$v \rightarrow\bullet\leftarrow w$$

on all internal edges.

Suppose that $H(\overline{C}_{\gamma'})$ is one-dimensional for each graph γ' with $n-1$ non hairy vertices, and let γ be a graph with n non-hairy vertices. Pick an internal edge $e = (v \longrightarrow w) \in E(\gamma)$, where v is hairy and w is not hairy. Consider the map

$$\begin{aligned} i_e : (\overline{C}_{\gamma/e}, \delta_{0,0}) &\to (\overline{C}_\gamma, \delta_{0,0}) \\ G &\mapsto G \otimes (v \rightarrow w). \end{aligned}$$

This is a map of dg vector spaces since

$$\delta_{0,0} i_e(G) = \delta_{0,0} G \otimes (v \rightarrow w) \pm \underbrace{G \otimes (v \rightarrow\bullet\leftarrow w)}_{=0 \text{ as } w \text{ is a source}} = i_e(\delta_{0,0} G).$$

We may take a filtration on $\overline{C}_\gamma$ and $\overline{C}_{\gamma/e}$ by the number of $a \rightarrow\bullet\leftarrow b$ substitutions, on the edges $(a,b) \in E(\gamma) \setminus \{e\}$. The associated graded complex $gr\overline{C}_\gamma \cong (\overline{C}_\gamma, \delta_{0,0,0})$ splits

$$(\overline{C}_\gamma, \delta_{0,0,0}) \cong \bigoplus_G (E_G, \delta_{0,0,0}),$$

where the sum runs over all possible choices of substitutions

$$a \rightarrow b, a \leftarrow b \text{ or } a \rightarrow\bullet\leftarrow b,$$

to the edges $(a,b) \in E(\gamma) \setminus \{e\}$, and

$$(E_G, \delta_{0,0,0}) \subset ((D_e, \delta_{0,0}) \otimes G)/B_\gamma$$

is the dg vector space spanned by the remaining possible substitutions to the edge $e = (v, w)$.

Remark 3.1.8. With a *possible choice of substitutions*

$$a \rightarrow b, a \leftarrow b \text{ or } a \rightarrow\bullet\leftarrow b,$$

we mean a choice of substitutions such that there still exists an oriented graph Γ, with $hs(\Gamma) = \gamma$, that is obtained from γ by substituting the edges (a,b) according to the choice of substitutions.

For the dg vector space $\overline{C}_{\gamma/e}$, we get that

$$(\overline{C}_{\gamma/e}, \delta_{0,0,0}) \cong \bigoplus_G (E'_G, 0),$$

where $E'_G \cong \mathbb{k}$ if w is a source vertex in G, and $E'_G = 0$ if w has an ingoing adjacent edge in G.

We need to show that each map

$$gr(i_e) : E'_G \to E_G$$

is a quasi-isomorphism.

If w has an ingoing adjacent edge in G, the dg vector space E_G is spanned by the substitutions

$$(v \rightarrow w), \quad (v \rightarrow\bullet\leftarrow w),$$

with $\delta_{0,0,0}(v \rightarrow w) = (v \rightarrow\bullet\leftarrow w)$. It is clear that this is acyclic, hence, $H(gr(i_e))$ is the isomorphism $0 \to 0$.

Otherwise, if w is a source vertex in G, then the only allowed configuration in E_G is $(v \rightarrow w) \otimes G = gr(i_e)(G)$.

It follows that $gr(i_e)$ is a quasi-isomorphism. By the mapping lemma, we have that i_e is a quasi-isomorphism

$$i_e : (\overline{C}_{\gamma/e}, \delta_{0,0}) \to (\overline{C}_\gamma, \delta_{0,0}).$$

By our induction hypothesis, we get that $H(\overline{C}_\gamma)$ is one-dimensional for every hairy graph γ. Also, for a hairy graph γ with k hairs, n vertices and m edges, we get that the cohomology class in $\overline{C}_\gamma$ is represented by any oriented graph $\Gamma \in \overline{C}_\gamma$ with $m-n+k$ bivalent target vertices, which are the graphs such that $\Phi(\Gamma) = \gamma$. This completes the proof.

□

3.2 Application to ribbon graphs and the moduli space of curves with punctures

In this section, we will follow [42] in order to define the ribbon graph complex $(\mathrm{RGC}, \delta + \Delta_1)$, as well as the map $F : (\mathsf{OGC}_1, \delta) \to (\mathrm{RGC}[1], \delta + \Delta_1)$. This will allow us to make the observation that Φ is also a map of dg vector spaces

$$F : (\mathsf{OGC}_1, \delta_0) \to (\mathrm{RGC}[1], \delta).$$

3.2.1 Ribbon Graphs

Definition 3.2.1. *A* ribbon graph Γ *is a triple* $(F_\Gamma, \iota_\Gamma, \sigma_\Gamma)$*, where* F_Γ *is a finite set,* $\iota_\gamma : F_\Gamma \to F_\Gamma$ *is an involution with no fixed points, i.e.*

$$\iota_\Gamma^2 = id, \quad \iota_\Gamma(f) \neq f \quad ,$$

for every $f \in F_\Gamma$*, and* $\sigma_\Gamma : F_\Gamma \to F_\Gamma$ *is a bijection.*

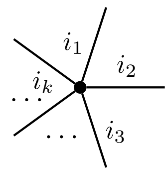
As in ordinary graphs, the elements of F_Γ are called *flags* or *half edges*. The orbits of the involution ι_Γ are called *edges*, and the set of all edges will be denoted by $E(\Gamma)$. The *vertices* of a ribbon graph Γ are orbits of the permutation σ_Γ, and the set of all vertices will be denoted by $V(\Gamma)$.

Definition 3.2.2. *We say that a cyclic ordering on a finite set A is a $\mathbb{Z}_{|A|} := \mathbb{Z}/|A|\mathbb{Z}$ action on A with precisely 1 orbit.*

The difference between an ordinary graph and a ribbon graph is that each vertex in a ribbon graph is equipped with a cyclic ordering of its adjacent (half) edges, given by $f + 1 = \sigma_\Gamma f$.

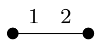
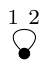
We may draw a picture of a ribbon graph Γ in the following way:

1. For each vertex $(i_1 i_2 \dots i_k) \in V(\Gamma)$, draw a dot with clockwise ordered lines labeled by $i_1, i_2, \dots i_k$ connected to the dot

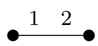
$(i_1 i_2 \dots i_k) \leftrightarrow$  .

2. For each edge $(i_a i_b) \in E(\Gamma)$, connect the lines labeled by i_a and i_b.

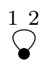
Two very basic examples of ribbon graphs are

$(\{1,2\}, (12), (1)(2)) =$  $\quad (\{1,2\}, (12), (12)) =$  .

We call the orbits of the permutation $\sigma_\Gamma^{-1} \circ \iota_\Gamma$ *boundaries* of Γ, and we denote the set of boundaries by $B(\Gamma)$. For example, the ribbon graph



has one boundary (12), while the ribbon graph



has two boundaries, (1) and (2).

3.2.2 A properad of ribbon graphs

Let $\mathrm{rgra}_{n,m,k}$ be the set of ribbon graphs with vertex set labeled by $[n]$, boundary set labeled by $[m]$ and edge set labeled by $[k]$. That is the set of ribbon graphs

$$\Gamma = ([k] \sqcup [k], \iota_k, \sigma_\Gamma),$$

where ι_k is the natural involution on $[k] \sqcup [k]$, together with bijections $v_\Gamma : [n] \to V(\Gamma)$, $\quad b_\Gamma : [m] \to B(\Gamma)$. Here, the edge labeled by $i \in [k]$ is the pair (i_1, i_2), $i = i_1 \in [k] \sqcup \emptyset$, $i = i_2 \in \emptyset \sqcup [k]$. We say that the edge (i_1, i_2) is *intrinsically oriented* from i_1 to i_2.

The group $\mathbb{P}_k = \mathbb{S}_k \ltimes \mathbb{S}_2^{\times k}$ acts naturally on $\mathrm{rgra}_{n,m,k}$ by permuting edges and flipping the intrinsic orientation. Let $\mathrm{RGra}_d(n, m)$ be the vector space

$$\mathrm{RGra}_d(n,m) := \begin{cases} \prod_{k\geq 0} \left(\mathbb{k}\langle \mathrm{rgra}_{n,m,k}\rangle \otimes \mathrm{sgn}_k\right)^{\mathbb{P}_k}[k(1-d)] & d \text{ even} \\ \prod_{k\geq 0} \left(\mathbb{k}\langle \mathrm{rgra}_{n,m,k}\rangle \otimes \mathrm{sgn}_2^{\otimes k}\right)^{\mathbb{P}_k}[k(1-d)] & d \text{ odd.} \end{cases}$$

Remark 3.2.3. By RGra without a subscript, we shall mean $\mathrm{RGra} := \mathrm{RGra}_0$, i.e. edges have degree 1.

The space

$$\mathrm{RGra}_d := \bigoplus_{n,m\geq 1} \mathrm{RGra}_d(n,m)$$

is an $\mathbb{S}$-bimodule, where $\mathbb{S}_n$ acts on $\mathrm{RGra}_d(n,m)$ by permuting vertex labels, and $\mathbb{S}_m$ acts on $\mathrm{RGra}_d(n,m)$ by permuting boundary labels.

In order to define the properadic composition maps $\circ : \mathrm{RGra}_d \otimes \mathrm{RGra}_d \to \mathrm{RGra}_d$, we have to make a few definitions.

Definition 3.2.4. *Let A and B be two finite sets with cyclic orderings. We say that an* ordered A-partition p of B *is a partition*

$$\bigsqcup_{a\in A} p_a = B,$$

where each

$$p_a = \{b_a^1, \ldots, b_a^k\} \subset B$$

is ordered with

$$b_a^i + 1 = \begin{cases} b_a^{i+1} & i < k \\ b_{a+r}^1 & i = k, \text{ where } r = \min\{j \in \mathbb{Z} | j \geq 1, p_{a+j} \neq \emptyset\}. \end{cases}.$$

We denote the set of all ordered A-partitions of B by $P(A, B)$.

Definition 3.2.5. *Let $\Gamma = (F, \iota, \sigma)$ be a ribbon graph, and let $v \in V(\Gamma)$, $b \in B(\Gamma)$ such that $v \cap b = \emptyset$. Then, for each ordered partition $p \in P(b, v)$, we define the* p-grafted *ribbon graph*

$$\circ_p\Gamma = (F, \iota, \circ_p\sigma),$$

by letting $\circ_p\sigma$ be the unique permutation such that:

1. $\circ_p\sigma|_{F\setminus(v\cup b)} = \sigma|_{F\setminus(v\cup v)}$.
2. *For each $j \in b$*

$$\circ_p\sigma(j) = \begin{cases} \min p_j & p_j \neq \emptyset, \\ \sigma(j) & p_j = \emptyset; \end{cases}$$

3. for each $i \in p_j$

$$\circ_p\sigma(i) = \begin{cases} \sigma(i) & i \neq \max p_j, \\ \sigma(j) & i = \max p_j. \end{cases}$$

For two ribbon graphs $\Gamma_1 = (F_1, \iota_1, \sigma_1)$ and $\Gamma_2 = (F_2, \iota_2, \sigma_2)$, $v \in V(\Gamma_1)$, $b \in B(\Gamma_2)$, and $p \in P(b, v)$, we set

$$\Gamma_1 \circ_p \Gamma_2 := (F_1 \sqcup F_2, \iota_1 \sqcup \iota_2, \circ_p(\sigma_1 \sqcup \sigma_2)) .$$

A picture of the ribbon graph $\circ_p\Gamma$ is obtained from a picture of the ribbon graph Γ by removing the vertex v and reconnecting its adjacent edges to the corners of the boundary b according to the partition p.

Lemma 3.2.6. *There is a bijection of vertices*

$$p_V : V(\Gamma) \setminus \{v\} \to V(\circ_p\Gamma)$$

and a bijection of boundaries

$$p_B : B(\Gamma) \setminus \{b\} \to B(\circ_p\Gamma),$$

such that $v' \subseteq p_V(v')$ and $b' \subseteq p_B(b')$ for every $v' \in V(\Gamma) \setminus \{v\}$ and $b' \in B(\Gamma) \setminus \{b\}$.

Furthermore, if $v' \cap b = \emptyset$ we have equality $v' = p_V(v')$. Similarly if $v \cap b' = \emptyset$, we have $b' = p_B(b')$.

Proof. Pick a boundary $b' \in B(\Gamma) \setminus \{b\}$, and a flag $j' \in b'$. Suppose that $(\circ_p\sigma)^{-1}\iota(j') \neq \sigma^{-1}\iota(j')$. Then we must have $\iota(j') = \sigma(j)$ or $\iota(j') = \min p_j$ for some $j \in b$. The first case implies that $\sigma^{-1}\iota(j') \in b$ and, therefore, we get $j' \in b$, which contradicts our choice of j'. If $\iota(j') = \min p_j$, then $(\circ_p\sigma)^{-1}\iota(j') \in b$. Furthermore, for $r = 1, 2, 3, \ldots$, we have that $((\circ_p\sigma)^{-1}\iota)^r(j') = (\sigma^{-1}\iota)^r(j') \in b$ until $\iota(\sigma^{-1}\iota)^{r-1}(j') = \sigma(j)$, in which case $\iota(\circ_p\sigma^{-1}\iota)^r(j') = \sigma^{-1}\iota(j')$. It follows that b' is a subset of the orbit of $\iota(\circ_p\sigma^{-1}\iota)^r(j')$. Thus, we may define the map

$$p_B : B(\Gamma) \setminus \{v\} \to B(\circ_p\Gamma),$$

by setting $p_2(b')$ to be the $(\circ_p\sigma)^{-1}\iota$ orbit of any $j' \in b'$. From the arguments above, it follows that p_2 is injective and $b' \subset p_2(b')$. Finally, we note that for any $j \in b$, there must exist an $r \geq 1$, such that $\iota(\circ_p\sigma^{-1}\iota)^r(j) \notin b$. Hence, p_2 must also be surjective.

If $v \cap b' = \emptyset$, then it is clear from the construction that $b' = p_B(b')$.

Similarly, we can define

$$p_V : V(\Gamma) \setminus \{v\} \to V(\circ_p\Gamma)$$

by setting $p_V(v')$ to be the $\circ_p\sigma$ orbit of any $i \in v'$. By the same arguments as above, we get that p_V is well defined, bijective, and $v' \subset p_V(v')$. $\square$

This lemma implies that, for mutually disjoint $v_1, v_2 \in V(\Gamma)$, $b_1, b_2 \in B(\Gamma)$, and partitions $p_1 \in P(b_1, v_1)$, $p_2 \in P(b_2, v_2)$, we may define

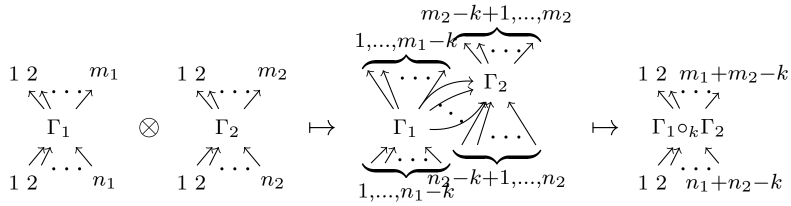
$$\circ_{p_1,p_2}\Gamma := \circ_{p_2}(\circ_{p_1}\Gamma).$$

It is clear that we have

$$\circ_{p_1,p_2}\Gamma = \circ_{p_2}(\circ_{p_1}\Gamma) = \circ_{p_1}(\circ_{p_2}\Gamma) = \circ_{p_2,p_1}\Gamma. \tag{3.5}$$

For each $k \leq m_1, n_2$, we define the properadic composition maps

$$\circ_k : \mathrm{RGra}_d(n_1, m_1) \otimes \mathrm{RGra}_d(n_2, m_2) \quad \to \quad \mathrm{RGra}_d(n_1 + n_2 - k, m_1 + m_2 - k),$$



composing the boundaries $m_1 - k + 1, \ldots, m_1$ of Γ_1 to the vertices $1, \ldots, k$ of Γ_2, by

$$\Gamma_1 \circ_k \Gamma_2 := \prod_{i=1}^{k} \left(\sum_{p \in P(b_{\Gamma_1}(n_1 - k + i), v_{\Gamma_2}(i))} \circ_p (\Gamma_1 \sqcup \Gamma_2) \right).$$

The element $\Gamma_1 \circ_k \Gamma_2$ is the sum of all graphs obtained from Γ_1 and Γ_2 by:

1. Removing each vertex $1, \ldots, k$ from Γ_2.
2. For each $i \in [k]$, reconnecting each half edge in $v_{\Gamma_2}(i)$ to a corner of the boundary $b_{\Gamma_1}(m_1 - k + i)$, respecting the cyclic orientations.

