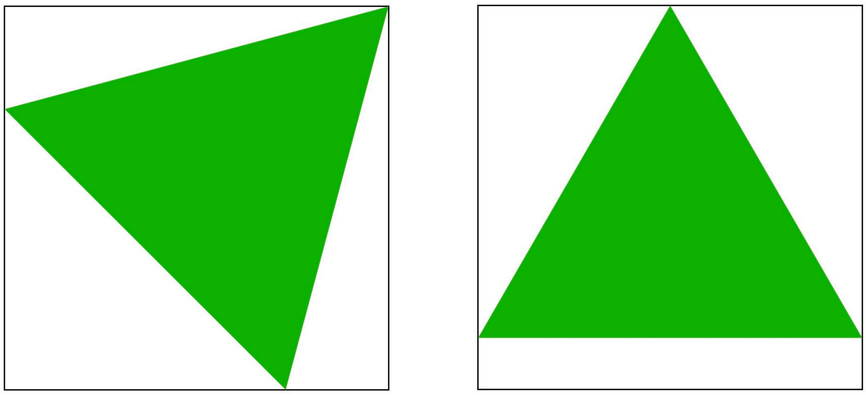
FUNNY GEOMETRICAL FACT: EQUILATERAL TRIANGLES INSCRIBED IN SQUARES

The largest equilateral triangle that is inscribed in a square has area slightly less than 50% of the area of the square. The difference of areas between the largest and smallest inscribed equilateral triangle is only roughly 3% of the square area.



The *smallest* equilateral triangle that is inscribed in a square has a side that is parallel to a square side. The side length for the triangle and for the square are the same.
We thus compare the areas of an equilateral triangle and of a square having the same side length. Suppose w.l.o.g. that the square length is 1. The triangle area is then $\sqrt{3}/4$ and the square area is 1. So the ratio is roughly $43\,\%$.

The *largest* equilateral triangle inscribed in a square is as follows: it has a vertex in common with the square, and the height at this vertex is on the square diagonal.
At that vertex, there are three angles that add up to a right triangle: $15^\circ + 60^\circ + 15^\circ = 90^\circ$.
We make use of the cosine value

$$\cos(15) = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Suppose w.l.o.g. that the square length is 1. The area of the triangle is then (relying on the above calculation up to a rescaling):

$$\left(\frac{1}{\cos(15)}\right)^2 \cdot \frac{\sqrt{3}}{4} = 2\sqrt{3} - 3$$

So the area of the triangle is roughly $46\,\%$ of the area of the square.

Challenges for the reader:

- Prove that the maximal (respectively, minimal) equilateral triangle inscribed in a square is as described.
- Determine precisely the one-parameter family of equilateral triangles inscribed in a square (visually, rotate and rescale the largest inscribed equilateral triangle so that the three vertices are still on the square sides; the area shrinks and eventually we obtain the smallest inscribed equilateral triangle).