ON THE ENTANGLEMENT OF RADICALS

CHI WA CHAN, ANTIGONA PAJAZITI, FLAVIO PERISSINOTTO AND ANTONELLA PERUCCA

ABSTRACT. In this work we make progress in the understanding of the so-called entanglement of radicals, showing that there are extremely few additive relations among radicals. Our results complete a famous theorem by Kneser from 1975 on the linear independence of radicals. Indeed, we determine all the radicals belonging to the Kneser field, that is a cyclotomic extension of the base field over which there is no entanglement anymore.

1. INTRODUCTION

Let K be a field (for which we fix an algebraic closure $\overline{K}$) and consider a multiplicative group G of radicals of K, that is a group generated by $K^\times$ and by elements in $\overline{K}$ that have some power in $K^\times$. Clearly, the multiplicative relations among the radicals in G are encoded in the group structure. We are interested in the additive relations among the radicals in G (also called *entanglement*) that become relevant when we consider the field $K(G)$. To study the additive relations among radicals we may suppose without loss of generality that the index $|G : K^\times|$ is finite. Then we can "measure" the additive relations by comparing this index and the degree of the extension $K(G)/K$. Indeed, for radicals that are dependent (in the sense that they give rise to additive relations that do not stem from multiplicative relations) the degree $[K(G) : K]$ is smaller than the index $|G : K^\times|$.

Roots of unity are radicals, and K-linear relations among them constitute one first type of entanglement, which we call *cyclotomic entanglement*. The basic relations are the following: if ζ_n is a root of unity of order n, then we have

$$1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1} = 0\,.$$

Over $\mathbb{Q}$ the above relations (and those generated by them) are all the additive relations among roots of unity, but there are more relations involving ζ_n for a field K such that the degree of the cyclotomic extension $K(\zeta_n)/K$ is less than $\varphi(n)$. For example, if $K = \mathbb{Q}(\sqrt{5}) \subset \mathbb{C}$ and $\zeta_5 = e^{2\pi i/5}$, then we have the K-linear relation

$$\sqrt{5} \cdot 1 - \zeta_5 + \zeta_5^2 + \zeta_5^3 - \zeta_5^4 = 0\,.$$

We remark that for a number field K there are only finitely many K-linear relations among roots of unity which generate all additive relations: this is because there is a constant c_K such that the intersection of K with $\mathbb{Q}(\zeta_\infty)$ (the largest cyclotomic extension of $\mathbb{Q}$) is contained in $\mathbb{Q}(\zeta_{c_K})$. In general, to understand the cyclotomic entanglement we have to analyze the intersection $K \cap F(\zeta_\infty)$, where F is $\mathbb{Q}$ or a prime field.

There can be further *entangled radicals*, for example the above relation

$$\sqrt{5} = \zeta_5 - \zeta_5^2 - \zeta_5^3 + \zeta_5^4$$

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is a $\mathbb{Q}$-linear relation for the radical $\sqrt{5}$ involving roots of unity. More generally, all square-roots of rational numbers are contained in a cyclotomic extension of $\mathbb{Q}$. To generate the corresponding entanglement we first take the relation

$$\sqrt{2} = \zeta_8 + \zeta_8^7$$

with compatible choices for the roots (after an embedding in $\mathbb{C}$ we can take $\sqrt{2}$ and $e^{\pm 2\pi i/8}$ or $-\sqrt{2}$ and $e^{\pm 2\pi i 3/8}$). Moreover, for any odd prime number p we express $\sqrt{p}$ as a $\mathbb{Q}$-linear combination of $4p$-th roots of unity with a Gauss sum that – with appropriate root choices – can be written as follows:

$$\sqrt{p} = (-1)^{(p-1)/4} \cdot \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) \zeta_p^i \,.$$

An example of additive relation that is explained by multiplicative relations is

$$\sqrt{6} = \zeta_{24}^{17} + \zeta_{24}^{11} - \zeta_{24} - \zeta_{24}^{19/24}$$

(with root choices $\sqrt{6} > 0$ and $\zeta_{24} = e^{2\pi i/24}$ in $\mathbb{C}$) because this is obtained by multiplying the additive relations for $\sqrt{2}$ and $\sqrt{3}$ presented above.

Over a field K different than $\mathbb{Q}$ there could be more entanglement of this type, which we call *Kummer entanglement*, because there can be further radicals that are contained in a cyclotomic extension of K. For example, over $K = \mathbb{Q}(\sqrt{5})$ the square root of $-\frac{5+\sqrt{5}}{8}$ is contained in $\mathbb{Q}(\zeta_5)$. For the Kummer entanglement, the entangled radicals generate abelian radical extensions of K and we can invoke Schinzel's Theorem on abelian radical extensions (Theorem 6): this entanglement is due to Kummer extensions of K that are contained in cyclotomic extensions and hence it is well-understood.

Over $\mathbb{Q}$, a *special entanglement* that is neither cyclotomic nor Kummer is given by the following $\mathbb{Q}$-linear relation (with the appropriate root choices):

$$\sqrt[4]{-4} = 1 + \zeta_4 \,.$$

This entanglement relation is due to the decomposition in (8), which in turn stems from the non-cyclicity of the extension $\mathbb{Q}(\zeta_8)/\mathbb{Q}$.

In fact, the relations that we presented completely describe the entanglement over $\mathbb{Q}$ (this is also a special case of our results below). Rather surprisingly, the entanglement is as limited as possible for any field. In a nutshell, for a general field there are no substantial differences with respect to $\mathbb{Q}$, and there may just be one element (of the form $1 + \zeta_{2^w}$) that plays the role that $1 + \zeta_4$ plays for $\mathbb{Q}$.

In this work we are able to bound the entanglement over any field K because we determine the radicals that are contained in its *Kneser field* (namely the field obtained by adding to K the roots of unity of order 4 or a prime number), over which there is no entanglement by a famous result by Kneser [6]. Our very general results are presented in the next section.

As explained by Lenstra in [8], beyond the theoretical interest, the understanding of entanglement is crucial for a designer of a computer algebra system who wishes to do computations with radicals e.g. over number fields or function fields.

