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DEFORMATION THEORY OF PROPERADS AND GRAPH COMPLEXES

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Abstract

In this thesis we study the deformation theory of the wheeled closure of the properad of strongly homotopy Lie bialgebras, and of its variations. Our main achievement is a computation of the cohomology groups of the derivation complexes of the aforementioned properads in terms of various Kontsevich graph complexes. This work can be seen as an extension of the results by S. Merkulov and T. Willwacher in the paper *Deformation theory of the Lie bialgebra properad* [MW1], where it has been proven that the deformation theory of the ordinary properad of homotopy Lie bialgebras $\mathcal{H}olieb$ is controlled by the Grothendieck-Teichmuller Lie algebra $\mathfrak{grt}$. We prove that in the wheeled case the deformation theory is much richer: the deformation theory of the wheeled closure of $\mathcal{H}olieb$ is controlled by two copies of $\mathfrak{grt}$. We illustrate this surprising result by showing explicitly two homotopy inequivalent actions of the famous tetrahedron class on the wheeled properad of homotopy Lie bialgebras. We also compute the cohomology groups of the derivation complexes of homotopy quasi- and pseudo-Lie bialgebras and of their wheeled closures.

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Till Morfar för att du visade mig musik.

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Contents

Chapter 1

Introduction

1.1 Background

Operads are a fairly modern addition to mathematics, first appearing in algebraic topology during the 1960s, and the study of them has since evolved into a theory in its own right, with multiple applications in algebra, in algebraic topology, in geometry and in mathematical physics.

To its core, the theory of operads is a theory of the compositions of operations, where an operad can be viewed as a space of operations together with a multiplication that composes these operations into new ones. The three classical objects in the center are associative algebras, Lie algebras and commutative algebras, but many other algebras are also studied, such as Poisson algebras, Jordan algebras, pre-Lie algebras and so on. The structure of an operad is similar to that of associative algebras, hence, many of the techniques and results about associative algebras can be transferred to operads.

Operads have also been useful in homotopy theory. Given a chain complex with an algebraic structure (associative algebra, Lie algebra, etc.), one asks if it is possible to transfer the same structure to the cohomology of this chain complex, without losing too much information about the original structure. Although the answer is no in general, for certain algebras it is possible to define another structure which transfers the desired algebraic structure to the cohomology groups. For associative algebras, such structures are called *homotopy associative algebras*, and are characterised by being dg algebras with a product that is associative up to homotopy. Similarly, the equivalents for Lie algebras are called *homotopy Lie algebras*.

While operads can be used to study operations with several inputs and one output (products), and to some degree co-operations (i.e., operations with one input and several outputs), they are not suitable to encode algebraic structures with both a product and a coproduct (or generally structural operations with several inputs and outputs). Examples of such structures have gathered more interest lately in the study of quantum groups, involving structures such as bialgebras, Lie bialgebras and Hopf algebras. The more general structure governing these are called properads.

Both operads and properads capture the idea of composing multilinear maps together in a linear fashion, composing one map after the other. Wheeled (pr)operads were introduced and studied in [M2, MMS], allowing the composition to be ”wheeled”, connecting the output of an operation to its input. Though seeming like a counterintuitive concept at first, these wheeled pr(operads) correspond to adding trace maps to the corresponding algebras for finite dimensional representations of pr(operads).

The main algebraic structures we study in this thesis are Lie bialgebras and homotopy Lie bialgebras. A Lie bialgebra is a vector space V together with a Lie bracket $[-,-] : V \otimes V \to V$ and a co-bracket $\Delta : V \to V \otimes V$ that satisfy a certain compability relation. Lie bialgebras were first introduced by V. Drinfeld [D1] in the context of Yang-Baxter equations and quantum group theory. Since then they have found many applications in many areas of pure mathematics and mathematical physics, such as the theory of Hopf algebra deformations of universal enveloping algebras [ES], quantization theory [EKa], string topology and symplectic field theory [CFL], Goldman-Turaev theory of free loops in Riemann surfaces with boundaries [G, Tu], Lagrangian Floer theory of higher genus [Tu], and the computation of cohomology groups of moduli spaces of algebraic curves with labeled punctures [MW2, AWZ]. The properad of Lie bialgebras is denoted by $\mathcal{L}ieb$ and the properad of homotopy Lie bialgebras by $\mathcal{H}olieb$. We will study the more general case of Lie bialgebras with a bracket of degree c and a cobracket of degree d. We denote the corresponding properad and homotopy properad by $\mathcal{L}ieb_{c,d}$ and $\mathcal{H}olieb_{c,d}$ respectively.

A derivation of an associative dg algebra A is a linear map $d : A \to A$ compatible with the differential such that $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$ and can be viewed as an infinitesimal deformation of A. The derivation complex $Der(A)$ is a dg Lie algebra where the bracket is given by the commutator. These notions can be generalized to (pr)operads. S. Merkulov and T. Willwacher previously studied the derivation complex $Der(\mathcal{H}olieb_{c,d})$ in [MW1] and related its cohomology to the oriented Kontsevich graph complex. We extend the study by considering the derivation complexes of the wheeled closure $\mathcal{H}olieb_{c,d}^{\circlearrowright}$ of $\mathcal{H}olieb_{c,d}$. These complexes are directly related to the deformation complexes of these properads.

The properad $\mathcal{H}olieb_{c,d}$ is generated by corollas with n inputs and m outputs such that m and n are greater than or equal to one and $m + n$ is greater than or equal to three. That is, it does not contain any "curvature" like generators where either m or n are equal to zero (see [M2] for full details). However, in the wheeled case, one can build from such generating elements of $\mathcal{H}olieb_{c,d}^{\circlearrowright}$ with no inputs ($n = 0$) or no outputs ($m = 0$), or both ($m = n = 0$). Such graphs can stay for non-trivial cohomology classes which control deformations of $\mathcal{H}olieb_{c,d}^{\circlearrowright}$ into its more general version $\mathcal{H}olieb^{\circlearrowright\bullet}$ in which curvature terms are allowed (such a phenomenon is impossible in the case of props that are not wheeled). It is therefore natural to consider two different deformation complexes of $\mathcal{H}olieb_{c,d}$, the one which *curved* operations with no inputs and outputs can be created, and another one where this is not the case. We study both cases for completeness, and hence consider the two derivation complexes $Der^{\bullet}(\mathcal{H}olieb_{c,d}^{\circlearrowright})$ and $Der(\mathcal{H}olieb_{c,d}^{\circlearrowright})$, the former controlling *all* possible deformations (i.e. with curvature terms allowed) of $\mathcal{H}olieb^{\circlearrowright}$ and the latter is the reduced version. In this thesis we will study the cohomology of the aforementioned derivation complexes and describe them in terms of Kontsevich graph complexes.

In the article [K2], M. Kontsevich introduced the *Kontsevich graph complexes* GC_k as a means to study the formality conjecture in the deformation quantization theory of Poisson structures, but have lately found use in the theory of (pr)operads [W1]. They consist of a family of dg Lie algebras parameterized by an integer k where each complex GC_k is generated by undirected graphs modulo the symmetries of vertices (when k is odd) or symmetries of edges (when k is even). Complexes where k is of the same parity are isomorphic up to degree shifts, and so it is enough to only focus on GC_2 and GC_3.

One important fact showed by T. Willwacher in [W1] is that there is an isomorphism of Lie algebras $H^0(\mathsf{GC}_2) \cong \mathfrak{grt}$, where $\mathfrak{grt}$ is the Grothendieck-Teichmuller Lie algebra. This is the Lie algebra of the pro-unipotent Grothendieck-Teichmuller group GRT, introduced by Drinfeld [D3].

Other versions of Kontsevich graph complexes have since been introduced and shown to be related to the original graph complexes GC_k. The *directed graph complex* dGC_k consisting of directed graphs modulo the same symmetries as the Kontsevich graph complex is quasi isomorphic to GC_k [W1], while the subcomplex $\mathsf{oGC}_k \subseteq \mathsf{dGC}_k$ of oriented graphs (graphs with no closed path of directed edges) is quasi-isomorphic to GC_{k-1} [W2]. Two other related subcomplexes are $\mathsf{dGC}_k^s, \mathsf{dGC}_k^t \subseteq \mathsf{dGC}_k$ of sourced and targeted graphs respectively (graphs containing at least one vertex with only outgoing edges and incoming edges respectively). They are isomorphic to each other by changing the direction of all edges and furthermore they are quasi-isomorphic to oGC_k [Z2]. The complex dGC_k^{s+t} of graphs with either a source vertex or a target vertex will also be of interest to us.

In chapter 3 we compute the cohomologies of $Der^\bullet(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$ and $Der(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$ by finding explicit quasi-isomorphisms to classical Kontsevich graph complexes. Let $\mathsf{dGC}_k^{=2,s+t} \subseteq \mathsf{dGC}_k$ be the subcomplex of graphs with at least one bivalent vertex or at least one source or target vertex. Then set $\mathsf{dGC}_k^{\geq 3,\circlearrowleft} := \mathsf{dGC}_k/\mathsf{dGC}_k^{=2,s+t}$ to be the quotient complex of graphs with neither sources nor targets, and all vertices at least trivalent.

Theorem 1.1.1 (Theorem 3.6.3)**.** *There is a quasi-isomorphism*

$$\mathbb{K} \oplus \mathsf{dGC}_{c+d+1}^{\geq 3,\circlearrowleft} \oplus \mathsf{dGC}_{c+d+1} \to Der^\bullet(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$$

where $\mathbb{K}$ *is a trivial complex corresponding to a rescaling class.*

The summand $\mathsf{dGC}_{c+d+1}^{\geq 3,\circlearrowleft}$ controls cohomology classes with neither inputs nor outputs. The presence of the latter complex is obvious, thus, the main claim above says that the cohomology classes with at least one input or one output are controlled by the directed graph complex dGC_{c+d+1}.

Let $\mathsf{dGC}_k^{no\ s} := \mathsf{dGC}_k/\mathsf{dGC}_k^s$ and $\mathsf{dGC}_k^{no\ t} := \mathsf{dGC}_k/\mathsf{dGC}_k^t$ be the quotient complexes of graphs with no sources and no targets respectively. There is a natural map $P : \mathsf{dGC}_k \to \mathsf{dGC}_k^{no\ s} \oplus \mathsf{dGC}_k^{no\ t}$ given by mapping a graph to the diagonal of the quotient maps.

Theorem 1.1.2 (Theorem 3.6.4)**.** *There is a quasi-isomorphism*

$$\mathbb{K} \oplus \mathrm{Cone}(P)[1] \to Der(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$$

where $\mathbb{K}$ *is a trivial complex corresponding to a rescaling class and* $\mathrm{Cone}(P)[1]$ *is the desuspended cone complex of* P*.*

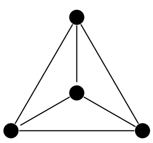
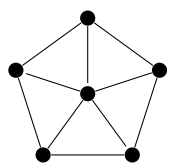
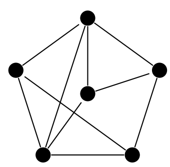
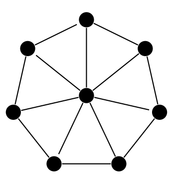
This theorem together with the observation that $H^0(\mathrm{Cone}(P)[1]) \cong \mathbb{K} \oplus H^0(\mathsf{dGC}_2^s \oplus \mathsf{dGC}_2^t)$ when $c = d = 1$ gives the following Corollary.

Corollary 1.1.3 (Corollary 3.6.5)**.** *There is an isomorphism of vector spaces*

$$H^0(Der(\mathcal{H}olieb_{1,1}^{\circlearrowleft})) \cong \mathbb{K} \oplus \mathfrak{grt} \oplus \mathfrak{grt}$$

where $\mathbb{K}$ *is a trivial complex corresponding to a rescaling class.*

The cohomollgy group $H^0(\mathsf{GC}_2)$ is conjectured to be a free Lie group generated by the so called wheel cohomology classes $\{\omega_{2n+1}\}_{n\geq 1}$, the first ones being

$\omega_3 =$  , $\omega_5 =$  $+ \frac{5}{2}$  , $\omega_7 =$  $+ \cdots$

We explicitly compute the two distinct cohomology classes in $Der(\mathcal{H}olieb_{1,1}^{\circlearrowleft})$ corresponding to the tetrahedron graph ω_3 (see section 3.7).

In the second article we extend the study to derivation complexes of the properads of quasi- and pseudo-Lie bialgebras.

A *quasi-Lie bialgebra* is a generalization of a Lie bialgebra and is a vector space V together with a Lie bracket $[-,-] : V \otimes V \to V$, an antisymmetric cobracket $\Delta : V \to V \otimes V$ which satisfy the co-Jacobi identity up to a natural relation with a skew-symmetric element $\phi : \mathbb{K} \to V \otimes V \otimes V$. Lie bialgebras are naturally quasi-Lie bialgebras where $\phi = 0$. Quasi-Lie bialgebras were first introduced by V. Drinfeld [D2] and have later found use in the study of quasi-surfaces [Tu], twisting operads [M3] and the theory of cohomology groups of the moduli spaces $\mathcal{M}_{g,n}$ of genus g algebraic curves with n punctures [M4]. The associated properad with a bracket of degree c and a cobracket of degree d is denoted by $\mathcal{QL}ieb_{c,d}$ and its homotopy properad, found by J. Granåker [G], is denoted by $\mathcal{QH}olieb_{c,d}$.

Theorem 1.1.4 (Theorem 4.5.4)**.** *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{oGC}_{c+d+1} \to Der(\mathcal{QH}olieb_{c,d})$$

where $\mathbb{K}$ *is a trivial complex corresponding to a rescaling class.*

Consider the subcomplex $\mathsf{dGC}_k^{=2,t} \subseteq \mathsf{dGC}_k$ of graphs with at least one bivalent vertex or at least one target vertex. Let $\mathsf{dGC}_k^{\geq 3,\ no\ t} := \mathsf{dGC}_k/\mathsf{dGC}_k^{=2,t}$ be the quotient complex of graphs whose vertices are at least trivalent and with no target vertices.

Theorem 1.1.5 (Theorem 4.5.5)**.** *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{dGC}_{c+d+1}^{\geq 3, no\ t} \oplus \mathsf{dGC}_{c+d+1} \to Der^{\bullet}(\mathcal{QH}olieb_{c,d}^{\circlearrowleft})$$

where $\mathbb{K}$ *is a trivial complex corresponding to a rescaling class.*

The complex $\mathsf{dGC}_{c+d+1}^{\geq 3, no\ t}$ corresponds to the trivial case of graphs with neither outputs nor inputs. The main claim above says that the cohomology classes with at least one output or input are controlled by the directed graph complex dGC_{c+d+1}.

Theorem 1.1.6 (Theorem 4.5.6)**.** *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{dGC}_{c+d+1}^{t} \to Der(\mathcal{QH}olieb_{c,d}^{\circlearrowleft}).$$

where $\mathbb{K}$ *is a trivial complex corresponding to a rescaling class.*

Pseudo-Lie bialgebras further extend quasi-Lie bialgebras in that they have an antisymmetric product and coproduct satisfying the Jacobi-relation up to a natural relation with a map $\eta : V \otimes V \otimes V \to \mathbb{K}$ and some other compability relations. Pseudo-Lie bialgebras has as of yet not found any applications, and they are studied in this paper for the sake of completion. We denote the properad of pseudo-Lie bialgebras with a bracket of degree c and a cobracket of degree d by $\mathcal{PL}ieb_{c,d}$ and its homotopy properad, studied by J. Granåker in [G], is denoted by $\mathcal{PH}olieb_{c,d}$.

Consider the subcomplex $\mathsf{oGC}_k^{=2} \subseteq \mathsf{oGC}_k$ of oriented graphs with at least one bivalent vertex. Let $\mathsf{oGC}_k^{\geq 3} := \mathsf{oGC}_k/\mathsf{oGC}_k^{=2}$ be the quotient complex of oriented graphs whose vertices are at least trivalent.

Theorem 1.1.7 (Theorem 4.5.7)**.** *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{oGC}_{c+d+1}^{\geq 3} \oplus \mathsf{oGC}_{c+d+1} \to Der(\mathcal{PH}olieb_{c,d})$$

where $\mathbb{K}$ *is a trivial complex corresponding to a rescaling class.*

The summand $\mathsf{oGC}^{\geq 3}_{c+d+1}$ corresponds to the trivial case of graphs with neither outputs nor inputs. The main claim above says that the cohomology classes with at least one output or input are controlled by the graph complex oGC_{c+d+1}.

For the wheeled closure of the properad $\mathcal{PH}olieb_{c,d}$, the two derivation complexes align ($Der^\bullet(\mathcal{PH}olieb_{c,d}) = Der(\mathcal{PH}olieb_{c,d})$) (see the introduction of article 2).

Theorem 1.1.8 (Theorem 4.5.8)**.** *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{dGC}^{\geq 3}_{c+d+1} \oplus \mathsf{dGC}_{c+d+1} \to Der^\bullet(\mathcal{PH}olieb^{\circlearrowright}_{c,d})$$

where $\mathbb{K}$ is a trivial complex corresponding to a rescaling class.

The complex $\mathsf{dGC}^{\geq 3}_{c+d+1}$ corresponds to the trivial case of graphs with neither outputs nor inputs. The main claim above says that the cohomology classes with at least one output or input are controlled by the graph complex dGC_{c+d+1}.

Remark 1.1.9. Even though both the derivation complexes and the graph complexes have dg Lie algebra structures, we do emphasize that the quasi-isomorphisms we show in this thesis are only on the level of chain complexes. The question regarding whether the Lie bracket is preserved under these maps or not is still open.

1.2 Structure of the manuscript

Chapter 2 serves as a background to the upcoming chapters. There we briefly define graphs and graphs with labeled hairs, acting as a foundation to define the notion of $\mathfrak{G}$-algebras. This concept generalizes and collects the notion of props, properads, operads, associative algebras and their wheeled correspondents with a single definition, and serves as a natural way to explain the relation between their wheeled and unwheeled versions. We proceed to give notable examples of operads and properads. Further, we give a brief reminder of derivation and deformation complexes of properads, and end the chapter by summarizing the most important facts about graph complexes.

Chapter 3 is based on the article *Deformation theory of the wheeled properad of strongly homotopy Lie bialgebras and graph complexes*, where we extend the result of S. Merkulov and T. Willwacher in [MW1]. In this paper they studied the deformation complex of the homotopy Lie bialgebra $\mathcal{H}olieb$ and related its cohomology to the oriented graph complex oGC. In our investigation, we study the deformation complex of the wheeled homotopy Lie bialgebra $\mathcal{H}olieb^{\circlearrowright}$ and describe its cohomology in terms of well known graph complexes. In particular we reach the result that the zeroth cohomology of the deformation complex is isomorphic to two copies of the Grothendieck-Teichmuller Lie algebra $\mathfrak{grt}$ (as a vector space).

Chapter 4 is based on the article *Graph complexes and Deformation theories of the (wheeled) properads of quasi- and pseudo-Lie bialgebras*. There we further extend the investigation of the first article by considering the deformation complex of the quasi-Lie bialgebra and pseudo-Lie bialgebra properads, and similarly relate their cohomology to well known graph complexes.

1.3 Notation

Let $\mathbb{S}_n$ denote the permutation group of the set $\{1, 2, ..., n\}$. The one dimensional sign representation of $\mathbb{S}_n$ is denoted by sgn_n. All vector spaces are assumed to be $\mathbb{Z}$ graded over a field $\mathbb{K}$ of characteristic zero. If $V = \bigoplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ denotes the graded vector space where $V[k]^i = V^{i+k}$. For a properad $\mathcal{P}$ we denote by

$\mathcal{P}\{k\}$ the properad being uniquely defined by the property: For any graded vector space V, a representation of $\mathcal{P}\{k\}$ in V is identical to a representation of $\mathcal{P}$ in $V[k]$.
Let Γ be a graph with v vertices and e edges. The *genus* of a graph is the number $e-v+1$. Further, for vertices in directed graphs, we consider the following conventions: A vertex is *passing* if it has exactly one incoming edge and one outgoing edge attached. It is a *source* if only outgoing edges are attached to it, and a *target* if there are only incoming edges attached. Finally, a vertex is called *generic* if it is at least trivalent, and has at least one outgoing edge and one incoming edge attached. The *loop number* of a graph Γ is the number $b = e - v + 1$, where e is the number of edges and v the number of vertices of Γ.

Chapter 2

Graph complexes, properads and their deformation theory

In the first two sections, we closely follow the paper [M5]. For the third section on deformation theory, we follow the paper [MV].

2.1 Graphs and $\mathfrak{G}$-algebras

2.1.1 Graphs

Definition 2.1.1. A *directed graph* G is a triple $(V(G), E(G), \nu_G)$ where $V(G)$ and $E(G)$ are finite sets and $\nu_G : E(G) \to V(G) \times V(G)$ a map such that $\alpha \mapsto (x^{out}, x^{in})$ for $\alpha \in E(G)$ and $x^{out}, x^{in} \in V(G)$. The elements of $E(G)$ are called *edges* and the elements of $V(G)$ are called *vertices*. An edge α is said to be *adjacent* to x^{out} and x^{in}, being *outgoing* from x^{out} and *incoming* to x^{in}. The *out-valency* $|x|^{out}$ of a vertex $x \in V(G)$ is the number of outgoing edges from x, the *in-valency* $|x|^{in}$ is the number of incoming edges to x, and the *valency* of x is the number $|x| := |x|^{out} + |x|^{in}$. If $|x|^{out} = 0$, then x is a *target* vertex, and if $|x|^{in} = 0$, then x is a *source* vertex. Vertices with valency one, two and three are respectively called *univalent*, *bivalent* and *trivalent*. Let G and H be directed graphs. A *morphism* $\varphi : H \to G$ is a pair of maps $\varphi_V : V(H) \to V(H)$ and $\varphi_E : E(H) \to E(G)$ such that $(\varphi_V \times \varphi_V) \circ \nu_H = \nu_G \circ \varphi_E$. The graph H is a *subgraph* of G if φ_V and φ_E are injective. The map φ is an *isomorphism* if φ_V and φ_E are bijective. We usually consider graphs up to isomorphisms.

Definition 2.1.2. Let $m, n \in \mathbb{N}$. A *directed hairy graph with m out-hairs and n in-hairs* $G(m, n)$ is a triple (G, h^{out}, h^{in}) where G is a directed graph and h^{out} and h^{in} are injective labeling maps

$$\begin{aligned} h^{out} &: [m] \to \{x \in V(G) \mid x \text{ univalent target}\} \\ h^{in} &: [n] \to \{x \in V(G) \mid x \text{ univalent source}\}. \end{aligned}$$

The elements of $H(G(m, n)) := \mathrm{Im}(h^{out}) \cup \mathrm{Im}(h^{in})$ are called *hairs*. The elements of the complement set $V(G(m, n)) = V(G) \backslash H(G(m, n))$ are called *hairy vertices* of $G(m, n)$. The set of *hairy edges* of $G(m, n)$ is the edges of $E(G)$ that are not adjacent to a hair. We will usually call hairy vertices and hairy edges just vertices and edges respectively, when there is no risk of confusion. Let $\mathfrak{U}(m, n)$ be the set of hairy graphs with m out-hairs and n in-hairs. There is a right action of the group $\mathbb{S}_m^{op} \times \mathbb{S}_n$ on $\mathfrak{U}(m, n)$ as

$$(G, h^{out}, h^{in})(\sigma, \tau) := (G, h^{out} \circ \sigma^{-1}, h^{in} \circ \tau)$$

permuting the labels of hairs. Note that graphs are not necessarily connected. Let $\mathfrak{U} = \bigcup_{m,n \geq 0} \mathfrak{U}(m, n)$.

Remark 2.1.3. When representing graphs pictorially, the direction of edges (and hairs) is normally indicated. To simplify certain pictures, the direction of edges (and hairs) has been omitted. In such a case, if an edge is attached to the upper part of a vertex, then it is assumed to be outgoing from that vertex. Similarly, an edge attached to the lower part of a vertex is assumed to be incoming to that edge (see figure 2.1). Similarly, the labeling of hairs is occasionally suppressed for simplification.

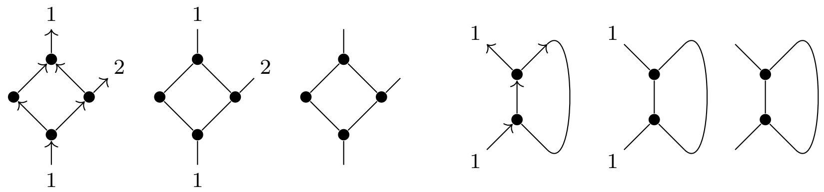
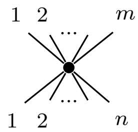


Figure 2.1: Convention of adding directions to undirected graphs

Remark 2.1.4. The hairy graph consisting of a single vertex with m out-hairs and n in-hairs is called an (m, n)*-corolla* or *corolla* in general. With the above mentioned convention, the (m, n)-corolla is visualized as  .

2.1.2 $\mathbb{S}$-bimodules and Decorated graphs

Definition 2.1.5. An $\mathbb{S}$*-bimodule* E is a collection of vector spaces $E = \{E(m, n)\}_{m,n\geq 0}$ such that $E(m, n)$ has a left $\mathbb{S}_m$-action and a right $\mathbb{S}_n$-action for all $m, n \geq 0$.

The following construction describes how to decorate the vertices of a general graph in $\mathfrak{U}$ with the elements of an $\mathbb{S}$-bimodule, respecting the symmetric action.

Definition 2.1.6. Let $E = \{E(m, n)\}_{m,n\geq 0}$ be an $\mathbb{S}$-bimodule and let $G \in \mathfrak{U}$. For each vertex $v \in V(G)$, let Out_v be the set of edges and hairs going away from v and In_v be the set of edges and hairs going into v. To each vertex v, we associate the vector space

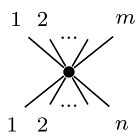
$$E(Out_v, In_v) := \langle Out_v\rangle \otimes_{\mathbb{S}_m} E(m, n) \otimes_{\mathbb{S}_n} \langle In_v\rangle$$

where $m = |Out_v|$, $n = |In_v|$ and $\langle Out_v\rangle$ is the vector space spanned by all bijections $[m] \to Out_v$ and $\langle In_v\rangle$ is the vector space spanned by all bijections $In_v \to [n]$. We remark that the space $E(Out_v, In_v)$ is non-canonically isomorphic to $E(m, n)$. There are natural actions of $\mathrm{Aut}(Out_v)$ and $\mathrm{Aut}(In_v)$ on $E(Out_v, In_v)$. Now suppose that $|V(G)| = k$. Then define

$$\bigotimes_{v\in V(G)} E(Out_v, In_v) := \Big(\bigoplus_{i:[k]\to V(G)} E(Out_{i(1)}, In_{i(1)}) \otimes \cdots \otimes E(Out_{i(k)}, In_{i(k)})\Big)_{\mathbb{S}_k}.$$

This is a representation space of $\mathrm{Aut}(G)$. Now let G be a graph and E an $\mathbb{S}$-bimodule. Then define the vector space

$$G\langle E\rangle := \Big(\bigotimes_{v\in V(G)} E(Out_v, In_v)\Big)_{\mathrm{Aut}(G)}.$$

This vector space represents the graph G with vertices decorated by elements of E. Suppose that (E, δ) is a dg $\mathbb{S}$-bimodule. Then for each graph G, we can define the chain complex $(G\langle E\rangle, \delta)$ whose differential is naturally induced from $E(Out_v, In_v)$. Note that when G is an (m, n)-corolla, i.e., $G =$  , then $G\langle E\rangle \cong E(m, n)$.

2.1.3 $\mathfrak{G}$-algebras

Definition 2.1.7. Let G and H be graphs of $\mathfrak{U}$ such that H is a connected subgraph of G. Then let G/H be the graph obtained by replacing the subgraph H of G with a single vertex having the same number of hairs as H, and reattaching the edges from G. Let $\mathfrak{G}$ be a subset of $\mathfrak{U}$ and G a graph in $\mathfrak{G}$. A subgraph H of G is *admissible* in $\mathfrak{G}$ if both $H \in \mathfrak{G}$ and $G/H \in \mathfrak{G}$.

Definition 2.1.8. Let $\mathfrak{G} \subseteq \mathfrak{U}$. A $\mathfrak{G}$-*algebra* is a pair $(\mathcal{P}, \{\mu_G\}_{G\in\mathfrak{G}})$ where

- $\mathcal{P} = \{\mathcal{P}(m,n)\}_{m,n\geq 0}$ is an $\mathbb{S}$-bimodule.
- $\{\mu_G\}_{G\in\mathfrak{G}}$ is a collection of linear $\mathbb{S}_m \times \mathbb{S}_n$-equivariant maps

$$\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{G\in\mathfrak{G}(m,n)},\ m,n \geq 0$$

  such that $\mu_G = \mu_{G/H} \circ \mu'_H$ for every admissible subgraph H of G in $\mathfrak{G}$. The map $\mu'_H : G\langle E\rangle \to (G/H)\langle E\rangle$ acts trivially on the decorations of vertices not in H, while the decoration of the collapsed vertex in G/H is induced by μ_H.

If $(\mathcal{P}, \delta)$ is a dg $\mathbb{S}$-bimodule with differential δ, then there is an induced differential on $G\langle\mathcal{P}\rangle$ for every graph G, which we denote by δ_G. A *dg* $\mathfrak{G}$-*algebra* is a triple $(\mathcal{P}, \{\mu_G\}_{G\in\mathfrak{G}}, \delta)$ such that $(\mathcal{P}, \{\mu_G\}_{G\in\mathfrak{G}})$ is a $\mathfrak{G}$-algebra, $(\mathcal{P}, \delta)$ is a dg $\mathbb{S}$-bimodule and $\delta \circ \mu_G = \mu_G \circ \delta_G$.

Remark 2.1.9. Strictly speaking, we have just defined $\mathfrak{G}$-algebras *without units.* A $\mathfrak{G}$-algebra with units is the $\mathfrak{G}$-algebra generated by the set of graphs $\mathfrak{G} \cup \mathfrak{I}$, where $\mathfrak{I}$ is the collection of graphs on the form $\{\uparrow \uparrow \cdots \uparrow \circlearrowleft \circlearrowleft \cdots \circlearrowleft\}$ with only edges and no vertices.

Example 2.1.10. We describe below the most frequent $\mathfrak{G}$-algebras. Their characteristic graphs are described in figure 2.1.3

- Let $\mathbf{Pr}^{\circlearrowleft} = \mathfrak{U}$. The $\mathbf{Pr}^{\circlearrowleft}$-algebras are called *wheeled props.*
- Let $\mathbf{Pr} \subseteq \mathbf{Pr}^{\circlearrowleft}$ be the subset of graphs with no closed directed paths. The $\mathbf{Pr}$-algebras are called *props.*
- Let $\mathbf{P}^{\circlearrowleft} \subseteq \mathbf{Pr}^{\circlearrowleft}$ be the subset of connected graphs. The $\mathbf{P}^{\circlearrowleft}$-algebras are called *wheeled properads.*
- Let $\mathbf{P} \subseteq \mathbf{Pr}$ of connected graphs. The $\mathbf{P}$-algebras are called *properads.*
- Let $\mathbf{O}^{\circlearrowleft} \subseteq \mathbf{P}^{\circlearrowleft}$ be the subset of graphs where each vertex has exactly one outgoing edge or hair. The $\mathbf{O}^{\circlearrowleft}$-algebras are called *wheeled operads.*
- Let $\mathbf{O} \subseteq \mathbf{P}$ be the subset of graphs where each vertex has exactly one outgoing edge or hair. The $\mathbf{O}$-algebras are called *operads.*
- Let $\mathbf{A} \subseteq \mathbf{O}$ be the subset of graphs where each vertex has exactly one incoming edge or hair. The $\mathbf{A}$-algebras are called *associative algebras.*

Remark 2.1.11. It was brought to the authors attention by one of the jury members of a definition of T-algebras (Section 5.2) in [BGHZ] analogous to that of wheeled props, but coming from the field of probability theory.

Example 2.1.12. Here follow some notable examples of $\mathfrak{G}$-algebras for a general set of graphs $\mathfrak{G}$.

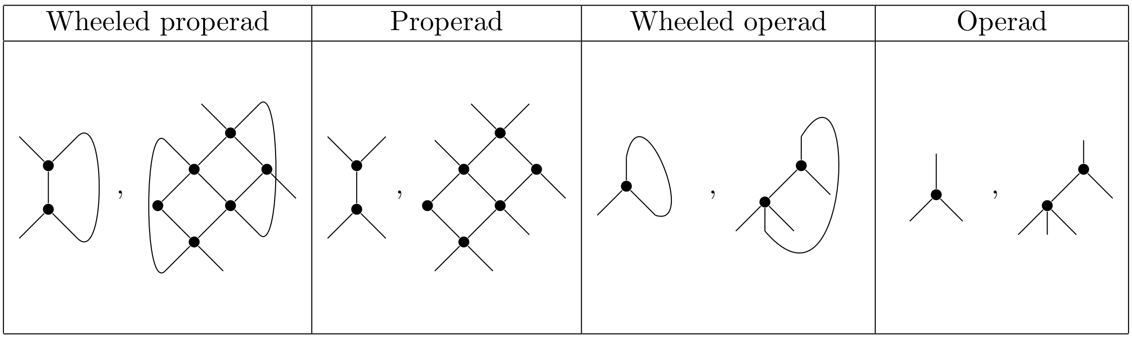


Figure 2.2: Characteristic graphs for different $\mathfrak{G}$-algebras.

- Let V be a finite-dimensional vector space. The $\mathbb{S}$-bimodule

$$\mathcal{E}nd_V = \{\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}_{m,n\geq 0}$$

  is a $\mathfrak{G}$-algebra where the contraction maps consist of the normal composition and trace maps. We call this the *endomorphism* $\mathfrak{G}$-algebra. The requirement that V is finite-dimensional can be dropped when $\mathfrak{G}$ does not contain any graphs with closed loops.

- Let $E = \{E(m,n)\}_{m,n\geq 0}$ be any $\mathbb{S}$-bimodule. The *free* $\mathfrak{G}$*-algebra over* E is the $\mathbb{S}$-bimodule $\mathcal{F}\langle E\rangle = \{\mathcal{F}(m,n)\}_{m,n\geq 0}$, where $\mathcal{F}(m,n) = \bigoplus_{G\in\mathfrak{G}(m,n)} G\langle E\rangle$. The contraction maps are canonically defined.

- Let $\mathcal{P}$ be a $\mathfrak{G}$-algebra. A submodule $\mathcal{Q}\subset\mathcal{P}$ is a $\mathfrak{G}$*-subalgebra of* $\mathcal{P}$ if the contraction maps μ_G restrict as $\mu_G : G\langle\mathcal{Q}\rangle \to \mathcal{Q}(m,n)$ for all $G\in\mathfrak{G}$.

- An *ideal* $\mathcal{I}$ *of* $\mathcal{P}$ is a $\mathfrak{G}$-subalgebra of $\mathcal{P}$ such that the contraction map μ_G restricts as $\mu_G : G\langle\mathcal{P}:\mathcal{I}\rangle \to \mathcal{I}(m,n)$ for all $G\in\mathfrak{G}$, where $G\langle\mathcal{P}:\mathcal{I}\rangle$ is the submodule of $G\langle\mathcal{P}\rangle$ of graphs where at least one vertex is decorated by an element of $\mathcal{I}$. The $\mathfrak{G}$-quotient algebra $\mathcal{P}/\mathcal{I}$ is then defined in the natural way.

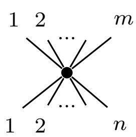
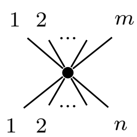
Definition 2.1.13. Let $\mathcal{P}$ and $\mathcal{Q}$ be $\mathfrak{G}$-algebras.

- A $\mathfrak{G}$*-algebra morphism* from $\mathcal{P}$ to $\mathcal{Q}$ is an $\mathbb{S}$-bimodule morphism $\rho:\mathcal{P}\to\mathcal{Q}$ such that for any graph G, the diagram commutes

$$\begin{array}{ccc} G\langle\mathcal{P}\rangle & \xrightarrow{\rho^{\otimes G}} & G\langle\mathcal{Q}\rangle \\ {\scriptstyle\mu_G}\downarrow & & \downarrow{\scriptstyle\mu_G} \\ \mathcal{P}(m,n) & \xrightarrow[\rho]{} & \mathcal{Q}(m,n) \end{array}$$

  where $\rho^{\otimes G} : G\langle\mathcal{P}\rangle \to G\langle\mathcal{Q}\rangle$ is the map where ρ is applied to each vertex in G.

- A *representation* of $\mathcal{P}$ in a graded vector space V is a morphism of $\mathfrak{G}$-algebras $\mathcal{P}\to\mathcal{E}nd_V$.

- A *free resolution* of $\mathcal{P}$ is a free dg $\mathfrak{G}$-algebra $(\mathcal{F}\langle E\rangle,\delta)$ together with a $\mathfrak{G}$-algebra morphism $(\mathcal{F}\langle E\rangle,\delta)\to\mathcal{P}$ that is an isomorphism on cohomology (when $\mathcal{P}$ is equipped with the trivial differential). A free resolution is *minimal* if the differential acts on corollas  such that $\delta($  $)\in\mathcal{F}_{\geq 2}\langle E\rangle$ where $\mathcal{F}_{\geq 2}\langle E\rangle$ is the subspace of $\mathcal{F}\langle E\rangle$ generated by graphs with at least two vertices. The resolution is called *quadratic* if the same image only contains graphs with exactly two vertices.

Remark 2.1.14. The notions of free resolution and minimal resolution stem from the theory of model categories, however, no such intuition is used when studying these objects in this thesis.

2.1.4 Wheeled closure of a $\mathfrak{G}$-algebra

Definition 2.1.15. A $\mathfrak{G}$-algebra $\mathcal{P}$ is *finitely generated by arity* if $\mathcal{P} = \mathcal{F}\langle E\rangle/\mathcal{I}$ where $E = \{E(m,n)\}_{m,n\geq 0}$ is an $\mathbb{S}$-bimodule and $\mathcal{I} = \{\mathcal{I}(m,n)\}_{m,n\geq 0}$ an ideal of $\mathcal{F}\langle E\rangle$ such that $E(m,n)$ is finite dimensional for all m,n.

Remark 2.1.16. All the $\mathfrak{G}$-algebras we encounter will be finitely generated by arity.

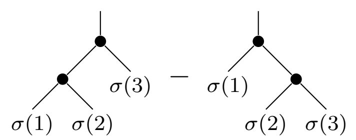
Definition 2.1.17. Let G be a hairy graph and let h_1 be an out-hair and h_2 an in-hair of G. A *joining* of h_1 and h_2 is the graph equivalent to G except that the hairs h_1 and h_2 have been removed and replaced by a directed edge. A *wheeling* of G is the graph where one or more pairs of out-hairs and in-hairs have been grafted. Let $\mathfrak{G}$ be a set of hairy graphs. The *wheeled closure* of $\mathfrak{G}$ is the set $\mathfrak{G}^{\circlearrowleft}$ containing both $\mathfrak{G}$ and all wheeling of graphs in $\mathfrak{G}$.

Remark 2.1.18. Let $\mathfrak{G}-\mathbf{alg}$ is the category of finitely generated by arity $\mathfrak{G}$-algebras and $\mathfrak{G}^{\circlearrowleft}-\mathbf{alg}$ the category of finitely generated by arity $\mathfrak{G}^{\circlearrowleft}$-algebras. Then the functor taking a $\mathfrak{G}$-algebra $\mathcal{F}/\mathcal{I}$ to $\mathcal{F}^{\circlearrowleft}/\mathcal{I}^{\circlearrowleft}$ is a left adjoint to the forgetful functor from $\mathfrak{G}^{\circlearrowleft}-\mathbf{alg}$ to $\mathfrak{G}-\mathbf{alg}$.

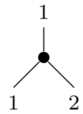
2.2 Examples of operads and properads

2.2.1 The operad of associative algebras

Example 2.2.1. Let $A_0 = \{A(m,n)\}_{m,n\geq 0}$ be the $\mathbb{S}$-bimodule where $A_0(1,2) = \mathbb{K}[\mathbb{S}_2]$ and $A_0(m,n) = 0$ for all other m,n. The free operad $\mathcal{F}\langle A_0\rangle$ is generated by oriented graphs whose vertices have two incoming edges and one outgoing edge. Consider the ideal $\mathcal{I} \subseteq \mathcal{F}\langle A_0\rangle$ generated by the set

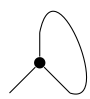
$I_0 := \Big\{$ 

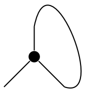
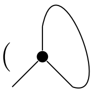
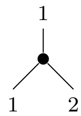
$: \sigma \in \mathbb{S}_3\Big\}.$ (2.1)

The *operad of associative algebras* is the quotient operad $\mathcal{A}ss := \mathcal{F}\langle A_0\rangle/\mathcal{I}$. If V is a vector space, then there is a 1-1 correspondence between associative algebra structures on V and representations $\rho : \mathcal{A}ss \to \mathcal{E}nd_V$. In the correspondence, the element $\rho($ 

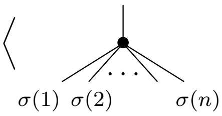
$) \in \mathcal{E}nd_V(1,2) = \mathrm{Hom}(V\otimes V, V)$ corresponds to an associative product on V.