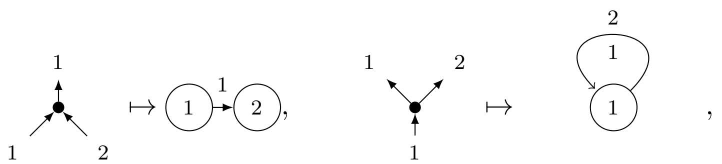
Proposition 3.2.7 ([42])**.** *The composition maps* $\circ_\bullet : \mathrm{RGra}_d \otimes \mathrm{RGra}_d \to \mathrm{RGra}_d$ *defines a prop(erad) structure on* RGra_d*.*

Proof. It follows by Lemma 3.2.6 and (3.5) that the maps are well defined and well behaved. □

Proposition 3.2.8 ([42])**.** *There is a map of properads*

$$s : \widehat{\mathsf{LieB}}_{d,d} \to \mathrm{RGra}_d$$

given by



Remark 3.2.9. The map s factors through the prop of involutive Lie bialgebras $\widehat{\mathsf{LieB}}^{\diamond}_{d,d}$ which is generated by the corollas (1.3) modulo the relations (2.3),(2.4), (2.5) plus the additional relation

 $= 0.$ (3.6)

3.2.3 The ribbon graph complex RGC

We define the *ribbon graph complex* $(\mathrm{RGC}_d, \delta + \Delta_1)$ to be the deformation complex

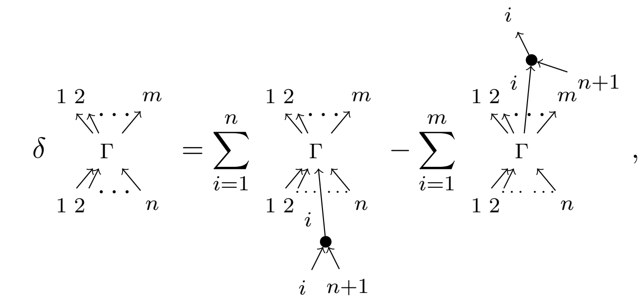
$$(\mathrm{RGC}_d, \delta + \Delta_1) := \mathsf{Def}(\widehat{\mathsf{hoLieB}}^{+}_{d,d} \overset{s \circ p}{\to} \mathrm{RGra}_d),$$

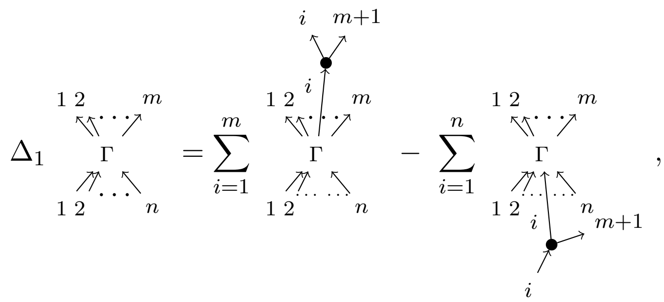
where p is the natural projection $p : \widehat{\mathsf{hoLieB}}^{+}_{d,d} \to \widehat{\mathsf{LieB}}_{d,d}$, and s is the map $\widehat{\mathsf{LieB}}_{d,d} \to \mathrm{RGra}_d$ from Proposition 3.2.8. As a graded vector space, we have that

$$\mathrm{RGC}_d \cong \prod_{n,m \geq 1} \left(\mathrm{RGra}_d(n,m)[d(n+m-2)] \otimes \mathrm{sgn}^d_{n,m} \right)^{\mathbb{S}_n \times \mathbb{S}_m}$$

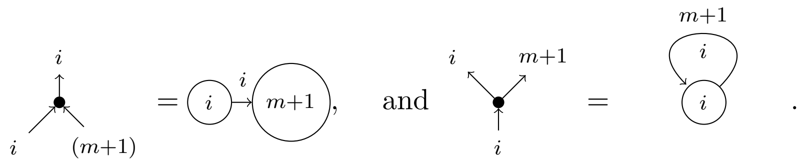
is spanned by ribbon graphs in RGra_d (co)invariant under the actions of permuting vertices and boundaries.

The differentials $\delta + \Delta_1$ acts on a ribbon graph Γ with n vertices and m boundaries by





where



In words, the first part of the differential, δ, splits vertices in a way that respects the cyclic ordering. The other part of the differential, Δ_1, adds an edge between each pair of corners of each boundary.

3.2.4 The map $F : \mathrm{OGC}_1 \to \mathrm{RGC}[1]$

The map

$$F : \mathrm{OGC}_1 \to \mathsf{Def}(\widehat{\mathsf{hoLieB}}^+_{0,0} \xrightarrow{\mathrm{id}} \widehat{\mathsf{hoLieB}}^+_{0,0})[1]$$

from Theorem 2.2.5, first defined in [43], may be extended to a map

$$\bar{F} : \mathrm{OGC}_1 \to \mathsf{Def}(\widehat{\mathsf{hoLieB}}^+_{0,0} \xrightarrow{sop} \mathrm{RGra})[1],$$

by setting $\bar{F}(\Gamma) := f \circ p \circ (F(\Gamma))$.

We note that $\bar{F}$ maps a graph in $\Gamma \in \mathrm{OGC}_1$ with n sources and m targets to a sum of ribbon graphs with n vertices and m boundaries. Hence, F maps a graph Γ with m target vertices to a sum of ribbon graphs with m boundaries. We may take a filtration on $\mathrm{RGC}(:= \mathrm{RGC}_0)$ by the number of boundaries, and a filtration on OGC_1 by the number of target vertices. Then

$$gr(\mathrm{RGC}[1], \delta + \Delta_1) = (\mathrm{RGC}[1], \delta),$$

and

$$gr(\mathsf{O}\mathrm{GC}_1, \delta) = (\mathsf{O}\mathrm{GC}_1, \delta_0^t),$$

where δ^t is the part of the differential that preserves the number of target vertices. It is clear that the map Φ in Theorem 3.1.6 has an analogue, say $\Psi : (\mathsf{O}\mathrm{GC}_1, \delta_0^t) \to \mathrm{HGC}_0$, which is also a quasi-isomorphism. As $\bar{F}$ maps a graph with m target vertices to a sum of ribbon graphs with precisely m boundaries, we get that

$$gr\bar{F} : gr(\mathsf{O}\mathrm{GC}_1, \delta) \to gr(\mathrm{RGC}[1], \delta + \Delta_1)$$

is given by the same map of vector spaces

$$\bar{F} : (\mathsf{O}\mathrm{GC}_1, \delta_0) \to (\mathrm{RGC}[1], \delta).$$

We have now established both maps mentioned in Corollary 3.0.2.

Theorem 3.2.10. *We have a zig-zag of morphisms*

$$(\mathrm{HGC}_0, \delta) \leftarrow (\mathsf{O}\mathrm{GC}_1, \delta_0^t) \to (\mathrm{RGC}[1], \delta),$$

where the left map is a quasi-isomorphism.

Chapter 4

A family of exotic automorphisms of polyvector fields

Introduction

In this chapter, we find a new application of the remarkable theory of differential forms with logarithmic singularities developed by A. Alekseev, C. A. Rossi, C. Torossian and T. Willwacher in [3]. That theory originated from M. Kontsevich's claim in [25], that his famous formality map introduced in [26] is also valid (i.e. all integrals converge) if one uses a propagator with logarithmic singularities on the collapsing strata of compactified configuration spaces. The reasons for that convergence were not clear until the authors of [3] proved a new version of Stokes' Theorem for differential forms with singularities of certain type at the boundary, called *the regularized Stokes' Theorem*.

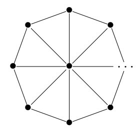
In this chapter, we construct a (regularized) De Rham field theory on the configuration space model for the 2-colored operad of Lie ∞-morphisms, using a two parametric propagator with logarithmic singularities

$$\omega^{t,\lambda}_{(i,j)} := \frac{1-t}{2\pi i} d\log\Big(\frac{z_i - z_j}{1+\lambda|z_i - z_j|}\Big) - \frac{t}{2\pi i} d\log\Big(\frac{\bar{z}_i - \bar{z}_j}{1+\lambda|z_i - z_j|}\Big), \qquad t,\lambda \in \mathbb{R},\ \lambda > 0, \tag{4.1}$$

(using a propagator which is smooth everywhere can only give a homotopy trivial theory). We use to construct an explicit two parametric family of exotic Lie ∞-automorphisms of Schouten-Nijenhuis Lie algebra of polyvector fields on an arbitrary affine space

$$\mathcal{F}^{t,\lambda} : T_{poly}(\mathbb{R}^d) \rightsquigarrow T_{poly}(\mathbb{R}^d).$$

The derivative $\frac{\partial \mathcal{F}^{t,\lambda}}{\partial t}|_{t=0}$ contains as summands of linear combinations of graphs of the form

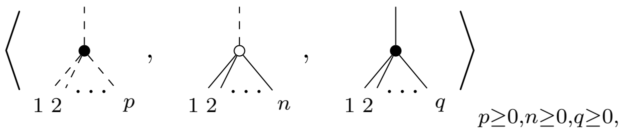
β_{2n+1}  $+ \ \dots$ (4.2) where the coefficients β_{2n+1}, $n \geq 1$ of the wheel-type summands are equal to zeta values $\frac{\zeta(2n+1)}{(2\pi)^{(2n+1)}}$ up to a non-zero rational factor [45], [19], [40]. These linear combinations of graphs represent non-trivial cohomology classes in $H(GC_2) = \mathfrak{grt}_1$.

4.1 Prerequisites

4.1.1 Morphism operads

In this chapter we will use a type of operads called *morphism operads* [29]. Using the framework of [8], and Chapter 1, we may define morphism operads as follows.

Let $\mathcal{T}^{\bullet,\circ}$ denote the category of trees generated by corollas



(4.3)

where a dashed outgoing leg may only be grafted to a dashed incoming leg, and a solid outgoing leg may only be grafted to a solid ingoing leg. Furthermore, we require all ingoing external legs of a tree to be either dashed or solid. A morphism operad is then a prop

$$(\mathcal{T}_A^{\bullet,\circ}, \sqcup) \to (\mathcal{P}, \otimes).$$

We shall denote the set of all $\bullet$ vertices with solid edges by $V_\infty^\bullet(T)$, the set of all $\bullet$ vertices with dashed edges by $V_0^\bullet(T)$, the set of all $\bullet$ vertices by $V^\bullet(T) = V_0^\bullet(T) \cup V_\infty^\bullet(T)$, and the set of all $\circ$ vertices (with solid inputs and a dashed output) by $V^\circ(T)$.

A morphism operad $\mathcal{P}$ consists of three $\mathbb{S}$-modules,

$$\mathcal{P} = (\mathcal{P}^{out}, \mathcal{P}^{mor}, \mathcal{P}^{in})$$

together with composition maps

$$\bigotimes_{v \in V_T} \mathcal{P}^{out}(|in(v)|) \to \mathcal{P}^{out}$$

for all trees T with dashed edges,

$$\bigotimes_{v \in V_T} \mathcal{P}^{in}(|in(v)|) \to \mathcal{P}^{in}$$

For all trees T with solid edges,

$$\bigotimes_{v \in V_T^{out}} \mathcal{P}^{out}(|in(v)|) \otimes \bigotimes_{v \in V_T^{mor}} \mathcal{P}^{mor}(|in(v)|) \bigotimes_{v \in V_T^{in}} \mathcal{P}^{in}(|in(v)|) \otimes \to \mathcal{P}^{out},$$

For all trees T with solid ingoing legs, and a dashed outgoing leg.

Example 4.1.1. For two vector spaces A and B

$$\underline{\mathrm{End}}_B^A := (\mathrm{End}_B, \mathrm{End}_B^A, \mathrm{End}_A),$$

where $\mathrm{End}_B^A(n) := \hom(A^{\otimes n}, B)$, is a morphism operad.

Example 4.1.2. As there is an forgetful functor $\mathcal{T}^{\circ,\bullet} \to \mathcal{T}$, any operad $\mathcal{P}$ "restricts" to a morphism operad $(\mathcal{P}, \mathcal{P}, \mathcal{P})$, where we simply do not allow composition if it goes against the colors.

An *algebra over a morphism operad* is a representation

$$\underline{\Phi} : (\mathcal{P}^{out}, \mathcal{P}^{mor}, \mathcal{P}^{in}) \to (\mathrm{End}_B, \mathrm{End}_B^A, \mathrm{End}_A).$$

That is a $\mathcal{P}^{in}$-algebra structure on A, a $\mathcal{P}^{out}$-algebra on B, and some sense of morphisms

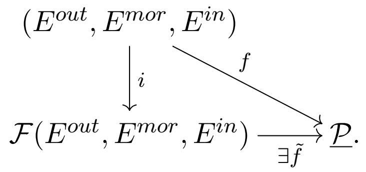
$$\Phi^{mor}(\mu_n) : A^{\otimes n} \to B,$$

for each $\mu_n \in \mathcal{P}^{mor}$.

For a triple of $\mathbb{S}$-modules $(E^{out}, E^{mor}, E^{in})$, we denote the *free morphism* over E^{out}, E^{mor}, E^{in} by

$$\mathcal{F}(E^{out}, E^{mor}, E^{in}),$$

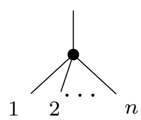
i.e. the morphism operad with the universal property, for any triple of $\mathbb{S}$-module morphisms f there exists a morphism operad morphism $\tilde{f}$, such that the following diagram commutes

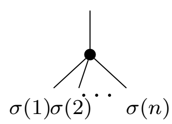


The underlying (triple of) $\mathbb{S}$-module(s) of $\mathcal{F}(E^{out}, E^{mor}, E^{in})$ is the vector space of all trees generated by the corollas in (4.3), where the vertices are decorated by elements in E^{out}, E^{mor}, E^{in} respectively, and the $\mathbb{S}$ action is permuting the labels of the leaves. The composition maps are given by gluing roots to leaves.

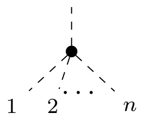
4.1.2 The operad of Lie ∞-morphisms

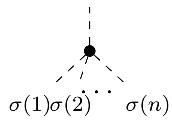
Consider the triple of $\mathbb{S}$-modules $\underline{\mathcal{E}_d} = (\mathcal{E}_d, \mathcal{E}_d^+[-1], \mathcal{E}_d)$, where $\mathcal{E}_d$ and $\mathcal{E}_d^+$ are the $\mathbb{S}$-modules that generate hoLie_d and hoLie_d^+ respectively. The free morphism operad $\mathcal{F}(\underline{\mathcal{E}_d})$ is spanned by trees generated by skew symmetric corollas



$= \mathrm{sgn}(\sigma)^d$


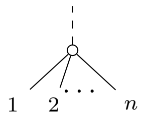
$, \quad n \geq 2$

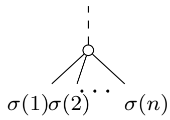


$= \mathrm{sgn}(\sigma)^d$


$, \quad n \geq 2$

and

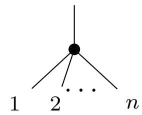


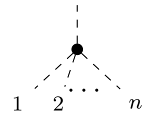
$= \mathrm{sgn}(\sigma)^d$


$, \quad n \geq 1,$

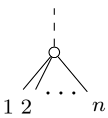
for all $\sigma \in \mathbb{S}_n$. The morphism operad governing Lie ∞-morphisms is the quasi-free morphism operad

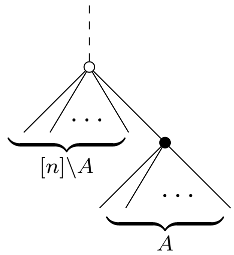
$$\mathcal{M}or(\mathsf{holie}_d) := (\mathcal{F}(\mathcal{E}_d, \mathcal{E}_d^+[-1], \mathcal{E}_d), \delta),$$

where δ acts as the holie_d differential on the corollas


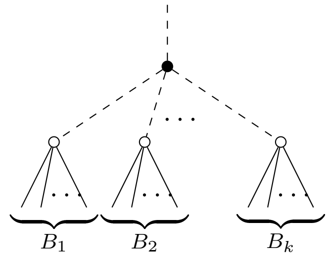
and


, while

δ


$= -\sum_{\substack{A \in [n] \\ |A| \geq 2}} \mathrm{sgn}([n] \setminus A, A)^d$


$+$

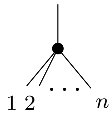
$+\sum_{k \geq 1} \sum_{\substack{B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k = [n] \\ \inf B_i < \inf B_{i+1}}} \mathrm{sgn}(B_1, B_2, \ldots, B_k)^d$


.

