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ON THE WEYL PROBLEM FOR UNBOUNDED SURFACES IN HYPERBOLIC AND ANTI-DE SITTER SPACE

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Abstract

In this thesis, we study quasi-Fuchsian manifolds and globally hyperbolic anti-de Sitter 3-manifolds.
Let S be a closed hyperbolic surface, and let M be convex co-compact sub-domain of a quasi-Fuchsian manifold Q homeomorphic to $S \times [0,1]$. The hyperbolic metric on M induces several invariants on its boundary (that is $S \times \{0,1\}$). These invariants migh be a metric h that has curvature strictly bigger than -1, which is relaized as induced metric on $S \times \{0\}$ or $S \times \{1\}$. It can be a metric h^*, that has curvature strictly smaller than 1, and has no contractible geodesic with weight bigger or equal to 2π, which realized as third fundamental form on $S \times \{0\}$ or $S \times \{1\}$. In other cases, the invariants can be a hyperbolic metric, a conformal structure, or a measured lamination. The nature of the data depends on the the convex domain M (whether it has a pleated boundary, a smooth boundary, or ideal boundary). Let L be measured lamination on S, that has no closed leaf of weight bigger or equal to π. Our main result in the first part, is showing that for any other realizable invariant on the boundary of M, there exists a quasi-Fuchsian manifold Q and a convex co-compact sub-domain M homeomorphic to $S \times [0,1]$ such that $S \times \{0\}$ is a pleated surface that has a bending lamination L, and it induces the other invariant that we ask on $S \times \{1\}$. If the bending lamination L is small enough, in a sense that we will define, and the invariant on $S \times \{1\}$ is a hyperbolic metric, we show the uniqueness of such convex co-compact sub-domain. Later in the first part, we study convex subsets of $\mathbb{H}^3$ that have ideal boundary equal to a quasi-circle, we will highlight their interaction with the universal Teichmüller space.
In the second part, we focus on globally hyperbolic anti-de Sitter 3-manifolds. We will define globally hyperbolic convex subsets that, in particular, serve as lifts of globally hyperbolic anti-de Sitter 3-manifolds. Let h^+ and h^- be two complete, conformal metrics on the disc $\mathbb{D}$. Assume that the derivatives of h^+ and h^- in any order are uniformly bounded with respect to the hyperbolic metric and that the curvatures are in the interval $(-\frac{1}{\epsilon}, -1-\epsilon)$ for some $\epsilon > 0$. Let f be a quasi-symmetric map. Our main result in the second part, is showing the existence of a globally hyperbolic convex subset Ω (see Definition 3.4.1) of the three-dimensional anti-de Sitter space, such that Ω has h^+ (respectively h^-) as the induced metric on the future boundary of Ω (respectively the past boundary of Ω) and has a gluing map Φ_Ω (see Definition 3.4.7) equal to f.

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Contents

Introduction

0.1 Convex subsets

We investigate the notion of hyperbolic convex co-compact 3-dimensional manifolds, and globally hyperbolic anti-de Sitter manifolds, which are the Lorentzian analogue of quasi-Fuchsian manifolds.
We define the deformation space of a hyperbolic manifold (Riemannian or Lorentzian) to be the space of isotopy classes of all the hyperbolic metrics on it. In particular we will focus on the interaction between the deformation space of these objects and the deformation space of their boundaries. This interplay between the deformation space of a convex 3-dimensional object and the deformation space of its boundary has been the subject of significant research interest.
The field can be traced back to Cauchy's proof (see [Cau]) regarding the rigidity of convex polyhedra. He proved that if two polyhedra have isometric faces glued together in the same combinatorial pattern, then they are the same polyhedron up to isometry.

A deep theorem came later through the work of Alexandrov [Ale05]. He realized that any Euclidean metric on the sphere with cone singularities, in which each cone angle is less than 2π, is isometric to the path metric on the boundary of a unique polyhedron. Alexandrov took that a step forward by solving the "Weyl problem", which was first brought by H.Weyl in 1915.

Theorem 0.1.1. *Any smooth metric of positive curvature on the sphere is induced on the boundary of a unique smooth strictly convex subset of* $\mathbb{R}^3$

Later, Alexandrov and Pogorolov extended this result to the hyperbolic space.

Theorem 0.1.2. *Any smooth metric in the sphere of curvature $K > -1$ is induced on the boundary of a unique convex subset of $\mathbb{H}^3$ with a smooth boundary.*

This "philosophy" of studying the geometric characteristics of geometric objects by studying their boundary can be applied in various contexts.

0.2 Hyperbolic convex cocompact 3-dimensional manifolds

Recall that any complete orientable 3-dimensional hyperbolic manifold is of the form $\mathbb{H}^3/\Gamma$, where Γ is a discrete torsion free subgroup of $\mathrm{PSL}(2, \mathbb{C})$ (recall that the group

of isometries of $\mathbb{H}^3$ that preserve orientation is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$). We say that $\mathbb{H}^3/\Gamma$ is complete convex co-compact if there is a convex subset $\Omega \subset \mathbb{H}^3$, such that Ω is invariant under the action of Γ, and Ω/Γ is compact. We can have a non complete convex co-compact hyperbolic manifolds, in that case they are the quotient of a convex subset $C \subset \mathbb{H}^3$ by Γ, a discrete torsion free subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and such that there is convex subset $\Omega \subset C$, which is invariant under the action of Γ, and Ω/Γ is compact.
In this thesis we will focus on quasi-Fuchsian manifolds, which are also 3-dimensional convex co-compact manifolds.
Let S be a closed surface of genus $g \geq 2$. We denote by M the three-dimensional manifold $S \times (0,1)$, and we denote by $\overline{M}$ the three-dimensional manifold with boundary $S \times [0,1]$. We also denote by $\mathcal{T}(S)$ the Teichmüller space of S, which is the space of isotopy classes of hyperbolic structures on S, or equivalently, due to uniformization, the space of isotopy classes of conformal structures on S.
We denote the isometry group of the hyperbolic space $\mathbb{H}^3$ as $Isom(\mathbb{H}^3)$, and the subgroup of isometries that preserve orientation as $Isom^+(\mathbb{H}^3)$. Let Γ be a subgroup of $Isom^+(\mathbb{H}^3)$. We say that Γ is Kleinian if it acts properly discontinuously on the hyperbolic space.
We say that a complete hyperbolic manifold (possibly with boundary) is geodesically convex if each arc is homotopic, relatively to its endpoints, to a geodesic arc.
Thurston has shown (see [Thu97, Proposition 8.3.2]) that if N is a convex hyperbolic manifold, then there exists a convex subset Ω of $\mathbb{H}^3$ and a Kleinian group Γ such that N is isometric to Ω/Γ (we assume that Ω is invariant under the action of Γ), or equivalently, Ω is a covering space of N. Moreover, if the quotient of the convex hull of the limit set of Γ by Γ is compact, we say that N is a hyperbolic convex cocompact manifold (see Section 2.1.1 for a definition of the limit set and convex hull).
We will focus on convex cocompact hyperbolic manifolds of the form Ω/Γ, where Γ is a Kleinian group with a limit set that is a Jordan curve (or equivalently a quasi-circle, see [Thu97, Proposition 8.7.2]). Here, Ω is a convex subset of $\mathbb{H}^3$ that is invariant under Γ, and Γ is isomorphic to $\pi_1(S)$, the fundamental group of S. Then, according to [Thu97, Proposition 8.7.2], the manifold Ω/Γ is diffeomorphic to either $S \times [0,1]$, $S \times]0,1]$, $S \times [0,1[$, or $S \times]0,1[$, depending on how many boundaries component Ω has (see Figure 3.2). Equivalently, if g is a convex co-compact hyperbolic metric on one of these manifolds, then the resulting manifold is identified with Ω/Γ, where Γ is a Kleinian group with a limit set equal to a quasi-circle, and Ω is a convex subset of $\mathbb{H}^3$ invariant under the action of Γ.

If Ω is the entire $\mathbb{H}^3$, we refer to $\mathbb{H}^3/\Gamma$ as quasi-Fuchsian (see Section 2.1 for more details). Note that in all cases, a hyperbolic convex cocompact metric on one of the four manifolds mentioned above is embedded in a unique quasi-Fuchsian manifold, which arises from the fact that Ω/Γ is embedded in $\mathbb{H}^3/\Gamma$.
If the manifold is complete and the boundary is open on one side or both sides, such as $S \times [0,1[$, $S \times]0,1[$, $S \times [0,1[$, the complete, convex cocompact hyperbolic manifold can still be embedded in $S \times [0,1]$ by adding the ideal boundary of Ω minus the limit set of Γ (see Section 2.1 for details). But when we extend in this way, the metric cannot be extended to this ideal boundary (because the hyperbolic metric cannot be extended to the ideal boundary of $\mathbb{H}^3$), but it induces a conformal structure (See Section 2.1 for details). In this case, we refer to the boundary as the boundary at infinity. It can be $S \times \{0\}$, $S \times \{1\}$ or $S \times \{0,1\}$ (see Section 2.1 for more details). In all cases, the hyper-

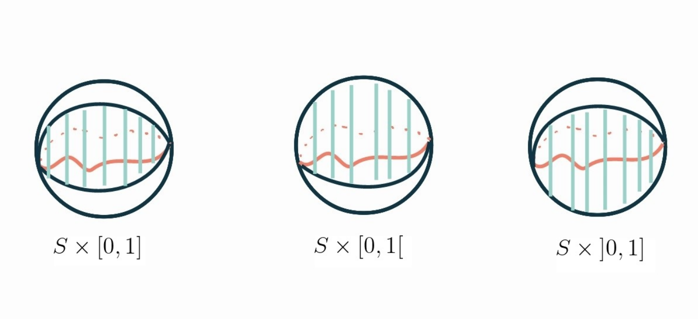


Figure 1: The shadowed region Ω. The number of boundary components of the quotient depends on the number of boundary components of Ω. We always have two boundary components when we include the ideal boundary.

bolic convex cocompact metric on $S \times]0,1[$ induces data on the boundary. It can be a conformal structure when we take the ideal boundary as explained above, or it can be an induced metric which may not be smooth, as we will see, when ever the induced data on the boundary is a metric, this metric has a Gaussian curvature bigger or equal to -1. If a boundary component of the hyperbolic convex cocompact manifold $S \times [0,1]$ is not ideal and is smoothly embedded in a quasi-Fuchsian manifold, then the induced metric has a Gaussian curvature strictly greater than -1. In this case, we refer to the induced metric as the first fundamental form. Additionally, we can define another metric on this boundary that measures how much the embedded surface is curved inside the quasi-Fuchsian manifold. This is referred to as the third fundamental form (see Section 1.4 for details on the third fundamental form). If both boundary components of the hyperbolic convex cocompact manifold $S \times [0,1]$ are smoothly embedded in a quasi-Fuchsian manifold, we say that the hyperbolic convex cocompact manifold $S \times [0,1]$ is strictly convex. In other words, the hyperbolic convex cocompact manifold $\overline{M}$ is considered strictly convex when there are no geodesic arcs that are fully contained within its boundary.

Let Γ be a Kleinian group, Λ_Γ be its limit set, and $CH(\Lambda_\Gamma)$ be the convex hull of the limit set (inside $\mathbb{H}^3$). If the limit set of a Kleinian group is a quasi-circle, which is not a geometric circle, then the quotient $CH(\Lambda_\Gamma)/\Gamma$ is a compact manifold homeomorphic to $S \times [0,1]$, endowed with a convex cocompact hyperbolic metric. Denoting $\mathbb{H}^3/\Gamma$ as Q (a quasi-Fuchsian manifold), we refer to $CH(\Lambda_\Gamma)/\Gamma$ as $C(Q)$, calling it the convex core of Q. $C(Q)$ is the smallest complete non-empty geodesically convex (in particular, it contains all the simple closed geodesics) submanifold of Q, as detailed in [Thu97, Proposition 8.1.2]. We see $C(Q)$ as a submanifold of Q. The boundary of $C(Q)$ consists of two components that are pleated surfaces (see Section 1.3 for the definition). The hyperbolic metric of Q induces a path metric on the boundary of $C(Q)$ (even if the boundary is not smoothly embedded. See [Thu97, Section 8.5] and [CME06, Chapter II.1]), which is isometric to a hyperbolic metric, and the pleating induces a measured lamination referred to as the bending lamination (see Section 1.2 for the definition of measured lamination and Section 2.1.2 for an explanation of how pleating induces a measured lamination).

As we can see, a hyperbolic convex cocompact metric on $S \times]0,1[$ can induce several types of data on the boundary (that is, $S \times \{0,1\}$). In particular, this data can include a metric with curvature strictly greater than -1, which is referred to as the first fundamental form or simply the induced metric. It can also include a metric with curvature strictly smaller

than 1, known as the third fundamental form and usually denoted as $I\!I\!I$. Furthermore, in the case where the boundary is ideal (meaning it lifts to the ideal boundary of $\mathbb{H}^3$), this data is a conformal structure. Alternatively, when the boundary is not smoothly embedded but is a pleated surface (for example the boundary of the convex core), the data can be a hyperbolic metric, or a measured lamination known as the bending lamination.
The study of convex cocompact hyperbolic metrics on M, and more precisely the correspondences between the metric g on M and the boundary data on M, has been the subject of significant research interest.
Let's consider a quasi-Fuchsian manifold Q, recall that this manifold has two ideal boundary components, and then it induces a conformal structure on $S \times \{0,1\}$. It was shown by Bers [Ber60] (see Theorem 2.1.1) that there is a homeomorphism (actually a biholomorphism if we consider the complex structure induced by the Character variety, see [Mar16a]) between the space of conformal structures on ∂M and the space of quasi-Fuchsian structures on M.
A key result about quasi-Fuchsian manifolds is that for any quasi-Fuchsian manifold Q, the set $Q \setminus C(Q)$ is foliated by k-surfaces (see [Lab91]). That is, $Q \setminus C(Q) = \bigcup_{k \in (-1,0)} S_k^{\pm}$, and the induced metric on each of S_k^+ and S_k^- has a constant Gaussian curvature equal to k (see Section 2.1.3).

Labourie, in [Lab92a], showed that for any metric h on $S \times \{0,1\}$ with Gaussian curvature strictly greater than -1, one can find a unique hyperbolic convex cocompact metric on M that induces h on the boundary. As previously explained, $(\overline{M}, g)$ is embedded in the interior of a quasi-Fuchsian manifold. Later, Schlenker, in [Sch06a], established the uniqueness of the hyperbolic convex cocompact metric g on M that induces h on the boundary of $\overline{M}$. He also showed a similar statement regarding the uniqueness and existence of the third fundamental form. Specifically, any metric h^* on $S \times \{0,1\}$ with Gaussian curvature strictly smaller than 1, and in which every contractible closed geodesic has length strictly greater than 2π, can be induced as the third fundamental form on the boundary of $\overline{M}$ by a unique hyperbolic convex cocompact metric g on M, where $(\overline{M}, g)$ is once again embedded in the interior of a quasi-Fuchsian manifold. If we assume that h or h^* has constant Gaussian curvature, then $S \times \{0,1\}$ corresponds to some k surfaces, while we observe $(\overline{M}, g)$ embedded within a quasi-Fuchsian manifold.
Moreover, in [Sch06a, Theorem 2.6] the author shows that for each $k \in (-1,0)$, the map that sends a quasi-Fuchsian structure on M to the induced metric on $S_k^- \bigsqcup S_k^+$ is a homeomorphism (in fact, a diffeomorphism. See [Sch06a] for details). For any quasi-Fuchsian manifold Q, when k goes to -1, the induced metrics on $S_k^{\pm}$ converge to the induced metrics on $\partial^{\pm} C(Q)$ (the boundary of the convex core) in the length spectrum, and the third fundamental forms induced on $S_k^{\pm}$ converge in the length spectrum to the bending lamination of $\partial^{\pm} C(Q)$ (see Theorem 2.7). Thurston has conjectured that the map that maps a quasi-Fuchsian structure on M to the induced metric on the boundary of its convex core is a homeomorphism. Even though it is known that any two hyperbolic metrics on S can be realized as the induced metrics on the convex core of some quasi-Fuchsian manifold Q (up to isotopy), the conjecture remains unproven, as injectivity is not established except for some special cases. Thurston made a similar conjecture for bending laminations, namely that the map that associates to a quasi-Fuchsian structure on M the bending lamination on the boundary of its convex core is a homeomorphism.

In [BO04], the authors characterize the measured laminations on ∂M that can be realized (up to isotopy) as the bending lamination of the convex core. Later Dular and Schlenker have etablished the uniqueness of such manifold (see [DS24]).
In the paper [CS22], the authors introduced a new parametrization of quasi-Fuchsian structures on M. This parametrization is obtained by considering the induced third fundamental form on S_k^+ and the induced metric on S_k^-, or by taking the third fundamental form on S_k^+ and the conformal structure on $S \times \{0\}$ (see Theorem 2.8). The main result of the paper [Mes23] is the extension of the surjectivity statements to the boundary of the convex core.
A convex hyperbolic metric on M is a metric g in which (M, g) is isometrically embedded in a quasi-Fuchsian manifold (which must be unique), such that the image is geodesically convex. For more details, we refer to [Thu97, Section 8.3].

Our main theorem in Chapter 2 is the following statement.

Theorem A. *Let S be a closed surface of genus g. Let h be Riemannian metric on S, and denote its curvature by k_h (which is not necessarily constant). We assume that $-1 < k_h$. Let μ be a measured lamination on S such that every closed leaf has weight strictly smaller than π. Then, there exists a convex hyperbolic metric g on $M = S \times]0,1[$, the interior of $\overline{M} = S \times [0,1]$, such that:*

- *g induces a metric on $S \times \{0\}$ which is isotopic to h.*
- *g induces on $S \times \{1\}$ a pleated surface structure in which its bending lamination is μ.*

Furthermore, when we see $\overline{M}$ as an embedded manifold in a quasi-Fuchsian manifold Q, then the boundary $S \times \{0\}$ is smoothly embedded.

We prove a similar statement for third fundamental forms.

Theorem A*. *Let h^* be Riemannian metrics on S, and denote their curvatures by k_{h^*} (which is not necessarily constant). We assume that $k_{h^*} < 1$. Moreover, we assume that every contractible closed geodesic with respect to h^* has length strictly bigger than 2π. Let μ be a measured lamination on S such that every closed leaf has weight strictly smaller than π. Then, there exists a convex hyperbolic metric g on $M = S \times (0,1)$, the interior of $\overline{M} = S \times [0,1]$, such that:*

- *g induces a third fundamental form on $S \times \{0\}$ which is isotopic to h^*.*
- *g induces on $S \times \{1\}$ a pleated surface structure in which its bending lamination is μ.*

Furthermore When we see $\overline{M}$ as an embedded manifold in a quasi-Fuchsian manifold Q, then the boundary $S \times \{0\}$ is smoothly embedded.

However, it is currently unknown whether the convex hyperbolic metric g on M is unique under the conditions stated in Theorem A.

Question. *Is the metric g given in Theorem A or in Theorem A* unique (up to isotopy) under the hypothesis of the theorem ?*

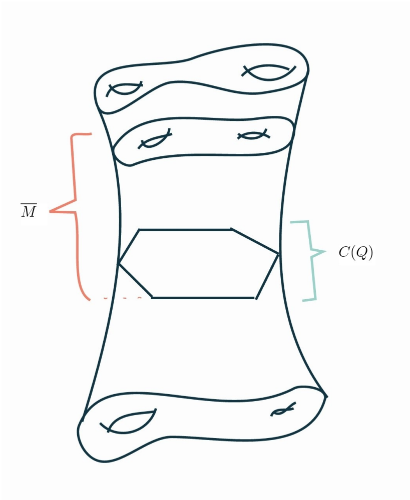


Figure 2: The manifold $\overline{M}$ is embedded in a quasi-Fuchsian manifold Q. One of its boundaries is smoothly embedded in Q, while the other one is a boundary component of $C(Q)$.

The proof leads us to show that the map that associates to a quasi-Fuchsian manifold the induced metric on $\partial^- C(Q)$ and the bending lamination on $\partial^+ C(Q)$ (call it M_{-1}), and the map that associates to a quasi-Fuchsian manifold the induced conformal structure on $S \times \{0\}$ and the bending lamination on $\partial^+ C(Q)$ (call it $M_{\infty,-1}$) are proper (see Section 2.1.4 for more details).

So, another theorem that we show in [Mes23] is the following:

Theorem B. *The maps M_{-1} and $M_{\infty,-1}$ are proper.*

In other words, Theorem B states the following. Consider a sequence $(Q_n)_{n\in\mathbb{N}}$ of quasi-Fuchsian manifolds. If the sequence $(h_n)_{n\in\mathbb{N}}$ of the hyperbolic metrics on $\partial^- C(Q_n)$ (up to isotopy) converges to a hyperbolic metric h, and if $(B_n^+)_{n\in\mathbb{N}}$, the sequence of bending laminations on $\partial^+ C(Q_n)$, converges to B_∞, where B_∞ is a measured lamination in which each closed leaf of it has weight strictly less than π, then $(Q_n)_{n\in\mathbb{N}}$ converges, up to a subsequence, to a quasi-Fuchsian manifold.
The proof of Theorem B employs techniques similar to those used in [LS14, Section 3], and the proof of this theorem took the most important part of [Mes23]. The proof consists of showing that the two induced metrics on $\partial^\pm C(Q)$ (up to isotopy) are in a compact subset of Teichmüller space. Then, we use Theorem 2.1.3 to deduce that the conformal structures at the boundary at infinity are in a compact subset of Teichmüller space. Finally, we conclude by the theorem of Bers (Theorem 2.1.1). After having proven Theorem B, we use the parametrization introduced in [CS22] (see Theorem 2.8) to approximate the metric and the measured lamination that we want to realize by k-surfaces. We use Theorem B to show that the sequence of quasi-Fuchsian manifolds that we constructed has a subsequence that converges, and finally, we conclude by Theorem 2.7.
As limiting case we find Theorem C, which is given in [Lec06a] (in unpublished notes). However, the proof that we give is completely independent from the one given in [Lec06a].

Theorem C. *Let μ be a measured lamination in which every closed leaf has weight strictly* *smaller than π, and let $h \in \mathcal{T}(S)$. Then:*

- *there exists a quasi-Fuchsian manifold Q such that the bending lamination of $\partial^+ C(Q)$ is L, and the induced metric on $\partial^- C(Q)$ is isotopic to h.*
- *There exists a quasi-Fuchsian manifold Q such that the bending lamination of $\partial^+ C(Q)$ is L, and the induced conformal structure on $S \times \{0\}$ is isotopic to h.*

We believe that our arguments can also be applied to quasi-Fuchsian manifolds with particles. Quasi-Fuchsian manifold with particles were introduced by Thurston (see [Thu82]), they have exactly the same geometric structure as a quasi-Fuchsian manifold, except along a finite number of infinite lines where they have cone singularities. These manifolds have been the subject of recent research [LS14][MS09][CS20], especially that they are related to the study of hyperbolic surfaces with conic singularities. In the convex core of a quasi-Fuchsian manifold with particles, the bending locus consistently stays away from singularities, depending on the conical angles. As a result, there's a strong indication that all the arguments in our proof of Theorem B can be extended. (for more details about these manifolds, see [MS09] and [LS14]). This gives us hope that Theorem C also holds for quasi-Fuchsian manifolds with particles. Nevertheless, it is currently unknown whether the methods used in Section 2.3 are applicable to this case.

A natural question that arises from Theorem C is the uniqueness of the quasi-Fuchsian manifold that realizes μ and h. We reformulate these questions in the following way.

Question. *Let μ be a measured lamination in which every closed curve has weight strictly smaller than π, and let $h \in T(S)$. Is there a unique quasi-Fuchsian manifold Q that induces L as the bending lamination on $\partial^+ C(Q)$ and induces h as the hyperbolic metric on $\partial^- C(Q)$?.*

In Section 2.5, we gave a partial answer to the last question 2.5 near the Fuchsian locus.

Theorem D. *For any $\mu \in \mathcal{ML}(S)$ and $h \in \mathcal{T}(S)$, there exists an $\delta_{h,\mu} > 0$ such that for any $0 < t < \delta_{h,\mu}$, there exists a unique quasi-Fuchsian manifold Q, such that the hyperbolic metric on $\partial CH^-(Q)$ is h and the bending lamination on $\partial CH^+(Q)$ is $t\mu$.*

Even if we don't give an answer on the following question, we believe that a partial answer similar to Theorem D, can be treated by similar arguments as the ones in Section 2.5.

Question. *Let L be a measured lamination in which every closed curve has weight strictly smaller than π, and let $h \in T(S)$. Is there a unique quasi-Fuchsian manifold Q that induces μ as the bending lamination on $\partial^+ C(Q)$ and induces h as the conformal structure on $S \times \{0\}$?*

0.3 Globally hyperbolic spacetimes

The three-dimensional anti-de Sitter space $\mathbb{ADS}^{2,1}$ is the Lorentzian analogue of hyperbolic space, that is, it is endowed with a Lorentzian metric having a constant sectional curvature equal to -1. The isometry elements of $\mathbb{ADS}^{2,1}$ that preserve orientation and time orientation form a group which is isomorphic to $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.

However if we take two Fuchsian representations ρ_l and ρ_r, then the representation $\rho = (\rho_l, \rho_r) : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ won't act properly discontinuously on $\mathbb{ADS}^{2,1}$ (see [Mes07]), but there is a maximal convex subset (see [Mes07]) $\Omega_\rho \subset \mathbb{ADS}^{2,1}$ in which the action of ρ on Ω_ρ is proper. In that case we call the quotient Ω_ρ/ρ by a globally hyperbolic maximal compact manifold, if we take $\Omega \subset \Omega_\rho$ which is a convex subset and which is invariant by ρ we call the quotient Ω/ρ globally hyperbolic compact manifold (note that $\Omega/\rho \subset \Omega_\rho/\rho$, it follows that any globally hyperbolic manifold is embedded inside a globally hyperbolic maximal manifold).
Since the work of Mess [Mes07], globally hyperbolic maximal compact (GHMC) manifolds, have received significant interest. These GHMC manifolds can be seen as the Lorentzian analogues of quasi-Fuchsian manifolds. As in the case of quasi-Fuchsian manifolds, they share many links to Teichmüler theory.
A globally hyperbolic compact (GHC) manifold is diffeomorphic to $S \times I$, where I can be $[0,1]$,$]0,1]$,$[0,1[$ or $]0,1[$. The holomomy representations of GHMC manifolds, which are of the form $(\rho_l, \rho_r) : \pi_1(S) \to PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ (where ρ_l and ρ_r are fuchsian representations), provide a bijection between the space of GHMC manifolds and $\mathcal{T}(S) \times \mathcal{T}(S)$. As in the case of quasi-Fuchsian manifolds, a GHMC manifold M has a convex core $C(M)$ whose boundary can be either one totally geodesic surface or can be the disjoint union of two pleated surfaces.
It follows that we can reformulate many questions about quasi-Fuchsian manifolds to the Lorentzian setting. For example we can ask if given two hyperbolic metrics h^+ and h^-, we can find a unique GHMC M such that the induced metrics on the boundary of $C(M)$ are h^+ and h^-. As in the case of quasi-Fuchsian manifolds the answer is positive for the existence (see [Dia13]), but the uniqueness is still open. Similarly to the hyperbolic case, we can ask whether for any two measured laminations L^+ and L^- that fill S, there exists a unique GHMC manifold M such that the bending laminations of the boundary of $C(M)$ are L^+ and L^-. The existence of such a manifold has been affirmed (see [BS12]), but the question of uniqueness remains unresolved. In the hyperbolic case, if L^+ and L^- are discrete, then there exists a unique quasi-Fuchsian manifold Q in which L^+ and L^- are the bending laminations of the boundary of $C(Q)$. However, for GHMC manifolds, an equivalent result is yet to be established. Another interesting question is, given two metrics h^+ and h^- on S with curvatures strictly less than -1, is there a unique globally hyperbolic manifold M such that the induced metrics on the boundary of M are h^+ and h^- ? The existence of such a manifold has been confirmed (see [Tam18]), yet unlike the hyperbolic case, the question of uniqueness remains unresolved.

0.4 Quasi-Fuchsian manifolds and globally hyperbolic spacetimes from the universal point of view

A new perspective, introduced by Bonsante, Danciger, Maloni, and Schlenker in [BDMS21a], involves investigating the boundaries of covers of globally hyperbolic manifolds and quasi-Fuchsian manifolds. Essentially, this approach delves into the study of convex subsets of $\mathbb{H}^3$ (or $\mathbb{ADS}^{2,1}$) whose ideal boundary forms a quasi-circle. This viewpoint extends the study of quasi-Fuchsian and GHC manifolds and also extends the Weyl problem to the case of unbounded convex subsets that intersect the ideal boundary in a quasi-circle.

When endeavoring to extend the results of Weyl and Alexandrov to unbounded convex subsets, one quickly encounters difficulties. However there is hope to extend these results to the case when the ideal boundary of the convex subset C is small enough. For example Rivin (see [Riv94]) proved that every complete hyperbolic metric on the n punctured sphere is isometric to a unique ideal polyhedron C thas has n vertices (an ideal polyhedron is a polyhedron that has its vertices on $\partial_\infty \mathbb{H}^3$).
In this thesis (and also in [BDMS21a]), we will focus on the case when the ideal boundary is a quasi-circle (a Jordan curve which is regular enough, see Section 1.11). For us, the boundary of C will consist of the disjoint union of two topological discs. In this scenario, the ambient metric of $\mathbb{H}^3$ induces a metric on the boundary of C (thus two metrics on the disc $\mathbb{D}$), and the asymptotic behavior of the metrics near the ideal boundary induces a homeomorphism from S^1 to S^1. This homeomorphism is quasi-symmetric (a homeomorphism that preserves orientation and exhibits certain regularity, see Section 1.8), we call this map the gluing map (see Figure 2.7). In the case where the convex subset C is equal to the convex hull of its ideal boundary (which we assume to be a quasi-circle), both of the induced metrics on the boundary of C are isometric to the hyperbolic metric on $\mathbb{D}$, and then the data on the boundary are reduced to the gluing map. In [BDMS21a], the authors extend the theorem of Alexandrov to this setting by showing that any quasi-symmetric map is the gluing map of the convex hull of some quasi-circle (Theorem [BDMS21a]).
As we will also explain, this can also be seen as an extension (if we show uniqueness) to Thurston's conjecture on the induced metrics on the boundary of the convex core of quasi-Fuchsian manifolds. In the same paper [BDMS21a], the authors show similar results when the induced metrics on the boundary components of C have constant Gaussian curvature $K \in (-1, 0)$ (see Theorem [BDMS21c, Theorem A]). Later in [CS22], the authors extend the results to some metrics of variable curvature metrics on $\mathbb{D}$.

As we will explain, the aforementioned results extend (provided we establish uniqueness) the results we mentioned regarding quasi-Fuchsian manifolds. However, we are still lacking a result concerning the bending laminations of the convex hull of a quasi-circle, which would extend Thurston's conjecture (and implicitly Bonahon and Otal's result). In Section 2.7.5, we will discuss a potential approach to address that.

