In the name of ALLAH

FEniCS 2024

Using random circular models to simulate stochastic anisotropic flow in aquifer systems with FEniCSx

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June 13, 2024

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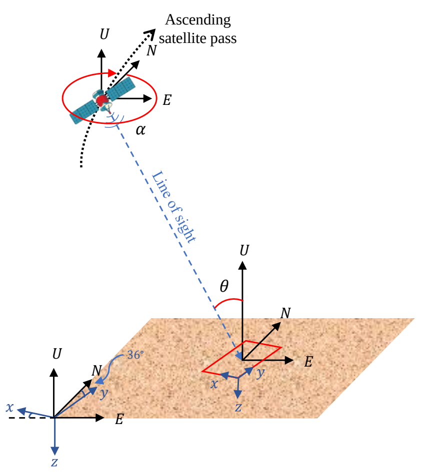
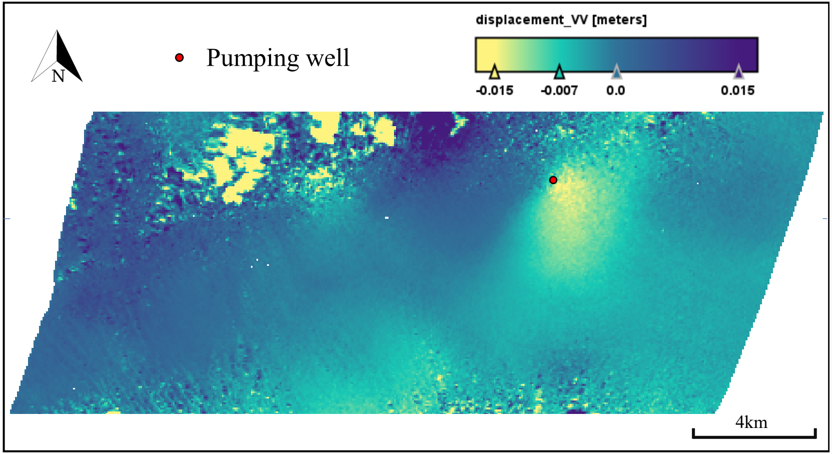
Objective

- The water flow in many aquifers is driven by strong anisotropy created by preferential flow features such as fractures and faults.
- **Overall goal**: assimilate InSAR surface displacement into an aquifer model to estimate aquifer properties.
- **In this work**: develop a flexible stochastic prior model of the anisotropic hydraulic conductivity (AHC) tensor that respects its underlying symmetry and positive definiteness.



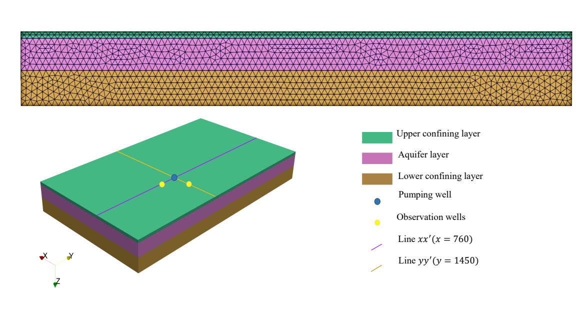
from (Salehian Ghamsari, Van Dam, & Hale, 2024)

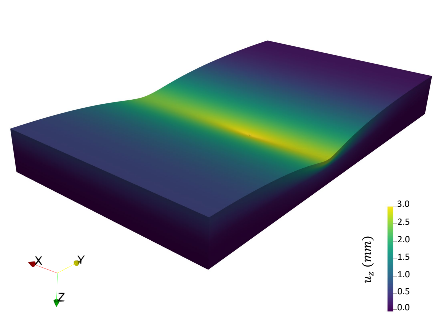
InSAR



InSAR-observed deformation of Nevada aquifer pumping test from (Salehian Ghamsari et al., 2024)

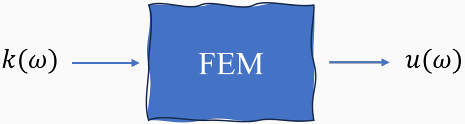
Poroelastic finite element model





from (Salehian Ghamsari et al., 2024)

Methodology



- *FEM* is the poroelastic finite element model,
- $u(\omega)$ (deformation, pressure, and flux) is a stochastic response due to the randomness in $k(\omega)$ (AHC):

$$k(\omega) = \begin{bmatrix} k_{xx}(\omega) & k_{xy}(\omega) \\ k_{yx}(\omega) & k_{yy}(\omega) \end{bmatrix}.$$