The morphism operad $\mathcal{M}or(\mathsf{holie}_d)$ governs ∞-morphisms of holie_d-algebras, as defined in e.g. [29]. That is, representations

$$\mu : \mathcal{M}or(\mathsf{holie}_d) \to \mathrm{End}_A^B$$

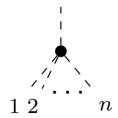
corresponds to a holie_d structure on A given by

$[a_1, a_2, \ldots, a_n] := \mu \Bigg($


$\Bigg) (a_1 \otimes a_2 \otimes \ldots \otimes a_n),$

a holie_d structure on B given by

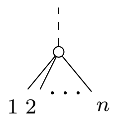
$$[b_1, b_2, \ldots, b_n] := \mu$$



$$(b_1 \otimes b_2 \otimes \ldots \otimes b_n),$$

and a Lie ∞-morphism $f : A \rightsquigarrow B$ given by

$$f_n(a_1, a_2, \ldots, a_n) := \mu$$



$$(a_1 \otimes a_2 \otimes \ldots \otimes a_n).$$

4.1.3 Homotopy between ∞-morphisms

In this section we will explain what we mean by exotic Lie ∞-morphisms.

Consider the dg vector space $\Bbbk[t, dt]$ of polynomial differential forms with the De Rham differential

$$d(f(t) + g(t)dt) = \frac{\partial f(t)}{\partial t} dt,$$

where $|dt|$ is set to be 1 or -1 depending if we consider cohomological or homological grading, and $dtdt = 0$.

For any dg vector space A, we have a natural inclusion

$$\begin{array}{rcl} i : A & \hookrightarrow & A \otimes \Bbbk[t, dt] \\ a & \mapsto & a \otimes 1, \end{array}$$

and a projection for every $t_0 \in \Bbbk$

$$\begin{array}{rcl} \mathrm{Eval}_{t_0} : A \otimes \Bbbk[t, dt] & \twoheadrightarrow & A \\ a \otimes (f(t) + g(t)dt) & \mapsto & f(t_0)a. \end{array}$$

Lemma 4.1.3. *There is an injective morphism of dg operads*

$$\iota : \mathrm{End}_A \hookrightarrow \mathrm{End}_{A \otimes \Bbbk[t,dt]}$$

by

$$(f : A^{\otimes n} \to A) \mapsto \left(\bar{f} : (A \otimes \Bbbk[t, dt])^{\otimes n} \to A^{\otimes n} \otimes \Bbbk[t, dt]^{\otimes n} \stackrel{f \otimes m}{\to} A \otimes \Bbbk[t, dt] \right),$$

where m is multiplication of the poly differential forms.

Proof. The map respects composition since m is associative. Also, it commutes with the differential, since m is a (co)cycle in $\mathrm{End}_{\Bbbk[t,dt]}$. □

By this lemma, any algebra structure on A extends to an algebra structure on $A \otimes \Bbbk[t, dt]$ by

$$\mu(a_1 \otimes \omega_1, \ldots, a_n \otimes \omega_n) = \pm\mu(a_1, \ldots, a_n) \otimes \omega_1 \cdots \omega_n,$$

where the sign is determined by the sign rules of graded vector spaces. Furthermore, the maps i and Eval_{t_0} are algebra morphisms.

Definition 4.1.4 ([13])**.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two* hoLie *algebras. Then two ∞-morphisms $f, f' : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ are said to be* homotopy equivalent *if there exists an ∞-morphism $H : \mathfrak{g} \rightsquigarrow \mathfrak{h} \otimes \Bbbk[t, dt]$ such that*

$$f = \mathrm{Eval}_0 \circ H,$$

$$f' = \mathrm{Eval}_1 \circ H.$$

We say that an *exotic automorphism* $\mathcal{F} : \mathfrak{g} \rightsquigarrow \mathfrak{g}$ is a Lie ∞-morphism that is not homotopy equivalent to the identity.

4.1.4 Graphs acting on polyvector fields

Let

$$T_{poly}^{(n)} := \Bbbk[x^1, \ldots, x^n, \xi_1, \ldots, \xi_n],$$

where $|x^i| = 0$ and $|\xi_i| = 1$, be the graded vector space of polyvector fields. Graphs $\Gamma \in \mathrm{Gra}_2(m)$ act on $T_{poly}^{(n)}$ by

$$\Gamma(\gamma_1, \ldots, \gamma_m) = mult\left(\prod_{(i,j)\in E(\Gamma)} \left(\sum_{k=1}^{n} \frac{\partial}{\partial x^k_{(i)}} \frac{\partial}{\partial \xi^{(j)}_k} + \frac{\partial}{\partial x^k_{(j)}} \frac{\partial}{\partial \xi^{(i)}_k}\right)(\gamma_1 \otimes \ldots \otimes \gamma_m)\right),$$

where the notation $\frac{\partial}{\partial x^k_{(i)}}$ means applying $\frac{\partial}{\partial x^k}$ to the i-th component of $\gamma_1 \otimes \ldots \otimes \gamma_n$.

From this action, we get an hoLie-structure on $T_{poly}^{(n)}$ for every MC-element in fGC_d. In particular, the graph •—• induces the Schouten–Nijenhuis bracket

$$[\gamma_1, \gamma_2] := \sum_{i=1}^{n} \frac{\partial \gamma_1}{\partial x^i} \frac{\partial \gamma_2}{\partial \xi_i} + \frac{\partial \gamma_1}{\partial \xi_i} \frac{\partial \gamma_2}{\partial x^i}.$$

We have a canonical map of dg Lie algebras

$$\mathrm{fGC}_2 = \mathsf{Def}(\mathsf{hoLie}_2^+ \to \mathrm{Gra}_2) \to \mathsf{Def}(\mathsf{hoLie}_2^+ \to \mathrm{End}_{T_{poly}}) \cong CE(T_{poly}^{(n)}, T_{poly}^{(n)})$$

for any n. In particular, $H^0(\mathrm{fGC}_2) = \mathfrak{grt}$ is mapped into the zero-th cohomology group of the Chevalley-Eilenberg complex $CE(T_{poly}^{(n)}, T_{poly}^{(n)})$. This map is proven in [50] to be a quasi-isomorphism in the stable limit $n \to \infty$.

4.1.5 A colored graph complex

Just as any operad, Gra_d extends to a morphism operad

$$\underline{\mathrm{Gra}} := (\mathrm{Gra}_d^{out}, \mathrm{Gra}_d^{mor}, \mathrm{Gra}_d^{in}).$$

Consider the deformation complex

$$\mathsf{Def}(\mathcal{M}or(\mathsf{holie}_2^+) \xrightarrow{0} \underline{\mathrm{Gra}_2}).$$

MC-elements in $\mathsf{Def}(\mathcal{M}or(\mathsf{holie}_2^+) \xrightarrow{0} \underline{\mathrm{Gra}_2})$ corresponds to morphisms of operads

$$\mathcal{M}or(\mathsf{holie}_2^+) \to \underline{\mathrm{Gra}_2}.$$

As the operad Gra_2 acts on polyvector fields, MC-elements in $\mathsf{Def}(\mathcal{M}or(\mathsf{holie}_2^+) \xrightarrow{0} \underline{\mathrm{Gra}_2})$ are also morphisms of operads

$$\mathcal{M}or(\mathsf{holie}_2^+) \to \underline{\mathrm{End}}_{T_{poly}^{(n)}}^{T_{poly}^{(n)}},$$

in particular, MC-elements on the form $(\bullet\!\!-\!\!\bullet, \mathcal{F}, \bullet\!\!-\!\!\bullet) \in \mathsf{Def}(\mathcal{M}or(\mathsf{holie}_2^+) \xrightarrow{0} \underline{\mathrm{Gra}_2})$, where

$$\mathcal{F} = \bullet + \text{graphs with more vertices},$$

corresponds to ∞-automorphisms of $T_{poly}^{(n)}$ with the Schouten–Nijenhuis bracket.

That is sums of graphs $\mathcal{F} = \bullet + \mathcal{F}_{\geq 1}$ such that

$$\delta\mathcal{F}_{\geq 1} - \frac{1}{2}\,\textcircled{\mathcal{F}_{\geq 1}}\!-\!\textcircled{\mathcal{F}_{\geq 1}} = 0,$$

where δ is the vertex splitting differential and $\textcircled{\mathcal{F}_{\geq 1}}\!-\!\textcircled{\mathcal{F}_{\geq 1}}$ is the sum of graphs obtained by adding an edge between graphs in $\mathcal{F}_{\geq 1}$.

Furthermore, if $\mathcal{F}_{\geq 1}$ consists of 1-vertex irreducible graphs (that is graphs such that removing a vertex will not disconnect the graph), we must have separately that

$$\frac{1}{2}\,\textcircled{\mathcal{F}_{\geq 1}}\!-\!\textcircled{\mathcal{F}_{\geq 1}} = 0,$$

and

$$\delta\mathcal{F}_{\geq 1} = 0,$$

i.e. $\mathcal{F}_{\geq 1}$ is a graph cocycle. For $\mathcal{F}_{\geq 1}$ to be homotopy equivalent to the identity map, this cycle must be a coboundary. We shall see below that our explicit map $\mathcal{F}_{\geq 1}$ is not a coboundary as it contains all odd wheel graphs so that our map can not homotopy trivial.

4.1.6 The Dual Co-operad of graphs

Since $\mathrm{Gra}_d(n)$ is finite dimensional in each degree (for $d \geq 2$), there exists a dual co-operad $\mathfrak{G}_d(n) := \mathrm{Hom}(\mathrm{Gra}_d(n), \Bbbk)$, where each $\mathfrak{G}_d(n) \cong \mathrm{Gra}_d(n)$ as vector spaces.

Here, we identify the graph $\Gamma \in \mathcal{G}_k(n)$ with the linear map $f_\Gamma : \mathrm{Gra}_d(n) \to \Bbbk$ by

$$f_\Gamma(\Gamma') := \begin{cases} 1 & \Gamma' = \Gamma \\ 0 & \text{otherwise,} \end{cases}.$$

The cocomposition map, for a tree $T \in \mathcal{T}$ with n leaves

$$\Delta_T : \mathfrak{G}(n) \quad \to \quad \bigotimes_{v \in V(T)} \mathfrak{G}(\mathrm{In}(v)),$$

(by $f_\Gamma \mapsto f_\Gamma(\mathrm{Gra}_d(con_T)-)$) is the map

$$\Gamma \mapsto (-1)^{\sigma_{\Gamma,T}} \bigotimes_{v \in V(T)} \Gamma_v / \{\Gamma_w\}_{w \in \mathrm{In}(v)},$$

where $\Gamma_v \subset \Gamma$ is the sub-graph that contains all vertices associated with leaves of T that are ancestors to v.

Similarly we have a morphism cooperad

$$\underline{\mathfrak{G}} := \mathrm{Hom}(\underline{\mathrm{Gra}}, \Bbbk) = (\mathfrak{G}^{out}, \mathfrak{G}^{mor}, \mathfrak{G}^{in}).$$

Lemma 4.1.5. *Let $\underline{\mathcal{P}}$ be a morphism operad, and let $\underline{\mathcal{P}}^*$ be its dual cooperad. Then given a morphism of cooperads*

$$\underline{F} : \underline{\mathfrak{G}} \to \underline{\mathcal{P}}^*,$$

we get a morphism of operads

$$\begin{array}{rcl} \underline{\mathcal{P}} & \to & \underline{\mathrm{Gra}} \\ \mu & \mapsto & \sum_\Gamma F(\Gamma)(\mu)\Gamma, \end{array}$$

where the sum runs over all isomorphism classes of graphs Γ.

In the following sections our efforts will go to constructing a morphism of co-operads

$$\underline{\mathfrak{G}} \to \mathcal{M}or(\mathsf{holie}_2)^*,$$

whose dual morphism of operads will induce an exotic automorphism of polyvector fields.

4.2 Operads of compactified configuration spaces

In this section, we follow [37] to construct two topological operads $\widehat{C} := \langle C_n \rangle_{n\geq 2}$ and

$$\widehat{\underline{\mathfrak{C}}} := \langle C_p, \mathfrak{C}_n, C_q \rangle_{p\geq 2, n\geq 1, q\geq 2},$$

such that

$$\mathcal{F}Chains(\widehat{C}) \cong \mathsf{holie}_2,$$

and

$$\mathcal{F}Chains(\widehat{\underline{\mathfrak{C}}}) \cong \mathcal{M}or(\mathsf{holie}_2).$$

4.2.1 The fundamental chains

Let $\bar{C} := \{\bar{C}_k\}$ be an operad in the category of smooth manifolds with corners, which satisfies the following conditions:

1. Each $\bar{C}_p$ is a compact oriented manifold.
2. $\bar{C}$ is free as an operad in the category of sets $\bar{C} = \mathcal{F}\langle C \rangle$.
3. The $\mathbb{S}$-space $C := \{C_k\}$, of generators is given by $C_p := \bar{C}_p \setminus \partial \bar{C}_p$.
4. Each manifold $\bar{C}_p$ is canonically stratified

   $$\bar{C}_p = \bigsqcup_{T\in\mathcal{T}_p} C_T$$

   as sets, where $C_T := \prod_{v\in V(T)} C_{|In(v)|}$. The inclusion map $C_T \hookrightarrow \bar{C}_p$ is assumed to be smooth.

The *Fundamental chains* of $\bar{C}$

$$\mathcal{F}Chains(\bar{C}) \subset Chains(\bar{C})$$

is an operad in the category of dg vector spaces, where $\mathcal{F}Chains_d(\bar{C})(n)$ is spanned by pairs (C_T, or), where T is a tree with n leaves, $\dim(C_T) = d$, and or is a orientation on C_T, with the relation

$$(C_T, or) = -(C_T, or^{opp}).$$

By the conditions above, $\mathcal{F}Chains(\bar{C})$ admits the structure of a graded free operad. Furthermore, we get a differential

$$\partial : \mathcal{F}Chains_d(\bar{C}) \to \mathcal{F}Chains_{d-1}(\bar{C})$$

by mapping a d dimensional component C_T to its $(d-1)$-dimensional boundary ∂C_T. The differential will have the structure

$$\partial(C_n, or) = \sum_{T\in\mathcal{T}_n^{boundary}} (C_T, or_T),$$

for some family of trees $\mathcal{T}^{boundary} \subset \mathcal{T}$, and or_T is the induced orientation on the boundary component C_T.

For simplicity, we will assume a standard orientation for each C_T and write

$$\partial C_n = \sum_{T \in \mathcal{T}_n^{boundary}} (-1)^{\sigma_T} C_T,$$

where the sign is negative if the induced orientation is opposite to the standard orientation.

4.2.2 A Model for holie_2

For any finite subset $A \subset \mathbb{Z}_+$, we define

$$Conf_A(\mathbb{C}) := \{A \hookrightarrow \mathbb{C}\}$$

to be the space of all injective maps $A \hookrightarrow \mathbb{C}$, and

$$\widetilde{Conf}_A(\mathbb{C}) := \{A \to \mathbb{C}\}$$

be the space of all maps $A \to \mathbb{C}$.

The 3-dimensional Lie group $\mathbb{R}^+ \ltimes \mathbb{C}$ acts freely on $Conf_A(\mathbb{C})$ ($|A| \geq 2$) by scaling and translating. We define

$$C_A := Conf(\mathbb{C})/(\mathbb{R}^+ \ltimes \mathbb{C})$$

and

$$\widetilde{C}_A := \widetilde{Conf}_A(\mathbb{C})/(\mathbb{R}^+ \ltimes \mathbb{C}).$$

For any $B \subseteq A$, we have a natural map $\pi_B : C_A \to C_B$, by restriction.

A topological compactification $\widehat{C}_A$ of C_A can be defined as the closure of the composition

$$C_A \xrightarrow{\prod \pi_B} \prod_{\substack{B \subseteq A \\ |B| \geq 2}} C_B \hookrightarrow \prod_{\substack{B \subseteq A \\ |B| \geq 2}} \widetilde{C}_B. \tag{4.4}$$

The compactified space $\widehat{C}_A$ admits a boundary component

$$C_{A/B} \times C_B$$

for each subset $B \subsetneq A$, $|B| \geq 2$, corresponding to the limit configurations where the points in B collapse to a single point. This is included into $\widehat{C}_A$ by the map

$$C_{A/B} \times C_B \xrightarrow{\phi_{B'}} \prod_{\substack{B' \subseteq A \\ |B| \geq 2}} \widetilde{C}_{B'},$$

where ϕ_B is the composition

$$C_{A/B} \times C_B \twoheadrightarrow C_B \xrightarrow{\pi_{B'}} \widetilde{C}_{B'},$$

if $B' \subseteq B$, and

$$C_{A/B} \times C_B \twoheadrightarrow C_{A/B} \xrightarrow{i} \widetilde{C}_A \xrightarrow{\pi_{B'}} \widetilde{C}_{B'},$$

otherwise. Here, i is the natural lift $i(z) : A \to A/B \to \mathbb{C}$.

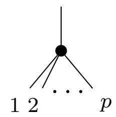
Similarly, $C_{A/B} \times C_B$ admits boundary components in $\widehat{C}_A$ corresponding to the limit configurations where points in $C_{A/B}$ or C_B collapse. We get that $\widehat{C}_n$ admits canonical stratification

$$\widehat{C}_n = \bigsqcup_{T \in \mathcal{T}_n} C_T,$$

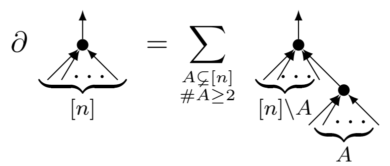
where $C_T := \prod_{v \in V(T)} C_{\mathrm{In}(v)}$ is a boundary component of co-dimension $|V(T)|$.