Example 2.2.2. The *wheeled operad of associative algebras* is the quotient operad $\mathcal{A}ss^{\circlearrowleft} := \mathcal{F}\langle A_0\rangle/\mathcal{I}$ where the generators are the same as above in the category of wheeled operads.

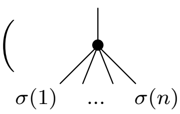
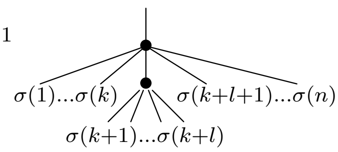
Then the wheeled operad $\mathcal{F}\langle A_0\rangle$ additionally includes wheeled graphs such as  .

Remark 2.2.3. If V is a finite dimensional vector space, then there is a 1-1 correspondence between associative algebra structures on V and representations $\rho : \mathcal{A}ss^{\circlearrowleft} \to \mathcal{E}nd_V$. A graph of $\mathcal{A}ss^{\circlearrowleft}$ containing a wheel, say , corresponds to an element $\rho($  $) \in$ $\mathrm{Hom}(V, \mathbb{K})$. This is effectively a trace map induced by the product .

Example 2.2.4. Let $A = \{A(m,n)\}_{m,n\geq 0}$ be the $\mathbb{S}$-bimodule where

$$A(1,n) = \mathbb{K}[\mathbb{S}_n] = \Big\langle$$  $$\Big\rangle_{\sigma\in\mathbb{S}_n}$$

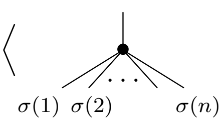
for $n \geq 2$ and $A(m,n) = 0$ otherwise. The free operad $\mathcal{F}\langle A\rangle$ together with the differential δ that act on generating corollas as

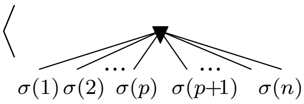
$$\delta\Big($$  $$\Big) = \sum_{k=0}^{n-2}\sum_{l=2}^{n-k}(-1)^{k+l(n-k-l)+1}$$ 

is a minimal resolution of $\mathcal{A}ss$, which we denote by $\mathcal{H}oass$.

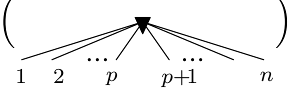
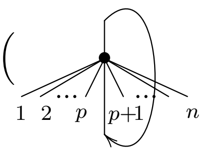
In general, the wheeled closure of a minimal resolution is not in general a minimal resolution, exemplified by the $\mathfrak{G}^{\circlearrowleft}$-algebra $\mathcal{A}ss^{\circlearrowleft}$ described below.

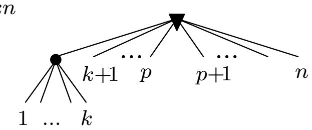
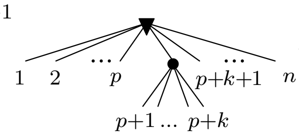
Example 2.2.5. Let $A^{\circlearrowleft} = \{A^{\circlearrowleft}(m,n)\}_{m,n\geq 0}$ be the $\mathbb{S}$-bimodule such that

$$A^{\circlearrowleft}(1,n) = \mathbb{K}[\mathbb{S}_n][n-2] = \Big\langle$$  $$\Big\rangle_{\sigma\in\mathbb{S}_n} \qquad \text{for } n \geq 2$$

$$A^{\circlearrowleft}(0,n) = \bigoplus_{p=1}^{n-1}\mathbb{K}[\mathbb{S}_n]_{C_p\times C_{n-p}}[n] = \bigoplus_{p=1}^{n-1}\Big\langle$$  $$\Big\rangle_{\sigma\in\mathbb{S}_n} \qquad \text{for } n \geq 2$$

and $A(m,n) = 0$ otherwise. Here $C_p \times C_{n-p}$ denotes the subgroup of $\mathbb{S}_n$ generated by two elements $(12\cdots p)$ and $(p+1\cdots n)$. The differential of the first kind of generator is the same as for $\mathcal{H}oass$, while it acts on the second type of generator as

$$\delta\Big($$  $$\Big) = \oint_{(12\cdots p)}\oint_{p+1\cdots n}\Big($$ 

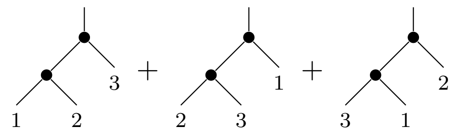
$$+\sum_{k=2}^{p}(-1)^{kn}$$  $$+\sum_{k=2}^{n-2}(-1)^{p+k(1+n-p)+1}$$  $$\Big).$$

2.2.2 The operad of Commutative algebras

Example 2.2.6. Let $C_0 = \{C_0(m,n)\}_{m,n\geq 0}$ be the $\mathbb{S}$-bimodule where $C_0(1,2) = \mathbb{K}$ and $C_0(m,n) = 0$ for all other m,n. The free operad $\mathcal{F}(C_0)$ is generated by oriented graphs whose vertices have two incoming edges and one outgoing edge. Consider the ideal $\mathcal{I} \subseteq \mathcal{F}(C_0)$ generated by the same set I_0 as in (2.1). The *operad of commutative algebras* is the quotient operad $\mathcal{C}om := \mathcal{F}(C_0)/\mathcal{I}$.

2.2.3 The operad of Lie algebras

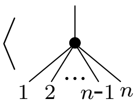
Example 2.2.7. Let $L_0 = \{L_0(m,n)\}_{m,n\geq 0}$ be the $\mathbb{S}$-bimodule where $L_0(1,2) = \mathrm{sgn}_2$ and $L_0(m,n) = 0$ for all other m,n. Consider the ideal $\mathcal{I}$ in the free operad $\mathcal{F}(L_0)$ generated by the linear combination

$$\mathcal{I} = \Big\langle$$


$$\Big\rangle.$$

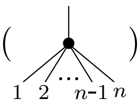
The *operad of Lie-algebras* is the quotient operad $\mathcal{L}ie := \mathcal{F}(L_0)/\mathcal{I}$.
If V is a vector space, then there is a 1-1 correspondence between Lie-algebra structures on V and representations $\rho : \mathcal{L}ie \to \mathcal{E}nd_V$.

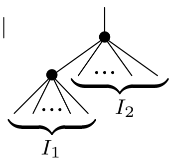
Example 2.2.8. Let $L = \{L(m,n)\}_{m,n\geq 0}$ be the $\mathbb{S}$-bimodule where

$$L(1,n) = \mathrm{sgn}_n[n-2] = \Big\langle$$


$$\Big\rangle$$

for $n \geq 2$ and $L(m,n) = 0$ otherwise. The free operad $\mathcal{F}\langle L\rangle$ together with the differential δ that act on generating corollas as

$$\delta\Big($$


$$\Big) = \sum_{\substack{[n]=I_1\sqcup I_2 \\ |I_1|\geq 2,\ |I_2|\geq 2}} (-1)^{\sigma(I_1,I_2)+(|I_1|+1)|I_2|} \tag{2.2}$$


is a minimal resolution of $\mathcal{L}ie$, which we denote by $\mathcal{H}olie$.

Remark 2.2.9. In contrast to the operad of associative algebras $\mathcal{A}ss$, the wheeled closure $\mathcal{H}olie^{\circlearrowleft}$ of $\mathcal{H}olie$ is a minimal resolution of $\mathcal{L}ie^{\circlearrowleft}$. That is $\mathcal{H}olie^{\circlearrowleft} := \mathcal{F}\langle L\rangle$ together with the differential (2.2) is a minimal resolution of $\mathcal{L}ie^{\circlearrowleft}$ (see Theorem 4.1.1 in [M2]).

Representations of $\mathcal{H}olie$ are called *homotopy Lie algebras* (or $\mathcal{L}_\infty$-algebras). A homotopy Lie algebra is a graded vector space V together with an infinite family of multilinear maps $Q = \{Q^{(n)} : V^{\wedge n} \to V\}_{n\geq 1}$ of degree $n-2$ satisfying the strong homotopy Jacobi identities

$$\sum_{i+j=n+1} \sum_{\sigma\in \mathrm{Shuff}(i,n-i)} \chi(\sigma, v_1, ..., v_n)(-1)^{i(j-1)} Q^{(j)}(Q^{(i)}(v_{\sigma(1)}, ..., v_{\sigma(i)}), v_{\sigma(i+1)}, ..., v_{\sigma(n)}) = 0$$

for every $n \geq 1$. Here $\mathrm{Shuff}(p,q)$ denotes the set of (p,q)-shuffles and χ is a signature of the permutation. In the first three instances, we get the equations

$$Q^{(1)} \circ Q^{(1)}(v_1) = 0$$
$$Q^{(1)}(v_1, v_2) = Q^{(2)}(Q^{(1)}(v_1), v_2) - (-1)^{|v_1||v_2|} Q^{(2)}(Q^{(1)}(v_2), v_1)$$
$$\oint_{(123)} Q^{(2)}(Q^{(2)}(v_3, v_1), v_2) = \oint_{(123)} Q^{(3)}(Q^{(1)}(v_1), v_2, v_3) + \oint_{(123)} Q^{(1)}(Q^{(3)}(v_1, v_2, v_3)).$$

The first one tells us that $Q^{(1)}$ squares to zero and the second that $Q^{(1)}$ is a derivative with respect to $Q^{(2)}$, hence $Q^{(1)}$ is a differential. The third equation shows that $Q^{(2)}$ satisfy the Jacobi identity up to some relations. In particular, when $Q^{(n)} = 0$ for $n \geq 3$, then this structure is exactly a dg Lie algebra. We say that a homotopy Lie algebra V is *filtered* if there is a sequence $V_0 = V \supseteq V_1 \supseteq V_2 \supseteq \cdots$ such that $\bigcap_{i=0}^{\infty} V_i = \{0\}$ and there is an $n_0 \in \mathbb{N}$ such that $Q^{(n)} : V^{\wedge n} \to V$ restricts to $Q^{(n)} : V^{\wedge n} \to V_n$ for $n \geq n_0$.

2.2.4 The properad of Lie bialgebras

Definition 2.2.10. Let $k \in \mathbb{Z}$. A *Lie k-bialgebra* is a vector space V together with a Lie bracket $[-,-] : V \wedge V \to V$ (satisfying the Jacobi identity) of degree $-k$ and a Lie cobracket $\Delta : V \to V \wedge V$ (satisfying the co-Jacobi identity) of degree 0 that satisfy the $([-,-],\Delta)$-compability relation

$$\Delta([a,b]) = \oint_{(ab)} \mathrm{Alt}_2(ad_a \otimes \mathrm{Id})\Delta(b). \tag{2.3}$$

for any $a, b \in V$. Here $ad_a(x) = [a,x]$, the map $\mathrm{Alt}_n : V^{\otimes n} \to V^{\otimes n}$ is the operator

$$\mathrm{Alt}_n(x_1 \otimes \cdots \otimes x_n) = \sum_{\sigma \in \mathbb{S}_n} (-1)^{|\sigma|}(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)})$$

for any linear map $f : V^{\otimes n} \to V^{\otimes m}$ and

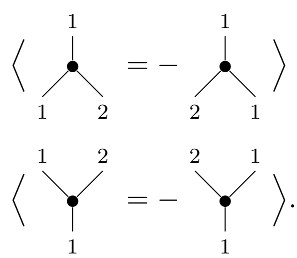
$$\oint_{(x_1 \cdots x_k)} f(x_1, \cdots, x_k) = \sum_{i=1}^{k} (-1)^{|\sigma^i|} f(x_{(\sigma^i)(1)}, \cdots, x_{(\sigma^i)(k)})$$

where $\sigma = (12\cdots k) \in \mathbb{S}_k$ and σ^i is the composition of σ with itself i times. The usual notion of a Lie bialgebra is obtained for $k = 0$.

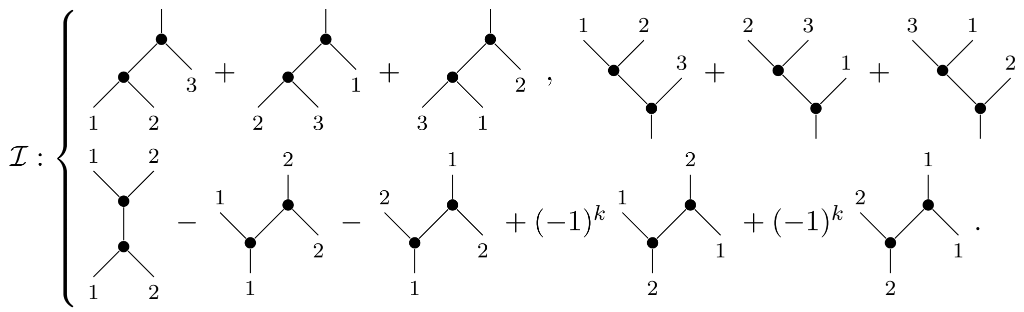
Definition 2.2.11. Let $Lb_0 = \{Lb_0(m,n)\}_{m,n\geq 0}$ be the $\mathbb{S}$-bimodule where $Lb_0(m,n) = 0$ for all m, n except

$Lb_0(1,2) = \mathbf{1}_1 \otimes \mathrm{sgn}_2[k] = \langle$ … $\rangle$

$Lb_0(2,1) = \mathrm{sgn}_2 \otimes \mathbf{1}_1 = \langle$ … $\rangle$.



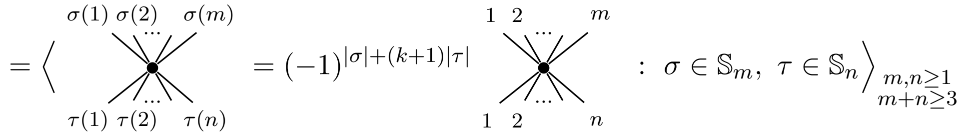
Further let $\mathcal{I}$ be the ideal of the free properad $\mathcal{F}\langle Lb_0 \rangle$ generated by the elements



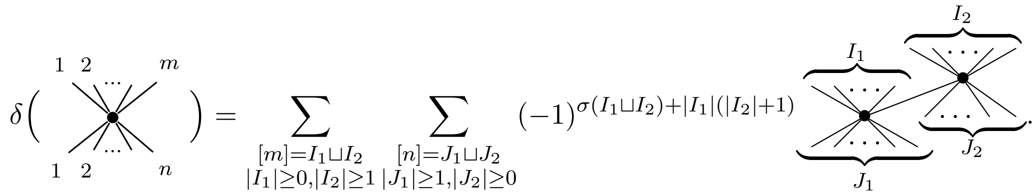
The properad of Lie k-bialgebras $\mathcal{L}ieb_k$ is the quotient properad $\mathcal{F}\langle Lb_0 \rangle / \mathcal{I}$.

Definition 2.2.12. Let $\mathcal{H}olieb_k$ be the free dg properad generated by the $\mathbb{S}$-bimodule $Lb = \{Lb(m,n)\}_{m,n\geq 0}$ where

$$Lb(m,n) = \mathrm{sgn}_m \otimes \mathrm{sgn}_n^{\otimes|k+1|}[(m-1)+(k+1)(n-1)-1]$$



and the differential is given on generators as



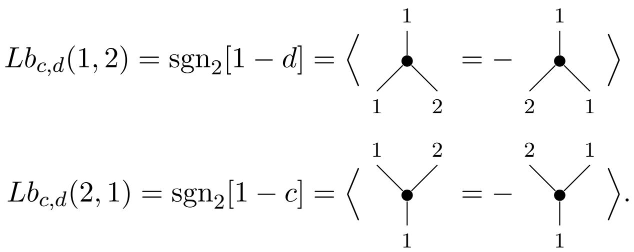
(2.4)

Proposition 2.2.13. *The properad $\mathcal{H}olieb_k$ is a minimal resolution of $\mathcal{L}ieb_k$.*

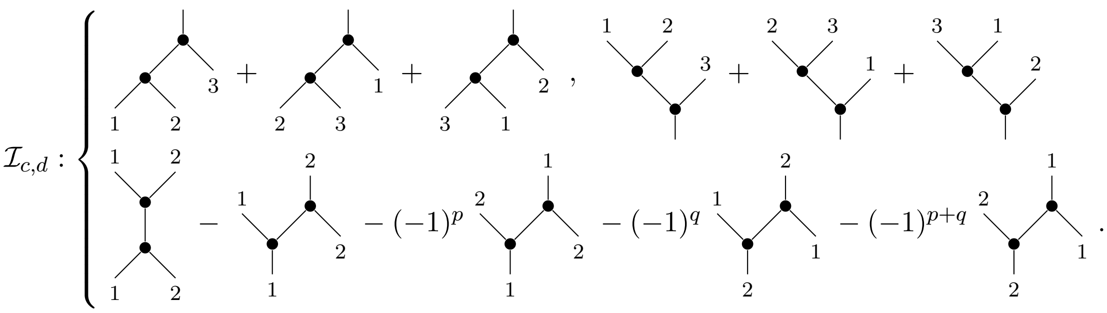
Proof. This was shown in [V, M1]. □

Remark 2.2.14. Let $c, d \in \mathbb{Z}$. We can generalize the notion of a Lie k-bialgebra to a (c,d)*-shifted Lie bialgebra* where the Lie bracket has degree $1-c$ and the Lie cobracket has degree $1-d$. Then the properad of (c,d)-shifted Lie-bialgebras is the properad $\mathcal{L}ieb_{c,d} := \mathcal{L}ieb_{c+d-2}\{1-c\}$ governing a degree shifted Lie-bialgebras where the Lie-bracket has degree $1-d$ and the Lie-cobracket has degree $1-c$. The standard Lie-bialgebra properad is regained when $c = d = 1$. We can equivalently define this properad as the quotient properad below.

Definition 2.2.15. Let $\mathcal{L}ieb_{c,d}$ be the quotient properad $\mathcal{F}\langle Lb_{c,d}\rangle/\mathcal{I}_{c,d}$ where $Lb_{c,d}$ is the $\mathbb{S}$-bimodule where

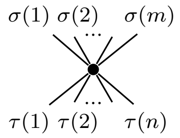
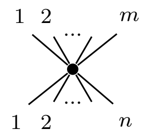


and $Lb_{c,d}(m,n) = 0$ for all other m,n and $\mathcal{I}_{c,d}$ is the ideal of $\mathcal{F}\langle Lb_{c,d}\rangle$ generated by the elements



Definition 2.2.16. Let $\mathcal{H}olieb_{c,d}$ free dg properad generated by the $\mathbb{S}$-bimodule $Lb_{c,d} = \{Lb_{c,d}(m,n)\}_{m,n\geq 0}$ where

$$Lb_{c,d}(m,n) = \mathrm{sgn}_m^{\otimes c} \otimes \mathrm{sgn}_n^{\otimes n}[c(m-1)+d(n-1)-1]$$

$= \Big\langle$  $= (-1)^{c|\sigma|+d|\tau|}$  $\mid \sigma \in \mathbb{S}_m,\ \tau \in \mathbb{S}_n \Big\rangle_{\substack{m,n\geq 1\\ m+n\geq 3}}$

and the differential is given as in equation (2.4).

Proposition 2.2.17. *The properad $\mathcal{H}olieb_{c,d}$ is a minimal resolution of $\mathcal{L}ieb_{c,d}$.*

Proof. This was shown in [V, M1]. □

Remark 2.2.18. Note that $\mathcal{H}olieb_k^{\circlearrowleft}$ is not a minimal resolution of $\mathcal{L}ieb_k^{\circlearrowleft}$, and the existence of such a minimal model is not shown.

Remark 2.2.19. It is still an open problem to find a minimal resolution of $\mathcal{L}ieb_k^{\circlearrowleft}$.

2.2.5 The properad of quasi-Lie bialgebras

Definition 2.2.20. A *quasi-Lie k-bialgebra* is a $\mathbb{Z}$-graded vector space V together with the maps

$$[-,-] : V^{\wedge 2} \to V[k]$$
$$\Delta : V \to V^{\wedge 2}$$
$$\phi : \mathbb{K} \to V^{\wedge 3}[-k]$$

of degree $-k$, 0 and k respectively, such that $[-,-]$ satisfies the Jacobi identity, Δ satisfies the modified co-Jacobi identity

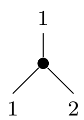
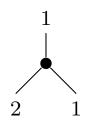
$$\frac{1}{2}\mathrm{Alt}_3(\Delta \otimes \mathrm{Id})\Delta(a) = [\mathrm{Alt}_3(a \otimes 1 \otimes 1), \phi]$$

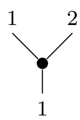
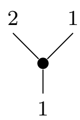
as well as the $([-,-],\delta)$-compability relation (2.3) and the (Δ,ϕ)-compability relation

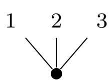
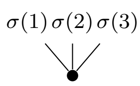
$$\mathrm{Alt}_4(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = 0.$$

Remark 2.2.21. A quasi-Lie bialgebra naturally is a Lie bialgebra when $\phi = 0$.

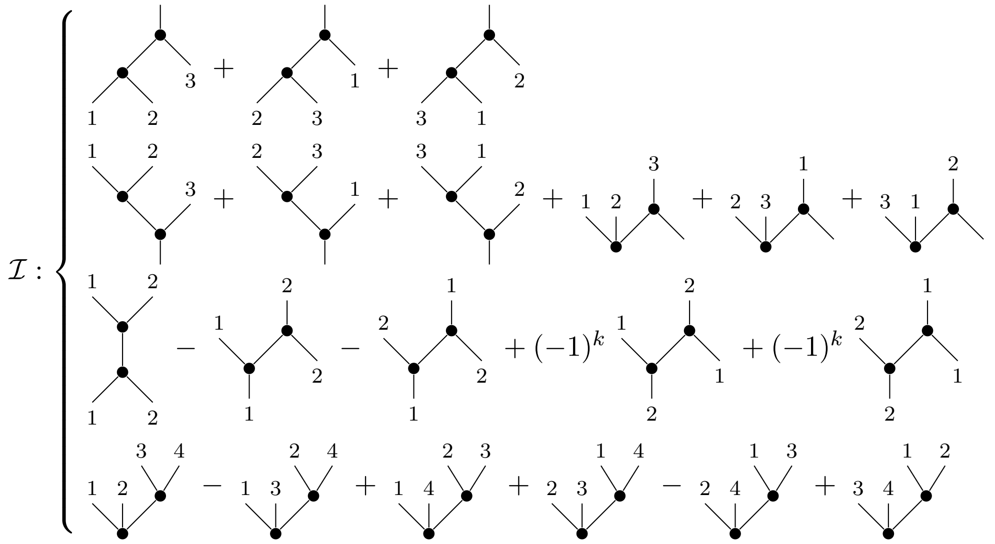
Definition 2.2.22. The properad of quasi-Lie k-bialgebras $\mathcal{Q}lieb_k$ is the quotient properad $\mathcal{F}\langle QLb_0\rangle/\mathcal{I}$ where $QLb_0 = \{QLb_0(m,n)\}_{m,n\geq 0}$ is the $\mathbb{S}$-bimodule with $QLb_0(m,n) = 0$ except for

$QLb_0(1,2) = \mathbf{1}_1 \otimes \mathrm{sgn}_2[k] = \Big\langle$  $= -$  $\Big\rangle$

$QLb_0(2,1) = \mathrm{sgn}_2 \otimes \mathbf{1}_1 = \Big\langle$  $= -$  $\Big\rangle$

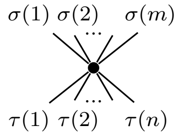
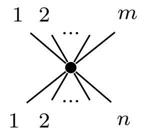
$QLb_0(3,0) = \mathrm{sgn}_3 \otimes \mathbf{1}_0[-k] = \Big\langle$  $= |\sigma|$  $\Big\rangle_{\sigma\in\mathbb{S}_3}$

and $\mathcal{I}$ is the ideal generated by the elements

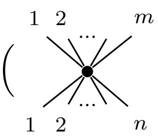
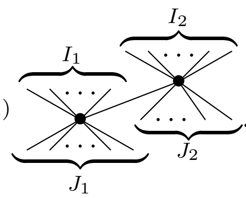


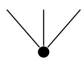
Definition 2.2.23. Let $\mathcal{QH}olieb_k$ be the free dg properad generated by the $\mathbb{S}$-bimodule $Qlb = \{QLb(m,n)\}_{m,n\geq 0}$ where

$$Lb(m,n) = \mathrm{sgn}_m \otimes \mathrm{sgn}_n^{\otimes |k+1|}[(m-1)+(k+1)(n-1)-1]$$

$= \Big\langle$  $= (-1)^{|\sigma|+(k+1)|\tau|}$  $\mid \sigma \in \mathbb{S}_m,\ \tau \in \mathbb{S}_n \Big\rangle_{\substack{m\geq 1, n\geq 0\\ m+n\geq 3}}$

and the differential is given on generators as

$\delta\Big($  $\Big) = \sum\limits_{\substack{[m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum\limits_{\substack{[n]=J_1\sqcup J_2\\ |J_1|\geq 0, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$ .

This minimal resolution contains an additional generator  compared to $\mathcal{H}olieb_k$.

Remark 2.2.24. We similarly define the degree shifted versions $\mathcal{Q}lieb_{c,d} := \mathcal{Q}lieb_{c+d-2}\{1-c\}$ and $\mathcal{QH}olieb_{c,d} := \mathcal{QH}olieb_{c+d-2}\{1-d\}$ of these properads.

Proposition 2.2.25. *The properads $\mathcal{QH}olieb_k$ and $\mathcal{QH}olieb_{c,d}$ are minimal resolutions of $\mathcal{Q}lieb_k$ and $\mathcal{Q}lieb_{c,d}$ respectively.*

Proof. This was shown in [Gr]. □

2.2.6 The properad of pseudo-Lie bialgebras

Definition 2.2.26. A *pseudo-Lie k-bialgebra* is a $\mathbb{Z}$-graded vector space together with the maps

$$\eta : V^{\wedge 3} \to V$$
$$[-,-] : V^{\wedge 2} \to V$$
$$\Delta : V \to V^{\wedge 2}$$
$$\phi : \mathbb{K} \to V^{\wedge 3}$$

of degree $-2k$, $-k$, 0 and k respectively such that the following equations are satisfied

1. (Modified Jacobi-relation)

$$\oint_{a,b,c} [[a,b],c] = \oint_{a,b,c} \eta(a,b,\Delta_1(c))\Delta_2(c)$$

2. (Modified co-Jacobi-relation)

$$\frac{1}{2}\mathrm{Alt}_3(\Delta \otimes \mathrm{Id})\Delta(a) = [\mathrm{Alt}_3(a \otimes 1 \otimes 1), \phi]$$

3. (Modified $([-,-],\Delta)$-compability)

$$\Delta([a,b]) = \oint_{a,b} \mathrm{Alt}_2(ad_a \otimes \mathrm{Id})\Delta(b) + \sum_{\phi} (\phi_1 \otimes \phi_2)\eta(\phi_3, a, b)$$

4. ((Δ,ϕ)-compability)

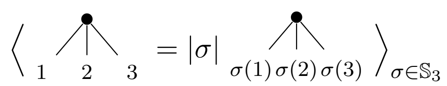
$$\mathrm{Alt}_4(\Delta \otimes \mathrm{Id} \otimes \mathrm{Id})(\phi) = 0$$

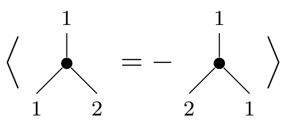
5. ($([-,-],\eta)$-compability)

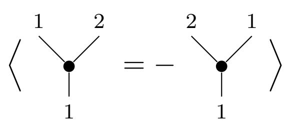
$$\eta(\mathrm{Alt}_4([a,b],c,d)) = 0$$

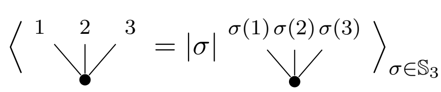
Remark 2.2.27. When $\eta = 0$, then we retain the definition of a quasi-Lie bialgebra, and when $\eta = \phi = 0$ we retain the definition of a Lie-bialgebra.

Definition 2.2.28. The properad of pseudo-Lie k-bialgebras $\mathcal{P}lieb_k$ is the quotient properad $\mathcal{F}\langle PLb_0\rangle/\mathcal{I}$ where $PLb_0 = \{PLb_0(m,n)\}_{m,n\geq 0}$ is the $\mathbb{S}$-bimodule where $PLb_0(m,n) = 0$ except for

$PLb_0(0,3) = \mathbf{1}_0 \otimes \mathrm{sgn}_3[2k] =$ 

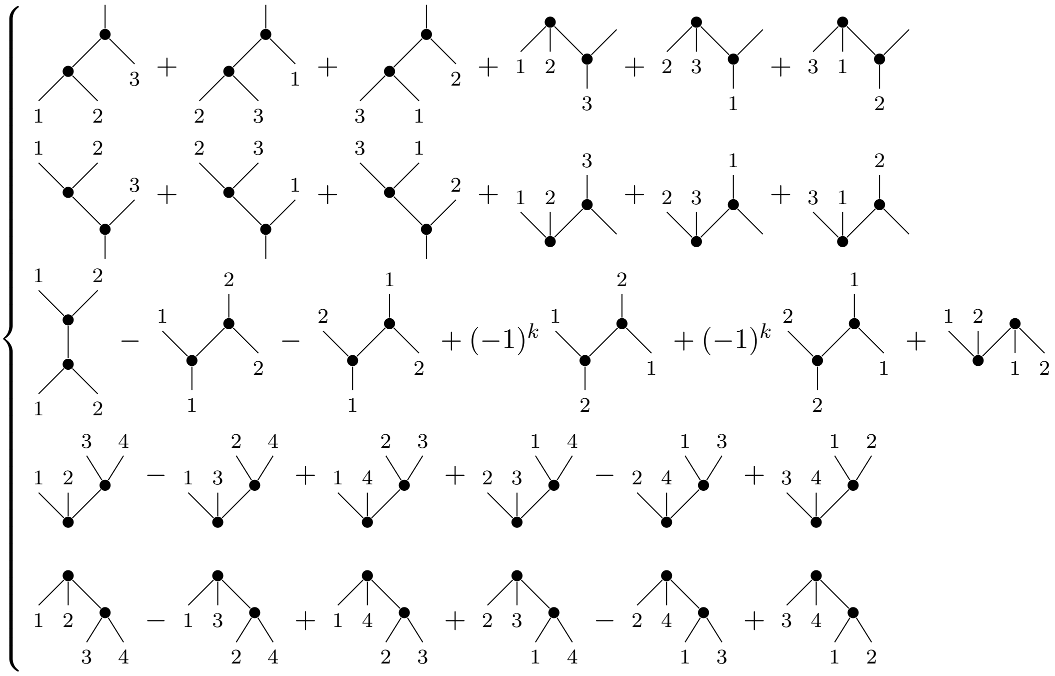
$PLb_0(1,2) = \mathbf{1}_1 \otimes \mathrm{sgn}_2[k] =$ 

$PLb_0(2,1) = \mathrm{sgn}_2 \otimes \mathbf{1}_1 =$ 

$PLb_0(3,0) = \mathrm{sgn}_3 \otimes \mathbf{1}_0[-k] =$ 

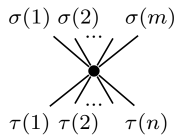
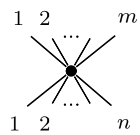
and $\mathcal{I}$ is the ideal generated by the elements

$\mathcal{I}$:

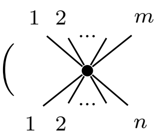
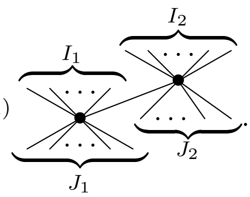


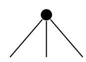
Definition 2.2.29. Let $\mathcal{PH}olieb_k$ be the dg free properad generated by the $\mathbb{S}$-bimodule $PLb = \{PLb(m,n)\}_{m,n\geq 0}$ where

$$PLb(m,n) = \mathrm{sgn}_m \otimes \mathrm{sgn}_n^{\otimes |k+1|}[(m-1)+(k+1)(n-1)-1]$$

$= \Big\langle$  $= (-1)^{|\sigma|+(k+1)|\tau|}$  $\mid \sigma \in \mathbb{S}_m,\ \tau \in \mathbb{S}_n \Big\rangle_{\substack{m,n\geq 0\\ m+n\geq 3}}$

and the differential is given on generators as

$\delta\Big($  $\Big) = \sum_{\substack{[m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 0}} \sum_{\substack{[n]=J_1\sqcup J_2\\ |J_1|\geq 0, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$  .

This properad contains the additional generator  compared to $\mathcal{QH}olieb_k$.

Remark 2.2.30. We similarly define the degree shifted versions $\mathcal{P}lieb_{c,d} := \mathcal{P}lieb_{c+d-2}\{1-c\}$ and $\mathcal{PH}olieb_{c,d} := \mathcal{PH}olieb_{c+d-2}\{1-c\}$ of these properads.

Proposition 2.2.31. *The properads $\mathcal{PH}olieb_k$ and $\mathcal{PH}olieb_{c,d}$ are minimal resolutions of $\mathcal{P}lieb_k$ and $\mathcal{P}lieb_{c,d}$ respectively.*

Proof. This was shown in [Gr]. □

2.3 Deformation theory

2.3.1 Derivation complexes

Definition 2.3.1. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are (dg) properads with composition maps $\{\mu_G^{\mathcal{P}} : G\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{G\in\mathfrak{G}}$ and $\{\mu_G^{\mathcal{Q}} : G\langle\mathcal{Q}\rangle \to \mathcal{Q}(m,n)\}_{G\in\mathfrak{G}}$. and let $\rho : \mathcal{P} \to \mathcal{Q}$ be a morphism of (dg) properads. Let $d \in \mathbb{Z}$ and $\mathcal{Q}[d] := \{\mathcal{Q}(m,n)[d]\}_{m,n\geq 0}$ be the degree shifted $\mathbb{S}$-bimodule and set $\mathcal{P} \oplus \mathcal{Q}[d] := \{\mathcal{P}(m,n) \oplus \mathcal{Q}[d]\}_{m,n\geq 0}$. Then there is a (dg) properad $\{\mu_G : G\langle \mathcal{P} \oplus \mathcal{Q}[d]\rangle \to \mathcal{P}(m,n) \oplus \mathcal{Q}[d](m,n)\}_{G\in\mathfrak{G}}$ induced by ρ such that for each decorated graph $g \in G\langle \mathcal{P} \oplus \mathcal{Q}[d]\rangle$ either

- $\mu_G(g) = \mu_G^{\mathcal{P}}(g)$ if g is only decorated by elements of $\mathcal{P}$.
- $\mu_G(g) = \mu_G^{\mathcal{Q}}(\rho(g))$ if g exactly one vertex is decorated by an element of both $\mathcal{Q}$ and the rest decorated by elements of $\mathcal{P}$, where $\rho(g)$ is the map that applies ρ to all elements of $\mathcal{P}$.
- $\mu_G(g) = 0$ if g has two or more vertices decorated by elements of $\mathcal{Q}$.

A *derivaton of degree d of ρ* is a morphism of $\mathbb{S}$-bimodules $D : \mathcal{P} \to \mathcal{Q}$ such that the associated map $\mathrm{Id} + D : \mathcal{P} \to \mathcal{P} \oplus \mathcal{Q}[d]$ is a morphism of properads. Denote the space of derivations of ρ by $Der(\mathcal{P} \xrightarrow{\rho} \mathcal{Q})$. This is a dg vector space with the differential

$$\delta D = \delta_{\mathcal{Q}} \circ D - (-1)^{|D|} D \circ \delta_{\mathcal{P}}. \tag{2.5}$$

When $\mathcal{P} = \mathcal{F}\langle E\rangle$ for some $\mathbb{S}$-bimodule E, then a derivation is uniquely determined by the images of generators, hence

$$Der(\mathcal{F}\langle E\rangle \xrightarrow{\rho} \mathcal{Q}) \cong \prod_{m,n\geq 0} \mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}(E(m,n), \mathcal{Q}(m,n))$$

where $\mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}(E(m,n), \mathcal{E}(m,n))$ is the space of all $\mathbb{S}_m \times \mathbb{S}_n$-equivariant linear maps (of any degree). In the special case when $\mathcal{P} = \mathcal{Q}$ and $\rho = \mathrm{id}$, then $Der(\mathcal{P}) := Der(\mathcal{P} \xrightarrow{\mathrm{id}} \mathcal{P})$ is a dg Lie algebra with Lie bracket

$$[D_1, D_2](a) = D_1(D_2(a)) - (-1)^{|D_1||D_2|} D_2(D_1(a)).$$

If $\mathcal{P} = \mathcal{F}\langle E\rangle$, then

$$Der(\mathcal{F}\langle E\rangle) \cong \prod_{m,n\geq 0} \mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}(E(m,n), \mathcal{F}\langle E\rangle(m,n)).$$

2.3.2 Deformation complexes

The following theorem is from [MV].

Theorem 2.3.2. *Let $(\mathcal{P} = \mathcal{F}\langle E\rangle, \delta)$ be any dg (wheeled) properad generated by an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 0}$. Let $\mathcal{Q} = \{\mathcal{Q}(m,n)\}_{m,n\geq 0}$ be any dg (wheeled) properad. Let $\rho : \mathcal{P} \to \mathcal{Q}$ be a morphism of dg (wheeled) properads. Then*

1. *The graded vector space*

$$Def(\mathcal{P} \xrightarrow{\rho} \mathcal{Q}) := \prod_{m,n\geq 0} \mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}(E(m,n), \mathcal{Q}(m,n))[1]$$

where $\mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}(E(m,n), \mathcal{Q}(m,n))$ is the space of all $\mathbb{S}_m \times \mathbb{S}_n$-equivariant linear maps (of any degree). Its differential is the same as equation 2.5. This is a canonically filtered homotopy Lie-algebra called the deformation complex of the morphism f.

2. *If $\delta(E(m,n)) \subseteq \mathcal{F}^{\leq 2}\langle E\rangle \subseteq \mathcal{P}$, then $Def(\mathcal{P} \xrightarrow{f} \mathcal{Q})$ is a dg Lie algebra.*

Remark 2.3.3. In this thesis, we will always encounter case (2), and will mainly focus on the chain complex structure rather than the Lie-algebra structure. For full details, including definitions of filtered homotopy Lie structures, we refer to [MV].

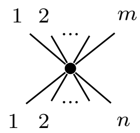
Remark 2.3.4. We have an isomorphism of complexes

$$Def(\mathcal{P} \xrightarrow{f} \mathcal{Q}) \cong \mathrm{Hom}(E, \mathcal{Q})[1] \cong Der(\mathcal{P} \xrightarrow{f} \mathcal{Q})[1].$$

This isomorphism does not, however, preserve the Lie bracket.

Example 2.3.5. Let us describe the complex $Def(\mathcal{P} \xrightarrow{f} \mathcal{Q})$ in the concrete example when $\mathcal{P} = \mathcal{Q} = \mathcal{H}olieb$ as seen in [MW1]. Then

$$Def(\mathcal{H}olieb_{c,d} \xrightarrow{id} \mathcal{H}olieb_{c,d}) = \prod_{\substack{m,n\geq 1\\ m+n\geq 3}} \mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}(\langle$$



$$\rangle, \mathcal{H}olieb_{c,d}(m,n))[1]$$

$$= (\mathcal{H}olieb_{c,d}(m,n) \otimes \mathrm{sgn}_m^{\otimes|c|} \otimes \mathrm{sgn}_n^{\otimes|d|})^{\mathbb{S}_m\times\mathbb{S}_n}[c(1-m)+d(1-n)]$$

where $\mathcal{H}olieb_{c,d}(m,n)$ is the subspace of oriented graphs with m out-hairs and n in-hairs. The differential acts on graphs by splitting a vertex into two vertices with an edge attached, and then summing over all possible reattachments of hairs and edges.