2. The main results

We denote as customary the roots of unity, and we let $\mathrm{char}(K)$ be the characteristic of K. We fix a prime number $\ell \neq \mathrm{char}(K)$ and denote by $\sqrt[\ell^n]{K^\times}$ the subgroup of $\overline{K}^\times$ consisting of the elements whose ℓ^n-th power is in $K^\times$ (and define $\sqrt[\ell^\infty]{K^\times}$ as the union of $\sqrt[\ell^n]{K^\times}$ for $n \geqslant 0$). Similarly, we write ζ_{ℓ^∞} to mean all roots of unity whose order is a power of ℓ. Moreover, we call $K(\zeta_{2\mathcal{P}})$ the extension of K that is generated by the roots of unity whose order is 4 or an odd prime number and it is not divisible by $\mathrm{char}(K)$.

Our results show the remarkable fact that the additive relations of radicals are extremely few. This is because by a famous result by Kneser (Theorem 4) there are no more additive relations over the Kneser field $K(\zeta_{2\mathcal{P}})$.

Theorem 1. *Suppose that ℓ is odd or that $\zeta_4 \in K$. Let $t \geqslant 1$ be the largest integer such that $\zeta_{\ell^t} \in K(\zeta_{2\mathcal{P}})$, or set $t = \infty$ if no such largest integer exists. If $\zeta_\ell \notin K$, then we have*

$$K(\zeta_{2\mathcal{P}}) \cap \sqrt[\ell^\infty]{K^\times} = \langle \zeta_{\ell^t}, K^\times \rangle \qquad \text{and} \qquad K(\zeta_{2\mathcal{P}}) \cap K(\sqrt[\ell^\infty]{K^\times}) = K(\zeta_{\ell^t})\,. \tag{1}$$

If there is some largest integer $w > 0$ such that $\zeta_{\ell^w} \in K$, we have

$$K(\zeta_{2\mathcal{P}}) \cap K(\sqrt[\ell^\infty]{K^\times}) = K(\zeta_{\ell^t}, K(\zeta_{2\mathcal{P}}) \cap \sqrt[\ell^w]{K^\times}) \tag{2}$$

and

$$K(\zeta_{2\mathcal{P}}) \cap \sqrt[\ell^\infty]{K^\times} = \langle \zeta_{\ell^t}, K(\zeta_{2\mathcal{P}}) \cap \sqrt[\ell^w]{K^\times} \rangle\,. \tag{3}$$

If $t \leqslant 2w$, then the field $K(\zeta_{2\mathcal{P}}) \cap K(\sqrt[\ell^\infty]{K^\times}) = K(\zeta_{2\mathcal{P}}) \cap K(\sqrt[\ell^w]{K^\times})$ is the largest subextension of $K(\zeta_{2\mathcal{P}})/K$ that is Kummer and with exponent a power of ℓ.

Notice that the case $\langle \zeta_{\ell^\infty} \rangle \subseteq K$ is not dealt with because setting $w = \infty$ in the above formulas would result in a trivial assertion. We denote by $\sqrt{K^\times}$ the subgroup of $\overline{K}^\times$ consisting of the elements whose square is in $K^\times$, noticing that $K(\sqrt{K^\times})/K$ is a Kummer extension.

Theorem 2. *Suppose that $\ell = 2$ and $\zeta_4 \notin K$. If $\langle \zeta_{2^\infty} \rangle \subseteq K(\zeta_{2\mathcal{P}})$, then we have*

$$K(\zeta_{2\mathcal{P}}) \cap \sqrt[2^\infty]{K^\times} = \langle \zeta_{2^\infty}, K(\zeta_{2\mathcal{P}}) \cap \sqrt{K^\times} \rangle \tag{4}$$

and

$$K(\zeta_{2\mathcal{P}}) \cap K(\sqrt[2^\infty]{K^\times}) = K(\zeta_{2^\infty}, K(\zeta_{2\mathcal{P}}) \cap \sqrt{K^\times})\,. \tag{5}$$

Else, call $w \geqslant 2$ the largest integer such that $\zeta_{2^w} + \zeta_{2^w}^{-1} \in K$ and let $t \geqslant w$ be the largest integer such that $\zeta_{2^t} \in K(\zeta_{2\mathcal{P}})$. Then we have

$$K(\zeta_{2\mathcal{P}}) \cap K(\sqrt[2^\infty]{K^\times}) = K(\zeta_{2^t}, K(\zeta_{2\mathcal{P}}) \cap \sqrt{K^\times}) \tag{6}$$

and

$$K(\zeta_{2\mathcal{P}}) \cap \sqrt[2^\infty]{K^\times} = \langle \zeta_{2^t}, 1 + \zeta_{2^w}, K(\zeta_{2\mathcal{P}}) \cap \sqrt{K^\times} \rangle\,. \tag{7}$$

If $t > w$, we may omit $1 + \zeta_{2^w}$ from the list of generators.

The further results of this paper are described in Section 4. We don't consider general radicals in $\sqrt[\infty]{K}$ but only radicals in $\sqrt[\ell^\infty]{K}$ where ℓ is prime. This is sufficient for understanding the additive relations among radicals because of the following property: for every extension K' of K and for every $\alpha \in \sqrt[\ell^\infty]{K'}$ that is not a root of unity, by Kneser theory (possibly applied over $K'(\zeta_4)$, see Theorem 4) the degree of $K'(\alpha)/K'$ is a power of ℓ, leading to pairwise coprime degrees for different ℓ's.

The proofs of our results rely on two famous theorems: Kneser's theorem on the linear independence of radicals and Schinzel's theorem on abelian radical extensions, see Theorems 4 and 6 respectively.

3. Classical theories of radicals

Kummer theory concerns the field extensions generated by radicals that satisfy the following condition: any finite subextension is Galois and the exponent of its Galois group is the order of a root of unity contained in the base field. We refer to [7, Ch. VI §8] or [3] for an introduction to Kummer theory.