Proposition 4.2.1. *[17] The fundamental chain operad of $\mathcal{F}Chains(\widehat{C})$ is a quasi-free operad isomorphic to* hoLie_2.

Proof. The underlying $\mathbb{S}$-module of $\mathcal{F}Chains(\widehat{C})$ is generated by

$\langle C_p \rangle_{p\geq 2} = \mathcal{F} \Big\langle$  $\Big\rangle_{p\geq 2,}$

where $\dim C_p = 2p - 3$. The $2p - 4$ dimensional boundary components of C_n consist of configurations where a group of points A has collapsed. Hence,



(4.5)

This coincides precisely with the hoLie_2 operad. ☐

Remark 4.2.2. If we compactify $\mathbb{C}(\mathbb{R}^d) := Conf(\mathbb{R}^d)/(\mathbb{R}^+ \ltimes \mathbb{R}^d)$ the analogous way we get

$$\mathcal{F}Chains(\widehat{C}(\mathbb{R}^d)) = \begin{cases} \mathsf{hoLie}_d & d \geq 2 \\ A_\infty & d = 1. \end{cases}$$

4.2.3 A Model for $\mathcal{M}or(\mathbf{hoLie}_2)$

Consider the space $\mathfrak{C}_A := Conf_A(\mathbb{C})/\mathbb{C}$, where $\mathbb{C}$ acts on $Conf_A(\mathbb{C})$ by translation. For $B \subset A$, there is again a natural projection $\pi_B : \mathfrak{C}_A \to \mathfrak{C}_B$ by forgetting the points that are not in A. We also have an isomorphism

$$\begin{array}{rcl} \psi : \mathfrak{C}_A & \overset{\cong}{\to} & C_A \times (0, +\infty) \\ (z_1, \ldots, z_n) & \mapsto & \left((z_1, \ldots, z_n), \sum_{i=1}^n |z_i - \zeta(z)|^2 \right), \end{array}$$

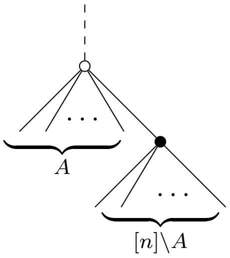
where $\zeta(z) = \sum_{i=1}^{n} \frac{z_i}{n}$ is the "center of mass" of the configuration.

A topological compactification $\widehat{\mathfrak{C}}_A$ of $\mathfrak{C}_A$ is defined to be the closure of the composition

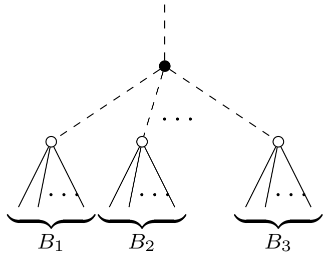
$$\mathfrak{C}_A \xrightarrow{\prod \pi_B} \prod_{\substack{B\subseteq A\\|B|\geq 2}} \mathfrak{C}_B \xrightarrow{\prod \Psi} \prod_{\substack{B\subseteq A\\|B|\geq 2}} C_B^{st} \times (0,+\infty) \hookrightarrow \prod_{\substack{B\subseteq [n]\\|A|\geq 2}} \widetilde{C}_B^{st} \times [0,+\infty]. \tag{4.6}$$

We get one codimension 1 boundary component of $\widehat{\mathfrak{C}}_n$ for each colored tree

$T_A :=$



$T^{B_1,\ldots,B_k} :=$



(4.7)

$|A| \geq 2$, and $B_1 \sqcup \ldots \sqcup B_k = [n]$ and $\inf B_i < \inf B_{i+1}$.

The component

$$\mathfrak{C}_{T_A} \cong \mathfrak{C}_{[n]/A} \times C_A$$

corresponds to the limit configurations where the points in A collapse. This is included into $\widehat{\mathfrak{C}}_n$ by

$$\mathfrak{C}_{[n]/A} \times C_A \twoheadrightarrow C_A \xrightarrow{(\pi_{B'},0)} \widetilde{C}_{B'} \times [0,\infty]$$

if $B' \subseteq A$, and

$$\mathfrak{C}_{[n]/A} \times C_A \twoheadrightarrow \mathfrak{C}_{[n]/A} \xrightarrow{\pi_{B'}\circ\psi\circ i} \widetilde{C}_{B'} \times [0,\infty]$$

otherwise.

The component

$$\mathfrak{C}_{T^{B_1,\ldots,B_k}} \cong C_{[n]/B_1\sqcup\ldots\sqcup B_k} \times \prod_{1\leq i\leq k} \mathfrak{C}_{B_i},$$

corresponds to the limit configurations where each set of points B_i, B_j goes infinitely far away from each other. This is included into $\widehat{\mathfrak{C}}_n$ by

$$C_{[n]/B_1\sqcup\ldots\sqcup B_k} \times \prod_{1\leq i\leq k} \mathfrak{C}_{B_i} \twoheadrightarrow \mathfrak{C}_{B_j} \xrightarrow{\psi\circ\pi_{B'}} \widetilde{C}_{B'} \times [0,\infty]$$

if $B' \subseteq B_j$, and

$$C_{[n]/B_1\sqcup\ldots\sqcup B_k} \times \prod_{1\leq i\leq k} \mathfrak{C}_{B_i} \twoheadrightarrow C_{[n]/B_1\sqcup\ldots\sqcup B_k} \xrightarrow{(\pi_{B'}\circ i,\infty)} \widetilde{C}_{B'} \times [0,\infty]$$

otherwise.

This extends to a boundary component

$$\mathfrak{C}_T \cong \prod_{v \in V^\circ_\infty(T)} C_{\mathrm{In}(v)} \times \prod_{v \in V^\bullet(T)} \mathfrak{C}_{\mathrm{In}(v)} \times \prod_{v \in V^\circ_0(T)} C_{\mathrm{In}(v)} \subset \widehat{\mathfrak{C}}_n,$$

of co-dimension $|V^\circ(T)|$, for each tree $T \in \mathcal{T}_n^{\circ,\bullet}$.

We define

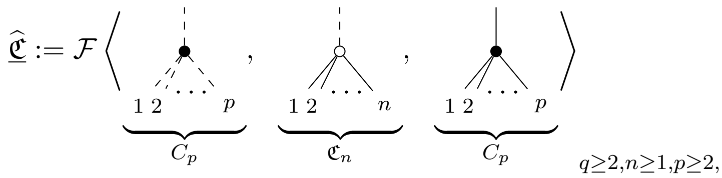
$$\widehat{\underline{\mathfrak{C}}} := (\widehat{C}, \widehat{\mathfrak{C}}, \widehat{C})$$

to be the morphism operad in the category of smooth manifolds with corners, with the natural smooth composition maps

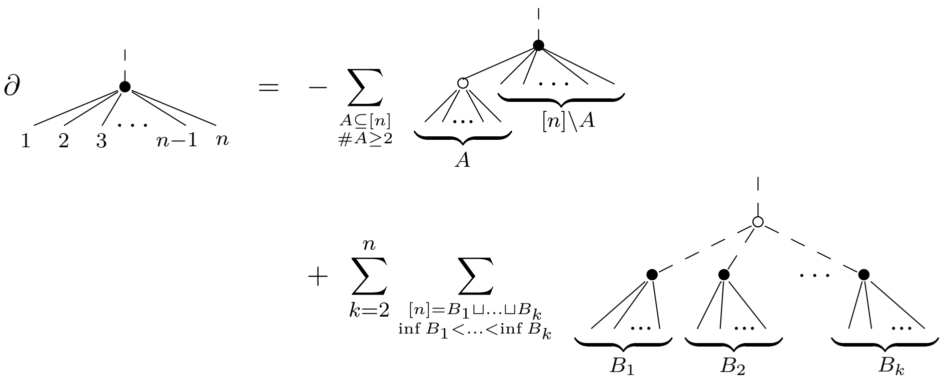
$$\widehat{\mathfrak{C}}_T \hookrightarrow \widehat{\mathfrak{C}}_n, \quad \text{for each } T \in \mathcal{T}_n^{\circ,\bullet}.$$

Theorem 4.2.3 (S. Merkulov [37]). *The fundamental chain operad $\mathcal{F}Chains(\widehat{\underline{\mathfrak{C}}}_\bullet)$ is a quasi-free morphism operad isomorphic to $\mathcal{M}or(\mathsf{hoLie}_2)$.*

Proof. We have that



and the differential on C_p, C_q coincides with the hoLie_2 differential. Furthermore, $\mathfrak{C}_n$ is of degree $2n-2$, and according to (4.7) we have



(4.8)

which coincides with the $\mathcal{M}or(\mathsf{hoLie}_2)$ operad. □

Remark 4.2.4. If we compactify $Conf(\mathbb{R}^d)/\mathbb{R}^d$ the analogous way, we get

$$\mathcal{F}Chains(\widehat{\mathfrak{C}}_\bullet(\mathbb{R})) \cong \mathcal{M}or(\mathsf{hoLie}_d).$$

for $d \geq 2$, and

$$\mathcal{F}Chains(\widehat{\mathfrak{C}}_\bullet(\mathbb{R})) \cong \mathcal{M}or(\mathsf{hoAss}_1).$$

4.2.4 De Rham field theories

Given an operad $\bar{C}$ in the category of smooth manifolds with corners, such that the fundamental chains are well defined. The De Rham complex $\Omega^\bullet(\bar{C})$ turns into a dg cooperad, where the co-composition map Δ_T is given by restricting a differential form to the boundary component associated to the tree T

$$\begin{array}{rcl}\Delta_T : \Omega^\bullet(\bar{C}) & \to & \Omega^\bullet(\bar{C}_T) = \bigotimes_{v \in V(T)} \Omega^\bullet(\bar{C}_{\mathrm{In}(v)}) \\ \omega & \mapsto & \omega|_{\bar{C}_T}.\end{array}$$

By Stokes' formula, we get a morphism of dg cooperads

$$\begin{array}{rcl}\Delta_T : \Omega^\bullet(\bar{C}) & \to & \mathcal{F}Chains(\bar{C})^* \\ \omega & \mapsto & \int_{-} \omega.\end{array}$$

Definition 4.2.5. *A* $\mathfrak{G}$-De Rham field theory *is a morphism of dg cooperads*

$$\Omega : (\mathfrak{G}, 0) \to (\Omega^\bullet(\bar{C}), d),$$

where $\mathfrak{G}$ *is the dual of some (colored) operad of graphs.*

Proposition 4.2.6. *Assume that* $\mathfrak{G}$ *acts on a (dg)-vector space* X*. Then, a De Rham field theory* $\Omega : \mathfrak{G} \to \Omega^\bullet(\bar{C})$ *induces a* $\mathcal{F}Chains(\bar{C})$*-algebra on* X*, by*

$$C_n(x_1 \otimes \cdots \otimes x_n) := \sum_\Gamma \left(\int_{C_n} \Omega(\Gamma) \right) \Gamma(x_1 \otimes \cdots \otimes x_n) \tag{4.9}$$

Proof. We obtain the formula by applying Lemma 4.1.5 to the map

$$\mathfrak{G} \xrightarrow{\Omega} \Omega^\bullet(\bar{C}) \xrightarrow{\int_{-} -} \mathcal{F}Chains(\bar{C}) .$$

□

4.3 Regularized De Rham field theories

In this section we follow A. Alekseev, C. A. Rossi, C. Torossian and T. Willwacher [3] to establish a version of Stokes' formula that allows certain well behaved singularities at the boundary. Furthermore, we construct a cooperad of differential forms on a smooth manifold with corners, that may have these well behaved singularities.

4.3.1 Trees as nested families

A *nested family of sets* is a collection of sets $\{A_i\}_{i\in I}$, where for all $i, j \in I$, we either have

$$A_i \cap A_j = A_i, \quad A_i \cap A_j = A_j, \text{ or } A_i \cap A_j = \emptyset.$$

In the following sections, it shall be useful to identify a tree T with a nested family $\{v\}_{v\in V(\Gamma)}$, where we identify the vertex v with a subset of the external legs (often called leaves) of the tree, consisting of all leaves i that are connected to v by a directed path.

For two trees T_1, T_2, we say that $T_1 \geq T_2$ if there exists a morphism of graphs $T_2 \to T_1$.

4.3.2 Regularized Stokes Formula

Let K be a smooth compact manifold with corners covered by a system chart $\{U_T\}_{T\in\mathcal{T}}$, where each U_T locally diffeomorphic to $\mathbb{R}^k_{\geq 0} \times \mathbb{R}^{n-k}$. Further assume that $\mathfrak{T}$ is a partially ordered set such that:

1. $U_{T_1} \cap U_{T_2} = \emptyset$ unless $T_1 \geq T_2$ or $T_2 \geq T_1$.
2. There exists a non-vanishing multi-vector field ξ_T on each chart U_T, which satisfies the compatibility condition
$$\xi_{T'}|_{U_T \cap U_{T'}} = \xi_T \wedge \xi'|_{U_T \cap U_{T'}}$$
for some multi-vector field ξ', whenever $T' < T$.
3. K admits a partition of unity $\{\rho_T\}_{T\in\mathcal{T}}$, subordinate to $\{U_T\}_{T\in\mathcal{T}}$, such that
$$\iota(\xi_v) d\rho_T = 0, \tag{4.10}$$
for the vector fields ξ_v, such that $\xi_T = \bigwedge_v \xi_v$.

Remark 4.3.1. In our application of this, the partially ordered set $\mathfrak{T}$ will be some sort of trees $\mathfrak{T} = \mathcal{T}$, or $\mathfrak{T} = \mathcal{T}^{\circ,\bullet}$ and we will have one vector field ξ_v for each vertex of T.

Remark 4.3.2. In [3], 2 and 3 are given as the conditions of the following Lemma.

Lemma 4.3.3. *Let $\{U_T\}_{T\in\mathcal{T}}$ be a system of charts that satisfies 1.*

Assume that each U_T carries a free torus action $f : \mathbb{T}_T \times K \to K$, that preserves the boundary components. If $T > T'$ we have an injective group homomorphism

$$\mathbb{T}_T \hookrightarrow \mathbb{T}_{T'}$$

such that the natural inclusions

$$U_T \cap U_{T'} \hookrightarrow U_T$$

are $\mathbb{T}_T$-equivariant. Then, the fundamental vector fields ξ_v of the circles $S^1_v \subset \mathbb{T}_T$ gives non-vanishing multi-vector fields

$$\xi_T = \bigwedge_{v=1}^{\dim T} \xi_v,$$

that satisfy 2. Furthermore, if each ρ_T is $\mathbb{T}_T$ invariant, then $\{\rho_T\}_{T\in\mathcal{T}}$ satisfy 3.

Definition 4.3.4. *Let ξ be a multi-vector field and α be a differential form on some topological space X. We say that α is ξ-basic if $\iota(\xi)\alpha = 0$ and $\iota(\xi)d\alpha = 0$.*

Definition 4.3.5. *Let $\omega \in \Omega^{\dim K-1}(K \setminus \partial K)$. We say that ω is regularizable if for every U_T, there exists a ξ_T-basic form α_T, called a* counter term*, such that $\omega - \alpha_T$ is regular on the boundary $\partial\bar{C} \cap U_T$. The regularized form $\mathrm{Reg}(\omega) \in \Omega^\bullet(\partial K)$ is the top degree form*

$$\mathrm{Reg}(\omega)|_{\partial K \cap U_T} = (\omega - \alpha_T)|_{\partial K \cap U_T}.$$

Proposition 4.3.6. *The regularization is well defined, i.e. it does not depend on the chart, nor the choice of counter terms.*

Proof. Let α_1, α_2 be two different counter terms for ω in U_{T_1} and U_{T_2} respectively. Then $\alpha_3 := (\alpha_2 - \alpha_1) = (\omega - \alpha_1) - (\omega - \alpha_2)$ is a $\xi_{T_1 \cap T_2}$-basic form on $U_{T_1} \cap U_{T_2}$ that restricts to a regular top degree on form the boundary $\partial(U_{T_1} \cap U_{T_2})$. Since the operator $\eta_{T_1 \cap T_2} \wedge \iota(\xi_{T_1 \cap T_2})$ is the identity on top degree forms, we have that

$$\alpha_3|_{\partial(U_{T_1} \cap U_{T_2})} = \eta \wedge \iota(\xi)\alpha_3|_{\partial(U_{T_1} \cap U_{T_2})} = 0,$$

which means that

$$(\omega - \alpha_1)|_{\partial(U_{T_1} \cap U_{T_2})} = (\omega - \alpha_2)|_{\partial(U_{T_1} \cap U_{T_2})} = \mathrm{Reg}(\omega).$$

□

Theorem 4.3.7 (Regularized Stokes' Theorem; A.Alekseev, C.A. Rossi, C. Torossian, T. Willwacher [3])**.** *Let $\omega \in \Omega^{top-1}(K \setminus \partial K)$ be a regularizable differential form. Then, $d\omega$ is regular on the boundary and*

$$\int_K d\omega = \int_{\partial K} \mathrm{Reg}(\omega).$$

Proof. Since $d\Omega$ is a top degree form in the interior of K, we have that

$$d\omega|_{U_T} = \eta_T \wedge \iota(\xi_T) d\omega|_{U_T} = \eta_T \wedge \iota(\xi_T) d(\omega - \alpha_T)|_{U_T} = d(\omega - \alpha_T)|_{U_T},$$

for each chart U_T. Since $\omega - \alpha_T$ is regular on the boundary, $d(\omega - \alpha_T) = d\omega$ is too.