2.4 Graph complexes

2.4.1 The Kontsevich graph complex

Definition 2.4.1. Let $\overline{\mathrm{V}}_v\overline{\mathrm{E}}_e$cgra be the set of connected directed graphs with e edges and v vertices. The edges and vertices are labeled from 1 to e and 1 to v respectively. Both tadpoles and multiple edges are allowed in the graphs. Let $k \in \mathbb{Z}$ and let $\mathrm{V}_v\mathrm{E}_e\mathrm{GC}_k$ be the graded $\mathbb{K}$ vector space concentrated in degree $(v-1)k + (1-k)e$ generated by $\overline{\mathrm{V}}_v\overline{\mathrm{E}}_e$cgra. There is a natural right action of $\mathbb{S}_v \times \mathbb{S}_e \times \mathbb{S}_2^{\times e}$ on the vector space permuting the labels of vertices, permuting the labels of edges, and reversing the direction of an edge. The *full and connected Kontsevich graph complex* (cfGC_k, d) is the chain complex where

$$\mathsf{cfGC}_k := \begin{cases} \prod_{e,v} \left(\bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\mathrm{GC}_k \otimes \mathrm{sgn}_e\right)_{\mathbb{S}_v\times\mathbb{S}_e\times\mathbb{S}_2^{\times e}} & \text{for } k \text{ even,}\\ \prod_{e,v} \left(\bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\mathrm{GC}_k \otimes \mathrm{sgn}_v \otimes \mathrm{sgn}_2^{\otimes e}\right)_{\mathbb{S}_v\times\mathbb{S}_e\times\mathbb{S}_2^{\times e}} & \text{for } k \text{ odd.} \end{cases}$$

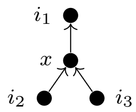
The subscript denotes taking the space of coinvariants under the group actions. The degree one differential d acts on graphs Γ as

$$d(\Gamma) := \delta(\Gamma) - \delta'(\Gamma) - \delta''(\Gamma) = \sum_{x\in V(\Gamma)} \delta_x(\Gamma) - \delta'_x(\Gamma) - \delta''_x(\Gamma)$$

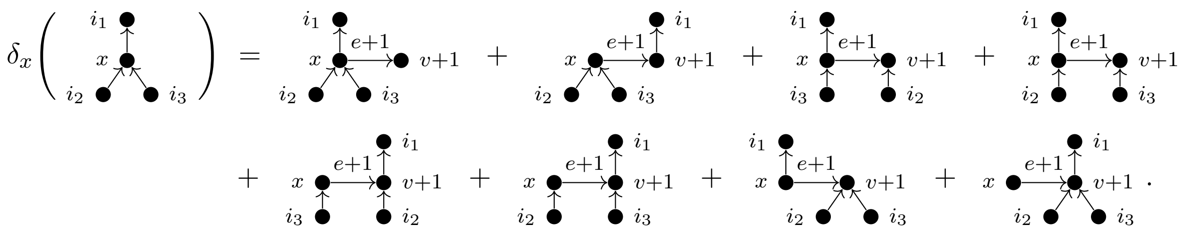
where $V(\Gamma)$ is the set of vertices of Γ, $\delta_x(\Gamma)$ is the sum of graphs where the vertex x has been replaced by two vertices with one edge between them, and the sum is over all reattachments of edges to these two vertices. The expression $\delta'_x(\Gamma)$ denotes the graph where an outgoing univalent vertex has been attached to x, and vice versa for $\delta''_x(\Gamma)$ with an incoming univalent vertex. The signs of the resulting graphs are determined so that the new edge is labeled $e+1$, the source vertex of the edge $e+1$ is labeled with the original vertex x, and the target vertex is labeled with $v+1$. Note that no univalent vertices are created under the action of the differential since any such graphs in δ_x and $\delta'_x + \delta''_x$ cancel each other.

Elements of cfGC_k can be viewed as equivalence classes of undirected graphs up to a sign depending on labelings on vertices or edges. When representing a graph, we normally pick a representative graph with fixed labelings on edges and vertices, as well as directions of edges.

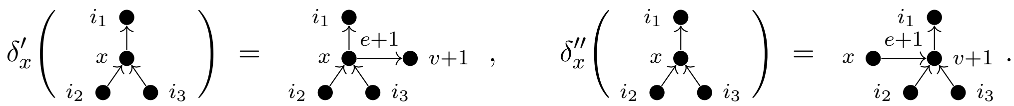
Example 2.4.2. Consider a vertex x with one outgoing edge and two incoming edges in some representative graph Γ in cfGC_k. Pictorially, we draw this as



where eventual vertices and edges not adjacent to x have been omitted. Then δ_x acts on Γ as



Similarly, δ'_x and δ''_x act on Γ as



Remark 2.4.3. The graph complex (fGC_k, d) of not necessarily connected graphs can be described in terms of the complex of connected graphs, so no information is lost by considering the smaller complex. More specifically, $\mathsf{cfGC}_k = S^+(\mathsf{fGC}_k[-k])[k]$, where $S^+(V)$ denotes the (completed) symmetric product space of the (dg) vector space V [W1].

Remark 2.4.4. From the definition of cfGC_k, we note that complexes of the same parity are isomorphic up to a degree shift. The only two crucial complexes to study are thus cfGC_2 and cfGC_3. As we will soon see, we know much more about the first one compared to the latter.

Definition 2.4.5. Let Γ be a graph with e edges and v vertices. The *loop number of* Γ is the integer $b = e - v + 1$.

Remark 2.4.6. The loop number of a graph is invariant under the differential.

The term *full* refers to that there is no restriction on which types of graphs we consider as generators. Let GC_k be the subcomplex of cfGC_k of graphs with no univalent vertices, and no bivalent vertices (except if the loop number of the graph is one). We call GC_k the *Kontsevich graph complex.* The inclusion

$$\mathsf{GC}_k \hookrightarrow \mathsf{cfGC}_k$$

is a quasi-isomorphism [W2]. The cohomology of the subcomplex $\mathsf{b}_1\mathsf{GC}_k$ of GC_k consisting of graphs with loop number one is fully described as

$$H(\mathsf{b}_1\mathsf{GC}_k) = \bigoplus_{\substack{i\geq 1 \\ i\equiv 2k+1 \mod 4}} \mathbb{K}[k-i]$$

where $\mathbb{K}[k-i]$ denotes the loop-graph containing i edges. The cohomology of GC_2 is only partially understood in negative degrees and degree zero. The following remarkable result was shown by T. Willwacher in [W1].

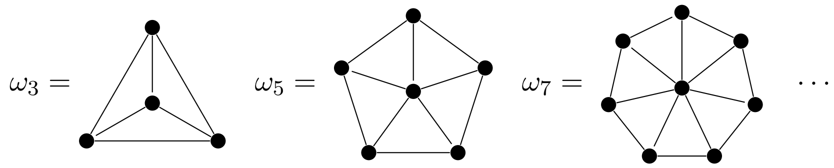
Theorem 2.4.7. *There is a Lie algebra structure on $H^0(\mathsf{GC}_2)$ that is isomorphic to the Grothendieck-Teichmuller Lie algebra $\mathfrak{grt}$.*

$$H^0(\mathsf{GC}_2) \cong \mathfrak{grt}$$

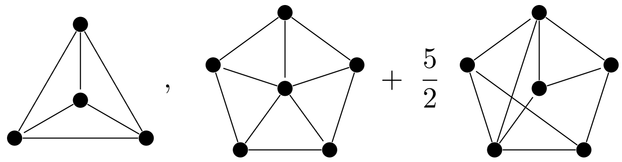
Furthermore the cohomology in negative degrees vanishes.

$$H^{<0}(\mathsf{GC}_2) = 0.$$

In the dual complex gc_k where the differential is defined by contracting edges, the zeroth homology contains the the wheel classes $\{\omega_{2n+1}\}_{n\in\mathbb{N}}$:



It is conjectured that these are the only classes of homology zero. In GC_2, the first wheel classes are represented by the following graphs:



We lack major results about the cohomology of GC_3 except that it is concentrated in negative degrees. This result can be derived from a more general statement using loop numbers.

Proposition 2.4.8. *Let $b \geq 2$ and $\mathsf{b}_b\mathsf{GC}_k$ be the subcomplex of GC_k of graphs with loop number b. Then $H^l(\mathsf{b}_b\mathsf{GC}_k) = 0$ for $(3-k)b - 3 < l$.*

Proof. The degree of a graph $\Gamma \in \mathsf{b}_b\mathsf{GC}_k$ can be rewritten as

$$|\Gamma| = (v-1)k - (k-1)e = (v-e-1)k + e = -bk + e.$$

Now $3v \leq 2e$ since the vertices of Γ are at least trivalent. Then we get the inequality

$$2b = 2e - 2v + 2 \geq v + 2 \implies v \leq 2b - 2.$$

Finally, the degree of $|\Gamma| \leq (3-k)b - 3$ since

$$|\Gamma| = -bk + e = -bk + (b + v - 1) = (1-k)b + v - 1 \leq (1-k)b + 2b - 3 = (3-k)b - 3.$$

Since b and k are independent from Γ, there are no graphs of such degrees in $\mathsf{b}_b\mathsf{GC}_k$. □

Corollary 2.4.9. $H^l(\mathsf{GC}_3) = 0$ *for* $l > -3$.

2.4.2 The directed graph complex and its subcomplexes

Definition 2.4.10. The *full and connected directed Kontsevich graph complex* (cfdGC_k, d) is the complex where

$$\mathsf{cfdGC}_k := \begin{cases} \prod_{e,v} \left(\bar{\mathrm{V}}_v \bar{\mathrm{E}}_e \mathrm{GC}_k \otimes \mathrm{sgn}_e\right)_{\mathbb{S}_v \times \mathbb{S}_e} & \text{for } k \text{ even,} \\ \prod_{e,v} \left(\bar{\mathrm{V}}_v \bar{\mathrm{E}}_e \mathrm{GC}_k \otimes \mathrm{sgn}_v\right)_{\mathbb{S}_v \times \mathbb{S}_e} & \text{for } k \text{ odd.} \end{cases}$$

The generators of this complex are equivalence classes of directed graphs. The differential is defined using the same formula $d(\Gamma) = \delta(\Gamma) - \delta'(\Gamma) - \delta''(\Gamma)$ in definition 2.4.1 and with the same sign conventions.

Remark 2.4.11. The directed graph complex similarly decomposes over loop numbers as $\mathsf{cfdGC}_k = \bigoplus_{i\geq 0} \mathsf{b}_i\mathsf{fdGC}_k$.

Definition 2.4.12. Let $\mathsf{b}_1\mathsf{dGC}_k \subseteq \mathsf{b}_1\mathsf{fdGC}_k$ be the subcomplex of graphs with no univalent vertices, and $\mathsf{b}_i\mathsf{dGC}_k \subseteq \mathsf{b}_i\mathsf{fdGC}_k$ the subcomplex of graphs with neither univalent nor passing vertices for $i \geq 2$. Set $\mathsf{dGC}_k = \bigoplus_{i\geq 1} \mathsf{b}_i\mathsf{dGC}_k$.

Proposition 2.4.13. *The inclusion $\mathsf{dGC}_k \hookrightarrow \mathsf{cfdGC}_k$ is a quasi-isomorphism.*

Proof. This was shown for the oriented graph complex in [Z1], but the same arguments apply here. □

Proposition 2.4.14. *Let $\mathsf{GC} \to \mathsf{dGC}_k$ be the chain morphism mapping an undirected graph Γ to the sum of directed graphs of the same underlying shape, summing over all possible directions on edges on Γ. Then this map is a quasi-isomorphism.*

Proof. See Appendix K of [W1]. □

2.4.3 Subcomplexes of dGC_k

There are several subcomplexes of dGC_k whose cohomology has been studied and shown to be related to other graph complexes. Here is an overview of the most important ones.

- The *oriented graph complex* oGC_k. This is the subcomplex of graphs that do not contain any closed paths of directed edges. M. Zivkovic found an explicit chain-map $\mathsf{dGC}_k \to \mathsf{oGC}_{k+1}$, which he also showed to be a quasi-isomorphism [Z1].
- The *sourced graph complex* dGC_k^s. It is the subcomplex of graphs that contain at least one source vertex. The inclusion $\mathsf{oGC}_k \hookrightarrow \mathsf{dGC}_k^s$ is a quasi-isomorphism [Z2].
- The *targeted graph complex* dGC_k^t. It is the subcomplex of graphs that contain at least one target vertex. It is naturally isomorphic to dGC_k^s by the map that reverses the direction of all edges of a graph. Similarly, the inclusion $\mathsf{oGC}_k \hookrightarrow \mathsf{dGC}_k^t$ is a quasi-isomorphism [Z2].
- The *sourced or targeted graph complex* dGC_k^{s+t}. It is the subcomplex of graphs with at least one source or one target vertex. We have that $H^l(\mathsf{dGC}_3^{s+t}) = 0$ for $l \leq 1$ [Z3].
- The *sourced and targeted graph complex* dGC_k^{st}. This is the subcomplex of graphs that contain at least one source and one target vertex. There is a short exact sequence

$$0 \longrightarrow \mathsf{dGC}_k^{st} \longrightarrow \mathsf{dGC}_k^s \oplus \mathsf{dGC}_k^t \longrightarrow \mathsf{dGC}_k^{s+t} \longrightarrow 0$$

$$\Gamma \longmapsto (\Gamma, \Gamma)$$

$$(\Gamma_1, \Gamma_2) \longmapsto \Gamma_1 - \Gamma_2$$

Since $H^l(\mathsf{dGC}_3^{s+t}) = 0$ for $l \leq 1$ one sees that $H^0(\mathsf{dGC}_3^{st}) = H^0(\mathsf{dGC}_3^s) \oplus H(\mathsf{dGC}_3^t)$ [Z3]. In particular, we get the remarkable result:

Corollary 2.4.15. $H^0(\mathsf{dGC}_3^{st}) = H^0(\mathsf{GC}_2) \oplus H^0(\mathsf{GC}_2) = \mathfrak{grt} \oplus \mathfrak{grt}$.

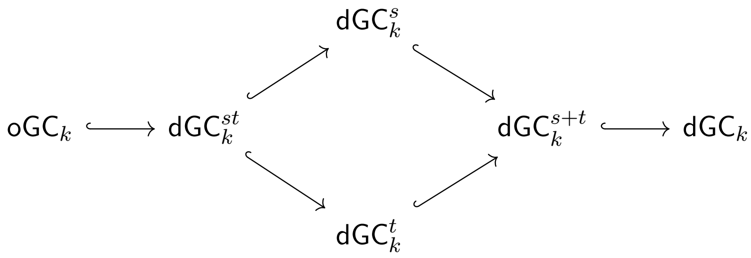


Figure 2.3: Diagram of the subcomplexes of dGC_k.

Additionally, we will be interested in the *completely wheeled graph complex* $\mathsf{dGC}_k^{\circlearrowleft}$. This complex is defined as the quotient complex $\mathsf{dGC}_k/\mathsf{dGC}_k^{s+t}$. Hence it consists of graphs where all vertices are at least trivalent, and each vertex has at least one incoming and one outgoing edge. We get the short exact sequence

$$0 \longrightarrow \mathsf{dGC}_k^{s+t} \longrightarrow \mathsf{dGC}_k \longrightarrow \mathsf{dGC}_k^{\circlearrowleft} \longrightarrow 0$$

From this short exact sequence, we can derive the following result.

Lemma 2.4.16. *$H^l(\mathsf{dGC}_3^{\circlearrowleft}) = 0$ for $-2 \leq l \leq 1$.*

Proof. This follows by noting that $H^l(\mathsf{dGC}_3^{s+t}) = 0$ for $l \leq 1$ and $H^l(\mathsf{dGC}_3) = 0$ for $l \geq -2$. □

26

Chapter 3

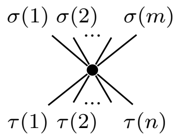
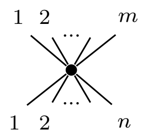
Derivations of Lie bialgebras

In this chapter we compute the cohomology of the two derivation complexes $Der(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$ and $Der^{\bullet}(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$ by establishing explicit quasi-isomorphisms to directed Kontsevich graph complexes. We end the chapter by explicitly computing two cohomology classes of $H^0(Der(\mathcal{H}olieb^{\circlearrowleft}_{1,1}))$. The content of this chapter is largely based on the article *Deformation theory of the wheeled properad of strongly homotopy Lie bialgebras and graph complexes.*

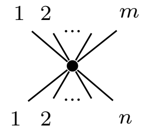
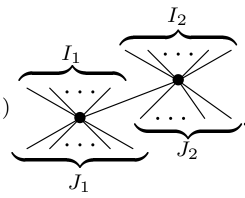
3.1 Derivation complexes of $\mathcal{H}olieb^{\circlearrowleft}_{c,d}$

Definition 3.1.1. Let $\mathcal{H}olieb^{\bullet}_{c,d}$ be the free dg properad generated by the $\mathbb{S}$-bimodule $Lb^{\bullet}_{c,d} = \{Lb^{\bullet}_{c,d}(m,n)\}_{m,n\geq 0}$ where

$$Lb^{\bullet}_{c,d}(m,n) = \mathrm{sgn}_m^{\otimes c} \otimes \mathrm{sgn}_n^{\otimes d}[c(m-1)+d(n-1)-1]$$

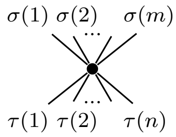
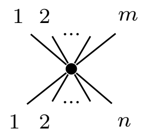
$= \Big\langle$  $= (-1)^{c|\sigma|+d|\tau|}$  $\mid \sigma \in \mathbb{S}_m,\ \tau \in \mathbb{S}_n \Big\rangle_{m,n\geq 0}$

and whose differential acts on generators as

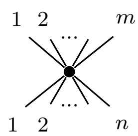
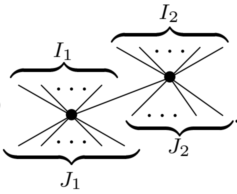
$\delta\Big($  $\Big) = \sum_{[m]=I_1\sqcup I_2} \sum_{[n]=J_1\sqcup J_2} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$  .

Definition 3.1.2. Let $\mathcal{H}olieb^{+}_{c,d}$ be the free dg properad generated by the $\mathbb{S}$-bimodule $Lb^{+}_{c,d} = \{Lb^{+}_{c,d}(m,n)\}_{m,n\geq 0}$ where

$$Lb^{+}_{c,d}(m,n) = \mathrm{sgn}_m^{\otimes c} \otimes \mathrm{sgn}_n^{\otimes d}[c(m-1)+d(n-1)-1]$$

$= \Big\langle$  $= (-1)^{c|\sigma|+d|\tau|}$  $\mid \sigma \in \mathbb{S}_m,\ \tau \in \mathbb{S}_n \Big\rangle_{m,n\geq 1}$

and whose differential acts on generators as

$\delta\Big($  $\Big) = \sum_{\substack{[m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$  .

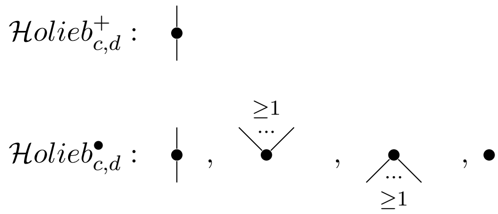


Figure 3.1: Additional generators of $\mathcal{H}olieb^{+}_{c,d}$ and $\mathcal{H}olieb^{\bullet}_{c,d}$ compared to $\mathcal{H}olieb_{c,d}$

Remark 3.1.3. The definition of these complexes only differs from that of $\mathcal{H}olieb_{c,d}$ in that they have additional generators (see Figure 3.1) and that their differentials are modified accordingly. Furthermore, there are natural projections

$$\pi^{\bullet} : \mathcal{H}olieb^{\bullet}_{c,d} \twoheadrightarrow \mathcal{H}olieb_{c,d} \text{ and } \pi^{+} : \mathcal{H}olieb^{+}_{c,d} \twoheadrightarrow \mathcal{H}olieb_{c,d}$$

which are dg morphisms. The morphism $\pi^{\bullet}$ factors through π^{+} as

$$\mathcal{H}olieb^{\bullet}_{c,d} \twoheadrightarrow \mathcal{H}olieb^{+}_{c,d} \twoheadrightarrow \mathcal{H}olieb_{c,d}.$$

Definition 3.1.4. Let the properad $\widehat{\mathcal{H}olieb}_{c,d}$ be the *loop number completion* of $\mathcal{H}olieb_{c,d}$. This properad has a complete topology such that the derivations are considered as continuous. We use the same notation for the loop number completion of other properads.

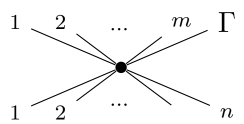
Definition 3.1.5. The complex $Der^{\bullet}(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$ is the derivation complex with respect to the morphism

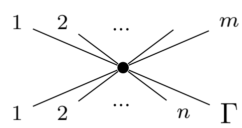
$$\pi^{\bullet,\circlearrowleft} : \widehat{\mathcal{H}olieb}^{\bullet,\circlearrowleft}_{c,d} \twoheadrightarrow \widehat{\mathcal{H}olieb}^{\circlearrowleft}_{c,d}$$

induced by $\pi^{\bullet}$. Similarly, let $Der(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$ be the derivation complex with respect to the morphism

$$\pi^{+,\circlearrowleft} : \widehat{\mathcal{H}olieb}^{+,\circlearrowleft}_{c,d} \twoheadrightarrow \widehat{\mathcal{H}olieb}^{\circlearrowleft}_{c,d}$$

induced by π^{+}. The differential d on both complexes is given by the vertex splitting differential d^{spl} from $\mathcal{H}olieb_{c,d}$ with the additional terms of attaching (m,n) corollas to every hair for all integers m,n:

$$d\Gamma = d^{spl}\Gamma \pm \sum_{m,n}$$


$$\mp \sum_{m,n}$$


(3.1)

The sign rule for this formula can be found in [MW1].

Remark 3.1.6. Since the properads of the derivation complexes are free, they can be described as

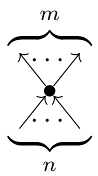
$$Der^{\bullet}(\mathcal{H}olieb^{\circlearrowleft}_{c,d}) \cong \prod_{m,n\geq 0} (\mathcal{H}olieb^{\circlearrowleft}_{c,d}(m,n) \otimes sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|})^{\mathbb{S}_m\times\mathbb{S}_n}[1+c(1-m)+d(1-n)]$$
$$Der(\mathcal{H}olieb^{\circlearrowleft}_{c,d}) \cong \prod_{m,n\geq 1} (\mathcal{H}olieb^{\circlearrowleft}_{c,d}(m,n) \otimes sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|})^{\mathbb{S}_m\times\mathbb{S}_n}[1+c(1-m)+d(1-n)].$$

where $\mathcal{H}olieb^{\circlearrowleft}_{c,d}(m,n)$ is the set of generating graphs of $\mathcal{H}olieb^{\circlearrowleft}_{c,d}$ with m outputs and n inputs.

The loop number of a graph in any of the derivation complexes remains invariant under the differential, and so the complex splits over loop numbers. The components of graphs with loop number zero in both complexes are identical, and we denote this complex by $Der_{b=0}(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$

Theorem 3.1.7. *The cohomology of* $Der_{b=0}(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$ *is generated by the series of single vertex graphs*

$$\sum_{\substack{m,n\geq 1\\ m+n\geq 3}} (m+n-2)$$



Proof. The component $Der_{b=0}(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$ is isomorphic to the component of graphs with loop number zero in the derivation complex $Der(\mathcal{H}olieb^{\uparrow}_{c,d})$ of the unwheeled properads. The cohomology of the latter was computed in [MW1] to be generated by the series of graphs above. □

3.2 The bi-weighted graph complex

When studying the complex $Der^{\bullet}(\mathcal{H}olieb^{*\circlearrowleft}_{c,d})$, one notes that the sign rules of the differential are rather complicated when expressed in terms of generating corollas using the defining formula (3.1). In this section we introduce the *bi-weighted graph complex* fwGC_k, and show that it is isomorphic to $Der^{\bullet}(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$. The sign rules of fwGC_k are the same as for the Kontsevich graph complexes, i.e., based on ordering of edges or vertices, and are generally easier to work with.

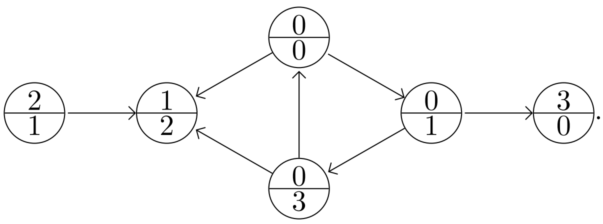
3.2.1 Definition of the bi-weighted graph complex

Let Γ be a directed graph and x a vertex of Γ. Let $|x|_{out}$ denote the number of outgoing edges from x, and similarly let $|x|_{in}$ denote the number of incoming edges to x. A *bi-weight* of a vertex x is a pair of non-negative integers (w^{out}_x, w^{in}_x) satisfying

1. $w^{out}_x + |x|_{out} \geq 1$,
2. $w^{in}_x + |x|_{in} \geq 1$,
3. $w^{out}_x + w^{in}_x + |x|_{out} + |x|_{in} \geq 3$.

We refer to w^{out}_x as the *out-weight* and w^{in}_x the *in-weight* of x respectively. A graph whose vertices all carry bi-weights is called a *bi-weighted* graph.

Example 3.2.1. We include the bi-weights of a vertex when drawing a bi-weighted graph as $\frac{w^{out}_x}{w^{in}_x}$. A bi-weighted graph might then look like



Let $\overline{\mathrm{V}}_v\overline{\mathrm{E}}_e\mathrm{wcgra}$ be the set of connected bi-weighted graphs with e edges and v vertices labeled from 1 to e and from 1 to v respectively. Let $k \in \mathbb{Z}$ and let

$$\overline{\mathrm{V}}_v\overline{\mathrm{E}}_e\mathrm{wGC}_k := \langle \overline{\mathrm{V}}_v\overline{\mathrm{E}}_e\mathrm{wcgra}\rangle[-(v-1)k-(1-k)e]$$

be the graded vector space over $\mathbb{K}$ of formal power series of bi-weighted graphs concentrated in degree $(v-1)k+(1-k)e$. We have an action of $\mathbb{S}_v \times \mathbb{S}_e$ on the vector space by permuting labels of vertices and edges respectively.

Definition 3.2.2. The *bi-weighted graph complex* (fwGC_k, d) is the chain complex where

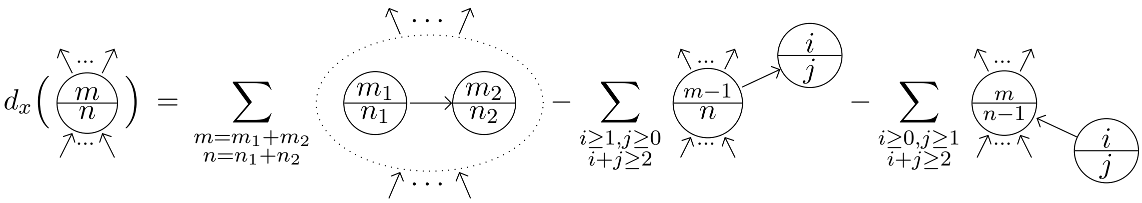
$$\mathsf{fwGC}_k := \begin{cases} \prod_{v,e}\left(\bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\mathrm{WGC}_d \otimes \mathrm{sgn}_e\right)_{S_v\times S_e} & \text{for } k \text{ even,} \\ \prod_{v,e}\left(\bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\mathrm{WGC}_d \otimes \mathrm{sgn}_v\right)_{S_v\times S_e} & \text{for } k \text{ odd.} \end{cases}$$

The differential d is defined on a graph Γ as

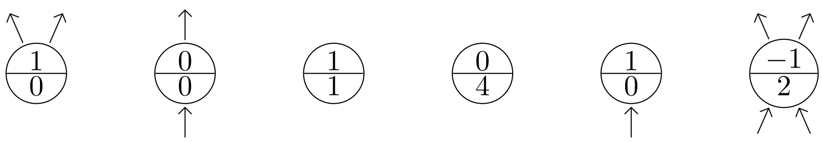
$$d(\Gamma) := \delta(\Gamma) - \delta'(\Gamma) - \delta''(\Gamma) = \sum_{x\in V(\Gamma)} \delta_x(\Gamma) - \delta'_x(\Gamma) - \delta''_x(\Gamma)$$

where $V(\Gamma)$ is the set of vertices of Γ, $\delta_x(\Gamma)$ denotes the splitting of the vertex x similar to the vertex splitting in dGC_k with the addition that we sum over all possible ways of redistributing the bi-weight of x to the two new vertices. The $\delta'_x(\Gamma)$ is the sum of graphs where the out-weight of x is decreased by one and has one outgoing edge added being attached to a new univalent vertex. The summation is over all possible bi-weights on the new vertex. The $\delta''_x(\Gamma)$ is defined similarly but where the in-weight is decreased by one and the edge is incoming to x. The signs of the graphs under the action of the differential are the same as for δ, δ', and δ'' in dGC_k.

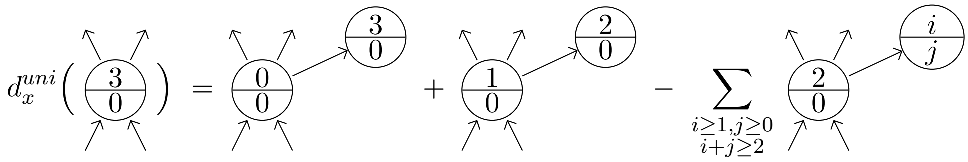
We pictorially represent the action of d_x on a vertex x as



We tacitly assume in this formula that any term (if any) with negative in- or out-weight is set to zero. Similarly, any graph with vertices of a bi-weight not satisfying condition 3 above is also set to zero. Some examples of vertices with invalid bi-weights are



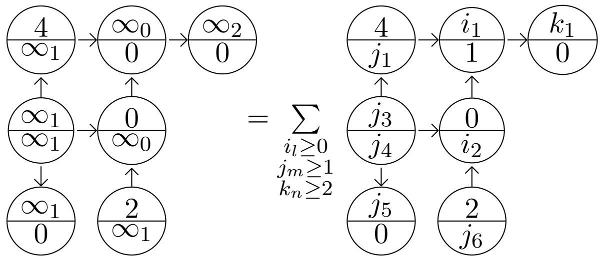
Remark 3.2.3. Contrary to dGC_k, the creation of new univalent vertices of graphs in fwGC_k do not in general cancel under the action of the differential. If d_x^{uni} is the part of the differential that increases the number of univalent vertices, then d_x^{uni} acting on a vertex can look like



3.2.2 A special kind of bi-weight

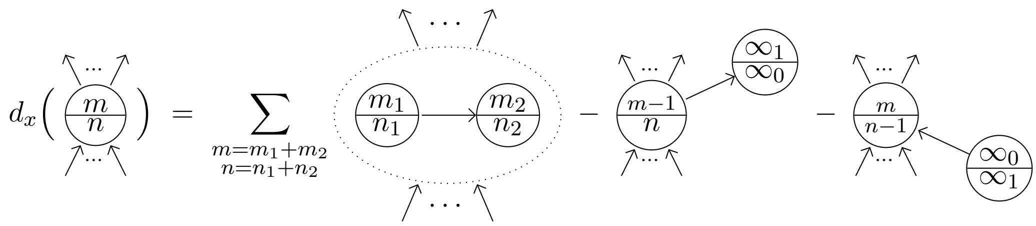
Definition 3.2.4. Let $r \geq 0$ be an integer. The symbol ∞_r when used as an in-weight or out-weight denotes the sum of graphs

For graphs with two or more of these symbols decorating vertices, the sum is distributed as in the example below:



Any term of such a sum containing a vertex of invalid bi-weight is set to zero.

Using this convention, the differential is described as



Remark 3.2.5. New univalent target vertices cancel under the differential when the out-weight of a vertex is 0 or ∞_1, and similarly for new univalent source vertices.

3.2.3 The bi-weighted graph complex and the deformation complex

Definition 3.2.6. Let fwGC_k^+ be the subcomplex of fwGC_k generated by graphs having at least one vertex with an out-weight greater than zero, and at least one vertex an with in-weight greater than zero. These two vertices are allowed to be the same vertex.

The complex fwGC_k is constructed to be isomorphic to the derivation complex $Der^\bullet(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$. We interpret the bi-weight of a vertex as the number of outgoing and incoming hairs that are attached to it. We define the map

$$F : Der^\bullet(\mathcal{H}olieb_{c,d}^{\circlearrowleft}) \to \mathsf{fwGC}_{c+d+1}$$

where a graph Γ with unlabeled hairs (up to symmetry/skew-symmetry) is mapped to the bi-weighted graph $F(\Gamma)$ of the same shape and where the bi-weights of vertices correspond to the number of in- and out-hairs of the vertices in Γ.

Proposition 3.2.7. *The map* $F : Der^\bullet(\mathcal{H}olieb_{c,d}^{\circlearrowleft}) \to \mathsf{fwGC}_{c+d+1}$ *is a chain map of degree 0 such that*

1. *the map F is an isomorphism of complexes,*
2. *the map F restricts to an isomorphism* $F^+ : Der(\mathcal{H}olieb_{c,d}^{\circlearrowleft}) \to \mathsf{fwGC}_{c+d+1}^+$.

Proof. Proving that F is an isomorphism is done by inspection. We do, however, need to show that F is of degree 0. Recall that the derivation complex decomposes as

$$Der^\bullet(\mathcal{H}olieb_{c,d}^{\circlearrowleft}) = \prod_{m,n\geq 0} \left(\mathcal{H}olieb_{c,d}^{\circlearrowleft}(m,n) \otimes sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|}\right)^{\mathbb{S}_m\times\mathbb{S}_n}[1+c(m-1)+d(n-1)]$$

where $\mathcal{H}olieb_{c,d}^{\circlearrowleft}(m,n)$ is the vector space of graphs with m out-hairs and n in-hairs. Let Γ be a graph in the derivation complex. For each vertex x of Γ, let $|x|_{out}$ denote the number of outgoing half-edges and $|x|_{in}$ the number of incoming half-edges. Then

$$\begin{aligned}
|\Gamma| &= \sum_{x\in V(\Gamma)} \big(1-(|x|_{out}-1)-(|x|_{in}-1)\big) - 1 - c(1-m) - d(1-n)\\
&= \sum_{x\in V(\Gamma)} \big(1+c+d\big) - (1+c+d) - \sum_{x\in V(\Gamma)} \big(c|x|_{out}+m|x|_{in}\big) + cm + dn\\
&= |V(\Gamma)-1|(c+d+1) - |E(\Gamma)|(c+d) = |F(\Gamma)|
\end{aligned}$$

One remarks that F^+ is a bijection by noting that bi-weights are symbolizing hairs and $Der(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$ can be seen as generated by graphs with at least one out-hair and one in-hair attached. □

3.2.4 Decomposition over decorations and loop numbers

Let fwGC_k^0 be the subcomplex of fwGC_k generated by graphs whose vertices are only decorated by $\frac{0}{0}$, and fwGC_k^* its complement. Then the complex fwGC_k split as

$$\mathsf{fwGC}_k = \mathsf{fwGC}_k^0 \oplus \mathsf{fwGC}_k^*.$$

Proposition 3.2.8. *Let* $\mathsf{dGC}_k^{\geq 3,\circlearrowleft} \subseteq \mathsf{dGC}_k^{\circlearrowleft}$ *be the subcomplex of graphs with all vertices at least trivalent. Then the complex* fwGC_k^0 *is isomorphic to* $\mathsf{dGC}_k^{\geq 3,\circlearrowleft}$ *of graphs with neither sources nor targets.*

Proof. By direct inspection of the graphs, where all vertices can be decorated by $\frac{0}{0}$, one easily sees that they need to be at least trivalent and have at least one incoming and one outgoing vertex. This corresponds to the graphs in $\mathsf{dGC}_k^{\geq 3,\circlearrowleft}$. One also notes that the differentials act in the same manner. □

Recall that the loop number of a graph is preserved under the differential. Consider the decompositions

$$\begin{aligned}
\mathsf{fwGC}_k &= \mathsf{b_0wGC}_k \oplus \mathsf{wGC}_k\\
\mathsf{fwGC}_k^* &= \mathsf{b_0wGC}_k^* \oplus \mathsf{wGC}_k^*\\
\mathsf{fwGC}_k^+ &= \mathsf{b_0wGC}_k^+ \oplus \mathsf{wGC}_k^+
\end{aligned}$$

where $\mathsf{b_0wGC}_k, \mathsf{b_0wGC}_k^*$ and $\mathsf{b_0wGC}_k^+$ are the subcomplexes of graphs with loop number zero and wGC_k, wGC_k^* and wGC_k^+ the subcomplex of graphs with loop number one and higher. We note that graphs with loop number zero cannot be completely bald, and so $\mathsf{b_0wGC}_k = \mathsf{b_0wGC}_k^*$. Further note that there are no closed loops in a graph with loop number zero, and so they contain at least one source and one target vertex. These vertices must have positive in-weight and out-weight respectively, and so $\mathsf{b_0wGC}_k = \mathsf{b_0wGC}_k^+$.

Proposition 3.2.9. *The cohomology of the complex of graphs with loop number zero* $\mathsf{b_0wGC}_k$ *is generated by the series*

$$\sum_{\substack{i,j\geq 1\\ i+j\geq 3}} (i+j-2) \bigcirc\!\!\!\!\!\!\tfrac{i}{j} \;\; .$$

Proof. These graphs correspond to the part of the deformation complex of graphs with loop number zero, whose cohomology was computed to be the counterpart of this graph in Theorem 3.1.7. □

3.3 Special in-vertices and special out-vertices

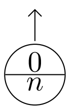
In this section we will define three subcomplexes $\mathsf{qGC}_k \subset \mathsf{wGC}_k$, $\mathsf{qGC}_k^* \subset \mathsf{wGC}_k^*$ and $\mathsf{qGC}_k^+ \subset \mathsf{wGC}_k^+$ consisting of graphs whose vertices are decorated by four types of decorations: $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$ and $\frac{0}{0}$. The main goal is to show that these inclusions are quasi-isomorphisms. We do this by considering two consecutive filtrations on the complexes over special-in and special-out vertices respectively. We then show that the associated spectral sequences agree on some page.

3.3.1 Convergence of filtrations and their associated spectral sequences

In this paper, we will consider many arguments where we consider the filtration of a chain complex and then study the associated spectral sequences. All the spectral sequences we construct in this way will converge. To see this, we do a similar trick of shifting the degrees of the complexes as seen in [W1]. Let the new degree of a graph be $k(v-1)-(k-1)e+(k-\frac{1}{2})(e-v) = \frac{1}{2}(v+e)-k$. The cohomology of both complexes agrees up to degree shifts. Further, any filtration in the old grading corresponds to a filtration with the new grading. We see that the number of underlying directed graphs of the bi-weighted graphs contained in each degree is finite. Any filtration we do consider will be over the number of vertices of certain types, and so the filtration will be bounded and hence converge to the desired cohomology. The cohomology of the original complex is then acquired from the shifted version.

3.3.2 Filtration over special in-vertices

Definition 3.3.1. Let Γ be a bi-weighted graph. A vertex x of Γ is a *special in-vertex* if

i) either x is a univalent vertex with one outgoing edge and out-weight zero, i.e on the form 

ii) or x becomes a univalent vertex of type i) after recursive removal of all special-in vertices of type i) from Γ (see figure 3.2).

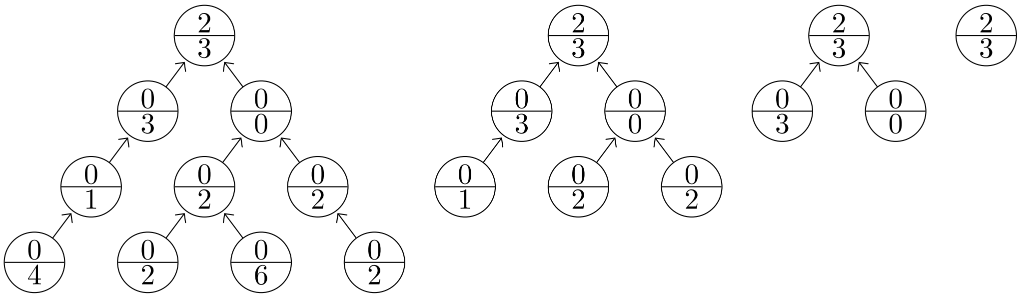


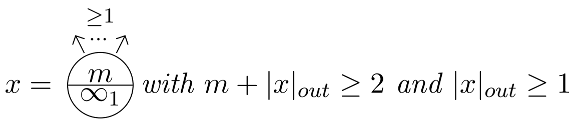
Figure 3.2: Example of recursive removal of special-in vertices of the rightmost graph. All vertices except the top one are special in-vertices.

Any vertex that is not a special in-vertex is called an *in-core vertex*. The special in-vertices of a graph form trees with a flow towards some *in-core vertex* and where the trees have no out-hairs. Given an arbitrary graph Γ in wGC_k, we define the associated *in-core graph* γ as the one spanned by in-core vertices with their in-weight forgotten. Then every vertex x in γ has three integer parameters associated to it, $|x|_{out}$, $|x|_{in}$ and w_x^{out}. Note that every graph contains at least one in-core vertex, and so every graph has an associated in-core graph.

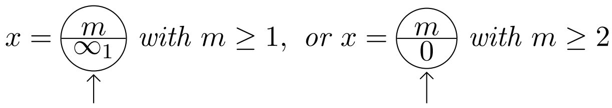
Consider the filtration of wGC_k over the number of in-core vertices, i.e., the sequence of complexes $\mathsf{wGC}_k = \mathcal{F}_0\mathsf{wGC}_k \subseteq \mathcal{F}_1\mathsf{wGC}_k \subseteq \mathcal{F}_2\mathsf{wGC}_k \subseteq ...$ where $\mathcal{F}_i\mathsf{wGC}_k$ is the subcomplex of wGC_k generated by graphs containing i or more special in-vertices. Let $\{S_r^{in}\mathsf{wGC}_k\}_{r\geq 0}$ be the associated spectral sequence.