Let K be a field, and $\overline{K}$ an algebraic closure of K. Let Γ be a subgroup of $\bar{K}^\times$ containing $K^\times$ such that $\Gamma/K^\times$ is finite and with order coprime to $\mathrm{char}(K)$ (in particular, the extension $K(\Gamma)/K$ is separable). We then have the following:

Theorem 3 (Kummer theory)**.** *Suppose that the exponent of the group $\Gamma/K^\times$ is the order of a root of unity in K. Then $K(\Gamma)/K$ is a Galois extension and we have*

$$\mathrm{Gal}(K(\Gamma)/K) \simeq \Gamma/K^\times\,.$$

On the other hand, Kneser theory is the theory about the linear independence of radicals that is based on the following result, see [6, Satz].

Theorem 4 (Kneser's theorem on the linear independence of radicals)**.** *Suppose that $\zeta_q \in K$ holds for every odd prime $q \neq \mathrm{char}(K)$ such that $\zeta_q \in \Gamma$. Moreover, if $\mathrm{char}(K) \neq 2$, suppose that $\zeta_4 \in K$ if $1+\zeta_4$ or $1-\zeta_4$ is in Γ. Then we have*

$$[K(\Gamma) : K] = |\Gamma : K^\times|\,.$$

The condition in Kneser's theorem relates to [7, Theorem 9.1, Ch.VI]:

Theorem 5. *Let $a \in K^\times$ and $n > 1$. The polynomial $x^n - a$ is irreducible in $K[x]$ if for all prime numbers $q \mid n$ we have $a \notin K^{\times q}$ and, in case $\mathrm{char}(K) \neq 2$ and $\zeta_4 \notin K$ and $4 \mid n$, we additionally have $a \notin -4K^{\times 4}$.*

The reason for the additional assumption for odd characteristic is the decomposition

$$(x^4+4) = (x^2+2x+2)(x^2-2x+2)\,. \tag{8}$$

Indeed, the roots of this polynomial are the fourth roots of -4: the squareroots of -4 generate the field $\mathbb{Q}(\zeta_4)$ and the fourth roots of -4 also generate that field because they are $\pm(1+\zeta_4)$ and $\pm(1-\zeta_4)$.

We also rely on the following result, see [12, Theorem 2]:

Theorem 6 (Schinzel's theorem on abelian radical extensions)**.** *Let $n \geqslant 1$ be not divisible by $\mathrm{char}(K)$. If $a \in K^\times$, the extension $K(\zeta_n, \sqrt[n]{a})/K$ is abelian if and only if $a^m = b^n$ holds for some $b \in K^\times$ and for some $m \mid n$ such that $\zeta_m \in K$.*

Finally, we apply [4, Satz B] by Halter-Koch, that states the following:

Theorem 7 (Halter-Koch's Theorem B)**.** *Suppose that $[K(\Gamma) : K] = |\Gamma : K^\times|$. If $\mathrm{char}(K) \neq 2$ and $\zeta_4 \notin K$ and 4 divides the order of $\Gamma/K^\times$ suppose moreover that the following condition holds: if $y \in K(\Gamma)$ and $(1+\zeta_4)y \in \Gamma$, then $\zeta_4 \in K(y)$ or $\zeta_4 \in K(y\zeta_4)$. Then every field F such that $K \subseteq F \subseteq K(\Gamma)$ is conjugated over K to a field of the form $K(\Gamma_F)$ where Γ_F is a subgroup of Γ that contains $K^\times$.*

We also mention Schinzel's theorem on the linear independence of radicals [13], that is concerned with the case of maximal degree $[K(\Gamma) : K] = |\Gamma : K^\times|$. Moreover, Halter-Koch proves further results with a focus on the case in which the degree is maximal (see e.g. [5, Satz 5]). There is also a vast literature on radical extensions, see for example [2] by Barrera Mora and Vélez, and the book [1] by Albu. Most importantly, there are results by Rybowicz [11, Theorems 2.3 and 2.4] which also complete Kneser's theorem.

The theory of *entanglement* was established by Lenstra [8] and it was later developed by Palenstijn [9], see also [10] (for number fields) by Perucca, Sgobba and Tronto.

4. Overview of the results

4.1. **Notation.** Let ℓ be a prime number different from $\mathrm{char}(K)$. We suppose that we have

$$\Gamma = \langle K^\times, W_\ell, \Gamma_\ell \rangle$$

where W_ℓ is a finite group generated by roots of unity of odd prime order different from ℓ and where Γ_ℓ is a subgroup of Γ containing $K^\times$ such that $\Gamma_\ell / K^\times$ has order a power of ℓ (this may include roots of unity of order a power of ℓ). For the ℓ-part of the index we have

$$|\Gamma : K^\times|_\ell = |\Gamma_\ell : K^\times| \,.$$

4.2. **Results for ℓ odd.** Suppose that ℓ is odd and different from the characteristic of K.

Theorem 8. *Suppose that $\zeta_\ell \in K$, and set $\Gamma_{\ell,E} := \Gamma_\ell \cap K(W_\ell)$. Then we have*

$$K(\Gamma_\ell) \cap K(W_\ell) = K(\Gamma_{\ell,E})$$

and

$$[K(\Gamma) : K] = \frac{|\Gamma_\ell : K^\times| \cdot [K(W_\ell) : K]}{|\Gamma_{\ell,E} : K^\times|} \,.$$

Supposing additionally that a root of unity ζ of order a power of ℓ is in $K(W_\ell)$ only if it is in K, then $K(\Gamma_{\ell,E})/K$ is a Kummer extension.

Remark 9. Theorem 8 still holds if we replace W_ℓ with $W'_\ell := \langle W_\ell, \zeta_4 \rangle$, the proof is completely analogous.