Quasi-circles can also be defined in $\mathbb{ADS}^{2,1}$ (see Section 3.2.1). Although Alexandrov and Weyl theory does not yet have a well-developed analogue in $\mathbb{ADS}^{2,1}$, the authors in [BDMS21a] has extended all their results established in hyperbolic space to $\mathbb{ADS}^{2,1}$.
Once again, as we will elaborate in Section 3.4, these results extend (provided we establish uniqueness) those mentioned earlier concerning globally hyperbolic manifolds ([Dia13], [Tam18]). Despite the case of the hyperbolic space, in [MS21] the authors extended the main result of [BS12], which is analogous to Bonahon and Otal in $\mathbb{ADS}^{2,1}$, to this setting in [BDMS21a]. In Section 3.4, we provide an extension of [Tam18] to that setting, but first we define what do we mean by bounded derivatives

Definition 0.4.1. *We denote by h_{-1} the conformal hyperbolic metric on the disc $\mathbb{D}$. Let $h = \rho h_{-1}$ be a complete conformal metric on the disc $\mathbb{D}$.*
We say that h has bounded derivatives at order p by $M_p > 0$ on the disc $\mathbb{D}$ if any derivative of ρ at order p is bounded at any point of the disc $\mathbb{D}$ by M_p with respect to the hyperbolic metric.

We say that h has bounded derivatives if its derivatives are bounded at any order.

Now we state the main theorem of Section 3.4.

Theorem 0.4.2. *Let h^+ and h^- be two complete, conformal metrics on the disc $\mathbb{D}$ that have curvatures in an interval of the form $(-\frac{1}{\epsilon}, -1-\epsilon)$, for some $\epsilon > 0$. Assume moreover that any derivative of h^+ or h^- of order p is bounded by some positive number M_p. Let f be a normalized quasi-symmetric map. Then there exists a globally hyperbolic convex subset Ω such that the induced metric on $\partial^+\Omega$ is isometric to $(\mathbb{D}, h^+)$, the induced metric on $\partial^-\Omega$ is isometric to $(\mathbb{D}, h^-)$, and the gluing map is equal to f.*

Chapter 1

Preliminaries

1.1 Teichmüller space

For more details see [FLP+21, Section 7]. Through the thesis S will be considered to be a smooth closed oriented surface of genus $g \geq 2$. The Teichmüller space of S, denoted by $\mathcal{T}(S)$, is the set of hyperbolic metrics on S modulo isotopy. That is, two metric h_1 and h_2 are identified if and only if there is an isometry $I : (S, h_1) \to (S, h_2)$ which is isotopic to the identity. Thanks to the uniformisation theorem, Teichmüller space can also be seen as the space of conformal structures on S modulo isotopy. That is, two conformal structures c_1 and c_2 are identified if and only if there is a conformal diffeomorphism $f : (S, c_1) \to (S, c_2)$ which is isotopic to the identity.
Let $\mathcal{S}$ be the set of free homotopy classes of simple closed curves not homotopic to one point on S. There is an embedding of $\mathcal{T}(S)$ into $\mathbb{R}^{\mathcal{S}}$ by the map, $h \mapsto \Phi(h)$, where h is a hyperbolic metric and Φ_h is the map

$$\begin{aligned} \Phi(h) : \mathcal{S} &\to \mathbb{R}_{>0} \\ \alpha &\mapsto l_h(\alpha') \end{aligned}$$

where $\ell_h(.)$ is the length function with respect to h, and α' is the unique simple closed geodesic in the free homotopy class α. This last function Φ can be defined for any metric with strictly negative curvature, and it is well defined modulo isotopy. That is, if there is an isometry which is isotopic to the identity between (S, h) and (S, h') then $\Phi(h) \equiv \Phi(h')$. We denote $\Phi(h)$ by ℓ_h.
Through this thesis when we mention a metric of negative curvature we mean its isotopy classe, that is, we identify any two metrics that are isometric to each other via an isometry isotopic to the identity. We say that a sequence $(h_n)_{n\in\mathbb{N}}$ of metrics of negative curvature converges to a metric h (also with negative curvature) in the length spectrum, if ℓ_{h_n} converge to ℓ_h point-wisely.
The embedding Φ determines a natural topology on $\mathcal{T}(S)$ in which a sequence of Teichmüller points h_n converge to a Teichmüller point h if and only if h_n converge to h in the length spectrum. (To be more precise, the embedding Φ is proper, when we endow $\mathcal{T}(S)$ with the quotient topology coming from seeing the hyperbolic metrics on S as tensors, see reference above for more details).

1.2 Measured laminations

For more details see [CME06, Chapter I.4], [LM01], [Thu97, Chapter 8.6]. Provide S with a hyperbolic metric h. We say that L is a geodesic lamination on S if it is a closed set which is a disjoint union of simple geodesics. The connected components of L are called its leaves, and the connected components of $S \setminus L$ together with the leaves of L are called strata. Even if the notion of geodesic lamination seems to depend on the hyperbolic metric provided on S, in fact it only depends on the topology of S. Indeed if we put a new hyperbolic metric h' on S each leaf of L became a quasi-geodesic in (S, h'), therefore it determines a unique geodesic.

A transverse measure λ on L is the assignment of a Radon measure λ_κ on each arc κ which is transversal to L such that:

- If κ' is a sub-arc of κ then $\lambda_\kappa|_{\kappa'} = \lambda_{\kappa'}$.
- If κ and κ' are two arcs transversal to L, and homotopic relative to L, then λ_κ and $\lambda_{\kappa'}$ are compatible. More precisely if $H : [0,1] \times [0,1] \to S$, is a homotopy between κ and κ' relative to L (that is, if $H(x,0)$ belongs to L, then $H(x,t)$ belongs to L for any t in $[0,1]$), we denote $H_t = H(.,t)$ and we assume that H_t is a continuous embedding for any t. We denote $tr_{\kappa',\kappa} := H_0^{-1} \circ H_1$ which a homeomorphisme between κ and κ', then $\lambda_{\kappa'} = tr^*_{\kappa',\kappa}\lambda_k$.

Throught this chapter, unless there is a confusion, we will refer to both of (L, λ) and L by λ. We denote the set of measured laminations on S by $\mathcal{ML}(S)$.
Note that we can assign to each transverse arc κ a positive number $i(\kappa, \lambda) := \int_\kappa d\lambda_\kappa$. In fact, $i(., \lambda)$ determines completely the measured lamination λ, see [Bon97b, Section 6].
The simplest example of a measured lamination, is in the case when the support of λ consists of finitely many closed leaves, and we associate to each leaf an atomic unit mass. In this case we say that λ is rational (or discrete).
Exactly as for Teichmüller space, there is a proper embedding of $\mathcal{ML}(S)$ into $\mathbb{R}^{\mathcal{S}}$ by the map $\lambda \in \mathcal{ML}(S) \mapsto \Phi(\lambda)$, such that $\Phi(\lambda)$ is the map

$$\begin{aligned} \Phi(\lambda) : \mathcal{S} &\to \mathbb{R}_{>0} \\ \alpha &\mapsto i(\alpha', \lambda) \end{aligned}$$

where α' is the unique closed geodesic in the free homotopy class α (the map Φ does not depend on the chosen metric). If α' is not transversal to λ then either it belongs to its leaves, or it is disjoint from λ, in this case we put $\Phi(\lambda)(\alpha) = 0$. This embedding gives a natural topology on the space of measured laminations in which $(\lambda_n)_{n\in\mathbb{N}}$ converge to λ if and only if $i(\cdot, \lambda_n))_{n\in\mathbb{N}}$ converge to $i(\cdot, \lambda)$ point-wisely. We call this topology the weak*-topology, for more details about this we refer to [FLP+21, Section 6].
We say that a sequence of metrics $(h_n)_{n\in\mathbb{N}}$ of negative curvature converge to a measured lamination λ in the length spectrum if for any $\alpha \in \mathcal{S}$, $\ell_{h_n}(\alpha)$ converge to $i(\alpha, \lambda)$.

1.3 Pleated surfaces

Let M be a hyperbolic 3-manifold. A pleated surface of M is a couple (S, f), where S is a complete hyperbolic surface, and $f : S \to M$ is a map that satisfies the following

property: Every point in S is contained in the interior of a geodesic arc of S which is mapped by f to a geodesic arc of M.
The pleating locus of a pleated surface (S, f) is the set of points of S that are contained in exactly one geodesic arc which is mapped to a geodesic arc of M. If S is connected, then the pleated locus of S is a geodesic lamination, see [CME06, Section I.5].

1.4 Third fundamental form

Let M be a hyperbolic 3-manifold, let S be an immersed surface. The Riemannian metric of M when restricted to the tangent bundle of S gives a Riemannian metric on S which is called the induced metric or the first fundamental form, and is denoted I. Let N be a unit normal vector field on S, and let ∇ be the Levi-Civita connection of M, then the shape operator $B : TS \to TS$ is defined to be $Bx = -\nabla_x N$.
The third fundamental form $I\!I\!I$ of S is defined by:

$$\forall x \in S,\ \forall u, v \in T_xS,\ I\!I\!I(u, v) = I(Bu, Bv)$$

The extrinsic curvature K_{ext} of the surface S is defined as the determinant of its shape operator B. This quantity is related to the Gaussian intrinsic curvature (or sectional curvature) K of S by the equation $K_{\text{ext}} = K - 1$.
Using [KS07a, Proposition 3.12], we get that the Gaussian curvature K^* of S endowed with the metric $I\!I\!I$ is given by:

$$K^* = \frac{K}{K+1}$$

1.5 Duality between hyperbolic and de Sitter geometry

In order to geometrically interpret the third fundamental form $I\!I\!I$, we need to introduce the de Sitter space dS^3 and clarify the connection between $\mathbb{H}^3$ and dS^3. Let $\mathbb{R}^{3,1}$ be the vector space $\mathbb{R}^4$ endowed with the Lorentzian scalar product $\langle ., . \rangle_{3,1}$ of signature $(3, 1)$. The hyperbolic space can be viewed as $\mathbb{H}^3 := \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ \langle x, x \rangle_{3,1} = -1,\ x_4 > 0 \right\}$ with the metric induced from $\langle ., . \rangle_{3,1}$. De Sitter space is defined to be the Lorentzian analogue of the sphere in the euclidean space, that is $dS^3 := \left\{ y \in \mathbb{R}^4 | \langle y, y \rangle_{3,1} = 1 \right\}$.
Let $P \subset \mathbb{H}^3$ be a totally geodesic plane, let n_P be the normal vector on P. Note that $\langle n_P, n_P \rangle_{3,1} = 1$. If we see n_P as a point in $\mathbb{R}^{3,1}$, it corresponds to a point in dS^3. It follows that each geodesic plane of $\mathbb{H}^3$ corresponds to a point in dS^3.
Let $\tilde{S}$ be a strictly convex surface of $\mathbb{H}^3$, and let $n : \tilde{S} \to dS^3$ be the Gauss map, that is, for any $x \in S$, $n(x)$ is the normal vector on T_xS pointing to the concave side. $n(\tilde{S}) = \tilde{S}^*$ is a convex space-like surface of dS^3, by space-like we mean that the restriction of $\langle ., . \rangle_{3,1}$ on $\tilde{S}^*$ induces a Riemannian metric call it I^*, and by convex we mean that the determinant of its shape operator is strictly positive. Moreover $(\tilde{S}^*, I^*)$ is isometric to $(\tilde{S}, I\!I\!I)$, we call $\tilde{S}^*$ the dual surface of $\tilde{S}$ (for more details see for example [Sch06a, Section 1]).
If $\tilde{S}$ is the lift of a strictly convex surface S embedded in a quasi-Fuchsian manifold Q (see Section 2 for a definition of quasi-Fuchsian manifolds) then $\tilde{S}^*$ determines a Cauchy

surface S^* inside a maximal global hyperbolic spatially compact de Sitter spacetime Q^* (see [Sca99, Section 5]). We call S^* the dual surface of S.
Let $\tilde{S}$ be a convex surface in $\mathbb{H}^3$ which is not necessarily smoothly embedded, then we can define the dual tree of $\tilde{S}$ (for definitions and details we refer to [Sca99, Section 5]). For any couple (x, P_x) where $x \in \tilde{S}$ and P_x is a support plane on $\tilde{S}$ at x we associate n_{P_x} the normal vector on P_x pointing to the concave side. We denote the tree obtained in this way as $\tilde{S}^*$, It corresponds to the dual tree of S ($\tilde{S}^*$ is the image of (x, P_x) by $n_.$ when x runs over $\tilde{S}$ and P_x runs over the support planes to $\tilde{S}$ at x).

1.6 Quasi-conformal maps

Let X and Y be Riemann surfaces (not necessarily compact). Let $f : X \to Y$ be a diffeomorphism. We define the Beltrami differential $\mu = \mu(f)$ by the equation $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$. We say that f is K quasi-conformal if the dilatation number $K(f) = \frac{1+|\mu|_\infty}{1-|\mu|_\infty}$ is less than or equal to K. Note that we don't need f to be a C^1 diffeomorphism to define the notion of quasi-conformal maps. In fact, all we need is for f to be a homeomorphism between X and Y that has derivatives in the sense of distribution that are L^2. For more details see [LV73]. The following proposition is well known, see for exampe [Tel07]

Proposition 1.6.1. *Any quasi-conformal homeomorphism $f : \mathbb{H}^2 \to \mathbb{H}^2$ has a continuous extension to a homeomorphism $\partial f : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$*

1.7 Quasi-isometries

Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $A \geq 1$ and $B \geq 0$. We say that a map $f : (X, d_X) \to (Y, d_Y)$ is a (A, B) quasi-isometric embedding if for any x_1 and x_2 in X the following inequalities hold:

$$\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B$$

Let $C \geq 0$. We say that f is an (A, B, C) quasi-isometry if it is a (A, B) quasi-isometric embedding and it is C-dense, that is:

$$\forall z \in Y, \exists\, x \in X, d_Y(f(x), z) \leq C$$

Any quasi-isometric embedding f between two δ-hyperbolic spaces (X, d_X) and (Y, d_Y) extends uniquely to a homeomorphism $\partial f : \partial_\infty X \to \partial_\infty Y$ of the visual boundary. The homeomorphism ∂f is called a quasi-symmetric map and has many interesting properties. We will discuss the notion of quasi-symmetric maps next.
It is worth to mention the following well known proposition, for a proof see for example [Tel07]

Proposition 1.7.1. *Any quasi-conformal map $f : \mathbb{H}^2 \to \mathbb{H}^2$ is a quasi-isometric embedding.*

Note that Proposition 1.7.1 implies Proposition 1.6.1.

1.8 Quasi-symmetric maps

For more details see [Hub16]. We denote $\mathbb{RP}^1 := \mathbb{R}\cup\{\infty\}$. Let $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ be a strictly increasing homeomorphism that satisfies $\phi(\infty) = \infty$. We say that ϕ is quasi-symmetric if there exists $k > 0$ such that,

$$\forall x \in \mathbb{R},\ \forall t \in \mathbb{R}^*_+,\ \frac{1}{k} \leq \frac{\phi(x+t) - \phi(x)}{\phi(x) - \phi(x-t)} \leq k.$$

In this case we say that k is quasi-symmetric, and we call k the quasi-symmetric constant of ϕ.
If ϕ does not fix ∞, then we say that ϕ is k quasi-symmetric if there exists an element $g \in PSL(2, \mathbb{R})$ (therefore many elements) such that $g \circ \phi(\infty) = \infty$ and $g \circ \phi$ is K quasi-symmetric.
The quasi-symmetric maps can be seen as the extension of quasi-conformal maps as shows the following proposition.

Proposition 1.8.1.

- *For any $k \geq 1$ there exists $k' > 0$ such that any K quasi-conformal map $f : \mathbb{H}^2 \to \mathbb{H}^2$ has a continuous extension to the boundary $\partial f : \mathbb{RP}^1 \to \mathbb{RP}^1$ which is k' quasi-symmetric.*
- *For any $M > 0$ there exists $M' > 1$ such that any k' quasi-symmetric map $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ has a continuous extension $f : \mathbb{H}^2 \to \mathbb{H}^2$ which is M' quasi-conformal.*

Proof. For the first point see for example [Hub16, Corollary 4.9.4].
For the second point see for example The Douady-Earle extension theorem, [Hub16, Theorem 5.1.2] □

Also, quasi-symmetric maps can be seen as the extension of quasi-isometric embeddings as shows the following proposition.

Proposition 1.8.2. *a map $f : \partial_\infty\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2$ is quasi-symmetric if and only if there exists a quasi-isometric embedding $F : \mathbb{H}^2 \to \mathbb{H}^2$ such that $f = \partial F$*

Proof. Since any quasi-conformal map is a quasi-isometry than we have one implication. For the secend implication we refer for example to [Hub16] or [Tel07]. □

Fix $k > 0$, an interesting property of k quasi-symmetric maps is the compactness as show the following proposition, see for example [Hub16, Corollary 4.9.7].

Proposition 1.8.3. *Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of k quasi-symmetric maps such that, for any n, $\phi_n(i) = i$, for any $i \in \{0, 1, \infty\}$ (We say that such a map is normalized). Then up to extract a subsequence, $(\phi_n)_{n\in\mathbb{N}}$ converge to a k quasi-symmetric map ϕ_∞ in the topology $C^0(\mathbb{RP}^1)$.*

1.9 Cross-ratio

Let $(a, b, c, d) \in (\partial_\infty \mathbb{H}^2)^4)$ be four points in that cyclic order (that is $a < b < c < d$). We define their cross-ratio by the formula (note that the order of (a, b, c, d) is important):

$$cr(a, b, c, d) = \frac{(c-a)(d-b)}{(b-a)(d-c)}$$

We recall that the cross-ratio is invariant under the action of $PSL(2, \mathbb{R})$, that is if $g \in PSL(2, \mathbb{R})$, then $cr(a, b, c, d) = cr(g(a), g(b), g(c), g(d))$.
A quadruple of points (a, b, c, d) is called symmetric if $cr(a, b, c, d) = -1$, or equivalently if there is $g \in PSL(2, \mathbb{R})$ such that $(g(a), g(b), g(c), g(d)) = (0, 1, -1, \infty)$. The following characterisation of quasi-symmetric maps using cross-ratio is well known, see [Tel07] for example.

Theorem 1.9.1. *For any $k \geq 1$, there exists $M \geq 1$ such that if $f : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$ is k quasi-symmetric, then for any symmetric quadruple (a, b, c, d) the following inequalities hold:*

$$-M \leq crf(a, b, c, d) \leq -\frac{1}{M}$$

Moreover, if k goes to infinity, then so does M.
Conversely for any $M \geq 1$ there exists $k \geq 1$ such that if $f : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$ is an orientation preserving homeomorphism that satisfies

$$-M \leq crf(a, b, c, d) \leq -\frac{1}{M}$$

then f is k quasi-symmetric. Again, if M goes to infinity, then so does k.

1.10 The universal Teichmüller space

The quasi-symmetric maps of $\partial_\infty \mathbb{H}^2$ form a group that we denote by $\mathcal{QS}(\partial_\infty \mathbb{H}^2)$. Recall that $PSL(2, \mathbb{R})$ acts on $\mathcal{QS}(\partial_\infty \mathbb{H}^2)$ by post-composition (since all the elements of $PSL(2, \mathbb{R})$ have extensions to the ideal boundary of $\mathbb{H}^2$). We define the universal Teichmüller space to be:

$$\mathcal{T} = \mathcal{QS}(\partial_\infty \mathbb{H}^2)/PSL(2, \mathbb{R})$$

Or equivalently, we can identify $\mathcal{T}$ with the set of normalized quasi-symmetric maps, that is, the set of quasi-symmetric maps f such that $f(p) = p$ for any $p \in \{0, 1, \infty\}$.
For any genus $g \geq 2$, the classical Teichmüller space $\mathcal{T}(S)$ is embedded in the universal Teichmüller space. Indeed, let us fix one Riemann structure X on S (which serves as a base point for $\mathcal{T}(S)$). $\tilde{X}$, the universal cover of X, is conformal to $\mathbb{H}^2$, so we can simply identify it with $\mathbb{H}^2$. This implies that we can identify X with $\mathbb{H}^2/\Gamma$.
Suppose Y is another Riemann structure on S, and let $f : X \to Y$ be a diffeomorphism. We can identify X with $\mathbb{H}^2/\Gamma$ and Y with $\mathbb{H}^2/\Gamma'$, where Γ and Γ' are Fuchsian groups. Let $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of f. Since f is compact, its lift $\tilde{f}$ is quasi-conformal. Hence, so is $\tilde{f}$, by Proposition 1.8.1. $\tilde{f}$ extends to a unique quasi-symmetric map $\partial\tilde{f}$. Note that adjusting X or Y by isotopy changes $\partial\tilde{f}$ up to Möbius translation. In conclusion, once we fix a base point X in $\mathcal{T}(S)$, we obtain a unique (up to composition by Möbius maps)

quasi-symmetric map for each $Y \in \mathcal{T}(S)$. This correspondence is an embedding because for each two Fuchsian representations $\rho_1, \rho_2 : \pi_1(S) \to PSL(2,\mathbb{R})$, there exists a unique quasi-symmetric map which is ρ_1-ρ_2 equivariant. As a consequence, each point Y in $\mathcal{T}(S)$ corresponds to a unique normalized quasi-symmetric map.

1.11 Quasi-circles in $\partial_\infty \mathbb{H}^3$

A quasi-circle is a special category of Jordan curves. Recall that $\partial_\infty \mathbb{H}^3$ is identified with $\mathbb{CP}^1$. A Jordan curve is the image of $\mathbb{RP}^1$ by a homeomorphism $f : \mathbb{CP}^1 \to \mathbb{CP}^1$. A simple way to define a quasi-circle is by saying that C is a quasi-circle if it is the image of $\mathbb{RP}^1$ by a quasi-conformal homeomorphism $f : \mathbb{CP}^1 \to \mathbb{CP}^1$. However, we will provide additional equivalent definitions that we will employ throughout this thesis.
Often, in the framework of hyperbolic geometry, we consider constructions that are $PSL(2,\mathbb{C})$ invariant. Therefore, we will frequently focus on Jordan curves that pass through $0, 1$, and ∞ in that specific cyclic order. We refer to Jordan curve that satisfies what we mentioned above as normalized.
Let $C \in \partial_\infty \mathbb{H}^3$ be a normalized Jordan curve, then $\partial_\infty \mathbb{H}^3 \setminus C$, consists of the disjoint union of two connected components Ω_C^+ and Ω_C^-. According to the Riemann mapping theorem, each of the component Ω_C^+ and Ω_C^- is conformally isomorphic to the hyperbolic plane $\mathbb{H}^2$. Furthermore, by using the Carathéodory theorem (for details see [Pom13, Section 21]), each of the conformal isomorphism mentioned above extends to a homeomorphism between C and $\partial_\infty \mathbb{H}^2$. Recall that $\partial_\infty \mathbb{H}^2$ is identified with $\mathbb{RP}^1$.
We assume that Ω_C^+ is compatible with the orientation of C. There is a unique conformally isomorphic map $U_C^+ : \mathbb{H}^2 \to \Omega_C^+$, whose extension $\partial U_C^+ : \partial \mathbb{H}^2 \to C$ satisfies $\partial U_C^+(p) = p$ for any $p \in \{0, 1, \infty\}$. Similarly, there is a unique conformally isomorphic map $U_C^+ : \mathbb{H}^2 \to \Omega_C^-$, whose extension $\partial U_C^- : \partial \mathbb{H}^2 \to C$ satisfies $\partial U_C^-(p) = p$ for any $p \in \{0, 1, \infty\}$, however, note that the orientations of Ω_C^+ and Ω_C^- are different. The gluing map associated to C is the map

$$\phi_C := (\partial U_C^-)^{-1} \circ \partial U_C^+ : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$$

A central result is the following (see [Ahl63] for example)

Lemma 1.11.1. *The following statements are equivalents.*

- *There is a k quasi-conformal map ψ such that $C = \psi(\mathbb{RP}^1)$.*
- *The map $U_C^+ : \mathbb{H}^2 \to \mathbb{CP}^1$ can be extended to a k-quasi-conformal map on $\mathbb{CP}^1$.*
- *The map ϕ_C is quasi-symmetric.*

Now we can give a definition of a quasi-circle.

Definition 1.11.2. *A k quasi-circle is a Jordan curve C in $\partial_\infty \mathbb{H}^3$ that satisfies one of the conditions in Lemma 1.11.1. We say that a quasi-circle is normalied if it is a normalized as Jordan curve. We denote the space of normalized quasi-circles in $\partial_\infty \mathbb{H}^3$ by QC.*

The following proposition is a geometric way to detect if a Jordan curve is a quasi-circle, this called the Ahlfors arc condition.

Proposition 1.11.3. *[Ahl06] A planar Jordan curve $C \subset \mathbb{C}$ is a k quasicircle if and only if it satisfies the k bounded turning condition: there is a constant $k \geq 1$ such that for each pair of points $x, y \in C$ we have that $diam(\Gamma[x, y]) \leq K|x - y|$, where $\Gamma[x, y]$ is the subarc of C joining x and y with smaller diameter.*

The following well known theorem shows that there is a one to one correspondence between normalized quasi-circles and normalized quasi-symmetric maps, this is called the conformal welding (see for example [Hub16]).

Theorem 1.11.4. *Let $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ be a normalized quasi-symmetric map, then there is a unique normalized quasi-circle C in $\mathbb{CP}^1$ such that $\phi_C = f$.*

The following continuity theorem is well known (see for example [Hub16]).

Theorem 1.11.5. *Let $k > 1$. Let C be the Hausdorff limit of a sequence $(C_n)_{n\in\mathbb{N}}$ of k quasi-circles. Then C is k quasi-circle. Moreover, the maps $U^{\pm}_{C_n}$ converge uniformly to $U^{\pm}_{C}$ on the closure of the hyperbolic plane $\mathbb{H}^2_{\pm} \cup \partial_\infty \mathbb{H}^2$.*

Similar to quasi-symmetric maps, the quasi-circles satisfy a compactness property (see for example [Hub16]) .

Theorem 1.11.6. *Let C_n be a sequence of k quasi-circles. Up to extract a subsequence C_n converge in the Hausdorff topology either to one point or to a k quasi-circle C.*

1.12 Hyperbolic ends

Let S be a closed surface of genus g greater than or equal to 2. A hyperbolic end E of type $S \times [0, \infty)$ is a hyperbolic manifold which is homeomorphic to $S \times (0, \infty)$ and has a metric completion $\overline{E} = S \times [0, \infty)$ which is obtained by adding to E a locally concave pleated surface $S \times \{0\} \subset S \times [0, \infty)$.
Two hyperbolic ends $(S \times [0, \infty), g)$ and $(S \times [0, \infty), g')$ are equivalent if there is an isometry $f : (S \times [0, \infty), g') \to (S \times [0, \infty), g')$ which is isotopic to the identity. We denote the set of hyperbolic ends on $S \times [0, \infty)$ up to equivalence by $\mathcal{E}(S)$.
The hyperbolic metric on a geometric end E induces a conformal structure at its boundary at infinity (which is $S\times\{\infty\}$). We can compactify a hyperbolic end by adding its boundary at infinity $S \times \{\infty\}$. We denote the pleated surface boundary of a hyperbolic end E by ∂E and its boundary at infinity by $\partial_\infty E$.
The universal cover $\widetilde{E}$ of a hyperbolic end E is locally described as the convex hull in $\mathbb{H}^3$ of a projective domain D in $\partial_\infty \mathbb{H}^3$. From that point of view, we see why a hyperbolic end induces a $\mathbb{CP}^1$ structure on $\partial_\infty E$. This induces a one-to-one correspondence between the hyperbolic ends $\mathcal{E}(S)$ and the $\mathbb{CP}^1$ structures on S. For more details, we refer to [KT92].
A hyperbolic end induces a measured lamination and a hyperbolic metric on its concave pleated surface boundary ∂E. Thurston has shown (in unpublished notes) that the map

$$Th : \mathcal{E}(S) \to \mathcal{T}(S) \times \mathcal{ML}(S)$$

that associates to a hyperbolic end the hyperbolic metric and the bending lamination on the locally concave pleated boundary is a homeomorphism (see [KT92] for a proof).

Moreover, since the hyperbolic end E induces a $\mathbb{CP}^1$ structure on its boundary at infinity $\partial_\infty E$, Thurston's parametrization induces the homeomorphism

$$Gr : \mathcal{CP}^1(S) \to \mathcal{T}(S) \times \mathcal{ML}(S)$$

The map Gr is called the grafting map.
A k-surface S_k embedded in $E \in \mathcal{E}(S)$ is a surface that its induced metric has a constant Gaussian curvature equal to k Labourie have proved that every hyperbolic end E is foliated by k surfaces

Theorem 1.12.1. *[Lab91, Theorem 2] Every hyperbolic end $E \in \mathcal{E}(S)$ is foliated by a family of k surfaces $(S_k)_{k\in(-1,0)}$. As k goes to -1 the surface S_k converge to ∂E and as k goes to 0 the surface S_k converge to $\partial_\infty E$.*

1.13 The length function

Let's assume that L is a discrete measured lamination, which is defined as a set of homotopy classes of disjoint simple closed curves, with each homotopy class being associated with a positive scalar referred to as its weight. Recall that discrete measured laminations are dense in the set of measured laminations (for precise definitions and references, see Section 1.2).
Let's denote by c_i the representatives of the simple closed curves for the homotopy classes that form L, and let's denote by a_i the associated weight for the homotopy class of c_i. Let h be a hyperbolic metric. Then we define

$$\ell_h(L) = \sum_i a_i \ell_h(c_i)$$

Here, $\ell_h(c_i)$ denotes the length of the unique simple closed geodesic in the hyperbolic surface (S, h) that is homotopic to c_i.
It was demonstrated in [Bon86] that $\ell_h(L)$ possesses a unique continuous extension to $\mathcal{ML}(S)$. This extension defines a function

$$\ell : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathbb{R}_+$$

referred to as the length function.
Fixing a measured lamination μ, this defines the map

$$\begin{aligned} \ell_\mu : \mathcal{T}(S) &\to \mathbb{R} \\ h &\mapsto \ell(\mu, h). \end{aligned}$$

The following theorem is given in [Ker92, Theorem 1.2]

Theorem 1.13.1. *Let $\mu, \mu' \in \mathcal{ML}(S)$ be two laminations that fill S. Then $\ell_\mu + \ell_{\mu'}$ has a unique critical point on $\mathcal{T}(S)$, which is necessarily a minimum.*

Under the hypothesis of Theorem 1.13.1, we denote the critical point of $\ell_\mu + \ell_{\mu'}$ by $\kappa(\mu, \mu')$.

Later, Bonahon established the injectivity of κ, as the following lemma illustrates

Lemma 1.13.2. *[Bon05, Lemma 4] Let μ, μ', and μ'' be measured laminations in $\mathcal{ML}(S)$. Then,*

$$\kappa(\mu, \mu') = \kappa(\mu, \mu'')$$

implies that $\mu' = \mu''$.