Proposition 3.3.2. *The page one complex $S_1^{in}\mathsf{wGC}_k$ is generated by directed graphs Γ whose vertices $V(\Gamma)$ are independently decorated by one out-weight m and with two possible symbols 0 or ∞_1 as in-decorations, subject to the following conditions:*

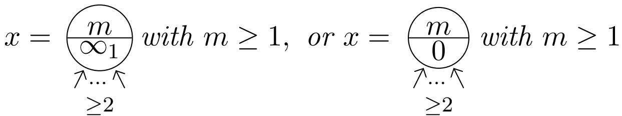
1. *If $x \in V(\Gamma)$ is a source, then*



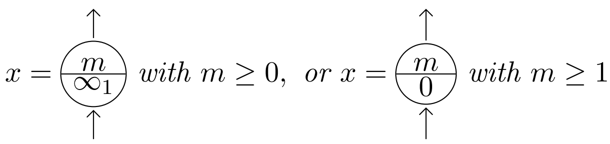
2. *If $x \in V(\Gamma)$ is a target with precisely one in-edge, then*



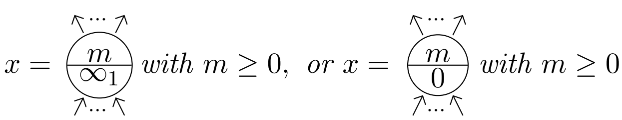
3. *If $x \in V(\Gamma)$ is a target with at least two in-edges, then*



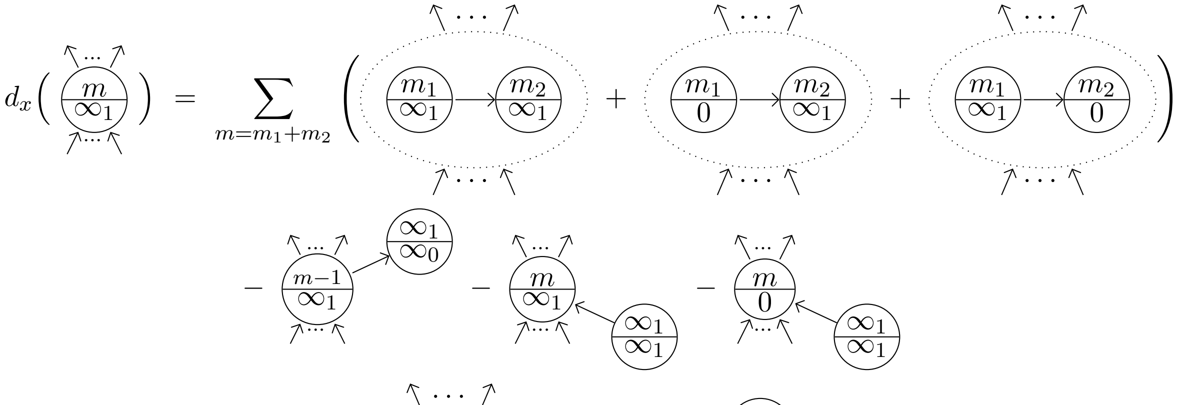
4. *If $x \in V(\Gamma)$ is passing (one in-edge and one out-edge), then*

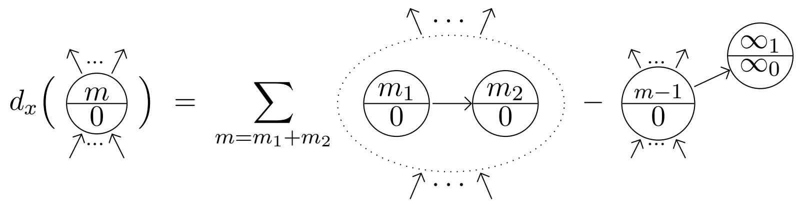


5. *If $x \in V(\Gamma)$ is of none of the types above (i.e., x is at least trivalent and has at least one in-edge and at least one out-edge), then*



The differential acts on a graph $\Gamma \in S_1^{in}\mathsf{wGC}_k$ with vertices of the types (1)-(5) above as $d(\Gamma) = \sum_{x\in V(\Gamma)} d_x(\Gamma)$. The map d_x acts on vertices with in-weight ∞_1 and 0 respectively *as*





Any term on the right-hand side containing at least one vertex not of the type (1) − (5) *is set to zero.*

The rule of the differential follows from the bi-weight types of the vertices mentioned above. So we only need to show that the cohomology of $S_0^{in}\mathsf{wGC}_k$ is generated by graphs with such bi-weights. The differential acts on the initial page $S_0^{in}\mathsf{wGC}_d$ by only creating special-in vertices and leaves the connected in-core graph unchanged. Hence the complex decomposes into a direct sum parameterized by the set of all possible in-core graphs

$$S_0^{in}\mathsf{wGC}_k \cong \bigoplus_{\gamma} \mathsf{inCore}(\gamma)$$

where $\mathsf{inCore}(\gamma)$ is the subcomplex of $S_0^{in}\mathsf{wGC}_k$ of all graphs whose associated in-core graph is γ. The complex $\mathsf{inCore}(\gamma)$ decomposes further into a tensor product of complexes

$$\mathsf{inCore}(\gamma) \cong \Big(\bigotimes_{x\in V(\gamma)} \mathcal{T}_x^{in} \Big)^{\mathrm{Aut}(\gamma)}$$

with one complex $\mathcal{T}_x^{in}$ for each x in $V(\gamma)$ and where $\mathrm{Aut}(\gamma)$ is the group of automorphisms of γ acting by permuting the complexes of the tensor complex to preserve signs. Each complex $\mathcal{T}_x^{in}$ consists of trees of special in-vertices attached to an in-core vertex x and the differential acts by only creating special in-vertices on x and in the trees. The complexes $\mathcal{T}_x^{in}$ depend on x only via the number of outgoing and incoming edges attached to x in the in-core graph as well as on the out-weight w_x^{out}. The in-weight of x is not fixed. The complexes $\mathcal{T}_x^{in}$ which have the same values of the parameters $|x|_{out}$, $|x|_{in}$ and w_x^{out} are isomorphic to each other, so we will often write $\mathcal{T}_v^{in} \cong \mathcal{T}_{|x|_{out},|x|_{in},w_x^{out}}^{in}$. So we have to study a family of complexes $\mathcal{T}_{a,b,c}^{in}$ parameterized by integers $a,b,c \geq 0$ such that $a+c \geq 1$ and $(a,b,c) \neq (1,0,0)$. The first condition guarantees that the in-core vertex has at least one out-edge or out-weight, and the second condition corresponds to the invalid configuration where the in-core vertex would be a special-in vertex. The in-weight of the core vertex in $\mathcal{T}_{a,b,c}^{in}$ is any number w_x^{in} satisfying the condition $w_x^{in} + b + \#(\text{in-edges from special in-vertices}) \geq 1$ and $w_x^{in} + a + b + c + \#(\text{in-edges from special in-vertices}) \geq 3$. Proposition 3.3.2 now follows from the following lemma.

Lemma 3.3.3. *The cohomology of $\mathcal{T}^{in}_{a,b,c}$ is generated by one or two classes containing the in-core vertex decorated by some bi-weights depending on the parameters a, b, c. More precisely*

$$\text{For } a \geq 1 \quad H(\mathcal{T}^{in}_{a,0,c}) = \left\langle \left(\tfrac{c}{\infty_1}\right) \right\rangle, \quad \text{when } c \geq 0 \text{ and } a + c \geq 2$$

$$H(\mathcal{T}^{in}_{0,1,c}) = \begin{cases} \left\langle \left(\tfrac{c}{\infty_1}\right) \right\rangle & \text{if } c = 1 \\ \left\langle \left(\tfrac{c}{\infty_1}\right), \left(\tfrac{c}{0}\right) \right\rangle & \text{if } c \geq 2 \end{cases}$$

$$H(\mathcal{T}^{in}_{1,1,c}) = \begin{cases} \left\langle \left(\tfrac{c}{\infty_1}\right) \right\rangle & \text{if } c = 0 \\ \left\langle \left(\tfrac{c}{\infty_1}\right), \left(\tfrac{c}{0}\right) \right\rangle & \text{if } c \geq 1 \end{cases}$$

$$\text{For } b \geq 2 \quad H(\mathcal{T}^{in}_{0,b,c}) = \left\langle \left(\tfrac{c}{\infty_1}\right), \left(\tfrac{c}{0}\right) \right\rangle, \quad \text{when } c \geq 1$$

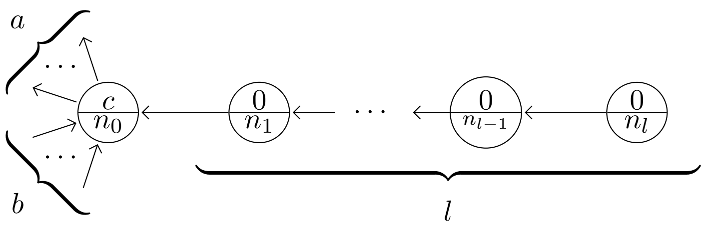
$$\text{For } a, b \geq 1 \text{ and } a + b \geq 3 \quad H(\mathcal{T}^{in}_{a,b,c}) = \left\langle \left(\tfrac{c}{\infty_1}\right), \left(\tfrac{c}{0}\right) \right\rangle, \quad \text{when } c \geq 0$$

The differential in $\mathcal{T}^{in}_{a,b,c}$ acts on graphs such that the number of univalent special in-vertices stays the same or is increased. Consider the filtration on $\mathcal{T}^{in}_{a,b,c}$ by the number of univalent vertices (considering the graph consisting of only the root vertex v as having one univalent vertex). Let $gr(\mathcal{T}^{in}_{a,b,c})$ be the associated graded complex. It decomposes as $gr(\mathcal{T}^{in}_{a,b,c}) = \bigoplus_{N\geq 1} u_N \mathcal{T}^{in}_{a,b,c}$ where $u_N \mathcal{T}^{in}_{a,b,c}$ is spanned by trees with precisely N univalent special in-vertices. Lemma 3.3.3 follows from the following results:

Lemma 3.3.4. *The cohomology $H(u_1 \mathcal{T}^{in}_{a,b,c})$ is generated by the same elements given in Lemma 3.3.3.*

Lemma 3.3.5. *The complex $u_N \mathcal{T}^{in}_{a,b,c}$ is acyclic for $N \geq 2$.*

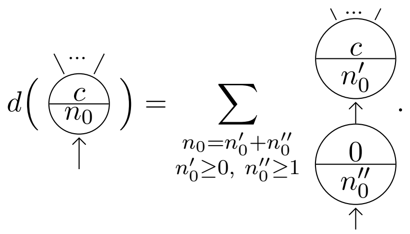
Proof of Lemma 3.3.4: We start by computing the cohomology of $u_1 \mathcal{T}^{in}_{a,b,c}$. The graphs in this complex are on the form



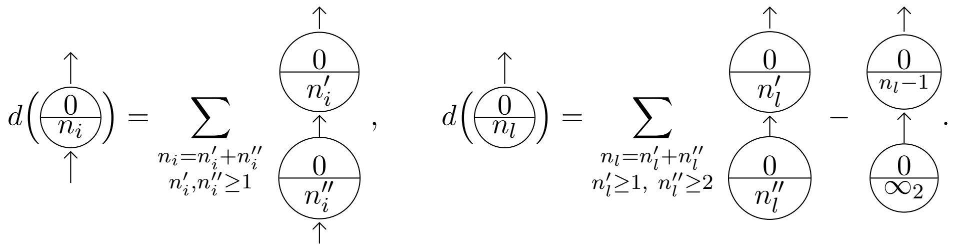
with $l \geq 0$. If $l = 0$, then $n_0 \geq 1$ when (a, b, c) is either of the three cases $(0, 1, 1)$, $(1, 0, 0)$ or $(a, 0, c)$ for $a \geq 1$ and $c \geq 0$. In any other case for (a, b, c) we have that $n_0 \geq 0$. If $l \geq 1$, then $n_0 \geq 0$, $n_i \geq 1$ for $1 \leq i \leq l - 1$ and $n_l \geq 2$. If $l = 0$, the induced differential acts on the root vertex as

$$d\left(\left(\tfrac{c}{n_0}\right)\right) = \sum_{\substack{n_0 = n_0' + n_0'' \\ n_0' \geq 0,\ n_0'' \geq 2}} \begin{array}{c} \left(\tfrac{c}{n_0'}\right) \\ \uparrow \\ \left(\tfrac{0}{n_0''}\right) \end{array} - \begin{array}{c} \left(\tfrac{c}{n_0 - 1}\right) \\ \uparrow \\ \left(\tfrac{0}{\infty_2}\right) \end{array}$$

and when $l \geq 1$, it acts on the root vertex as



Also for $l \geq 1$ there are passing vertices and a univalent vertex in the graph. The differential acts on these vertices as



We can already identify the first cohomology classes generated by the graph consisting of one core vertex on the form $\left(\frac{c}{0}\right)$ (whenever this in-weight is possible). The remaining graphs to study are then on the form above with $n_0 \geq 1$ if $l = 0$, and $n_0 \geq 0, n_1, ..., n_{l-1} \geq 1$ and $n_l \geq 2$ if $l \geq 1$. Let $(\mathcal{I}, d)$ be the complex generated by these graphs (i.e., the complex where we exclude the graphs $\left(\frac{c}{0}\right)$ from $u_1\mathcal{T}^{in}_{a,b,c}$ when present). It is easy to verify that that $\left(\frac{c}{\infty_1}\right)$ is a cycle in $(\mathcal{I}, d)$. The lemma follows if we can show that the cohomology of $\mathcal{I}$ is one-dimensional. There is a filtration of $\mathcal{I}$ over the total in-weight of a graph, as the total in-weight can only remain invariant or increase under the action of the differential. Let $\{T_r\mathcal{I}\}_{r\geq 0}$ be the associated spectral sequence. The following two results conclude the proof. □

Lemma 3.3.6. *The cohomology of* $(T_0\mathcal{I}, d)$ *is generated by the graph consisting of a single in-core vertex with in-weight* 1. *That is* $H(\mathcal{I}, \delta) = \left\langle \left(\frac{c}{1}\right) \right\rangle$.

Proof. We construct an isomorphic complex using the bar and cobar-construction. Let $V = V_{-1} \oplus V_0$ be a graded vector space with $V_0 = \mathbb{K}$ and $V_{-1} = \mathbb{K}a$. We consider the augmented dga-algebra structure on V where the product is defined by $\mu(a, a) = 0$, and the differential d is zero. The bar complex $B(V) = (T^c(\overline{V}), d_B)$ satisfies $d_B = 0$. Recall that $T^c(\overline{V}) = \mathbb{K} \oplus sVa \oplus (sVa)^{\otimes 2} \oplus ...$. All elements are of zero degree and we denote the generator of $(sVa)^{\otimes n}$ by $sa \otimes \cdots \otimes sa = [sa]^n$. Also recall the coproduct is defined as $\Delta([sa]^n) = \sum_{i=1}^{n-1} [sa]^i \otimes [sa]^{n-i}$. Next consider the cobar complex $\Omega(B(V)) = (T(s^{-1}\overline{T^c(s\overline{V})}), d_\Omega)$. The degrees of the elements are ranging from 0 to $-\infty$. A general element of degree $-k$ is on the form $s^{-1}[sa]^{p_1} \otimes \cdots \otimes s^{-1}[sa]^{p_k}$. The differential d_Ω is now defined as $d_\Omega(s^{-1}[sa]^{p_1} \otimes \cdots \otimes s^{-1}[sa]^{p_k}) = \sum_{i=1}^{k} (-1)^{1-i} s^{-1}[sa]^{p_1} \otimes \cdots \otimes \Delta(s^{-1}[sa]^{p_i}) \otimes \cdots \otimes s^{-1}[sa]^{p_k}$. If we reverse the grading of $\Omega(B(V))$ and consider the reduced complex, it is easy to check that $\overline{\Omega(B(V))}$ is isomorphic $(T_0\mathcal{I}, d)$ as a complex. In this isomorphism the element $s^{-1}[sa]^{p_1} \otimes \cdots \otimes s^{-1}[sa]^{p_k}$ maps to the graph with k vertices, the in-core vertex having in-weight $p_1 - 1$ and the $i : th$ vertex having in-weight p_i for $i \geq 2$. Now $\Omega(B(V))$ is quasi-isomorphic to V (for example, see [LV]). $H(V)$ is generated by $1_{\mathbb{K}}$ and a. Hence we note that $[s^{-1}sa]$ is the only cohomology class of $\overline{\Omega(B(V))}$, which corresponds in $(T_0\mathcal{I}, d)$ to the single vertex graph with in-weight one. □

Corollary 3.3.7. *The cohomology group of $(\mathcal{I}, d)$ is generated by the graph*

$$\frac{c}{\infty_1} = \sum_{i=1}^{\infty} \frac{c}{i} .$$

Proof. By the above argument, we already know that $H(\mathcal{I}, d)$ is one-dimensional and that its generating class given by $\frac{c}{1}$ plus higher in-weight terms. It is straight-forward to see that the sum $\frac{c}{\infty_1}$ is a cycle that satisfies this property. □

Proof of Lemma 3.3.5. We need to show that $u_N\mathcal{T}^{in}_{a,b,c}$ is acyclic for $N \geq 2$. We say that a vertex x is a *branch vertex* if there are at least two paths starting at two different univalent special in-vertices and ending at x (see figure 3.3). The number of branch vertices either remains the same or increases under the action of the differential, and so we consider the filtration over the number of branch vertices. In the associated graded complex $gr(u_N\mathcal{T}^{in}_{a,b,c})$ the non branch vertices of a graph are attached to branch vertices as strings of passing vertices, and the differential acts by prolonging these strings. The differential can not increase the in-weight of a branch vertex. We consider the filtration over the total sum of the in-weights of the branch vertices. The differential of the associated graded complex $gr(gr(u_N\mathcal{T}^{in}_{a,b,c}))$ now acts only on the non branch vertices. The branch vertices and the number of non branch vertices that are attached to a branch vertex are invariant under the differential. By contracting the strings of non branch vertices in a graph Γ into N hairs, we get a *branch graph* Γ_{br} whose vertices are branch vertices. Then the complex splits as

$$gr(gr(u_N\mathcal{T}^{in}_{a,b,c})) = \bigoplus_{\Gamma_{br}} \mathsf{branchGraph}(\Gamma_{br})$$

summed over the set of all branch graphs Γ_{br}. These complexes decompose as

$$\mathsf{branchGraph}(\Gamma_{br}) = \Big(\bigotimes_{h\in H(\Gamma_{br})} \mathcal{I}\Big)^{\mathrm{Aut}(\Gamma_{br})}$$

where $H(\Gamma_{br})$ is the set of hairs in Γ_{br} and $\mathcal{I}$ is the complex from Lemma 3.3.7 and $\mathrm{Aut}(\Gamma_{br})$ is the group of symmetries of Γ_{br} acting with the appropriate signs. Since this group is finite, Maschke's Theorem gives

$$H\Big(\big(\bigotimes_{h\in H(\Gamma_{br})} \mathcal{I}\big)^{\mathrm{Aut}(\Gamma_{br})}\Big) \cong \Big(H\big(\bigotimes_{h\in H(\Gamma_{br})} \mathcal{I}\big)\Big)^{\mathrm{Aut}(\Gamma_{br})} \cong \big(\bigotimes_{h\in H(\Gamma_{br})} H(\mathcal{I})\big)^{\mathrm{Aut}(\Gamma_{br})}$$

Now $H(\mathcal{I})$ is generated by one element by Corollary 3.3.7. In a branch graph, there is at least one vertex with two or more hairs. Hence the cohomology classes are zero in $\mathsf{branchGraph}(\Gamma_{br})$ due to symmetries, finishing the proof. □

Lastly, we consider the complexes wGC^*_k and wGC^+_k. Let $\{S^{in}_r\mathsf{wGC}^*_k\}_{r\geq 0}$ and $\{S^{in}_r\mathsf{wGC}^+_k\}_{r\geq 0}$ be the spectral sequences associated to the filtration over the number of in-core vertices.

Proposition 3.3.8. *The page one complex $S^{in}_1\mathsf{wGC}^*_k$ is a subcomplex of $S^{in}_1\mathsf{wGC}_k$ generated by directed graphs whose vertices are independently decorated by the two types $\left\{\frac{m}{\infty_1}\right\}_{m\geq 0}$ and $\left\{\frac{m}{0}\right\}_{m\geq 0}$ subject to the conditions of Proposition 3.3.2 as well as the additional condition that either*

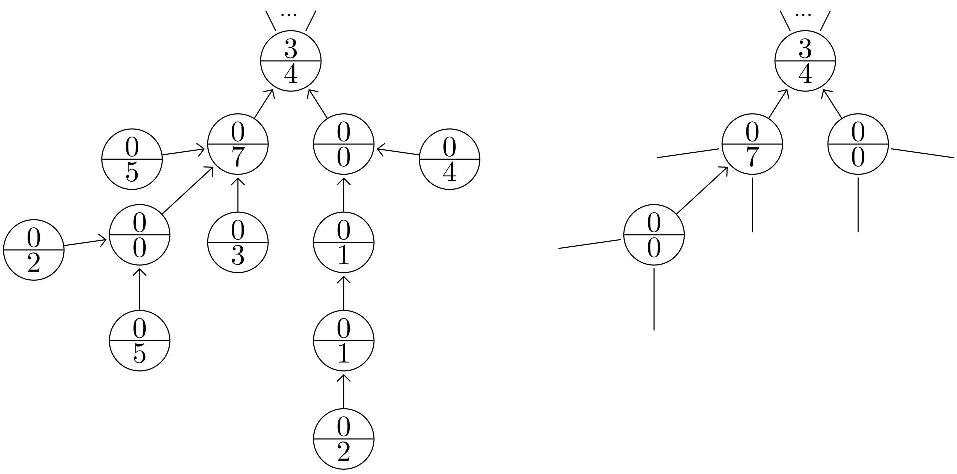


Figure 3.3: Picture of a graph and its corresponding branch graph.

- *at least one vertex is decorated with the bi-weight* $\frac{m}{\infty_1}$ *for some* $m \geq 0$, *or*
- *at least one vertex is decorated with the bi-weight* $\frac{m}{0}$ *for some* $m \geq 1$.

Furthermore, the page one complex $S_1^{in}\mathsf{wGC}_k^+$ *is a subcomplex of* $S_1^{in}\mathsf{wGC}_k^*$ *where each graph additionally satisfies that either*

- *one vertex is decorated with the bi-weight* $\frac{m}{\infty_1}$ *for some* $m \geq 1$, *or*
- *two vertices are decorated with the bi-weights* $\frac{m}{0}$ *and* $\frac{0}{\infty_1}$ *respectively for some* $m \geq 1$.

Proof. Similar to the case of $S_0^{in}\mathsf{wGC}_k$, both $S_0^{in}\mathsf{wGC}_k^*$ and $S_0^{in}\mathsf{wGC}_k^+$ decompose over in-core graphs γ as

$$S_0^{in}\mathsf{wGC}_k^* \cong \bigoplus_\gamma \mathsf{inCore}^*(\gamma)$$
$$S_0^{in}\mathsf{wGC}_k^+ \cong \bigoplus_\gamma \mathsf{inCore}^+(\gamma)$$

where $\mathsf{inCore}^*(\gamma)$ and $\mathsf{inCore}^+(\gamma)$ are the complex generated by graphs having γ as their in-core graph. These complexes do not however decompose into a tensor product over the tree complexes $\mathcal{T}_x^{in}$, since some of the tensors will not represent graphs in these complexes. For example, a graph having all vertices decorated by $\frac{0}{0}$ or a graph having only zero out-weights or zero in-weights respectively. The tree complexes $\mathcal{T}_x^{in}$ split into (at most) four complexes as $\mathcal{T}_x^{in} = \mathcal{T}_x^{in}(0) \oplus \mathcal{T}_x^{in}(out) \oplus \mathcal{T}_x^{in}(in) \oplus \mathcal{T}_x^{in}(out \wedge in)$, where $\mathcal{T}_x^{in}(0)$ is the complex where all bi-weights are zero, $\mathcal{T}_x^{in}(out)$ is the complex where all in bi-weights are zero and at least one vertex is decorated with positive out bi-weight, $\mathcal{T}_x^{in}(in)$ is the complex where all out bi-weights are zero and at least one vertex is decorated with positive in bi-weight, and $\mathcal{T}_x^{in}(out \wedge in)$ is the complex where at least one vertex is decorated with positive out bi-weight and at least one vertex with positive in bi-weight. Depending on the vertex x, some of the three first complexes might be zero in the decomposition of $\mathcal{T}_x^{out}$.

It is easy to verify that the differential preserves the decomposition. We then get that

$$\mathsf{inCore}^*(\gamma) = \bigoplus_I \left(\mathcal{T}^{out}_{x_1}(I_1) \otimes \mathcal{T}^{out}_{x_2}(I_2) \otimes \dots \otimes \mathcal{T}^{out}_{x_k}(I_k)\right)$$
$$\mathsf{inCore}^+(\gamma) = \bigoplus_J \left(\mathcal{T}^{out}_{x_1}(J_1) \otimes \mathcal{T}^{out}_{x_2}(J_2) \otimes \dots \otimes \mathcal{T}^{out}_{x_k}(J_k)\right)$$

where the first sum runs over signatures $I = (I_1, I_2, ..., I_k) \in \{0,\ in,\ out,\ in \wedge out\}^k$ such that there is an i such that $I_i \neq 0$. Similarly the second sum runs over signatures $J = (J_1, J_2, ..., J_k) \in \{0,\ in,\ out,\ in \wedge out\}^k$ such that either there is an i such that $J_i = out \wedge in$, or there are i, j such that $J_i = in$ and $J_j = out$. Lemma 3.3.3 gives us that

$$\begin{aligned}
H(\mathcal{T}^{out}_x(out \wedge in)) &= \left\langle \tfrac{m}{\infty_1}\ :\ m \geq 1 \right\rangle \\
H(\mathcal{T}^{out}_x(out)) &= \left\langle \tfrac{m}{0}\ :\ m \geq 1 \right\rangle \\
H(\mathcal{T}^{out}_x(in)) &= \left\langle \tfrac{0}{\infty_1} \right\rangle \\
H(\mathcal{T}^{out}_x(0)) &= \left\langle \tfrac{0}{0} \right\rangle
\end{aligned}$$

The proposition immediately follows by comparing the condition for the signatures I and J with the proposition statement. □

3.3.3 Filtrations over special out-vertices

We define special out-vertices by analogy to the special in-vertices introduced above.

Definition 3.3.9. A vertex x in a graph Γ is called a *special out-vertex* if

i) either x is a univalent vertex with one incoming edge and in-weight zero, i.e on the form $\underset{\uparrow}{\frac{m}{0}}$

ii) or v becomes a univalent vertex of type i) after recursive removal of all special-out vertices of type i) from Γ.

Vertices that are not special out-vertices are called *out-core vertices* or just *core vertices.* Note that there are no special-in vertices in any graph of $S^{in}_1\mathsf{wGC}_k$, but there are graphs with special out-vertices (see figure 3.4). Given an arbitrary graph Γ in $S^{in}_1\mathsf{wGC}_k$, we define the associated *core graph* γ as the graph spanned by core vertices with their out-weight forgotten. Similar to before, we consider the filtration of $S^{in}_1\mathsf{wGC}_k$ over the number of core-vertices and let $\{S^{out}_r\mathsf{wGC}_k\}_{r\geq 0}$ be the associated spectral sequence. The differential acts by only creating special out-vertices and leaves the connected core-graph unchanged.

Proposition 3.3.10. *The page one complex $S^{out}_1\mathsf{wGC}_d$ is generated by graphs Γ whose vertices $V(\Gamma)$ are independently decorated with four possible bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{0}{\infty_1}$, $\frac{\infty_1}{0}$, and $\frac{0}{0}$ subject to the following conditions:*

1. *If $x \in V(\Gamma)$ is univalent, then*

$$x = \underset{\uparrow}{\frac{\infty_1}{\infty_1}} \quad or \quad x = \overset{\uparrow}{\frac{\infty_1}{\infty_1}}$$

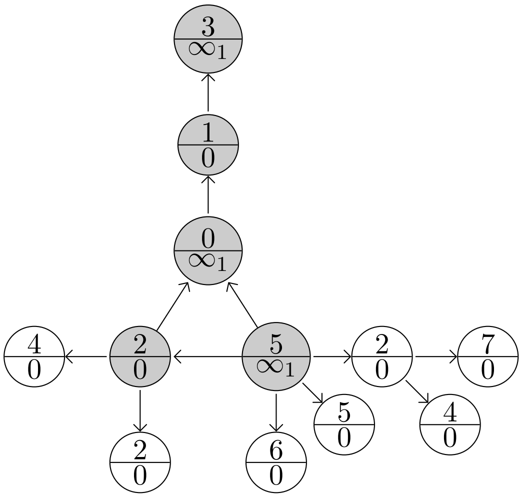
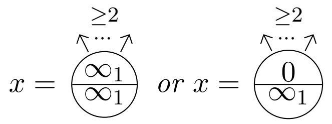
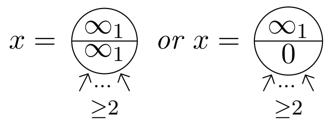


Figure 3.4: Example of a graph in $S_1^{in}\mathsf{wGC}_k$ where out-core vertices are colored gray and the special out-vertices are colored white.

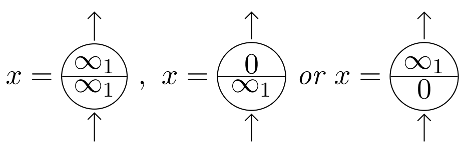
2. *If $x \in V(\Gamma)$ is a source with at least two out-edges, then*



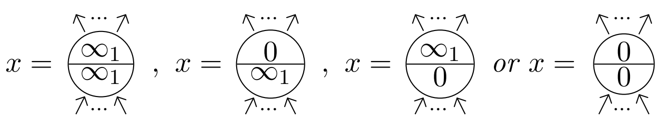
3. *If $x \in V(\Gamma)$ is a target with at least two in-edges, then*



4. *If $x \in V(\Gamma)$ is passing (one in-edge and one out-edge), then*

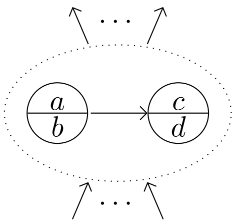


5. *If $x \in V(\Gamma)$ is none of the above types (i.e., x is at least trivalent and has at least one in-edge and at least one out-edge), then*



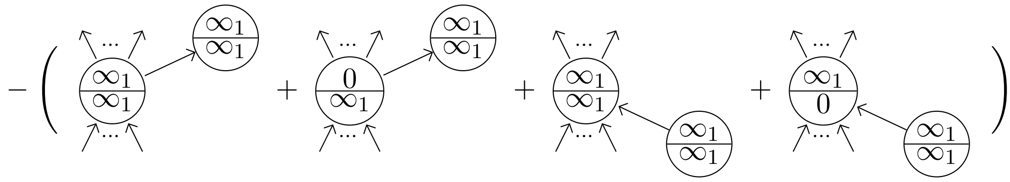
Let Γ be a graph in $S_1^{out}\mathsf{wGC}_d$ and x some vertex of Γ. Further, let a, b, c and d be either of the symbols ∞_1 or 0. Then we set $\left(\frac{a}{b}, \frac{c}{d}\right)_x$ to denote the sum over all possible reattachments of the edges attached to x among two new vertices x' and x'' (connected by a single edge going from x' to x'') of bi-weight $\frac{a}{b}$ and $\frac{c}{d}$ respectively. The reattachments *are done in such a way that the resulting vertices are of the types $(1)-(5)$ described above. Using earlier notation for vertex splitting, we get*

$$\left(\frac{a}{b},\frac{c}{d}\right)_x :=$$

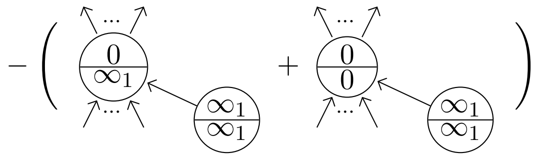


The differential d acts on graphs $\Gamma \in S_1^{out}\mathsf{wGC}_k$ as $d(\Gamma) = \sum_{x\in V(\Gamma)} d_x(\Gamma)$. The map d_x acts on vertices with the four different bi-weights in the following way:

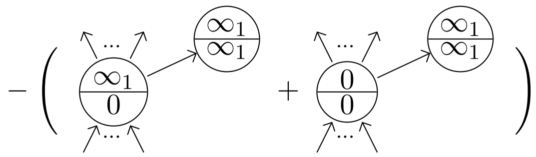
$$d_x\left(\frac{\infty_1}{\infty_1}\right) = \left(\frac{\infty_1}{\infty_1},\frac{\infty_1}{\infty_1}\right)_x + \left(\frac{0}{\infty_1},\frac{\infty_1}{\infty_1}\right)_x + \left(\frac{\infty_1}{0},\frac{\infty_1}{\infty_1}\right)_x + \left(\frac{\infty_1}{\infty_1},\frac{0}{\infty_1}\right)_x + \left(\frac{\infty_1}{\infty_1},\frac{\infty_1}{0}\right)_x$$
$$+ \left(\frac{0}{0},\frac{\infty_1}{\infty_1}\right)_x + \left(\frac{\infty_1}{0},\frac{0}{\infty_1}\right)_x + \left(\frac{0}{\infty_1},\frac{\infty_1}{0}\right)_x + \left(\frac{\infty_1}{\infty_1},\frac{0}{0}\right)_x$$



$$d_x\left(\frac{0}{\infty_1}\right) = \left(\frac{0}{\infty_1},\frac{0}{\infty_1}\right)_x + \left(\frac{0}{0},\frac{0}{\infty_1}\right)_x + \left(\frac{0}{\infty_1},\frac{0}{0}\right)_x$$



$$d_x\left(\frac{\infty_1}{0}\right) = \left(\frac{\infty_1}{0},\frac{\infty_1}{0}\right)_x + \left(\frac{0}{0},\frac{\infty_1}{0}\right)_x + \left(\frac{\infty_1}{0},\frac{0}{0}\right)_x$$



$$d_x\left(\frac{0}{0}\right) = \left(\frac{0}{0},\frac{0}{0}\right)_x$$

The first page $S_0^{out}\mathsf{wGC}_k$ decomposes into a directed sum parameterized by the set of all possible core graphs

$$S_0^{out}\mathsf{wGC}_k \cong \bigoplus_{\gamma} \mathsf{outCore}(\gamma)$$

where $\mathsf{outCore}(\gamma)$ is the subcomplex of $S_0^{out}\mathsf{wGC}_k$ of graphs with associated core graph γ. Further $\mathsf{outCore}(\gamma)$ decomposes into a tensor product of complexes

$$\mathsf{outCore}(\gamma) \cong \Big(\bigotimes_{x \in V(\gamma)} \mathcal{T}_x^{out} \Big)^{\mathrm{Aut}(\gamma)}$$

with one complex $\mathcal{T}_x^{out}$ for each vertex x in $V(\gamma)$, and $\mathrm{Aut}(\gamma)$ is the group of automorphisms of γ acting on the tensor product. Each complex $\mathcal{T}_x^{out}$ consists of trees of special out-vertices attached to a core vertex x, and the differential acts by only creating special out-vertices on x and in the tree. The complex $\mathcal{T}_x^{out}$ depends on x only via the number of outgoing and incoming edges attached to x in the core graph as well as on the in-weight w_x^{in}. Note that here the in-weight can be assigned one of the symbols 0 or ∞_1. The complexes $\mathcal{T}_x^{out}$ having the same values of the parameters $|x|_{out}$, $|x|_{in}$, and w_x^{in} are isomorphic, and we often write $\mathcal{T}_x^{out} \cong \mathcal{T}^{out}_{|x|_{out},|x|_{in},w_x^{in}}$. Hence we study the family of complexes $\mathcal{T}_{a,b,c}^{out}$ parameterized by integers $a, b \geq 0$ and $c \in \{0, \infty_1\}$ such that $a \geq 0$, $b \geq 1$ and $a + b \geq 2$ when $c = 0$, and $a, b \geq 0$ and $a + b \geq 1$ when $c = \infty_1$. The out-weight of the core vertex in $\mathcal{T}_{a,b,c}^{out}$ is any number w_x^{out} satisfying the conditions

$$w_x^{out} + a + \#(\text{out-edges from special out-vertices}) \geq 1$$
$$a + b + w_x^{out} + |c| + \#(\text{out-edges from special out-vertices}) \geq 3$$

where $|0| := 0$ and $|\infty_1| := 1$. Proposition 3.3.10 follows from these remarks together with the following lemma.

Lemma 3.3.11. *The cohomology of $\mathcal{T}_{a,b,c}^{out}$ is generated by one or two classes containing the out-core vertex decorated by some bi-weights depending on the parameters a, b, c. More precisely*

$$H(\mathcal{T}^{out}_{1,0,\infty_1}) = \left\langle \frac{\infty_1}{\infty_1} \right\rangle$$

$$H(\mathcal{T}^{out}_{0,1,\infty_1}) = \left\langle \frac{\infty_1}{\infty_1} \right\rangle$$

$$H(\mathcal{T}^{out}_{1,1,\infty_1}) = \left\langle \frac{\infty_1}{\infty_1}, \frac{0}{\infty_1} \right\rangle, \qquad H(\mathcal{T}^{out}_{1,1,0}) = \left\langle \frac{\infty_1}{0} \right\rangle$$

$$\textit{For } a \geq 2 \quad H(\mathcal{T}^{out}_{a,0,\infty_1}) = \left\langle \frac{\infty_1}{\infty_1}, \frac{0}{\infty_1} \right\rangle$$

$$\textit{For } b \geq 2 \quad H(\mathcal{T}^{out}_{0,b,\infty_1}) = \left\langle \frac{\infty_1}{\infty_1} \right\rangle, \qquad H(\mathcal{T}^{out}_{0,b,0}) = \left\langle \frac{\infty_1}{0} \right\rangle$$

$$\textit{For } a, b \geq 1 \textit{ and } a + b \geq 3 \quad H(\mathcal{T}^{out}_{a,b,\infty_1}) = \left\langle \frac{\infty_1}{\infty_1}, \frac{0}{\infty_1} \right\rangle, \qquad H(\mathcal{T}^{out}_{a,b,0}) = \left\langle \frac{\infty_1}{0}, \frac{0}{0} \right\rangle$$

Proof. The proof is similar to the proof for Proposition 3.3.10, using the same filtrations and decompositions together with Corollary 3.3.7. □

Now we turn to the complexes $S_1^{in}\mathsf{wGC}_k^*$ and $S_1^{in}\mathsf{wGC}_k^+$. Let $\{S_r^{out}\mathsf{wGC}_k^*\}_{r\geq 0}$ and $\{S_r^{out}\mathsf{wGC}_k^+\}_{r\geq 0}$ be the spectral sequences associated to the filtrations over the number of out-core vertices.

Proposition 3.3.12. *The page one complex $S_1^{out}\mathsf{wGC}_k^*$ is a subcomplex of $S_1^{out}\mathsf{wGC}_k$ generated by directed graphs whose vertices are independently decorated by the bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{0}{\infty_1}$, $\frac{\infty_1}{0}$, and $\frac{0}{0}$ subject to the conditions of Proposition 3.3.10 as well as the additional condition that at least one vertex is not decorated by $\frac{0}{0}$. Furthermore, the* *page one complex $S_1^{out}\mathsf{wGC}_k^+$ is a subcomplex of $S_1^{out}\mathsf{wGC}_k^*$ where each graph additionally satisfies that either*

- *at least one vertex is decorated with the bi-weight* $\frac{\infty_1}{\infty_1}$, *or*
- *at least two vertices are decorated with the bi-weights* $\frac{0}{\infty_1}$ *and* $\frac{\infty_1}{0}$ *respectively.*

Proof. The proof is analogous to that of Proposition 3.3.8. □

3.3.4 The 4-type graph complex qGC_k

Definition 3.3.13. Let qGC_k be the subcomplex of wGC_k consisting of graphs whose vertices can independently be decorated by four types of decorations $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$ and $\frac{0}{0}$. The possible decorations of a vertex depend on its type. More concretely

- A univalent vertex can only be decorated by $\frac{\infty_1}{\infty_1}$.
- A source vertex can be decorated by $\frac{\infty_1}{\infty_1}$ and $\frac{0}{\infty_1}$.
- A target vertex can be decorated by $\frac{\infty_1}{\infty_1}$ and $\frac{\infty_1}{0}$.
- A passing vertex can be decorated by $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$.
- A generic vertex can be decorated by $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$ and $\frac{0}{0}$.

Let qGC_k^* and qGC_k^+ be the subcomplexes of qGC_k where

- qGC_k^* is generated by graphs where at least one vertex is not decorated by $\frac{0}{0}$.
- qGC_k^+ is generated by graphs with at least one vertex decorated by $\frac{\infty_1}{\infty_1}$ or two vertices decorated by $\frac{0}{\infty_1}$ and $\frac{\infty_1}{0}$ respectively.

The differential acts on vertices in the same way as the differential in Proposition 3.3.10. Note that $\mathsf{qGC}_k^* \subset \mathsf{wGC}_k^*$ and $\mathsf{qGC}_k^+ \subset \mathsf{wGC}_k^+$.

Remark 3.3.14. We will use the following convention: when specifying the decorations of a general graph, the univalent vertices are excluded from this specification. For example, "A graph where all vertices are decorated by $\frac{\infty_1}{0}$" should be interpreted as a graph where all non-univalent vertices are decorated by $\frac{\infty_1}{0}$ and the univalent vertices decorated by their only possible decoration $\frac{\infty_1}{\infty_1}$.