Theorem 10. *Suppose that $\zeta_\ell \notin K$.*

- *If $\zeta_\ell \notin \Gamma$, then we have*

$$[K(\Gamma) : K] = |\Gamma_\ell : K^\times| \cdot [K(W_\ell) : K] \,.$$

- *If $\zeta_\ell \in \Gamma$, then we have*

$$[K(\Gamma) : K] = \frac{|\Gamma_\ell : K^\times| \cdot [K(W_\ell, \zeta_\ell) : K]}{\ell^\varepsilon \cdot [K(\zeta_{\ell^\tau}) : K(\zeta_\ell)]} \,,$$

  where $\tau \geqslant 1$ is the largest integer such that $\zeta_{\ell^\tau} \in \Gamma \cap K(W_\ell, \zeta_\ell)$, and ε is the largest integer such that $\zeta_{\ell^\varepsilon} \in \Gamma \cap K(\zeta_\ell)$.

4.3. **Results for $\ell = 2$.** Suppose that the characteristic of K is different from 2. We call *special case of Kneser's theorem* the following case: $\zeta_4 \notin K$, and $1+\zeta_4 \in \Gamma$ or $1-\zeta_4 \in \Gamma$.

Theorem 11. *Exclude the special case of Kneser's theorem, and set $\Gamma_{2,E} := \Gamma_2 \cap K(W_2)$. Then we have*
$$K(\Gamma_2) \cap K(W_2) = K(\Gamma_{2,E})$$
and
$$[K(\Gamma) : K] = \frac{|\Gamma_2 : K^\times| \cdot [K(W_2) : K]}{|\Gamma_{2,E} : K^\times|}\,.$$
Supposing additionally that a root of unity ζ of order a power of 2 is in $K(W_2)$ only if it is in K, then $K(\Gamma_{2,E})/K$ is a Kummer extension.

Theorem 12. *Consider the special case of Kneser's theorem. Then we have*
$$[K(\Gamma_2) : K] = 2|\langle \Gamma_2, K(\zeta_4)^\times\rangle : K(\zeta_4)^\times|\,.$$
Moreover, setting $\Gamma_{2,E} := \langle \Gamma_2, K(\zeta_4)^\times\rangle \cap K(W_2, \zeta_4)$, we have
$$K(\Gamma_2) \cap K(W_2, \zeta_4) = K(\Gamma_{2,E})$$
and
$$[K(\Gamma) : K] = \frac{[K(\Gamma_2) : K] \cdot [K(W_2, \zeta_4) : K]}{2 \cdot |\Gamma_{2,E} : K(\zeta_4)^\times|}\,.$$

Also consider the following:

Remark 13. Let K be a field of characteristic zero, ℓ a prime number, and W a finite group of roots of unity of order coprime to ℓ. Suppose that ℓ does not ramify in any finite subextension of $K/\mathbb{Q}$. Then a root of unity ζ of order a power of ℓ is in $K(W)$ only if it is already in K. If K is a number field, this is because $K(\zeta)/K$ is totally ramified at ℓ while $K(W)/K$ is unramified at ℓ. In general, we may reduce to number fields: firstly we may restrict to consider the subfield of K consisting of algebraic elements, secondly we may reduce to a finitely generated field.

5. Proof of the results for the case ℓ odd

Let ℓ be an odd prime number different from $\mathrm{char}(K)$.

Proof of Theorem 8. Let w be the largest integer such that $\zeta_{\ell^w} \in K$, or set $w = \infty$ if $\zeta_{\ell^n} \in K$ holds for every $n \geqslant 1$. To prove that $K(\Gamma_{\ell,E})/K$ is a Kummer extension, it suffices to show that $K(\alpha)/K$ is a Kummer extension for every $\alpha \in \Gamma_{\ell,E}$. Fix $\alpha \in \Gamma_{\ell,E}$, and let n be the smallest non-negative integer such that $\alpha^{\ell^n} \in K$. We have to prove that $n \leqslant w$, so suppose instead that $n > w$. By Theorem 6, as α is contained in an abelian extension of K, we have $\alpha^{\ell^w} \in \langle K^\times, \zeta_{\ell^m}\rangle$ for some minimal non-negative integer $m \leqslant n$, and we must have $m > w$ because $\alpha^{\ell^w} \notin K^\times$. We deduce that $\zeta_{\ell^m} \in \langle K^\times, \alpha^{\ell^w}\rangle$ and hence $\zeta_{\ell^m} \in K(W_\ell)$. The additional assumption implies $\zeta_{\ell^m} \in K$ and hence $m \leqslant w$, contradiction.

Now consider the general case. Since $\zeta_\ell \in K$, by Kneser's theorem we have

(9) $$[K(\Gamma_\ell) : K] = |\Gamma_\ell : K^\times| \quad \text{and} \quad [K(\Gamma_{\ell,E}) : K] = |\Gamma_{\ell,E} : K^\times|\,.$$

So we are left to prove

(10) $$K(\Gamma_\ell) \cap K(W_\ell) = K(\Gamma_{\ell,E})\,,$$

the inclusion $\supseteq$ being clear. Letting $F = K(\Gamma_{\ell,E})$, it suffices to prove that $F(\Gamma_\ell) \cap F(W_\ell) = F$. By (9) the degree of $F(\Gamma_\ell)/F$ is a power of ℓ. Since $F(W_\ell)/F$ is abelian, it suffices to prove that $F(\Gamma_\ell)/F$ has no subextension L/F of degree ℓ contained in $F(W_\ell)$. As $\zeta_\ell \in F$, such an extension would be a Kummer subextension of $F(\Gamma_\ell)/F$ and hence we would have $L = F(\gamma)$ for some $\gamma \in \Gamma_\ell$. Since $\gamma \in L \subseteq K(W_\ell)$, this would contradict $F(\Gamma_{\ell,E}) = F$. $\square$