1.14 Tangentiability

For more details see [Bon98]. A map $f : U \to V$ between two open subsets of $\mathbb{R}^n$ is said to be tangentiable if, for each $x \in U$ and $v \in \mathbb{R}^n$, the limit

$$\frac{\partial}{\partial t}\bigg|_{t=0^+} f(x+tv) = \lim_{t\to 0^+} \frac{f(x+tv)-f(x)}{t}$$

exists, and the convergence is locally unifrom on v. In other words, a tangentiable map can be seen as a a one sided differentiable map with directional derivatives everywhere. If f is tangentiable, we can define the tangent map of f at x as follows

$$\begin{aligned} T_x f &: \mathbb{R}^n \to \mathbb{R}^n \\ T_x f(v) &\mapsto \frac{\partial}{\partial t}\bigg|_{t=0^+} f(x+tv) \end{aligned}$$

Note that when f is tangentiable, $T_x f$ is both continuous and homogeneous (see[Bon98, Section 1]), in other words, for any $\lambda > 0$, we have $T_x f(\lambda v) = \lambda T_x f(v)$.
A tangentiable manifold is locally modeled on $\mathbb{R}^n$ and has transition maps that are tangentiable, the smooth manifolds are examples of tangentiable manifolds. If M is a tangentiable manifold, we can refer to its tangent space as $T_x M$. However, it's important to note that $T_x M$ has the structure of a cone rather than a vector space. Notably, the space of measured laminations $\mathcal{ML}(S)$ possesses a natural tangentiable manifold structure [PH92] and [Thu22]
The notion of a tangentiable map extends to tangentiable manifolds using local charts. A homeomorphism between two tangentiable manifolds is called a bitangentiable homeomorphism if it and its inverse are tangentiable, and if the tangent maps are homeomorphisms everywhere. Bonahon in [Bon98, Lemma 4] gave a criterion for this

Lemma 1.14.1. *Let $f : M \to N$ be a homeomorphism between tangentiable manifolds. If f is tangentiable, and all of its tangent maps are injective, then f is a bitangentiable homeomorphism.*

1.15 Grafting and quasi-Fuchsian manifolds

For details, refer to [KP], [KT92], and [Dum09]. A projective complex structure on S consists of a maximal atlas on S, in which the charts map open sets of S to $\mathbb{CP}^1$, and the transitions are restrictions of Möbius maps. We denote the set of complex projective structures on S by $\mathcal{P}(S)$.
Note that if $Q \in \mathcal{QF}(S)$ is a quasi-Fuchsian manifold, then it induces projective structures on each of $S\times\{0\}$ and $S\times\{1\}$. Indeed, $S\times\{0\}$ (resp $S\times\{1\}$) is identified with Ω_Γ^+/Γ (resp Ω_Γ^-/Γ), here we assume that Q is identified with $\mathbb{H}^3/\Gamma$, where Γ is a quasi-Fuchsian group and $\Omega_\Gamma^\pm$ are the connected component of the domain of discontinuity of Γ (see Section 2.1 for notations and definitions).
A natural homeomorphism, called grafting, exists between $\mathcal{P}(S)$ and $\mathcal{ML}(S)\times\mathcal{T}(S)$. We will elucidate this mapping in the situation where the $\mathbb{CP}^1$ structure is induced from a quasi-Fuchsian manifold (we will clarify this in the next lines). For a general definition, we refer to the mentioned references ([KP], [KT92], and [Dum09]).

The space quasi-Fuchsian manifolds can be embedded in the space of $\mathbb{CP}^1$ structures via the following correspondence. Let $Q \in \mathcal{QF}(S)$ be a quasi-Fuchsian manifold. Note that knowing the $\mathbb{CP}^1$ structure on $S \times \{0\}$ (resp $S \times \{1\}$), then we know the quasi-Fuchsian structure on Q (because we will know the quasi-Fuchsian representation from $\pi_1(S)$ to $PSL_2(\mathbb{C})$). Then this defines an embedding of the quasi-Fuchsian manifolds $\mathcal{QF}(S)$ into $\mathcal{P}(S)$ the space of $\mathbb{CP}^1$ structures (the embedding depends on whether we see the space of the of $\mathbb{CP}^1$ structure on $S \times \{1\}$ or on $S \times \{0\}$).
Also, we have an embedding of the space quasi-Fuchsian manifolds $\mathcal{QF}(S)$ into $\mathcal{T}(S) \times \mathcal{ML}(S)$ by considering the hyperbolic metric and the bending lamination on $\partial^+ C(Q)$ (or on $\partial^- C(Q)$). These two maps coincide with the grafting map mentioned above when we embed $\mathcal{QF}(S)$ into $\mathcal{P}(S)$ (the grafting map is defined on hyperbolic ends, see Section 1.12), we denote these maps by

$$Gr^{\pm} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$$

where the sign $+$(res $-$) is when we consider the $\mathbb{CP}^1$ structure on $S \times \{1\}$ (resp on $S \times \{0\}$).

Bonahon has shown the following Theorem [Bon98, Theorem 3]

Theorem 1.15.1. *The grafting map $Gr : \mathcal{P}(S) \to \mathcal{ML}(S) \times \mathcal{T}(S)$ is a bitangentiable homeomorphism.*

We refer to Section 2.1 to recall the definitions of $\mathcal{F}(S)$ the Fuchsian manifolds and $\mathcal{QF}(S)$ the Quasi-Fuchsian manifolds. We also recall that M_{-1} is the map defined from $\mathcal{QF}(S)$ to $\mathcal{T}(S) \times \mathcal{ML}(S)$, which associates to a quasi-Fuchsian manifold Q the hyperbolic metric on $\partial^- C(Q)$ and the bending lamination on $\partial^+ C(Q)$ (see Section 2.1.4).

1.16 Earthquakes in $\mathbb{H}^2$

Let L be a measured geodesic lamination in $\mathbb{H}^2$. However, since we are working in $\mathbb{H}^2$, every geodesic corresponds to two points in $\partial_\infty \mathbb{H}^2$, so we may think of the space of unoriented geodesics of $\mathbb{H}^2$ as $S^1 \times S^1 \setminus \Delta / (\mathbb{Z}/2\mathbb{Z})$, where Δ is the diagonal of $S^1 \times S^1$. Let's denote the space of geodesics of $\mathbb{H}^2$ by $\mathcal{G}$. This gives an alternative point of view for measured geodesic laminations. Measured geodesic laminations can be thought as Radon measures on $\mathcal{G}$ whose support is simple (that is, its support forms a geodesic lamination). The space of measured geodesic laminations is endowed with the weak-* topology. A sequence of measured laminations $(L_n)_{n \in \mathbb{N}}$ in $\mathcal{ML}(\mathbb{H}^2)$ converges to a lamination L in $\mathcal{ML}(\mathbb{H}^2)$ if for any compactly supported function $f : \mathcal{G} \to \mathbb{R}$, we have

$$\int_{\mathcal{G}} f dL_n(l) \to \int_{\mathcal{G}} f dL(l).$$

Let $L \in \mathcal{ML}(\mathbb{H}^2)$ be a measured lamination, the Thurston norm of L, that we denote by $\|L\|_{Th}$ is defined to be

$$\|L\|_{Th} := \sup_{\alpha \in \mathcal{U}} i(\alpha, L)$$

where $\mathcal{U}$ is the set of geodesic arcs that have length equal to 1.
The Thurston norm of a given measured lamination can be infinite. If the Thurston norm

$\|L\|_{Th}$ of measured lamination L is finite, then we say that the lamination L is bounded. Recall that we call the elements of the geodesic lamination L by leaf. A measured lamination L in $\mathbb{H}^2$ gives a stratification of $\mathbb{H}^2$ into geodesics (the leaves of L) and flat regions, which are the components of $\mathbb{H}^2 \setminus \bigcup_{l\in L} l$. We call each such component or a leaf of L a stratum.

Definition 1.16.1. *A left (resp. right) earthquake along a geodesic lamination L (called the fault locus) is a (possibly discontinuous) bijective map $E : \mathbb{H}^2 \to \mathbb{H}^2$ such that*

- *the restriction of E to any stratum F of L (that is, a geodesic of L or a connected component of $\mathbb{H}^2 \setminus \bigcup_{l\in L} l$) extends to a global isometry $A(F)$ of $\mathbb{H}^2$,*
- *for any pair of strata F_0 and F_1, the comparison map $A(F_0)^{-1}A(F_1)$ is a hyperbolic transformation whose axis weakly separates F_0 from F_1, and which moves F_1 on the left (resp. right) as seen from F_0.*

It was shown by Thurston [Thu97, Proposition III.6.1] that it is possible to associate to every earthquake E a measured geodesic lamination L, whose support is equal to the fault locus of the earthquake E, and the amount of shearing of E defines a measure on L (for more details, see also [CME06, Chapter III]). An example is when the fault locus of E consists of a geodesic lamination of finite leaves, which we denote as $l_1, \ldots, l_n$. To each leaf l_i, we associate the translation number of the hyperbolic isometry $A(F_i)^{-1}A(F_{i+1})$, where F_i and F_{i+1} are the flat regions adjacent to l_i. We call this measured lamination that we have associated to E by the shearing lamination of E. An earthquake is determined by its shearing lamination up to post composition by an element of $\mathrm{PSL}(2,\mathbb{R})$ (see [Thu97, Proposition III.6.1]).
Thurston (see [Thu97, Theorem III.3.1]) has shown that although an earthquake is not necessarily continuous, it always extends uniquely to an orientation preserving homeomorphism

$$E\mid_{\partial_\infty\mathbb{H}^2}: \partial_\infty\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2.$$

Reciprocally, Thurston has also shown that any homeomorphism $f : \partial_\infty\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2$ that preserves the orientation is the extension of a unique earthquake (see [Thu97, Proposition III.6.1]).
We say that the earth quake E is normalized if $E\mid_{\partial_\infty\mathbb{H}^2}(p) = p$ for any $p \in \{0, 1, \infty\}$.
There is a strong relation between bounded measured laminations of $\mathbb{H}^2$ and between quasi-symmetric maps as the following proposition shows.

Proposition 1.16.2. *[GHL02][Šar06, Theorem 1] Let f be a homeomorphism of the circle that preserves the orientation, the following statements are equivalent*

- *f is quasi-symmetric*
- *the shearing measure L of the earthquake E that extends f is bounded.*

Moreover, for any M, there exists a constant $C = C(M)$ such that if f is M-quasisymmetric, then $\|L\|_{Th} \le C$.
Conversely, any bounded lamination can serve as the shearing measure of an earthquake E such that $E|_{\mathbb{RP}^1}$ is quasisymmetric. Moreover, for any constant C, there exists a $M = M(C)$ such that if $\|L\|_{Th} \le C$, then the earthquake $E : \mathbb{H}^2 \to \mathbb{H}^2$ is an M-quasi-isometry.

Moreover the correspondence between bounded laminations and quasi-symmetric maps stated in Theorem 1.16.2 is continuous, more precisely the following lemma holds

Lemma 1.16.3. *[CME06, Lemma II.3.11.5] If the shearing laminations L_n of a sequence of normalized earthquakes E_n weakly-* converge to the shearing locus L_∞ of an earthquake E_∞, then E_n restricted to $\partial_\infty\mathbb{H}^2$ converges uniformly to E_∞ restricted to $\partial_\infty\mathbb{H}^2$, while E_n converges pointwise to E_∞ on $\mathbb{H}^2$ except on the weighted leaf set λ_∞.*

1.17 Harmonic maps

Let h_1 and h_2 be two hyperbolic metrics on a surface S. Let $f : (S, h_1) \to (S, h_2)$ be a smooth map. We define the energy of f as follows

$$E(f) = \frac{1}{2}\int_S \|df\|^2 \omega_{h_1}$$

where ω_{h_1} is the area form associated with (S, h_1).
We say that f is harmonic if it is a critical point of the energy map.
One can notice that the harmonicity of f depends on the conformal class of h_1 rather than the metric itself. More precisely, if $f : (S, h_1) \to (S, h_2)$ is harmonic, then $f : (S, h'_1) \to (S, h_2)$ is harmonic for any metric h'_1 (not necessarily hyperbolic) that is conformal to h_1. Therefore, it makes more sense to consider a conformal structure c on S rather than the metric h_1.

Theorem 1.17.1. *[Sam78] Let c be a conformal class on S, and let $h \in \mathcal{T}(S)$ be a hyperbolic metric. There is a unique harmonic map $f : (S, c) \to (S, h)$ isotopic to the identity. Moreover, f is a diffeomorphism.*

Let c be a conformal class on S, and let h_1 be any metric in the conformal class c. For a C^1 map $f : (S, h_1) \to (S, h_2)$, we define the Hopf differential $\Phi = \Phi(f)$ of f as follows

$$f^* = eh_1 + \Phi + \overline{\Phi}$$

where $e = \frac{1}{2}\mathrm{tr}_{h_1}(f^*h)$. The Hopf differential of f is a quadratic differential that measures the traceless part of the pullback of h by f. If f is harmonic, then its Hopf differential is holomorphic; conversely, if f is a C^2 map and its Hopf differential is holomorphic, then f is harmonic.

Theorem 1.17.2. *see [Wol89] Given Φ a quadratic holomorphic differential in (S, c), there exists a unique hyperbolic metric h in S such that the identity map $Id : (S, c) \to (S, h)$ is harmonic with a Hopf differential equal to Φ.*

1.18 Minimal Lagrangian maps between hyperbolic surfaces

Definition 1.18.1. *Let h_1 and h_2 be two hyperbolic metrics on S. A diffeomorphism*

$$m : (S, h_1) \to (S, h_2)$$

is minimal Lagrangian if

- *It is area-preserving and orientation preserving.*
- *Its graph is a minimal surface in* $(S \times S)$.

Minimal Lagrangian maps have another equivalent definition

Proposition 1.18.2. *[Lab92b] If* $m : (S, h) \to (S, h^*)$ *is minimal Lagrangian, then* $m^*(h) = h(b_*, b_*)$*, where* $b : TS \to TS$*:*

- *is self-adjoint for* h*,*
- *has determinant* 1*,*
- *satisfies the Codazzi equation:* $d^\nabla b = 0$*, where* ∇ *is the Levi-Civita connection of* h*.*

Conversely, if $m : S \to S$ *is a diffeomorphism satisfying those properties, then it is minimal Lagrangian.*

Theorem 1.18.3. *Let* h_1, h_2 *be two hyperbolic metrics on* S*. There exists a unique minimal Lagrangian diffeomorphism*

$$m : (S, h_1) \to (S, h_2)$$

isotopic to the identity.

As a corollary

Corollary 1.18.4. *Let* h_1*,* h_2 *be two hyperbolic metrics on* S*. There exists a unique bundle morphism* $b : TS \to TS$ *which is self-adjoint for* h_1*, has determinant equal to* 1 *everywhere, satisfies the Codazzi equation* $d\nabla b = 0$*, where* ∇ *is the Levi-Civita connection of* h*, and such that* $h_1(b_*, b_*)$ *is isotopic to* h_2*.*

It follows that for any two points in Teichmüller space p_1, p_2 $\mathcal{T}(S)$ (recall that each of p_1 and p_2 is a class of hyperbolic metrics), we can find two hyperbolic metrics $h_1 \in p_1$ and $h_2 \in p_2$ such that $h_2 = h_1(b, b)$ where b is self-adjoint for h, has determinant 1, and satisfies the Codazzi equation $d\nabla b = 0$.

Chapter 2

Quasi-Fuchsian manifolds

2.1 Quasi-Fuchsian manifolds

In this section, we define the objects we will study in this chapter and give some key theorems concerning them.
We denote by $\mathbb{H}^3$ the hyperbolic space, and we denote by $Isom^+(\mathbb{H}^3)$ the group of isometries of $\mathbb{H}^3$ that preserve orientation. A Kleinian group is a discrete subgroup of $Isom^+(\mathbb{H}^3)$. Recall that every complete Riemannian manifold with a metric of constant sectional curvature equal to -1 is isometric to $\mathbb{H}^3/\Gamma$, where Γ is a Kleinian group. For any Kleinian group Γ, and any $p \in \mathbb{H}^3$, we denote by $\Gamma(P) := \{\gamma(p) \mid \gamma \in \Gamma\}$ the orbit set. The set $\Gamma(P)$ has accumulation points on the boundary at infinity $\partial_\infty \mathbb{H}^3$, and these points are the limit points of Γ (the intersection of the closure of $\Gamma(P)$ with $\partial_\infty \mathbb{H}^3$). The set of these points does not depend on the choice of the point p, and is called the limit set of Γ, we denote it by Λ_Γ. The complement of Λ_Γ in $\partial_\infty \mathbb{H}^3$ is referred to as the discontinuity domain, which we denote by Ω_Γ, that is:

$$\Omega_\Gamma := \partial_\infty \mathbb{H}^3 \setminus \Lambda_\Gamma.$$

The group Γ acts properly discontinuously in Ω_Γ [Thu97, Proposition 8.2.3].
Assume that Γ is a finitely generated torsion-free Kleinian group. We say that Γ is a quasi-Fuchsian group if its limit set Λ_Γ is a closed Jordan curve (then a quasi-circle [Thu97, Proposition 8.7.2]) and each component of Ω_Γ is invariant under the action of Γ. If we add the assumption that Λ_Γ is a geometric circle, we say that Γ is Fuchsian. it follows that $M_\Gamma = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold, which we call quasi-Fuchsian, with conformal boundary Ω_Γ/Γ that consists of the disjoint union of two surfaces. In the case when Γ is a Fuchsian group we say that $M_\Gamma = \mathbb{H}^3/\Gamma$ is a Fuchsian manifold.
Assume that Γ is quasi-Fuchsian, and let $\Omega_\Gamma^\pm$ be the components of Ω_Γ. It was shown by Marden in [Mar74] that $M_\Gamma = \mathbb{H}^3/\Gamma$ is diffeomorphic to $(\Omega_\Gamma^+/\Gamma) \times]0,1[$ and that $\bar{M}_\Gamma = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$ is diffeomorphic to $(\Omega_\Gamma^+/\Gamma) \times [0,1]$. In this chapter we assume that Γ is a quasi-Fuchsian group that has no parabolic element, it yields that Ω_Γ^+/Γ is a closed surface of genus $g \geq 2$.

Let $M = S \times]0,1[$, and let g be a Riemannian metric on M. We say that (M, g) is quasi-Fuchsian if it is isometric to $\mathbb{H}^3/\Gamma$ for some quasi-Fuchsian group Γ, and we say that (M, g) is Fuchsian if Γ is a Fuchsian group. We identify two metrics g_1 and g_2 if there is an isometry $\Phi : (M, g_1) \to (M, g_2)$ which is isotopic to the identity. We denote by

$\mathcal{QF}(S)$ the classes of quasi-Fuchsian metrics on M up to identification, and we denote by $\mathcal{F}(S)$ the subset of Fuchsian manifolds. Note that $\mathcal{QF}(S)$ has a topology induced from the set of representations $Hom(\pi_1(S), \mathbb{P}SL_2(\mathbb{C}))$. Through this chapter when we mention a quasi-Fuchsian metric on M we mean its isotopy classe, that is, we identify any two metrics that are isometric to each other via an isometry isotopic to the identity.

A quasi-Fuchsian metric on M induces a conformal structure on $\partial\bar{M} = S \times \{0,1\}$. Denote by c^+(resp c^-) the conformal structure on $S \times \{1\}$ (resp $S \times \{0\}$), more precisely (S, c^-) is identified with Ω_Γ^-/Γ and (S, c^+) is identified with Ω_Γ^+/Γ. Then we have a well defined map

$$\begin{aligned} B : \mathcal{QF}(S) &\to \mathcal{T}(S) \times \mathcal{T}(S) \\ (M, g) &\mapsto (c^+, c^-) \end{aligned}$$

A well known Theorem of Bers [Ber60] is the following:

Theorem 2.1.1. *The map*

$$\begin{aligned} B : \mathcal{QF}(S) &\to \mathcal{T}(S) \times \mathcal{T}(S) \\ Q &\mapsto (c^+, c^-) \end{aligned}$$

is a homeomorphism.

2.1.1 Convex core

For more details see [Thu97, Section 8.5], [CME06, Section II.1]. Let $Q = (M, g) \in \mathcal{QF}(S)$, then Q is isometric to $\mathbb{H}^3/\Gamma$. Let $CH(\Lambda_\Gamma)$ be the convex hull of Λ_Γ. We call the convex core of Q the set $C(Q) := CH(\Lambda_\Gamma)/\Gamma$ (we think of $C(Q)$ as a subset of M after identification by isometry). The set $C(Q)$ is the smallest non-empty geodesically convex subset of Q, that is, if C is a geodesically convex subset of Q then $C(Q) \subset C$. Moreover the inclusion map $\iota : C(Q) \to \bar{M}$ is a homotopy equivalence.
Except in the case when Q is Fuchsian, the convex core $C(Q)$ has nonempty interior and its boundary $\partial C(Q)$ consists of the disjoint union of two surfaces $\partial^+ C(Q)$, and $\partial^- C(Q)$ and each of the two is homeomorphic to S. If Q is Fuchsian then $C(Q)$ is a totally geodesic surface homeomorphic to S.
Note that $\bar{M} \setminus C(Q)$ consists of two connected components. We denote $E^+(Q)$ the connected component that contains $S \times \{1\}$ and we denote $E^-(Q)$ the connected component that contains $S \times \{0\}$. We assume by convention that $\partial^+ C(Q) \subset \bar{E}^+(Q)$ and $\partial^- C(Q) \subset \bar{E}^-(Q)$. It was shown by Thurston (see references above) that if Q is not Fuchsian, then $\partial^+ C(Q)$ and $\partial^- C(Q)$ are pleated surfaces and Q induces a path metric on each component which is hyperbolic, this gives two hyperbolic metrics h^+ and h^-.
Then we have a well defined map,

$$\begin{aligned} T : \mathcal{QF}(S) &\to \mathcal{T}(S) \times \mathcal{T}(S) \\ Q &\mapsto (h^+, h^-) \end{aligned}$$

Thurston made the following conjecture,

Conjecture 2.1.2. *(Thurston) For any two hyperbolic metrics $h^{\pm}$ on S (up to isotopy) there exists a unique quasi-Fuchsian manifold Q (up to isotopy) that induce $h^{\pm}$ on $\partial^{\pm}C(Q)$ (up to isotopy).*

The existence part of this statement is known since work of Labourie [Lab92a], Epstein and Marden [Eps87] and Sullivan [Sul06]. In other words, the conjecture says that the map T is bijective, for the moment we only know that it is surjective.
The conformal structures at infinity and the induced metrics on the boundary of the convex core are related by the following theorem [CME06, Section II.2].

Theorem 2.1.3. *(Sullivan, Epstein-Marden) There exists a universal constant K such that c^{+} (respectively c^{-}) is K quasi-conformal to h^{+} (respectively h^{-}).*

Combining Theorem 2.1.3 with Theorem 2.1.1, we obtain the following corollary.

Corollary 2.1.4. *Let $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{QF}(S)$ be a sequence of quasi-Fuchsian manifolds such that $(T(Q_n))_{n\in\mathbb{N}}$ is contained in a compact subset of $\mathcal{T}(S)\times\mathcal{T}(S)$. Then $(Q_n)_{n\in\mathbb{N}}$ has a convergent subsequence.*

Proof. Recall that if h and h' are two hyperbolic metrics on S such that there is a K-quasi-conformal map which is homotopic to the identity between (S,h) and (S,h'), then by [Wol79, Lemma 3.1] we have the following inequality on the length spectrum

$$\forall \alpha \in \mathcal{S},\ \frac{\ell_{h'}(\alpha)}{K} \leq \ell_h(\alpha) \leq K\ell_{h'}(\alpha).$$

Therefore, if $(T(Q_n))_{n\in\mathbb{N}}$ is contained in a compact subset of $\mathcal{T}(S)\times\mathcal{T}(S)$, Theorem 2.1.3 implies that $(B(Q_n))_{n\in\mathbb{N}}$ is also contained in a compact subset of $\mathcal{T}(S)\times\mathcal{T}(S)$. This implies that $(B(Q_n))_{n\in\mathbb{N}}$ has a convergent subsequence, and since B is a homeomorphism by Theorem 2.1.1, we deduce that $(Q_n)_{n\in\mathbb{N}}$ has a convergent subsequence. $\square$

As a consequence of Corollary 2.1.4, we get that each fiber of T is compact.

2.1.2 Bending laminations of the convex core

Let $Q = (M,g) \in \mathcal{QF}(S)$. Assume that Q is not Fuchsian. Although the induced metrics on the boundary of the convex hull $C(Q)$ are hyperbolic, the boundary surfaces of $C(Q)$ are not smoothly embedded, in particular they are not totally geodesic. If Q is Fuchsian, $C(Q)$ turns to be a totally geodesic surface.
Thurston has noticed that $\partial^{\pm}C(Q)$ are convex pleated surfaces, and the pleating locus gives a measured lamination (see [Thu97] section 8.5 and [CME06] in section II.1). We have seen in Section 1.3 that when S is connected the pleating locus is a geodesic lamination, so $\partial^{+}C(Q)$ (resp $\partial^{-}C(Q)$) gives a geodesic lamination L^{+}(resp L^{-}) on S. Moreover, the amount of bending gives a transverse measure on each of $L^{\pm}$. The simplest case is when $L^{\pm}$ are rational, so we associate to each leaf an atomic weight which is equal to the exterior dihedral angle of the bending at that leaf. In the case when the lamination is not rational, it was shown in [CME06] Section II.1.11 that we can always approximate by rational ones. One remark is that any closed leaf of $L^{\pm}$ has an atomic weight strictly

smaller than π.
It follows that we have a well defined map:

$$\begin{aligned} L : \mathcal{QF}(S) &\to \mathcal{MGL}(S) \times \mathcal{MGL}(S) \\ Q &\mapsto (L^+, L^-) \end{aligned}$$

This map is clearly not surjective, but we know exactly what its image is.
Let L^+ and L^- be two measured laminations on S, we say that they fill S if there exists $\epsilon > 0$ such that for any simple closed curve α,

$$i(\alpha, L^+) + i(\alpha, L^-) > \epsilon.$$

We define the set $\mathcal{ML}_\pi(S)$ to be the subset of $\mathcal{ML}(S)$ that consists of measured laminations in which every closed leaf has weight strictly smaller than π, and we define $\mathcal{L}$ to be

$$\mathcal{L} := \left\{(L^+, L^-) \subset \mathcal{ML}_\pi(S) \times \mathcal{ML}_\pi(S) | L^+ \textit{ and } L^- \textit{fill } S\right\}$$

Thurston made the following conjectures.

Conjecture 2.1.5. *(Thurston) Given any pair of measured laminations $L^+, L^- \in \mathcal{MGL}_\pi(S)$ that fill S. There is a unique quasi-Fuchsian manifold Q (up to isotopy), such that L^+ is the bending lamination of $\partial^+ C(Q)$ and L^- is the bending lamination of $\partial^- C(Q)$.*

Bonahon and Otal ([BO04]) showed the existence part, but we still don't know if Q is unique except for some particular cases.
In other words the conjecture says that the map

$$\begin{aligned} L : \mathcal{Q}(S) \setminus F(S) \to& \mathcal{L} \\ Q \mapsto& (L^+, L^-). \end{aligned}$$

is bijective. Bonahon and Otal showed that the image of the map L is exactly the set $\mathcal{L}$, but injectivity is still open.
We also know that the maps T and L are continuous [KS95].

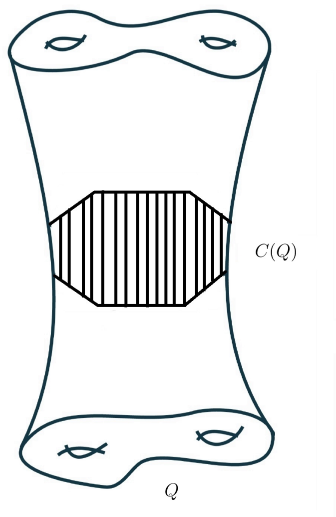


Figure 1. The shadowed region is the convex core of this quasi-Fuchsian manifold.

2.1.3 Foliation by k-surfaces

Let $Q \in \mathcal{QF}(S)$, and let $E^+(Q)$ and $E^-(Q)$ be the upper and the lower connected components of $M \setminus C(Q)$. It was shown by Labourie in [Lab91] that $E^+(Q)$ (resp $E^-(Q)$) is foliated by surfaces $(S_k^+)_{-1<k<0}$ (resp $(S_k^-)_{-1<k<0}$) such that the induced metric on each $S_k^\pm$ has constant Gaussian curvature equal to k, and each $S_k^\pm$ is homeomorphic to S. Each of $S_k^\pm$ is the unique surface in $E^\pm(Q)$ that has constant Gaussian curvature equal to k, and in which the projection $r_k^\pm : S \times \{0,1\} \to S_k^\pm$ is a homotopy equivalence.
Moreover, for any $-1 < k < 0$, and $Q \in \mathcal{QF}(S)$ there is a unique convex subset $C_k(Q) \subset M$, such that the inclusion map $\iota : C_k(Q) \to \bar{M}$ is a homotopy equivalence and $\partial C_K(Q) = S_k^+ \cup S_k^-$.
From that, we deduce that any quasi-Fuchsian manifold induces two metrics h_k^+ and h_k^- on S of constant sectional curvature equal to k by taking the induced metrics on S_k^+ and S_k^- respectively. The quasi-Fuchsian manifold also induces two metrics g_k^+ and g_k^- of curvature $\frac{k}{1+k}$ by taking the induced third fundamental forms on S_k^+ and S_k^- respectively (recall that the Gaussian curvature of $(S, I\!I\!I_k)$ is equal to $\frac{k}{1+k}$, see Section 1.4).
This defines two new maps,

$$\begin{aligned} T_k :& \mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(S) \\ & Q \mapsto (-kh_k^+, -kh_k^-) \end{aligned}$$

and

$$\begin{aligned} L_k :& \mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(S) \\ & Q \mapsto (\frac{-k}{1+k} g_k^+, \frac{-k}{1+k} g_k^-). \end{aligned}$$

Then there is a theorem which can be seen as a smooth version of Thurston's conjecture.

Theorem 2.1.6.

- *Given any two metrics h_k^+ and h_k^- on S of curvature k, where $k \in \,]-1, 0[$, there is a unique (up to isotopy) quasi-Fuchsian manifold that induces h_k^+ on S_k^+ and h_k^- on S_k^- as the first fundamental form (up to isotopy).*
- *Given any two metrics g_k^+ and g_k^- on S of curvature $\frac{k}{1+k}$, where $k \in \,]-1, 0[$, there is a unique (up to isotopy) quasi-Fuchsian manifold that induces g_k^+ on S_k^+ and g_k^- on S_k^- as the third fundamental form (up to isotopy).*

The existence part of the first point of Theorem 2.1.6 was proved by Labourie in [Lab92a], and the uniqueness part was proved by Schlenker in [Sch06a]. The second point of Theorem 2.1.6 was proved by Schlenker in [Sch06a]. In other words the maps T_k and L_k are bijective.
The maps T_k and L_k can be seen as an extension of the maps T and L when $k > -1$, this is because of the following theorem.