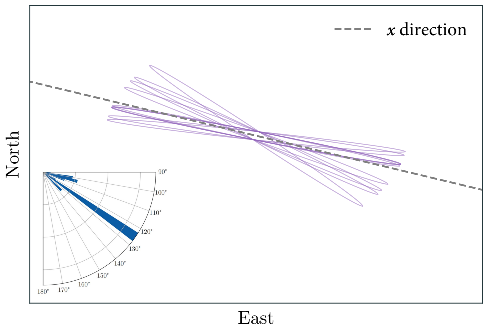
Methodology

- We use random physical symmetry and positive definiteness (SPD) matrix because of the nature of hydraulic conductivity tensor (Shivanand, Rosić, & Matthies, 2022)
- We apply spectral decomposition, enabling separation of size/strength encoded in eigenvalues and orientation encoded in eigenvectors
- We ensure positive definiteness using an exponential map

$$k = \exp(H) = Q\Lambda Q^T.$$

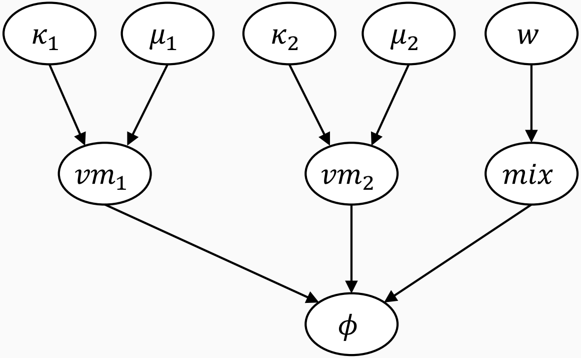
Random orientation with fixed scaling

1. $k_r(\omega_r) = R(\omega_r)\widehat{k}R(\omega_r)^T$,
   where index "r" signifies that only the eigenvectors are random.
2. In order to arrive at a vector space setting:
   $$R := \exp(W)$$
   $$W = \begin{bmatrix} 0 & -\phi \\ \phi & 0 \end{bmatrix}$$
   .
3. ϕ is rotation angle $\rightarrow$ circular random variable



Random rotation angle: Mixture of 2 von Mises model

(Lark, Clifford, & Waters, 2014) applied the mixture von Mises distribution to datasets consisting of observations of the dip direction of bedding planes.



$\kappa_1, \kappa_2 \sim Gamma$

$\mu_1, \mu_2 \sim VonMises$

$vm_1, vm_2 \sim VonMises$

$w \sim Uniform$

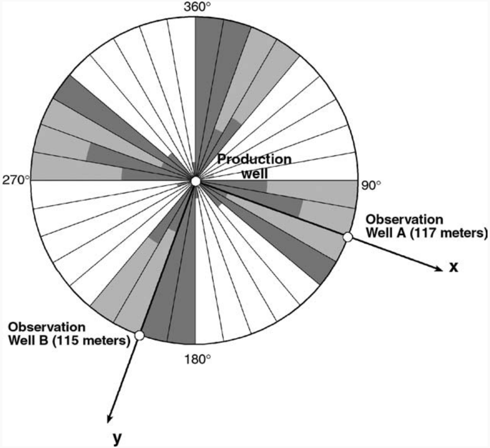
$mix \sim Categorical$

$\phi \sim Mixture$

Rose diagram

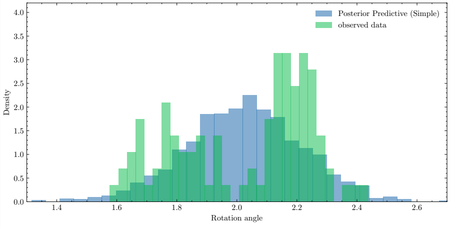
We use the NUTS algorithm in the MCMC method to estimate the posterior using the generated angles from the rose diagram as likelihood.

From (Heilweil & Hsieh, 2006)

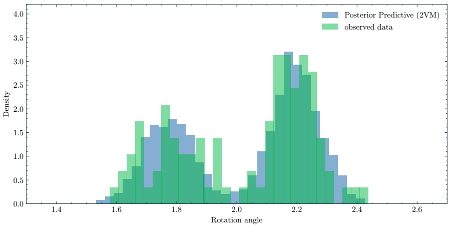


Random rotation angle: Model selection

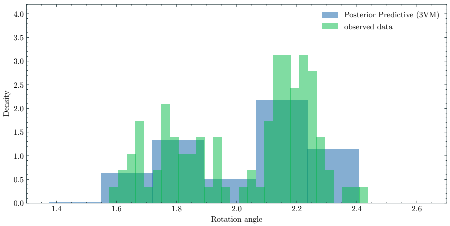
model	rank	$elpd_{loo}$	p_{loo}
2vm	0	29.581509	4.947668
3vm	1	29.473026	6.092196
simple	2	3.591472	1.886592



Simple von Mises model (simple)



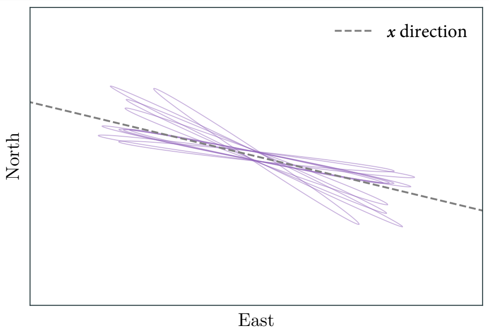
Mixture of 2 von Mises model (2vm)



Mixture of 3 von Mises model (3vm)