Lemma 14. *Suppose that $\zeta_\ell \notin K$ and $\zeta_\ell \in \Gamma$. Letting $\tau \geqslant 1$ be the largest integer such that $\zeta_{\ell^\tau} \in \Gamma \cap K(W_\ell, \zeta_\ell)$, we have*

$$K(\Gamma_\ell) \cap K(W_\ell, \zeta_\ell) = K(\zeta_{\ell^\tau})\,. \tag{11}$$

Moreover, let $\varepsilon \geqslant 1$ be the largest integer such that $\zeta_{\ell^\varepsilon} \in \Gamma \cap K(\zeta_\ell)$. We have

$$[K(\Gamma_\ell) : K(\zeta_\ell)] = |\Gamma_\ell : K^\times| \cdot \ell^{-\varepsilon}$$

and

$$|\Gamma_\ell \cap K(\zeta_\ell)^\times : K^\times| = \ell^\varepsilon\,. \tag{12}$$

Proof. We first prove (11). The inclusion $\supseteq$ holds because $\zeta_{\ell^\tau} \in \Gamma_\ell$, so it suffices to consider $F := K(\zeta_{\ell^\tau})$ and prove

$$F(\Gamma_\ell) \cap F(W_\ell) \subseteq F\,.$$

Over F we can apply Kneser's theorem to $\langle \Gamma_\ell, F^\times\rangle$. If $F(\Gamma_\ell) \cap F(W_\ell)$ is larger than F, it contains a subfield L such that L/F has degree ℓ (hence it is a Kummer extension). So we have $L = F(\gamma)$ for some $\gamma \in \Gamma_\ell$. Notice that $\gamma \in F(W_\ell) = K(W_\ell, \zeta_\ell)$. Let $m \geqslant 1$ be minimal such that $\gamma^{\ell^m} \in K^\times$. Since $K(\zeta_{\ell^m}, \gamma)$ is abelian, by Theorem 6 we deduce that $\gamma \in \langle \zeta_{\ell^n}, K^\times\rangle$ holds for some minimal positive integer $n \leqslant m$. So we have

$$\zeta_{\ell^n} \in \langle \gamma, K^\times\rangle \subseteq \Gamma \cap K(W_\ell, \zeta_\ell)$$

and hence $n \leqslant \tau$. We deduce that $\gamma \in F$ and $L = F$, contradiction.

By Kneser's theorem over $K(\zeta_\ell)$ we have

$$[K(\Gamma_\ell) : K(\zeta_\ell)] = |\langle \Gamma_\ell, K(\zeta_\ell)^\times\rangle : K(\zeta_\ell)^\times| = |\Gamma_\ell : \Gamma_\ell \cap K(\zeta_\ell)^\times|\,.$$

So to conclude it suffices to prove (12). Notice that ℓ^ε divides the index in (12) because $\zeta_{\ell^\varepsilon} \in \Gamma_\ell \cap K(\zeta_\ell)^\times$ and $\zeta_\ell \notin K^\times$. It then suffices to prove that for every $\alpha \in \Gamma_\ell \cap K(\zeta_\ell)^\times$ we have $\alpha \in \langle \zeta_{\ell^n}, K^\times\rangle$ for some integer $n \geqslant 0$ (taking n minimal, we have $n \leqslant \varepsilon$ because $\zeta_{\ell^n} \in \langle \alpha, K^\times\rangle$). This is a consequence of Theorem 6 because we have $\alpha^{\ell^m} \in K^\times$ for some $m \geqslant 0$ and α is contained in an abelian extension of K (hence $\alpha^{\ell^m} \in K^{\times \ell^m}$). $\square$

Proof of Theorem 10. Suppose first that $\zeta_\ell \notin \Gamma$. By Kneser's theorem we have $[K(\Gamma_\ell) : K] = |\Gamma_\ell : K^\times|$ hence it suffices to prove that $K(W_\ell) \cap K(\Gamma_\ell) = K$. The extension $K(W_\ell)/K$ is abelian while $K(\Gamma_\ell)/K$ has degree a power of ℓ by Kneser's theorem applied to Γ_ℓ. So it suffices to prove that $K(\Gamma_\ell)$ has no subextension L/K of degree ℓ that is abelian. By Theorem 7 applied to Γ_ℓ the field L is conjugated and thus equal to $K(\gamma)$ for some $\gamma \in \Gamma_\ell \setminus K^\times$. By Kneser's theorem applied to $\langle \gamma, K^\times\rangle$ we deduce that $\gamma^\ell \in K^\times$. Since the extension $K(\zeta_\ell, \gamma)/K$ is abelian, Theorem 6 implies $\gamma^\ell \in K^{\times \ell}$. So we have $\gamma \in \Gamma_\ell \cap \langle \zeta_\ell, K^\times\rangle$, contradicting that $\gamma \notin K^\times$ and $\zeta_\ell \notin \Gamma_\ell$.

Now consider the case $\zeta_\ell \in \Gamma$ (equivalently, $\zeta_\ell \in \Gamma_\ell$). We can apply Theorem 8 to $\Gamma'_\ell := \langle \Gamma_\ell, K(\zeta_\ell)^\times\rangle$ over $K(\zeta_\ell)$, setting $\Gamma'_{\ell,E} := \Gamma'_\ell \cap K(W_\ell, \zeta_\ell)$. We get

$$[K(\Gamma) : K(\zeta_\ell)] = \frac{|\Gamma'_\ell : K(\zeta_\ell)^\times| \cdot [K(W_\ell, \zeta_\ell) : K(\zeta_\ell)]}{|\Gamma'_{\ell,E} : K(\zeta_\ell)^\times|}$$

and hence

$$[K(\Gamma):K]=\frac{|\Gamma_\ell:K^\times|\cdot[K(W_\ell,\zeta_\ell):K]}{|\Gamma'_{\ell,E}:K(\zeta_\ell)^\times|\cdot|\Gamma_\ell\cap K(\zeta_\ell)^\times:K^\times|}\,.$$

Recalling (12) it suffices to show $|\Gamma'_{\ell,E}:K(\zeta_\ell)^\times|=[K(\zeta_{\ell^\tau}):K(\zeta_\ell)]$. By Kneser's theorem over $K(\zeta_\ell)$ we have

$$|\Gamma'_{\ell,E}:K(\zeta_\ell)^\times|=[K(\Gamma'_{\ell,E}):K(\zeta_\ell)]$$

so we may conclude by proving $K(\zeta_{\ell^\tau})=K(\Gamma'_{\ell,E})$. The inclusion $\subseteq$ is because $\zeta_{\ell^\tau}\in\Gamma_\ell\cap K(W_\ell,\zeta_\ell)$. The other inclusion is because $K(\Gamma'_{\ell,E})\subseteq K(\Gamma_\ell)\cap K(W_\ell,\zeta_\ell)$ (as $K(\Gamma'_\ell)=K(\Gamma_\ell)$) and this intersection equals $K(\zeta_{\ell^\tau})$ by (11). $\square$

Proof of Theorem 1 for ℓ odd. The last assertion concerning the special case $t\leqslant 2w$ follows from (3) and (14), considering that $\zeta_{\ell^t}\in\sqrt[\ell^w]{K^\times}$.