Theorem 2.1.7. *Let $(k_n)_{n\in\mathbb{N}}$ be a decreasing sequence of negative numbers that converges to -1. Let $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{QF}(S)$ be a sequence of quasi-Fuchsian manifolds that converge to $Q_\infty \in \mathcal{QF}(S)$. We denote $T_{k_n}(Q_n) = (-k_n h_{k_n}^+, -k_n h_{k_n}^-)$ and $L_{k_n}(Q_n) = (-\frac{k_n}{1+k_n} g_{k_n}^+, -\frac{k_n}{1+k_n} g_{k_n}^-)$. Then the following assertions are true:*

1. *$(h^+_{k_n}, h^-_{k_n})$ converge to $T(Q_\infty)$ in the length spectrum.*

2. *$(g^+_{k_n}, g^-_{k_n})$ converge to $L(Q_\infty)$ in the length spectrum.*

Proof. The first point is classical statement see for example [BMS13, Proposition 6.6]. The second point has been shown in details in [BMS13, Section 6], for the reader's convenience we sketch the proof. Let's work on $S^+_{k_n}$ and denote it just by S_{k_n}, $S^-_{k_n}$ is treated in a similar way. As explained in Section 1.5, for each n, each $E^+(Q_n)$ and S_{k_n} determine a maximal globally hyperbolic spatially compact de Sitter spacetime manifold Q^*_n and a Cauchy surface $S^*_{k_n}$ which is dual to S_{k_n}. The foliation of $E^+(Q_n)$ by k-surfaces determines a foliation of Q^*_n by surfaces S^*_k of constant Gaussian curvature equal to $\frac{k}{1+k}$ on each leaf. Recall that $(S_{k_n}, I\!I\!I_{k_n})$ is isometric by the duality explained in Section 1.4 to $(S^*_{k_n}, I^*_n)$. Moreover the initial singularity of Q^*_n is dual to the bending lamination of $\partial^+ C(Q_\infty)$ (see [Sca96, Section 9] for proofs and definitions).
As a corollary of [Bel14, Theorem 2.10], the intrinsic metrics of the surfaces S^*_k converge, with respect to the Gromov equivariant topology, to the real tree dual of the bending lamination of $\partial^+ C(Q_\infty)$. It follows that $(S_{k_n}, I\!I\!I_{k_n})$, which are isometric to $(S^*_{k_n}, I^*_n)$, converge to the length spectrum of of the bending lamination of $\partial^+ C(Q_\infty)$.

□

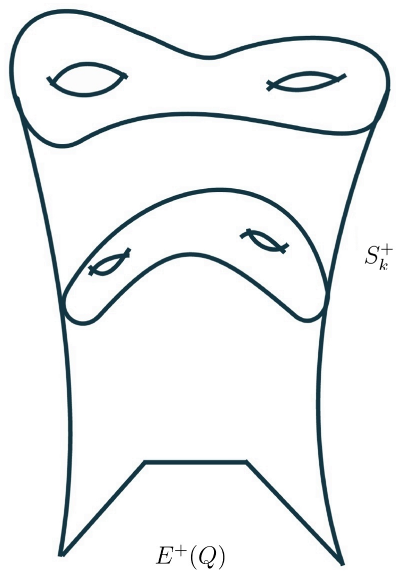


Figure 2. The connected components of $Q \setminus C(Q)$ are foliated by k-surfaces.

2.1.4 The mixed boundary data of the convex core

Let $Q \in \mathcal{QF}(S)$, and denote by h^-_k and g^+_k the metrics on S coming from the first fundamental form of S^-_k and the third fundamental form of S^+_k, this defines the map

$$M_k : \mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(S)$$
$$Q \mapsto \left(-\frac{k}{1+k} g^+_k, -k h^-_k\right)$$

and also the map

$$\begin{aligned} M_{\infty,k} :& \mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(S) \\ Q &\mapsto (-\frac{k}{1+k} g_k^+, c^-) \end{aligned}$$

where c^- is the conformal structure on $S \times \{0\}$ induced by Q. The following theorem was shown by Chen and Schlenker [CS22, Theorem 1.6].

Theorem 2.1.8. *Let h and h^* respectively be Riemannian metrics on S, and denote their curvatures by k_h and k_{h^*} respectively (which are not necessarily constant). We assume that $-1 < k_h$ and $k_{h^*} < 1$. Moreover, we assume that every contractible closed geodesic with respect to h^* has length strictly bigger than 2π. Let L be a measured lamination on S such that every closed leaf has weight strictly smaller than π. Then, there exists a convex hyperbolic metric g on $M = S \times (0,1)$, the interior of $\overline{M} = S \times [0,1]$, such that:*

- *g induces a metric (respectively third fundamental form) on $S \times \{0\}$ which is isotopic to h (respectively h^*).*
- *g induces on $S \times \{1\}$ a pleated surface structure in which its bending lamination is L.*

Furthermore, the surface $S \times \{0\}$ is smoothly embedded in M.

In particular, the authors show that the maps M_k and $M_{\infty,k}$ are bijective.
In this chapter we will define two maps that are extensions of M_k and $M_{\infty,k}$ to the boundary of the convex core. Let

$$\begin{aligned} M_{-1} :& \mathcal{QF}(S) \to \mathcal{MGL}_\pi(S) \times \mathcal{T}(S) \\ Q &\mapsto (L^+, h^-) \end{aligned}$$

Where L^+ is the bending lamination on $\partial^+ C(Q)$ and h^- is the induced metric on $\partial^- C(Q)$.
Another map is

$$\begin{aligned} M_{\infty,-1} :& \mathcal{QF}(S) \to \mathcal{MGL}_\pi(S) \times \mathcal{T}(S) \\ Q &\mapsto (L^+, c^-) \end{aligned}$$

Where L^+ is the bending lamination on $\partial^+ C(Q)$ and c^- is the induced conformal structure on $S \times \{0\}$.
The main theorem of this chapter is proving that the maps M_{-1} and $M_{\infty,-1}$ are surjective.
We prefer to state the theorem in the following way:

Theorem 2.1.9. *Let $L \in \mathcal{MGL}_\pi(S)$ and let $h \in \mathcal{T}(S)$, then:*

- *There exists a quasi-Fuchsian manifold Q such that the bending lamination of $\partial^+ C(Q)$ is L, and the induced metric on $\partial^- C(Q)$ is h.*
- *There exists a quasi-Fuchsian manifold Q such that the bending lamination of $\partial^+ C(Q)$ is L, and the induced conformal structure on $S \times \{0\}$ is h.*

Before proving the surjectivity of the maps $M_{\infty,-1}$ and M_{-1}, we will need to show that they are proper. So we give the following theorem

Theorem 2.1.10. *The maps $M_{\infty,-1}$ and M_{-1} are proper.*

Note that the properness of these maps is equivalent.

Lemma 2.1.11. *The map $M_{\infty,-1}$ is proper if and only if M_{-1} is proper.*

Proof. Let $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{QF}(S)$, then the sequence $M_{-1}(Q_n) = (L_n^+, h_n^-)$ lies on a compact subset if and only if $M_{\infty,-1}(Q_n) = (L_n^+, c_n^-)$ lies on a compact subset. Indeed, by Theorem 2.1.3 h_n^- lies on a compact subset if and only if c_n^- does. □

2.2 Closing lemma

In this section, we will show that the maps $M_{\infty,-1}$ and M_{-1} are proper. To show that, we need to give a compactness statement that we will call the closing lemma. The name "Closing lemma" was originally given by Bonahon and Otal in their work on laminations [BO04, Section 2]. Where the author proved the properness of the map that associates to a quasi-Fuchsian manifolds the bending lamination of the boundary of the convex core. Before giving the main statement of the this section, we need to explain the notion of pleated annulus.

2.2.1 Pleated annulus

For more details see [Bon86, proof of Lemma 2.1]. Let M be a 3-dimensional hyperbolic manifold, let α be a closed geodesic and let α^* be a piece-wise geodesic closed curve which is homotopic to α by the homotopy $A : S^1 \times [0,1] \to M$. We want to make $Im(A)$ hyperbolically simplicial. Let $a_1, ..., a_n$ be the points on $S^1 \times \{1\}$, in this order, that are sent by A to the vertices of α^*. Choose $b_1, ..., b_n$ on $S^1 \times \{0\}$, in this order. We triangulate the annulus $S^1 \times [0,1]$ by joining a_i to b_i and b_{i+1} ($b_{n+1} = b_1$). We can assume that each triangle is sent by A to a totally geodesic triangle in M. Then the hyperbolic metric on M induces a path metric on $Im(A)$ which is hyperbolic, and makes $Im(A)$ into a hyperbolic surface with piece-wise geodesic boundary.
An important fact for us is that by the Gauss-Bonnet Theorem the area of $Im(A)$ is equal to the sum of the exterior angles of α^*.
In particular if $Q \in \mathcal{QF}(S)$ and α^* lies on $\partial C(Q)$, then the area of the annulus $Im(A)$ is smaller than $i(\alpha, B)$, where B is the bending lamination of the component of $\partial C(Q)$ on which α^* lies on. Indeed, this is true because the sum of the exterior angels of α^* is smaller than the sum of the exterior dihedral angles of $\partial C(Q)$, see [HR93, Lemma 3.2].
If α^* is not finitely bent, then we approximate it by finitely piece-wise geodesic closed curves see [CME06, Chapter II].

2.2.2 Closing lemma

The following Lemma, that we call the closing lemma, is the main statement of this section.

Lemma 2.2.1. *Let $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{QF}(S)$ be a sequence of quasi-Fuchsian manifolds, denote by h_n^- the induced metric on $\partial^- C(Q_n)$, and by B_n^+ the bending lamination of $\partial^+ C(Q_n)$. Assume that:*

- $(h_n^-)_{n\in\mathbb{N}}$ *belongs to a compact subset of* $\mathcal{T}(S)$.
- $(B_n^+)_{n\in\mathbb{N}}$ *converges in the length spectrum to a measured lamination* B_∞, *and every closed leaf of* B_∞ *has weight strictly smaller than* π.

Then up to extracting a subsequence, the sequence $(Q_n)_{n\in\mathbb{N}}$ *converges to a quasi-Fuchsian manifold* Q_∞.

Note that the statement of Lemma 2.2.1 means exactly that the map M_{-1} (and therefore $M_{\infty,-1}$) is proper. The proof of this lemma is mainly based on the arguments given by [LS14] in Section 3. Nevertheless there are some differences between the two proofs, that's why we prefer to give the full arguments here.
Recall that by Corollary 2.1.4 $(Q_n)_{n\in\mathbb{N}}$ converges (up to extracting a subsequence) if and only if both of h_n^+ and h_n^-, the induced metrics on $\partial^+C(Q_n)$ and $\partial^-C(Q_n)$ respectively, belong to a compact subset of $\mathcal{T}(S)$. Then the proof of Lemma 2.2.1 is based on showing for any $\alpha\in\mathcal{S}$, the sequence of lengths $(\ell_{h_n^+}(\alpha))_{n\in\mathbb{N}}$ is bounded (recall that $\mathcal{S}$ is the set of free homotopy classes of simple closed curves not homotopic to a point on S).
In what follows we will show that if $\ell_{h_n^+}(\alpha)\to\infty$, then B_∞ must have a leaf of weight bigger than or equal to π. The fact that $(h_n^-)_{n\in\mathbb{N}}$ belongs to a compact subset of $\mathcal{T}(S)$ plays a crucial role.

The previous discussion leads us to show the following lemma.

Lemma 2.2.2. *Let* $(Q_n)_{n\in\mathbb{N}}\subset\mathcal{QF}(S)$ *be a sequence of quasi-Fuchsian manifolds, denote by* h_n^- *(resp* h_n^+*) the induced metric on* $\partial^-C(Q_n)$ *(resp* $\partial^+C(Q_n)$*), and denote by* B_n^+ *the bending lamination of* $\partial^+C(Q_n)$. *If we assume that.*

- $\exists\ \alpha\in\mathcal{S}$ *such that* $\ell_{h_n^+}(\alpha)\to\infty$
- $(h_n^-)_{n\in\mathbb{N}}$ *belongs to a compact subset of* $\mathcal{T}(S)$
- $(B_n^+)_{n\in\mathbb{N}}$ *converge in the length spectrum to a measured lamination* B_∞,

then B_∞ *has at least one closed leaf of weight bigger than or equal to* π.

The key technical lemma for showing Lemma 2.2.2 is the following.

Lemma 2.2.3. *Under the hypothesis of Lemma 2.2.2, there is a sequence of geodesic arcs* $(\kappa_n)_{n\in\mathbb{N}}$ *(each* k_n *is a geodesic arc of* Q_n*), such that each* k_n *has endpoints in* $\partial^+C(Q_n)$, $\ell_{m_n}(\kappa_n)\to 0$, *and* $\ell_{h_n^+}(\kappa_n')\to\infty$, *where* κ_n' *is the geodesic arc in* $\partial^+C(Q_n)$ *which is homotopic to* κ_n *relative to its endpoints.*
Moreover, if we denote by α_n *the unique simple closed geodesic representative of* α *in* $\partial^+C(Q_n)$, *then we can choose the endpoints of* κ_n *on* α_n, *and* κ_n' *to be a subarc of* α_n.

Proof. Let $\alpha\in\mathcal{S}$ be such that $\ell_{h_n^+}(\alpha)\to\infty$, we denote by α_n the unique simple closed geodesic in $\partial^+C(Q_n)$ that belongs to α, and we denote by α_n^* the unique simple closed geodesic of Q_n which is freely homotopic to α_n. Since $\ell_{h_n^-}(\alpha)\geq\ell_{m_n}(\alpha_n^*)$ (because α_n^* minimizes the lengths on its free homotopy class), and since the sequence $(\ell_{h_n^-}(\alpha))_{n\in\mathbb{N}}$ is bounded (by hypothesis), we deduce that the sequence $(\ell_{m_n}(\alpha_n^*)_{n\in\mathbb{N}})$ is also bounded by some $L>0$.
To continue the proof, we need to show the following technical sublemma.

Sublemma 2.2.4. *There is a sequence* $(\eta_n)_{n\in\mathbb{N}}$ *such that each* η_n *is a subarc of* α_n *(*α_n *is the geodesic in* $\partial^+C(Q_n)$ *with homotopy class* α*),* $\ell_{m_n}(\eta_n)\to\infty$*, and* $i(\eta_n, B_n^+)\to 0$*.*

Proof. We argue by contradiction, if the lemma were not true then there would exist $h, r>0$ such that if a subarc η_n of α_n satisfies $\ell_{m_n}(\eta_n)\geq h$, then $i(\eta_n, B_n^+)\geq r$. Since $\ell_{m_n}(\alpha_n)\to\infty$ this would imply that $i(\alpha_n, B_n^+)\to\infty$ but by hypothesis $i(\alpha_n, B_n^+)\to i(\alpha, B_\infty)$, which is absurd. □

Let η'_n be the geodesic arc in Q_n which is homotopic to η_n relative to its endpoints. By [Lec06b, Lemma A.1], we know that the Hausdorff distance between η_n and η'_n converges to 0 and that $\ell_{m_n}(\eta'_n)\to\infty$.
Let A_n be the annulus bounded by α_n and α_n^* as explained in Section 2.2.1. The area of A_n is bounded by $i(\alpha_n, B_n^+)$. Let A'_n be the same annulus as A_n, except that we replace η_n by η'_n, that is, A'_n is a pleated annulus bounded by α_n^* and $\alpha'_n := (\alpha_n\setminus\eta_n)\cup\eta'_n$ (see Figure 3). Note that the sums of the angles of α'_n are bounded independently on n (because the sums of the angles of α_n are bounded), this ensures that the areas of A'_n are bounded independently on n by a constant $K>0$.

Let $a_n, b_n\in\eta'_n$ be the endpoints of η'_n, let $a'_n, b'_n\in\eta'_n$ such that $d_{\eta'_n}(a_n, a'_n)=\frac{\ell_{m_n}(\eta'_n)}{3}$ and $d_{\eta'_n}(b_n, b'_n)=\frac{\ell_{m_n}(\eta'_n)}{3}$, where $d_{\eta'_n}$ is the distance induced from the metric m_n of Q_n restricted to the arc η'_n. Denote by ν_n the subarc of η'_n that has a'_n and b'_n as endpoints, then $\ell_{m_n}(\nu_n)=\frac{\ell_{m_n}(\eta'_n)}{3}$ (that is ν_n is the second third segment of η'_n).

Let $\delta_n := \sinh^{-1}(\frac{9K}{\ell_{m_n}(\eta'_n)}+\frac{1}{n})$. Denote by E_n the set of segments in A'_n that are orthogonal on ν_n, have one endpoint in ν_n, and either have length equal to δ_n and exactly one endpoint on $\partial A'_n$, or have length strictly smaller than δ_n and their second endpoints belong to $\partial A'_n$ (the first endpoint belongs to $\partial A'_n$ by definition). We denote by D_n the subset of E_n that consists of segments having length equal to δ_n, and we denote by Z_n the set of their endpoints on ν_n.

Sublemma 2.2.5. *If for any* $n\in\mathbb{N}$ *there exists a segment* $\xi_n\in E_n\setminus D_n$ *that has both endpoints on* α'_n*, then Lemma 2.2.3 holds.*

Proof. Assume the existence of a segment $\xi_n\in E_n\setminus D_n$ that has both endpoints on α'_n. Because ξ_n is orthogonal to η'_n, it has one endpoint on ν_n and the other endpoint will be on $\alpha'_n\setminus\eta'_n=\alpha_n\setminus\eta_n$. Denote these two endpoints by x_n and y_n respectively ($x_n\in\nu_n$ and $y_n\in\alpha'_n\setminus\eta'_n$). Recall that the Hausdorff distance between η_n and η'_n goes to 0. Because $x_n\in\nu_n\subset\eta'_n$, we can find a sequence of points $z_n\in\eta_n$ such that $\lim\limits_{n\to\infty} d_{m_n}(z_n, x_n)=0$. Then $\lim\limits_{n\to\infty} d(z_n, y_n)=0$, it yields that there exists a geodesic arc κ_n in Q_n joining z_n and y_n such that $\lim\limits_{n\to\infty}\ell_{m_n}(\kappa_n)=0$. On the other hand, we have that κ'_n, the subarc of α_n that joins z_n and y_n which is homotopic to κ_n, relative to its end points, satisfies $\ell_{m_n}(\kappa'_n)\geq\frac{\ell_{m_n}(\eta'_n)}{3}-d_{m_n}(x_n, z_n)$. Also note that κ'_n is a geodesic arc of $\partial^+C(Q_n)$. Moreover, note that κ_n and κ'_n can be chosen in a way that the endpoints of κ_n are on α_n and κ'_n is a subarc of α_n.
We conclude that κ_n and κ'_n satisfy the statement of Lemma 2.2.3.

□

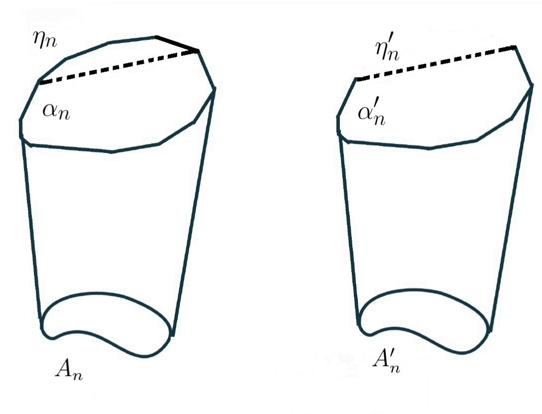


Figure 3. We get the annulus A'_n from the annulus A_n by replacing the piece-wise geodesic arc η_n by the geodesic arc η'_n.

In what follows we will use the fact that the lengths $\ell_{m_n}(\alpha_n^*)$ are bounded to show that we are always in the case of Sublemma 2.2.5. We argue by contradiction, so we assume that every segment $\xi_n \in E_n$ that has length strictly smaller than δ_n intersects α_n^*. Let $\mathcal{D}_n = \cup_{d\in D_n} d$, note that $\mathcal{D}_n$ is a hyperbolic strip (maybe disconnected), so $\mathrm{Area}(\mathcal{D}_n) = \ell_{\eta'_n}(z_n).\sinh(\delta_n)$ (where $\ell_{\eta'_n}$ is the length measure induced on η'_n by m_n), and since it is a subset of A'_n and the $(\mathrm{Area}(A'_n))_{n\in\mathbb{N}}$ are bounded by some constant K, it follows that,

$$K \geq \mathrm{Area}(A'_n) \geq \mathrm{Area}(\mathcal{D}_n) = \ell_{\eta'_n}(Z_n)\cdot\sinh(\delta_n) > \ell_{\eta'_n}(Z_n)\cdot\frac{9K}{\ell_{m_n}(\eta'_n)}$$

Then $\frac{\ell_{m_n}(\eta'_n)}{9} > \ell_{\eta'_n}(Z_n)$. In particular we can find two segments $\xi_n, \xi'_n \in E_n \setminus D_n$ with endpoints $\xi_n(0), \xi'_n(0) \in \nu_n$, such that ν'_n, the subarc of ν_n with endpoints $\xi_n(0)$ and $\xi'_n(0)$, satisfies $\ell_{\eta'_n}(\nu'_n) \geq \frac{\ell_{m_n}(\eta'_n)}{9}$ (for example take ξ_n and ξ'_n such that $\xi_n(0)$ is in the first third of ν_n and $\xi'_n(0)$ is in the last third of ν_n). Let $\bar{\alpha}_n$ be the subarc of α_n^* with endpoints $\xi_n(1)$ and $\xi'_n(1)$ and such that the arc $\kappa_n = \xi_n \cup \bar{\alpha}_n \cup \xi'_n$ is homotopic to ν'_n relatively end points. This yields a contradiction, because on one hand the lengths $\ell_{m_n}(\kappa_n)$ are bounded, and on the other hand $\ell_{m_n}(\nu'_n) \to \infty$, this can not happen because ν'_n is a geodesic arc so we must have $\ell_{m_n}(\nu'_n) \leq \ell_{m_n}(\kappa_n)$.
It follows that the hypothesis of Sublemma 2.2.5 holds, and then Lemma 2.2.2 is true. □

The rest of the proof of Lemma 2.2.2 follows from Claim 3.6, Claim 3.8 and Claim 3.9 from [LS14]. Nevertheless, we provide a proof for the reader's convenience. The proof that we provide is a bit simpler because we are dealing with quasi-Fuchsian manifolds without particles.
It will happen that we will use $i(\kappa,\lambda) = \int_\kappa d\lambda_\kappa$ as defined in Subsection 1.2 instead of taking the map Φ (see Section 1.2). When we use it, we will point it out.

Lemma 2.2.6. *Let κ_n and κ'_n be the arcs constructed in Lemma 2.2.3, let y_n and z_n be the endpoints of κ_n (therefore the endpoints of κ'_n). Let l_n be a simple closed curve on $\partial^+ C(Q_n)$ which is based at y_n and which is geodesic in $\partial^+ C(Q_n)$ (except at y_n). Let f_n be the simple closed curve based at z_n which is geodesic in $\partial^+ C(Q_n)$ (except at z_n) and freely homotopic to l_n. Choose $(l_n)_{n\in\mathbb{N}}$ such that they have a bounded length (independently on n). Then $\lim_{n\to\infty} i(l_n, B_n^+) = 0$ and $\lim_{n\to\infty} i(f_n, B_n^+) = 0$, where here we mean by $i(l_n, B_n^+)$*

and $i(f_n, B_n^+)$ the weight associated to l_n and f_n as transverse arcs (instead of taking the weight associated to the simple closed geodesic that represents them).

Proof. Assume that $Q_n \approx \mathbb{H}^3/G_n$, where G_n is a quasi-Fuchsian group. Let $\tilde{y}_n$ and $\tilde{z}_n$ be lifts of y_n and z_n respectively, let $\tilde{\kappa}_n$ be a lift of κ_n and $\tilde{\kappa}'_n$ be a lift of κ'_n, such that both of $\tilde{\kappa}_n$ and $\tilde{\kappa}'_n$ have $\tilde{y}_n$ and $\tilde{z}_n$ as endpoints, and let $\partial^+\tilde{C}(Q_n)$ be the lift of $\partial^+C(Q_n)$. Up to composition by isometries of $\mathbb{H}^3$, we can assume that $\tilde{y}_n$ is constant independently on n. Up to moving $\tilde{y}_n$ and $\tilde{z}_n$ slightly we can assume that they are away from the pleating locus of $\partial^+\tilde{C}(Q_n)$.
Let $P(\tilde{y}_n)$ be the support plane at $\tilde{y}_n$ and let $P(\tilde{z}_n)$ be the support plane at $\tilde{z}_n$. The fact that $\ell_{\partial^+C(Q_n)}(\kappa'_n) \to \infty$ implies that either $P(\tilde{y}_n) \cap P(\tilde{z}_n)$ is empty or it diverges to the boundary at infinity (that is, for any K, a compact subset of $\mathbb{H}^3$, the set $\{n \in \mathbb{N}|\ P(\tilde{y}_n) \cap P(\tilde{z}_n) \cap K \neq \emptyset\ \}$ is finite). And since $\ell_{m_n}(\tilde{\kappa}_n) \to 0$, we deduce that $P(\tilde{y}_n)$ and $P(\tilde{z}_n)$ converge (up to extracting a subsequence) to the same plane, call it P_∞. We denote by $E(\tilde{y}_n)$ (respectively $E(\tilde{z}_n)$) the half space determined by $P(\tilde{y}_n)$ (respectively $P(\tilde{z}_n)$) and contains $\partial^+\tilde{C}(Q_n)$. Let $\tilde{l}_n$ be a lift of l_n with endpoints $\tilde{y}_n$ and $\tilde{y}'_n$, let $g_n \in G_n$ such that $\tilde{y}'_n = g_n\tilde{y}_n$. Let $\tilde{f}_n$ be a lift of f_n with endpoints $\tilde{z}_n$ and $\tilde{z}'_n$ such that $\tilde{z}'_n = g_n\tilde{z}_n$.
Let $P(\tilde{y}'_n)$ be the support plane at $\tilde{y}'_n$ ($\tilde{y}'_n$ is away from the pleating locus, because the pleating locus is preserved by the action of the group G_n). Note that $d_{\partial^+\tilde{C}(Q_n)}(\tilde{y}'_n, \tilde{z}_n) \to \infty$ (because the lengths of $(\ell_{\partial^+C(Q_n)}(l_n))_{n\in\mathbb{N}}$ are bounded independently on n). Then $P(\tilde{y}'_n) \cap P(\tilde{z}_n)$ is empty or converge to the boundary at infinity. Since $\tilde{y}_n$ is fixed and since $\tilde{y}'_n$ is at a bounded distance from $\tilde{y}_n$ it follows that up to extracting a subsequence $P(\tilde{y}'_n)$ converge to P_∞ (Indeed, since $E(\tilde{y}_n) \cap E(\tilde{z}_n)$ converge to an empty set, it follows that if $P(\tilde{y}'_n)$ does not converge to the same plane as $P(\tilde{z}_n)$, then $P(\tilde{y}'_n) \cap P(\tilde{z}_n)$ will not be empty and will not go to the boundary at infinity). Now it follows that both of $P(\tilde{y}_n)$ and $P(\tilde{y}'_n)$ converge to P_∞, and since $i(l_n, B_n^+)$ is dominated by the dihedral angle of the intersection between $P(\tilde{y}_n)$ and $P(\tilde{y}'_n)$ [CME06, Section II.1.10], we deduce that $i(l_n, B_n^+) \to 0$. Finally, recall that $\tilde{y}'_n = g_n\tilde{y}_n$ and $\tilde{z}'_n = g_n\tilde{z}_n$, then $d_{\mathbb{H}^3}(\tilde{y}'_n, \tilde{z}'_n) = d_{\mathbb{H}^3}(\tilde{y}_n, \tilde{z}_n)$ and $d_{\partial^+\tilde{C}(Q_n)}(\tilde{y}'_n, \tilde{z}'_n) = d_{\partial^+\tilde{C}(Q_n)}(\tilde{y}_n, \tilde{z}_n)$. So for the same reasons as above $P(\tilde{z}'_n)$ converge up to extracting a subsequence to the same limit as $P(\tilde{y}'_n)$, therefor to the same limit as $P(\tilde{z}_n)$, therefore we deduce that $i(f_n, B_n^+) \to 0$.

□

Lemma 2.2.7. *Let κ_n and κ'_n be the arcs constructed in Lemma 2.2.3, then* $\liminf i(\kappa'_n, B_n^+) \geq \pi$.

Proof. Up to approximating κ'_n by piece-wise geodesic segments, we can assume that the curve $\kappa_n \cup \kappa'_n$ is a skew polygon. Note that the curve $\kappa_n \cup \kappa'_n$ bounds a disk in $C(Q_n)$. Let $\tilde{\kappa}_n$ and $\tilde{\kappa}'_n$ be lifts of κ_n and κ'_n respectively, such that $\tilde{\kappa}_n \cup \tilde{\kappa}'_n$ is a lift of $\kappa_n \cup \kappa'_n$. Denote by $\tilde{y}_n$ and $\tilde{z}_n$ the end points of $\tilde{\kappa}_n$, and let D_n be the geodesic cone starting from y_n to $\tilde{\kappa}_n \cup \tilde{\kappa}'_n$. Since $\tilde{\kappa}_n \cup \tilde{\kappa}'_n$ is piece-wise geodesic, it follows that D_n is a finite union of hyperbolic triangles, therefore the induced metric on D_n is hyperbolic. Since $\ell_{\mathbb{H}^3}(\tilde{\kappa}_n)$ goes to 0, the support planes at $\tilde{x}_n$ and $\tilde{y}_n$ intersect and converge to the same support plane. It follows that the sum of the angles at $\tilde{y}_n$ and $\tilde{y}_n$ converge to π, therefore by applying Gauss Bonnet Theorem on D_n we get that $\liminf i(\kappa'_n, B_n^+) \geq \pi$. □

Now we are ready to give a proof of Lemma 2.2.2.

Proof. We identify $Q_n \cong \mathbb{H}^3/G_n$. Let κ_n,κ'_n,l_n and f_n as defined in Lemma 2.2.6. Let $\tilde{\kappa}_n$ be a lift of κ_n with endpoints $\tilde{y}_n$ and $\tilde{z}_n$, let $\tilde{\kappa}'_n$ be the lift of κ'_n with endpoints $\tilde{y}_n$ and $\tilde{z}_n$. Take $\tilde{l}_n$ (respectively $\tilde{f}_n$) to be a connected components of the preimage of l_n (respectively of f_n) such that $\tilde{l}_n$ and $\tilde{f}_n$ are connected by $\tilde{\kappa}'_n$. The curves $\tilde{l}_n$ and $\tilde{f}_n$ are disjoint broken geodesics that bound an infinite band B_n in $\partial^+\tilde{C}(Q_n)$ (where $\partial^+\tilde{C}(Q_n)$ is the lift of $\partial^+C(Q_n)$). Let $\langle g_n \rangle \subset G_n$ be the subgroup generated by g_n, such that for all n, the action of g_n leaves the elements $\tilde{l}_n$ invariant. Since f_n is homotopic to l_n it follows that $\tilde{f}_n$ is also invariant under the action of $\langle g_n \rangle$. Let c_n be the closed simple geodesic in the homotopy class of l_n and f_n and let $\tilde{c}_n$ be the component of the preimage of c_n that has same endpoints as $\tilde{l}_n$ and $\tilde{f}_n$.
Let e_n be a simple closed geodesic and let $\tilde{e}_n$ be a lift of e_n, let $\tilde{a}_n$ be the arc $\tilde{e}_n \cap B_n$, assume that $\tilde{a}_n$ connects $\tilde{l}_n$ to $\tilde{f}_n$, then we have the inequality

$$i(\tilde{a}_n, \tilde{B}_n^+) \geq i(\tilde{\kappa}'_n, \tilde{B}_n^+) - (\sharp \{\tilde{a}_n \cap \langle g_n \rangle \tilde{\kappa}'_n\} + 1)(i(l_n, B_n^+) + i(f_n, B_n^+))$$

Where $\sharp X$ denotes the cardinal of the set X.