Proposition 3.3.15. *The three inclusions $\mathsf{qGC}_k \hookrightarrow \mathsf{wGC}_k$, $\mathsf{qGC}_k^* \hookrightarrow \mathsf{wGC}_k^*$, and $\mathsf{qGC}_k^+ \hookrightarrow \mathsf{wGC}_k^+$ are quasi-isomorphims.*

Proof. Consider the filtration over special in-vertices as seen in the previous section of both wGC_k and qGC_k. The filtration is preserved by the inclusions since no graph in qGC_k contains special in-vertices, and furthermore, the differential is trivial in the associated complex of qGC_k. On the second page, we consider a second filtration over special out-vertices. The two complexes agree on the second page of the associated spectral sequence by Proposition 3.3.10. This proves that the inclusion is a quasi-isomorphism. The proof is analogous for the other two inclusions with the help of Proposition 3.3.12 □

Remark 3.3.16. We notice the splitting of $\mathsf{qGC}_k = \mathsf{qGC}_k^0 \oplus \mathsf{qGC}_k^*$ where qGC_k^0 is the complex of graphs where all vertices are decorated by $\frac{0}{0}$ and qGC_k^* is the complex of graphs with at least one vertex not decorated by $\frac{0}{0}$. This is analogous to the splitting $\mathsf{fwGC}_k = \mathsf{fwGC}_k^0 \oplus \mathsf{fwGC}_k^*$ in section 3.2.4, and it is immediate by Proposition 3.2.8 that $\mathsf{qGC}_k^0 \cong \mathsf{dGC}_k^{\circlearrowleft}$. In the remainder of the paper we focus on studying qGC_k^* and qGC_k^+.

3.4 Reducing qGC_k^* to the mono-decorated graph complex fM_k^*

3.4.1 Removing $\frac{0}{0}$ decorations from qGC_k^*

Let $\mathsf{qGC}_k^{*,0} \subset \mathsf{qGC}_k^*$ be the subcomplex generated by graphs with at least one vertex decorated by $\frac{0}{0}$. Consider the short exact sequence

$$0 \to \mathsf{qGC}_k^{*,0} \hookrightarrow \mathsf{qGC}_k^* \to \mathsf{tGC}_k^* \to 0$$

where the quotient complex tGC_k^* is generated by graphs with no decoration $\frac{0}{0}$. Similarly, let $\mathsf{qGC}^{+,0} \subset \mathsf{qGC}_k^+$ be the complex spanned by graphs with at least one decoration $\frac{0}{0}$ and $\mathsf{tGC}_k^+ := \mathsf{qGC}_k^+/\mathsf{qGC}_k^{+,0}$ the quotient complex of graphs with at least one vertex decorated by $\frac{\infty_1}{\infty_1}$, or a pair of vertices decorated by $\frac{0}{\infty_1}$ and $\frac{\infty_1}{0}$, with no vertices decorated by $\frac{0}{0}$.

Proposition 3.4.1. *The two projections* $\mathsf{qGC}_k^* \to \mathsf{tGC}_k^*$ *and* $\mathsf{qGC}_k^+ \to \mathsf{tGC}_k^+$ *are quasi-isomorphisms.*

Proof. It is enough to show that $\mathsf{qGC}_k^{*,0}$ and $\mathsf{qGC}_k^{+,0}$ are acyclic. Consider the filtration of $\mathsf{qGC}_k^{*,0}$ over the number of non-passing vertices, and let $P_0\mathsf{qGC}_k^{*,0}$ be the first page of the associated spectral sequence. The differential acts by only creating passing vertices. Consider the filtration of $P_0\mathsf{qGC}_k^{*,0}$ over the number of vertices *not* decorated by $\frac{\infty_1}{\infty_1}$. The differential on the first page of the d spectral sequence $D_0P_0\mathsf{qGC}_k^{*,0}$ only creates passing vertices decorated by $\frac{\infty_1}{\infty_1}$. To each graph in this complex, we can associate an $\frac{\infty_1}{\infty_1}$-*skeleton graph* by removing passing vertices decorated by $\frac{\infty_1}{\infty_1}$ and replacing them by a single edge. The $\frac{\infty_1}{\infty_1}$-skeleton graph is invariant under the action of the differential, hence the complex split as

$$gr(P_0\mathsf{qGC}_k^{*,0}) = \bigoplus_\gamma \mathcal{C}^*(\gamma)$$

where $\mathcal{C}^*(\gamma)$ is the associated complex of graphs with $\frac{\infty_1}{\infty_1}$-skeleton γ. Note that no graph has the associated $\frac{\infty_1}{\infty_1}$-skeleton graph with one single vertex and one edge, since any such graph has no vertex decorated by $\frac{0}{0}$. We claim the following:

- If γ has at least one vertex decorated by $\frac{\infty_1}{\infty_1}$, then $H(\mathcal{C}^*(\gamma)) = 0$.
- If γ has no vertices decorated by $\frac{\infty_1}{\infty_1}$ and at least one decorated by $\frac{\infty_1}{0}$ or $\frac{0}{\infty_1}$, then $H(\mathcal{C}^*(\gamma)) = \langle\gamma\rangle$.
- If γ is only decorated by $\frac{0}{0}$ and has at least one univalent vertex, then $H(\mathcal{C}^*(\gamma)) = \langle\gamma\rangle$.
- If γ is only decorated by $\frac{0}{0}$, then $H(\mathcal{C}^*(\gamma)) = 0$.

In the three first cases, the complex can be written as

$$\mathcal{C}^*(\gamma) \cong \Big(\bigotimes_{e\in E(\gamma)} \mathcal{E}_e\Big)^{\mathrm{Aut}(\gamma)} \tag{3.2}$$

where $\mathcal{E}_e$ is the associated complex of passing vertices decorated by $\frac{\infty_1}{\infty_1}$ on the edge e in the skeleton. In the first case, there is one edge e' in γ such that one of its adjacent vertices is decorated by $\frac{\infty_1}{\infty_1}$ and the other by another decoration. One computes that $\mathcal{E}_{e'}$ is acyclic. In the second and third cases, all $\mathcal{E}_e$ are isomorphic, consisting of the complex of passing vertices decorated by $\frac{\infty_1}{\infty_1}$. One sees that $H(\mathcal{E}_e)$ is generated by the graph of two vertices and an edge with no passing vertices, giving the desired result. In the fourth case, the complex does not decompose as a tensor product. Instead it split as a direct sum of tensor products, where in each product at least one complex $\mathcal{E}_e$ has at least one passing vertex decorated by $\frac{\infty_1}{\infty_1}$. This complex is equivalent to the tensor above when removing the initial graph with no passing vertices, which we saw was a cycle in the second case. Hence the complex is acyclic. The cohomology of $D_0P_0\mathsf{qGC}^{*,0}$ now consists of graphs only decorated by $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$ and $\frac{0}{0}$ with at least one vertex decorated by $\frac{0}{0}$. The differential acts by creating passing vertices. It is easy to see that these graphs form a subcomplex of $P_0\mathsf{qGC}_k^{*,0}$, and so the second page $D_1P_0\mathsf{qGC}^{*,0}$ of the spectral sequence is described with the full differential. We claim that $D_1P_0\mathsf{qGC}^{*,0}$ is acyclic. By considering a similar filtration over the number of vertices not decorated by $\frac{\infty_1}{0}$, one finds that this page of the spectral sequence is acyclic, finishing the proof. The proof to show that $\mathsf{qGC}_k^{+,0}$ is acyclic follows the same argument above using the same filtrations. □

3.4.2 Subcomplex of monodecorated graphs

Definition 3.4.2. Let Γ be an undecorated directed graph. We define $\Gamma^{\frac{0}{\infty_1}}$ to be the graph Γ where all non-univalent vertices are decorated by $\frac{0}{\infty_1}$ and the univalent vertices are decorated by $\frac{\infty_1}{\infty_1}$. If Γ contains a target vertex (which cannot be decorated by $\frac{0}{\infty_1}$), then we set $\Gamma^{\frac{0}{\infty_1}} = 0$. Similarly, define $\Gamma^{\frac{\infty_1}{0}}$.

Definition 3.4.3. Let Γ be a directed graph without any univalent vertices. Define $\Gamma^\omega \in \mathsf{tGC}_k^+$ to be the sum of graphs

$$\Gamma^\omega = \sum_d \Gamma_d$$

where d is a decoration of all vertices of Γ, Γ_d is the bi-weighted graph with underlying graph Γ decorated by the bi-weights of b, and the sum is over all possible decorations d of Γ where at least one vertex is decorated by $\frac{\infty_1}{\infty_1}$ or a pair of vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$. Equivalently, if $\Gamma^{\frac{\infty_0}{\infty_0}}$ denotes the graph Γ where all vertices are decorated by $\frac{\infty_0}{\infty_0}$ (when expanded, terms of graphs including vertices decorated by $\frac{0}{0}$ are considered zero), then $\Gamma^\omega = \Gamma^{\frac{\infty_0}{\infty_0}} - \Gamma^{\frac{\infty_1}{0}} - \Gamma^{\frac{0}{\infty_1}}$.

Remark 3.4.4. Let $d = d_s + d_u$ be the decomposition of the differential of tGC_k^+ where d_u is the part of where a new univalent vertex is created and d_s the part where no new univalent vertices are created (also known as splitting). We see that $d_s(\Gamma^\omega) = (d_s\Gamma)^\omega$. Further, we see that $d_u(\Gamma^\omega) = -(d_u(\Gamma^{\frac{\infty_1}{0}}) + d_u(\Gamma^{\frac{0}{\infty_1}}))$. Due to $\Gamma^{\frac{\infty_1}{0}}$ and $\Gamma^{\frac{0}{\infty_1}}$ being zero for some graphs, we have more specifically that

- If Γ contains both at least one source and one target, then $d_u(\Gamma^\omega) = 0$.
- If Γ contains at least one source but no targets, then $d_u(\Gamma^\omega) = -d_u(\Gamma^{\frac{0}{\infty_1}})$.
- If Γ contains at least one target but no sources, then $d_u(\Gamma^\omega) = -d_u(\Gamma^{\frac{\infty_1}{0}})$.
- If Γ contains neither sources nor targets, then $d_u\Gamma^\omega = -(d_u\Gamma^{\frac{0}{\infty_1}} + d_u\Gamma^{\frac{\infty_1}{0}})$.

Definition 3.4.5. The subcomplex $\mathsf{mGC}_k^+ \subset \mathsf{tGC}_k^+$ is the complex generated by graphs on three forms:

1. Graphs on the form Γ^ω with Γ a directed graph with no univalent vertices.
2. Graphs on the form $\Gamma^{\frac{0}{\infty_1}}$ with Γ a directed graph with at least one univalent source, and no targets of any valency.

3. Graphs on the form $\Gamma^{\frac{\infty_1}{0}}$ with Γ a directed graph with at least one univalent target, and no sources of any valency.

Similarly, the subcomplex $\mathsf{mGC}^*_k \subset \mathsf{tGC}^*_k$ is the complex generated by graphs in three forms:

1. Graphs on the form Γ^ω with Γ a directed graph with no univalent vertices.
2. Graphs on the form $\Gamma^{\frac{0}{\infty_1}}$ with Γ a directed graph containing no targets of any valency.
3. Graphs on the form $\Gamma^{\frac{\infty_1}{0}}$ with Γ a directed graph containing no sources of any valency.

Proposition 3.4.6. *The two inclusions* $\mathsf{mGC}^+_k \to \mathsf{tGC}^+_k$ *and* $\mathsf{mGC}^*_k \to \mathsf{tGC}^*_k$ *are quasi-isomorphisms.*

Consider the short exact sequences

$$0 \to \mathsf{mGC}^+_k \to \mathsf{tGC}^+_k \to \mathsf{Q}^+_k \to 0$$
$$0 \to \mathsf{mGC}^*_k \to \mathsf{tGC}^*_k \to \mathsf{Q}^*_k \to 0$$

where $\mathsf{Q}^*_k = \mathsf{tGC}^*_k/\mathsf{mGC}^*_k$ and $\mathsf{Q}^+_k = \mathsf{tGC}^+_k/\mathsf{mGC}^+_k$. We observe that $\mathsf{Q}^*_k = \mathsf{Q}^+_k$. The complex Q^+_k is generated by graphs $\Gamma \in \mathsf{tGC}^+_k$ on the forms

1. Γ has no univalent vertices and has at least one vertex not decorated by $\frac{\infty_1}{\infty_1}$.
2. $\Gamma = \Gamma^{\frac{\infty_1}{0}}$ and has at least one univalent target.
3. $\Gamma = \Gamma^{\frac{0}{\infty_1}}$ and has at least one univalent source.
4. Γ has at least one univalent vertex and at least two non-univalent vertices with different decorations.

The proposition follows if we show that Q^+_k is acyclic. Consider the subcomplex $\mathsf{Q}^{+,1}_k \subset \mathsf{Q}^+_k$ of graphs with at least one univalent vertex. We get the induced short exact sequence

$$0 \to \mathsf{Q}^{+,1}_k \to \mathsf{Q}^+_k \to \mathsf{Q}^{+,\geq 2}_k \to 0.$$

The complex $\mathsf{Q}^{+,\geq 2}_k$ is spanned by graphs on the form (1) as above, and $\mathsf{Q}^{+,1}_k$ is spanned by graphs on the form (2)-(4). The acyclicity of Q^+_k follows from the following proposition.

Proposition 3.4.7. *The complexes* $\mathsf{Q}^{+,1}_k$ *and* $\mathsf{Q}^{+,\geq 2}_k$ *are acyclic.*

Proof. First consider the filtration of $\mathsf{Q}^{+,\geq 2}_k$ over the number of non-passing vertices and let $P_0\mathsf{Q}^{+,\geq 2}_k$ be the first page of the spectral sequence. On this page, the differential acts by only creating passing vertices. We will show that this page is acyclic. Consider the filtration on $P_0\mathsf{Q}^{+,\geq 2}_k$ over the number of vertices not decorated by $\frac{\infty_1}{\infty_1}$ and let $D_0P_0\mathsf{Q}^{+,\geq 2}_k$ be the first page of the associated spectral sequence. Here the differential acts by only creating passing vertices decorated by $\frac{\infty_1}{\infty_1}$. Similar to Proposition 3.4.1, the complex decompose over $\frac{\infty_1}{\infty_1}$-skeleton graphs as

$$D_0P_0\mathsf{Q}^{+,\geq 2}_k = \bigoplus_\gamma \mathcal{C}^+(\gamma)$$

where $\mathcal{C}^+(\gamma)$ is the complex of graphs with $\frac{\infty_1}{\infty_1}$-skeleton γ. We claim the following

- If γ has at least one vertex decorated by $\frac{\infty_1}{\infty_1}$, then $\mathcal{C}^+(\gamma) \simeq 0$.

- If γ is only decorated by $\frac{\infty_1}{0}$, then $\mathcal{C}^+(\gamma) \simeq 0$.
- If γ is only decorated by $\frac{0}{\infty_1}$, then $\mathcal{C}^+(\gamma) \simeq 0$.
- If γ is decorated by both $\frac{\infty_1}{0}$ and $\frac{0}{\infty}$ (but not $\frac{\infty_1}{\infty_1}$), then $H(\mathcal{C}^+(\gamma)) = \langle\gamma\rangle$.

These results follow by using similar arguments as in Proposition 3.4.1. Let $D_1P_0\mathsf{Q}_k^{+,\geq 2}$ be the second page of the spectral sequence. By the result above, it consists of graphs with at least a pair of vertices decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$. Consider the filtration over the number of vertices not decorated by $\frac{\infty_1}{0}$. Using similar arguments as above on the associated spectral sequence, we get that $D_1P_0\mathsf{Q}_k^{+,\geq 2}$ is acyclic. The proof showing that $\mathsf{Q}_k^{+,1}$ is acyclic is similar, noting that the differential acting on univalent sources only creates passing vertices decorated by $\frac{0}{\infty_1}$ and univalent targets only create passing vertices decorated by $\frac{\infty_1}{0}$. □

3.4.3 Removing long antennas

Consider the two subcomplexes $\mathsf{mGC}_k^*(\frac{\infty_1}{0})$ and $\mathsf{mGC}_k^*(\frac{0}{\infty_1})$ of mGC_k^* of graphs whose non-univalent vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively.

Proposition 3.4.8. *The complexes* $\mathsf{mGC}_k^*(\frac{\infty_1}{0})$ *and* $\mathsf{mGC}_k^*(\frac{0}{\infty_1})$ *are acyclic.*

Proof. The two complexes are isomorphic by swapping decorations and reversing the orientation of all edges; hence, it is enough to show that $\mathsf{mGC}_k^*(\frac{\infty_1}{0})$ is acyclic. Let the core of a graph be the graph remaining after iterative removal of univalent vertices. Any vertex in the core graph is called a core-vertex. Vertices that are not core vertices are called antenna vertices. The number of core vertices can only remain the same or increase under the action of the differential. Consider the filtration of $\mathsf{mGC}_k^*(\frac{\infty_1}{0})$ over the number of core-vertices in a graph. Let $gr(\mathsf{mGC}_k^*(\frac{\infty_1}{0}))$ be the associated graded complex. The differential acts by creating antenna vertices, but the core graph is invariant. Hence we get the decomposition

$$gr(\mathsf{mGC}_k^*(\frac{\infty_1}{0})) = \bigoplus_{\gamma} \mathsf{Core}(\gamma)$$

where the summation is over all possible core graphs γ and $\mathsf{Core}(\gamma)$ is the complex of graphs with core graph γ. This complex further decomposes as

$$\mathsf{Core}(\gamma) \cong \Big(\bigotimes_{x\in V(\gamma)} \mathcal{T}_x\Big)^{\mathrm{Aut}(\gamma)}$$

where $\mathcal{T}_x$ is the associated complex of antenna-vertices attached to the vertex x. The complex $\mathcal{T}_x$ is composed of directed trees with a designated core vertex x. All edges in any such graph are directed away from x. The proof follows if we show that $\mathcal{T}_x$ is acyclic. Consider the filtration on $\mathcal{T}_x$ over the number of univalent vertices (considering the core vertex x to be univalent only when it is the only vertex in the graph). Let $gr\ \mathcal{T}_x$ be the associated graded complex. The differential acts by splitting vertices such that no new univalent vertices are created (except in the case of the one vertex graph). Hence it decomposes as $gr\ \mathcal{T}_x = \bigoplus_{N\geq 1} u_N\mathcal{T}_x$ where $u_N\mathcal{T}_x$ is the complex of graphs with N univalent vertices. It is easy to see that $u_1\mathcal{T}_x$ is acyclic. Consider $u_N\mathcal{T}_x$ for $N \geq 2$. Similar to the proof of Lemma 3.3.5, call a vertex y of $\Gamma \in u_N\mathcal{T}_x$ a *branch vertex* if there are at least two outgoing edges from y, or if there is a directed path from y to a vertex z that has at least two outgoing edges from it. The number of such vertices remains the same or is increased under the action of the differential. Consider the filtration over the number of branch vertices on $u_N\mathcal{T}_x$. Following the same arguments as in the proof of Lemma 3.3.5, we get that $u_N\mathcal{T}_x$ is acyclic, finishing the proof. □

We have the following decomposition

$$\mathsf{mGC}_k^* = \mathsf{fM}_k^* \oplus \mathsf{fM}_k^{a\geq 2}$$
$$\mathsf{mGC}_k^+ = \mathsf{fM}_k^+ \oplus \mathsf{fM}_k^{a\geq 2}$$

where $\mathsf{fM}_k^{a\geq 2}$ is the subcomplex of graphs containing at least one antenna with two or more vertices.

Proposition 3.4.9. *The two injections $\mathsf{fM}_k^* \hookrightarrow \mathsf{mGC}_k^*$ and $\mathsf{fM}_k^+ \hookrightarrow \mathsf{mGC}_k^+$ are quasi-isomorphisms.*

Proof. The statement follows if we show that $\mathsf{fM}_k^{a\geq 2}$ is acyclic. Note that $\mathsf{fM}_k^{a\geq 2}$ decompose over graphs whose non-univalent vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively, i.e $\mathsf{fM}_k^{a\geq 2} = \mathsf{fM}_k^{a\geq 2}(\frac{\infty_1}{0}) \oplus \mathsf{fM}_k^{a\geq 2}(\frac{0}{\infty_1})$. Further note that $\mathsf{mGC}_k^*(\frac{\infty_1}{0}) = \mathsf{fM}_k^{*,a=0,1}(\frac{\infty_1}{0}) \oplus \mathsf{fM}_k^{a\geq 2}(\frac{\infty_1}{0})$. Hence acyclicity of $\mathsf{mGC}_k^*(\frac{\infty_1}{0})$ from Proposition 3.4.8 gives that $\mathsf{fM}_k^{a\geq 2}(\frac{\infty_1}{0})$ is also acyclic. □

3.5 Cohomology of fM_k^* and fM_k^+

3.5.1 A commutative diagram of graph complexes

Let $\mathsf{fM}_k^{+,1}$ be the subcomplex of fM_k^+ of graphs with at least one univalent vertex. Further let $\mathsf{fM}_k^*(\frac{\infty_1}{0})$ and $\mathsf{fM}_k^*(\frac{0}{\infty_1})$ be the subcomplexes of fM_k^* of graphs whose non-univalent vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively. Note that $\mathsf{fM}_k^{+,1}$ is a subcomplex of $\mathsf{fM}_k^*(\frac{\infty_1}{0}) \oplus \mathsf{fM}_k^*(\frac{0}{\infty_1})$ and can be rewritten as $\mathsf{fM}_k^{+,1} = \mathsf{fM}_k^{*,1}(\frac{\infty_1}{0}) \oplus \mathsf{fM}_k^{*,1}(\frac{0}{\infty_1})$ where $\mathsf{fM}_k^{*,1}(\frac{\infty_1}{0})$ is the subcomplex of fM_k^* of graphs with at least one univalent vertex and where all non-univalent vertices are decorated by $\frac{\infty_1}{0}$, and similarly for $\mathsf{fM}_k^{*,1}(\frac{0}{\infty_1})$. We get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathsf{fM}_k^{+,1} & \to & \mathsf{fM}_k^+ & \to & \mathsf{fM}_k^{+,\geq 2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow {\scriptstyle id} & & \\
0 & \to & \mathsf{fM}_k^*(\frac{\infty_1}{0}) \oplus \mathsf{fM}_k^*(\frac{0}{\infty_1}) & \to & \mathsf{fM}_k^* & \to & \mathsf{fM}_k^{+,\geq 2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathsf{fM}_k^{*,\geq 2}(\frac{\infty_1}{0}) \oplus \mathsf{fM}_k^{*,\geq 2}(\frac{0}{\infty_1}) & \xrightarrow{id} & \mathsf{fM}_k^{*,\geq 2}(\frac{\infty_1}{0}) \oplus \mathsf{fM}_k^{*,\geq 2}(\frac{0}{\infty_1}) & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.3}$$

where $\mathsf{fM}_k^{*,\geq 2}(\frac{\infty_1}{0})$ is the quotient complex of graphs with no univalent vertices, and all vertices are decorated by $\frac{\infty_1}{0}$, and similarly for $\mathsf{fM}_k^{*,\geq 2}(\frac{0}{\infty_1})$). Further, $\mathsf{fM}_k^{+,\geq 2}$ is the quotient complex of graphs with no univalent vertices, and each graph is of the form Γ^ω. We already showed in Proposition 3.4.8 that $\mathsf{fM}_k^*(\frac{\infty_1}{0})$ and $\mathsf{fM}_k^*(\frac{0}{\infty_1})$ are acyclic, and so we get the following corollaries.

Corollary 3.5.1.

- *The projection $\mathsf{fM}_k^* \to \mathsf{fM}_k^{+,\geq 2}$ is a quasi-isomorphism.*
- *The connecting morphisms $\delta : H^\bullet(\mathsf{fM}_k^{*,\geq 2}(\frac{\infty_1}{0}) \oplus \mathsf{fM}_k^{*,\geq 2}(\frac{0}{\infty_1})) \to H^{\bullet+1}(\mathsf{fM}_k^{+,1})$ is an isomorphism.*

3.5.2 The cohomology of fM_k^*

Let $\mathsf{fM}_k^* = \mathsf{b}_1\mathsf{M}_k^* \oplus \mathsf{M}_k^*$ where $\mathsf{b}_1\mathsf{M}_k^*$ is the subcomplex of graphs with loop number one, and M_k^* is the subcomplex of graphs with loop number two and higher. All complexes in the commutative diagram (3.3) above split in the same manner, and we use an analogous notation for their splittings.

Proposition 3.5.2. *The map $a : \mathsf{dGC}_k \to \mathsf{fM}_k^{+,\geq 2}$ where $a(\Gamma) = \Gamma^\omega$ is a quasi-isomorphism. In particular, the restriction to graphs of loop number one $a_1 : \mathsf{b}_1\mathsf{dGC}_k \to \mathsf{b}_1\mathsf{M}_k^{+,\geq 2}$ is an isomorphism.*

Proof. Consider a filtration over $\mathsf{fM}_k^{\geq 2}$ over the number of non-passing vertices. The differential on the associated graded sequence acts on edges by creating new passing vertices, and it is straightforward to control that the cohomology is generated by graphs with no passing vertices. Hence on the second page the map is an isomorphism, giving that a is a quasi-isomorphism. By direct inspection, one sees that a_1 is already an isomorphism. □

3.5.3 The cohomology of fM_k^+

Proposition 3.5.3. *Let $f^s : \mathsf{dGC}_k/\mathsf{dGC}_k^s \to \mathsf{fM}_k^{*,\geq 2}(\frac{\infty_1}{0})$ be the map where $f^s(\Gamma) = \Gamma^{\frac{\infty_1}{0}}$, and let $f^t : \mathsf{dGC}_k/\mathsf{dGC}_k^t \to \mathsf{fM}_k^{*,\geq 2}(\frac{0}{\infty_1})$ be the map where $f^t(\Gamma) = \Gamma^{\frac{0}{\infty_1}}$. These maps are quasi-isomorphisms. In particular, the restriction $f_1^s : \mathsf{b}_1\mathsf{dGC}_k/\mathsf{dGC}_k^s \to \mathsf{b}_1\mathsf{M}_k^{*,\geq 2}$ is an isomorphism. The analogously defined map f_1^t is also an isomorphism.*

Proof. The first part follows by the filtration over the number of non-passing vertices. The latter part follows by noting that graphs with loop number one and no univalent vertices have either both a source and a target or neither. □

We can already get the following proposition.

Proposition 3.5.4. *$H^\bullet(\mathsf{fM}_3^+) = \mathfrak{grt} \oplus \mathfrak{grt}$.*

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathsf{fM}_3^{+,\geq 1} \longrightarrow \mathsf{fM}_3^+ \longrightarrow \mathsf{fM}_3^{+,\geq 2} \longrightarrow 0\,.$$

We extract the following exact sequence from the induced long exact sequence

$$H^{-1}(\mathsf{dGC}_3) \longrightarrow H^{-1}(\mathsf{dGC}_3/\mathsf{dGC}_3^s) \oplus H^{-1}(\mathsf{dGC}_3/\mathsf{dGC}_3^t) \longrightarrow H^0(\mathsf{fM}_k^+) \longrightarrow H^0(\mathsf{dGC}_3)$$

by using Proposition 3.5.3 and the quasi-isomorphisms induced from the diagram (3.3). Now since $H^k(\mathsf{dGC}_3) = 0$ for $k \geq -2$, we get the isomorphism $H^0(\mathsf{mGC}_3) \cong H^{-1}(\mathsf{dGC}_3/\mathsf{dGC}_3^s) \oplus H^{-1}(\mathsf{dGC}_3/\mathsf{dGC}_3^t)$. Using the same trick on the long exact sequence on cohomology induced by the short exact sequence

$$0 \to \mathsf{dGC}_3^s \to \mathsf{dGC}_3 \to \mathsf{dGC}_3/\mathsf{dGC}_3^s \to 0$$

we get that $H^{-1}(\mathsf{dGC}_3/\mathsf{dGC}_3^s) \cong H^0(\mathsf{dGC}_3^s)$. Similarly, $H^{-1}(\mathsf{dGC}_3/\mathsf{dGC}_3^t) \cong H^0(\mathsf{dGC}_3^t)$. It was shown in [Z2] that there are quasi-isomorphisms $\mathsf{dGC}_k \to \mathsf{dGC}_{k+1}^s$ and $\mathsf{dGC}_k \to \mathsf{dGC}_{k+1}^t$, and so in particular

$$H^0(\mathsf{dGC}_3^s) = H^0(\mathsf{dGC}_3^t) = H^0(\mathsf{dGC}_2) = \mathfrak{grt}$$

which gives the proposition. □

From the same short exact sequence of complexes of restricted to graphs of loop number one

$$0 \longrightarrow \mathsf{b}_1\mathsf{M}_k^{+,1} \longrightarrow \mathsf{b}_1\mathsf{M}_k^{+} \longrightarrow \mathsf{b}_1\mathsf{M}_k^{+,\geq 2} \longrightarrow 0\ .$$

we can derive the cohomology of $\mathsf{b}_1\mathsf{M}_k^{+}$.

Proposition 3.5.5. *The cohomology of* $\mathsf{b}_1\mathsf{M}_k^{+}$ *is given by*

$$H^\bullet(\mathsf{b}_1\mathsf{M}_k^{+}) = \bigoplus_{\substack{i\geq 1 \\ i\equiv 1 \mod 2}} \mathcal{C}_i$$

$$\textit{where } \mathcal{C}_i = \begin{cases} \mathbb{K}[k-(i+1)] & \textit{if } i \equiv 2k+1 \mod 4 \\ \mathbb{K}[k-(i+1)] \oplus \mathbb{K}[k-(i+1)] & \textit{if } i \equiv 2k+3 \mod 4 \end{cases}$$

Proof. We first show that the connecting morphism

$$\delta : H^i(\mathsf{b}_1\mathsf{M}_k^{+,\geq 2}) \to H^{i+1}(\mathsf{b}_1\mathsf{M}_k^{+,1})$$

is injective. The injection $b_1 : \mathsf{b}_1\mathsf{GC}_k \to \mathsf{b}_1\mathsf{M}_k^{+,\geq 2}$ mapping an undirected graph Γ to the sum of graphs over all possible ways of adding directions on the edges of Γ is a quasi-isomorphism. Let $[\Gamma] \in H^i(\mathsf{b}_1\mathsf{GC}_k)$ be a non-zero equivalence class, where Γ is a loop graph with i vertices. Then $i \geq 1$ and $i \equiv 2k+1 \mod 4$, since otherwise $[\Gamma] = 0$. The sum $b_1(\Gamma)$ contains the loop graph with only passing vertices, call it $\Gamma^{\circlearrowleft}$, while all other terms are graphs with at least one source and one target. The connecting morphism

$$\delta : H^i(\mathsf{b}_1\mathsf{M}_k^{+,\geq 2}) \to H^{i+1}(\mathsf{b}_1\mathsf{M}_k^{+,1})$$

is the univalent part d_u of the differential, and is hence zero on graphs with both a source and a target. Hence we gather $\delta \circ b_1(\Gamma) = d_u(\Gamma^{\circlearrowleft})$. We can clearly see that this element belongs in the diagonal of

$$H^{i+1}(\mathsf{b}_1\mathsf{M}_k^{+,1}) = \mathsf{b}_1\mathsf{M}_k^{+,1}(\frac{\infty_1}{0}) \oplus \mathsf{b}_1\mathsf{M}_k^{+,1}(\frac{0}{\infty_1}),$$

that it is non-zero and this assignment is unique. Hence the long exact sequence on cohomology splits as short exact sequences

$$0 \longrightarrow H^{i-k}(\mathsf{b}_1\mathsf{M}_k^{+,\geq 2}) \longrightarrow H^{i-k+1}(\mathsf{b}_1\mathsf{M}_k^{+,1}) \longrightarrow H^{i-k+1}(\mathsf{b}_1\mathsf{M}_k^{+}) \longrightarrow 0\ .$$

When i is even, the whole sequence is zero. When $i \equiv 2k+3 \mod 4$, then $H^{i-k}(\mathsf{b}_1\mathsf{M}_k^{+,\geq 2})$ acyclic, and so

$$H^{i-k+1}(\mathsf{b}_1\mathsf{M}_k^{+}) \cong H^{i-k+1}(\mathsf{b}_1\mathsf{M}_k^{+,1}) = \mathbb{K}[k-i-1] \oplus \mathbb{K}[k-i-1].$$

When $i \equiv 2k+1 \mod 4$, then we get the short exact sequence

$$0 \longrightarrow \mathbb{K}[k-i] \longrightarrow \mathbb{K}[k-i-1] \oplus \mathbb{K}[k-i-1] \longrightarrow H^{i-k+1}(\mathsf{b}_1\mathsf{M}_k^{+}) \longrightarrow 0$$

where the first map is the connecting morphism δ, being identified with the suspended diagonal map. Hence we gather that

$$H^{i-k+1}(\mathsf{b}_1\mathsf{M}_k^{+}) \cong (\mathbb{K}[k-i-1] \oplus \mathbb{K}[k-i-1])/\delta(\mathbb{K}[k-i]) \cong \mathbb{K}[k-i-1].$$

□

3.5.4 Describing fM_k^+ as a mapping cone

So far we have only been able to describe the cohomology of fM_k^+ via a short exact sequence. In this section we will show where this short exact sequence originates from, and that there is fM_k^+ quasi-isomorphic to a mapping cone of directed graph complexes. Let

$$P_1 : \mathsf{dGC}_k \to \mathsf{dGC}_k/\mathsf{dGC}_k^s \quad \text{and} \quad P_2 : \mathsf{dGC}_k \to \mathsf{dGC}_k/\mathsf{dGC}_k^t$$

be the projections and

$$P = P_1 \oplus P_2 : \mathsf{dGC}_k \to \mathsf{dGC}_k/\mathsf{dGC}_k^s \oplus \mathsf{dGC}_k/\mathsf{dGC}_k^t$$

their sum. The suspended mapping cone of P is the complex $\mathrm{Cone}(P)[1] = \mathsf{dGC}_k \oplus \big(\mathsf{dGC}_k/\mathsf{dGC}_k^s[1] \oplus \mathsf{dGC}_k/\mathsf{dGC}_k^t[1]\big)$ with differential d_c where

$$d_c(\Gamma, (\Gamma_1, \Gamma_2)) = (d_s\Gamma, (-P_1(\Gamma) - d_s\Gamma_1, -P_2(\Gamma) - d_s\Gamma_2)).$$

The mapping cone naturally fits in the short exact sequence

$$0 \longrightarrow \mathsf{dGC}_k/\mathsf{dGC}_k^s[1] \oplus \mathsf{dGC}_k/\mathsf{dGC}_k^t[1] \longrightarrow \mathrm{Cone}(P)[1] \longrightarrow \mathsf{dGC}_k \longrightarrow 0$$

which resembles the short exact sequence

$$0 \longrightarrow \mathsf{fM}_k^{+,1} \longrightarrow \mathsf{fM}_k^+ \longrightarrow \mathsf{fM}_k^{+,\geq 2} \longrightarrow 0$$

in that the cohomology of the leftmost and rightmost complexes is the same. We will relate these two short exact sequences with three injective chain maps a, b and $a \oplus b$, giving us the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{dGC}_k/\mathsf{dGC}_k^s[1] \oplus \mathsf{dGC}_k/\mathsf{dGC}_k^t[1] & \longrightarrow & \mathrm{Cone}(P)[1] & \longrightarrow & \mathsf{dGC}_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a\oplus b} & & \downarrow{\scriptstyle a} & & \\
0 & \longrightarrow & \mathsf{fM}_k^{+,1} & \longrightarrow & \mathsf{fM}_k^+ & \longrightarrow & \mathsf{fM}_k^{+,\geq 2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Q}_1 & \longrightarrow & \mathsf{Q}_2 & \longrightarrow & \mathsf{Q}_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If we show that a and b are quasi-isomorphisms, then the lower short exact sequence is acyclic, implying main Theorem 2.

Theorem 3.5.6. *The map $a \oplus b : \mathrm{Cone}(P)[1] \to \mathsf{fM}_k^+$ is a quasi-isomorphism.*

The proof follows from the two following propositions.

Proposition 3.5.7. *Let $b : \mathsf{dGC}_k/\mathsf{dGC}_k^s[1] \oplus \mathsf{dGC}_k/\mathsf{dGC}_k^t[1] \to \mathsf{fM}_k^{+,1}$ be the map defined by $b(\Gamma_1, \Gamma_2) = \big(d_u(\Gamma_1^{\frac{0}{\infty_1}}) + d_u(\Gamma_2^{\frac{\infty_1}{0}})\big)$. Then b is a chain map, and furthermore it is a quasi-isomorphism.*

Proof. For abbreviation, let $\mathcal{C}_k = \mathsf{dGC}_k/\mathsf{dGC}_k^s \oplus \mathsf{dGC}_k/\mathsf{dGC}_k^t$, and $\mathcal{C}_k[1]$ be the suspension. We first check that the differentials commute. The differential of the degree shifted complex $\mathcal{C}[1]$ is of the form $-d_s$. So

$$\begin{aligned} b \circ d(\Gamma_1, \Gamma_2) &= b\big(- d_s(\Gamma_1), -d_s(\Gamma_2)\big) \\ &= -d_u\big((d_s(\Gamma_1))^{\frac{0}{\infty_1}} + (d_s(\Gamma_2))^{\frac{\infty_1}{0}}\big) \\ &= -d_u d_s\big(\Gamma_1^{\frac{0}{\infty_1}} + \Gamma_2^{\frac{\infty_1}{0}}\big). \end{aligned}$$

On the other hand, $d_u \circ d_u = 0$, and $d_s d_u + d_u d_s = 0$ in fM_k^+, so

$$\begin{aligned} d \circ b(\Gamma_1, \Gamma_2) &= d\big(d_u(\Gamma_1^{\frac{0}{\infty_1}}) + d_u(\Gamma_2^{\frac{\infty_1}{0}})\big) \\ &= d_s d_u(\Gamma_1^{\frac{0}{\infty_1}} + \Gamma_2^{\frac{\infty_1}{0}}) + d_u d_u(\Gamma_1^{\frac{0}{\infty_1}} + \Gamma_2^{\frac{\infty_1}{0}}) \\ &= -d_u d_s(\Gamma_1^{\frac{0}{\infty_1}} + \Gamma_2^{\frac{\infty_1}{0}}). \end{aligned}$$

Hence $b \circ d = d \circ b$. Lastly, we show that $b^* : H^\bullet(\mathcal{C}_k[1]) \to H^\bullet(\mathsf{fM}_k^{+,1})$ is an isomorphism. We note that the chain map $h : \mathcal{C}_k \to \mathsf{fM}_k^{*,\geq 2}(\frac{0}{\infty_1}) \oplus \mathsf{fM}_k^{*,\geq 2}(\frac{\infty_1}{0})$ where $(\Gamma_1, \Gamma_2) \mapsto (\Gamma_1(\frac{0}{\infty_1}), \Gamma_2(\frac{\infty_1}{0}))$ is an isomorphism. Furthermore, Corollary 3.5.1 gives that the connecting morphism $\delta : H^\bullet(\mathsf{fM}_k^{*,\geq 2}(\frac{0}{\infty_1}) \oplus \mathsf{fM}_k^{*,\geq 2}(\frac{\infty_1}{0})) \to H^{\bullet+1}(\mathsf{fM}_k^{+,1})$ is an isomorphism, and is given by $\delta[\Gamma] = [d_u\Gamma]$. Hence we get the following chain of isomorphisms of cohomology groups

$$\begin{aligned} 0 \longrightarrow H^\bullet(\mathcal{C}_k[1]) \xrightarrow{id[1]} H^{\bullet-1}(\mathcal{C}_k) \xrightarrow{h} H^{\bullet-1}(\mathsf{fM}_k^{*,\geq 2}(\tfrac{0}{\infty_1}) \oplus \mathsf{fM}_k^{*,\geq 2}(\tfrac{\infty_1}{0})) \\ \xrightarrow{-\delta} H^\bullet(\mathsf{fM}_k^{+,1}) \longrightarrow 0 \end{aligned}$$

We can easily see that the composition of these maps is equal to b^*, finishing the proof. ☐

Let $a : \mathsf{dGC}_k \to \mathsf{mGC}_k^{+,\geq 2}$ be the quasi-isomorphism from Proposition 3.5.2 where $a(\Gamma) = \Gamma^\omega$. From the short exact sequence

$$0 \longrightarrow \mathsf{fM}_k^{+,1} \longrightarrow \mathsf{fM}_k^+ \longrightarrow \mathsf{fM}_k^{+,\geq 2} \longrightarrow 0$$

we see that $\mathsf{fM}_k^+ \cong \mathsf{fM}_k^{+,\geq 2} \oplus \mathsf{fM}_k^{+,1}$ as a vector space. Let $a \oplus b : \mathrm{Cone}(P)[1] \to \mathsf{fM}_k^+$ be the linear map defined using a and b the decomposition of fM_k^+.