Now, we have

$$\int_{\partial K} \mathrm{Reg}(\omega) = \sum_T \int_{\partial K \cap U_T} \rho_T \,\mathrm{Reg}(\omega) = \sum_T \int_{U_T} d(\rho_T \,\mathrm{Reg}(\omega)) =$$

$$= \sum_T \int_{U_T} d(\rho_T(\omega - \alpha_T)) = \sum_T \int_{U_T} d(\rho_T \omega) = \int_K d(\sum_T \rho_T \omega) = \int_K d\omega,$$

where we used (4.10) to conclude that

$$d(\rho_T \alpha_T) = \eta_T \wedge \iota(\xi_T)(d\rho_T \wedge \alpha_T + \rho_T d\alpha_T) = 0.$$

□

4.3.3 A cooperad of regularizable differential forms

Let $\bar{C} := \{\bar{C}_k\}$ be an operad in the category of smooth manifolds with corners, which satisfies the conditions of Section 4.2.1, so that the fundamental chains $\mathcal{F}Chains(\bar{C})$ are well defined.

Furthermore, we assume that $\bar{C}_p$ is covered by charts $\{U_T\}_{T\in\mathcal{T}_n}$, as in the previous section. Also, assume that each boundary component $\bar{C}_T$ is covered by $\{U_{T'/T} := U'_T \cap \bar{C}_T\}_{T'\leq T}$, and we have multi-vector fields $\xi_{T'/T}$, such that

$$\xi_{T'} = \xi_T \wedge \xi_{T'/T} \tag{4.11}$$

on each chart $U_{T'/T}$.

Remark 4.3.8. Here, we see ξ_T as a multi-vector field defined on $\bigcup_{T'\leq T} U_{T'}$ (not just U_T).

We also fix a choice of dual forms η_T, such that $\iota(\xi_T)\eta_T = 1$ with the compatibility condition

$$\eta_{T'} = \eta_T \wedge \eta_{T'/T}|_{U_{T'}}, \tag{4.12}$$

for each subtree T of T'.

Definition 4.3.9. *We denote by $\Omega^\bullet_{reg}(C_n)$ the sub-dg vector space of $\Omega^\bullet(C_n)$ consisting of differential forms ω on C_n such that, for each $T \in \mathcal{T}_n$;*

$$\eta_T \wedge \iota(\xi_T)\omega \quad \text{and} \quad \eta_T \wedge \iota(\xi_T)d\omega$$

are regular on the boundary component $\partial_T C_n \cap U_T$, and

$$\eta_T \wedge \iota(\xi_T)d\omega|_{\partial_T C_n} = d\left(\eta_T \wedge \iota(\xi_T)\omega|_{\partial_T C_n}\right). \tag{4.13}$$

Furthermore, if ω is a top-1 degree form, we require ω to be regularizable.

It is evident that $\Omega^\bullet_{reg}(C_n)$ is indeed a sub-dg vector space of $\Omega^\bullet(C_n)$. Since $\eta_T \wedge \iota(\xi_T)\omega$ is the identity on top degree forms, we have that $\Omega^{top}_{reg}(C_n) = \Omega^{top}(C_n)$. In $\Omega^{top-1}_{reg}(C_n)$ we have that all forms must satisfy

$$\eta_T \wedge \iota(\xi_T)d\omega|_{\partial_T C_n} = d\left(\eta_T \wedge \iota(\xi_T)\omega|_{\partial_T C_n}\right) (= 0)$$

on all boundary components. Thus, we get that $\Omega^{top-1}_{reg}(C_n)$ consists of all regularizable forms.

Proposition 4.3.10. *We have a dg cooperad $\left(\Omega^\bullet_{reg}(\bar{C}), \Delta, d\right)$, where*

$$\Omega^\bullet_{reg}(\bar{C}) := \left\langle \Omega^\bullet_{reg}(C_n) \right\rangle,$$

the co-composition is given by regularizing and restricting

$$\begin{aligned} \Delta_T : \Omega^\bullet_{reg}(C_n) &\to \Omega^\bullet_{reg}(C_T) \\ \omega &\mapsto \eta_T \wedge \iota(\xi_T)\omega|_{C_T}, \end{aligned}$$

and the differential is given by

$$\begin{aligned} d_T : \Omega^k_{reg}(C_T) &\to \Omega^{k+1}_{reg}(C_T) \\ \omega &\mapsto d_{C_T}(\omega), \end{aligned}$$

where d_{C_T} is the De Rham differential on C_T.

Proof. The assumptions (4.11) and (4.12) assure co-associativity of Δ. The computability between the co-composition and the differential is assured by (4.13). □

Remark 4.3.11. the co-composition Δ is, in general, not independent of the choice of ξ_T and η_T. However, if $\Delta_T\omega$ is a top degree form, then it is independent of the choice of ξ_T and η_T.

Theorem 4.3.12. *Regularization and integration gives a morphism of dg cooperads*

$$\begin{array}{rcl}\Omega^{\bullet}_{reg}(\underline{C}) & \to & \mathcal{F}Chains(\underline{C})^* \\ \omega & \mapsto & \int_{-} \operatorname{Reg}\omega.\end{array}$$

Proof. By Theorem 4.3.7, we have that this is map commutes with the differentials. It is also clear that it is a morphism of cooperads. □

Definition 4.3.13. *A* regularized $\mathfrak{G}$-De Rham field theory *is a morphism of dg cooperads*

$$\Omega : (\mathfrak{G}, 0) \to (\Omega^{\bullet}_{reg}(\bar{C}), d_{DR}),$$

where $\mathfrak{G}$ is the dual of some operad of graphs G.

Proposition 4.3.14. *A regularized $\mathfrak{G}$-De Rham field theory*

$$\begin{array}{rcl}\Omega : (\mathfrak{G}, 0) & \to & (\Omega^{\bullet}_{reg}(\bar{C}), d_{DR}) \\ \Gamma & \mapsto & \omega_\Gamma\end{array}$$

gives a morphism of dg operads

$$\begin{array}{rcl}\rho : \mathcal{F}Chains(\bar{C}) & \to & G \\ C_T & \mapsto & \sum_{\Gamma} \left(\int_{C_T} \omega_\Gamma \right) \Gamma.\end{array}$$

4.4 Regularization on $\mathfrak{C}_n$

In this section, we shall define the framework to define a cooperad of regularized differential forms on $\mathfrak{C}_n$. More precisely, for each tree $T \in \mathcal{T}_n^{\circ,\bullet}$, we will define charts U_T, covering the boundary component $\widehat{\mathfrak{C}}_T$, and vector fields ξ_T and a partition of unity ρ_T on $\widehat{\mathfrak{C}}_n$. These charts U_T, and vector fields ξ_T, will be shown to satisfy the conditions of the previous sections, so that we get a cooperad of regularizable differential forms $\Omega^{\bullet}_{reg}(\widehat{\mathfrak{C}})$.

4.4.1 The multi vector fields

Following [45], we will start by defining closed sets $\widehat{V}^c_T$, where we can define our multi-vector fields. In the next section we will define a partition of unity $\{\rho_T\}_{T\in\mathcal{T}^{\circ,\bullet}}$, and open sets $U_T \subset \widehat{V}^c_T$ that satisfy all the conditions in Section 4.3, so that $\Omega^{\bullet}_{reg}(\widehat{\mathfrak{C}})$ is well defined.

Given a configuration $z \in Conf_n(\mathbb{C})$ and a subset $A \subseteq [n]$, we denote the center of mass of the points in A by

$$\zeta_A := \frac{1}{|A|} \sum_{a \in A} z_a. \tag{4.14}$$

For a tree $T \in \mathcal{T}_n^{\circ,\bullet}$, we may parametrize $\mathfrak{C}_n$ so that

$$\zeta_v = \zeta_{\mathrm{Out}(v)} + r_{\mathrm{Out}(v)} z'_v,$$

$$\zeta_{[n]} = 0,$$

where $r_v > 0$ and

$$(z_w)_{w \in \mathrm{In}(Out(v))} \in C^{st}_{\mathrm{In}(Out(v))}.$$

Here

$$C^{st}_A := \{z' \in Conf_A(\mathbb{C}) : \zeta(z') = 0 \text{ and } \max_{w \in A} |z'_w| = 1\} \cong C_A$$

is the "standard position" of configurations in C_A. Note that we have

$$\mathfrak{C}_n \cong \prod_{v \in V(T)} C^{st}_{\mathrm{In}(v)} \times (0, \infty) \setminus \{\text{some configurations where points collide}\}.$$

For $c \in (0, 1)$, we define a set $V_T^c \subset \widehat{\mathfrak{C}}_n$ by

$$V_T^c := \prod_{v \in V_\infty^\bullet(T)} C^{st}_{\mathrm{In}(v)} \times (1/c, \bar{c}_v) \times \prod_{\substack{v \in V^\circ(T) \\ |\mathrm{In}(v)| \geq 2}} C^{st}_{\mathrm{In}(v)} \times (0, \bar{c}_v) \times \prod_{v \in V_0^\bullet(T)} C^{st}_{\mathrm{In}(v)} \times (0, \bar{c}_v) \setminus \{\text{colliding points}\}$$

where

$$\bar{c}_v := c r_{\mathrm{Out}(v)} \min_{v'' \in C^{st}_{\mathrm{In}(\mathrm{Out}(v)) \setminus \{v\}}} |z'_v - z'_{v''}|,$$

and $c_{v_{root}} := \infty$.

Furthermore, let $\widehat{V}_T^c$ be the closure of V_T^c in $\widehat{\mathfrak{C}}_n$

$$\widehat{V}_T^c := \prod_{v \in V_\infty^\bullet(T)} \widehat{C}^{st}_{\mathrm{In}(v)} \times [1/c, \bar{c}_v] \times \prod_{\substack{v \in V^\circ(T) \\ |\mathrm{In}(v)| \geq 2}} \widehat{C}^{st}_{\mathrm{In}(v)} \times [0, \bar{c}_v] \times \prod_{v \in V_0^\bullet(T)} \widehat{C}^{st}_{\mathrm{In}(v)} \times [0, \bar{c}_v] \setminus \{\text{colliding points}\}.$$

Lemma 4.4.1. *We have that:*

1. *The sets $\widehat{V}_T^c$ covers $\widehat{\mathfrak{C}}_n$.*
2. *Each $\widehat{V}_T^c$ contains the boundary component $\mathfrak{C}_T$.*
3. *For c small enough there are no colliding points in $\widehat{V}_T^c$.*
4. *Let T_1, T_2 be non-ancestors. Then, for any $c' > 0$ there exists $c > 0$, such that*

$$\widehat{V}_{T_1}^c \cap \widehat{V}_{T_2}^c \subset \widehat{V}^{c'}_{\mathrm{gcd}(T_1, \mathbb{T}_2)}.$$

Proof. The first statement is obvious, since $\widehat{V}^c_{T_{top}}$ covers $\widehat{\mathfrak{C}}_n$.

For the second statement, we note that the boundary component $\mathfrak{C}_T$ corresponds to

$$\prod_{v\in V^\bullet_\infty(T)} C^{st}_{\mathrm{In}(v)} \times \{\infty\} \times \prod_{v\in V^\bullet(T)} C^{st}_{\mathrm{In}(v)} \times (0,\infty) \times \prod_{v\in V^\circ_0(T)} C^{st}_{\mathrm{In}(v)} \times \{0\} \subset \widehat{V}^c_T.$$

For the third statement, let a, b be leaves of T, and let k_a, k_b be the number of vertices between a and $\mathrm{lca}(a,b)$, and b and $\mathrm{lca}(a,b)$ respectively. The points $z_a, z_b \in V^c_T$ can only collide if

$$\sum_{i=1}^{k_a} \bar{c}_{\mathrm{Out}^i(a)} + \sum_{i=1}^{k_b} \bar{c}_{\mathrm{Out}^i(b)} \geq r_{\mathrm{lca}(a,b)}|z'_{\mathrm{Out}^{k_a}(a)} - z'_{\mathrm{Out}^{k_b}(b)}|.$$

However, for $c < \frac{1}{4}$, we get that $\bar{c}_v < \frac{1}{2}\bar{c}_{\mathrm{Out}(v)}$. Hence,

$$\sum_{i=1}^{k_a} \bar{c}_{\mathrm{Out}^i(a)} + \sum_{i=1}^{k_b} \bar{c}_{\mathrm{Out}^i(b)} < 2c_{\mathrm{lca}(a,b)} \sum_{k=1}^{\infty} \frac{1}{2^k} = 2c_{\mathrm{lca}(a,b)} \leq r_{\mathrm{lca}(a,b)}|z'_{\mathrm{Out}^{k_a}(a)} - z'_{\mathrm{Out}^{k_b}(b)}|$$

which implies that the points z_a and z_b cannot collide with each other.

For the last statement, we note that

$$\lim_{c\to 0} \widehat{V}^c_{T_1} \cap \widehat{V}^c_{T_2} = \widehat{\mathfrak{C}}_{T_1} \cap \widehat{\mathfrak{C}}_{T_2} = \widehat{\mathfrak{C}}_{\mathrm{gcd}(T_1,T_2)} \subset \widehat{V}^{c'}_{\mathrm{gcd}(T_1,T_2)}$$

for all $c' > 0$. $\square$

For c small enough, so that there are no colliding points in V^c_T, the torus $\mathbb{T}_T := \prod_{v\in V^\circ_0(T)} S^1_v$ acts on $\widehat{V}^c_T$ by rotating the points associated to a vertex v around their center of mass. This torus action induces a multi-vector field

$$\xi_T := \bigwedge_{v\in V^\circ_0(T)} \xi_v,$$

where

$$\xi_v := \sum_{i\in v} \frac{\partial}{\partial\,\mathrm{Arg}(z_i - \zeta_v)}.$$

Furthermore, for a subtree T of a tree T' it is clear that

$$\xi'_T = \xi_T \wedge \xi_{T'/T}$$

on $\widehat{V}^c_T \cap V^c_{T'}$.

4.4.2 Partition of unity and open cover

In this section, we will use the sets $\widehat{V}^c_T$ to construct our open cover $\{U_T\}_{T\in\mathcal{T}^{\circ,\bullet}_A}$, and partition of unity $\{\rho_T\}_{T\in\mathcal{T}^{\circ,\bullet}_A}$ of $\widehat{\mathfrak{C}}_n$.

For each tree $T \in \mathcal{T}_n^{\circ,\bullet}$, pick c_T, $0 < c_T < \tilde{c}_T$, such that

$$\widehat{V}_{T_1}^{\tilde{c}_{T_1}} \cap \widehat{V}_{T_2}^{\tilde{c}_{T_2}} \subset \widehat{V}_{\gcd(T_1,T_2)}^{c_{\gcd(T_1,R_2)}} \tag{4.15}$$

for non-ancestors T_1, T_2, and the torus actions are defined. We are able to do this because of Lemma 4.4.1. Next we choose functions $\chi_T : \widehat{\mathfrak{C}}_A \to \mathbb{R}$ such that:

1. $\chi_T \equiv 1$ on a neighborhood around $\widehat{V}_T^{c_T}$,
2. $\operatorname{supp} \chi_T \subset \widehat{V}_T^{\tilde{c}_T}$,
3. χ_T is invariant under the torus actions.

This allows us to recursively define the partition of unity $\{\phi_T\}_{T \in \mathcal{T}_n^{\circ,\bullet}}$ by

$$\rho_T = \chi_T(1 - \sum_{\substack{T' \in \mathcal{T}_n \\ |V(T')| > |V(T)|}} \rho_{T'}).$$

Lemma 4.4.2. *The functions $\{\rho_T\}_{T \in T \in \mathcal{T}_n^{\circ,\bullet}}$ are a partition of unity of $\widehat{\mathfrak{C}}_A$. Furthermore, we have that* $\operatorname{supp} \rho_{T_1} \cap \operatorname{supp} \rho_{T_2} = \emptyset$*, unless $T_1 \geq T_2$ or $T_2 \geq T_1$.*

Proof. We will show by reverse induction that the following properties hold for each n:

1. $0 \leq \rho_T \leq 1$,
2. $\sum_{|V(T)| \geq n} \rho_T \leq 1$,
3. $\sum_{|V(T)| \geq n} \rho_T = 1$ on a neighborhood of each $\widehat{V}_{T'}^{c_{T'}}$ such that $|V(T')| \geq n$,
4. If $|V(T_1)|, |V(T_2)| \geq n$, and T_1, T_2 are non-ancestors, then $\operatorname{supp} \rho_{T_1} \cap \operatorname{supp} \rho_{T_2} = \emptyset$.