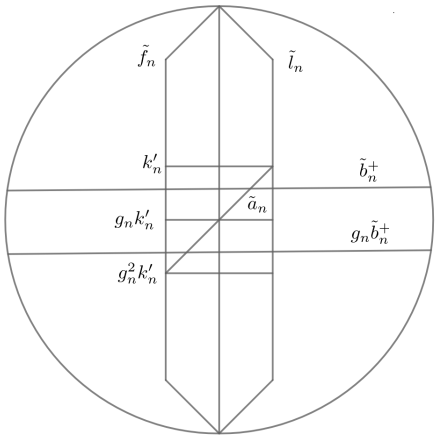


Figure 4. Each leaf b_n^+ of B_n^+ that intersects $\tilde{a}_n$, but does not intersect k_n, must intersect the broken geodesics $\tilde{f}_n$ and $\tilde{l}_n$. This happens at most the intersection number of $\tilde{a}_n$ with the lifts of $k'_n + 1$ times.

Let's show that up to extract a subsequence, the homotopy classes of c_n don't depend on n. Indeed, otherwise we can find a simple closed curve e such that $i(e, c_n) \to \infty$. Since e belongs to a fixed homotopy class we get that $\sharp \{a_n \cap \langle g_n \rangle \tilde{\kappa}'_n\}$ is bounded (because κ'_n belongs to d_n which has a free homotopy class independent on n). Then the inequality $\liminf i(e, B_n^+) \geq \liminf i(e, c_n)\pi$ holds. It follows that $i(e_n, B_n^+) \to \infty$ which is absurd, because we assumed that B_n^+ converge, therefore $\liminf i(e, B_n^+)$ must be bounded.
In conclusion we can assume that c_n is in a fixed homotopy class.
Then, for any simple closed geodesic e that is not equal to c the inequality $\liminf i(e, B_n^+) \geq i(e, c)\pi$ holds. It follows that c is a leaf of B_∞ with weight bigger than or equal to π. and the lemma holds. $\square$

We end the section by giving a proof of Lemma 2.2.1.

Proof. Since by hypothesis of Lemma 2.2.1 the lamination B_∞ has no closed leaf of weight bigger than π, we deduce that by Lemma 2.2.2 that $(h_n^\pm)_{n\in\mathbb{N}}$, the sequences of induced metrics on $\partial^\pm C(Q_n)$, belong to a compact subset of $\mathcal{T}(S)$. The conclusion follows from Corollary 2.1.4. □

2.3 Approximation by first and third fundamental forms

In this section we give a proof of Theorem 2.1.9. Let $Q \in \mathcal{QF}(S)$ be a quasi-Fuchsian manifold, recall that $Q \setminus C(Q)$ is foliated by constant Gaussian curvature surfaces (see Section 2.3). The proof is based on using Theorem 2.1.8 to realize abstract metrics as first and third fundamental forms of the k-surfaces in a sequence of quasi-Fuchsian manifolds. We choose these abstract metrics in a way that they converge to the couple $(h, L) \in \mathcal{T}(S) \times \mathcal{ML}(S)$ that we want to realize. Lemma 2.2.1 will ensure the convergence of the sequence of quasi-Fuchsian manifolds. Finally, we conclude by Theorem 2.1.7 that they converge to what we want to realize.
In the next lemma we construct the abstract metrics that we will use for approximation.

Lemma 2.3.1. *Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of decreasing real numbers that converge to -1. Let h be a hyperbolic metric on S, and let μ be a measured lamination on S. Then for any n we can find a metric h_{k_n} on S that has constant Gaussian curvature equal to k_n such that the sequence $(h_{k_n})_{n\in\mathbb{N}}$ converge in the length spectrum to h. Also for any n we can find a metric g_{k_n} on S that has constant Gaussian curvature equal to $\frac{k_n}{1+k_n}$ such that the sequence $(g_{k_n})_{n\in\mathbb{N}}$ converges in the length spectrum to μ.*

By Theorem 2.1.8 we can find a sequence of quasi-Fuchsian manifolds $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{QF}(S)$, such that the induced metric on $S^-_{k_n}$ is isotopic to h_{k_n} and the third fundamental form on $S^+_{k_n}$ is isotopic to g_{k_n}. Denote the induced metric on $\partial^- C(Q_n)$ by h_n^- and the bending lamination of $\partial^+ C(Q_n)$ by B_n^+. We know that I_{k_n} and $I\!I\!I_{k_n}$ converge, but we don't have any information about the convergence of B_n^+ and h_n^- because for the moment it is unclear whether the sequence $(Q_n)_{n\in\mathbb{N}}$ converges or not. For this purpose, we need to find upper bounds on the length spectrum of B_n^+ and h_n^-.
Recall that $\mathcal{S}$ is the set of free homotopy classes of simple closed curves of S not homotopic to a point.

Lemma 2.3.2. *Let $\alpha \in \mathcal{S}$,then*

- $\ell_{I_{k_n}}(\alpha) \geq l_{h_n^-}(\alpha)$
- $\ell_{I\!I\!I_{k_n}}(\alpha) \geq i(\alpha, B_n^+)$

Proof. By [CME06] Lemma II.1.3.4, the projection $r : \tilde{S}^-_{k_n} \to \partial^- \tilde{C}(Q_n)$ is a 1-lipschitz equivariant map, where $\tilde{S}^-_{k_n}$ and $\partial^- \tilde{C}(Q_n)$ are respectively the lifts of $S^-_{k_n}$ and $C^+(Q_n)$. Then the first point follows.
The second point comes from the existence of an equivarient 1-lipschitz map from $(\tilde{S}^*)^+_{k_n}$ into $\partial^+ \tilde{C}^*(Q_n)$ [Bel14], where $(\tilde{S}^*)^+_{k_n}$ and $\partial^+ \tilde{C}^*(Q_n)$ are the dual of $\tilde{S}^-_{k_n}$ and $\partial^- \tilde{C}(Q_n)$ respectively in de Sitter space. For more details see [BMS13, Lemma 6.5]. □

By Lemma 2.3.2, we deduce that $\forall\alpha \in \mathcal{S}$, the sequence $(l_{h_n^-}(\alpha))_{n\in\mathbb{N}}$ is bounded, this implies that $(h_n^-)_{n\in\mathbb{N}}$ belongs to a compact subset of $\mathcal{T}(S)$. Also by the same lemma we deduce that the sequence $(i(\alpha, B_n^+))_{n\in\mathbb{N}}$ is bounded, this implies that $(B_n)_{n\in\mathbb{N}}$ converges in the length spectrum (up to extracting a subsequence) to a transversal measured lamination B_∞. The next lemma gives us control on the weight of closed leaves of B_∞.

Lemma 2.3.3. *Let L and λ be two measured laminations such that:*

- *λ is a discrete lamination in which the weight of every closed leaf is strictly smaller than π.*
- *$\forall\alpha \in \mathcal{S}$, $i(\alpha, L) \leq i(\alpha, \lambda)$.*

Then any closed leaf of L has weight strictly smaller than π.

Proof. Let L' be the sublamination of L that consists of closed simple geodesics, then the following inequality holds

$$\forall\alpha \in \mathcal{S},\ i(\alpha, L') \leq i(\alpha, L) \leq i(\alpha, \lambda).$$

Let $\gamma_1, .., \gamma_k$ be the leaves of λ, let $\gamma_{k+1}, ..., \gamma_{k'}$ be simple closed curves such that $\{\gamma_1, ..., \gamma_k, \gamma_{k+1}, ..., \gamma_{k'}\}$ forms pants decomposition (it may be that λ is already maximal, then we don't need to add leaves).
Note that for any j, $i(\gamma_j, \lambda) = 0$, this implies that $i(\gamma_j, L') = 0$, so for any j, γ_j does not transversely intersect the leaves of L'. Let β be a leaf of L', if $\beta \notin \{\gamma_1, ..., \gamma_{k'}\}$ then β intersects some γ_j, what is absurd since γ_j does not transversely intersect the leaves of L', we deduce that the leaves of L' belong to the set $\{\gamma_1, ..., \gamma_{k'}\}$. Let σ_j be a simple closed curve dual to γ_j, that is σ_j is a simple closed curve such that $i(\gamma_j, \sigma_j) = 1$ or 2 (depending on the position of γ_j) and $i(\gamma_r, \sigma_j) = 0$ for any $r \neq j$. Note that the weight of γ_j in λ (respectively L') is equal to $\frac{i(\sigma_j,\lambda)}{i(\sigma_j,\gamma_j)}$ (respectively $\frac{i(\sigma_j,L')}{i(\sigma_j,\gamma_j)}$), also note that if γ_j does not belong to the lamination then its weight is equal to 0.
Since $\frac{i(\sigma_j,L')}{i(\sigma_j,\gamma_j)} \leq \frac{i(\sigma_j,\lambda)}{i(\sigma_j,\gamma_j)}$, we deduce that any closed leaf of L is a closed leaf of λ that has a bigger weight. Then the proof of the lemma follows. □

Now we can give a proof for Theorem 2.1.9

Proof. Let's start proving the first point of the theorem. Let $(h, L) \in \mathcal{T} \times \mathcal{MGL}_\pi(S))$. We start by the case when L is discrete.
Let k_n be a sequence of decreasing numbers converging to -1, by Lemma 2.3.1 there exists a sequence of metrics I_{k_n} on S with a Gaussian curvature equal to k_n, and that converge to h in the length spectrum. Also by Lemma 2.3.1 there exists metrics $I\!I\!I_{k_n}$ on S with Gaussian curvature equal to $\frac{k_n}{1+k_n}$, and that converge to L in the length spectrum.
By Theorem 2.1.8 we can find a sequence of quasi-Fuchsian manifolds $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{QF}(S)$, such that the induced third fundamental form on $S_{k_n}^+$ is isotopic to $I\!I\!I_{k_n}$, and the induced first fundamental form on $S_{k_n}^-$ is isotopic to I_{k_n}.
We denote the induced metric on $\partial^- C(Q_n)$ by h_n^-, and we denote the bending lamination on $\partial^+ C(Q_n)$ by B_n^+.

Sublemma 2.3.4. *The following assertions are true:*

- *The sequence $(h_n^-)_{n\in\mathbb{N}}$ belongs to a compact subset of $\mathcal{T}(S)$.*

- *There exists $B_\infty \in \mathcal{GML}_\pi$, such that there is a subsequence of $(B_n^+)_{n\in\mathbb{N}}$ that converge to B_∞ in the length spectrum.*

Proof. By Lemma 2.3.2, for any $\alpha \in \mathcal{S}$, $\ell_{I_{k_n}}(\alpha) \geq \ell_{h_n^-}(\alpha)$, what implies that $(h_n^-)_{n\in\mathbb{N}}$ belongs to a compact subset of $\mathcal{S}$ (recall that $(I_{k_n})_{n\in\mathbb{N}}$ converge in the length spectrum). Also, Lemma 2.3.2 implies that for any $\alpha \in \mathcal{S}$, $\ell_{I\!I\!I_{k_n}}(\alpha) \geq i(\alpha, B_n^+)$. Since $(I\!I\!I_{k_n})_{n\in\mathbb{N}}$ converge to L in the length spectrum, we deduce that $(B_n^+)_{n\in\mathbb{N}}$ belongs to compact subset of $\mathcal{GML}$. Then we can say that there exists $B_\infty \in \mathcal{GML}(S)$ such that $(B_n^+)_{n\in\mathbb{N}}$ converge to B_∞ in the length spectrum(up to extracting a subsequence). Moreover, we have that for any $\alpha \in \mathcal{S}$, $i(\alpha, L) \geq i(\alpha, B_\infty)$. The fact that $L \in \mathcal{GML}_\pi(S)$ and that L is discrete, allows us to use Lemma 2.3.3 to deduce that $B_\infty \in \mathcal{GML}_\pi(S)$. □

Sublemma 2.3.4 ensures that we are in the hypothesis of Lemma 2.2.1. Then up to extract a subsequence the quasi-Fuchsian manifolds Q_n converge to some quasi-Fuchian manifold Q_∞. To conclude, we will show

Sublemma 2.3.5. *The induced hyperbolic metric on $\partial^- C(Q_\infty)$ is h, and the bending lamination of $\partial^+ C(Q_\infty)$ is L.*

Proof. By Theorem 2.1.7 the induced metric on $\partial^- C(Q_\infty)$ is the limit of the metrics I_{k_n}, so it is equal to h.
Also by Theorem 2.1.7 the bending lamination of $\partial^+ C(Q_\infty)$ is the limit of the metrics $I\!I\!I_{K_n}$, then it is L. □

The following sublemma follows

Sublemma 2.3.6. *The first point of Theorem 2.1.9 is true when L is discrete.*

To conclude, let L be any lamination in $\mathcal{GML}_\pi(S)$, recall that the discrete transverse geodesic measured laminations are dense in the set of measured laminations. Let $(L_n)_{n\in\mathbb{N}} \subset \mathcal{GML}_\pi(S)$ be a sequence of discrete laminations that converge to L. Then by Sublemma 2.3.6 there is a sequence of quasi-Fuchsian manifolds $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{QF}(S)$, such that the induced metric on $\partial^- C(Q_n)$ is h and the bending lamination of $\partial^+ C(Q_n)$ is L_n. We assumed that L_n converge to L which is in $\mathcal{GML}_\pi(S)$, then we are in the hypothesis of Lemma 2.2.1. We deduce that the sequence $(Q_n)_{n\in\mathbb{N}}$ converge up to a subsequence to $Q_\infty \in \mathcal{QF}(S)$. By [KS95, Theorem 4.6], the bending lamination of $\partial^+ C(Q_\infty)$ is the limit of $(L_n)_{n\in\mathbb{N}}$, so it is equal to L. And by [KS95, Corollary 4.4], the induced metric on $\partial^- C(Q_\infty)$ is h. This finishes the proof of the first point of Theorem 2.1.9.

The proof of the second point of Theorem 2.1.9 is similar to the proof of the first point. Let $(c, L) \in \mathcal{T}(S) \times \mathcal{GML}_\pi(S)$. Assume first that L is discrete. Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of decreasing numbers that converge to -1. Take $I\!I\!I_{k_n}$ to be metrics on S, of constant Gaussian curvature equal $\frac{k_n}{1+k_n}$, and assume that $(I\!I\!I_{k_n})_{n\in\mathbb{N}}$ converge to L. By Theorem 2.1.8 there exists a sequence of quasi-Fuchsian manifolds $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{QF}(S)$ such that the third fundamental form on $S_{k_n}^+ \subset Q_n$ is isotopic to $I\!I\!I_{k_n}$ and the induced conformal metric on $S \times \{0\}$ is c. Denote the induced metric on $\partial^- C(Q_n)$ by h_n^-. By Theorem 2.1.3 $(h_n)_{n\in\mathbb{N}}$ is in a compact subset of $\mathcal{T}(S)$.

Sublemma 2.3.7. *The following assertions are true:*

- *The sequence* $(h_n^-)_{n\in\mathbb{N}}$ *belongs to a compact subset of* $\mathcal{T}(S)$*.*
- *There exists* $B_\infty \in \mathcal{GML}_\pi$*, such that there is a subsequence of* $(B_n^+)_{n\in\mathbb{N}}$ *that converge to* B_∞ *in the length spectrum.*

Proof. The first point is a direct consequence of Theorem 2.1.3. The second point was already shown in Sublemma 2.3.4. □

We have shown the conditions of Lemma 2.2.1 are satisfied. Then up to extracting a subsequence, $(Q_n)_{n\in\mathbb{N}}$ converge to $Q_\infty \in \mathcal{QF}(S)$.

Sublemma 2.3.8. *The conformal structure induced on* $S \times \{0\}$ *by* Q_∞ *is* c*, and the bending lamination of* $\partial^+ C(Q_\infty)$ *is* L*.*

Proof. The first point in because of Theorem 2.1.1, in particular it is true because of the fact that the map $B : \mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(S)$ is continuous. □

The following sublemma follows,

Sublemma 2.3.9. *Let* $(c, L) \in \mathcal{T}(S) \times \mathcal{GML}_\pi(S)$ *such that* L *is discrete. Then, there exists a quasi-Fuchsian manifold such that the conformal structure on* $S \times \{0\}$ *induced by* Q *is* c*, and the the bending lamination of* $\partial^+ C(Q)$ *is* L*.*

To finish the proof, let $L \in \mathcal{GML}_\pi(S)$, take a sequence $(L_n)_{n\in\mathbb{N}} \subset \mathcal{GML}_\pi(S)$ of discrete laminations that converge to L. By Sublemma 2.3.9 we can find a sequence of quasi-Fuchsian manifolds $(Q_n)_{n\in\mathbb{N}}$ such that each Q_n induces c as a conformal structure on $S \times \{0\}$, and induces L_n as the bending lamination on $\partial^+ C(Q_n)$. By Lemma 2.2.1 $(Q_n)_{n\in\mathbb{N}}$ converge, up to subsequence, to a quasi-Fuchsian manifold Q_∞. By [KS95], the bending lamination of $\partial^+ C(Q_\infty)$ is L, and by Theorem 2.1.1 the conformal structure on $S \times \{0\}$ induced by Q_∞ is c.

□

2.4 The proof of Theorem A and Theorem A*

In this section, we will provide a proof of Theorem A and Theorem A*. Let us first recall the statement of the theorems

Theorem 2.4.1. *Let* h *and* h^* *respectively be Riemannian metrics on* S*, and denote their curvatures by* k_h *and* k_{h^*} *respectively (which are not necessarily constant). We assume that* $-1 < k_h$ *and* $k_{h^*} < 1$*. Moreover, we assume that every contractible closed geodesic with respect to* h^* *has length strictly bigger than* 2π*. Let* L *be a measured lamination on* S *such that every closed leaf has weight strictly smaller than* π*. Then, there exists a convex hyperbolic metric* g *on* $M = S \times (0, 1)$*, the interior of* $\overline{M} = S \times [0, 1]$*, such that:*

- g *induces a metric (respectively third fundamental form) on* $S \times \{0\}$ *which is isotopic to* h *(respectively* h^**).*
- g *induces on* $S \times \{1\}$ *a pleated surface structure in which its bending lamination is* L*.*

Furthermore, the surface $S \times \{0\}$ is smoothly embedded in M.

Before proceeding with the proof, we need to provide some theorems and lemmas that we will use. We will begin with a theorem that Theorem 2.1.8 is a particular case of, and which can also be found in the same reference.

Theorem 2.4.2. *[CS22, Theorem 1.6] Let h be a Riemannian metric on S of curvature k_h, that satisfies $-1 < k_h$, and let h^* be a Riemannian metric on S of curvature $k_{h^*} < 1$ and every contractible closed geodesic with respect to h^* has length strictly bigger than 2π. Then there exists a unique convex hyperbolic metric g on M such that:*

- *The induced metric on $S \times \{0\}$ is isotopic to h.*

The induced third fundamental form on $S \times \{1\}$ is isotopic to h^*.

The following corollary is a direct consequence of Theorem 2.4.2 and [Thu97, Proposition 8.3.2].

Corollary 2.4.3. *Let h be a Riemannian metric on S of curvature k_h, that satisfies $-1 < k_h$, and let h^* be a Riemannian metric on S of curvature $k_{h^*} < 1$ such that every contractible closed geodesic with respect to h^* has length strictly bigger than 2π.*
There exists a unique quasi-Fuchsian manifolds Q (up to isotopy), and a unique geodesically convex domain $C_{h,h^*}(Q)$ with boundary such that:

- *The inclusion map $\iota : C_{h,h^*}(Q) \to Q$ is homotopically equivalent to the identity.*
- *The boundary of $C_{h,h^*}(Q)$ consists of two components $\partial^+ C_{h,h^*}(Q)$ and $\partial^- C_{h,h^*}(Q)$. The induced metric on $\partial^- C_{h,h^*}(Q)$ is isotopic to h, and the induced third fundamental form on $\partial^+ C_{h,h^*}(Q)$ is isotopic to h^*.*

As in the proof of Theorem 2.1.9, we will use Corollary 2.4.2 to construct a sequence of quasi-Fuchsian manifolds, and we will show that this sequence converges to a quasi-Fuchsian manifold that satisfies Theorem 2.4.1.
For more details about the following two lemmas we refer to [Sch06a, Section 8].

Lemma 2.4.4. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic metrics on M with smooth, strictly convex boundary. Let $(h_n)_{n\in\mathbb{N}}$ be the sequence of the first fundamental forms on the boundary. If $(g_n)_{n\in\mathbb{N}}$ converges to a convex hyperbolic metric g on M and if $(h_n)_{n\in\mathbb{N}}$ converge to a smooth metric h on ∂M with curvature strictly bigger than -1. Then the induced first fundamental form on ∂M by g is h.*

Lemma 2.4.5. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic metrics on M with smooth, strictly convex boundary. Let $(h_n^*)_{n\in\mathbb{N}}$ be the sequence of the third fundamental forms on the boundary. If $(g_n)_{n\in\mathbb{N}}$ converges to a convex hyperbolic metric g on M and if $(h_n^*)_{n\in\mathbb{N}}$ converges to a smooth metric h^* on ∂M with curvature strictly smaller than -1 and in which every contractible closed geodesic has length bigger then 2π. Then the induced third fundamental form on ∂M by g is h^*.*

Also, we will need a similar lemma to Lemma 2.3.2.

Lemma 2.4.6. *Let h, h^*, Q, and $C_{h,h^*}(Q)$ be as given in Corollary 2.4.3. The projection map $r : \partial^- C_{h,h^*}(Q) \to \partial^- C(Q)$ is a 1-Lipschitz map homotopic to the identity.*

Proof. Let $\tilde{\partial}^- C_{h,h^*}(Q)$ and $\tilde{\partial}^- C(Q)$ be lifts of $\partial^- C_{h,h^*}(Q)$ and $\partial^- C(Q)$ respectively. As it was shown in [CME06, Lemma II.1.3.4] the projection map $r : \tilde{\partial}^- C_{h,h^*}(Q) \to \tilde{\partial}^- C(Q)$ is a 1-Lipschitz equivariant map. Therefore, the statement follows. □

To prove Theorem A for the metric h^*, we will need to show that, given the Gaussian curvature of the third fundamental form on ∂M, the first fundamental form is bilipschitz with respect to the third fundamental form as a function of this curvature. To do this we will use some compactness statements given by Labourie [Lab89] and we use similar techniques as in [BDMS21b].
Let's recall statements that we will use.

Theorem 2.4.7. *[Lab89, Theorem D] Let $f_n : S \to \mathbb{H}^3$ be a sequence of immersions of a surface S such that the pullbacks $f_n^*(g_{\mathbb{H}^3})$ of the hyperbolic metric $g_{\mathbb{H}^3}$ converge smoothly to a metric h. If*

- *$(f_n)_{n\in\mathbb{N}}$ converge in C^0 to a map f.*
- *There exists $k_0 > -1$ such that for any n the Gaussian curvature of $f_n^*(g_{\mathbb{H}^3})$ is bigger than k_0.*
- *The integral of the mean curvature is uniformly bounded.*

Then a subsequence of $(f_n)_{n\in\mathbb{N}}$ converges smoothly to an isometric immersion f that satisfies $f^*(\mathbb{H}^3) = h$.

Lemma 2.4.8. *[BDMS21b, Lemma 3.7] Let $f : \mathbb{D} \to \mathbb{H}^3$ be a convex embedding (that is the image is the boundary of a convex subset of $\mathbb{H}^3$) and R be the extrinsic diameter of $f(\mathbb{D})$. Denote by H the mean curvature and by da the area form induced by f. Then we have*

$$\int_{\mathbb{D}} H da < \frac{A(R+1)}{sinh(1)}$$

where we denote by $A(\rho)$ the area of the sphere that has radius ρ in the hyperbolic space.

Now we prove the following

Lemma 2.4.9. *Let (M, g) be a convex hyperbolic manifold, and let h^* be the third fundamental form induced on $S \times \{1\}$ (one component of the boundary of M). Then there exist r and r', both greater than 0 and dependent only on h^*, such that the principal curvatures of $S \times \{1\}$ belong to the interval $[r, r']$.*
In particular, the first fundamental form of $S \times \{1\}$ is bi-Lipschitz to the third fundamental form, where the bi-Lipschitz constant depends only on h^*.

Proof. Recall from Section 1.4 that the Gaussian curvature of the first fundamental form of $S \times \{1\}$, denoted by k_I, satisfies $k_I = \frac{k_{h^*}}{1+k_{h^*}}$, where k_{h^*} is the curvature of h^*. Since S is compact and $k_{h^*} < 1$, we can deduce that k_I is bounded away from -1 and ∞ by constants that depend only on k_{h^*}, therefore it depends only on h^*.
We will now argue by contradiction. Assume that the lemma is false. Then, there exists a sequence of convex hyperbolic manifolds $(M, g_n)_{n\in\mathbb{N}}$ such that the induced third fundamental form on $S\times\{1\}$ is isotopic to h^*, and there is a sequence of points $x_n \in (S\times\{1\}, g_n)$ such that the principal curvatures at x_n diverge. (Note that their product is equal to k_I+1,

so one goes to zero and the other goes to infinity).
Let $(D, \tilde{h}_n)$ be a covering space of $(S \times \{1\}, g_n)$ and let Γ_n be a quasi-Fuchsian representation (that is $\Gamma_n(\pi_1(S))$ is a quasi-Fuchsian group) such that (M, g_n) is embedded in $\mathbb{H}^3/\Gamma_n$. Then there is an equivariant isometric embedding $f_n : (D, \tilde{h}_n) \to \mathbb{H}^3$ that descends to $(S \times \{1\}, g_n)$.
Up to applying isometries of $\mathbb{H}^3$, we can assume that x_n lifts to a constant $\tilde{x}_0 \in \mathbb{H}^3$.
Since $(f_n)_{n\in\mathbb{N}}$ are isometric embeddings, they are 1-Lipschitz and hence equicontinuous. Up to normalization, we can assume the existence of $o \in \mathbb{D}$ such that $f_n(0) = x_0$ for any n. Then by the Ascoli-Arzela theorem, up to extracting a subsequence, $(f_n)_{n\in\mathbb{N}}$ converges in C^0 topology to some function f.
Since k_{h_n}, the curvature of h_n, belongs to a compact interval (independent of n), then, up to extracting a subsequence, h_n converge to a metric h.
Note that the Gaussian curvature of h_n is bounded away from -1 independently on n. Also by Lemma 2.4.8 there is a neighborhood U of the fixed point o such that the integral of the mean curvatures of $f_n(U)$ is bounded. Then by Theorem 2.4.7 $f_n|_U$ converge to $f|_U$ in C^∞ topology, and moreover, the metrics $(f_n|_U)^*(g_{\mathbb{H}^3})$ converge to $(f|_U)^*(g_{\mathbb{H}^3})$. In particular, this contradicts the fact that the principal curvature of $f_n(U)$ at x_0 diverge. □

To prove Theorem 2.4.1, we will follow a similar approach to the one used in the proof of Theorem 2.1.9. Namely, we will construct a sequence of quasi-Fuchsian manifolds $(Q_n)n \in \mathbb{N}$ and show that it converges (up to extracting of a subsequence). Proposition 2.4.4 and Theorem 2.1.7 ensure that the limit quasi-Fuchsian manifold satisfies the statement of the Theorem 2.4.1.
Let us start the proof.

Proof. We start by proving the statement for the first fundamental form.
Let h be a Riemannian metric on S that has curvature k_h strictly bigger than -1. Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of decreasing real numbers that converge to -1. Let L be a measured lamination on S in which every closed leaf has weight strictly smaller than π. Let h_n^* be a Riemannian metric on S with a constant Gaussian curvature $k_{h_n^*}$ satisfying $k_{h_n^*} = \frac{k_n}{1+k_n}$, and such that the sequence $(h_n^*)_{n\in\mathbb{N}}$ converges to L in the length spectrum.
Using Corollary 2.4.3, we construct a sequence of quasi-Fuchsian manifolds $(Q_n)_{n\in\mathbb{N}}$, each containing a geodesically convex subset $C_{h,h_n^*}(Q_n)$ as described in the corollary (Corollary 2.4.3).

Sublemma 2.4.10. *Assume that L is a discrete measured lamination, then the sequence $(Q_n)_{n\in\mathbb{N}}$ converges, up to extracting a subsequence, to a quasi-Fuchsian manifold Q_∞.*

Proof. Using Lemma 2.4.6, the second point of Lemma 2.3.2, Lemma 2.2.3, and the argument in the proof of Sublemma 2.3.4, we can conclude that the sequence of hyperbolic metrics on $\partial^- C(Q_n)$ belongs to a compact subset of $\mathcal{T}(S)$, and that the sequence $(B_n^+)_{n\in\mathbb{N}}$ of bending laminations on $\partial^+ C(Q_n)$ converges (up to extracting a subsequence) to a measured lamination B_∞ in which every closed leaf has weight strictly less than π.
Since the hypothesis of Lemma 2.2.1 are satisfied, we can conclude that the Sublemma holds. □

The next Sublemma is a direct consequence of Theorem 2.1.7 and Proposition 2.4.4.

Sublemma 2.4.11. *If L is discrete, then the bending lamination of $\partial^+ C(Q_\infty)$ is equal to L. Moreover, there exists a strictly convex surface S^-_∞ that is smoothly embedded in $E^-(Q\infty)$, such that S^-_∞ bounds, together with $\partial^+ C(Q\infty)$, a geodesically convex subset of Q_∞ in which the inclusion is homotopic to the identity. And the induced metric on S^-_∞ is isotopic to h.*

Proof. The fact that bending lamination of $\partial^+ C(Q_\infty)$ is equal to L follows directly from Theorem 2.1.7.
The second point follows from Proposition 2.4.4. Specifically, it follows from the fact that each geodesically convex hyperbolic manifold (M, g) is embedded in a unique quasi-Fuchsian manifold, as shown in [Thu97]Proposition 8.3.2. □

We conclude that

Sublemma 2.4.12. *When L is discrete, Theorem 2.4.1 holds for the first fundamental form part.*

Next, suppose that L is not necessarily discrete. Let $(L_n)n \in \mathbb{N}$ be a sequence of discrete elements of $\mathcal{ML}\pi(S)$ (recall that $\mathcal{ML}\pi(S)$ is the set of measured laminations in which closed leaf has weight strictly smaller then π) that converges to L. By Sublemma 2.4.12, we can find a sequence of geodesically convex hyperbolic manifolds (M, g_n) such that the induced metric on $S \times \{0\}$ is isotopic to h, and g_n induces on $S \times \{1\}$ a structure of pleated surface of bending lamination L_n. Since such a submanifold is embedded in a unique quasi-Fuchsian manifold, we can apply the same argument given in Sublemma 2.4.10 to deduce that the metrics g_n converge to a hyperbolic metric g in which the induced metric on $S \times \{0\}$ is isotopic to h. Moreover, by the continuity of the bending lamination [KS95], the metric g induces on $S \times \{0\}$ a structure of pleated surface of bending lamination L.