Random orientation and scaling

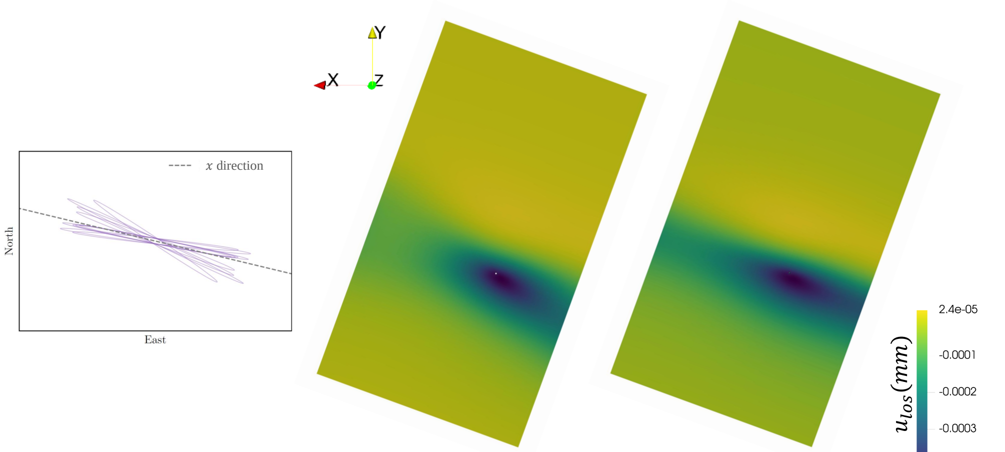
1. $k_s(\omega) = \widehat{Q}\Lambda_s(\omega)\widehat{Q}^T$ where index "s" shows that only random anisotropic scaling was used.
2. $R(\omega_r)$, which is a random orientation
3. We consider these two approaches (fixed orientation and fixed scaling) independent, so we can combine them.
4. $k_{rs}(\omega_{rs}) = R(\omega_r)k_s(\omega_s)R(\omega_r)^T$, where index "rs" denotes a combined rotational-scaling uncertainty.



Implementation

Random Hydraulic Conductivity Tensor ($k_{rs}(\omega_{rs})$) → Running Parallel Poroelastic Finite Element Model in HPC → Outputs (line of sight deformation) → Statistical Analysis (mean, standard deviation, ...)

Surface displacement



a) Elliptical representation of random AHC tensor

b) Mean of surface displacement of random AHC

c) Surface displacement of fixed AHC in x direction

Take-home messages

Aim:

- understanding aquifer system and estimating aquifer properties
- using InSAR technique instead of digging wells to make the process easier and cheaper

Contribution:

✓ we built a poroelastic finite element model to simulate an aquifer system with anisotropic hydraulic conductivity (AHC)

☞ we developed a flexible stochastic model of the AHC tensor

Next step:

✈ we will solve an inverse problem using InSAR data to estimate AHC

Acknowledgement

I would like to thank **Damian Ndiwago** from the University of Luxembourg for his helping for model selection.

This work was funded in whole, or in part, by the Luxembourg National Research Fund (FNR), grant reference PRIDE/17/12252781.

☞ If you are interested in my research it would be great to chat with you further at this email address: sona.salehianghamsari@uni.lu

Any Question?

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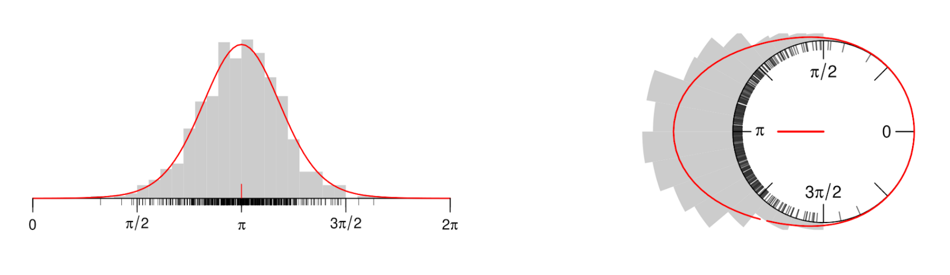
The von-Mises disrtibution

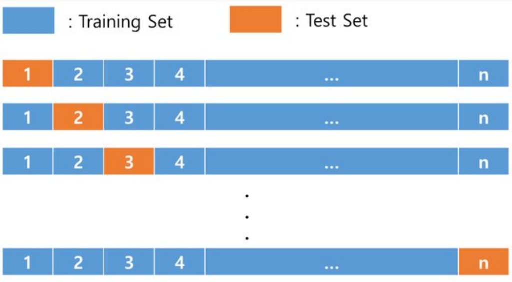


From (Lang et al., 2020)

$$f(\phi(\omega_r)|\mu,\kappa) = \frac{\exp(\kappa\cos(\phi(\omega_r)-\mu))}{2\pi I_0(\kappa)}$$

where I_0 is the modified Bessel function of order 0.

Leave-one-out cross-validation



From (Cha et al., 2020)

- $ELPD_{loo}$: expected log pointwise predictive density.
  Higher ELPD indicates higher out-of-sample predictive fit ("better" model).
- P_{loo}: Estimated effective number of parameters.