To avoid a case distinction, we set $w=0$ if $\zeta_\ell\notin K$. We first prove

$$K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^\infty]{K^\times}\subseteq\langle\zeta_{\ell^{t+w}},\sqrt[\ell^w]{K^\times}\rangle \tag{13}$$

$$K(\zeta_{2\mathcal{P}})\cap K(\sqrt[\ell^\infty]{K^\times})=K(K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^\infty]{K^\times})\,. \tag{14}$$

Let $\alpha\in K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^\infty]{K^\times}$. Since α is contained in an abelian extension of K, by Theorem 6 there is some non-negative integer n such that $\alpha^{\ell^w}\zeta_{\ell^n}\in K^\times$. We deduce that $\zeta_{\ell^n}\in K(\zeta_{2\mathcal{P}})$ and hence $n\leqslant t$, so (13) follows. If $w=0$, (13) implies $K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^\infty]{K^\times}=\langle\zeta_{\ell^t},K^\times\rangle$. The second equality in (1) will then follow from (14).

The inclusions $\supseteq$ in (2), (3) and (14) are immediate, and to prove the other inclusions we may replace $\sqrt[\ell^\infty]{K^\times}$ by a subgroup $\Gamma_\ell\supseteq K^\times$ such that $\Gamma_\ell/K^\times$ is finite. Moreover, we may replace $K(\zeta_{2\mathcal{P}})$ by a subfield $K(W_\ell,\zeta_4,\zeta_\ell)\ni\zeta_{\ell^t}$ where W_ℓ is a group generated by finitely many roots of unity of odd prime order different from ℓ. Set $W'_\ell:=\langle W_\ell,\zeta_4\rangle$. In view of Remark 9, Theorem 8 applied to $\langle\Gamma_\ell,K(\zeta_{\ell^t})^\times\rangle$ over $K(\zeta_{\ell^t})$ gives

$$K(W'_\ell,\zeta_\ell)\cap K(\Gamma_\ell)\subseteq K(\langle\Gamma_\ell,K(\zeta_{\ell^t})^\times\rangle\cap K(W'_\ell,\zeta_\ell))\,.$$

Over $K(\zeta_{\ell^t})^\times$, the elements of $\langle\Gamma_\ell,K(\zeta_{\ell^t})^\times\rangle\cap K(W'_\ell,\zeta_\ell)$ are generated by elements in $\Gamma_\ell\cap K(W'_\ell,\zeta_\ell)\subseteq K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^\infty]{K^\times}$ and we conclude the proof of (14) because $\zeta_{\ell^t}\in K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^\infty]{K^\times}$.

We now prove (3), where we may suppose that t is finite (the case $t=\infty$ being obvious) and hence K has characteristic zero by Remark 15. Notice that the containment $\supseteq$ is clear. From (13) we deduce that

$$K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^\infty]{K^\times}\subseteq\langle\zeta_{\ell^{t+w}},K(\zeta_{\ell^{t+w}},\zeta_{2\mathcal{P}})\cap\sqrt[\ell^w]{K^\times}\rangle\,.$$

Let $\alpha\in K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^\infty]{K^\times}$ and write $\alpha=\zeta^m_{\ell^{t+w}}\beta$ where $\beta\in K(\zeta_{\ell^{t+w}},\zeta_{2\mathcal{P}})\cap\sqrt[\ell^w]{K^\times}$ and $m\geqslant 1$.

By Kummer theory (because $K(\zeta_{2\mathcal{P}},\zeta_{\ell^{t+w}})/K$ is abelian and we investigate a Kummer subextension) we may write $\beta=\beta'\gamma$ so that $\beta'\in K(\zeta_{\ell^{t+w}})\cap\sqrt[\ell^w]{K^\times}$ and $\gamma\in K(\zeta_{2\mathcal{P}})\cap\sqrt[\ell^w]{K^\times}$. We may suppose w.l.o.g. that $\gamma=1$. So we have

$$K(\alpha)\subseteq K(\zeta_{\ell^{t+w}})\cap K(\zeta_{2\mathcal{P}})=K(\zeta_{\ell^t})\,.$$

We have $\beta\in K(\zeta_{\ell^{2w}})$ because $K(\beta)$ is a subextension of $K(\zeta_{\ell^{t+w}})/K$ with exponent dividing ℓ^w. From $K(\alpha)\subseteq K(\zeta_{\ell^t})$ we deduce that $t+w-v_\ell(m)\leqslant\max(t,2w)$. If $t\geqslant 2w$ we may conclude because $\alpha \in \langle \zeta_{\ell^t}, \beta\rangle \cap K(\zeta_{2\mathcal{P}}) = \langle \zeta_{\ell^t}, \langle\beta\rangle \cap K(\zeta_{2\mathcal{P}})\rangle$. Else, we conclude because $\alpha \in \langle \zeta_{\ell^{2w}}, \beta\rangle \cap K(\zeta_{2\mathcal{P}}) \subseteq \sqrt[\ell^w]{K} \cap K(\zeta_{2\mathcal{P}})$.

Notice that (2) can be obtained by combining (14) and (3). □

6. Proof of the results for the case ℓ=2

Now we consider the results for $\ell = 2$.