For the third fundamental form, the statement follows in a similar way. First, we assume that L is discrete. Let $(M, g_n)_{n\in\mathbb{N}}$ be a sequence of convex hyperbolic manifolds in which the induced third fundamental form on $S \times \{1\}$ is isotopic to h^* and the third fundamental form on $S \times \{0\}$ is isotopic to h^*_n, where h^*_n has constant Gaussian curvature equal to $\frac{kn}{1+k_n}$ and converges to L in the length spectrum.
By Lemma 2.4.9, there is a positive constant that depends only on the curvature of h^* such that h_n, the first fundamental form on $S \times \{0\}$ induced by g_n, is bi-Lipschitz to h^*
Let Q_n be the unique quasi-Fuchsian manifold in which (M, g_n) is embedded. By Lemma 2.4.6, we deduce that the sequence of induced metrics on $\partial^- C(Q_n)$ converges, up to extracting a subsequence. Moreover, by Lemma 2.3.2 and Lemma 2.2.3, the bending lamination B^+_n on $\partial^+ C(Q_n)$ converges, up to extracting a subsequence, to a lamination B_∞ without any closed leaf of weight bigger than π.
We deduce, by Lemma 2.2.1, that the metrics g_n converge, up to extracting a subsequence, to a convex hyperbolic metric g. Then, by Lemma 2.4.5, the third fundamental form on $S \times \{0\}$ is isotopic to h^*, and by Lemma 2.3.1, the bending lamination on $S \times 1$ is L.
If L is not discrete, we approximate it by a sequence of discrete laminations $(L_n)_{n\in\mathbb{N}}$. We can then construct a sequence of convex hyperbolic metrics $(g_n)_{n\in\mathbb{N}}$ on M such that the third fundamental form on $S \times \{0\}$ is isotopic to h^* and the bending lamination on $S \times \{1\}$ is L_n. Applying Lemma 2.4.9, Lemma 2.4.6, and Lemma 2.2.1, we find that the

metrics g_n converge,up to extracting a subsequence, to a convex hyperbolic metric g, and so (M, g) is the desired hyperbolic convex hyperbolic manifold. □

2.5 Uniqueness near the Fuchsian locus

It was shown by Bonahon in [Bon05] that if we assume that two measured laminations L^+ and L^- are small enough (for the meaning of small, we refer to see Bonahon's paper [Bon05], or see the statement of Theorem 2.5.1 of this section), then there exists a unique quasi-Fuchsian manifold Q that has L^+ (resp L^-) as the bending lamination on $\partial^+ C(Q)$ (resp on $\partial^- C(Q)$).
In this section, we will establish the uniqueness in the first part of the statement of Theorem 2.1.9, close to the Fuchsian locus (we will provide a definition for "close to the Fuchsian locus" later in this section). This uniqueness holds when h is considered as the hyperbolic metric on the boundary of the convex core, given that we assume the lamination L to be sufficiently small. The precise notion of small enough will be established in the main theorem of this section's statement.
The main theorem of this section is

Theorem 2.5.1. *For any $\mu \in \mathcal{ML}(S)$ and $h \in \mathcal{T}(S)$, there exists an $\delta_{h,\mu} > 0$ such that for any $0 < t < \delta_{h,\mu}$, there exists a unique quasi-Fuchsian manifold Q, such that the hyperbolic metric on $\partial CH^-(Q)$ is h and the bending lamination on $\partial CH^+(Q)$ is $t\mu$.*

We recall that the Teichmüller space $\mathcal{T}(S)$ defined in Section 1.1, and the space of quasi-Fuchsian manifolds $\mathcal{QF}(S)$ defined in Section 2, possess the structure of smooth (actually analytic) manifolds.
Bonahon has shown the following Theorem [Bon98, Theorem 3]

Theorem 2.5.2. *The grafting map $Gr : \mathcal{P}(S) \to \mathcal{ML}(S) \times \mathcal{T}(S)$ is a bitangentiable homeomorphism.*

We refer to Section 2 to recall the definitions of $\mathcal{F}(S)$ the Fuchsian manifolds and $\mathcal{QF}(S)$ the Quasi-Fuchsian manifolds. We also recall that M_{-1} is the map defined from $\mathcal{QF}(S)$ to $\mathcal{T}(S) \times \mathcal{ML}(S)$, which associates to a quasi-Fuchsian manifold Q the hyperbolic metric on $\partial^- C(Q)$ and the bending lamination on $\partial^+ C(Q)$ (see Section 2.4).
To initiate the proof of the main statement of this section, we will first establish the existence of a neighborhood of $\mathcal{F}(S)$ denoted as V, in which the map $M_{-1} |_V$ is injective. Later we will show that $(M_{-1})^{-1}(M_{-1}(V)) = V$.

We will employ a strategy similar to the one used in [Bon05], where Bonahon shows the uniqueness of the map L in a neighborhood of the Fuchsian locus (See Section 2.2 for the definition of the map L). That is, when the bending laminations are sufficiently small, they uniquely determine the quasi-Fuchsian manifold.
Hereafter, let $\mu \in \mathcal{ML}(S)$ be a measured lamination, and $h \in \mathcal{T}(S)$ be a point in the Teichḿuller space of S. We denote by $\mathcal{B}^+(\mu)$ the subset of quasi-Fuchsian manifolds Q that possess a bending lamination equal to $t\mu$ on $\partial^+\mathcal{C}(Q)$ for some $t \geq 0$. Similarly, $\mathcal{M}^-(h)$ is the subset of quasi-Fuchsian manifolds Q that feature h as the induced metric on $\partial^-\mathcal{C}(Q)$.
We denote the dimension of $\mathcal{T}(S)$ by θ.

The subset $\mathcal{B}^+(\mu)$ is a submanifold with a boundary, as demonstrated by the following lemma.

Lemma 2.5.3. *[Bon05, Lemma 7] The set $\mathcal{B}^+(\mu)$ is a submanifold of $\mathcal{QF}(S)$ with boundary of dimension $\theta+1$. Furthermore, the boundary $\partial\mathcal{B}^+(\mu)$ is $\mathcal{F}(S)$. (Recall that θ is the dimension of $\mathcal{T}(S)$ and that $\mathcal{F}(S)$ denotes the space of Fuchsian manifolds).*

We will also show that $\mathcal{M}^-(h)$ is a C^1 submanifold of $\mathcal{QF}(S)$ (although we do not know if it is of higher regularity than C^1). We can deduce this as a corollary from the following theorem.
Denote by $gr_{\mathcal{T}}$ the composition of the projection map $\mathcal{ML}(S)\times\mathcal{T}(S)\to\mathcal{T}(S)$ with the grafting map $Gr:\mathcal{P}(S)\to\mathcal{T}(S)\times\mathcal{ML}(S)$

Theorem 2.5.4. *[Bon98, Theorem 3] The map*

$$gr_{\mathcal{T}}:\mathcal{P}(S)\to\mathcal{T}(S)$$

is C^1.

As it has been observed in [DW08, Corollary 2.5] the map $gr_{\mathcal{T}}$ is a C^1 submersion.

Corollary 2.5.5. *The map $gr_{\mathcal{T}}:\mathcal{P}(S)\to\mathcal{T}(S)$ is a C^1 submersion.*

Proof. Denote by $Gr:\mathcal{P}(S)\to\mathcal{T}(S)\times\mathcal{ML}(S)$ the grafting map. Note that for each $\lambda\in\mathcal{ML}(S)$, the relation $gr_{\mathcal{T}}\circ Gr^{-1}(\lambda,.)=Id$ holds. Then for each $\lambda\in\mathcal{T}(S)$, the map $Gr^{-1}(\lambda,.)$ is a smooth section (smoothness follows from the proof of [SW02, Theorem A]) of $gr_{\mathcal{T}}$, and since the map Gr is a homeomorphism (in particular surjective), the sections fill up $\mathcal{P}(S)$, so we conclude ($gr_{\mathcal{T}}$ is surjective, and its differentials are surjective at each point). □

Then a direct consequence of Corollary 2.5.5 is the following.

Corollary 2.5.6. *The submanifold $\mathcal{M}^-(h)$ is C^1 submanifold of $\mathcal{QF}(S)$*

Proof. Note that $\mathcal{M}^-(h)=gr_{\mathcal{T}}^{-1}(h)$. The statement follows because $gr_{\mathcal{T}}$ is a C^1 submersion. □

Bonahon showed a proposition which is crucial in this section. Before stating it, we would like to draw the reader's attention to the fact that when $\mu_0=0\in\mathcal{ML}(S)$, the tangent space $T_{\mu_0}\mathcal{ML}(S)$ in $\mathcal{ML}(S)$ corresponds to the set of measured laminations $\mathcal{ML}(S)$ itself (This was mentioned in [Bon05], for more details we refer to [Bon97a]).

As in Section 2.2, we denote by

$$L:\mathcal{QF}(S)\to\mathcal{ML}(S)\times\mathcal{ML}(S)$$

the map that associates to a quasi-Fuchsian manifold Q the bending laminations on $\partial^+C(Q)$ and $\partial^-C(Q)$.

Lemma 2.5.7. *[Bon05, Proposition 6] Let $\mu, \nu \in \mathcal{ML}(S)$ be two measured geodesic laminations, and let $t \mapsto q_t$, $t \in [0, T]$, be a differentiable curve in $\mathcal{QF}(S)$, originating from a Fuchsian metric q_0, such that the derivative $\frac{\partial}{\partial t} L(q_t)|_{t^+=0}$ of the bending measured lamination is equal to $(\mu, \nu) \neq 0$. Then μ and ν fill up the surface S, and $q_0 \in \mathcal{F}(S) = \mathcal{T}(S)$ (we identify the Fuchsian manifolds with Teichmüller space) is equal to the minimum $\kappa(\mu, \nu)$ of the length function $l_\mu + l_\nu : \mathcal{T}(S) \to \mathbb{R}$ (See Section 6.1 for the definition of $\kappa(\mu, \nu)$).*

The following lemma follows

Lemma 2.5.8. *Let $(q_t)_{t \in [0,\epsilon[}$ be a differentiable curve in $\mathcal{B}^+(\mu)$ which is originating from a Fuchsian point*
$q_0 = h_0 \in \mathcal{F}(S) = \mathcal{T}(S)$. Let (h_t, l_t) be respectively the metric and the bending lamination on $\partial^- C(q_t)$. Recall that $q_t \in \mathcal{B}^+(\mu)$, means that the bending lamination on the other side of the convex core , $\partial^+ C(q_t)$, is of the form $\alpha(t)\mu$, where $\alpha(t) \geq 0$. Then there exists $\mu' \in \mathcal{ML}(S)$, that depends only on μ and q_0, such that $\frac{d}{dt}(h_t, l_t)|_{t=0}$ is of the form $(\dot{h}, k\mu')$ where $k \in \mathbb{R}$ and $\dot{h} \in T_{h_0}\mathcal{T}(S)$.

Proof. First, consider (m_t, μ_t) to be the metric and the bending lamination on $\partial^+ C(q_t)$, respectively. As q_t belongs to $\mathcal{B}^+(\mu)$, there exists a real function (with values in $\mathbb{R}$) denoted as $\alpha(t)$ such that $(m_t, \mu_t) = (m_t, \alpha(t)\mu)$. This leads to $\frac{d}{dt}(m_t, \alpha(t)\mu)|_{t=0} = (\dot{m}, \dot{\alpha}\mu)$.

Using Lemma 1.13.1 and Lemma 1.13.2, we establish the existence of a unique $\mu' \in \mathcal{ML}(S)$ such that $q_0 = h_0 = \kappa(\mu, \mu')$.

Now, let (h_t, l_t) represents the metric and the bending lamination on $\partial^- C(q_t)$, respectively. Applying Lemma 2.5.7, we find $\dot{l} = \dot{\alpha}\mu'$. Consequently, we deduce that the differential of the coordinates on $\partial^- C(q_t)$ for any differentiable curve in $\mathcal{B}^+(\mu)$ takes the form $(\dot{h}, k\mu')$, where $\dot{h} \in T_{h_0}\mathcal{T}(S)$ and k is a non-negative scalar. ☐

Now we can deduce that the intersection between $\mathcal{M}^-(h)$ and $\mathcal{B}^+(\mu)$ at the Fuchsian locus is transverse.

Lemma 2.5.9. *The intersection $\mathcal{M}^-(h) \cap \partial\mathcal{B}^+(\mu)$ consists of exactly one point, namely $h \in \mathcal{F}(S)$. Moreover the intersection $T_h\mathcal{M}^-(h) \cap T_h\mathcal{B}^+(\mu)$ consists of exactly one line.*

Proof. The intersection $\mathcal{M}^-(h) \cap \partial\mathcal{B}^+(\mu)$ consists of the Fuchsian manifold that have h as the induced metric on the convex core (which is a totally geodesic surface in this case), there is a unique such Fuchsian manifold, namely $h \in \mathcal{F}(S)$. Let q_t^+ be a differentiable curve in $\mathcal{B}^+(\mu)$ and let q_t^- be a differentiable curve in $\mathcal{M}^-(h)$.

Denote by Gr^- the map that associates the metric and the bending lamination on $\partial^- C(Q)$ to a quasi-Fuchsian manifold Q. This map is bi-tangentiable into its image, as Theorem 2.5.2 states. As we have seen in Lemma 2.5.8, $\frac{d}{dt}Gr^-(q_t^+)|_{t=0}$ is of the form $(\dot{m}, k\mu')$, and $\frac{d}{dt}Gr^-(q_t^-)|_{t=0}$ is of the form $(0, \lambda)$ (Since the metric component is constant, the variation is null).
Therefore, the intersection $T_h\mathcal{M}^-(h) \cap T_h\mathcal{B}^+(\mu)$ is the line generated by $(Gr^-)^{-1}(0, \mu')$. ☐

Next, we deduce the following:

Corollary 2.5.10. *There exists a neighborhood $U_{h,\mu}$ of $h \in \mathcal{F}(S)$ such that the intersection $U_{h,\mu} \cap \mathcal{B}^+(\mu) \cap \mathcal{M}^-(h)$ forms a submanifold with boundary of dimension 1 in $\mathcal{QF}(S)$.*

Proof. According to Lemma 2.5.8, the two submanifolds $\mathcal{M}^-(h)$ and $\mathcal{B}^+(\mu)$ intersect transversely at the point h. Consequently, there exists a neighborhood $U_{h,\mu}$ of $h \in \mathcal{F} \subset \mathcal{QF}(S)$, such that the intersection $U_{h,\mu} \cap \mathcal{B}^+(\mu) \cap \mathcal{M}^-(h)$ forms a one dimensional submanifold with boundary $U_{h,\mu} \cap \partial\mathcal{B}^+(\mu) \cap \mathcal{M}^-(h)$. □

Corollary 2.5.11. *For any $h \in \mathcal{T}(S)$ and $\mu \in \mathcal{ML}(S)$, there exists $\epsilon_{\mu,h} > 0$ and a C^1 embedded curve*

$$\begin{aligned} q_{h,\mu} : [0, \epsilon_{\mu,h}[&\to \mathcal{QF}(S) \\ t &\mapsto q_{h,\mu}(t) \end{aligned}$$

such that $M_{-1}(q(t)) = (h, t\mu)$.
Recall that M_{-1} is the map that associates to a quasi-Fuchsian manifold Q the hyperbolic metric on $\partial CH^-(Q)$ and the bending lamination on $\partial CH^+(Q)$

Proof. From Corollary 2.5.10, there exists a neighborhood $U_{h,\mu}$ of the Fuchsian locus such that $U_{h,\mu} \cap \mathcal{B}^+(\mu) \cap \mathcal{M}^-(h)$ forms a C^1 submanifold with a boundary.

By definition, if $Q \in U_{h,\mu} \cap \mathcal{B}^+(\mu) \cap \mathcal{M}^-(h)$, then $M_{-1}(Q) = (h, t\mu)$ for some $t \geq 0$. This establishes a mapping:

$$\begin{aligned} \pi_{h,\mu} : U_{h,\mu} \cap \mathcal{B}^+(\mu) \cap \mathcal{M}^-(h) &\to \mathbb{R}^+ \\ Q &\mapsto t \end{aligned}$$

where $M_{-1}(Q) = (h, t\mu)$.
The differentiability of the map $\pi_{h,\mu}$ is shown in the proof of [Bon05, Lemma 7].
Let $q(t)$ be a curve in $U_{h,\mu} \cap \mathcal{B}^+(\mu) \cap \mathcal{M}^-(h)$ originating from the Fuchsian locus.
We will show that if $\frac{\partial}{\partial t}\pi_{h,\mu}(q(t)) = 0$, then $\frac{\partial}{\partial t}q(t) = 0$. Indeed, the variation of the hyperbolic metric on $\partial CH^-(q(t))$ is 0 (due to the constancy of the metrics). And the variation of the bending laminations on $\partial CH^-(q(t))$ is also 0. This latter assertion is based on our assumption that if the variation of the bending lamination on $\partial CH^+(q(t))$ is 0, then the variation on $\partial CH^-(q(t))$ is also 0. This follows from Lemma 2.5.7 and the fact that the curve q originates from the Fuchsian point h. Therefore, using the bitangentiablity of Gr^-, we can deduce that $\frac{\partial}{\partial t}q(t) = 0$.
Then, by the inverse function theorem, we conclude that

$$\pi_{h,\mu} : U_{h,\mu} \cap \mathcal{B}^+(\mu) \cap \mathcal{M}^-(h) \to \mathbb{R}^+$$

is a local diffeomorphism near $U_{h,\mu} \cap \partial\mathcal{B}^+(\mu) \cap \mathcal{M}^-(h)$. This implies the existence of $\epsilon_{h,\mu}$ such that $\pi_{h,\mu}^{-1} : \{(h, t\mu), 0 \leq t < \epsilon_{h,\mu}\} \to \mathcal{QF}(S)$ forms an embedded C^1 curve. By construction, this curve satisfies the statement of the Corollary.

□

For $h \in \mathcal{T}(S)$ and $\mu \in \mathcal{ML}(S)$, the set $\{(h, t\mu), 0 \leq t < \epsilon_{h,\mu}\}$ is denoted as $[0, \epsilon_{h,\mu}[\,(h, \mu)$. Note that from Corollary 2.5.11, we can define a map

$$\phi_{h,\mu} : [0, \epsilon_{h,\mu}[\,(h, \mu) \to \mathcal{QF}(S)$$

such that $M_{-1} \circ \phi_{h,\mu} = \mathrm{Id}$ (the map M_{-1} is the one defined in Section 2.1.4). The two submanifolds $\mathcal{M}^-(h)$ and $\mathcal{B}^+(\mu)$ intersect transversely at each point in the image of $\phi_{h,\mu}$.

Choose $\epsilon_{h,\mu}$ in a way that the ray $[0, \epsilon_{h,\mu})\,(h,\mu)$ with the aforementioned properties is maximal (i.e., satisfies $M_{-1} \circ \phi_{h,\mu} = \mathrm{Id}$ and ensures transverse intersection between $\mathcal{M}^-(h)$ and $\mathcal{B}^+(\mu)$ along the image of $\phi_{h,\mu}$). For such chosen $\epsilon_{h,\mu}$ that maximizes the ray, we denote $\mathcal{R}_{h,\mu} := [0, \epsilon_{h,\mu}[\,(h,\mu)$.

Bonahon has shown that when $\mu_n \in \mathcal{ML}(S)$ converge to μ, then $\mathcal{B}^+(\mu_n)$ converges to $\mathcal{B}^+(\mu)$ with respect to the C^∞ uniform topology on compact subsets.

Lemma 2.5.12. *[Bon05, Lemma 12] Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measured laminations in $\mathcal{ML}(S)$. As $(\mu_n)_{n\in\mathbb{N}}$ converge to μ with respect to the topology of $\mathcal{ML}(S)$, the submanifolds $\mathcal{B}^+(\mu_n)$ converge to $\mathcal{B}^+(\mu)$ in the C^∞ uniform topology on compact subsets.*

Recall that $\mathcal{M}^-(h)$ is defined as the reciprocal image of a submersion. This leads to the following lemma

Lemma 2.5.13. *Let $(h_n)_{n\in\mathbb{N}} \subset \mathcal{T}(S)$ be a sequence of hyperbolic metrics. When $(h_n)_{n\in\mathbb{N}}$ converges to h in the topology of $\mathcal{T}(S)$ (described in Section 1.1), then $\mathcal{M}^-(h_n)$ tends to $\mathcal{M}^-(h)$ in the C^1 uniform topology on compact subsets.*

Proof. Recall that the map $gr_{\mathcal{T}} : \mathcal{P}(S) \to \mathcal{T}(S)$ represents a surjective C^1 submersion, and that $\mathcal{M}^-(h) = gr_{\mathcal{T}}^{-1}(h)$. Thus, when h_n converges to h, the sets $\mathcal{M}^-(h_n) = gr_{\mathcal{T}}^{-1}(h_n)$ also converge in the C^1 uniform topology on compact subsets to $\mathcal{M}^-(h) = gr_{\mathcal{T}}^{-1}(h)$. □

Let $\mathcal{U} \subset \mathcal{T}(S) \times \mathcal{ML}(S)$ be the union of all the sets $\mathcal{R}_{h,\mu}$ as (h,μ) varies over $\mathcal{T}(S) \times \mathcal{ML}(S)$.
Let $\phi : \mathcal{U} \to \mathcal{QF}(S)$ be the map that restricts to $\phi_{h,\mu}$ at each $\mathcal{R}_{h,\mu}$ to $\phi_{h,\mu}$.
Our aim next is to show that $\mathcal{U}$ is an open set, and that the map ϕ is a homeomorphism into its image.

Proposition 2.5.14. *The set $\mathcal{U}$ is an open neighborhood of $\mathcal{T}(S)\times\{0\}$ in $\mathcal{T}(S)\times\mathcal{ML}(S)$, and ϕ is a homeomorphism from $\mathcal{U}$ into an open neighborhood V of $\mathcal{F}(S)$ in $\mathcal{QF}(S)$.*

Proof. The restriction $\phi_{h,\mu}$ of ϕ to $\mathcal{R}_{h,\mu}$ was constructed by considering the transverse intersection of the manifolds $\mathcal{M}^-(h)$ and $\mathcal{B}^+(\mu)$ in a neighborhood of the Fuchsian locus $\mathcal{F}(S)$. By Lemma 2.5.13 and Lemma 2.5.12, the manifold $\mathcal{M}^-(h)$ depends continuously on h for the topology of C^1-convergence, and $\mathcal{B}^+(\mu)$ depends continuously on μ for the topology of C^∞-convergence (therefore C^1-convergence). It follows that $\epsilon_{h,\mu}$ (the length of $\mathcal{R}_{h,\mu} = [0, \epsilon_{h,\mu}[\,(h,\mu))$ depends continuously on (h,μ), and that $\phi_{h,\mu}$ depends continuously on (h,μ). We conclude that $\mathcal{U}$, the union of $\mathcal{R}_{h,\mu}$, is open in $\mathcal{T}(S)\times\mathcal{ML}(S)$ and that the map ϕ is continuous.
Recall that $M_{-1} \circ \phi = Id$, it follows that ϕ is injective. It follows by the Theorem of the invariance of domain that ϕ is a homeomorphism into its image. □

Recall that M_{-1} is the map that associates to a quasi-Fuchsian manifold Q the hyperbolic metric on $\partial CH^-(Q)$ and the bending lamination on $\partial CH^+(Q)$.
The following theorem, is a consequence of Proposition 2.5.14.

Corollary 2.5.15. *There is a neighborhood O of the Fuchsian locus, such that $M_{-1}\,|_O$ is injective.*

Proof. Denote $O := \phi(\mathcal{U})$. From Proposition 2.5.14 the map

$$\phi : \mathcal{U} \to O$$

is a homeomorphism and $M_{-1} \circ \phi = Id$
It follows that $M_{-1} \mid O = \phi^{-1}$, then in particular it is injective. $\square$

Next, let's show the following

Proposition 2.5.16. *Let $h \in \mathcal{T}(S)$, $\mu \in \mathcal{ML}(S)$, and θ_n be a sequence of real numbers converging to 0. Let Q_n be a sequence of quasi-Fuchsian manifolds such that $M_{-1}(Q_n) = (h, \theta_n \mu)$. Then $(Q_n)_{n\in\mathbb{N}}$ converges to the Fuchsian manifold h.*

Proof. Recall that by Lemma 2.2.1 the map M_{-1} is proper.
We will show that each subsequence of $(Q_n)_{n\in\mathbb{N}}$ has a subsequence that converge to h.
Let $(Q_{\psi(n)})_{n\in\mathbb{N}}$ be a subsequence of $(Q_n)_{n\in\mathbb{N}}$. By definition $M_{-1}(Q_{\psi(n)})$ converge to $(h, 0)$.
Since M_{-1} is proper, $(Q_{\psi(n)})_{n\in\mathbb{N}}$ has a subsequence that converge to some quasi-Fuchsian manifold $Q_{\psi(\infty)}$, by continuity of M_{-1} we get that $M_{-1}(Q_{\psi(\infty)}) = (h, 0)$, this implies that $Q_{\psi(\infty)} = h$. $\square$

Lemma 2.5.17. *There is O', a neighborhood of the Fuchsian locus in $\mathcal{QF}(S)$, such that $M_{-1}^{-1}(M_{-1}(O')) = O'$.*

Proof. We argue by contradiction. If the lemma were not true, then we would find a sequence $(Q_n)_{n\in\mathbb{N}}$ that does not converge to some point in $\mathcal{F}(S)$, while $(M_{-1}(Q_n))_{n\in\mathbb{N}}$ converge to some point of $\mathcal{T}(S) \times \{0\}$. This contradicts Proposition 2.5.16 $\square$

Then we show Theorem 2.5.1

Proof. Let $V := O' \cap \mathcal{U}$. Note that $M_{-1} \mid_V$ is injective, and that $M_{-1}^{-1}(M_{-1}(V)) = V$. Then for any $(h, \mu) \in \mathcal{T}(S) \times \mathcal{ML}(S)$, it suffices to take an $\epsilon > 0$, such that for any $0 < t < \epsilon$, the couple $(h, t\mu)$ belongs to V. $\square$

2.6 More about quasi-Fuchsian manifolds

2.6.1 Measured foliation at infinity

Let $\mathcal{D}$ be a simply connected domain of $\mathbb{CP}^1$ which is biholomorphic to the unit disc $\mathbb{D}$. We say that a function is univalent if it is an injective holomorphic function on an open subset of the complex plane. Let $f : \mathbb{D} \to \mathcal{D}$ be the Riemann uniformization map.
We define the Schwarzian of f to be,

$$S(f) := ((\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2)dz^2$$

The Schwartzian derivative satisfies

- $S(f) \approx 0$ if and only if f is the restriction of a Möbius transformation of $\mathbb{CP}^1$.
- if f and g are two univalent functions for which $f \circ g$ is well defined, then

$$S(f \circ g) = S(g) + g^* S(f)$$

We refer to [Dum09, Section 3] for details and proves. Let σ and σ' be two $\mathbb{CP}^1$ structure on the surface S. Let's $\tilde{S}$ be the universal cover of S, and let $D : \tilde{S} \to \mathbb{CP}^1$ (resp $D' : \tilde{S} \to \mathbb{CP}^1$) be developing map associated to σ (resp σ'). The holomorphic quadratic differentials $(D^{-1})^*(D' \circ D^{-1})$ is well defined (does not depend on the local choices of D^{-1}), then it defines a holomorphic quadratic differential on $\tilde{S}$ which is invariant under the deck transformations of the cover $\pi : \tilde{S} \to S$, we denote that holomorphic quadratic differential by $S(\sigma', \sigma)$. Moreover for any complex projective structue σ and for any quadratic holomorphic differential q there exists a unique projective structure σ' such that $S(\sigma', \sigma) = q$.

Definition 2.6.1. *Let M be a hyperbolic end. The Schwarzian at infinity of $\partial_\infty \mathcal{M}$ is the holomorphic quadratic differential $q_\infty := S(\sigma_F, \sigma_\infty)$, where σ_∞ stands for the natural complex projective structure of $\partial_\infty \mathcal{M} = \Omega_\Gamma/\Gamma$, and σ_F is the Fuchsian uniformization of the conformal structure at infinity $c_\infty := \pi(\sigma_\infty)$ of $\mathcal{M}$.*

Given a conformal structure c on the surface S, and a holomorphic quadratic differential q on S with respect to the conformal structure c. We can always find a local coordinates $z \mapsto w$ in which q has the form dw^2

Definition 2.6.2. *The horizontal measured foliation $hor_c(q)$ (resp. vertical measured foliation $hor_c(q)$) of q on (S, c) is a smooth singular measured foliation, with singularities at the zeroes of q, which is obtained locally by pulling back the horizontal measured foliations (resp. vertical measured foliation) of dw^2 under the change of coordinate $z \mapsto w := R\sqrt{q}$ defined above. The transverse measure for the horizontal measured foliation (resp. vertical measured foliation) is given by $\left|\mathcal{I}\sqrt{q}\right|$ (resp $\left|\mathcal{R}\sqrt{q}\right|$).*

We define a pair of measured foliation that fill in the same way how we define a pair of measured laminations that fill

Definition 2.6.3. *A pair of measured foliations F, G is said to fill S if for any measured foliation H on S we have,*

$$i(H, F) + i(H, G) > 0$$

We define the measured foliations at infinity of a quasi-Fuchsian manifold as the horizontal measured foliations associated to the Schwartzian at its boundary at infinity.

Definition 2.6.4. *Let $Q \in \mathcal{QF}(S)$. Let σ^+ (resp σ^-) be the Shwartzian associated to $S \times \{1\}$ (resp $S \times \{1\}$). We define F_Q^+ (resp F_Q^-) as the horizontal measured foliation associated to σ^+ (resp σ^-). We call $F_Q^\pm$ by the measured foliations at infinity of Q.*

The author in [Cho21] has shown the following theorem

Theorem 2.6.5. *[Cho21, Theorem 1.1] For every pair of measured foliations (F^+, F^-) which are arational and fill S, there exists an $\epsilon_{F^\pm} > 0$ such that for every $t \in (0, \epsilon_{F^\pm})$ there exists a unique quasi-Fuchsian manifold $Q \in \mathcal{QF}(S)$ sufficiently close to the Fuchsian locus, whose measured foliations at infinity are given by tF^+ and tF^-.*

A more general question about foliations at infinity is the following,

Question 2.6.6. *Let F^+ and F^- be two measured foliations that fill S. Is there a unique quasi-Fuchsian manifold $Q \in \mathcal{QF}(S)$ that has F^+ and F^- as the measured foliations at infinity ?*

2.6.2 Foliation of a quasi-Fuchsian manifold by CMC surfaces

In this section we will talk about constant mean curvatures foliation, or just CMC foliation. We begin the section by the definition of a CMC foliation.