Proposition 3.5.8. *The map $a \oplus b$ is a chain map.*

Proof. Let $(\Gamma, (\Gamma_1, \Gamma_2)) \in \mathrm{Cone}(P)[1]$. Then

$$\begin{aligned} (a \oplus b) \circ d_c\big(\Gamma, (\Gamma_1, \Gamma_2)\big) &= a(\big(d_s(\Gamma)\big) + b\big(- P_1(\Gamma) - d_s(\Gamma_1), -P_2(\Gamma) - d_s(\Gamma_2)\big) \\ &= \big(d_s(\Gamma)\big)^\omega - d_u\big((P_1(\Gamma))^{\frac{0}{\infty_1}} + ((d_s(\Gamma_1))^{\frac{0}{\infty_1}}\big) - d_u\big((P_2(\Gamma))^{\frac{\infty_1}{0}} + ((d_s(\Gamma_2))^{\frac{\infty_1}{0}}\big) \\ &= d_s(\Gamma^\omega) - d_u\big((\Gamma^{\frac{0}{\infty_1}}) + d_s(\Gamma_1^{\frac{0}{\infty_1}})\big) - d_u\big((\Gamma^{\frac{\infty_1}{0}}) - d_s(\Gamma_2^{\frac{\infty_1}{0}})\big) \\ &= d_s(\Gamma^\omega) + d_u(\Gamma^\omega) + d_s d_u(\Gamma_1^{\frac{0}{\infty_1}}) + d_s d_u(\Gamma_2^{\frac{\infty_1}{0}}) \\ &= d\big(\Gamma^\omega + d_u(\Gamma_1^{\frac{0}{\infty_1}}) + d_u(\Gamma_2^{\frac{\infty_1}{0}})\big) \\ &= d \circ (a \oplus b)\big(\Gamma, (\Gamma_1, \Gamma_2)\big), \end{aligned}$$

showing that g is a chain map. ☐

3.5.5 A second long exact sequence with $H^\bullet(\mathsf{fM}_k^+)$

We finish this chapter by describing a second long exact sequence with $H^\bullet(\mathsf{fM}_k^+)$. Let us use the abbreviation C_k for dGC_k, and similarly for other subcomplexes of dGC_k. Further, if $A \to B$ is a chain map of two chain complexes A, B, we write $A \oplus_c B[1]$ for the corresponding suspended mapping cone. We remark that C_k^{st} is the kernel of the map $P : \mathsf{C}_k \to \mathsf{C}_k/\mathsf{C}_k^s \oplus \mathsf{C}_k/\mathsf{C}_k^t$, and hence it is a subcomplex of $\mathrm{Cone}(P)[1]$. Further the mapping cone $\mathsf{C}_k^{s+t} \oplus_c (\mathsf{C}_k^{s+t}/\mathsf{C}_k^{st})[1]$ is a subcomplex of $\mathrm{Cone}(P)[1]$. We can check that the following diagram commutes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathsf{C}^{st} \oplus_c 0 & \to & \mathsf{C}_k^{s+t} \oplus_c (\mathsf{C}_k^{s+t}/\mathsf{C}_k^{st})[1] & \to & \mathsf{C}_k^{s+t}/\mathsf{C}_k^{st} \oplus_c (\mathsf{C}_k^{s+t}/\mathsf{C}^{st})[1] & \to & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
0 & \to & \mathsf{C}^{st} \oplus_c 0 & \to & \mathsf{C}_k \oplus_c (\mathsf{C}_k/\mathsf{C}_k^s \oplus \mathsf{C}_k/\mathsf{C}_k^t)[1] & \to & \mathsf{C}_k/\mathsf{C}_k^{st} \oplus_c (\mathsf{C}_k/\mathsf{C}_k^s \oplus \mathsf{C}_k/\mathsf{C}_k^t)[1] & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & \mathsf{C}_k^{\circlearrowleft} \oplus_c (\mathsf{C}_k^{\circlearrowleft} \oplus \mathsf{C}_k^{\circlearrowleft})[1] & \xrightarrow{id} & \mathsf{C}_k^{\circlearrowleft} \oplus_c (\mathsf{C}_k^{\circlearrowleft} \oplus \mathsf{C}_k^{\circlearrowleft})[1] & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where we recall that $\mathsf{C}_k^{\circlearrowleft} := \mathsf{C}_k/\mathsf{C}_k^{s+t}$.

Proposition 3.5.9.

1. $\mathsf{C}_k^{st} \oplus_c 0 \cong \mathsf{C}^{st}$.
2. *The complex* $\mathsf{C}_k^{\circlearrowleft} \oplus_c (\mathsf{C}_k^{\circlearrowleft} \oplus \mathsf{C}_k^{\circlearrowleft})[1]$ *is quasi-isomorphic to* $\mathsf{C}_k^{\circlearrowleft}[1]$.
3. *The complex* $\mathsf{C}_k^{s+t}/\mathsf{C}_k^{st} \oplus_c (\mathsf{C}_k^{s+t}/\mathsf{C}^{st})[1]$ *is acyclic.*

Proof. (1) follows by direct inspection. (2) follows by first considering the short exact sequence

$$0 \to \mathsf{C}_k^{\circlearrowleft}[1] \oplus \mathsf{C}_k^{\circlearrowleft}[1] \to \mathsf{C}_k^{\circlearrowleft} \oplus_c (\mathsf{C}_k^{\circlearrowleft} \oplus \mathsf{C}_k^{\circlearrowleft})[1] \to \mathsf{C}_k^{\circlearrowleft} \to 0$$

noting that the connecting morphism is the diagonal map $H^\bullet(\mathsf{C}_k^{\circlearrowleft}) \to H^\bullet(\mathsf{C}_k^{\circlearrowleft}) \oplus H^\bullet(\mathsf{C}_k^{\circlearrowleft})$, and thus it is injective. In particular, the induced map $H^\bullet(\mathsf{C}_k^{\circlearrowleft} \oplus_c (\mathsf{C}_k^{\circlearrowleft} \oplus \mathsf{C}_k^{\circlearrowleft})[1]) \to \mathsf{C}_k^{\circlearrowleft}$ is zero. By exactness of the long exact sequence, we get $H^\bullet(\mathsf{C}_k^{\circlearrowleft} \oplus_c (\mathsf{C}_k^{\circlearrowleft} \oplus \mathsf{C}_k^{\circlearrowleft})[1]) \cong H^\bullet(\mathsf{C}_k^{\circlearrowleft}[1] \oplus \mathsf{C}_k^{\circlearrowleft}[1])/H^\bullet(\mathsf{C}_k^{\circlearrowleft}[1]) \cong \mathsf{C}_k^{\circlearrowleft}[1]$. (3) follows from the fact that a mapping cone $\mathrm{Cone}(f)$ is acyclic if and only if f is a quasi-isomorphism. This complex is the mapping cone of the identity morphism, and hence the cone is acyclic. □

By reading the diagram, we get the following corollary by considering the induced long exact sequence produced by the middle column or row.

Corollary 3.5.10. *There is a long exact sequence on cohomology*

$$\cdots \to H^{\bullet-2}(\mathsf{dGC}_k^{\circlearrowleft}) \to H^\bullet(\mathsf{dGC}_k^{st}) \to H^\bullet(\mathsf{fM}_k^+) \to H^{\bullet-1}(\mathsf{dGC}_k^{\circlearrowleft}) \to H^{\bullet+1}(\mathsf{dGC}_k^{st}) \to \cdots$$

By Lemma 2.4.16, $H^l(\mathsf{dGC}_3^{\circlearrowleft}) = 0$ for $-2 \leq l \leq 1$. In particular, $H^0(\mathsf{fM}_3^+) \cong H^0(\mathsf{dGC}_3^{st})$. Finally, $H^0(\mathsf{dGC}_3^{st}) = \mathfrak{grt} \oplus \mathfrak{grt}$ by Proposition 3.4.6, giving us a second proof of the main theorem.

3.6 Main theorems

In this section, we first give a review of the bi-weighted complexes that have been defined in the previous sections, followed by stating and proving the main theorems of this chapter.

3.6.1 Summary of bi-weighted graph complexes

The complex fwGC_k is the complex of all possible bi-weighted graphs. The complex fwGC_k^+ is the subcomplex of fwGC_k of graphs that have at least one vertex with positive out-weight and one vertex with positive in-weight.

The full bi-weighted graph complex fwGC_k split as

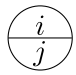
$$\mathsf{fwGC}_k = \mathsf{fwGC}_k^0 \oplus \mathsf{fwGC}_k^*$$

where fwGC_k^0 is the subcomplex of graphs where all vertices are decorated by $\frac{0}{0}$ and wGC_k^* its complement. The complex fwGC_k^0 is isomorphic to the complex $\mathsf{dGC}_k^{\geq 3,\circlearrowleft}$ (see Proposition 3.2.8.

The complexes wGC_k^* and wGC_k^+ split as

$$\mathsf{wGC}_k^* = \mathsf{b}_0\mathsf{wGC}_k \oplus \mathsf{wGC}_k^* \quad \text{and} \quad \mathsf{wGC}_k^+ = \mathsf{b}_0\mathsf{wGC}_k \oplus \mathsf{wGC}_k^+$$

where $\mathsf{b}_0\mathsf{wGC}_k$ is the subcomplex of graphs with loop number zero, and wGC_k^* and wGC_k^+ are the subcomplexes of graphs with loop number greater than or equal to one. The cohomology of $\mathsf{b}_0\mathsf{wGC}_k$ is generated by the following series (see Proposition 3.2.9):

$$\sum_{\substack{i,j\geq 1\\ i+j\geq 3}} (i+j-2)$$ 

The complex qGC_k^* is the subcomplex of wGC_k^* of graphs whose vertices are independently decorated by the four bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$ and $\frac{0}{0}$ such that no graph is only decorated by $\frac{0}{0}$ and that univalent vertices are only decorated by $\frac{\infty_1}{\infty_1}$. The complex qGC_k^+ is the subcomplex of qGC_k^+ of graphs where at least one vertex is decorated by $\frac{\infty_1}{\infty_1}$ or a pair of vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$. It follows that qGC_k^+ is a subcomplex of wGC_k^+.

The complex tGC_k^* is the quotient complex $\mathsf{qGC}_k^*/\mathsf{qGC}_k^{*,0}$ where $\mathsf{qGC}_k^{*,0} \subseteq \mathsf{qGC}_k^*$ is the subcomplex of graphs with at least one vertex decorated by $\frac{0}{0}$. In other words, tGC_k^* is generated by graphs whose vertices are independently decorated by the three bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$. The complex tGC_k^+ is the subcomplex of tGC_k^* of graphs where at least one vertex is decorated by $\frac{\infty_1}{\infty_1}$ or a pair of vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$. It follows that tGC_k^+ is a quotient complex of qGC_k^+.

We recall two definitions.

- Let Γ be an undecorated directed graph. Set $\Gamma^{\frac{\infty_1}{0}}$ to be the decorated graph in tGC_k^* of the same shape as Γ where all univalent vertices are decorated by $\frac{\infty_1}{\infty_1}$ and all non-univalent vertices are decorated by $\frac{\infty_1}{0}$. We similarly define $\Gamma^{\frac{0}{\infty_1}}$.

- Let Γ be an undecorated directed graph *without* any univalent vertices. We then set $\Gamma^\omega \in \mathsf{tGC}_k^+$ to be the sum of graphs

$$\Gamma^\omega = \sum_d \Gamma_d.$$

where d is a full decoration of Γ, Γ_d is the graph Γ decorated by d, and the sum is over all possible decorations d of Γ such that at least one vertex has positive out-weight one has positive in-weight.

The complex mGC_k^* is a subcomplex of tGC_k^* generated by graphs on the forms Γ^ω, $\Gamma^{\frac{\infty_1}{0}}$ and $\Gamma^{\frac{0}{\infty_1}}$ for all directed graphs Γ. The complex mGC_k^+ is a subcomplex of mGC_k^* such that no graph is only decorated by $\frac{\infty_1}{0}$ or only $\frac{0}{\infty_1}$. It follows that mGC_k^+ is a subcomplex of tGC_k^+.

We recall that an *antenna vertex* of a directed graph Γ is either univalent, or becomes univalent at some stage after recursively removing univalent vertices from Γ. An *antenna* of Γ is a connected subgraph γ of Γ consisting of antenna vertices such that there are no other antenna vertex adjacent to γ. The complex fM_k^* is the subcomplex of mGC_k^* of graphs whose antennas contain at most one vertex. Similarly, fM_k^+ is the subcomplex of mGC_k^+ of graphs whose antennas only contain one vertex.
The complexes that have been used in this chapter are related as follows:

$$\mathsf{wGC}_k^* \hookleftarrow \mathsf{qGC}_k^* \twoheadrightarrow \mathsf{tGC}_k^* \hookleftarrow \mathsf{mGC}_k^* \hookleftarrow \mathsf{fM}_k^* \longleftarrow \mathsf{dGC}_k$$

$$\mathsf{wGC}_k^+ \hookleftarrow \mathsf{qGC}_k^+ \twoheadrightarrow \mathsf{tGC}_k^+ \hookleftarrow \mathsf{mGC}_k^+ \hookleftarrow \mathsf{fM}_k^+ \longleftarrow \mathrm{Cone}(P)[1]$$

where $\mathrm{Cone}(P)[1]$ is the desuspended cone complex of the chain map

$$P : \mathsf{dGC}_k \to \mathsf{dGC}_k^{no\ s} \oplus \mathsf{dGC}_k^{no\ t}.$$

Proposition 3.6.1. *Let* $\mathsf{dGC}_k \to \mathsf{wGC}_k^*$ *be the map where a graph* Γ *is mapped to the sum of all possible bi-weights to put on* Γ*, excluding the decoration with only* $\frac{0}{0}$*. Then this map is a quasi-isomorphism.*

Proof. This map restricts to chain maps to all of the complexes in the diagram above, making it commute. One checks that all of these maps are quasi-isomorphisms, starting with that the restriction to dGC_k is an isomorphism. □

Proposition 3.6.2. *Consider the decomposition*

$$\mathrm{Cone}(P)[1] = \mathsf{dGC}_k \oplus \big(\mathsf{dGC}_k/\mathsf{dGC}_k^s[1] \oplus \mathsf{dGC}_k/\mathsf{dGC}_k^t[1]\big).$$

Let $\mathrm{Cone}(P)[1] \to \mathsf{wGC}_k^+$ *be the map where*

- *A graph* $\Gamma \in \mathsf{dGC}_k$ *is mapped to the sum of all possible bi-weights to put on* Γ *such that at least one vertex has positive in-weight and one vertex has positive out-weight.*

- *A graph* $\Gamma \in \mathsf{dGC}_k/\mathsf{dGC}_k^t[1]$ *is mapped to* $d_u(\Gamma^{\frac{0}{\infty_0}})$*, where* $\Gamma^{\frac{0}{\infty_0}}$ *is the graph* Γ *with all vertices decorated by* $\frac{0}{\infty_0}$ *and* d_u *is the differential of* wGC_k^* *only creating univalent vertices.*

- *A graph* $\Gamma \in \mathsf{dGC}_k/\mathsf{dGC}_k^t[1]$ *is mapped to* $d_u(\Gamma^{\frac{\infty_0}{0}})$*, where* $\Gamma^{\frac{\infty_0}{0}}$ *is the graph* Γ *with all vertices decorated by* $\frac{\infty_0}{0}$ *and* d_u *is the differential of* wGC_k^* *only creating univalent vertices.*

Then this map is a quasi-isomorphism.

Proof. The argument is equivalent to that of Proposition 3.6.1. □

3.6.2 Main theorems and proofs

Theorem 3.6.3. *There is a quasi-isomorphism*

$$\mathbb{K} \oplus \mathsf{dGC}_{c+d+1}^{\geq 3, \circlearrowleft} \oplus \mathsf{dGC}_{c+d+1} \to Der^{\bullet}(\mathcal{H}olieb_{c,d}^{\circlearrowleft}).$$

Proof. This follows from the decomposition of $Der^{\bullet}(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$ over graphs of loop number zero and bi-weighted graphs only decorated by $\frac{0}{0}$, together with the quasi-isomorphism of Proposition 3.2.9, Proposition 3.2.8 and Proposition 3.6.1. □

Theorem 3.6.4. *There is a quasi-isomorphism*

$$\mathbb{K} \oplus \mathrm{Cone}(P)[1] \to Der(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$$

where $\mathrm{Cone}(P)[1]$ *is the desuspended cone complex of the chain map* $P : \mathsf{dGC}_k \to \mathsf{dGC}_k^{no\ s} \oplus \mathsf{dGC}_k^{no\ t}$.

Proof. This follows from the decomposition of $Der(\mathcal{H}olieb_{c,d}^{\circlearrowleft})$ over graphs of loop number zero, together with the quasi-isomorphisms of Proposition 3.2.9 and Proposition 3.6.2. □

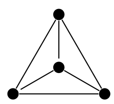
Corollary 3.6.5. *There is an isomorphism of vector spaces*

$$H^0(Der(\mathcal{H}olieb_{1,1}^{\circlearrowleft})) \cong \mathbb{K} \oplus \mathfrak{grt} \oplus \mathfrak{grt}.$$

Proof. This follows directly from Proposition 3.5.4. □

3.7 Explicit example of two cohomology classes of $Der(\mathcal{H}olieb_{1,1}^{\circlearrowleft})$

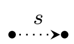
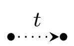
One of the simplest cohomology classes in $H^0(\mathsf{GC}_2) = \mathfrak{grt}$ is given by the tetrahedron class



which has loop number three. We have an explicit morphism of complexes $H^0(\mathsf{dGC}_3^{st}) \to H^0(Der(\mathcal{H}olieb_{1,1}^{\circlearrowleft}))$ given by attaching hairs to a cohomology class of dGC_3^{st} as described in Corollary 3.5.10. Therefore, to see the above mentioned two homotopy inequivalent actions of the tetrahedron class on $\mathcal{H}olieb_{1,1}^{\circlearrowleft}$, we have to find *explicit* incarnations of the tetrahedron class in dGC_3^{st}. In this section we explicitly describe these two incarnations denoted by α^s and α^t, both having loop number three.

3.7.1 A reduced version of dGC_k^{st}

Following S. Merkulovs paper [M3], we consider a "smaller" version $\widehat{\mathsf{dGC}}_k^{st}$ of the complex dGC_k^{st}, which is quasi-isomorphic to it. First, define the graph complex $\overline{\mathsf{dGC}}_k$ generated by graphs whose vertices are at least trivalent and which have four kinds of edges:

- Solid edges of degree zero
- Dotted s-edges of degree one 
- Dotted t-edges of degree one 
- wavy edges of degree two

Let Γ be a graph and let e be the total number of edges, v the number of vertices, e_1 the number of s-edges and t-edges, and e_2 the number of wavy edges. Then the degree of Γ is

$$|\Gamma| = (v-1)k + (1-k)e + e_1 + 2e_2.$$

The space is defined as the graphs invariant under permutations of vertices or edges depending on the parity of k, with the change that labels of non-solid edges can only be permuted with edges of the same type. Further permutations of dotted edges give the sign of the permutation for k odd. Further, the non-solid edges satisfy the relations

- $\bullet\overset{s}{\cdots\!\!\rightarrow}\bullet = (-1)^{k+1} \bullet\overset{s}{\leftarrow\!\!\cdots}\bullet$
- $\bullet\overset{t}{\cdots\!\!\rightarrow}\bullet = (-1)^{k+1} \bullet\overset{t}{\leftarrow\!\!\cdots}\bullet$
- $\bullet\!\leadsto\!\bullet = (-1)^{k+1} \bullet\!\leftarrow\!\sim\!\bullet$

The differential is defined as $d = d_V + d_E$, where d_V acts by splitting vertices so that neither univalent nor bivalent vertices are created, and with the same sign rule as in dGC_d. The term d_E acts on edges accordingly:

- $d_E\ \bullet\!\longrightarrow\!\bullet\ =\ \bullet\overset{t}{\cdots\!\!\rightarrow}\bullet\ -\ \bullet\overset{s}{\cdots\!\!\rightarrow}\bullet$
- $d_E\ \bullet\overset{s}{\cdots\!\!\rightarrow}\bullet\ =\ \bullet\!\leadsto\!\bullet$
- $d_E\ \bullet\overset{t}{\cdots\!\!\rightarrow}\bullet\ =\ \bullet\!\leadsto\!\bullet$
- $d_E\ \bullet\!\leadsto\!\bullet\ = 0$

We say that a vertex in such a graph is a *solid source* if the attached edges are solid and outgoing or t-dotted. A vertex is a *solid target* if the attached edges are solid and incoming or s-dotted. Consider the subcomplex $\overline{\mathsf{dGC}}_k^{st}$ of $\overline{\mathsf{dGC}}_k$ generated by graphs that either

- have at least one solid source and one solid target,
- have at least one dotted s-edge and one solid target,
- have at least one dotted t-edge and one solid source,
- have at least one dotted s-edge and one dotted t-edge,
- have at least one wavy edge.

Consider the map $f : \overline{\mathsf{dGC}}_k^{st} \to \mathsf{dGC}_k^{st}$ where a graph is mapped to the graph in dGC_k^{st} where solid edges remain the same, but s-dotted, t-dotted, and wavy edges are replaced by the following edges:

- $\bullet\overset{s}{\cdots\!\!\rightarrow}\bullet \mapsto \bullet\!\longleftarrow\!\bullet\!\longrightarrow\!\bullet$
- $\bullet\overset{t}{\cdots\!\!\rightarrow}\bullet \mapsto \bullet\!\longrightarrow\!\bullet\!\longleftarrow\!\bullet$
- $\bullet\!\leadsto\!\bullet \mapsto \bullet\!\longrightarrow\!\bullet\!\longleftarrow\!\bullet\!\longrightarrow\!\bullet + (-1)^k\ \bullet\!\longleftarrow\!\bullet\!\longrightarrow\!\bullet\!\longleftarrow\!\bullet$

Then the map f is a quasi-isomorphism [M3]. Consider the subcomplex Z_d of $\overline{\mathsf{dGC}}_k^{st}$ of graphs that either

- have at least one s-edge and one t-edge,
- have at least one s-edge and one wavy edge,
- have at least one $t$edge and one wavy edge,

- have at least two wavy edges.

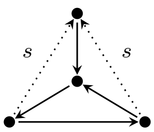
This subcomplex is acyclic, hence the projection $\overline{\mathsf{dGC}}_k^{st} \to \overline{\mathsf{dGC}}_k^{st}/\mathsf{Z}_d$ is a quasi-isomorphism. Let $\widehat{\mathsf{dGC}}_k^{st} = \overline{\mathsf{dGC}}_k^{st}/\mathsf{Z}_d$. This complex consists of graphs that either

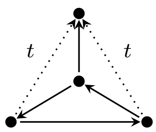
- have only solid edges such that there is at least one solid source and one solid target,
- have only solid and dotted s-edges such that there is at least one solid target and one dotted s-edge,
- have only solid and dotted t-edges such that there is at least one solid source and one dotted t-edge,
- have only solid and wavy edges such that there is at least one wavy edge.

We will use this complex when finding the cohomology classes.

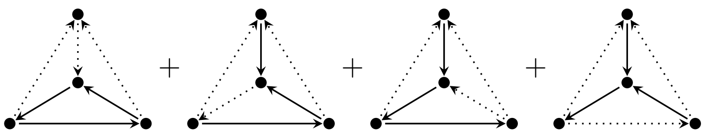
3.7.2 Calculating the cohomology classes

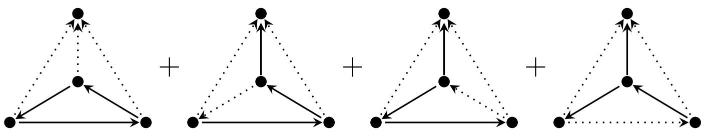
Let γ^s and γ^t be the graphs

$\gamma^s =$ 

$\gamma^t =$ 

Note that γ^s is in $\widehat{\mathsf{dGC}}_3^{s}$ but not in $\widehat{\mathsf{dGC}}_3^{st}$. Similarly, γ^t is in $\widehat{\mathsf{dGC}}_3^{t}$ but not in $\widehat{\mathsf{dGC}}_3^{st}$. The vertices of γ^s are all trivalent, and so the differential acts by only changing solid edges to s-edges. We have omitted the s on s-edges for clarity in the following pictures. Hence

$\alpha^s := d(\gamma^s) =$ 

$\alpha^t := d(\gamma^t) =$ 

Note that each graph contains a source and a target vertex, and hence α^s and α^t are cycles in $\widehat{\mathsf{dGC}}_3^{st}$.

Theorem 3.7.1. *The elements α^s and α^t are non-trivial cycles of $\widehat{\mathsf{dGC}}_3^{st}$. Furthermore, they represent two different cohomology classes in $H(\widehat{\mathsf{dGC}}_3^{st})$.*

Let X and A be the spaces of graphs of loop number three with at least one source and target vertex of degree -1 and 0 respectively. Then the differential is a linear map $d : X \to A$ and we want to show that there is no $\beta^s \in X$ so that $d(\beta^s) = \alpha^s$. To prove this statement at this stage, we would need to compute the differential of hundreds of graphs in X and check that α^s is linearly independent of them. Instead, we can reduce this problem to giving us only a handful of graphs to study. Let A^s be the subspace of A of graphs spanned by tetrahedron graphs with three s-edges. Note that $\alpha^s \in A^s$. The graphs $a_1, ..., a_{10}$ in figure 3.5 form a basis for A^s.
Let $\overline{A}$ be the orthogonal complement of A^s so that $A = A^s \oplus \overline{A}$ and let $p_s : A^s \oplus \overline{A} \to A^s$ be the projection. Let $\overline{X}$ be the kernel of the map $p_s \circ d : X \to A^s$ and X^s its orthogonal complement so that $X = X^s \oplus \overline{X}$. The space X^s is spanned by the graphs $x_1, ..., x_{11}$ in

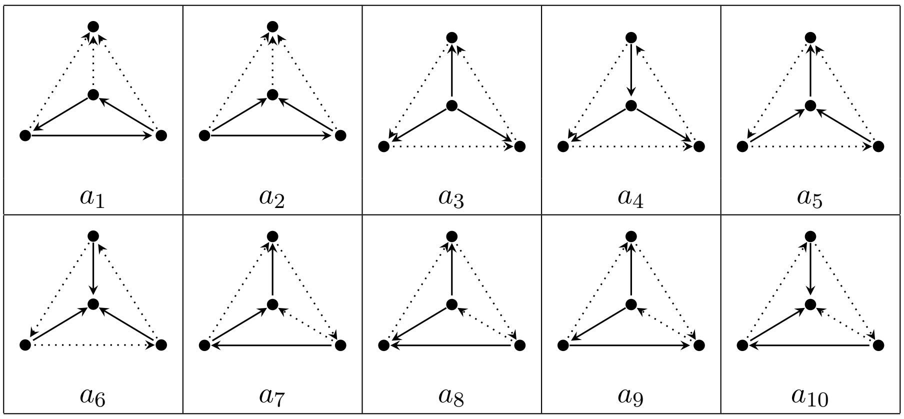


Figure 3.5: Basis of A^s

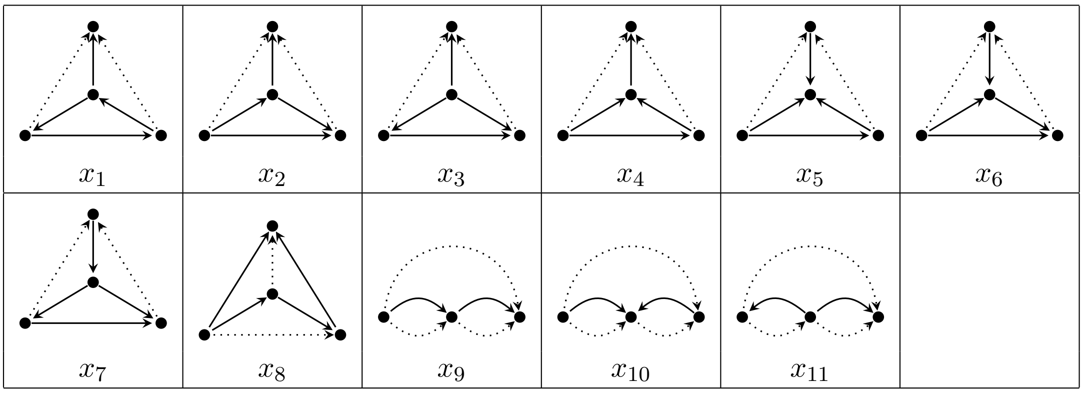


Figure 3.6: Basis of X^s

figure 3.6. Let $\iota^s : X^s \to X^s \oplus \overline{X}$ be the inclusion map. We get the following commutative diagram:

$$\begin{array}{ccc} X^s \oplus \overline{X} & \xrightarrow{\ d\ } & A^s \oplus \overline{A} \\ \uparrow{\scriptstyle \iota_s} & & \downarrow{\scriptstyle p_s} \\ X^s & \xrightarrow{\ p_a \circ\ d\ \circ \iota_s\ } & A^s \end{array}$$

Lemma 3.7.2. *Let $a^s \in A^s$. If there is an element $b \in X = X^s \oplus \overline{X}$ such that $d(b) = a^s$, then there is $b^s \in X^s$ such that $\iota_s \circ d \circ p_s(b^s) = a^s$.*

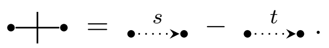
Proof. Suppose that such an element b exists. Then $b = b^s + \overline{b}$ where $b^s \in X^s$ and $\overline{b} \in \overline{X}$. Now $d(b^s + \overline{b}) = a$. The space $\overline{X}$ is the kernel space of $p_s \circ d$, and so $p_s \circ d(\overline{b}) = 0$. Hence $p_s \circ d(b) = p_s \circ d(b^s) = a^s$. Finally, $b^s \in X^s$ and so $p_s \circ d \circ \iota_s(b^s) = a$. □

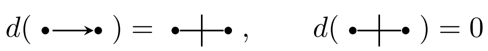
Proof of Theorem 3.7.1. By the contrapositive statement of Lemma 3.7.2, it is sufficient for us to show that there is no $\beta^s \in X^s$ so that $p_s \circ d \circ \iota_s(\beta^s) = \alpha^s$. We have already established the basis $a_1 ... a_{10}$ of A^s and $x_1 ... x_{11}$ of X^s. In this basis, $\alpha^s = a_1 + a_5 + a_7 - a_{10}$. We compute the image of each vector $x_1 ... x_{11}$ under the differential and get a matrix representation of the map $d_s \circ d \circ p_s : X^s \to A^s$ as seen in figure 3.7. By using the application wolfram-alpha, we can, with Gaussian elimination, see that α^s is not in the image of $d_s \circ d \circ p_s$. Hence we conclude that α^s is a non-trivial cycle of $\widehat{\mathsf{dGC}}_3^{st}$. The argument to show that α^t is a non-trivial cycle is analogous by considering the same graphs as for α^t but with all solid edges reversed and all s-edges turned into t-edges. Finally, let us show that α^s and α^t represent different equivalence classes in $H(\widehat{\mathsf{dGC}}_3^{st})$. For a contraction,

$$\begin{cases} d_s(x_1) &= a_1 + a_4 - a_7 - a_9 \\ d_s(x_2) &= a_2 + a_4 - a_8 + a_9 \\ d_s(x_3) &= a_2 + a_3 - a_8 - a_9 \\ d_s(x_4) &= a_2 + a_5 - a_7 + a_9 \\ d_s(x_5) &= a_2 + a_6 + a_8 - a_{10} \\ d_s(x_6) &= a_2 + a_5 + a_8 + a_{10} \\ d_s(x_7) &= a_2 + a_4 + a_7 + a_{10} \\ d_s(x_8) &= a_7 + a_8 - a_9 + a_{10} \\ d_s(x_9) &= -a_4 + a_5 - a_7 + a_8 \\ d_s(x_{10}) &= -a_5 + a_6 - 2a_{10} \\ d_s(x_{11}) &= -a_3 + a_4 + 2a_9 \end{cases} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

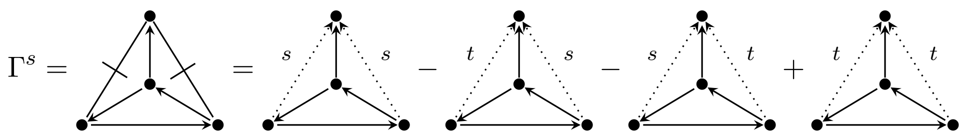
Figure 3.7: Matrix representation of the map $d_s = p_s \circ d \circ \iota_s : X^s \to A^s$.

suppose that there exists $\beta \in X$ such that $d(\beta) = \alpha^s - \alpha^t$. Let A^t and X^t be "targeted" versions of the spaces A^s and X^s. More concretely, they are the vector spaces generated by the graphs in Figure 4 and 5 but where the solid edges have the opposite direction and the s-edges have been replaced by t-edges. It is clear that $A^s \cap A^t = X^s \cap X^t = \{0\}$. Let $\widehat{A}$ be the obvious (say orthogonal) complement of the subspace $A^s \oplus A^t$ of A, and let $\widehat{X}$ be the similar complement of the subspace $X^s \oplus X^t$ in X so that $A = A^s \oplus A^t \oplus \widehat{A}$ and $X = X^s \oplus X^t \oplus \widehat{X}$. Let $p : A^s \oplus A^t \oplus \widehat{A} \to A^s \oplus A^t$ be the projection map. We note that $\widehat{X}$ is the kernel of the map $p \circ d : X \to A^s \oplus A^t$. By Lemma 3.7.2, there is a $\beta' \in X^s \oplus X^t$ so that $p \circ d(\beta') = \alpha^s - \alpha^t$. Now $\beta' = \beta^s + \beta^t$ for some $\beta^s \in X^s$ and $\beta^t \in X^t$. Further, X^t is a subspace of $\overline{X}$, the kernel of the map $p_s \circ d : X \to A^s$. Hence $p \circ d(\beta^s) = \alpha^s$. But this contradicts α^s being a non-trivial cycle, finishing the proof. ☐

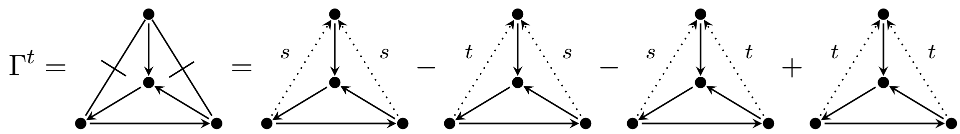
The epimorphism $\mathsf{dGC}_3^{st} \to \widehat{\mathsf{dGC}}_3^{st}$ is a quasi-isomorphism, so it remains to find a lift of α^s and α^t to cycles in dGC_3^{st}. Let us introduce a new kind of edge, defined as a linear combination of an s-edge and a t-edge . The differential split as $d = d_V + d_E$ where d_V acts on vertices by vertex splitting and d_E acts on edges accordingly:



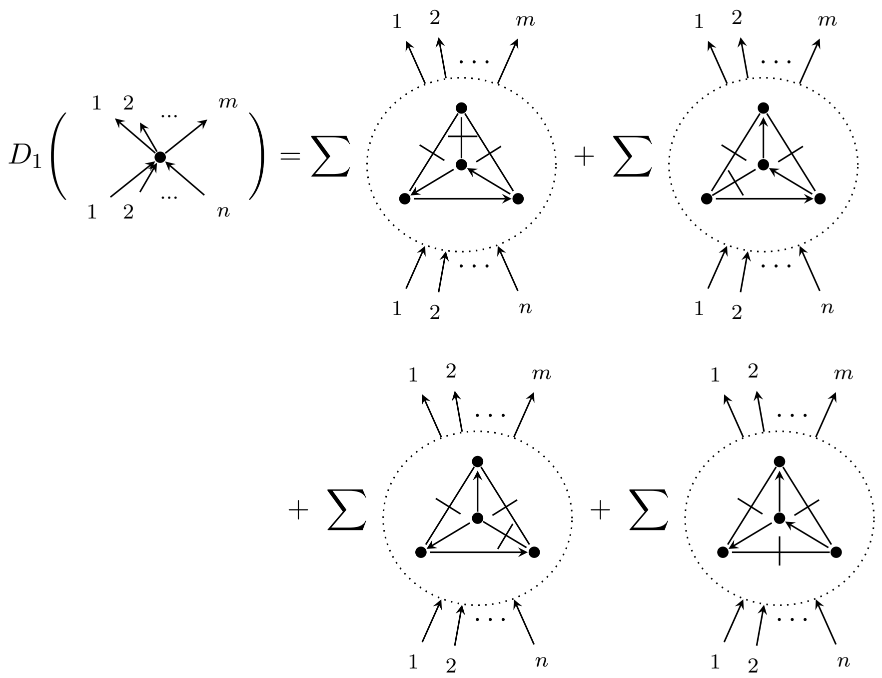
Consider the graph

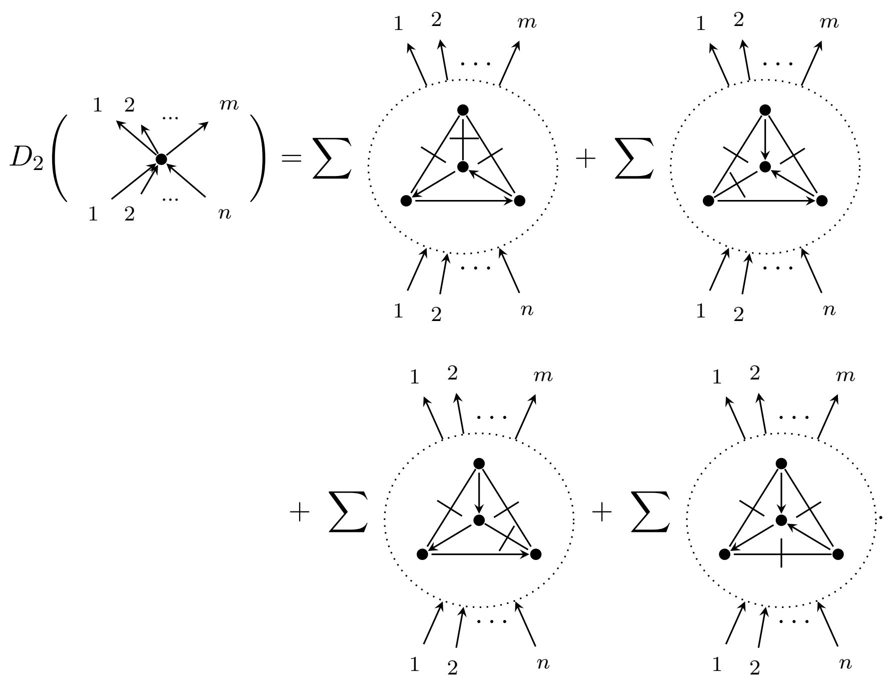


Note that γ^s is the first term, while the other terms $\overline{\gamma}^s$ are in dGC_3^{st}. Hence $d(\Gamma^s) = d(\gamma^s) + d(\overline{\gamma}^s) = \alpha^s + d(\overline{\gamma}^s)$ is a cycle, and it is a lift of α^s. The corresponding lift of α^t is $d(\Gamma^t)$, where



Now we get an explicit action of these classes on $\mathcal{H}olieb_{1,1}^{\circlearrowleft}$ as the derivations D_1 and D_2, which respectively act on (m, n) corollas as





These formulae give us the required explicit homotopy inequivalent actions of the tetrahedron class in the Kontsevich graph complex GC_2 as derivations of the wheeled properad $\mathcal{H}olieb_{1,1}^{\circlearrowleft}$.

Chapter 4

Derivations of quasi- and pseudo-Lie bialgebras

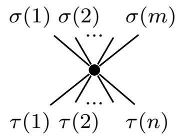
In this chapter we compute the cohomology of the derivation complexes of the quasi-Lie bialgebra properad and the pseudo-Lie bialgebra properad. We do this in both the wheeled and unwheeled cases. The cohomology is computed by establishing explicit quasi-isomorphisms to directed Kontsevich graph complexes. The content of this chapter is largely based on the article *Graph complexes and Deformation theories of the (wheeled) properads of quasi- and pseudo-Lie bialgebras* [F2].

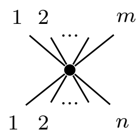
4.1 Derivation complexes of the quasi- and pseudo-Lie bialgebra properads

4.1.1 The derivation complexes of wheeled quasi-Lie bialgebra properad

Definition 4.1.1. Let $\mathcal{QH}olieb^{+}_{c,d}$ be the free dg properad generated by the $\mathbb{S}$-bimodule $QLb^{+}_{c,d} = \{QLb^{\bullet}_{c,d}(m,n)\}_{m,n\geq 0}$ where

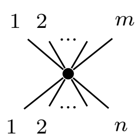
$$QLb^{+}_{c,d}(m,n) = \mathrm{sgn}_m^{\otimes c} \otimes \mathrm{sgn}_n^{\otimes d}[c(m-1)+d(n-1)-1]$$

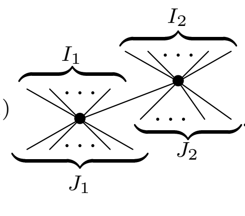
$= \Big\langle$ 

$= (-1)^{c|\sigma|+d|\tau|}$ 

$\mid \sigma \in \mathbb{S}_m,\ \tau \in \mathbb{S}_n \Big\rangle_{\substack{m\geq 1\\ n\geq 0}}$

and whose differential acts on generators as

$\delta\Big($ 

$\Big) = \sum_{\substack{[m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{[n]=J_1\sqcup J_2} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$ 

.