Proof of Theorem 11. This is the analogue of Theorem 8. Beyond the special case of Kneser's theorem, we may reason as done in the proof of Theorem 8 for the case $\zeta_\ell \in K$. □

Proof of Theorem 1 in case $\ell = 2$. Since $\zeta_4 \in K$, we may proceed as in the case ℓ odd and $\zeta_\ell \in K$, relying on Theorem 11 in place of Theorem 8. □

Proof of Theorem 12. Since we are in the special case of Kneser's theorem we have in particular $\zeta_4 \notin K$ and $\zeta_4 \in K(\Gamma_2)$. By Theorem 11 applied to $\Gamma_2' := \langle \Gamma_2, K(\zeta_4)^\times\rangle$ over $K(\zeta_4)$ we obtain

$$K(\Gamma_2) \cap K(W_2, \zeta_4) = K(\Gamma_{2,E})$$

and

$$[K(\Gamma) : K(\zeta_4)] = \frac{|\Gamma_2' : K(\zeta_4)^\times| \cdot [K(W_2, \zeta_4) : K(\zeta_4)]}{|\Gamma_{2,E} : K(\zeta_4)^\times|}\,.$$

It then suffices to prove

$$[K(\Gamma_2) : K(\zeta_4)] = |\Gamma_2' : K(\zeta_4)^\times|$$

which follows by Kneser's Theorem applied to Γ_2' over $K(\zeta_4)$. □

Remark 15. If p is a prime and $\ell \neq p$ is a prime, then $\mathbb{F}_p(\zeta_{2\mathcal{P}})$ contains $\mathbb{F}_p(\zeta_{\ell^\infty})$. Indeed, the field $\mathbb{F}_p(\zeta_{\ell^\infty})$ is the compositum of $\mathbb{F}_p(\zeta_\ell)$ and of all extensions of $\mathbb{F}_p$ whose degree is a power of ℓ. Moreover, by Zygsmondy's Theorem [14], for every $m \geqslant 3$ there is a prime $q \neq p$ such that the multiplicative order of $(p \bmod q)$ equals m, which implies $[\mathbb{F}_p(\zeta_q) : \mathbb{F}_p] = m$.

Remark 16. With the notation of Theorem 2, let $w \geqslant 2$ and suppose that $\zeta_{2^w} + \zeta_{2^w}^{-1} \in K$. Consider the radical

$$\eta := \zeta_{2^{w+1}}\sqrt{\zeta_{2^w} + \zeta_{2^w}^{-1} + 2} \in \sqrt[2^\infty]{K^\times}\,.$$

We have $\eta^2 = (1 + \zeta_{2^w})^2$ hence $\eta \in \{\pm(1 + \zeta_{2^w})\}$ and $K(\eta) = K(\zeta_4)$. Notice that

$$\zeta_{2^{w+1}}^{-1}\sqrt{\zeta_{2^w} + \zeta_{2^w}^{-1} + 2} \in \langle \eta, \zeta_{2^w}, K^\times\rangle$$

and that, in general, the ratio between a radical and its negative is in $K^\times$.

Example 17. With the notation of Theorem 2, if $K = \mathbb{Q}(\sqrt{6})$, then $t = 3$ and $\zeta_8 \notin K(\zeta_4)$.

Proof of Theorem 2. Let s be the largest element in $\mathbb{Z} \cup \{\infty\}$ such that $\langle \zeta_{2^s}\rangle \subseteq K(\zeta_{2\mathcal{P}})$ (and let $s + 1 = \infty$ if $s = \infty$). We first prove

$$K(\zeta_{2\mathcal{P}}) \cap \sqrt[2^\infty]{K^\times} \subseteq \langle \zeta_{2^{s+1}}, \sqrt{K^\times}\rangle\,. \tag{15}$$

Fix $\alpha \in K(\zeta_{2\mathcal{P}}) \cap \sqrt[2^\infty]{K^\times}$. To investigate α we may replace $\sqrt[2^\infty]{K^\times}$ by a subgroup $\Gamma_2 \supseteq K^\times$ such that Γ_2/K is finite and contains $1 \pm \zeta_4$, and we may replace $K(\zeta_{2\mathcal{P}})$ by an extension of the form $K(\zeta_4, W_2)$ where W_2 is generated by finitely many roots of unity that have odd prime order. Then $\alpha \in \Gamma_2 \cap K(W_2, \zeta_4)$. Since α is contained in an abelian extension of K by Theorem 6 (since $\zeta_4 \notin K$) we have $\alpha^2 \cdot \zeta_{2^n} \in K^\times$ for some minimal $n \geqslant 0$. We deduce that

$\zeta_{2^n} \in \langle \alpha^2, K^\times \rangle \subseteq \Gamma_2 \cap K(W_2, \zeta_4)$ hence $n \leqslant s$. We deduce that $\alpha \in \langle \zeta_{2^{s+1}}, \sqrt{K^\times} \rangle$. We now prove

$$K(\zeta_{2\mathcal{P}}) \cap K(\sqrt[2^\infty]{K^\times}) \subseteq K(\zeta_{2^{s+1}}, \sqrt{K^\times})\,. \tag{16}$$

It suffices to show that, if W_2 and Γ_2 are as above, we have

$$K(W_2, \zeta_4) \cap K(\Gamma_2) \subseteq K(\zeta_{2^{s+1}}, \sqrt{K^\times})\,.$$

By Theorem 12 we have

$$K(W_2, \zeta_4) \cap K(\Gamma_2) = K(\Gamma_{2,E}) \quad \text{where} \quad \Gamma_{2,E} := \langle \Gamma_2, K(\zeta_4)^\times \rangle \cap K(W_2, \zeta_4)\,.$$

We may conclude because the group $\Gamma_{2,E}$ is generated by $K(\zeta_4)^\times \subseteq K(\sqrt{K^\times})$ and by elements in $\Gamma_2 \cap K(W_2, \zeta_4)$ which, as shown above, are in $\langle \zeta_{2^{s+1}}, \sqrt{K^\times} \rangle$.