Definition 2.6.7. *Let M be a three dimensional Riemmanian manifold which is homeomorphic to $S \times \mathbb{R}$. We say that M is smoothly monotically foliated by CMC surfaces with mean curvatures H ranging in the interval (a, b) if there exists a diffeomorphism between M and $S \times (a, b)$, such that for every $H \in (a, b)$, the restriction of the diffeomorpism on $S \times \{H\}$ is an embedding of constant mean curvature H.*

One can ask if a quasi-Fuchsian manifold $Q \in \mathcal{QF}(S)$ has CMC foliation. The answer in general is no. Indeed the authors in [And83] and [HW15] have shown the existence of quasi-Fuchsian manifolds that have several closed minimal surfaces (where a minimal surface is a surface that has mean curvature equal to 0). Such a quasi-Fuchsian manifold does not admit any CMC foliation, because if it admits a CMC foliation, then by the principle maximum it would have a unique closed minimal surface (see the introduction of of [CMS23] for more details).
The author in [Uhl83] highlighted the importance of a special class of quasi-Fuchsian manifolds called almost-Fuchsian quasi-Fuchsian manifolds. A quasi-Fuchsian manifold $Q \in \mathcal{QF}(S)$ is called almost-Fuchsian if it contains a closed minimal surface that has principal curvatures contained in the interval $(-1, 1)$, this condition is enough to deduce that this closed minimal surface is unique. An example of an almost-Fucshain manifold in a Fuchsian manifold, in that case the equidistant surfaces form the closed minimal surface defines a foliation which is CMC. However in the more general case that foliation by equidistant surfaces does not necessarily define a CMC foliation.

Conjecture 2.6.8 (Thurston)**.** *Let $Q \in \mathcal{QF}(S)$ be an almost-Fuchsian manifold. Then Q has a CMC foliation.*

The authors in [CMS23] showed the following theorem

Theorem 2.6.9. *[CMS23, Theorem A] Let S be a closed surface of genus bigger or equal to 2. Then there exists a neighborhood $\mathcal{U}$ of the Fuchsian locus in the space $\mathcal{QF}(S)$ such that every quasi-Fuchsian manifold in $\mathcal{U}$ is smoothly monotically foliated by CMC surfaces, with mean curvatures ranging in $(-1, 1)$.*

2.6.3 More on K-surfaces

As we have stated before, any hyperbolic end is foliated by K-surfaces, where K varies in $(-1, 0)$. The K surfaces induces a path between the conformal structure at infinity at the hyperbolic metric of the pleated surface. When we consider the third fundamental forms on the K surfaces, then they induce a path between the conformal structure at infinity and the bending lamination of the pleated boundary. The authour in [Maz19] has studied more invariant on these K surfaces.
Let $E \in \mathcal{E}(S)$ be a hyperbolic end, and let S_K be a K surface. As the authour in [Maz19] did, we introduce the following notations. Let $I_k, I\!I_K, I\!I\!I_K$ be respectively the first, the second, and the third fundametal form on S_K. We also denote by h_K and h^*_K

the hyperbolic metrics $-kI_k$ and $-\frac{k}{1+k}I\!I\!I_K$, we also denote by c_k the conformal structure induced by $I\!I_K$. Finally we denote by q_K the holomorphic quadratic differential

$$-\frac{2\sqrt{k+1}}{k}Hopf(Id:(S_K,c_K)\to(S_K,h_k)=\frac{2\sqrt{k+1}}{k}Hopf(Id:(S_K,c_K)\to(S_K,h_k^*)$$

Labourie in [Lab92b] has defined the following map, which is a parametrezation of the space of hyperbolic ends

$$\begin{aligned}\Phi_K:\mathcal{E}(S)\to&T^*\mathcal{T}(S)\\ E\mapsto&(c_K,q_K)\end{aligned}$$

Theorem 2.6.10. *[Lab92b, Theorem 3.1] The map Φ_k is a diffeomorphism for every $K\in(-1,0)$.*

The authour in [Qui20] has proved the following theorem.

Theorem 2.6.11. *For every hyperbolic end $E\in\mathcal{E}(S)$ we have*

$$\lim_{K\to 0}h_K=\lim_{K\to 0}(-k)I\!I_k=h_0$$

where h_0 is the hyperbolic metric in the conformal class at infinity of E.
Moreover

$$\dot{h}_0=-\frac{1}{2}h_0-Re\ q_0,\ \frac{d}{dK}(-K)I\!I_K\mid_{K=0}=0$$

where q_0 is the Schwartzian at infinity of E.

The authour in [Maz19] deduced the following corollary

Corollary 2.6.12. *[Maz19, Corollary 2.4] The maps $(\Phi_K)_{K\in(-1,0)}$ converge to the Schwartzian C^1-uniformly over compact subsets when K goes to 0.*

2.7 Convex subsets of $\mathbb{H}^3$ that have a quasi-circle as their ideal boundary.

In this section, we will study convex subsets of $\mathbb{H}^3$ that are homeomorphic to the sphere and have a quasi-circle as the ideal boundary. An example of such subsets is the convex hull in $\mathbb{H}^3$ of a quasi-circle $C\subset\partial_\infty\mathbb{H}^3$. For any such convex subset Ω, its boundary in the hyperbolic space $\partial\Omega\cap\mathbb{H}^3$ consists of the disjoint union of two topological discs (except when $\partial_\infty\Omega$ is a geometric circle). For our purposes, we will assume that even $\partial\Omega$, the boundary of Ω, is smooth, or we assume that Ω is the convex hull in $\mathbb{H}^3$ of a quasi-circle $C\subset\partial_\infty\mathbb{H}^3$ (see Figure 2.2), in that last case, the boundary of Ω is not smooth, but it is a pleated surface (see [CME06] and [Thu97]). In all the cases mentioned above, the metric of $\mathbb{H}^3$ induces a path metric on the boundary of Ω. The asymptotic behavior of the path metrics on $\partial\Omega\cap\mathbb{H}^3$ near the ideal boundary of Ω gives a quasi-symmetric map f, this map is called the gluing map (see Figure 2.7). Then the data that we have on the boundary of Ω gives two metrics $h^+$and h^- on the disc $\mathbb{D}$ (because the boundary of Ω consists of the disjoint union of two topological discs) and a quasi-symmetric map f, which is the gluing map. In the next pages we will define carefully the gluing map in the case when the induced metric on $\partial\Omega\cap\mathbb{H}^3$ has constant curvature (we will give the arguments of [BDMS21a]). For the more general case (when the curvature is not necessarily constant), we refer to [CS22]. Before that, we need to recall (From Section 2.4) some statements about the compactness of convex subsets.

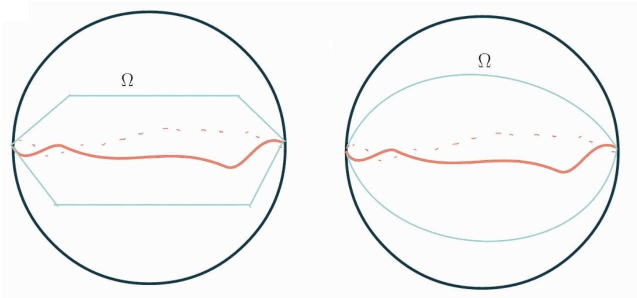


Figure 2.1: Ω is a convex subset of $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ this has a quasi-circle as ideal boundary. In this thesis we will consider when Ω is the convex hull of its ideal boundary, or when Ω has a smooth boundary.

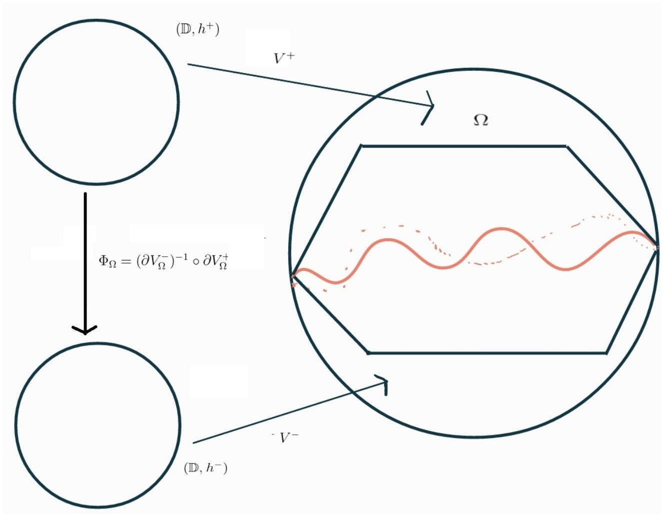


Figure 2.2: The asymptotic behavior of Ω near its ideal boundary induces a quasi-symmetric map called the gluing map.

2.7.1 Labourie compactness statements

Theorem 2.7.1. *[Lab89] Denote by $f_n : S \to \mathbb{H}^3$ a sequence of immersions of a surface S. Assume that the metrics $f_n^*(h)$ converge smoothly to a metric g_0 (where $f_n^*(h)$ is the pullback of the hyperbolic metric h by f_n). We also assume that the integrals of the mean curvatures are uniformly bounded, so, up to extracting a subsequence, f_n converge smoothly to an isometric immersion f_∞ such that $f_\infty^*(h) = g_0$.*

Proposition 2.7.2. *We denote by K a real number in the interval $(-1, 0)$. Assume that $f_n : \mathbb{H}^2_K \to \mathbb{H}^3$ is a sequence of proper isometric embeddings. We also assume the existence of a point $z \in \mathbb{H}^2_k$ such that the points $f_n(z)$ belong a compact subset of $\mathbb{H}^3$, so up to extracting a subsequence, f_n converges smoothly on compact subsets to an isometric immersion $f : \mathbb{H}^2_k \to \mathbb{H}^3$.*

A locally convex immersion $f : S \to \mathbb{H}^3$ is termed a convex embedding if it satisfies two conditions: first, it must be an embedding; second, the image $f(S)$ must lie within the boundary of its convex hull CH(f(S)). Alternatively, this condition can be interpreted as there existing a convex subset $C \subset \mathbb{H}^3$ such that f maps S onto the boundary of C.
In particular, if f is a proper embedding, then it qualifies as a convex embedding if and only if it bounds a convex region within $\mathbb{H}^3$. Importantly, any proper locally convex embedding inherently also functions as a convex embedding, and restricting a convex embedding to an open subset preserves its convex nature.
Lastly, if $h : S \to \mathbb{H}^3$ represents a convex embedding with N denoting the normal vector directed towards the concave side. In this case, the map $h_t : S \to \mathbb{H}^3$ defined by $h_t(x) = \exp_{h(x)}(tN(x))$ also constitutes a convex embedding.

The proof of Proposition 2.7.2 is based on Theorem 2.7.1 and the following Lemma.

Lemma 2.7.3. *We denote by $V : S \to \mathbb{H}^3$ a convex embedding and we denote by R the extrinsic diameter of $V(S)$ (that is the diameter of $V(S)$ with respect to the metric of $\mathbb{H}^3$). We denote by H the mean curvature and by da the area form induced by V. So we have*

$$\int_S H da < \frac{1}{\sinh 1} A(R+1)$$

where $A(\rho)$ denotes the area of the sphere of radius ρ in the hyperbolic space.

Proof. For a proof we refer to [BDMS21a, Lemma 3.7] □

Now we prove Proposition 2.7.2

Proof. First, we prove that V_n uniformly converges on compact sets of $\mathbb{D}$. Gauss equation implies that the map V_n is locally convex, so by the properness umwumption, V_n is in fact a convex embedding. Let U be a bounded open subset in $\mathbb{D}$ with diameter R. Notice that V_n restricts to a convex isometric embedding of U. In particular, the extrinsic diameter of U is bounded by R. By Lemma 3.7, then the integral of the mean curvature of V_n over U is uniformly bounded. By Theorem 3.4, we conclude that, up to a subsequence, V_n converges over U to an isometric immersion. □

The most important fact for us deduced from Proposition 2.7.2 is the following.

Proposition 2.7.4. *Let Ω be a convex subset of $\mathbb{H}^3$ as described at the beginning of this chapter. Assume that the induced metric on $\partial\Omega \cap \mathbb{H}^3$ has constant Gaussian curvature equal to $-1 < k < 0$. Then there exists an N that depends on the induced metrics on $\partial^{\pm}\Omega$, such that the principal curvatures of $\partial^{\pm}\Omega$ are contained in the interval $\left[\frac{1}{N}, N\right]$. In particular, it follows that the third fundamental form on $\partial^{\pm}\Omega$ is complete.*

Proof. Since there is $\epsilon > 0$ such that the product of the principal curvatures is greater than ϵ, it suffices to show that the principal curvatures have an upper bound. Assume, by contradiction, the existence of a sequence of points $(x_n)_{n\in\mathbb{N}} \subset \partial^+\Omega$ such that one of the principal curvatures at x_n is greater than n. Up to applying isometries of $\mathbb{H}^3$, we can assume that $x_n = x_0 \in \mathbb{H}^3$ (this will change the set Ω, but it will not change the induced metrics on the boundary of Ω), call the image of Ω under these isometries Ω_n. We then have a sequence of isometries $V_n : (\mathbb{D}, h) \to \partial^+\Omega_n$ sending a fixed point $p_0 \in \mathbb{D}$ to x_0. Proposition 2.7.2 implies that, up to a subsequence, V_n converges smoothly to an isometric immersion. But this contradicts the fact that one of the principal curvatures at x_0 diverges to ∞. $\square$

2.7.2 The gluing maps

For simplicity we will Begin by the case when the boundary of Ω is assumed to be smooth. The first crucial proposition that we need to show is the following.

Proposition 2.7.5. *Let Ω be a convex subset of $\mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ which is homeomorphic to S^2, and let its ideal boundary $\partial_\infty\Omega$ be a Jordan curve. In this case, $\partial\Omega \cap \mathbb{H}^3$ consists of the disjoint union of two topological disks $\partial^{\pm}\Omega$.*
Assume that we have the following isometries

$$V^{\pm}_{\partial\Omega} : (\mathbb{D}, h^{\pm}) \to \partial^{\pm}\Omega$$

Where the curvatures of $h^{\pm}$ is constant and equal to k, for some $-1 < k < 0$.
Then $V^{\pm}_{\partial\Omega}$ extend to a homeomorphism of $\mathbb{D} \cup \partial_\infty\mathbb{H}^2 \to \Omega \cup \partial^{\pm}\Omega$.

In what follows we will moreover assume that $\partial_\infty\Omega$ (which is a Jordan curve) goes thought $\{0, 1, \infty\}$. Assume at the moment that Proposition 2.7.5 is true. In that case we have unique isometries
$V^{\pm}_{\partial\Omega}(\mathbb{D}, h^{\pm}) \to \partial^{\pm}\Omega$ such that the extension $\partial V^{\pm}_{\partial\Omega} : \partial_\infty\mathbb{H}^2 \to \partial_\infty\Omega$ satisfies $\partial V^{\pm}_{\Omega}(i) = i$ for any $i \in \{0, 1, \infty\}$.

Definition 2.7.6. *The gluing map Φ_Ω associated with Ω is defined to be the comparison map between V^+_Ω and V^-_Ω, that is:*

$$\Phi_\Omega = (\partial V^-_\Omega)^{-1} \circ \partial V^+_\Omega$$

In [BDMS21b, Theorem B] the authors show the following

Theorem 2.7.7. *Assume that the induced metric on the boundary $\partial\Omega \cap \mathbb{H}^3$ is constant and equal to $-1 < k < 0$. The Jordan curve $\partial_\infty\Omega$ is a quasi-circle if and only if Φ_Ω is a quasi-symmetric map.*

Remark 2.7.8. *In [BDMS21a, Theorem A] the authors also showed that Theorem 2.7.7 holds in the case when $k=-1$, that is when Ω is equal to the convex hull of a Jordan curve in $\partial_\infty \mathbb{H}^3$*

We will begin by justifying the existence of the gluing map. This will be done by proving Proposition 2.7.5. However, We recall that we are giving a sketch of proof in the case when the induced metric on $\partial\Omega \cap \mathbb{H}^3$ has a constant curvature, for more details we refer to [BDMS21a]. For a proof in the general case where the curvature of the induced metric on $\partial\Omega \cap \mathbb{H}^3$ is not necessarily constant, we refer to [CS22].

2.7.3 The existence of the gluing map

Let Ω be a convex subset of $\mathbb{H}^3$ as described in Proposition 2.7.5. We define a natural 1-Lipschitz retraction

$$r_\Omega : \mathbb{H}^3 \to \Omega$$

The map r_Ω send a point $x \in \mathbb{H}^3$ to the nearest point in Ω. The map r_Ω induces a map $r_\Omega : \mathbb{H}^3 \setminus \Omega \to \partial\Omega$ which is a 1-Lipschitz retraction see [CME06, Section II.1.3]. Moreover the map r_Ω extends to a map $r_\Omega : \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3 \to \Omega$. For every point $x \in \partial_\infty \mathbb{H}^3$, $r_\Omega(x)$ is the intersection point of the smallest horosphere centered at x which meets $\mathcal{C}$. It is also worth to mention that the r_Ω behaves well under limits, in the sense that if a sequence of convex subsets $(\Omega_n)_{n\in\mathbb{N}}$ converge in the Hausdorff topology to Ω then r_{Ω_n} converge uniformly to r_Ω in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$.
The convex subset Ω induces a natural metric on $\partial_\infty \mathbb{H}^3 \setminus \partial_\infty \Omega$. Let's explain how. Let $\mathbb{B}$ be the space of horospheres of $\mathbb{H}^3$, recall that a horosphere is nothing but a sphere in $\mathbb{H}^3$ (in the Poincaré ball model) which is tangent to one point in $\partial_\infty \mathbb{H}^3$, this tangent point is called the center of the horosphere. Let $\pi : \mathbb{B} \to \partial_\infty \mathbb{H}^3$ be the natural projection map that send each horosphere to each center.
We define $\sigma_\Omega : \partial_\infty \mathbb{H}^3$ to be the section that maps the point p to the horosphere centered at p and tangent to $\partial\Omega$ at $r(p)$.
Recall that each point $x \in \mathbb{H}^3$ induces a Riemannian conformal metric on $\partial_\infty \mathbb{H}^3$ which is called the visual metric. The visual metric induced by two points $x_1, x_2 \in \mathbb{H}^3$ at a point $z \in \partial_\infty \mathbb{H}^3$ agree if and only if x_1 and x_2 lie on the same horosphere centered at z. Then we have an identification between the space of metrics on $T_z \partial_\infty \mathbb{H}^3$ that are compatible with the conformal structure and $\pi^{-1}(z)$.
We notice then that the section σ_Ω determine a conformal metric on $\partial_\infty \mathbb{H}^3 \setminus \partial_\infty \Omega$, we denote this conformal metric by I^*_Ω, we call this metric by the horospherical metric.
In the case when Ω is the convex hull in $\mathbb{H}^3$ of a Jordan curve in $\partial_\infty \mathbb{H}^3$, we still can define the conformal metric I^*_Ω, in this case it is called the Thurston metric on $\partial_\infty \mathbb{H}^3 \setminus \partial_\infty \mathbb{H}^3$ (for more details see [BC10]).
Note that if Ω_1 and Ω_2 are two convex subsets that share the same ideal boundary (that is $\partial_\infty \Omega_1 = \partial_\infty \Omega_2$), then $I^*_{\Omega_1 | CP1 \setminus \partial\infty\Omega_2} \leq I^*_{\Omega_2}$ if and only if $\Omega_1 \subset \Omega_2$.
The following lemma summarize some important properties about the horospherical metric, (for a proof see [Sch02]).

Lemma 2.7.9. *The following statements are true:*

- *We denote by Ω_s the set of points at distance less than or equal to s from Ω. So Ω_s is a convex set $I^*_{\Omega_s} = e^s I^*_\Omega$*

- *If $\partial\Omega$ is of class C^2, then $r_\Omega : \partial_\infty\mathbb{H}^3 \setminus \partial_\infty\Omega \to \partial\Omega$ is a C^1 diffeomorphism and $(r_\Omega^{-1})^*(I^*_\Omega) = I + 2I\!I + I\!I\!I$*

- *Asuume that $\partial\Omega$ is smooth. So the curvature of I^*_Ω at $z \in \partial_\infty\mathbb{H}^3 \setminus \partial_\infty\Omega$ is*

$$K^*(z) = \frac{K(r_\Omega(z))}{(1+\mu_1(r_\Omega(z)))(1+\mu_2(r_\Omega(z)))}$$

Where K is the intrinsic curvature of $\partial\Omega$ and μ_1 and μ_2 are the principal curvatures.

Lets denote the the components of $\partial_\infty\mathbb{H}^3 \setminus \partial_\infty\Omega$ by $\partial^\pm_\infty\Omega$, we assume that $r_\Omega(\partial^+_\infty\Omega) = \partial^+\Omega$ and that $r_\Omega(\partial^-_\infty\Omega) = \partial^-\Omega$.
Using Lemma 2.7.9 we deduce the following propostion.

Proposition 2.7.10. *Let Ω be a convex subset as described in the beginning of this chapter. There exists an $L > 0$ that depends on the induced metric on $\partial^\pm\Omega$ such that each of the maps*

$$r^\pm_\Omega : \partial^\pm_\infty\Omega \to \partial^\pm\Omega$$

is L-bilipschitz, where $\partial^\pm_\infty\Omega$ is equipped with the hyperbolic metric coming from the uniformization.

We need first to show the following lemma

Lemma 2.7.11. *There is a constant M that depends on the induced metric on the boundary of Ω, such that the conformal hyperbolic metric on $\partial^+_\infty\Omega$ (resp $\partial^-\Omega$) is M bilipschitz to $I^*_\Omega \mid \partial^+_\infty\Omega$ (resp $I^*_\Omega \mid \partial^-_\infty\Omega$)*

Proof. See [BDMS21a, Lemma 3.13] □

Now we prove the proposition

Proof. By the second formula of Lemma 2.7.9, we have $(r_\Omega^{-1})^*(I^*_\Omega) = I + 2I\!I + I\!I\!I$. It follows from Lemma 2.7.4 that $r_\Omega : (\partial^+_\infty\Omega, I^*_\Omega) \to \partial^+\Omega$ is uniformly bilipchitz. Then, it follows from Lemma 2.7.11 that $r_\Omega : \partial^+_\infty\Omega \to \partial^+\Omega$ is uniformly bilipchitz (where $\partial^+_\infty\Omega$ is endowed with its unique hyperbolic metric). □

Proposition 2.7.12. *The isometries $V^\pm_\Omega : (\mathbb{D}, h^\pm) \to \partial^\pm\Omega$ can be extended to a homeomorphism of $\mathbb{D} \cup \partial_\infty\mathbb{H}^2$ onto $\partial^\pm\Omega \cup \partial_\infty\Omega$.*

Proof. By Proposition 2.7.10 the nearest retraction map $r_\Omega : \partial^\pm_\infty\Omega \to \partial^\pm\Omega$. Let $U^\pm_\Omega : \mathbb{H}^2 \to \partial^\pm_\infty\Omega$ be the uniformization maps. Hence $V^{-1} \circ r^\pm_\Omega \circ U^\pm_\Omega : (\mathbb{D}, h^\pm) \to \mathbb{H}^2$ is L bilipschitz, then in particular it is quasi-conformal. Then it has a homeomorphism extension to $\mathbb{D} \cup \partial_\infty\mathbb{H}^2 \to \mathbb{H}^2 \cup \partial_\infty\mathbb{H}^2$.
Consider a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{H}^2$ converging to $x_\infty \in \partial_\infty\mathbb{H}^2$. Consider another sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{H}^2$ such that $r_\Omega(U^+_\Omega(y_n)) = V^+(x_n)$, which implies $x_n = V^{-1} \circ r_{\pm C,K} \circ U_{\pm C}(y_n)$. Since $x_n \to x_\infty$, and $V^{-1} \circ r_\Omega \circ U^+_\Omega$ is a homeomorphism from $\partial_\infty\mathbb{H}^2$ to $\partial_\infty\mathbb{H}^2$, we conclude that $(y_n)_{n\in\mathbb{N}}$ converges to $y_\infty \in \partial_\infty\mathbb{H}^2$. Therefore, the formula $\partial V(x_\infty) = r^\pm_{C,K}(U_{\pm C}(y_\infty)) = U^\pm_C(y_\infty)$ defines the desired extension of V to a homeomorphism from $\mathbb{D}$ to $\partial^+\Omega \cup \partial_\infty\Omega$.
We denote the extension of $V^\pm$ to the boundary at infinity by $\partial_\infty V^\pm$. □

We call the map

$$\Phi_\Omega := (\partial_\infty V^-)^{-1} \circ \partial_\infty V^+$$

by the gluing map of Ω.
We have justified the existence of Φ_Ω when the boundary of Ω is smooth and the induced metric on it is of constant curvatures. However we refer to [CS22] to justify the existence of gluing maps when the curvature is not necessarily constant.
For the case when the boundary of Ω is not smooth (then a pleated surface), the path metric on the boundary is hyperbolic (has constant curvature equal to -1), the gluing map still exists in this case, for more details we refer to [BDMS21a]. However it is worth to mention that the steps of the proof are the same as in the smooth case, except that insted of Proposition 2.7.10, we have the following proposition, for a proof see [BC10][Corollary 1.3]

Proposition 2.7.13. *Assume that the boundary of Ω is a pleated surface. The nearest point retraction map $r_\Omega : \partial^\pm_\infty\Omega \to \partial^\pm\Omega$ is a quasi-isometry for some constant independent on Ω.*

2.7.4 Quasi-Fuchsian manifolds from universal point of view

Let $Q \in \mathcal{QF}(S)$ be a quasi-Fuchsian manifold. Recall that the ideal boundary of Q, denoted by $\partial_\infty Q$, consists of the disjoint union of two copies of S, that is, $\partial_\infty Q = S \times \{0, 1\}$. The hyperbolic metric on Q (as explained in 2.1) induces a conformal structure on $\partial_\infty Q$, which then induces two conformal structures on S. We denote the conformal structure on $S \times \{0\}$ by c^- and the conformal structure on $S \times \{1\}$ by c^+.
The Riemann surface (S, c^+) (respectively (S, c^-)) is identified with $\mathbb{H}^2/\rho^+$ (respectively $\mathbb{H}^2/\rho^-$), where ρ^+ (resp. ρ^-) is a Fuchsian representation.
Recall that the quasi-Fuchsian manifold Q is identified with $\mathbb{H}^3/\rho$, where ρ is a quasi-Fuchsian representation (see 2.1). The limit set of $\rho(\pi_1(S))$ (denoted by Λ_ρ) is a quasi-circle. Thus, $\partial_\infty\mathbb{H}^3 \setminus \Lambda_\rho$ consists of the disjoint union of two topological discs, which we denote by D^+ and D^-. Recall that (S, c^+) is identified with D^+/ρ and (S, c^-) is identified with D^-/ρ.
Up to normalizing, we can assume that the uniformization maps $U^\pm : \mathbb{H}^2 \to D^\pm$ are normalized. That is, their extension $\partial U^\pm : \partial_\infty\mathbb{H}^2 \to \partial D^\pm$ satisfies $\partial U^\pm(i) = i$ for any $i \in \{0, 1, \infty\}$. Since D^+/ρ is identified with (S, c^+) which is identified with $\mathbb{H}^2/\rho$, we get that the map U^+ (resp. U^-) is ρ, ρ^+ (resp. ρ, ρ^-) equivariant.
This implies that the map

$$(\partial U^-)^{-1} \circ \partial U^+ : \partial_\infty\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2$$

is ρ^+, ρ^- equivariant.
Lets recall the following theorem, which is known as the conformal welding, for proof see [Bis07] for example.

Theorem 2.7.14. *For any normalized quasi-symmetric map f, there is a unique normalized quasi circle C, such that*

$$f = (\partial U^-)^{-1} \circ \partial U^+$$

Where $U^\pm$ are the uniformization maps of $\partial_\infty\mathbb{H}^3 \setminus C$ that satisfy $\partial U^\pm(i) = i$ for any $i \in \{0, 1, \infty\}$.

As one can observe, Theorem 2.7.14 implies Bers' theorem (Theorem 2.1.1). In particular, for any quasi-symmetric map f which is ρ_1, ρ_2 equivariant, there exists a unique quasi-Fuchsian manifold such that the gluing map of its limit set (after normalizing) is the map f. Thus, by Theorem 2.7.14, any such equivariant map corresponds to a unique quasi-Fuchsian manifold. We won't delve into the details of why Theorem 2.7.14 implies Bers' theorem, but we will present a "universal" version of Thurston's conjecture about the prescribed metrics on the boundary of the convex core of a quasi-Fuchsian manifold. We will then show why, if this conjecture is true, it will imply Thurston's conjecture. The arguments are similar to show why Theorem 2.7.14 implies Bers' theorem.

Let C be a quasi-circle in $\partial\mathbb{H}^3$, and let $\Omega = CH(C)$ be the convex hull of C in $\mathbb{H}^3$. Then (unless if C is a geometric circle) the boundary of $\Omega \cap \mathbb{H}^3$ consists of the disjoint union of two topological disks $\partial^{\pm}\Omega$, and $\mathbb{H}^3$ induces a path metric which is hyperbolic on each of them. As we have seen (for more detail see [BDMS21c]),the normalized isometries $V^{\pm} : \mathbb{H}^2 \to \partial^{\pm}\Omega$ extend to a homemorphism $\partial V : \partial_\infty\mathbb{H}^2 \to \partial_\infty\Omega$, and this induces a normalized quasi-symmetric map $(\partial V^-)^{-1} \circ \partial V^+$. The following statement can be seen (we will explain why) as an universal version of Thurston conjecture

Question 2.7.15. *Let f be a normalized quasi-symmetric map, is there a unique normalized quasi-circle C in $\partial_\infty\mathbb{H}^3$ such that f is the gluing map of* CH(C) *?.*

The existence part of Question 2.7.15 has been confirmed to be true. For a detailed proof, we refer to [BDMS21a, Theorem A] . The main idea behind the proof of this existence theorem involves an approximation by the lifts of the convex cores of quasi-Fuchsian manifolds.

Theorem 2.7.16. *[BDMS21a, Theorem A] Let f be a normalized quasi-symmetric map, then there exists a normalized quasi-circle C such that f is the gluing map of $CH(C)$.*

Next, we will show that if the uniqueness part of Question 2.7.15 holds, then Thurston's conjecture also holds.

Proposition 2.7.17. *Assume that Question 2.7.15 holds, then Thurston conjecture (Conjecture 2.1.2) also holds*

Proof. Let h^+ and h^- be two hyperbolic metrics on S. Let $Q \in \mathcal{QF}(S)$ be a quasi-Fuchsian manifold such that the induced metric on $\partial^+C(Q)$ (resp $\partial^-C(Q)$) is identified with (S, h^+) (resp (S, h^-)). Then there is two Fuchsian representations $\rho^+, \rho^- : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ such that (S, h^+) is identified with $\mathbb{H}^2/\rho^+$ and (S, h^-) is identified with $\mathbb{H}^2/\rho^-$. Also, the quasi-Fuchsian manifold Q is identified with $\mathbb{H}^3/\rho$ for some quasi-Fuchsian representation ρ.

We denote the limit set of $\rho(\pi_1(S))$ by Λ_ρ. Up to normalization, there is a normalized isometry $V^+ : \mathbb{H}^2 \to \partial^+CH(\Lambda_\rho)$ (resp $V^- : \mathbb{H}^2 \to \partial^-CH(\Lambda_\rho)$) which is ρ^+, ρ (resp ρ^-, ρ) equivariant. It follows that the gluing map $\Phi_{CH(\Lambda_\rho)} := (\partial V^-)^{-1} \circ \partial V^+$ is ρ^+, ρ^- equivariant.