Remark 4.1.2. The complexes $\mathcal{QH}olieb^{+}_{c,d}$ and $\mathcal{H}olieb^{\bullet}_{c,d}$ only differ from $\mathcal{QH}olieb_{c,d}$ in that they have additional generators (see Figure 4.1.1) and that the differential is modified accordingly. Furthermore, the natural projections

$$Q\pi^{+} : \mathcal{QH}olieb^{+}_{c,d} \twoheadrightarrow \mathcal{QH}olieb_{c,d} \text{ and } P\pi^{\bullet} : \mathcal{H}olieb^{\bullet}_{c,d} \twoheadrightarrow \mathcal{QH}olieb_{c,d}$$

are morphisms of dg properads. The morphism $Q\pi^{\bullet}$ factors through $Q\pi^{+}$ as

$$\mathcal{H}olieb^{\bullet}_{c,d} \twoheadrightarrow \mathcal{QH}olieb^{+}_{c,d} \twoheadrightarrow \mathcal{QH}olieb_{c,d}.$$

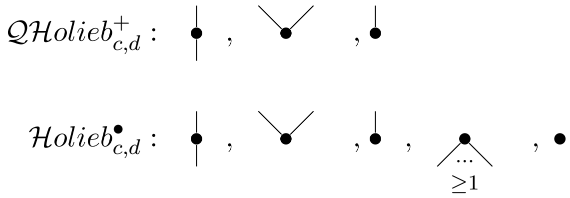


Figure 4.1: Additional generators of $\mathcal{QH}olieb^+_{c,d}$ and $\mathcal{H}olieb^\bullet_{c,d}$ compared to $\mathcal{QH}olieb_{c,d}$.

Recall that we use the notation $\widehat{\mathcal{QH}olieb}_{c,d}$ to denote the loop number completion of $\mathcal{QH}olieb_{c,d}$ (and similarly for any other properad). We define the following derivation complexes of $\mathcal{QH}olieb_{c,d}$ analogously to those in Definition 3.1.5.

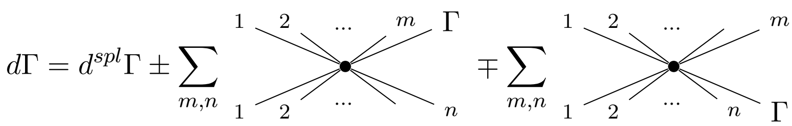
Definition 4.1.3. The complex $Der^\bullet(\mathcal{QH}olieb^\circlearrowleft_{c,d})$ is the derivation complex with respect to the morphism

$$Q\pi^{\bullet,\circlearrowleft} : \widehat{\mathcal{H}olieb}^{\bullet,\circlearrowleft}_{c,d} \twoheadrightarrow \widehat{\mathcal{QH}olieb}^{\circlearrowleft}_{c,d}$$

induced by $Q\pi^\bullet$. Similarly, let $Der(\mathcal{QH}olieb^\circlearrowleft_{c,d})$ be the derivation complex with respect to the morphism

$$Q\pi^{+,\circlearrowleft} : \widehat{\mathcal{QH}olieb}^{+,\circlearrowleft}_{c,d} \twoheadrightarrow \widehat{\mathcal{QH}olieb}^{\circlearrowleft}_{c,d}$$

induced by $Q\pi^+$. The differential d on both complexes is given by the vertex splitting differential d^{spl} from $\mathcal{QH}olieb_{c,d}$ with the additional terms of attaching (m,n) corollas to every hair for all integers m,n:



(4.1)

The sign rule for this formula can be found in [MW1].

Definition 4.1.4. Consider the natural projection morphism $P\pi^\bullet : \mathcal{H}olieb^\bullet_{c,d} \twoheadrightarrow \mathcal{PH}olieb_{c,d}$ of dg properads. Let $Der(\mathcal{PH}olieb^\circlearrowleft_{c,d})$ be the derivation complex with respect to the morphism

$$P\pi^{\bullet,\circlearrowleft} : \widehat{\mathcal{H}olieb}^{\bullet,\circlearrowleft}_{c,d} \twoheadrightarrow \widehat{\mathcal{PH}olieb}^{\circlearrowleft}_{c,d}$$

induced by $P\pi^\bullet$. The differential is defined as in Equation 4.1 above.

Remark 4.1.5. Since the derivation complexes are induced by maps from a free properad, they can be described as

$$Der^\bullet(\mathcal{QH}olieb^\circlearrowleft_{c,d}) \cong \prod_{m,n\geq 0} (\mathcal{QH}olieb^\circlearrowleft_{c,d}(m,n) \otimes sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|})^{\mathbb{S}_m\times\mathbb{S}_n}[1 + c(1-m) + d(1-n)]$$

$$Der(\mathcal{QH}olieb^\circlearrowleft_{c,d}) \cong \prod_{\substack{m\geq 1\\ n\geq 0}} (\mathcal{QH}olieb^\circlearrowleft_{c,d}(m,n) \otimes sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|})^{\mathbb{S}_m\times\mathbb{S}_n}[1 + c(1-m) + d(1-n)]$$

$$Der(\mathcal{PH}olieb^\circlearrowleft_{c,d}) \cong \prod_{m,n\geq 0} (\mathcal{PH}olieb^\circlearrowleft_{c,d}(m,n) \otimes sgn_m^{\otimes|c|} \otimes sgn_n^{\otimes|d|})^{\mathbb{S}_m\times\mathbb{S}_n}[1 + c(1-m) + d(1-n)]$$

where $\mathcal{QH}olieb^\circlearrowleft_{c,d}(m,n)$ is the set of generating graphs of $\mathcal{QH}olieb^\circlearrowleft_{c,d}$ with m outputs and n inputs, and respectively for $\mathcal{PH}olieb^\circlearrowleft_{c,d}(m,n)$.

4.1.2 The derivation complexes of the unwheeled quasi- and pseudo-Lie bialgebra properads

Definition 4.1.6. Let $Der(\mathcal{QH}olieb^{\uparrow}_{c,d})$ be the derivation complex with respect to the morphism

$$Q\pi^+ : \widehat{\mathcal{QH}olieb}^{+}_{c,d} \twoheadrightarrow \widehat{\mathcal{QH}olieb}_{c,d}.$$

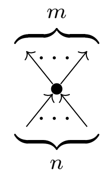
Similarly let $Der(\mathcal{PH}olieb^{\uparrow}_{c,d})$ be the derivation complex with respect to the morphism

$$P\pi^\bullet : \widehat{\mathcal{H}olieb}^{\bullet}_{c,d} \twoheadrightarrow \widehat{\mathcal{PH}olieb}_{c,d}.$$

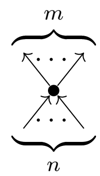
The differential is defined as in Equation 4.1 above.

Similar to the derivation complexes of $\mathcal{H}olieb^{\circlearrowleft}_{c,d}$, the loop number of a graph in the derivation complexes above remains invariant under the differential. The components of graphs with loop number zero in the derivation complexes of quasi-Lie bialgebras above are all the same, and we denote it by $Der_{b=0}(\mathcal{QH}olieb_{c,d})$. The analogous statement is true for the quasi-Lie bialgebra complexes and we denote the corresponding component by $Der_{b=0}(\mathcal{PH}olieb_{c,d})$.

Theorem 4.1.7. *The cohomology of $Der_{b=0}(\mathcal{QH}olieb_{c,d})$ is generated by the series of single vertex graphs*

$$\sum_{\substack{m\geq 1, n\geq 0\\ m+n\geq 3}} (m+n-2)$$  .

Furthermore, the cohomology of $Der_{b=0}(\mathcal{PH}olieb_{c,d})$ is generated by the series of single vertex graphs

$$\sum_{\substack{m,n\geq 0\\ m+n\geq 3}} (m+n-2)$$  .

Proof. This can be proven using similar methods as for Theorem 3.1.7. □

4.2 Graph complexes

4.2.1 The quasi and pseudo bi-weighted graph complex

In Chapter 3, we defined the *bi-weighted graph complex* fwGC_k as a tool to easier compute the cohomology of $Der(\mathcal{H}olieb^{\circlearrowleft}_{c,d})$. Here we follow the same idea and define two new complexes fwQGC_k and fwPGC_k to study the derivations of the quasi- and pseudo-Lie bialgebra properads. We refer to our previous paper for full details. Let Γ be a directed graph, and let x be a vertex of Γ. Recall that a bi-weight on x is a pair of non-negative integers (w_x^{out}, w_x^{in}) satisfying

$$\begin{aligned} &w_x^{out} + |x|_{out} \geq 1 \\ &w_x^{in} + |x|_{in} \geq 1 \\ &w_x^{out} + w_x^{in} + |x|_{out} + |x|_{in} \geq 3. \end{aligned}$$

Further recall that a *bi-weighted graph* is a graph Γ with a bi-weight on each vertex and that fwGC_k is the chain complex spanned by all bi-weighted graphs.

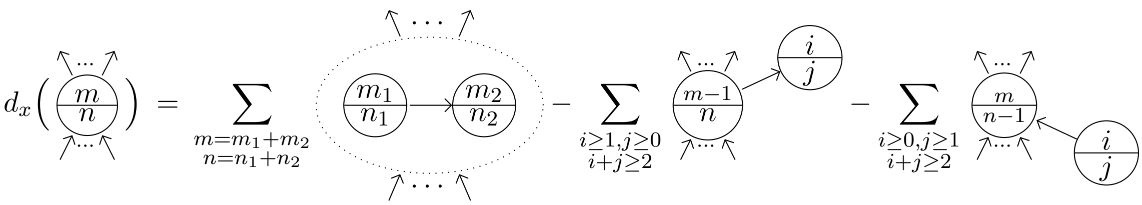
Definition 4.2.1. Let Γ be a directed graph and x a vertex of Γ. A *quasi bi-weight* $(w_x^{out}, w_x^{in})_q$ on x and a *pseudo bi-weight* $(w_x^{out}, w_x^{in})_p$ on x respectively, is a pair of non-positive integers satisfying

$(w_x^{out}, w_x^{in})_q$	$(w_x^{out}, w_x^{in})_p$
$w_x^{out} + \lvert x\rvert_{out} \geq 1$ $w_x^{in} + \lvert x\rvert_{in} \geq 0$ $w_x^{out} + w_x^{in} + \lvert x\rvert_{out} + \lvert x\rvert_{in} \geq 3$	$w_x^{out} + \lvert x\rvert_{out} \geq 0$ $w_x^{in} + \lvert x\rvert_{in} \geq 0$ $w_x^{out} + w_x^{in} + \lvert x\rvert_{out} + \lvert x\rvert_{in} \geq 3$

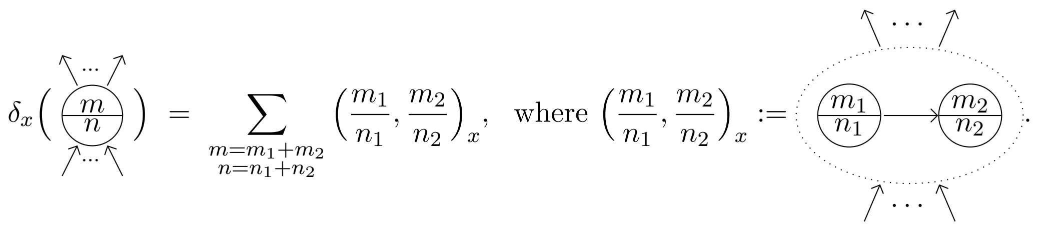
A *quasi bi-weighted* graph is a directed graph whose vertices are all decorated by quasi bi-weights. We similarly define a *pseudo bi-weighted graph*. Quasi and pseudo bi-weighted vertices are represented in the same manner as bi-weighted vertices, and the type of bi-weight will be understood from its context. The *quasi bi-weighted graph complex* is the chain complex fwQGC_k generated by quasi bi-weighted graphs. Similarly the *pseudo bi-weighted graph complex* is the chain complex fwPGC_k generated by pseudo bi-weighted graphs. The differentials of both complexes act similar to the differential in wGC_k by splitting vertices and attaching univalent vertices, but where the bi-weight limits of each complex are considered. Hence in all three cases the differential acts on a graph Γ as

$$d(\Gamma) := \delta(\Gamma) - \delta'(\Gamma) - \delta''(\Gamma) = \sum_{x\in V(\Gamma)} \delta_x(\Gamma) - \delta'_x(\Gamma) - \delta''_x(\Gamma).$$

where we pictorially represent $d_x(\Gamma) = \delta_x(\Gamma) - \delta'_x(\Gamma) - \delta''_x(\Gamma)$ in both complexes as



Remark 4.2.2. We will sometimes use the following notation for brevity when describing the splitting term δ_x of the differential:



Any invalid assignment of bi-weights to a vertex makes the whole graph zero. Note that this can be different in each of the three complexes (see Figure 4.2).

Remark 4.2.3. Similar to the differential of fwGC_k, the differentials of fwQGC_k and fwPGC_k do not in general cancel the creation of new univalent vertices in any of the bi-weighted complexes. However, vertices on the form $\frac{\infty_0}{\infty_0}$ do not create any new univalent vertices under the action of the differential. Here we are once again using the convention of decorations introduced in Definition 3.2.4.

Proposition 4.2.4. *Let the maps*

$$F : Der^\bullet(\mathcal{H}olieb_{c,d}^{\circlearrowleft}) \to \mathsf{fwGC}_{c+d+1}$$
$$qF : Der^\bullet(\mathcal{QH}olieb_{c,d}^{\circlearrowleft}) \to \mathsf{fwQGC}_{c+d+1}$$
$$pF : Der^\bullet(\mathcal{PH}olieb_{c,d}^{\circlearrowleft}) \to \mathsf{fwPGC}_{c+d+1}$$

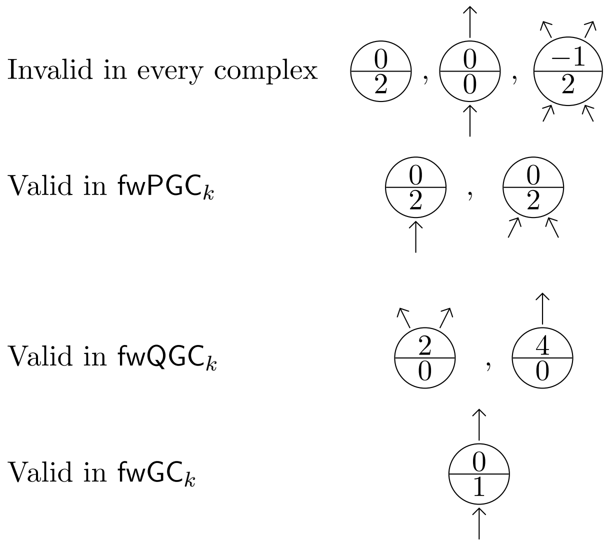


Figure 4.2: Examples of valid and invalid decorations of graphs in the three bi-weighted complexes.

be defined by mapping the graph representation of an element with out-hairs and in-hairs to a bi-weighted graphs of the same shape where the hairs have been interpreted as bi-weights. Then these maps are chain maps of degree 0*, and furthermore they are isomorphisms.*

Proof. The proof is equivalent to that of Proposition 3.2.7. □

4.2.2 Subcomplexes of fwGC_k, fwQGC_k and fwPGC_k

Definition 4.2.5. Let fwGC_k^+ be the subcomplex of fwGC_k of graphs having at least one vertex with a positive out-weight and at least one vertex with a positive in-weight. Let fwQGC_k^+ be the subcomplex of fwQGC_k of graphs having at least one vertex with positive out-weight.

Proposition 4.2.6. *The isomorphisms from Proposition 4.2.4 restrict to isomorphisms*

$$F : Der(\mathcal{H}olieb_{c,d}^{\circlearrowleft}) \to \mathsf{fwGC}_{c+d+1}^+$$
$$qF : Der(\mathcal{QH}olieb_{c,d}^{\circlearrowleft}) \to \mathsf{fwQGC}_{c+d+1}^+$$

Proof. This follows by inspection. □

Definition 4.2.7. Let fowGC_k be the subcomplex of fwGC_k of graphs containing no closed paths. Similarly, define fowQGC_k and fowPGC_k.

Proposition 4.2.8. *The isomorphisms from Proposition 4.2.4 restrict to isomorphisms*

$$F : Der(\mathcal{H}olieb_{c,d}^{\uparrow}) \to \mathsf{fowGC}_{c+d+1}$$
$$qF : Der(\mathcal{QH}olieb_{c,d}^{\uparrow}) \to \mathsf{fowQGC}_{c+d+1}$$
$$pF : Der(\mathcal{PH}olieb_{c,d}^{\uparrow}) \to \mathsf{fowPGC}_{c+d+1}$$

Proof. This follows by inspection. □

4.2.3 Loop number zero

All graph complexes above split over their loop number $b = e - v + 1$. Let $\mathsf{b_0wGC}_k$ be the subcomplex of fwGC_k of graphs with loop number $b = 0$ and let wGC_k denote the subcomplex of graphs with looper number one and higher. Hence $\mathsf{fwGC}_k = \mathsf{b_0wGC}_k \oplus \mathsf{wGC}_k$. Similarly, let $\mathsf{b_0wQGC}_k$ and $\mathsf{b_0wPGC}_k$ be the subcomplexes of graphs with loop number $b = 0$, and wQGC_k and wPGC_k the subcomplexes of graphs with loop number greater than zero.

Proposition 4.2.9. *The cohomologies of* $\mathsf{b_0wGC}_k$*,* $\mathsf{b_0wQGC}_k$ *and* $\mathsf{b_0wPGC}_k$ *are generated by the single vertex graph*

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 3}} (i+j-2) \bigcirc\!\!\!\!\!\!\frac{i}{j}$$

where any graph containing a vertex of invalid bi-weight is zero.

Proof. The proof has already been done for the complex $\mathsf{b_0wGC}_k$. The proof naturally extends to the quasi- and pseudo cases. ☐

The oriented graph complexes decompose in a similar manner, where the loop number zero component are equivalent to the complexes above. Denote their subcomplexes of graphs whose loop number greater than zero by owGC_k, owQGC_k and owPGC_k respectively.

4.3 Special in-vertices and special out-vertices

4.3.1 Special in-vertices

Recall that a special in-vertex is a vertex on the form $\overset{\uparrow}{\bigcirc\!\!\!\!\!\!\frac{0}{n}}$ or is on this form after a recursive removal of special-in vertices of this form. Any vertex that is not a special-in vertex is called an in-core vertex.

Definition 4.3.1. Let SwGC_k be the subcomplex of wGC_k of graphs Γ whose vertices $V(\Gamma)$ are independently decorated by the modified bi-weights $\frac{m}{\infty_1}$ and $\frac{m}{0}$ for $m \in \mathbb{N}$ subject to the following conditions:

(1) If $x \in V(\Gamma)$ is a source, then

$$x = \overset{\geq 1}{\overset{\nwarrow \cdots \nearrow}{\bigcirc\!\!\!\!\!\!\frac{m}{\infty_1}}} \text{ with } m + |x|_{out} \geq 2$$

(2) If $x \in V(\Gamma)$ is a univalent target, then

$$x = \underset{\uparrow}{\bigcirc\!\!\!\!\!\!\frac{m}{\infty_1}} \text{ with } m \geq 1, \text{ or } x = \underset{\uparrow}{\bigcirc\!\!\!\!\!\!\frac{m}{0}} \text{ with } m \geq 2$$

(3) If $x \in V(\Gamma)$ is a target with at least two in-edges, then

$$x = \underset{\geq 2}{\underset{\nearrow \cdots \nwarrow}{\bigcirc\!\!\!\!\!\!\frac{m}{\infty_1}}} \text{ with } m \geq 1, \text{ or } x = \underset{\geq 2}{\underset{\nearrow \cdots \nwarrow}{\bigcirc\!\!\!\!\!\!\frac{m}{0}}} \text{ with } m \geq 1$$

(4) If $x \in V(\Gamma)$ is passing (one in-edge and one out-edge), then

$$x = \underset{\uparrow}{\overset{\uparrow}{\left(\frac{m}{\infty_1}\right)}} \text{ with } m \geq 0, \text{ or } x = \underset{\uparrow}{\overset{\uparrow}{\left(\frac{m}{0}\right)}} \text{ with } m \geq 1$$

(5) If $x \in V(\Gamma)$ is of none of the types above (i.e., x is at least trivalent and has at least one in-edge and at least one out-edge), then

$$x = \underset{\nearrow \cdots \nwarrow}{\overset{\nwarrow \cdots \nearrow}{\left(\frac{m}{\infty_1}\right)}} \text{ with } m \geq 0, \text{ or } x = \underset{\nearrow \cdots \nwarrow}{\overset{\nwarrow \cdots \nearrow}{\left(\frac{m}{0}\right)}} \text{ with } m \geq 0$$

Definition 4.3.2. Let SwQGC_k be the subcomplex of wQGC_k of graphs Γ whose vertices $V(\Gamma)$ are independently decorated by the modified bi-weights $\frac{m}{\infty_1}$ and $\frac{m}{0}$ for $m \in \mathbb{N}$ subject to the same conditions $(2)-(5)$ as in Definition 4.3.1 together with the additional modified condition

(1′) If $x \in V(\Gamma)$ is a source, then

$$x = \overset{\overset{\geq 1}{\nwarrow \cdots \nearrow}}{\left(\frac{m}{\infty_1}\right)} \text{ with } m + |x|_{out} \geq 2, \text{ or } x = \overset{\overset{\geq 1}{\nwarrow \cdots \nearrow}}{\left(\frac{m}{0}\right)} \text{ with } m + |x|_{out} \geq 3.$$

Definition 4.3.3. Let SwPGC_k be the subcomplex of wPGC_k of graphs Γ whose vertices $V(\Gamma)$ are independently decorated by the modified bi-weights $\frac{m}{\infty_1}$, $\frac{m}{0}$ and $\frac{0}{\infty_2}$ for $m \in \mathbb{N}$ subject to the conditions (1′) and $(4)-(5)$ as in Definition 4.3.2 and 4.3.1, together with the additional modified conditions

(2′) If $x \in V(\Gamma)$ is a univalent target, then

$$x = \underset{\uparrow}{\left(\frac{m}{\infty_1}\right)} \text{ with } m \geq 1, \quad x = \underset{\uparrow}{\left(\frac{m}{0}\right)} \text{ with } m \geq 2, \text{ or } x = \underset{\uparrow}{\left(\frac{0}{\infty_2}\right)}$$

(3′) If $x \in V(\Gamma)$ is a target with at least two in-edges, then

$$x = \underset{\underset{\geq 2}{\nearrow \cdots \nwarrow}}{\left(\frac{m}{\infty_1}\right)} \text{ with } m \geq 0, \text{ or } x = \underset{\underset{\geq 2}{\nearrow \cdots \nwarrow}}{\left(\frac{m}{0}\right)} \text{ with } m + |x|_{in} \geq 3$$

Remark 4.3.4. The differentials of the complexes above are induced by the differential from the complex they are embedded in and act on a graph Γ as $d(\Gamma) = \sum_{x \in V(\Gamma)} d_x(\Gamma)$. The map d_x acts differently on vertices depending on which of the three complexes above we consider. It can generally be described using the formulas of Figure 4.3, where any term containing a vertex with an invalid bi-weight for the specific complex the computation is done in is set to zero. Note for example that the decoration $\frac{0}{\infty_2}$ is only valid in SwPGC_k.

Proposition 4.3.5. *The three inclusions*

$$\mathsf{SwGC_k} \hookrightarrow \mathsf{wGC}_k \ , \quad \mathsf{SwQGC_k} \hookrightarrow \mathsf{wQGC}_k \quad \textit{and} \quad \mathsf{SwPGC_k} \hookrightarrow \mathsf{wPGC}_k$$

are quasi-isomorphisms.

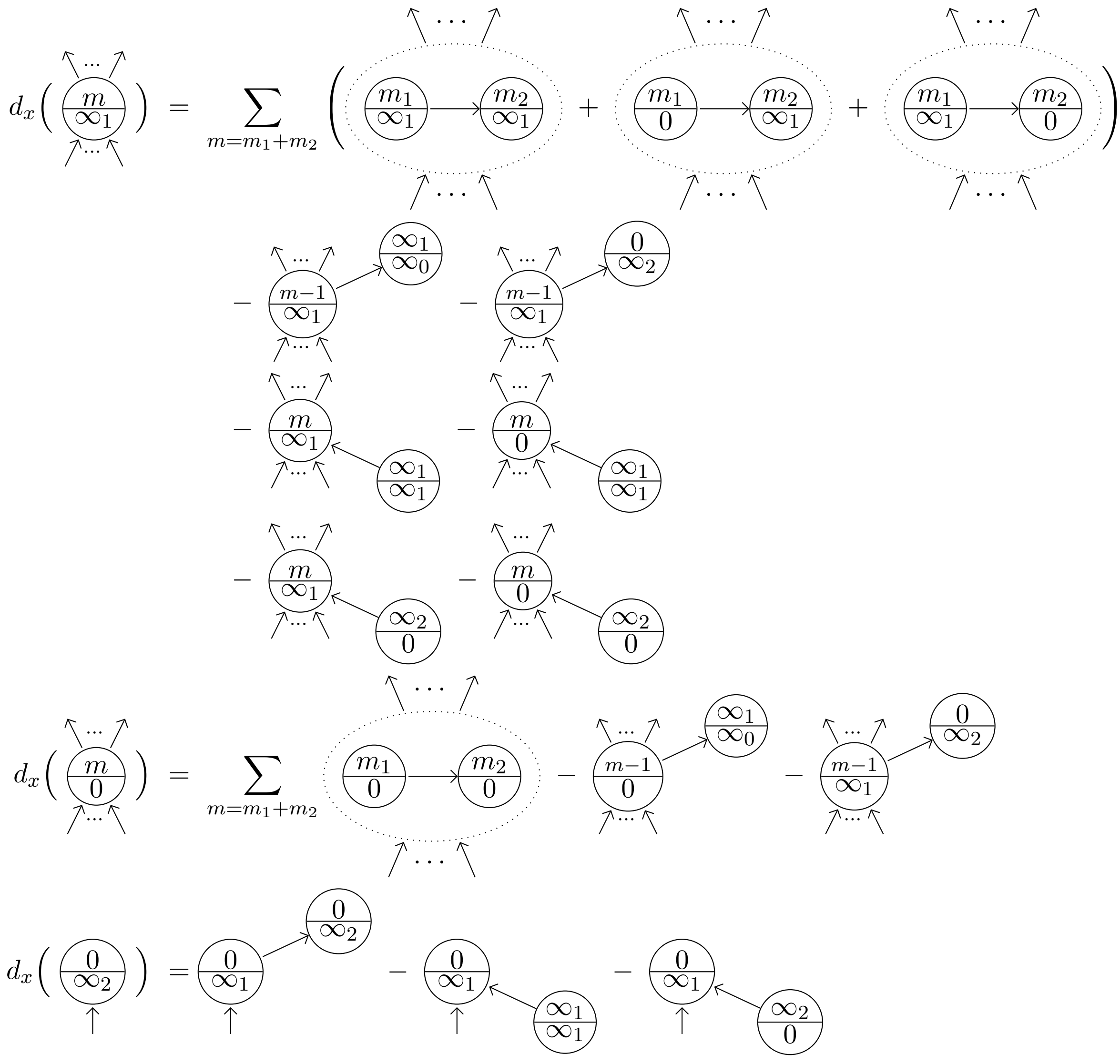


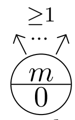
Figure 4.3: General formula describing how the differentials of SwGC_k, SwQGC_k and SwPGC_k act on a vertex x. Any term containing an invalid decoration for the considered complex is set to zero.

Proof. The statement was already proved for the first inclusion in Proposition 3.3.2, and the proof for the other two inclusions follows the same line of argument with some extra cases to consider.
To show the second statement, let gr SwQGC_k be the associated graded of the filtration over the number of in-core vertices in a graph. This complex decompose over in-core graphs as

$$gr\ \mathsf{SwQGC}_k = \bigoplus \mathsf{QinCore}(\gamma)$$

where $\mathsf{QinCore}(\gamma)$ is the subcomplex of graphs with in-core graph γ. This complex further decomposes as

$$\mathsf{QinCore}(\gamma) \cong \Big(\bigotimes_{x\in V(\gamma)} q\mathcal{T}_x^{in} \Big)^{\mathrm{Aut}(\gamma)}$$

where $q\mathcal{T}_x^{in}$ is the complex of in-core trees attached to the vertex x in γ. We first note that if $|x|_{in} \geq 1$, then $q\mathcal{T}_x^{in} \cong \mathcal{T}_x^{in}$, which shows cases $(2)-(5)$. Suppose that $|x|_{in} = 0$. Then $q\mathcal{T}_x^{in}$ contains a subcomplex $\mathcal{C}$ generated by single vertex graphs on the form 
for $m \geq 0$ such that $m + |x|_{out} \geq 3$. These are all non-trivial cycles, and the complex split as $q\mathcal{T}_x^{in} = \mathcal{C} \oplus \mathcal{T}_x^{in}$. The cohomology of $\mathcal{T}_x^{in}$ is given by condition (1), and so this gives the modified condition $(1')$ and proves the second statement.
Finally to show the last statement, Let gr SwPGC_k be the associated graded of the filtration of wPGC_k over the number of core-vertices in a graph. This complex decomposes over in-core graphs γ as

$$gr\ \mathsf{SwPGC}_k = \bigoplus_{\gamma} \mathsf{PinCore}(\gamma)$$

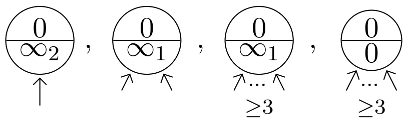
and further

$$\mathsf{PinCore}(\gamma) \cong \Big(\bigotimes_{x\in V(\gamma)} p\mathcal{T}_x^{in} \Big)^{\mathrm{Aut}(\gamma)}$$

where $p\mathcal{T}_x^{in}$ is the complex of in-core trees attached to the vertex x in γ. Recall that $|x|_{out}$, $|x|_{in}$ and w_x^{out} are invariant in $p\mathcal{T}_x^{in}$. By observation, we gather the following:

- If $|x|_{out}, |x|_{in} \geq 1$, then $p\mathcal{T}_x^{in} = \mathcal{T}_x^{in}$.
- If $|x|_{in} = 0$, then $p\mathcal{T}_x^{in} = q\mathcal{T}_x^{in}$.
- If $|x|_{out} = 0$ and $w_x^{out} \geq 1$, then $p\mathcal{T}_x^{in} = \mathcal{T}_x^{in}$.

These cases correspond to the conditions $(4)-(5)$, $(1')$ and $(2)-(3)$ respectively. One last case remains. Suppose that $|x|_{out} = 0$ and $w_x^{out} = 0$. We will show that $H(p\mathcal{T}_x^{in})$ is generated by the following one-vertex graphs



depending on the value of $|x|_{in}$. The proof of this statement is similar to that of Lemma 3.3.4 and Lemma 3.3.5 so we do only give a brief outline here.
Consider the filtration of $p\mathcal{T}_x^{in}$ over the number of univalent special-in vertices, where we consider the graph containing a single vertex as having one such vertex. The associated graded complex split as $gr\ p\mathcal{T}_x^{in} = \bigoplus_{N\geq 1} u_N p\mathcal{T}_x^{in}$ where $u_N p\mathcal{T}_x^{in}$ is the subcomplex of graphs with N univalent special-in vertices. We can show that $u_N p\mathcal{T}_x^{in}$ is acyclic for $N \geq 2$ by considering the filtration over the number of *branch vertices*. A vertex y is

a branch vertex if there are at least two directed paths from a univalent vertex to y. Secondly, we consider the filtration over the total in-weights of the branch vertices. The in-core vertex is always a branch vertex, and the differential does not depend on the bi-weight of the core-vertex. Hence the proof now is equivalent to that of Lemma 3.3.5. Lastly, we note that the one vertex graphs above are non-trivial cycles of $u_1 p\mathcal{T}_x^{in}$. By the same methods as in the proof of Lemma 3.3.4 we show that the cohomology of $u_1 p\mathcal{T}_x^{in}$ is one or two-dimensional, finishing the proof. □

Definition 4.3.6. Let SowGC_k be the subcomplex of SwGC_k of oriented graphs. Similarly, define SowQGC_k and SowPGC_k as subcomplexes of SwQGC_k and SwPGC_k respectively. These complexes are subcomplexes of owGC_k, owQGC_k and owPGC_k.

Corollary 4.3.7. *The inclusions*

$$\mathsf{SowGC}_k \hookrightarrow \mathsf{owGC}_k \ , \quad \mathsf{SowQGC}_k \hookrightarrow \mathsf{owQGC}_k \quad \textit{and} \quad \mathsf{SowPGC}_k \hookrightarrow \mathsf{owPGC}_k$$

are quasi-isomorphisms.

Proof. Let $gr\mathsf{owGC}_k$ be the associated graded complex to the filtration of owGC_k over the number of core-vertices in a graph. The complex decompose over oriented core graphs γ as

$$gr \ \mathsf{owGC}_k = \bigoplus_{\gamma} \mathsf{inCore}(\gamma)$$

where $\mathsf{inCore}(\gamma)$ is the subcomplex of graphs with in-core γ. These are the same complexes as in Proposition 3.3.2, giving the same cohomology classes, which gives the result for the first inclusion. The same argument applies to the two other cases. □

Definition 4.3.8. Let SwGC_k^+ be the subcomplex of SwGC_k of graphs that contain at least one vertex with out-decoration $m \geq 1$ and one vertex with in-decoration ∞_1. Further, let SwQGC_k^+ be the subcomplex of SwQGC_k of graphs that contain at least one vertex with out-decoration $m \geq 1$. These complexes are subcomplexes of wGC_k^+ and wQGC_k^+ respectively.

Proposition 4.3.9. *The inclusions*

$$\mathsf{SwGC}_k^+ \hookrightarrow \mathsf{wGC}_k^+ \quad \textit{and} \quad \mathsf{SwQGC}_k^+ \hookrightarrow \mathsf{wQGC}_k^+$$

are quasi-isomorphisms.

Proof. The first case was already proven in Proposition 3.3.8, but we do here give an alternate proof that can be generalized to the second case. Consider the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{SwGC}_k^+ & \longrightarrow & \mathsf{SwGC}_k & \longrightarrow & \mathsf{SwGC}_k^\sim & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{wGC}_k^+ & \longrightarrow & \mathsf{wGC}_k & \longrightarrow & \mathsf{wGC}_k^\sim & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Q}_k^+ & \longrightarrow & \mathsf{Q}_k & \longrightarrow & \mathsf{Q}_k^\sim & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where Q_k^+, Q_k and $\mathsf{Q}_k^\sim$ are the appropriate quotient complexes. Note in particular that $\mathsf{wGC}_k^\sim$ is the quotient complex of bi-weighted graphs that do not both have a positive out-weight and positive in-weight at the same time. It decomposes as

$$\mathsf{wGC}_k^\sim = \mathsf{wGC}_k^{out} \oplus \mathsf{wGC}_k^{in} \oplus \mathsf{wGC}_k^0$$

where wGC_k^{out} is the complex of graphs with at least one vertex with positive out-weight, wGC_k^{out} the complex of graphs with at least one vertex with positive in-weight, and wGC_k^0 the complex of graphs with neither positive out-weights nor in-weights. The complex $\mathsf{SwGC}_k^\sim$ is a subcomplex of $\mathsf{wGC}_k^\sim$ of graphs whose vertices are either only decorated by $\frac{0}{\infty_1}$ and $\frac{0}{0}$, or $\frac{m}{0}$ and $\frac{0}{0}$ for $m \geq 1$. It similarly decomposes as

$$\mathsf{SwGC}_k^\sim = \mathsf{SwGC}_k^{out} \oplus \mathsf{SwGC}_k^{in} \oplus \mathsf{SwGC}_k^0$$

where is the subcomplex of graphs with at least one vertex decorated by $\frac{m}{0}$ $m \geq 1$. By Proposition 3.3.2, the inclusion $\mathsf{SwGC}_k \hookrightarrow \mathsf{wGC}_k$ is a quasi-isomorphism, and so Q_k is acyclic. If we show that the inclusion $\mathsf{SwGC}_k^\sim \hookrightarrow \mathsf{wGC}_k^\sim$ is a quasi-isomorphism, we get that both $\mathsf{Q}_k^\sim$ and Q_k^+ are acyclic, and that the inclusion $\mathsf{SwGC}_k^+ \hookrightarrow \mathsf{wGC}_k^+$ is a quasi-isomorphism. Equivalently, we show that the inclusions

$$\mathsf{SwGC}_k^{out} \hookrightarrow \mathsf{wGC}_k^{out}\ , \quad \mathsf{SwGC}_k^{in} \hookrightarrow \mathsf{wGC}_k^{out} \quad \text{and} \quad \mathsf{SwGC}_k^0 \hookrightarrow \mathsf{wGC}_k^0$$

are quasi-isomorphisms. Consider the filtration of these complexes over the number of in-core vertices. The associated graded $gr\ \mathsf{wGC}_k^{out}$ contain no special-in vertices since any such vertex necessarily has a positive in-weight, and so the differential is trivial and easily seen to be equal to SwGC_k. Consider the associated graded of $\mathsf{wGC}_k^{in} \oplus \mathsf{wGC}_k^0$. It decomposes over in-core graphs γ as

$$gr(\mathsf{wGC}_k^{in} \oplus \mathsf{wGC}_k^0) = \bigoplus_\gamma \mathsf{inCore}^{out,0}(\gamma)$$

where $\mathsf{inCore}^{out,0}(\gamma)$ is the complex of graphs with in-core γ. The complex further decomposes as

$$\mathsf{inCore}^{out,0}(\gamma) \cong \Big(\bigotimes_{x \in V(\gamma)} \mathcal{T}_x^{in,\sim} \Big)^{\mathrm{Aut}(\gamma)}$$

where $\mathcal{T}_x^{in,\sim}$ is the complex of special-in trees attached to x. This complex is isomorphic to $\mathcal{T}^{in}_{|x|_{out},|x|_{in},0}$ as seen in Lemma 3.3.3. Hence $\mathcal{T}_x^{in,\sim}$ is acyclic when x is a target vertex, and otherwise generated by the single vertex graph decorated by $\frac{0}{\infty_1}$ and $\frac{0}{0}$ (when x is generic). This completes the first part of the proof. The argument for the second proof is similar, instead using the tree complexes $q\mathcal{T}_x^{in}$ from Proposition 3.3.2. □

4.3.2 Special out-vertices

Recall that a special out-vertex is a vertex on the form $\textcircled{\frac{m}{0}}$ or becomes on this same form after an iterative removal of such vertices from the graph. Vertices that are not special out-vertices are called out-core vertices or just core vertices.

Definition 4.3.10. Let qGC_k be the subcomplex of SwGC_k of graphs Γ whose vertices $V(\Gamma)$ are independently decorated by the bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$ and $\frac{0}{0}$ subject to the following conditions:

(1) If $x \in V(\gamma)$ is a univalent vertex, then it is decorated by $\frac{\infty_1}{\infty_1}$.

(2) If $x \in V(\gamma)$ is a source, then it is either decorated by $\frac{\infty_1}{\infty_1}$ or $\frac{0}{\infty_1}$.

(3) If $x \in V(\gamma)$ is a target, then it is either decorated by $\frac{\infty_1}{\infty_1}$ or $\frac{\infty_1}{0}$.

(4) If $x \in V(\gamma)$ is a passing vertex, then it is either decorated by $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$ or $\frac{0}{\infty_1}$.

(5) If $x \in V(\gamma)$ is a generic vertex, then it is either decorated by $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$ or $\frac{0}{0}$.

Definition 4.3.11. Let qQGC_k be the subcomplex of SwQGC_k of graphs Γ whose vertices $V(\Gamma)$ are independently decorated by the bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$, $\frac{0}{0}$ and $\frac{\infty_2}{0}$ subject to the conditions (3)-(5) of Definition 4.3.10 in addition to the following modified conditions:

(1′) If $x \in V(\gamma)$ is a univalent source, then it is decorated by $\frac{\infty_1}{\infty_1}$ or $\frac{\infty_2}{0}$.

(1″) If $x \in V(\gamma)$ is a univalent target, then it is decorated by $\frac{\infty_1}{\infty_1}$.

(2′) If $x \in V(\gamma)$ is a bivalent source, then it is either decorated by $\frac{\infty_1}{\infty_1}$ or $\frac{0}{\infty_1}$.

(2″) If $x \in V(\gamma)$ is an at least trivalent source, then it is either decorated by $\frac{\infty_1}{\infty_1}$, $\frac{0}{\infty_1}$ or $\frac{0}{0}$.

Definition 4.3.12. Let qPGC_k be the subcomplex of SwPGC_k of graphs Γ whose vertices $V(\Gamma)$ are independently decorated by the bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$, $\frac{0}{0}$, $\frac{\infty_2}{0}$ and $\frac{0}{\infty_2}$ subject to the conditions of Definition 4.3.11, except where condition (3) has been replaced by the two modified conditions:

(3′) If $x \in V(\gamma)$ is a bivalent target, then it is either decorated by $\frac{\infty_1}{\infty_1}$ or $\frac{\infty_1}{0}$.

(3″) If $x \in V(\gamma)$ is an at least trivlanet target, then it is either decorated by $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$ or $\frac{0}{0}$.