The assertion for $s = \infty$ is a consequence of (15) and (16). Now suppose that s is finite. By Remark 15 the field K has characteristic zero. Notice that (15) implies

$$K(\zeta_{2\mathcal{P}}) \cap \sqrt[2^\infty]{K^\times} \subseteq \langle \zeta_{2^{s+1}}, \sqrt{K^\times} \cap K(\zeta_{2\mathcal{P}}, \zeta_{2^{s+1}}) \rangle\,. \tag{17}$$

Fix an embedding $\mathbb{Q}(\zeta_{2^\infty}) \hookrightarrow \overline{K}$ and write $K_0 := K \cap \mathbb{Q}(\zeta_{2^\infty})$. Let $\alpha \in K(\zeta_{2\mathcal{P}}) \cap \sqrt[2^\infty]{K^\times}$ and write $\alpha = \zeta_{2^{s+1}}^m \beta$ where $\beta \in \sqrt{K^\times} \cap K(\zeta_{2\mathcal{P}}, \zeta_{2^{s+1}})$ and $m \geqslant 1$.

By Kummer theory (since $K(\zeta_{2\mathcal{P}}, \zeta_{2^{s+1}})/K$ is abelian, a subextension of degree 2 is contained in the compositum of two subextensions of degree at most 2 of $K(\zeta_{2\mathcal{P}})/K$ and $K(\zeta_{2^{s+1}})/K$ respectively) we may write $\beta = \beta'\gamma$ so that $\beta' \in \sqrt{K^\times} \cap K(\zeta_{2^{s+1}})$ and $\gamma \in \sqrt{K^\times} \cap K(\zeta_{2\mathcal{P}})$.

We now prove (7), noticing that the containment $\supseteq$ holds by Remark 16. We may suppose w.l.o.g. that $\gamma = 1$. So we have

$$K(\alpha) \subseteq K(\zeta_{2^{s+1}}) \cap K(\zeta_{2\mathcal{P}}) = K(\zeta_{2^s})\,.$$

If $K(\zeta_{2^s})$ strictly contains $K(\zeta_4)$ or if K_0 is not totally real, then the exponent of $K(\zeta_{2^{s+1}})/K$ is divisible by 4. We deduce that $\beta \in K(\zeta_{2^s})$ because β is contained in a subextension of exponent 2 of $K(\zeta_{2^{s+1}})/K$. From $K(\alpha) \subseteq K(\zeta_{2^s})$ we deduce that m must be even and we may easily conclude.

Now we may suppose that $K(\zeta_{2^s}) = K(\zeta_4)$, that K_0 is totally real, and w.l.o.g. that $\alpha \notin \sqrt{K^\times}$. So we have $K(\alpha) = K(\zeta_4)$ and $s = w$. By Remark 16, the radical $\eta \in \sqrt[2^\infty]{K^\times}$ is such that $K(\eta) = K(\zeta_4)$, and the same holds for η/ζ_{2^s}.

If $1 \pm \zeta_4 \notin \langle \alpha, K^\times \rangle$, then by Kneser's theorem the degree of $K(\alpha)/K$ is 2^n, where $n \geqslant 2$ is minimal such that $\alpha^{2^n} \in K$. This contradicts $\alpha \in K(\zeta_4)$. From this we also deduce that $R \in \langle \eta, K^\times \rangle$ and $R' \in \langle \eta/\zeta_{2^s}, K^\times \rangle$ hold for some $R, R' \in \{1 \pm \zeta_4\}$, where $R \neq R'$ because η and η/ζ_{2^s} are complex conjugates (for any embedding of the involved radicals inside $\mathbb{C}$).

Finally suppose that $R \in \langle \alpha, K^\times \rangle$ for some $R \in \{1 \pm \zeta_4\}$. If $\alpha \in \langle R, K^\times \rangle$, we may conclude because $\alpha \in \langle \zeta_{2^s}, \eta, K^\times \rangle$. Else, up to replacing α by an odd power of it, or replacing α by its reciprocal, we can write $\alpha^{2^d} = Rk_0$ for some $k_0 \in K^\times$ and for some $d \geqslant 1$. Writing $R = \zeta_8^{\pm 1}\sqrt{2}$ we get $\alpha = \zeta_{2^{3+d}}^x \sqrt[2^d]{\sqrt{2}k_0}$ for some odd integer x. Since $\sqrt[2^d]{\sqrt{2}k_0}$ is contained in an abelian extension of K, by Theorem 6 we have $2k_0^2 \in K^{\times 2^d}$ and hence $\sqrt{2} \in K$. Then we have $\alpha = \zeta_{2^{3+d}}^y \sqrt{k_1}$ for some $k_1 \in K^\times$ and for some odd integer y. If $3 + d \leqslant s$ we deduce that $\alpha \in \langle \zeta_{2^s}, K(\zeta_{2\mathcal{P}}) \cap \sqrt{K^\times} \rangle$ and we conclude. Moreover, we cannot have $3 + d \geqslant s + 2$ because $K(\alpha, \sqrt{k_1})/K$ has exponent 2 while $K(\zeta_{2^{s+2}})/K$ has exponent at least 4. Finally suppose that $3 + d = s + 1$. The conditions $K(\alpha) = K(\zeta_4)$ and $\zeta_{2^{s+1}} \in K(\alpha, \sqrt{k_1})$ imply that

$K(\sqrt{k_1})$ is either $K(\zeta_{2^{s+1}}+\zeta_{2^{s+1}}^{-1})$ or $K(\zeta_4(\zeta_{2^{s+1}}+\zeta_{2^{s+1}}^{-1}))$. Remarking that $(\zeta_{2^{s+1}}+\zeta_{2^{s+1}}^{-1})^2 = \zeta_{2^s}+\zeta_{2^s}^{-1}+2$, we conclude because $\alpha \in \langle \zeta_{2^s}, \eta, K^\times\rangle$.

To show (6), consider the proof of (16) and observe that by (7) we know that $\Gamma_2 \cap K(W_2, \zeta_4)$ is contained in $K(\zeta_{2^s}, K(\zeta_{2\mathcal{P}}) \cap \sqrt{K^\times})$. □

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