For $n = top$, we have that $\rho_T = \chi_T$ and 1 - 4 holds by assumption. Now assume that they hold for $n + 1$.

Assertion 1 (for n) follows immediately from the induction assumption 2 (for $n + 1$).

To show assertion 4, let T_1, T_2 be non-ancestors with $V(T_1), V(T_2) \geq n$. By (4.15), we have that

$$\operatorname{supp} \rho_{T_1} \cap \operatorname{supp} \rho_{T_2} \subset \operatorname{supp} \chi_{T_1} \cap \operatorname{supp} \chi_{T_2} \subset V_{\gcd(T_1,T_2)}^{c_{\gcd(T_1,T_2)}}. \tag{4.16}$$

Further, we have that

$$1 \geq \sum_{|V(T)| > |V(T_1)|} \rho_T \geq \sum_{|V(T)| \geq |V(\gcd(T_1,T_2))|} \rho_T = 1$$

in a neighborhood of $V_{\gcd(T_1,T_2)}^{c_{\gcd(T_1,T_2)}}$, where the first inequality is assured by the induction assumption 2, the second inequality is assured by assertion 1 with $|V(\gcd(T_1, T_2))| > |V(T_1)|$, and the equality is assured by assumption 3. Thus, we have

$$\rho_{T_1}|_{V_{\gcd(T_1,T_2)}^{c_{\gcd(T_1,T_2)}}} = \chi_{T_1}(1 - \sum_{\substack{T' \in \mathcal{T}_n \\ |V(T')| > |V(T)|}} \rho_{T'})|_{V_{\gcd(T_1,T_2)}^{c_{\gcd(T_1,T_2)}}} = 0, \tag{4.17}$$

and combined with (4.16), it is clear that we have assertion 4 for n.

From assertion 4, it follows that for $T_1 \neq T_2$ with n vertices, ρ_{T_1} and ρ_{T_2} have disjoint support. Assume that $p \in \mathrm{supp}(T')$ for some T' with n vertices, then

$$\sum_{|V(T)|\geq n} \rho_T = \chi_T(1 - \sum_{|V(T)|>n} \rho_T) + \sum_{|V(T)|>n} \rho_T \leq 1,$$

which is assertion 2.

Finally, in a neighborhood of $\widehat{V}_{T'}^{c_{T'}}$, we have that $\chi_{T'} \equiv 1$, and there we have

$$\sum_{|V(T)|\geq n} \rho_T = (1 - \sum_{|V(T)|>n} \rho_T) + \sum_{|V(T)|>n} \rho_T = 1,$$

which is assertion 3. □

Lemma 4.4.3. *Each ρ_T is invariant of the torus action in U_T.*

Proof. Since $\mathrm{supp}\,\rho_{T_1} \cap \mathrm{supp}\,\rho_{T_2} = \emptyset$ if T_1, T_2 are non-ancestors, we have that

$$\rho_T = \chi_T(1 - \sum_{\substack{T' \in \mathcal{T}_n \\ |V(T')|>|V(T)|}} \rho_{T'}) = \chi_T(1 - \sum_{\substack{T' \in \mathcal{T}_n \\ T'>T}} \rho_{T'})$$

on $\mathrm{supp}\,\rho_T$. This is a sum of products of $\{\chi_{T'}\}_{T' \geq T}$, and all such $\chi_{T'}$ are also invariant of the $\mathbb{T}_T$ action. □

We can now choose open sets $\mathrm{supp}(\rho_T) \subset U_T \subset V_T^{\tilde{c}_T}$, such that:

1. The collection $\{U_T\}_{T \in \mathcal{T}_n^{\circ,\bullet}}$ is an open cover of $\widehat{\mathfrak{C}}_n$.
2. The intersection $U_T \cap U_{T'} = \emptyset$, unless $T \geq T'$ or $T' \geq T$.
3. The torus $\mathbb{T}_T$ acts freely on U_T, preserving all boundary components.

4.5 Constructing the automorphisms

In this section, we will finally construct our family of morphisms of dg operads

$$\mathcal{F}^{t,\lambda} : \mathcal{M}or(\mathsf{hoLie}_2) \to \underline{\mathrm{Gra}}.$$

We will do so with a regularized De Rham field theory

$$\Omega^{t,\lambda} : \underline{\mathfrak{G}} \to \Omega^\bullet_{reg}(\underline{\mathfrak{C}}).$$

This section closely follows [3] Section 4 and [45] Sections 4 and 5.

4.5.1 A logarithmic propagator

We define a two parameter family of logarithmic propagators

$$\omega^{t,\lambda}_{(i,j)} := \frac{1-t}{2\pi i} d\log\left(\frac{z_i - z_j}{1+\lambda|z_i - z_j|}\right) - \frac{t}{2\pi i} d\log\left(\frac{\bar{z}_i - \bar{z}_j}{1+\lambda|z_i - z_j|}\right) \in \Omega^\bullet(\mathfrak{C}_n), \tag{4.18}$$

where $t \in \mathbb{R}$ and $\lambda > 0$. Indeed, $\omega^{t,\lambda}_{(i,j)}$ has singularities when z_i and z_j collapse, and is therefore not a member of $\Omega^\bullet(\widehat{\mathfrak{C}}_n)$.

Proposition 4.5.1. *The propagator $\omega^{t,\lambda}_{(i,j)}$ is a member of $\Omega^\bullet_{reg}(\underline{\widehat{\mathfrak{C}}}_n)$.*

Proof. In polar coordinates $z_i - z_j = r_{(i,j)} e^{i\theta_{(i,j)}}$, we get that

$$\omega^{t,\lambda}_{(i,j)} = \frac{1-2t}{2\pi i} d\log\left(\frac{r_{(i,j)}}{1+\lambda r_{(i,j)}}\right) + \frac{d\theta_{(i,j)}}{2\pi}. \tag{4.19}$$

On the boundary components where $r_{(i,j)} = \infty$, we get that $\omega^{t,\lambda}_{(i,j)}$ restricts to $\frac{d\theta_{(i,j)}}{2\pi}$, which is regular.

In a chart U_T around a boundary component where $r_{(i,j)} = 0$, we have that $\iota(\xi_v)\xi_v\omega^{t,\lambda}_{(i,j)} = 1$ whenever $i, j \in v$. Hence, we get that

$$\eta_T \wedge \iota(\xi_T)\omega^{t,\lambda}_{i,j} = \begin{cases} \eta_T & |V_0^\circ(T)| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We also need to show that $\omega^{t,\lambda}_{(1,2)}$ is regularizable on $\mathfrak{C}_2$, where $\omega^{t,\lambda}_{(1,2)}$ is a top-1 degree form. We note that the vector field around the inner boundary component $\mathfrak{C}_2|_{r=0}$ is $\frac{\partial}{\partial\theta}$, and that

$$\alpha := \frac{1-2t}{2\pi i} d\log\left(\frac{r}{1+\lambda r}\right)$$

is a $\frac{\partial}{\partial\theta}$-basic form. Hence, $\omega^{t,\lambda}_{(1,2)}$ is regularizable on $\widehat{\mathfrak{C}}_2$, and

$$\mathrm{Reg}(\omega^{t,\lambda})|_{r_{(1,2)}=0} := (\omega^{t,\lambda}_{(1,2)} - \alpha)|_{r_{(1,2)}=0} = \frac{d\theta}{2\pi}.$$

□

Theorem 4.5.2. *We get a morphism of dg cooperads*

$$\Omega : \underline{\mathfrak{G}} \to \Omega^\bullet_{reg}(\underline{\widehat{\mathfrak{C}}})$$

by

$$\begin{array}{rcl} \Omega : \mathfrak{G}^{mor}(n) & \to & \Omega^\bullet_{reg}(\widehat{\mathfrak{C}}_n) \\ \Gamma & \mapsto & \omega_\Gamma^{t,\lambda} := \bigwedge_{e\in E(\Gamma)} \omega_e^{t,\lambda}; \\ \Omega : \mathfrak{G}^{\infty}(n) & \to & \Omega^\bullet_{reg}(\widehat{C}^\infty_n) \\ \Gamma & \mapsto & \bigwedge_{e\in E(\Gamma)} \frac{\mathrm{Arg}_e}{2\pi}, \\ \Omega : \mathfrak{G}^{0}(n) & \to & \Omega^\bullet_{reg}(\widehat{C}^0_n) \\ \Gamma & \mapsto & \eta_n \wedge \iota(\xi_n)\omega_\Gamma^{t,\lambda}\Big|_{C^0_n}. \end{array}$$

Since $\Omega^\bullet_{reg}(\widehat{\underline{\mathfrak{C}}})$ is not a dg algebra, it is not clear that this is well defined. We need to show that:

1. $\iota(\xi_T)\omega_\Gamma^{t,\lambda}$ is regular in U_T.
2. Top-1 degree forms $\omega_\Gamma^{t,\lambda}$ are regularizable.
3. $\omega_{\Delta\Gamma}^{t,\lambda} = \Delta\omega_\Gamma^{t,\lambda}$, and $\omega_{d\Gamma(=0)}^{t,\lambda} = d\omega_\Gamma^{t,\lambda}$.

The following forms of $\omega_\Gamma^{t,\lambda}$ will be useful. In polar coordinates $z_1 - z_2 = re^{i\theta}$, we get that

$$\omega^{t,\lambda}(r,\theta) = \frac{1-2t}{2\pi i}\left(\frac{dr}{r} - \frac{\lambda dr}{1+\lambda r}\right) + \frac{d\theta}{2\pi}. \tag{4.20}$$

In a chart U_T, we can write $z_i = r_v z_i^v + \zeta_v$ if $i \in v$, and $z_i - z_j = r_v(r_e^v e^{i\theta_e})$, where $e = (i,j)$ and $i,j \in v$. With this notation, we get

$$\begin{aligned} \omega_e^{t,\lambda}(r_v, r_e^v, \theta_e) &= \frac{1-2t}{2\pi i}\left(\frac{d(r_v r_e^v)}{(r_v r_e^v)} - \frac{\lambda d(r_v r_e^v)}{1+\lambda(r_v r_e^v)}\right) + \frac{d\theta_e}{2\pi} = \\ &= \frac{1-2t}{2\pi i}\frac{dr_v}{r_v} + \underbrace{\frac{1-2t}{2\pi i}\left(\frac{dr_e^v}{r_e^v} - \frac{\lambda r_e^v dr_v}{1+\lambda(r_v r_e^v)} - \frac{\lambda r_v dr_e^v}{1+\lambda(r_v r_e^v)}\right) + \frac{d\theta_e}{2\pi}}_{\text{regular as } r_v\to 0}, \end{aligned} \tag{4.21}$$

when endpoints of e are in v.

Proposition 4.5.3. *The form $\omega_\Gamma^{t,\lambda}$ admits a decomposition*

$$\omega_\Gamma^{t,\lambda} = \frac{dr_v}{r_v} \wedge \alpha_v + \textit{terms regular in } r_v \tag{4.22}$$

in every chart in U_T and $v \in V_0^\circ(T)$, where α_v is independent of r_v, $\iota(\xi_v)\alpha_v = 0$ and $d\alpha_v = 0$.

Proof. We have that

$$\omega_\Gamma^{t,\lambda} = \pm \underbrace{\bigwedge_{\substack{(i,j)\in E(\Gamma)\\ i,j\in v}} \omega_{(i,j)}^{t,\lambda}}_{\omega_{\Gamma_v}^{t,\lambda}} \wedge \underbrace{\bigwedge_{\substack{(i,j)\in E(\Gamma)\\ i \text{ or } j\notin v}} \omega_{(i,j)}^{t,\lambda}}_{\omega_{\Gamma\setminus\Gamma_v}^{t,\lambda}},$$

where all the possible singularities are in $\omega_{\Gamma_v}^{t,\lambda}$. From (4.21), we see that $\omega_{\Gamma_v}^{t,\lambda}$ admits a decomposition

$$\omega_{\Gamma_v}^{t,\lambda} = \frac{dr_v}{r_v} \wedge \alpha_{\Gamma_v} + \text{terms regular in } r_v.$$

Next, let $\alpha_{\Gamma\setminus\Gamma_v}$ be an r_v independent form such that $\alpha_{\Gamma\setminus\Gamma_v} := \omega_{\Gamma\setminus\Gamma_v}^{t,\lambda}|_{r_v=0}$. Then, $\omega_{\Gamma\setminus\Gamma_v}^{t,\lambda}$ admits a decomposition

$$\omega_{\Gamma\setminus\Gamma_v}^{t,\lambda} = \underbrace{\alpha_{\Gamma\setminus\Gamma_v}}_{\text{independent of } r_v} + \underbrace{\left(\omega_{\Gamma\setminus\Gamma_v}^{t,\lambda} - \alpha_{\Gamma\setminus\Gamma_v}\right)}_{\text{proportional to } r_v}.$$

Hence, $\omega_\Gamma^{t,\lambda}$ admits a decomposition

$$\omega_\Gamma^{t,\lambda} = \frac{dr_v}{r_v} \wedge \alpha_v + \text{terms regular in } r_v, \tag{4.23}$$

where $\alpha_v := \alpha_{\Gamma_v} \wedge \alpha_{\Gamma\setminus\Gamma_v}$ is independent of r_v.

It remains to show that $\iota(\xi_v)\alpha_v = 0$. In order to see this, consider the form

$$\iota\left(\frac{\partial}{\partial r_v}\right)\iota(\xi_v)\omega_\Gamma^{t,\lambda} = \frac{1}{r_v}\iota(\xi_v)\alpha_v + \text{terms regular in } r_v.$$

This form is regular as $r_v \to 0$, if and only if $\iota(\xi_v)\alpha_v = 0$. Here, we note that $\omega_\Gamma^{t,\lambda}$ is regular for $t = 1/2$, and for $t \neq 1/2$ we note that

$$\frac{\partial}{\partial r_v} \wedge \xi_v = \frac{\partial}{\partial r_v} \wedge \left(\xi_v - \frac{ir_v}{1-2t}\frac{\partial}{\partial r_v}\right).$$

Thus, it is sufficient to show that

$$\iota\left(\xi_v - \frac{ir_v}{1-2t}\frac{\partial}{\partial r_v}\right)\omega_\Gamma^{t,\lambda} = \sum_{e\in E(\Gamma)} \pm \left(\iota\left(\xi_v - \frac{ir_v}{1-2t}\frac{\partial}{\partial r_v}\right)\omega_e^{t,\lambda}\right) \wedge \omega_{\Gamma\setminus e}^{t,\lambda}$$

is regular as $r_v \to 0$.

If both endpoints of e are outside of v, then it is clear that

$$\left(\iota\left(\xi_v - \frac{ir_v}{1-2t}\frac{\partial}{\partial r_v}\right)\omega_e^{t,\lambda}\right) = 0.$$

If both endpoints of e are in v, then $\iota(\xi_v)\omega_e^{t,\lambda} = \frac{ir_v}{1-2t}\iota(\frac{\partial}{\partial r_v})\omega_e^{t,\lambda}$. Hence, we get again that

$$\left(\iota\left(\xi_v - \frac{ir_v}{1-2t}\frac{\partial}{\partial r_v}\right)\omega_e^{t,\lambda}\right) = 0.$$

Finally if one endpoint of e is in v but the other is not, then both $\iota(\xi_v)\omega_e^{t,\lambda}$ and $\frac{ir_v}{1-2t}\iota(\frac{\partial}{\partial r_v})\omega_e^{t,\lambda}$ are proportional to r_v. This cancels the singularity in $\omega_{\Gamma\setminus e}^{t,\lambda}$. □

In particular, this proposition implies that $\iota(\xi_T)\omega^{t,\lambda} = \iota\left(\bigwedge_{v\in V_0^\circ(T)}\right)\omega_\Gamma^{t,\lambda}$ is regular in U_T, which is 1 in our list of things to show in this section.

Lemma 4.5.4. *Let ω be a top-1 degree form on $\mathfrak{C}_n$. If ω admits a decomposition*

$$\omega = d\alpha_v + \textit{terms regular in } r_v, \tag{4.24}$$

where $\iota(\xi_v)d\alpha_v = 0$, in every chart in U_T and every $v \in V^\circ(T)$.

Proof. We need to show that ω admits a decomposition

$$\alpha_T + \text{regular terms},$$

where α_T is ξ_T-basic.

Let

$$\xi_T^v := \bigwedge_{v'\in V_0^\circ(T)\setminus v} \xi_{v'},$$

and let η_T^v be the dual form $\iota(\xi_T^v)\eta_T^v = 1$.