Remark 4.3.13. The differentials of the complexes above are induced by the differential from the complex they are embedded in and act on a graph Γ as $d(\Gamma) = \sum_{x \in V(\Gamma)} d_x(\Gamma)$. The latter map further decomposes as $d_x(\Gamma) = \delta_x(\Gamma) - \delta'_x(\Gamma) - \delta''_x(\Gamma)$, where the first map denotes vertex splitting, the second attaching a univalent target to x, and the third attaching a univalent source to x. These maps act differently on vertices depending on which of the three complexes above we consider. It can generally be described using the formulas of figure 4.3.2, where any term containing a vertex with an invalid bi-weight for the specific complex the computation is done in is set to zero.

Proposition 4.3.14. *The three inclusions*

$$\mathsf{SwGC}_k \hookrightarrow \mathsf{qGC}_k \ , \quad \mathsf{SwQGC}_k \hookrightarrow \mathsf{qQGC}_k \quad \textit{and} \quad \mathsf{SwPGC}_k \hookrightarrow \mathsf{qPGC}_k$$

are quasi-isomorphisms.

Proof. The first inclusion was shown to be a quasi-isomorphism in Proposition 3.3.10. For the second inclusion, consider the filtration of SwQGC_k over the number of out-core vertices in a graph. Then the associated graded split over out-core graphs

$$gr\ \mathsf{SwQGC}_k = \bigoplus_{\gamma} \mathsf{outQCore}(\gamma)$$

where $\mathsf{outQCore}(\gamma)$ is the complex of graphs with out-core γ. This complex further split as

$$\mathsf{outQCore}(\gamma) \cong \Big(\bigotimes_{x \in V(\gamma)} q\mathcal{T}_x^{out} \Big)^{\mathrm{Aut}(\gamma)}$$

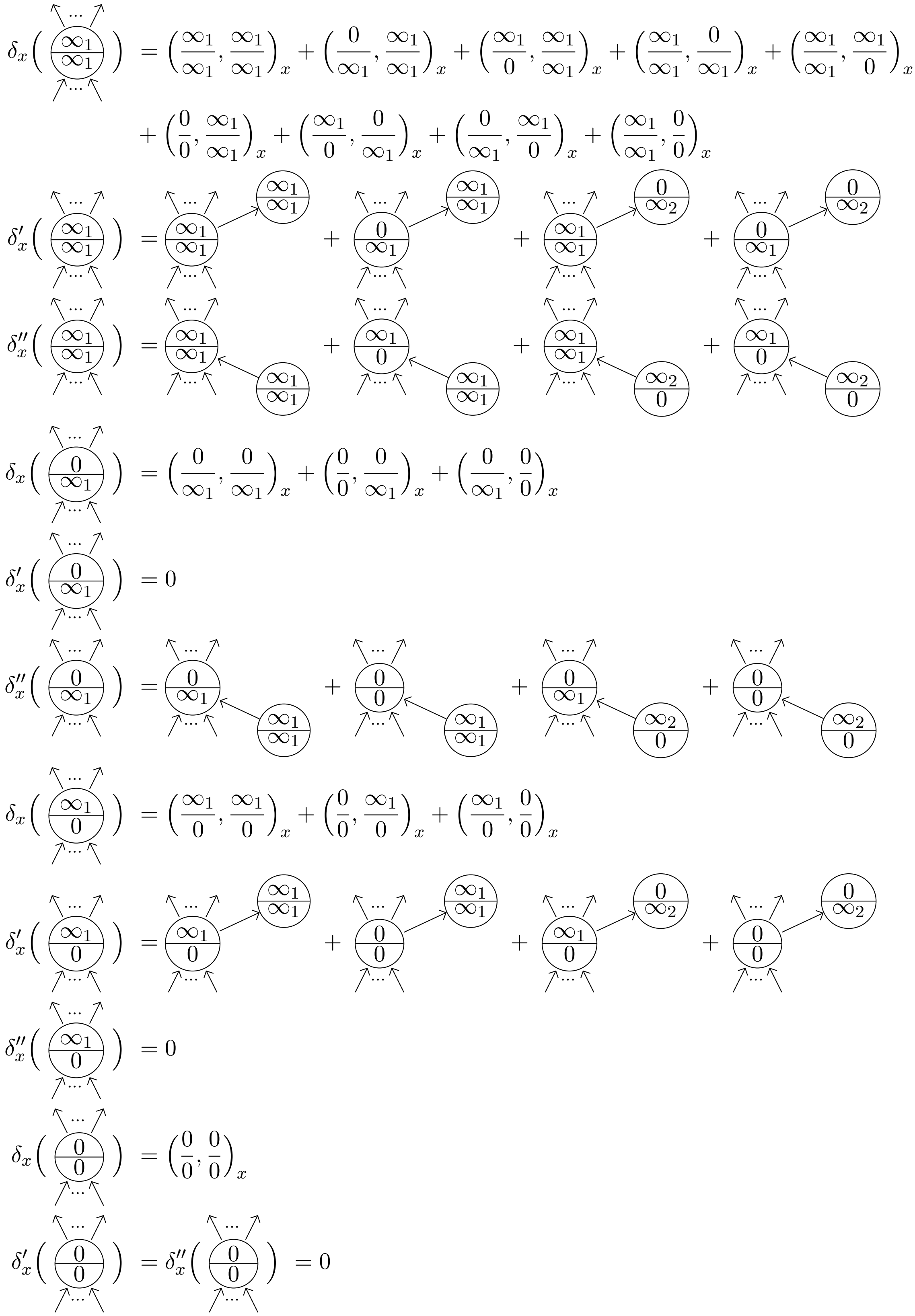


Figure 4.4: General formula describing how the differentials of qGC_k, qQGC_k and qPGC_k act on vertices. Any term containing an invalid decoration for the considered complex is set to zero.

where $q\mathcal{T}_x^{out}$ is the complex of out-tree graphs attached to x. We note that $q\mathcal{T}_x^{out} = \mathcal{T}_x^{out}$ when x is not a source or has in-weight ∞_1, and so these results follow from Proposition **??**. Assume $|x|_{in} = 0$ and $w_x^{in} = 0$ and consider the three cases $|x|_{out} = 1$, $|x|_{out} = 2$, and $|x|_{out} \geq 3$. By the same arguments as in Proposition **??**, we compute the cohomology of $q\mathcal{T}_x^{out}$ and see that it agrees with the proposition. For the third inclusion, consider the filtration of SwPGC_k over the number of out-core vertices. The associated graded complex split over out-core graphs as

$$gr\ \mathsf{SwPGC}_k = \bigoplus_{\gamma} \mathsf{outPCore}(\gamma)$$

where $\mathsf{outPCore}(\gamma)$ is the complex of graphs with out-core γ. This complex further decompose as

$$\mathsf{outPCore}(\gamma) \cong \Big(\bigotimes_{x \in V(\gamma)} p\mathcal{T}_x^{out} \Big)^{\mathrm{Aut}(\gamma)}$$

where $p\mathcal{T}_x^{out}$ is the complex of special-out trees attached to x. If x is a passing or generic vertex, then $p\mathcal{T}_x^{out} = \mathcal{T}_x^{out}$. If x is a source, then $p\mathcal{T}_x^{out} = q\mathcal{T}_x^{out}$. The last case to consider is when $|x|_{out} = 0$. In the three cases $|x|_{out} = 1$, $|x|_{out} = 2$ and $|x|_{out} \geq 3$ we can show with the same arguments as in Proposition **??** that the cohomology of $p\mathcal{T}_x^{out}$ agrees with the proposition. □

Definition 4.3.15. Let oqGC_k be the subcomplex of qGC_k of oriented graphs. Similarly, define oqQGC_k and oqPGC_k as subcomplexes of qQGC_k and qPGC_k. These complexes are subcomplexes of SowGC_k, SowQGC_k and SowPGC_k respectively.

Corollary 4.3.16. *The inclusions*

$$\mathsf{oqGC}_k \hookrightarrow \mathsf{SowGC}_k\ , \quad \mathsf{oqQGC}_k \hookrightarrow \mathsf{SowQGC}_k \quad \text{and} \quad \mathsf{oqPGC}_k \hookrightarrow \mathsf{SowPGC}_k$$

are quasi-isomorphisms.

Proof. This follows by considering the same filtrations as in Proposition 4.3.14 and noting that the decompositions are preserved as in Corollary 4.3.7. □

Definition 4.3.17. Let qGC_k^+ be the subcomplex of SwGC_k of graphs that have either at least one vertex with bi-weight $\frac{\infty_1}{\infty_1}$, or a pair of vertices decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively. Further, let qQGC_k^+ be the subcomplex of SwQGC_k of graphs with at least one vertex decorated by $\frac{\infty_1}{\infty_1}$, or a vertex decorated by $\frac{\infty_1}{0}$ (or $\frac{\infty_2}{0}$). These complexes are subcomplexes of SwGC_k^+ and SwQGC_k^+ respectively.

Proposition 4.3.18. *The inclusions*

$$\mathsf{qGC}_k^+ \hookrightarrow \mathsf{SwGC}_k^+ \qquad \text{and} \qquad \mathsf{qQGC}_k^+ \hookrightarrow \mathsf{SwQGC}_k^+$$

are quasi-isomorphisms.

Proof. By setting up a similar diagram as in Proposition 4.3.9, one can prove the above statements with similar arguments. We leave the details to the reader. □

4.4 Further reductions of the derivation complexes

4.4.1 $\frac{0}{0}$-decorations

The complex qQGC_k split as

$$\mathsf{qQGC}_k = \mathsf{qQGC}_k^0 \oplus \mathsf{qQGC}_k^*$$

where qQGC_k^0 is the complex of graphs whose all vertices are decorated by $\frac{0}{0}$, and qQGC_k^* is the complex of graphs with at least one vertex with a positive out-weight or a positive in-weight. The complex qPGC_k split similarly as

$$\mathsf{qPGC}_k = \mathsf{qPGC}_k^0 \oplus \mathsf{qPGC}_k^*.$$

Proposition 4.4.1. *Let* $\mathsf{dGC}_k^2 \subset \mathsf{dGC}_k$ *be the subcomplex of graphs with at least one bivalent vertex, and let* $\mathsf{dGC}_k^{2+t} \subseteq \mathsf{dGC}_k$ *be the complex of graphs with at least one bivalent vertex or one target vertex. Then*

$$\mathsf{qQGC}_k^0 \cong \mathsf{dGC}_k/\mathsf{dGC}_k^{2+t} \qquad \textit{and} \qquad \mathsf{qPGC}_k^0 \cong \mathsf{dGC}_k/\mathsf{dGC}_k^2$$

Proof. Consider a graph in qQGC_k^0. Only vertices that are at least trivalent and not sources can be decorated by $\frac{0}{0}$. Let $f : \mathsf{dGC}_k \to \mathsf{qQGC}_k^0$ be the map where a graph Γ is mapped to the graph $f(\Gamma)$ where all vertices of Γ have been decorated by the bi-weight $\frac{0}{0}$. Any graph where such an assignment is not possible is mapped to zero. One readily checks that this is a chain map, and that the kernel of the map is spanned by graphs with at least one bivalent vertex or at least one target vertex, completing the proof. A similar argument can be made for qPGC_k^0. □

The complex oqQGC_k does not contain any graphs that are only decorated by $\frac{0}{0}$, but the complex oqPGC_k does contain such graphs.

Proposition 4.4.2. *Let* $\mathsf{oGC}_k^2 \subseteq \mathsf{oGC}_k$ *be the subcomplex of graphs with at least one bivalent vertex and* $\mathsf{oGC}_k^{\geq 3} := \mathsf{oGC}_k/\mathsf{oGC}_k^2$ *be the quotient complex of graphs where all vertices are at least trivalent. Then* $\mathsf{oqPGC}_k^0 \cong \mathsf{oGC}_k^{\geq 3}$.

Proof. This can be shown using the same argument as in Proposition 4.4.1. □

Definition 4.4.3. Let $\mathsf{qQGC}_k^{0,*} \subseteq \mathsf{qQGC}_k^*$ be the subcomplex of graphs with at least one vertex decorated by $\frac{0}{0}$ and set $\mathsf{tQGC}_k^* = \mathsf{qQGC}_k^*/\mathsf{qQGC}_k^{*,0}$. Similarly, define $\mathsf{qPGC}_k^{0,*} \subseteq \mathsf{qPGC}_k^*$ as the analogous subcomplex and tPGC_k^* the corresponding quotient.

Proposition 4.4.4. *The projections*

$$\mathsf{qQGC}_k^* \to \mathsf{tQGC}_k^* \quad \textit{and} \quad \mathsf{qPGC}_k^* \to \mathsf{tPGC}_k^*$$

are quasi-isomophisms.

Proof. It is enough to show that $\mathsf{qQGC}_k^{0,*}$ and $\mathsf{qPGC}_k^{0,*}$ are acyclic. The argument is equivalent to the proof of Proposition 3.4.1 by considering a filtration over the number of non-passing vertices. The details are left to the reader. □

Remark 4.4.5. There are similar propositions for qQGC_k^+ and the oriented subcomplexes with the same proofs. Denote their corresponding quotient complex of graphs with three decorations: $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ by tQGC_k^+, otGC_k, otQGC_k and otPGC_k.

4.4.2 Monodecorated graphs

We remind the reader of the special decorations of graphs seen in Definition 3.4.2. The decorated graphs here are in tQGC_k^*. If Γ is a directed graph without univalent vertices, then $\Gamma^{\frac{\infty_1}{0}}$ is the decorated graph of the shape Γ where all vertices are decorated by $\frac{\infty_1}{0}$. We similarly define $\Gamma^{\frac{0}{\infty_1}}$. Further, Γ^ω is the sum of all possible decorations of Γ (in tQGC_k^*) such that at least one vertex is decorated by $\frac{\infty_1}{\infty_1}$ or a pair of vertices is decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$.

Remark 4.4.6. The differential of tQGC_k^* decomposes as $d = d_s + d_u$ where d_u increases the number of univalent vertices of a graph by one, and d_s splits vertices without creating univalent ones. One can verify that $d_u(\Gamma^\omega + \Gamma^{\frac{\infty_1}{0}} + \Gamma^{\frac{0}{\infty_1}}) = 0$. One can further see that $\Gamma^{\frac{0}{\infty_1}} = 0$ if Γ contains a target vertex. Hence we gather that

$$d_u(\Gamma^\omega) = \begin{cases} -d_u(\Gamma^{\frac{\infty_1}{0}}) & \text{if there is a target vertex in } \Gamma. \\ -d_u(\Gamma^{\frac{\infty_1}{0}}) - d_u(\Gamma^{\frac{0}{\infty_1}}) & \text{if there is no target vertex in } \Gamma. \end{cases}$$

The graphs in $d_u(\Gamma^{\frac{\infty_1}{0}})$ have the shape of Γ together with a "special" univalent vertex attached on the form: $\underset{\uparrow}{\left(\frac{\infty_1}{\infty_1}\right)} + (-1)^{k+1}\underset{\downarrow}{\left(\frac{\infty_2}{0}\right)}$. Similarly the graphs in $d_u(\Gamma^{\frac{0}{\infty_1}})$ have the shape of Γ together with another "special" univalent vertex attached on the form $\underset{\downarrow}{\left(\frac{\infty_1}{\infty_1}\right)} +$ $\underset{\downarrow}{\left(\frac{\infty_2}{0}\right)}$. This motivates the next definition.

Definition 4.4.7. The subcomplex $\mathsf{mQGC}_k^* \subseteq \mathsf{tQGC}_k^*$ is the complex on the form

$$\mathsf{mQGC}_k^* = \mathcal{C}^{\geq 2} \oplus \mathcal{C}^1(\frac{\infty_1}{0}) \oplus \mathcal{C}^1(\frac{0}{\infty_1}).$$

where

- $\mathcal{C}^{\geq 2}$ is the subspace (not subcomplex) of graphs of the forms $\Gamma^{\frac{\infty_1}{0}}$, $\Gamma^{\frac{0}{\infty_1}}$ and Γ^ω for Γ with no univalent vertices.
- $\mathcal{C}^1(\frac{\infty_1}{0})$ be the subcomplex of tGC_k^* of graphs whose non-univalent vertices are decorated by $\frac{\infty_1}{0}$ and where the only univalent vertices are attached to non-antenna vertices and are of the special type $\underset{\uparrow}{\left(\frac{\infty_1}{\infty_1}\right)} + (-1)^{k+1}\underset{\downarrow}{\left(\frac{\infty_2}{0}\right)}$.
- $\mathcal{C}^1(\frac{0}{\infty_1})$ be the similar subcomplex of tGC_k^* of graphs only decorated by $\frac{0}{\infty_1}$ with univalent vertices of the type $\underset{\downarrow}{\left(\frac{\infty_1}{\infty_1}\right)} + \underset{\downarrow}{\left(\frac{\infty_2}{0}\right)}$.

Remark 4.4.8. By the previous remark, we see that the differential is closed on this subspace, and it is thus a chain complex.

Proposition 4.4.9. *The inclusion $\mathsf{mQGC}_k^* \hookrightarrow \mathsf{tQGC}_k^*$ is a quasi-isomorphism.*

Proof. First consider the subcomplex $\mathsf{tQGC}_k^{1,\frac{\infty_1}{0},\frac{0}{\infty_1}} \subseteq \mathsf{tQGC}_k^*$ of graphs with at least one univalent vertex and a pair of non-univalent vertices decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively. One can show that this complex is acyclic by considering the filtration over the number of non-passing vertices. Hence the quotient complex $\mathsf{s}_1\mathsf{tQGC}_k$ is quasi-isomorphic to tQGC_k^*. One also notes that mQGC_k^* is also a subcomplex of this quotient complex. Next, consider the subcomplex $\mathsf{s}_2\mathsf{tQGC}_k$ of $\mathsf{s}_1\mathsf{tQGC}_k$ of graphs with non-univalent vertices as in mQGC_k^*, and all other types of graphs with at least one univalent vertex as in $\mathsf{s}_1\mathsf{tQGC}_k$. The quotient complex consists of graphs with no univalent vertices and where at least two vertices have different decorations. This complex is acyclic, as seen by considering the filtration over the number of non-passing vertices. Next, consider the subcomplex $\mathsf{s}_3\mathsf{tQGC}_k$ of $\mathsf{s}_2\mathsf{tQGC}_k$ containing the same graphs having no univalent vertices, and additionally graphs with at least one univalent vertex but no non-univalent vertices decorated by $\frac{\infty_1}{\infty_1}$. The quotient complex is generated by graphs with at least one univalent vertex, and all non-univalent vertices are decorated by $\frac{\infty_1}{\infty_1}$. This complex is acyclic, as seen by considering the filtration over the number of non-passing vertices. Let $\mathsf{s}_4\mathsf{tQGC}_k$ be the subcomplex of $\mathsf{s}_3\mathsf{tQGC}_k$ containing the same graphs having no univalent vertices, and additionally graphs with univalent vertices on the following forms: Let Γ be a graph without univalent vertices. Then we consider graphs $\Gamma^{\frac{\infty_1}{0}}$ where at least one vertex has the univalent vertex $\underset{\uparrow}{\textcircled{\frac{\infty_1}{\infty_1}}} + (-1)^{k+1} \underset{\downarrow}{\textcircled{\frac{\infty_2}{0}}}$ attached. Similarly, we also consider graphs $\Gamma^{\frac{0}{\infty_1}}$ where at least one vertex has the univalent vertex $\underset{\downarrow}{\textcircled{\frac{\infty_1}{\infty_1}}} + \underset{\downarrow}{\textcircled{\frac{\infty_2}{0}}}$ attached. Let $\mathsf{s}_4\mathsf{tQGC}_k^1$ be the corresponding complex of graphs with at least one univalent vertex, and similarly for $\mathsf{s}_3\mathsf{tQGC}_k$. We get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{s}_4\mathsf{tQGC}_k^1 & \longrightarrow & \mathsf{s}_4\mathsf{tQGC}_k & \longrightarrow & \mathsf{s}_4\mathsf{tQGC}_k^{\geq 2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{s}_3\mathsf{tQGC}_k^1 & \longrightarrow & \mathsf{s}_3\mathsf{tQGC}_k & \longrightarrow & \mathsf{s}_3\mathsf{tQGC}_k^{\geq 2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By showing that the inclusion $\mathsf{s}_4\mathsf{tQGC}_k^1 \hookrightarrow \mathsf{s}_3\mathsf{tQGC}_k^1$ is a quasi-isomorphism, we also show that the inclusion for the full complexes is a quasi-isomorphism. The complex $\mathsf{s}_3\mathsf{tQGC}_k^1$ split as $\mathsf{s}_3\mathsf{tQGC}_k^1(\frac{\infty_1}{0}) \oplus \mathsf{s}_3\mathsf{tQGC}_k^1(\frac{0}{\infty_1})$ where each complex contains the graphs where all non-univalent vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively. We consider a vertex an *antenna-vertex* if it is either univalent or becomes univalent after iterative removal of univalent vertices in a graph. Consider the filtration of both complexes over the number of non-bivalent non-antenna vertices. In the associated graded, the differential acts on by creating bivalent antenna vertices. All graphs of *gr* $\mathsf{s}_4\mathsf{tQGC}_k^1$ are non-trivial cycles, and so we need to show that the cohomology of *gr* $\mathsf{s}_3\mathsf{tQGC}_k^1$ is generated by the same graphs. Consider first the filtration over the number of non-passing vertices. We see that $H(gr\ \mathsf{s}_3\mathsf{tQGC}_k^1(\frac{0}{\infty_1}))$ is generated exactly by the graphs in $\mathsf{s}_4\mathsf{tQGC}_k^1(\frac{0}{\infty_1})$. We further get that $H(gr\ \mathsf{s}_3\mathsf{tQGC}_k^1(\frac{\infty_1}{0}))$ is generated by graphs with no passing antenna-vertices and univalent vertices of two types, either $\underset{\uparrow}{\textcircled{\frac{\infty_1}{\infty_1}}}$ or $\underset{\downarrow}{\textcircled{\frac{\infty_2}{0}}}$. Note that there is only one possible decoration for a univalent vertex, depending on if it is a source or a target. On the next page of the spectral sequence, the antenna-vertices form trees, where chains of bivalent vertices are composed of edges alternating directions. Call such an edge a *zig-zag* edge.

The case when zig-zag edges are adjacent to two at least trivalent vertices and how the differential acts on them has been studied in [Z2] (where they are called skeleton edges). The more important case here is the case when a zig-zag edge is adjacent to a univalent vertex. They are on the form $\rightarrowtail\overset{m}{\rightsquigarrow}\bullet := \rightarrowtail\overbrace{\bullet\!\!\longrightarrow\cdots\rightarrow\bullet\!\!\longleftarrow\!\!\bullet\!\!\longrightarrow\!\!\bullet}^{m}$ where m is the number of normal edges in the zig-zag edge and its direction is the direction of the last edge attached to the univalent vertex. The differential acts on zig-zag edges as

$$
\begin{array}{lcllcll}
d(\rightarrowtail\overset{2m+1}{\rightsquigarrow}\bullet) & = & (-1)^{k+1} & \rightarrowtail\overset{2m+2}{\rightsquigarrow}\bullet & + & (-1)^{k+1} & \rightarrowtail\overset{2m+2}{\leftsquigarrow}\bullet \\
d(\rightarrowtail\overset{2m+1}{\leftsquigarrow}\bullet) & = & - & \rightarrowtail\overset{2m+2}{\rightsquigarrow}\bullet & - & & \rightarrowtail\overset{2m+2}{\leftsquigarrow}\bullet \\
d(\rightarrowtail\overset{2m}{\rightsquigarrow}\bullet) & = & & \rightarrowtail\overset{2m+1}{\rightsquigarrow}\bullet & + & (-1)^{k+1} & \rightarrowtail\overset{2m+1}{\leftsquigarrow}\bullet \\
d(\rightarrowtail\overset{2m}{\leftsquigarrow}\bullet) & = & - & \rightarrowtail\overset{2m+1}{\rightsquigarrow}\bullet & + & (-1)^{k} & \rightarrowtail\overset{2m+1}{\leftsquigarrow}\bullet
\end{array}
$$

We see that its cohomology is generated by $\rightarrowtail\!\bullet\!\!\longrightarrow\!\!\bullet + (-1)^{k+1} \rightarrowtail\!\bullet\!\!\longleftarrow\!\!\bullet$. The complex decomposes over graphs with the same skeleton graphs, i.e., the resulting graph obtained by replacing sequences of bivalent vertices with one single edge. These complexes decompose into tensor products over the edges of the skeleton graphs, similar to the decompositions in Proposition **??**, we get that the cohomology of $gr\ \mathsf{s}_3\mathsf{tQGC}_k^1(\frac{\infty_1}{0}))$ is generated the desired one. □

Definition 4.4.10. Let mQGC_k^+ be the subcomplex of mQGC_k^* on the form

$$
\mathsf{mQGC}_k^+ = \mathcal{C}^{\geq 2,+} \oplus \mathcal{C}^1(\frac{\infty_1}{0}) \oplus \mathcal{C}^1(\frac{0}{\infty_1})
$$

where $\mathcal{C}^1(\frac{\infty_1}{0})$ and $\mathcal{C}^1(\frac{0}{\infty_1})$ are as in Definition 4.4.7 and $\mathcal{C}^{\geq 2,+} \subseteq \mathcal{C}^{\geq 2}$ is the subspace generated by graphs of the forms Γ^ω and $\Gamma^{\frac{\infty_1}{0}}$.

Corollary 4.4.11. *The inclusion* $\mathsf{mQGC}_k^+ \hookrightarrow \mathsf{tQGC}_k^+$ *is a quasi-isomorphism.*

Proof. Consider the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{mQGC}_k^+ & \longrightarrow & \mathsf{mQGC}_k^* & \longrightarrow & \mathsf{mQGC}_k^\sim & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{tQGC}_k^+ & \longrightarrow & \mathsf{tQGC}_k^+ & \longrightarrow & \mathsf{tQGC}_k^\sim & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Q}_k^+ & \longrightarrow & \mathsf{Q}_k^* & \longrightarrow & \mathsf{Q}_k^\sim & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

induced by the inclusions, and where the other complexes are the appropriate quotient complexes. One checks that it commutes, and further that $\mathsf{mQGC}_k^\sim = \mathsf{tQGC}_k^\sim$ and so $\mathsf{Q}_k^* \cong \mathsf{Q}_k^+$. Proposition 4.4.9 gives that Q_k^* is acyclic since the inclusion $\mathsf{mQGC}_k^* \hookrightarrow \mathsf{tQGC}_k^*$ is a quasi-isomorphism. □

Definition 4.4.12. Let $\mathsf{omQGC}_k \subseteq \mathsf{mQGC}_k$ be the subcomplex of oriented graphs.

Proposition 4.4.13. *The inclusion* $\mathsf{omQGC}_k \hookrightarrow \mathsf{otQGC}_k$ *is a quasi-isomorphism.*

Proof. The argument is the same as the proof of Proposition 4.4.9, noting that all the arguments are independent of the orientation of a graph since the filtrations are done with respect to passing vertices and antennas. □

4.4.3 Final steps

Definition 4.4.14. Let $\mathcal{C}(\frac{\infty_1}{0})$ and $\mathcal{C}(\frac{0}{\infty_1})$ be the subcomplexes of mQGC_k^* of graphs whose non-univalent vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively.

Proposition 4.4.15. *Consider the short exact sequence*

$$0 \longrightarrow \mathcal{C}(\tfrac{\infty_1}{0}) \oplus \mathcal{C}(\tfrac{0}{\infty_1}) \longrightarrow \mathsf{mQGC}_k^* \longrightarrow \mathsf{mQGC}_k(\omega) \longrightarrow 0 .$$

Then

1. *The complexes* $\mathcal{C}(\frac{\infty_1}{0})$ *and* $\mathcal{C}(\frac{0}{\infty_1})$ *are acyclic.*
2. *The complex* $\mathsf{mQGC}_k(\omega)$ *is isomorphic to* dGC_k*.*

In particular, there is a quasi-isomorphism $\mathsf{mQGC}_k^* \to \mathsf{dGC}_k$*.*

Proof. The proof is the same as the proof of Proposition 3.4.8 in Chapter 3, by considering a filtration over the number of non-passing vertices. □

Corollary 4.4.16. *Let* $\mathsf{omQGC}_k(\omega)$ *be the subcomplex of* $\mathsf{mQGC}_k(\omega)$ *of oriented graphs. Then the induced projection* $\mathsf{omQGC}_k \to \mathsf{omQGC}_k(\omega)$ *is a quasi-isomorphism. Furthermore* $\mathsf{omQGC}_k(\omega) \cong \mathsf{oGC}_k$*.*

Proof. The same arguments as in Proposition 4.4.15 give the result since it is independent of the orientation of a graph. □

Definition 4.4.17. Let $\mathcal{C}(\frac{\infty_1}{0})$ be the subcomplex of mQGC_k^+ of graphs where all non-univalent vertices are decorated with $\frac{\infty_1}{0}$, and let $\mathcal{Q} := \mathsf{mQGC}_k^+/\mathcal{C}(\frac{\infty_1}{0})$.

Proposition 4.4.18. *The complex* $\mathcal{C}(\frac{\infty_1}{0})$ *is acyclic. In particular the projection* $\mathsf{mQGC}_k^+ \to \mathcal{Q}$ *is a quasi-isomorphism.*

Proof. The proof follows by considering the filtration over the number of non-univalent vertices. □

Proposition 4.4.19. *Let* $\mathcal{Q}^t$ *be the subcomplex of* $\mathcal{Q}$ *of graphs with at least one target vertex when excluding antenna-vertices. Then the inclusion is a quasi-isomorphism. Furthermore, the complex* $\mathcal{Q}^t$ *is isomorphic to* dGC_k^t*.*

Proof. One shows that the quotient $\mathcal{Q}/\mathcal{Q}^t$ is acyclic by the same argument as in Proposition 4.4.18. The second part follows by inspection. □

Definition 4.4.20. Let $\mathcal{C}$ be the subcomplex of tPGC_k^* of graphs with at least one univalent vertex and graphs with univalent on the forms $\Gamma^{\frac{\infty_1}{0}}$ and $\Gamma^{\frac{0}{\infty_1}}$.

Proposition 4.4.21. *Consider the short exact sequence*

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathsf{tPGC}_k^* \longrightarrow \mathsf{tPGC}_k(\omega) \longrightarrow 0$$

where $\mathsf{tPGC}_k(\omega)$ *is the complex of graphs without univalent vertices on the form* Γ^ω*. Then projection* $\mathsf{tPGC}_k^* \to \mathsf{tPGC}_k(\omega)$ *is quasi-isomorphism. Furthermore, the complex* $\mathsf{tPGC}_k(\omega)$ *is isomorphic to* dGC_k*.*

Proof. It is enough to show that $\mathcal{C}$ is acyclic. The proof is similar to that of Proposition 4.4.9. First consider the subcomplex $\mathsf{s}_1\mathsf{tQGC}_k$ of graphs with at least one non-univalent vertex decorated by $\frac{\infty_1}{0}$ or $\frac{0}{\infty_1}$. The quotient complex is acyclic. Secondly, consider the subcomplex of graphs with at least a pair of non-univalent vertices decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively. This complex is acyclic, and let $\mathsf{s}_2\mathsf{tPGC}_k$ be the corresponding quotient complex. It splits as $\mathcal{C}(\frac{\infty_1}{0}) \oplus \mathcal{C}(\frac{0}{\infty_1})$ of graphs where all non-univalent vertices are decorated by $\frac{\infty_1}{0}$ and $\frac{0}{\infty_1}$ respectively. Consider the subcomplex of $\mathcal{C}(\frac{\infty_1}{0})$ of graphs with at least one univalent vertex $\underset{\uparrow}{\left(\frac{0}{\infty_2}\right)}$. This complex is acyclic. The quotient complex has only two types of univalent vertices on the form $\underset{\uparrow}{\left(\frac{\infty_1}{\infty_1}\right)}$ and $\underset{\downarrow}{\left(\frac{\infty_2}{0}\right)}$. By considering the filtration over the number of non-bivalent non-antenna vertices, we can show that this complex is acyclic on the first page. The proof is analogous to showing that $\mathcal{C}(\frac{0}{\infty_1})$ is acyclic. □

Corollary 4.4.22. *Let* $\mathsf{otPGC}_k(\omega)$ *be the subcomplex of* $\mathsf{tPGC}_k(\omega)$ *of oriented graphs. Then the induced projection* $\mathsf{optGC}_k \to \mathsf{optGC}_k(\omega)$ *is a quasi-isomorphism. Furthermore,* $\mathsf{otPGC}_k(\omega) \cong \mathsf{oGC}_k$.

Proof. The same argument as in Proposition 4.4.21 gives the results since it is independent of the orientation of a graph. □

4.5 Main results

In this section, we first give a review of the quasi bi-weighted complexes that have been defined in the previous sections, followed by stating and proving the main theorems of this chapter.

4.5.1 Summary of quasi bi-weighted graph complexes

The complex fwQGC_k is the complex of all possible quasi bi-weighted graphs. The complex fwQGC_k^+ is the subcomplex of fwQGC_k of graphs that have at least one vertex with positive out-weight.
The complexes fwQGC_k and fwQGC_k^+ split as

$$\mathsf{fwQGC}_k = \mathsf{b_0wQGC}_k \oplus \mathsf{wQGC}_k \quad \text{and} \quad \mathsf{fwQGC}_k^+ = \mathsf{b_0wQGC}_k \oplus \mathsf{wQGC}_k^+$$

where $\mathsf{b_0wQGC}_k^0$ is the subcomplex of graphs with loop number zero and wQGC_k and wQGC_k^+ are the respective complements. The cohomology of $\mathsf{b_0wQGC}_k^0$ is generated by the graph following graph (see Proposition 4.2.9)

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 3}} (i+j-2) \left(\frac{i}{j}\right)$$

The complex qQGC_k^* is the subcomplex of wQGC_k^* of graphs whose vertices are independently decorated by the five bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$, $\frac{0}{0}$ and $\frac{\infty_2}{0}$ such that univalent vertices are only decorated by $\frac{\infty_1}{\infty_1}$ (or $\frac{\infty_2}{0}$ when they are sources). The complex qQGC_k^+ is the subcomplex of qQGC_k^+ of graphs where at least one vertex is decorated by $\frac{\infty_1}{\infty_1}$ or $\frac{\infty_1}{0}$ (or $\frac{\infty_2}{0}$). It follows that qQGC_k^+ is a subcomplex of wQGC_k^+.

The complex qQGC_k split as

$$\mathsf{qQGC}_k = \mathsf{qQGC}_k^0 \oplus \mathsf{qQGC}_k^*$$

where qQGC_k^0 is the subcomplex of graphs whose vertices are only decorated by $\frac{0}{0}$ and qQGC_k^* its complement. The complex qQGC_k^0 is isomorphic to the complex $\mathsf{dGC}_k/\mathsf{dGC}_k^{2+t}$, where $\mathsf{dGC}_k^{2+t} \subseteq \mathsf{dGC}_k$ is the subcomplex of graphs with at least one bivalent vertex or one target vertex (see Proposition 4.4.1).

The complex tQGC_k^* is the quotient complex $\mathsf{qQGC}_k^*/\mathsf{qQGC}_k^{*,0}$ where $\mathsf{qQGC}_k^{*,0} \subseteq \mathsf{qGC}_k^*$ is the subcomplex of graphs with at least one vertex decorated by $\frac{0}{0}$. In other words, tQGC_k^* is generated by graphs whose vertices are independently decorated by the four bi-weights $\frac{\infty_1}{\infty_1}$, $\frac{\infty_1}{0}$, $\frac{0}{\infty_1}$ and $\frac{\infty_2}{0}$. The complex tQGC_k^+ is the subcomplex of tQGC_k^* of graphs where at least one vertex is decorated by $\frac{\infty_1}{\infty_1}$ or $\frac{\infty_1}{0}$ (or $\frac{\infty_2}{0}$). It follows that tQGC_k^+ is a quotient complex of qQGC_k^+.

The complex mQGC_k^* is a subcomplex of tQGC_k^* on the form

$$\mathsf{mQGC}_k^* = \mathcal{C}^{\geq 2} \oplus \mathcal{C}^1(\frac{\infty_1}{0}) \oplus \mathcal{C}^1(\frac{0}{\infty_1}).$$

The complex mGC_k^+ is a subcomplex of mGC_k^*. (see full details of these complexes in Definition 4.4.7).

The complex $\mathsf{mQGC}_k^*(\omega)$ is a quotient complex of mQGC_k^* (see Proposition 4.4.15). The complex $\mathcal{Q}$ is a quotient complex of mQGC_k (see Definition 4.4.17), and $\mathcal{Q}^t$ is a subcomplex of $\mathcal{Q}$ (see Proposition 4.4.19).

The complexes that have been used in this chapter are related as follows:

$$\mathsf{qQGC}_k^* \twoheadrightarrow \mathsf{tQGC}_k^* \hookleftarrow \mathsf{mQGC}_k^* \twoheadrightarrow \mathsf{mQGC}_k^*(\omega) \longleftarrow \mathsf{dGC}_k$$

$$\mathsf{qQGC}_k^+ \twoheadrightarrow \mathsf{tQGC}_k^+ \hookleftarrow \mathsf{mQGC}_k^+ \twoheadrightarrow \mathsf{mQGC}_k^+(\omega) \twoheadrightarrow \mathcal{Q} \hookleftarrow \mathcal{Q}^t \longleftarrow \mathsf{dGC}_k^t$$

Proposition 4.5.1. *Let $\mathsf{dGC}_k \to \mathsf{wQGC}_k$ be the map where a graph Γ is mapped to the sum of all possible bi-weights to put on Γ excluding the decoration with only $\frac{0}{0}$. Then this map is a quasi-isomorphism up to the subcomplex of graphs with loop number zero and graphs only decorated by $\frac{0}{0}$.*

Proof. This map restricts to chain maps to all of the complexes in the diagram above, making it commute. One checks that all of these maps are quasi-isomorphisms, starting with that the restriction to dGC_k is an isomorphism. □

Proposition 4.5.2. *Let $\mathsf{dGC}_k^t \to \mathsf{wQGC}_k^+$ be the map where a graph Γ is mapped to the sum of all possible bi-weights to put on Γ excluding the decoration with only $\frac{0}{0}$. Then this map is a quasi-isomorphism up to the subcomplex of graphs with loop number zero.*

Proof. The argument is equivalent to that of Proposition 4.5.1. □

Proposition 4.5.3. *Let $\mathsf{oGC}_k^t \to \mathsf{owQGC}_k$ be the map where a graph Γ is mapped to the sum of all possible bi-weights to put on Γ excluding the decoration with only $\frac{0}{0}$. Then this map is a quasi-isomorphism up to the subcomplex of graphs with loop number zero.*

Proof. The argument is equivalent to that of Proposition 4.5.1. □

4.5.2 Deformation theory of quasi-Lie bialgebras

Theorem 4.5.4. *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{oGC}_{c+d+1} \to Der(\mathcal{QH}olieb_{c,d}).$$

Proof. This follows from the decomposition of $Der(\mathcal{QH}olieb_{c,d})$ over graphs of loop number zero together with the quasi-isomorphism of Theorem 4.1.7 and Corollary 4.4.22. □

Theorem 4.5.5. *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{dGC}^{\geq 3, no\ t}_{c+d+1} \oplus \mathsf{dGC}_{c+d+1} \to Der^{\bullet}(\mathcal{QH}olieb^{\circlearrowleft}_{c,d}).$$

Proof. This follows from the decomposition of $Der^{\bullet}(\mathcal{QH}olieb^{\circlearrowleft}_{c,d})$ over graphs of loop number zero and quasi bi-weighted graphs only decorated by $\frac{0}{0}$, together with the quasi-isomorphism of Theorem 4.1.7, Proposition 4.4.1 and Proposition 4.4.15. □

Theorem 4.5.6. *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{dGC}^{t}_{c+d+1} \to Der(\mathcal{QH}olieb^{\circlearrowleft}_{c,d}).$$

Proof. This follows from the decomposition of $Der(\mathcal{QH}olieb^{\circlearrowleft}_{c,d})$ over graphs of loop number zero together with the quasi-isomorphism of Theorem 4.1.7 and Proposition 4.4.19. □

4.5.3 Deformation theory of pseudo-Lie bialgebras

The results and arguments from the previous section also apply to the subcomplexes of wPGC_k and owPGC_k. We get the following results.

Theorem 4.5.7. *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{oGC}^{\geq 3}_{c+d+1} \oplus \mathsf{oGC}_{c+d+1} \to Der(\mathcal{PH}olieb_{c,d}).$$

Proof. This follows with the above remark together with Theorem 4.1.7, Proposition 4.4.2 and Corollary 4.4.22. □

Theorem 4.5.8. *There is a quasi-isomorphism of complexes*

$$\mathbb{K} \oplus \mathsf{dGC}^{\geq 3}_{c+d+1} \oplus \mathsf{dGC}_{c+d+1} \to Der^{\bullet}(\mathcal{PH}olieb^{\circlearrowleft}_{c,d}).$$

Proof. This follows with the above remark together with Theorem 4.1.7, Proposition 4.4.1 and Proposition 4.4.21. □

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