Since ω is a top-1 degree form, it admits a decomposition

$$\omega = \sum_{v\in V(T)} \eta_T^v \wedge \iota(\xi_T^v)\omega - \underbrace{(k-1)\eta_T \wedge \iota(\xi_T)\omega}_{\text{regular by (4.24)}},$$

where $k = |V^\circ(T)|$. Furthermore, we have

$$\sum_{v\in V(T)} \eta_T^v \wedge \iota(\xi_T^v)\omega = \sum_{v\in V(T)} \eta_T^v \wedge \iota(\xi_T^v)\left(d\alpha_v + \text{terms regular in } r_v\right).$$

Here, $\eta_T^v \wedge \iota(\xi_T^v)$(terms regular in r_v) must be completely regular. Now,

$$\alpha_T := \sum_{v\in V(T)} \eta_T^v \wedge \iota(\xi_T^v)d\alpha_v$$

is our ξ_T-basic form.

Indeed, since $d\alpha_v$ is a top-1 degree form and $\iota(\xi_v)d\alpha_v = 0$, we must have that $\eta_T^v\wedge\iota(\xi_T^v)d\alpha_v = d\alpha_v$. Hence,

$$\alpha_T = \sum_{v\in V(T)} d\alpha_v,$$

and it is clear that $\iota(\xi_T)\alpha_v = 0$ and $d\alpha_T = 0$. $\square$

Note that $\frac{dr}{r} \wedge \alpha = d(\log(r)\alpha)$ for a closed form α. Hence, Lemma 4.5.4 with Proposition 4.5.3 implies that top-1 degree forms $\omega_\Gamma^{t,\lambda}$ are regularizable, which is 2.

Our next proposition implies 3.

Proposition 4.5.5. *Let $\Gamma \in \mathfrak{G}(n)$ be a graph. Then, for every tree $T \in \mathcal{T}^{\circ,\bullet}$ and $v \in V_s^\circ(T)$, we have*

$$\eta_v \wedge \iota(\xi_v)\omega_\Gamma^{t,\lambda}\Big|_{\partial_v U_T} = \omega_{\Delta_v\Gamma}^{t,\lambda}$$

Proof. We have that

$$\eta_v \wedge \iota(\xi_v)\omega_\Gamma^{t,\lambda} = \underbrace{(-1)^{\sigma(v)}\eta_v \wedge \iota(\xi_v)\omega_{\Gamma_v}^{t,\lambda} \wedge \omega_{\Gamma\backslash\Gamma_v}^{t,\lambda}}_{\omega_{\Delta_v\Gamma}^{t,\lambda}} \pm \eta_v \wedge \omega_{\Gamma_v}^{t,\lambda} \wedge \iota(\xi_v)\omega_{\Gamma\backslash\Gamma_v}^{t,\lambda}.$$

We need to show that

$$\eta_v \wedge \omega_{\Gamma_v}^{t,\lambda} \wedge \iota(\xi_v)\omega_{\Gamma\backslash\Gamma_v}^{t,\lambda}\Big|_{\partial_v U_T} = 0.$$

Indeed, $\omega_{\Gamma_v}^{t,\lambda}$ admits a decomposition

$$\omega_{\Gamma_v}^{t,\lambda} = \frac{dr_v}{r_v} \wedge \alpha + \gamma,$$

where α and γ are regular in r_v, and $\iota(\xi_v)\omega_{\Gamma\backslash\Gamma_v}^{t,\lambda}$ admits a decomposition

$$\iota(\xi_v)\omega_{\Gamma\backslash\Gamma_v}^{t,\lambda} = r_v\beta,$$

where β is regular in r_v. Since dr_v and r_v disappear at the boundary component $\partial_v U_T$, we get that

$$\eta_v \wedge \omega_{\Gamma_v}^{t,\lambda} \wedge \iota(\xi_v)\omega_{\Gamma\backslash\Gamma_v}^{t,\lambda}\Big|_{\partial_v U_T} = \eta_v \left(dr_v \wedge \alpha \wedge \beta + r_v\gamma \wedge \beta\right)\big|_{\partial_v U_T} = 0.$$

□

4.5.2 Interpreting the automorphisms

We now have established a morphism of dg cooperads

$$\Omega^{t,\lambda} : \underline{\mathfrak{G}} \to \Omega^\bullet_{reg}(\underline{\widehat{\mathfrak{C}}}).$$

By Proposition 4.3.14, we get a morphism of dg operads

$$\begin{aligned}
\mathcal{F}^{t,\lambda} : \mathcal{F}Chains(\underline{\widehat{\mathfrak{C}}}) &\to \underline{\text{Gra}},\\
\mathfrak{C}_n &\mapsto \sum_{\Gamma\in \text{Gra}^{mor}(n)} \varpi_\Gamma^{t,\lambda}\Gamma,\\
C_n^{out} &\mapsto \sum_{\Gamma\in \text{Gra}^{out}(n)} \left(\int_{C_n^{out}} \omega_\Gamma^{t,\lambda}\right)\Gamma,\\
C_n^{in} &\mapsto \sum_{\Gamma\in \text{Gra}^{in}(n)} \left(\int_{C_n^{in}} \text{Reg}(\omega_\Gamma^{t,\lambda})\right)\Gamma.
\end{aligned}$$

Since $\mathcal{F}Chains(\underline{\widehat{\mathfrak{C}}}) \cong \mathcal{M}or(\mathsf{holie}_2)$, this is equivalent to an MC-element

$$(\phi_{t,\lambda}^{out}, \mathcal{F}^{t,\lambda}, \phi_{t,\lambda}^{in}) \in \mathsf{Def}(\mathcal{M}or(\mathsf{holie}_2) \xrightarrow{0} \underline{\text{Gra}_2}).$$

Let us first take a look at the MC-elements

$$\phi_{t,\lambda}^{out/in} = \prod_{n\geq 0} \phi_{t,\lambda,(n)}^{out/in} \in \mathsf{Def}(\mathsf{holie}_2 \to \text{Gra}^{out/in}),$$

where $\phi^{out/in}_{(n)}$ is a sum of graphs with $n+1$ vertices

$$\phi^{in}_{t,\lambda,(n)} := \sum_{\Gamma\in \mathrm{Gra}(n+1)} \left(\int_{C_{n+1}} \mathrm{Reg}(\omega^{t,\lambda}_\Gamma) \right) \Gamma,$$

and

$$\phi^{out}_{t,\lambda,(n)} := \sum_{\Gamma\in \mathrm{Gra}(n+1)} \left(\int_{C_{n+1}} \left(\frac{d\theta}{2\pi}\right)_\Gamma \right) \Gamma.$$

Proposition 4.5.6. *Let C_n be the boundary strata where all points collapse, and let $\Gamma \in \mathfrak{G}^0(n)$ be a graph with $|E(\Gamma)| = 2n-3$. Then,*

$$\mathrm{Reg}(\omega^{t,\lambda}_\Gamma)\Big|_{C_n} = \bigwedge_{e\in E(\Gamma)} \frac{d\theta_e}{2\pi}.$$

Proof. As $\omega^{t,\lambda}_\Gamma$ is regular, the singularities must disappear. The formula (4.19) gives us the desired result. □

Lemma 4.5.7 (Kontsevich vanishing lemma, [26])**.** *Let $\Gamma \in \mathfrak{G}^0(n)$ be a graph with $|E(\Gamma)| = 2n-3$. Then,*

$$\int_{C_n} \left(\frac{d\theta}{2\pi}\right)_\Gamma = \begin{cases} 1 & \Gamma = \bullet\!\!-\!\!\bullet \\ 0 & \textit{otherwise.} \end{cases}$$

As a direct consequence of Kontsevich vanishing lemma, we get that the MC-elements $\phi^{in}_{t,\lambda}$, $\phi^{out}_{t,\lambda}$ are concentrated in weight degree 1, and are equal to the graph

$$\bullet\!\!-\!\!\bullet,$$

for all t, λ. It follows that the induced MC-elements in

$$\mathsf{Def}(\mathsf{holie}_2 \to \mathrm{End}_{T_{poly}(\mathbb{R}^d)})$$

induce the Schouten–Nijenhuis Lie bracket on T_{poly}.

Next, let us look at

$$\mathcal{F}^{t,\lambda}_{(n)} := \sum_\Gamma \varpi^{t,\lambda}_\Gamma \Gamma, \tag{4.25}$$

where $\varpi^{t,\lambda}_\Gamma := \int_{\mathfrak{C}_{n+1}} \omega^{t,\lambda}_\Gamma$.

Proposition 4.5.8. *We have*

$$\mathcal{F}^{t,\lambda}_{(0)} = \bullet$$

for all t, λ. For $t = 1/2$, we have

$$\mathcal{F}^{1/2,\lambda} = \bullet.$$

Proof. The first statement is clear. For the second statement, we note that $\omega^{1/2,\lambda}(z_i, z_j) = \frac{1}{2\pi} d\,\mathrm{Arg}(z_i - z_j)$. As the space $\mathfrak{C}_n$ cannot be spanned by just the angles between the points, the weights $\varpi^{t,\lambda}_\Gamma$ must vanish for all graphs $\Gamma \neq \bullet$. □

It follows that the induced ∞-morphism

$$\mathcal{F}^{t,\lambda} : T_{poly}(\mathbb{R}^d) \to T_{poly}(R^d)$$

is the identity for $t = 1/2$, and an automorphism for all t, λ.

Our aim now is to show that $\mathcal{F}^{t,\lambda}$ is exotic for $t \neq 1/2$.

4.5.3 The partial derivative $\partial_t \omega^{t,\lambda}$

Let us look at the partial derivative $\partial_t \omega_\Gamma^{t,\lambda}$. We have that

$$\partial_t \omega^{t,\lambda}(z_1, z_2) = \frac{i}{\pi} d \log \left(\frac{|z_1 - z_2|}{1 + \lambda |z_1 - z_2|} \right). \tag{4.26}$$

Let

$$\widetilde{\omega}_\Gamma^{t,\lambda} := \sum_{e \in \Gamma} (-1)^{e+1} \beta_e^\lambda \omega_{\Gamma \setminus \{e\}}^{t,\lambda},$$

where

$$\beta_{(i,j)}^\lambda := \frac{i}{\pi} \log \left(\frac{|z_i - z_j|}{1 + \lambda |z_i - z_j|} \right),$$

so that

$$d\widetilde{\omega}_\Gamma^{t,\lambda} = \partial_t \omega_\Gamma^{t,\lambda}.$$

Proposition 4.5.9. *For any graph $\Gamma \in \mathfrak{G}(n)$, we have that $\widetilde{\omega}_\Gamma^{t,\lambda}$ is a regularizable form on $\widehat{\underline{\mathfrak{C}}}_n$.*

Lemma 4.5.10. *Let $\Gamma \in \mathfrak{G}^{mor}$ be a graph, let $T \in \mathcal{T}_n^{\circ,\bullet}$, and let $v \in V_0^\bullet(T)$. Then $\widetilde{\omega}_\Gamma^{t,\lambda}$ admits a decomposition*

$$\widetilde{\omega}_\Gamma^{t,\lambda} = \frac{dr_v}{r_v} \wedge \alpha_1 + \log(r_v)\alpha_2 + \textit{terms regular in } r_v$$

in U_T, where α_1 and α_2 are independent of r_v, and $\iota(\xi_v)\alpha_1 = \iota(\xi_v)\alpha_2 = 0$, $\iota(\xi_v)d\alpha_1 = d\alpha_2 = 0$.

Proof. We see that β_e^λ is regular in r_v, if $e \notin E(\Gamma_v)$, and

$$\beta_e^\lambda = \frac{i}{\pi} \log(r_v) + \text{terms regular in } r_v$$

if $e \in E(\Gamma_v)$. Hence, $\widetilde{\omega}_\Gamma^{t,\lambda}$ admits a decomposition

$$\underbrace{\left(\frac{i}{\pi} \log(r_v) \underbrace{\sum_{e \in E(\Gamma_v)} (-1)^{e+1} \omega_{\Gamma_v \setminus \{e\}}^{t,\lambda}}_{= \iota(\xi_v)\omega_{\Gamma_v}^{t,\lambda}(\text{regular in } r_v)} \right) \wedge \omega_{\Gamma \setminus \Gamma_v}^{t,\lambda}}_{= \log(r_v)\alpha_2 + \text{terms regular in } r_v} + \underbrace{\sum (\text{terms regular in } r_v)_e \omega_{\Gamma \setminus \{e\}}^{t,\lambda}}_{= \frac{dr_v}{r_v} \wedge \alpha_1 + \text{terms regular in } r_v},$$

where we can choose α_1 and α_2 to be independent of r_v. Furthermore, we get that

$$\iota(\xi_v)\widetilde{\omega}_\Gamma^{t,\lambda} = \frac{i}{\pi}\log(r_v)\iota(\xi_v)\omega_{\Gamma_v}^{t,\lambda} \wedge \underbrace{\iota(\xi_v)\omega_{\Gamma\setminus\Gamma_v}^{t,\lambda}}_{\text{proportional to } r_v} + \underbrace{\text{terms regular in } r_v}_{\text{by Proposition 4.5.3}}$$

is regular in r_v. Hence, $\iota(\xi_v)\alpha_1 = \iota(\xi_v)\alpha_2 = 0$. Since

$$\partial_t\omega_\Gamma^{t,\lambda} = d\widetilde{\omega}_\Gamma^{t,\lambda}$$

does not have a $\log(r_v)$ singularity, we have that $d\alpha_2 = 0$. Furthermore, since

$$\partial_t\iota(\xi_v)\omega_\Gamma^{t,\lambda} = \iota(\xi_v)d\widetilde{\omega}_\Gamma^{t,\lambda} = \frac{dr_v}{r_v}\wedge\iota(\xi_v)d\alpha_1 - \frac{dr_v}{r_v}\wedge\iota(\xi_v)\alpha_2 + \text{terms regular in } r_v$$

is regular in r_v, we must have that $\iota(\xi_v)d\alpha_1 = \iota(\xi_v)\alpha_2 = 0$. □

It follows that each $\eta_T\wedge\iota(\xi_T)\widetilde{\omega}_\Gamma^{t,\lambda}$ is regular on each boundary component $\partial_T\mathfrak{C}_n$. We also have that

$$\eta_T\wedge\iota(\xi_T)d\widetilde{\omega}_\Gamma^{t,\lambda} = \partial_t\left(\eta_T\wedge\iota(\xi_T)\omega_\Gamma^{t,\lambda}\right)$$

is regular in $\partial_T\mathfrak{C}_n$.

Lemma 4.5.11. *Top−1 degree forms $\widetilde{\omega}_\Gamma^{t,\lambda}$ are regularizable.*

Proof. Let $\widetilde{\omega}_\Gamma^{t,\lambda}$ be a top−1 degree form. Then, $d\widetilde{\omega}_\Gamma^{t,\lambda} = \partial_t\omega_\Gamma^{t,\lambda}$ is a top degree regular form. Thus, we must have that

$$d\left(\frac{dr_v}{r_v}\wedge\alpha_1 + \log(r_v)\alpha_2\right) = \frac{dr_v}{r_v}\wedge(\alpha_2 - d\alpha_1)$$

is regular. Since α_1 and α_2 are independent of r_v, we must have that $\alpha_2 = d\alpha_1$, and thus

$$\frac{dr_v}{r_v}\wedge\alpha_1 + \log(r_v)\alpha_2 = d(\log(r_v)\alpha_1).$$

We can now apply Lemma 4.5.4 to conclude that $\widetilde{\omega}_\Gamma^{t,\lambda}$ is regularizable. □

Lemma 4.5.12. *For a collapsing boundary strata*

$$\Delta_{T_A}\widetilde{\omega}_\Gamma^{t,\lambda} = \pm\left(\eta_A\wedge\iota(\xi_A)\omega_{\Gamma_A}^{t,\lambda}\right)\wedge\widetilde{\omega}_{\Gamma\setminus\Gamma_A}^{t,\lambda}\Big|_{\partial_A\mathfrak{C}_n} \pm \left(\eta_A\wedge\iota(\xi_A)\widetilde{\omega}_{\Gamma_A}^{t,\lambda}\right)\wedge\omega_{\Gamma\setminus\Gamma_A}^{t,\lambda}\Big|_{\partial_A\mathfrak{C}_n}.$$

For an infinity boundary strata, we denote $\Gamma_\infty := \Gamma/\{\Gamma_{B_1},\ldots,\Gamma_{B_k}\}$. Then we have

$$\begin{aligned}\Delta_{T^{B_1,\ldots,B_k}}\widetilde{\omega}_\Gamma^{t,\lambda} := \widetilde{\omega}_\Gamma^{t,\lambda}\Big|_{\partial_{T^{B_1,\ldots,B_k}}\mathfrak{C}_n} &= (d\widetilde{\mathrm{Arg}})_{\Gamma_\infty}\wedge\omega_{\Gamma_{B_1}}^{t,\lambda}\wedge\ldots\wedge\omega_{\Gamma_{B_k}}^{t,\lambda}\pm \\ &\pm (d\,\mathrm{Arg})_{\Gamma_\infty}\wedge\sum_{i=1}^k\pm\omega_{\Gamma_{B_1}}^{t,\lambda}\wedge\ldots\wedge\widetilde{\omega}_{\Gamma_{B_i}}^{t,\lambda}\wedge\ldots\wedge\omega_{\Gamma_{B_k}}^{t,\lambda},\end{aligned}$$

where

$$(d\widetilde{\mathrm{Arg}})_\Gamma := \frac{i}{\pi}\log\left(\frac{1}{\lambda}\right)\sum_{e\in E(\Gamma)}(-1)^{1+e}(d\,\mathrm{Arg})_{\Gamma\setminus\{e\}}.$$

Proof. We have that

$$\iota(\xi_A)\widetilde{\omega}_\Gamma^{t,\lambda} = (-1)^{\sigma_A}\iota(\xi_A)\widetilde{\omega}_{\Gamma_A}^{t,\lambda} \wedge \omega_{\Gamma\setminus\Gamma_A}^{t,\lambda} \pm \underbrace{\widetilde{\omega}_{\Gamma_A}^{t,\lambda} \wedge \iota(\xi_A)\omega_{\Gamma\setminus\Gamma_A}^{t,\lambda}}_{\to 0 \text{ as } r_A \to 0} +$$
$$+(-1)^{\sigma_A+|A|}\iota(\xi_A)\omega_{\Gamma_A}^{t,\lambda} \wedge \widetilde{\omega}_{\Gamma\setminus\Gamma_A}^{t,\lambda} \pm \underbrace{\omega_{\Gamma_A}^{t,\lambda} \wedge \iota(\xi_A)\widetilde{\omega}_{\Gamma\setminus\Gamma_A}^{t,\lambda}}_{\to 0 \text{ as } r_A \to 0}.$$

The second statement is clear, since $\omega_{\Gamma_{B_i}}^{t,\lambda}$ and $\widetilde{\omega}_{\Gamma_{B_i}}^{t,\lambda}$ are independent of r_v, and

$$\widetilde{\omega}_{\Gamma_\infty}^{t,\lambda}\Big|_{r_v=\infty} = (d\widetilde{\operatorname{Arg}})_{\Gamma_\infty}.$$

□

Lemma 4.5.13. *For any graph* Γ, *we have that* $d\Delta\widetilde{\omega}_\Gamma^{t,\lambda} = \Delta d\widetilde{\omega}_\Gamma^{t,\lambda}$.

Proof. It is sufficient to show this for codimension 1 boundary components. From the formulas in the previous lemma, we can see that $d\Delta\widetilde{\omega}_\Gamma^{t,\lambda} = \partial_t\Delta\omega_\Gamma^{t,\lambda}$. It follows that

$$d\Delta\widetilde{\omega}_\Gamma^{t,\lambda} = \partial_t\Delta\omega_\Gamma^{t,\lambda} = \Delta\partial_t\omega_\Gamma^{t,\lambda} = \Delta d\widetilde{\omega}_\Gamma^{t,\lambda}.$$

□

Proposition 4.5.9 follows from Lemma 4.5.10, Lemma 4.5.11 and Lemma 4.5.13.

4.5.4 A family of graph cocycles $x^{t,\lambda}$

Looking at the formulas of Lemma 4.5.12, we see that

$$\begin{aligned}\int_{\mathfrak{C}_{T^{B_1,\dots,B_k}}} \operatorname{Reg}(\widetilde{\omega}_\Gamma^{t,\lambda}) &= \int_{C_{[n]/\{B_1,\dots,B_k\}}} (d\operatorname{Arg})_{\Gamma_\infty} \prod_{i\le k}\int_{\mathfrak{C}_{B_i}} \omega_{\Gamma_{B_i}}^{t,\lambda} \pm \\ &\pm \int_{C_{[n]/\{B_1,\dots,B_k\}}} (d\widetilde{\operatorname{Arg}})_{\Gamma_\infty} \sum_j \left(\int_{\mathfrak{C}_{B_j}} \widetilde{\omega}_{\Gamma_{B_j}}^{t,\lambda} \prod_{i\ne j}\int_{\mathfrak{C}_{B_i}} \omega_{\Gamma_{B_i}}^{t,\lambda}\right).\end{aligned}$$

Applying Kontsevich vanishing lemma, we get

$$\int_{\mathfrak{C}_{T^{B_1,\dots,B_k}}} \operatorname{Reg}(\widetilde{\omega}_\Gamma^{t,\lambda}) = \begin{cases} \int_{\mathfrak{C}_{B_1}} \widetilde{\omega}_{\Gamma_{B_1}}^{t,\lambda} \int_{\mathfrak{C}_{B_2}} \omega_{\Gamma_{B_2}}^{t,\lambda} \pm \int_{\mathfrak{C}_{B_1}} \omega_{\Gamma_{B_1}}^{t,\lambda} \int_{\mathfrak{C}_{B_2}} \widetilde{\omega}_{\Gamma_{B_2}}^{t,\lambda} & \Gamma_\infty = \bullet\!\!-\!\!\!-\!\!\bullet \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}$$

For a graph Γ with $|E(\Gamma)| = 2|V(\Gamma)| - 2$, this is

$$\int_{\partial_\infty\mathfrak{C}_n} \widetilde{\omega}_\Gamma^{t,\lambda} = \sum_{\substack{|E(\Gamma_A)|=2|A|-2 \\ |E(\Gamma_B)|=2|B|-1}} \varpi_{\Gamma_A}^{t,\lambda}\widetilde{\varpi}_{\Gamma_B}^{t,\lambda}. \tag{4.28}$$

On the collapsing boundary strata, we have

$$\int_{\mathfrak{C}_{T_A}} \operatorname{Reg}(\widetilde{\omega}_\Gamma^{t,\lambda}) = \begin{cases} \widetilde{\varpi}_{\Gamma/\Gamma_A}^{t,\lambda} & \Gamma_A = \bullet\!\!-\!\!\bullet \\ \bar{\beta}_{\Gamma_A}^{t,\lambda} \varpi_{\Gamma/\Gamma_A}^{t,\lambda} & |E(\Gamma/\Gamma_A)| = 2(n-|A|) \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\bar{\beta}_{\Gamma_A}^{t,\lambda} := \int_{C_A} \operatorname{Reg}(\widetilde{\omega}_{\Gamma_A}^{t,\lambda}).$$

This gives us

$$\int_{\partial_0 \mathfrak{C}_n} \widetilde{\omega}_\Gamma^{t,\lambda} = \widetilde{\varpi}_{d\Gamma}^{t,\lambda} + \sum_{E(\Gamma_A)=2|A|-2} \bar{\beta}_{\Gamma_A}^{t,\lambda} \varpi_{\Gamma/\Gamma_A}^{t,\lambda}, \tag{4.29}$$

where $d\Gamma := \sum_{e \in E(\Gamma)} (-1)^{e+n} \Gamma/e$.

Theorem 4.5.14. *There exists a family of graph cocycles $x^{t,\lambda} \in \mathrm{fGC}_2$, such that*

$$\partial_t \mathcal{F}^{t,\lambda} = x^{t,\lambda} \cdot \mathcal{F}^{t,\lambda} + \delta \widetilde{\mathcal{F}}^{t,\lambda} + [\mathcal{F}^{t,\lambda}, \widetilde{\mathcal{F}}^{t,\lambda}].$$

Here, $x \cdot \Gamma$ is the action of summing over all ways to insert the graph x into a vertex of Γ, and the Lie bracket is given by $[\Gamma_1, \Gamma_2] := \bigcirc\!\!\!\!\!\!\;\Gamma_1 \!-\!\!\!-\! \bigcirc\!\!\!\!\!\!\;\Gamma_2$.

Proof. By the regularized Stokes' formula, we have that

$$\partial_t \mathcal{F}_n^{t,\lambda} = \sum_\Gamma \left(\int_{\partial \mathfrak{C}_n} \operatorname{Reg} \widetilde{\omega}_\Gamma^{t,\lambda} \right) \Gamma =$$
$$= \sum_\Gamma \left(\int_{\partial_0 \mathfrak{C}_n} \operatorname{Reg} \widetilde{\omega}_\Gamma^{t,\lambda} - \int_{\partial_\infty \mathfrak{C}_n} \operatorname{Reg} \widetilde{\omega}_\Gamma^{t,\lambda} \right) \Gamma.$$

According to (4.28), we have

$$\sum_\Gamma \left(\int_{\partial_\infty \mathfrak{C}_n} \operatorname{Reg} \widetilde{\omega}_\Gamma^{t,\lambda} \right) \Gamma = \sum_{\Gamma_1, \Gamma_2} \varpi_{\Gamma_2}^{t,\lambda} \widetilde{\varpi}_{\Gamma_1}^{t,\lambda} [\Gamma_1, \Gamma_2] = [\mathcal{F}^{t,\lambda}, \widetilde{\mathcal{F}}^{t,\lambda}].$$

The sum runs over all graphs Γ_1, Γ_2 such that $|V(\Gamma_1)| + |V(\Gamma_2)| = n$, $|E(\Gamma_1)| = 2|V(\Gamma_1)| - 1$ and $|E(\Gamma_2)| = 2|V(\Gamma_2)| - 2$. According to (4.29), we have

$$\begin{aligned} \sum_\Gamma \left(\int_{\partial_0 \mathfrak{C}_n} \operatorname{Reg} \widetilde{\omega}_\Gamma^{t,\lambda} \right) \Gamma &= \sum_\Gamma \widetilde{\varpi}_{d\Gamma}^{t,\lambda} \Gamma + \sum_\Gamma \left(\sum_{|E(\Gamma_A)|=2|A|-2} \bar{\beta}_{\Gamma_A}^{t,\lambda} \varpi_{\Gamma/\Gamma_A}^{t,\lambda} \right) \Gamma = \\ &= \underbrace{\sum_{\Gamma'} \varpi_\Gamma^{t,\lambda} \delta \Gamma'}_{\delta \widetilde{\mathcal{F}}^{t,\lambda}} + \underbrace{\sum_{\Gamma_1, \Gamma_2} \bar{\beta}_{\Gamma_1}^{t,\lambda} \varpi_{\Gamma_2}^{t,\lambda} \Gamma_1 \cdot \Gamma_2}_{x^{t,\lambda} \cdot \mathcal{F}^{t\lambda}} \end{aligned}$$

where

$$x^{t,\lambda} := \sum_\Gamma \bar{\beta}_\Gamma^{t,\lambda} \Gamma.$$

Since $\mathcal{F}^{t,\lambda}$ is a Lie ∞-morphism, we have that

$$0 \;=\; \delta\mathcal{F}^{t,\lambda} + \frac{1}{2}[\mathcal{F}^{t,\lambda}, \mathcal{F}^{t,\lambda}].$$

Hence

$$\begin{aligned}
0 &= \partial_t(\delta\mathcal{F}^{t,\lambda} + \frac{1}{2}[\mathcal{F}^{t,\lambda}, \mathcal{F}^{t,\lambda}]) \\
&= \delta(x^{t,\lambda}\cdot\mathcal{F}^{t,\lambda} + \delta\widetilde{\mathcal{F}}^{t,\lambda} + [\mathcal{F}^{t,\lambda}, \widetilde{\mathcal{F}}^{t,\lambda}]) + [x^{t,\lambda}\cdot\mathcal{F}^{t,\lambda} + \delta\widetilde{\mathcal{F}}^{t,\lambda} + [\mathcal{F}^{t,\lambda}, \widetilde{\mathcal{F}}^{t,\lambda}], \mathcal{F}^{t,\lambda}] = \\
&= (\delta x^{t,\lambda})\cdot\mathcal{F}^{t,\lambda} + \underbrace{x^{t,\lambda}\cdot(\delta\mathcal{F}^{t,\lambda} + \frac{1}{2}[\mathcal{F}^{t,\lambda}, \mathcal{F}^{t,\lambda}])}_{=0} + \delta[\mathcal{F}^{t,\lambda}, \widetilde{\mathcal{F}}^{t,\lambda}] + [\delta\widetilde{\mathcal{F}}^{t,\lambda} + [\mathcal{F}^{t,\lambda}, \widetilde{\mathcal{F}}^{t,\lambda}], \mathcal{F}^{t,\lambda}] = \\
&= (\delta x^{t,\lambda})\cdot\mathcal{F}^{t,\lambda} + [\delta\widetilde{\mathcal{F}}^{t,\lambda}, 0] + \underbrace{[\delta\mathcal{F}^{t,\lambda} + \frac{1}{2}[\mathcal{F}^{t,\lambda}, \mathcal{F}^{t,\lambda}], \widetilde{\mathcal{F}}^{t,\lambda}]}_{=0} = \\
&= (\delta x^{t,\lambda})\cdot\mathcal{F}^{t,\lambda}.
\end{aligned}$$

Since $\mathcal{F}^{t,\lambda}$ is a group-like element, i.e. $\mathcal{F}^{t,\lambda} = \bullet +$ graphs with more vertices, we get that

$$(\delta x^{t,\lambda})\cdot\mathcal{F}^{t,\lambda} = \delta x^{t,\lambda} + \text{graphs with more vertices},$$

which vanishes if and only if $\delta x^{t,\lambda} = 0$. □

4.5.5 About the graph cocycles $x^{t,\lambda}$

We now have constructed a family of graph cocycles

$$x_k^{t,\lambda} := \sum_{\Gamma}\left(\int_{C_k}\operatorname{Reg}(\widetilde{\omega}_\Gamma^{t,\lambda})\right)\Gamma \tag{4.30}$$

We note first that $\widetilde{\omega}_\bullet^{t,\lambda} = 0$. Since there are no connected graphs with $|E(\Gamma)| = 2|V(\Gamma)| - 2$ for $|V(\Gamma)| = 2, 3$, we have that $x_k^{t,\lambda} = 0$ for $k = 2, 3$.

Lemma 4.5.15. *Let C_k be the boundary component of $\widehat{\mathfrak{C}}_k$ where all points collapse. Then*

$$\int_{C_k}\operatorname{Reg}(\widetilde{\omega}_\Gamma^{t,\lambda}) = \frac{i}{\pi}\sum_{e\in E(\Gamma)}(-1)^e\int_{C_k}\log(r_e)\left(\frac{d\theta}{2\pi}\right)_{\Gamma\setminus\{e\}}.$$

Proof. We will use the coordinates $z_i - z_j = Rr_{i,j}e^{i\theta_{i,j}}$ on $\mathfrak{C}_k$. We have that

$$\widetilde{\omega}_\Gamma^{t,\lambda} = \sum(-1)^e\left(\log(R) + \log\left(\frac{r_e}{1+\lambda Rr_e}\right)\right)\omega_{\Gamma\setminus\{e\}}^{t,\lambda}.$$

Since $\iota(\xi_k)\widetilde{\omega}_\Gamma^{t,\lambda}$ is regular as $R\to 0$, we must have that the $\log(R)$ singularity cancels, and thus we have

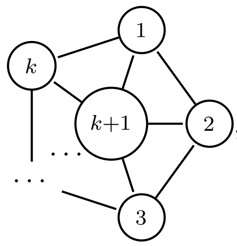
$$\begin{aligned}
&\eta_T\wedge\iota(\xi_k)\widetilde{\omega}^{t,\lambda}|_{R=0} = \\
&= \sum_{e\in E(\Gamma)}(-1)^e\log(r_e)\eta_T\wedge\iota(\xi_k)\omega_{\Gamma\setminus\{e\}}^{t,\lambda}\Big|_{R=0} = \sum_{e\in E(\Gamma)}(-1)^e\log(r_e)\operatorname{Reg}\left(\omega_{\Gamma\setminus\{e\}}^{t,\lambda}\right).
\end{aligned}$$

The statement follows from Proposition 4.5.6. □

The integral weights

$$\sum_{e\in E(W_n)} (-1)^e \int_{C_{k+1}} \log(r_e) \left(\frac{d\theta}{2\pi}\right)_{\Gamma\setminus\{e\}}$$

are computed for the wheel graphs

$W_k =$ ,

by A. B. Goncharov [19] and A. Rossi, T. Willwacher [45].

Theorem 4.5.16 ([19],[45]). *For odd $k \geq 3$, we have that*

$$\sum_{e\in E(W_n)} (-1)^e \int_{C_{k+1}} \log(r_e) \left(\frac{d\theta}{2\pi}\right)_{\Gamma\setminus\{e\}} = q_k \frac{\zeta(k)}{(2\pi)^k},$$

for some rational number q_k, and ζ is Riemann's zeta function.

In particular, As the graphs W_n cannot be coboundaries in $H(\mathrm{fGC}_2)$, we have that the cohomology class $[x^{t,\lambda}]$ is non-trivial. It follows that $\mathcal{F}^{t\neq 1/2,\lambda}$ and $\mathcal{F}^{1/2,\lambda} = \mathrm{id}$ are not homotopy equivalent, i.e. $\mathcal{F}^{t\neq 1/2,\lambda}$ is an exotic Lie ∞-automorphism.

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