We argue by contradiction and we assume the existence of a quasi-Fuchsian manifold $Q' \in \mathcal{QF}(S)$ such that the induced metric on $\partial^+C(Q')$ (resp $\partial^-C(Q')$) is identified with (S, h^+) (resp (S, h^-)). This quasi-Fuchsian manifold Q' is identified with $\mathbb{H}^3/\rho'$, for some quasi-Fuchsian representationρ'.

We denote the limit set of $\rho'(\pi_1(S))$ by $\Lambda_{\rho'}$. By the same argument as above $\Phi_{\mathrm{CH}(\Lambda_{\rho'})}$ the gluing map of $\mathrm{CH}(\Lambda_{\rho'})$ is ρ^+, ρ^- equivariant, there is a unique such quasi-symmetric map, it follows that $\Phi_{\mathrm{CH}(\Lambda_{\rho'})} = \Phi_{\mathrm{CH}(\Lambda_\rho)}$ And then it follows that $\Lambda_\rho = \Lambda_{\rho'}$.
As the action of ρ on $\partial^+\mathrm{CH}(\Lambda_\rho)$ (resp $\partial^-\mathrm{CH}(\Lambda_\rho)$) is isomorphic to the action of ρ' on $\partial^+\mathrm{CH}(\Lambda_{\rho'})$ (resp $\partial^-\mathrm{CH}(\Lambda_{\rho'})$), we deduce that the action of ρ on Λ_ρ is isomorphic to the action of ρ' on $\Lambda_{\rho'}$. It follows that the action of ρ on $\partial_\infty\mathbb{H}^3 \setminus \Lambda_\rho$ is isomorphic to the action of ρ' on $\partial_\infty\mathbb{H}^3 \setminus \Lambda_{\rho'}$.
Then we deduce by Bers theorem that $Q = Q'$. □

We can also state a universal version of Theorem 2.1.6.

Question 2.7.18. *Let h^+ and h^- be two metrics on $\mathbb{D}$ that have curvatures in $\left[-1+\epsilon, \frac{1}{\epsilon}\right]$, for some $\epsilon > 0$. And let f be a normalized quasi-symmetric map. Is there a unique covex set $\Omega \subset \mathbb{H}^3$ thas has a quasi-circle as ideal boundary, such that the induced metric on $\partial^+\Omega$ is isometric to h^+, the metric on $\partial^-\Omega$ is isometric to h^- and the gluing map Φ_Ω is equal to f.*

In [BDMS21a, Theorem B] the authors show the existence part of Question 2.7.18 when the metrics h^+ and h^- have constant curvatures equal to k, for $k \in (-1, 0)$. And later in [CS22, Theorem I.2] the authors proved the existence part of Question 2.7.18 in the general case.

It is also possible to look at the third fundamental form on $\partial\Omega \cap \mathbb{H}^3$, and the gluing map that corresponds to the third fundamental form. More explicitly, the third fundamental form on $\partial\Omega \cap \mathbb{H}^3$ is a metric that has curvature strictly smaller than 1. Then we have isometries

$$V^{*\pm}_\Omega : (\mathbb{D}, h^{*\pm}) \to (\partial^\pm\Omega, I\!I\!I)$$

This isometries extend homemorphically to a homeomorphism between $\mathbb{D} \cup \partial_\infty\mathbb{H}^2$ and $\partial^\pm\Omega \cup \partial_\infty\Omega$ After normalization we can assume that $\partial V^{*\pm}(i) = i$ for any $i \in \{0, 1, \infty\}$ and we define the gluing map with respect the third fundamental form to be $\Phi^*_\Omega := (\partial V^{*-}_\Omega)^{-1} \circ \partial V^{*+}_\Omega$.

We can also state a universal version of Theorem 2.1.6 about the third fundamental forms.

Question 2.7.19. *Let h^{*+} and h^{*-} be two metrics on $\mathbb{D}$ that have curvatures in $\left[-\frac{1}{\epsilon}, 1-\epsilon\right]$ for some $\epsilon > 0$, and in which every contractible closed geodesic has length strictly smaller than 2π. And let f be a normalized quasi-symmetric map. Is there a unique convex subsets of Ω such that the induced third fundamental form on $\partial^+\Omega$ is isometric to h^{*+}, the induced third fundamental form on $\partial^-\Omega$ is isometric to h^{*-} and the gluing map $\Phi_{I\!I\!I,\Omega}$ is equal to Ω.*

In [BDMS21a, Theorem B], the authors show the existence part of Question 2.7.19 when the metrics h^{*+} and h^{*-} have constant curvatures equal to k, for $k \in (-\infty, -1)$. Later in [CS22, Theorem I.2], the authors proved the existence part of Question 2.7.19 when the curvatures of h^{*-} and h^{*+} are strictly negative (but might be variable).

It is worth to mention that the complement of the convex hull of a quasi-circle in $\mathbb{H}^3$ is foliated by k-surfaces by a Theorem of Rosenberg and Spruck (see [RS94, Theorem 4]), this theorem extends Labourie theorem about the foliation of the complement of the convex core by k surfaces to the universal setting, we will explain why after giving the following theorem

Theorem 2.7.20. *[RS94, Theorem 4] Consider a Jordan curve $C \subset \mathbb{CP}^1$, and let K be a real number in $(-1, 0)$. There exists precisely two properly embedded K-surfaces in $\mathbb{H}^3$ that spine C. Each of these surfaces is topologically a disk, they are disjoint, and together they form the boundary a closed convex subset $C_K(C) \subset \mathbb{H}^3$. Moreover, $C_K(C)$ contains a neighborhood of the convex hull $CH(C)$. Furthermore, these K-surfaces, when K vary in $(-1, 0)$, form a foliation of $\mathbb{H}^3 \setminus CH(C)$.*

As one can see, Theorem 2.7.20 implies Labourie theorem in [Lab91]. Indeed, let $\mathbb{H}^3/\rho$ be a quasi-Fuchsia manifold, Let Λ_ρ be the limit set of ρ (then it is a quasi-circle). By Theorem 2.7.20 each connected component of $\mathbb{H}^3 \setminus \mathrm{CH}(\Lambda_\rho)$ is foliated by k-surfaces as mentioned in the statement of Theorem 2.7.20. For each k there is a unique k surface, and since Λ_ρ is invariant under the action of ρ, then for each k, the surfaces $S_k^{\pm}(\Lambda_\rho)$ are invariant under the action of ρ, it follows that the quotient $S_k^{\pm}(\Lambda_\rho)/\rho$ is a k surface in $\mathbb{H}^3/\rho$.

In [BDMS21a] the authors have shown that equivariant quasi-symmetric maps are dense in the set of quasi-symmetric maps as the following Proposition shows. We say that a normalized quasi-symmetric map f is quasi-Fuchsian if there is a closed hyperbolic surface S and two Fuchsian-representations $\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ such that f is $\rho_1.\rho_2$ equivariant.

Proposition 2.7.21. *[BDMS21a, Proposition 9.1] Any normalized quasi-symmetric map f is the C^0 limit of a sequence of normalised uniformly quasisymmetric quasi-Fuchsian homeomorphisms f_n.*

As a consequence of Proposition2.7.21, we deduce that the quasi-circles that are limit sets of some quasi-Fuchsian groups, are dense in the set of quasi-circles with respect to the Hausdorff topology. There is a possible approach to show Theorem 2.7.20 by using the main theorem of [Lab89]. This can be done by approximating a quasi-circle C by a sequence $(C_n)_{n\in\mathbb{N}}$ of quasi-Fuchsian quasi-circles, which the complement of their convex hull in $\mathbb{H}^3$ has a foliation by k-surface by Labourie Theorem (see [Lab91]), and then it remains to show that the foliation of $\mathbb{H}^3 \setminus \mathrm{CH(C_n)}$ by k-surfaces converge to a foliation of $\mathbb{H}^3 \setminus \mathrm{CH(C)}$ by k-surfaces.

2.7.5 The bending lamination of the boundary of the convex hull of a quasi-circle in $\mathbb{H}^3$

As stated earlier, the boundary $\partial C(Q)$ of the convex core of a quasi-Fuchsian manifold $Q \in \mathcal{QF}(S)$ is the disjoint union of two pleated surfaces homeomorphic to S. The bending locus, plus the amount of bending of $\partial C(Q)$, defines two measured laminations on the surface S. Thurston made the conjecture that any two measured laminations L^+ and L^- that fill the surface S and don't have any closed leaf of weight equal to or greater than π can be realized as the bending lamination of the boundary of the convex core of a unique quasi-Fuchsian manifold Q. Bonahon and Otal [BO04] have shown the existence of such a manifold, and later, Dular and Schlenker [DS24] have shown the uniqueness of such a

manifold.
As we have seen, all the statements regarding prescribing data on the boundary of a convex co-compact manifold of the form $S\times[0,1]$ have a universal analogue using quasi-circles and quasi-symmetric maps. Therefore, we aim to formulate a statement that generalizes the Thurston conjecture about bending laminations to this setting.
Let C be a quasi-circle in $\partial_\infty\mathbb{H}^3$, which is not a geometric circle. Let CH(C) be the convex hull of the quasi-circle C in $\mathbb{H}^3$. Then, $\partial\mathrm{CH(C)}\cap\mathbb{H}^3$, the boundary of CH(C), is the disjoint union of two topological disks denoted by $\partial^\pm\mathrm{CH(C)}$, which are pleated surfaces in $\mathbb{H}^3$. Each of $\partial^\pm\mathrm{CH(C)}$ has a path metric induced from the metric of $\mathbb{H}^3$. Consequently, each of $\partial^\pm\mathrm{CH(C)}$, with its path metric, is isometric to $\mathbb{H}^2$. Since they are pleated surfaces each of them defines a measured lamination in $\mathbb{H}^2$.

Definition 2.7.22. *Let $\phi:\partial_\infty\mathbb{H}^2\to\partial_\infty\mathbb{H}^3$ be a continuous embedding map such that $Im(\phi)=C$ is a quasi-circle.*
Let $\mathrm{CH}(C)$ be the convex hull of C in $\mathbb{H}^3$, and let $\partial^\pm\mathrm{CH}(C)$ be the components of $\partial\mathrm{CH}(C)\cap\mathbb{H}^3$.
We say that ϕ is quasi-symmetric if there are isometries (or, equivalently, for any isometry)

$$V^\pm:\partial^\pm\mathrm{CH}(C)\to\mathbb{H}^2$$

such that both of the maps $\partial V^\pm\circ\phi$ are quasi-symmetric maps.
Then we say that $\phi:\partial_\infty\mathbb{H}^2\to C$ is a parametrized quasi-circle.

We have seen that all the data of a measured lamination in $\mathbb{H}^2$ is in the ideal boundary $\partial_\infty\mathbb{H}^2$ (indeed, it can be prescribed as a measure on $\partial_\infty\mathbb{H}^2\times\partial_\infty\mathbb{H}^2\setminus\Delta$). Then, given any homeomorphism that preserves the orientation $f:\partial_\infty\mathbb{H}^2\to\partial_\infty\mathbb{H}^2$, we can pull back a measured lamination.

Definition 2.7.23. *Let $\phi:\partial_\infty\mathbb{H}^2\to C$ be a parametrized quasi-circle. We say that $L^+,L^-\in\mathcal{ML}(\mathbb{H}^2)$ are the measured lamination of the parametrized quasi-circle ϕ if L^+ (resp L^-) is the pull back of the bending lamination of $\partial^+\mathrm{CH(C)}$ (resp $\partial^-\mathrm{CH(C)}$) by ϕ.*

In this section we will be interested on prescribing the subset of $\mathcal{ML}(\mathbb{H}^2)\times\mathcal{ML}(\mathbb{H}^2)$ that can be the bending lamination of the boundary of the convex hull of some quasi-circle.

Question 2.7.24. *Let $L^\pm\in\mathcal{ML}(\mathbb{H}^2)$ be two measured laminations.*
Under what conditions are $L^\pm$ the bending laminations of some parametrized quasi-circle ?

The first thing that one should think about is what is the lift of two measured laminations that fill a closed surface S. This gives raise to the following notion.

Definition 2.7.25. *Let $L^\pm\in\mathcal{ML}(\mathbb{H}^2)$. We say that $L^\pm$ strongly fill if for any $A>0$ there exists $L>0$ such that for any geodesic arc α that has length $\ell(\alpha)\geq L$ the following inequality holds*

$$i(\alpha,L^+)+i(\alpha,L^-)>A$$

As an example of a pair of measured laminations of $\mathbb{H}^2$ that strongly fill, consider a closed hyperbolic surface S and take any two measured laminations that fill S. Then, the

lift of these two measured laminations will strongly fill $\mathbb{H}^2$.
Recall a measured lamination in $\mathbb{H}^2$ can be seen as a measure on $\partial_\infty\mathbb{H}^2 \times \partial_\infty\mathbb{H}^2 \setminus \Delta/(\mathbb{Z}/2\mathbb{Z})$ (where Δ is the diagonal of $\partial_\infty\mathbb{H}^2 \times \partial_\infty\mathbb{H}^2$). One can detect if two bounded measured laminations strongly fill from the boundary at infinity by the following proposition

Proposition 2.7.26. *Let L^+ and L^- be two bounded measured laminations that strongly fill. Let $a, b, c, d \in \partial_\infty\mathbb{H}^2$ be four point in that cyclic order. For any $A > 0$, there is $L > 0$ such that if the distance between the geodesics (a, d) and (b, c) is bigger than L, then the measure of $L^+([a, b] \times [c, d]) + L^-([a, b] \times [c, d]) > A$.*

A pair of bounded measured laminations behave well with quasi-symmetric maps as show the following Proposition

Proposition 2.7.27. *Let $f : S_1 \to S_1$ be a homeomorphism that preserves orientation. Let L^+ and L^- be two bounded measured laminations that strongly fill in $\mathbb{H}^2$. Assume that both $f_*(L^+)$ and $f_*(L^-)$ are bounded. Then f is quasi-symmetric.*

Proof. We argue by contradiction. If f is not quasi-symmetric, then there exists a sequence of points (a_n, b_n, c_n, d_n) that are in symmetric position, and $cr(f^{-1}(a_n), f^{-1}(b_n), f^{-1}(c_n), f^{-1}(d_n))$ goes to ∞. In particular the distances between the two geodesics (a_n, b_n) and (c_n, d_n) are bounded, and the distances between $(f^{-1}(a_n), f^{-1}(b_n))$ and $(f^{-1}(c_n), f^{-1}(d_n))$ go to ∞. Since $L^\pm$ are bounded and strongly fill, by proposition 2.7.26

$$L^+\left(\left[f^{-1}(a_n), f^{-1}(b_n)\right] \times \left[f^{-1}(c_n), f^{-1}(d_n)\right]\right) + L^-\left(\left[f^{-1}(a_n), f^{-1}(b_n)\right] \times \left[f^{-1}(c_n), f^{-1}(d_n)\right]\right)$$

go to ∞ but since we assumed that $f_*(L^+)$ and $f_*(L^-)$ are bounded, also

$$L^+([a_n, b_n] \times [c_n, d_n]) + L^-([a_n, b_n] \times [c_n, d_n])$$

must be bounded. This is a contradition, then f must be quasi-symmetric. $\square$

Bridgman (see [Bri98]) has shown that the Thurston norm of the bending laminations of $\partial^\pm\mathrm{CH(C)}$ must be bounded by some universal constant c (that is c does not depend on the Jordan-curve C)

Theorem 2.7.28. *[Bri98] There is $c > 0$, such that for any Jordan curve C, the Thurston norm of the bending lamination of $\partial^+\mathrm{CH(C)}$ and $\partial^-\mathrm{CH(C)}$ is bounded by c.*

In the quasi-Fuchsian case, leaves with angles greater than π are not allowed, this lead to the condition that the measured laminations are bounded by π. We give the following claim.

Claim 2.7.29. *Let $L^\pm \in \mathcal{ML}(\mathbb{H}^2)$ be two measured laminations that strongly fill, and each of them has a Thurston norm bounded by π. Then there exists a parameterized quasi-circle $\pi : \partial_\infty\mathbb{H}^2 \to C$ such that the pullback of the bending lamination of $\partial^+\mathrm{CH(C)}$ by ϕ is L^+ and the pullback of the bending lamination of $\partial^-\mathrm{CH(C)}$ by ϕ is L^-.*

As we will explain later, the pair of laminations described in Claim 2.7.29 are not a full characterization of pairs of measured laminations that can be realized as bending laminations of some parameterized quasi-circle. However, we hope that is a sufficient condition of the pair of measured laminations of $\mathbb{H}^2$ to be bending laminations of some parameterized quasi-circle, this condition is enough to generalize the quasi-Fuchsian case, since the lift of two measured laminations that fill a closed surface, strongly fill $\mathbb{H}^2$.
A possible approach to prove Claim 2.7.29 is by approximating by the lifts of quasi-Fuchsian manifolds. However we will need to the following claim

Claim 2.7.30. *Let L^+ and L^- be two measured laminations in $\mathbb{H}^2$ that are strongly filled and bounded by π. Let $\phi_n : \partial_\infty \mathbb{H}^2 \to C_n$ be a sequence of parameterized quasi-circles such that the pullback of the bending lamination of $\partial^+\mathrm{CH}(\mathrm{C_n})$ (respectively $\partial^-\mathrm{CH}(\mathrm{C_n})$) by ϕ_n is L_n^+ (respectively L_n^-).*
Assume that $(L_n^+)_{n\in\mathbb{N}}$ (respectively $(L_n^-)_{n\in\mathbb{N}}$) converge in the weak- topology to L^+ (respectively L^-). Then, up to extracting a subsequence, the parameterized quasi-circles ϕ_n converge uniformly to a parameterized quasi-circle $\phi : \partial_\infty\mathbb{H}^2 \to C$ such that the pullback of the bending lamination of $\partial^+\mathrm{CH}(\mathrm{C})$ (respectively $\partial^-\mathrm{CH}(\mathrm{C})$) by ϕ is L^+ (respectively L^-).*

Note that Claim 2.7.29 is a generalization of the closing lemma in the quasi-Fuchsian manifolds given by Bonahon and Otal in [BO04]. The most obvious necessary condition for a pair of laminations L^+ and L^- to be the bending laminations of parameterized quasi-circles is filling (a weaker version than strongly filled), but it is not sufficient. At the same time, the condition of strong filling is not necessarily, as shown in the following examples.
First lets define filling and show why it is a necessarily condition

Proposition 2.7.31. *Let $\phi : \partial_\infty\mathbb{H}^2 \to C$ be parameterized quasi-circles, and let L^+ and L^- in $\mathcal{ML}(\mathbb{H}^2)$ be its bending laminations (that is the pullback of the bending laminations of $\partial^+\mathrm{CH}(\mathrm{C})$ and $\partial^-\mathrm{CH}(\mathrm{C})$ by ϕ). Than the pair of laminations L^+ and L^- fill $\mathbb{H}^2$, that is for any γ, a complete geodesic of $\mathbb{H}^2$, the following holds*

$$i(\gamma, L^+) + i(\gamma, L^+) > 0$$

Proof. We argue by contradiction. Assume the existence of γ, a geodesic that is realized as a geodesic of $\mathbb{H}^3$ (without bending) in both of $\partial^+\mathrm{CH}(\mathrm{C})$ and $\partial^-\mathrm{CH}(\mathrm{C})$. Therefore, those two boundary components $\partial^+\mathrm{CH}(\mathrm{C})$ and $\partial^-\mathrm{CH}(\mathrm{C})$ would intersect along a complete geodesic, which we can still denote as γ. But then if P^- is a support plane of $\partial^-\mathrm{CH}(\mathrm{C})$ along γ, and P^+ is a support plane of $\partial^+\mathrm{CH}(\mathrm{C})$ along γ. This can only happen if $P^+ = \partial^+\mathrm{CH}(\Gamma) = \partial^-\mathrm{CH}(\Gamma) = P^-$, that is, if C is a geometric circle. ☐

Next we give an example about a pair of laminations that fill but cannot be realized as bending laminations of any parameterized quasi-circle.
Start by considering a complete convex surface Σ^- contained in $\mathbb{H}^3$, which is the union of two totally geodesic half-planes that meet along their common boundary. Let $C = \mathrm{CH}(\Sigma^-)$ be its convex core. By construction, Σ^- is the lower boundary component of C. A simple symmetry argument shows that the upper boundary component of C, ∂^+C, is bent along a measured foliation which is invariant under a one-parameter subgroup of translations, and each leaf of the foliation is orthogonal to the common axis of the translations. We denote by λ^+, λ^- the measured bending laminations on the future and past boundary components of C, so that the support of λ^- contains only one line, while the support of λ^+ is the whole hyperbolic plane, see Figure 2.7.5. Clearly, the pair (λ^-, λ^+) does not strongly fill, since a long segment of a bending line of λ^+ can be disjoint from the support of λ^-.
Now consider a modified lamination λ_0^+ obtained from λ^+ by adding an atomic weight to one leaf of the foliation. It is then clear that the pair (λ^-, λ_0^+) weakly fill. However, it cannot be realized as measured bending laminations of a parameterized quasi-circle, since in this case λ^- by itself determines the curve and therefore the upper bending lamination,

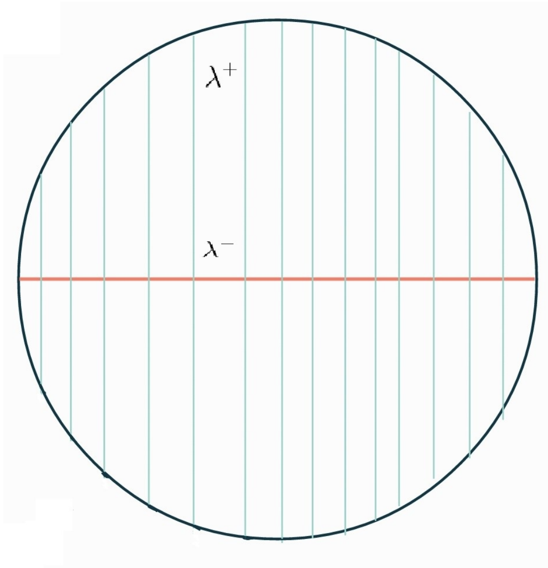


Figure 2.3: The bending lamination of ∂^-CH(C) is one leaf, and the bending lamination of ∂^+CH(C) forms a foliation of $\mathbb{H}^2$.

which needs to be equal to λ^+.
However, we can show that if the bending laminations of a Jordan curve are bounded and strongly fill then this Jordan curve is a quasi-circle

Lemma 2.7.32. *Let $f : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$ be a parametrized Jordan curve, that is f is injective and its image is a Jordan curve denoted by C. Assume that the pull back of the bending laminations of $\partial^\pm$CH(C) by f are bounded and strongly fill, then C is a quasi-circle*

2.7.6 Some facts about the hyperbolic convex hull of a Jordan curve in $\partial_\infty \mathbb{H}^3$

Let $C \in \partial_\infty \mathbb{H}^3$ be a Jordan curve, and as usual denote its hyperbolic convex hull by CH(C). We will show that C is a quasi-circle if and only if one of the isometric embeddings (then both) $v^\pm : \mathbb{H}^2 \to \partial^\pm \mathrm{CH(C)} \subset \mathbb{H}^3$ is a quasi-isometry from $\mathbb{H}^2$ to $\mathbb{H}^3$.

Lemma 2.7.33. *[BDMS21a, Lemma 4.4] Let P_n be a sequence of convex sets of $\mathbb{H}^3$, and let S_n denote their boundaries. Assume that P_n converge to a convex subset of $\mathbb{H}^3$, P, in the Hausdorff topology of closed sets of $\mathbb{H}^3$ and denote by S the boundary of P. If $x_n, y_n \in S_n$ converge to $x, y \in S$, then $d_{S_n}(x_n, y_n)$ converge to $d_S(x, y)$.*

Proposition 2.7.34. *Let C be a k quasi-circle.*

- *$\exists M(k) > 0$ a constant depending only on k such that if $x, y \in \partial^+$CH(C) and $d_{\mathbb{H}^3}(x, y) = 1$, then $d_{\partial^+ \mathrm{CH(C)}}(x, y) \leq M(k)$.*
- *$\exists m(k) > 0$ a constant depending only on k such that if $x, y \in \partial^+ CH(C)$ and $d_{\partial^+ CH(C)}(x, y) = 1$, then $d_{\mathbb{H}^3}(x, y) \geq m(k)$.*

Proof. Let's prove the first point, we argue by contradiction. Assume the existence of a sequence C_n of k quasi-circles such that, $\exists x_n, y_n \in \partial^+\mathrm{CH(C_n)}$ such that $d_{\mathbb{H}^3}(x_n, y_n) = 1$ and $d_{\partial^+\mathrm{CH(C_n)}}(x_n, y_n) \geq n$.
Up to applying isometries on C_n, we can assume that x_n and y_n are both constant points in $\mathbb{H}^3$, denote them by x and y respectively. By Ahlfors's lemma, Up to extracting a subsequence of C_n, we can assume that C_n converge to a k quasi-circle C in the Hausdorff topology of closed sets of $\mathbb{H}^3$. Then $\mathrm{CH}(C_n)$ converge to $\mathrm{CH}(C)$ in the Hausdorff topology of closed sets of $\mathbb{H}^3$. Then, $d_{\partial^+\mathrm{CH(C_n)}}(x, y)$ (which is bigger than n) converge to $d_{\partial^+\mathrm{CH(C)}}(x, y)$, which is absurd. Thus, we take $M(k)$ to be:

$$M(k) := inf\left\{d_{\partial^+CH(C)}(x,y) | C \in QC_k, x, y \in \partial^+CH(C), d_{\mathbb{H}^3}(x,y) = 1\right\}$$

We also prove the second point by arguing by contradiction. Assume the existence of a sequence C_n of quasi-circles such that, $\exists x_n, y_n \in \partial^+\mathrm{CH(C_n)}$ such that $d_{\mathbb{H}^3}(x_n, y_n) \leq \frac{1}{n}$ and $d_{\partial^+\mathrm{CH(C_n)}}(x_n, y_n) = 1$.
Up to applying an isometry, we can assume that $x_n = x$ a constant point, note that y_n converge to x. Again by Ahlfors's lemma, Up to extracting a subsequence of C_n, we can assume that C_n converge to a k quasi-circle C in the Hausdorff topology of closed sets of $\mathbb{H}^3$. Then $d_{\partial^+\mathrm{CH(C_n)}}(x_n, y_n)$ (which is equal to 1) converge to $d_{\partial^+\mathrm{CH(C)}}(x, y)$ (which is equal to 0), which is absurd. Thus, we take $m(k)$ to be

$$m(k) := \inf\left\{d_{\mathbb{H}^3}(x,y), C \in QC_k, x, y \in \partial^+\mathrm{CH(C)}, \mathrm{d_{\partial^+CH(C)}(x,y)} = 1\right\}$$

$\square$

We deduce then the following Corollary

Corollary 2.7.35. *Let $C \in \partial_\infty\mathbb{H}^3$ be a quasi-circle. Then the maps $V^\pm : \mathbb{H}^2 \to \partial^\pm\mathrm{CH(C)}$ are quasi-isometries from $\mathbb{H}^2$ to $\mathbb{H}^3$.*

Theorem 2.7.36. *[Ahl63] We identify $\partial_\infty\mathbb{H}^3$ with $\hat{\mathbb{C}}$. Let C be a Jordan curve. Then, C is a k quasi-circle if and only if there is a constant $C(k)$ that depends only on k, such that for all $a, b, c, d \in C$ such that ac separates bd, the following inequality holds:*

$$\frac{|b-a| \cdot |c-a|}{|b-d| \cdot |c-d|} \leq C(k)$$

Proposition 2.7.37. *Let $C \subset \partial_\infty\mathbb{H}^3$ be a Jordan curve. Assume that one of the maps $: \mathbb{H}^2 \to \partial^\pm\mathrm{CH(C)}$ is a quasi-isometry from $\mathbb{H}^2$ to $\mathbb{H}^3$, then C is a quasi-circle.*

Proof. We argue by contradiction. Assume that C is not a quasi-circle. Then, there exists a sequence of points $a_n, b_n, c_n, d_n \in C$ such that

$$\frac{|b_n-a_n| \cdot |c_n-a_n|}{|b_n-d_n| \cdot |c_n-d_n|} > n$$

Without loss of generality, we can assume that $a_n = 1$, $c_n = -1$, $d_n = i$, and $V_+(p) = p$ for all $p \in \{1, i, -1\}$. From our assumption, b_n converges to d, which implies that the Hyperbolic Hausdorff distance between the geodesics $(-1,1)_{\mathbb{H}^3}$ and $(b_n, i)_{\mathbb{H}^3}$ goes to infinity. However, the distance between the two geodesics $(-1,1)_{\mathbb{H}^2}$ and $((V_+)^{-1}(b_n), i)_{\mathbb{H}^2}$ is equal to 0, which is absurd. $\square$

Chapter 3

Globally hyperbolic manifolds

3.1 Anti-de Sitter geometry

3.1.1 Hyperboloid model

In this chapter we will focus more on Anti-de Sitter space. The anti-de Sitter space is one of the model spaces in the Lorentzian geometry, it is the Lorentzian analogue of the the hyperbolic space, for more details about Anti-de Sitter space we refer to [BS20].
Let $\mathbb{R}^{n,2}$ be the vector space $\mathbb{R}^{n+2}$ endowed with the quadratic form

$$q_{n,2}(x) = x_1^2 + ... + x_n^2 - x_{n+1}^2 - x_{n+2}^2$$

The linear transformations that preserve $q_{n,2}$ forms a group that we denote by $O(n,2)$ and we denote by $\langle .,. \rangle_{n,2}$ the scalar product associated to $q_{n,2}$.
We define

$$\mathbb{H}^{n,1} = \left\{ x \in \mathbb{R}^{n,2}, q_{n,2}(x) = -1 \right\}.$$

The subset $\mathbb{H}^{n,1}$ is a submanifold of $\mathbb{R}^{n+2}$ of dimension $n+1$, the restriction of $q_{n,2}$ to each tangent space $T_x\mathbb{H}^{n,1}$ has a signature $(n,1)$, then it induce on $\mathbb{H}^{n,1}$ a Lorentzian geometry. That model is called the quadratic model, we refer to [BS20, Section 2.1] fore more details and for a proof why the Lorentzian manifold $\mathbb{H}^{n,1}$ has a constant sectional curvature equal to -1.
Now we will introduce the projective model to be

$$\mathbb{ADS}^{n,1} := \mathbb{H}^{n,1} / \{\pm 1\}$$

We refer to [BS20, Section 2.2] to see proofs why $\mathbb{ADS}^{n,1}$ has a constant sectional curvature equal to -1.
Note that $\mathbb{ADS}^{n,1}$ is identified with the subset of $\mathbb{RP}^{n+1}$ which is defined as

$$\mathbb{ADS}^{n,1} := \left\{ [x] \in \mathbb{RP}^1, q_{n,2}(x) < 0 \right\}$$

We define the ideal boundary $\partial_\infty \mathbb{ADS}^{n,1}$ to be the projectivization of the set of lightlike vectors in $\mathbb{R}^{n,2}$, that is

$$\partial_\infty \mathbb{ADS}^{n,1} := \left\{ [x] \in \mathbb{RP}^{n+1}, q_{n,2}(x) = 0 \right\}$$

We refer to [BS20, Section2.2] to see why $\mathbb{ADS}^{n,1}$ induces a Lorentzian conformal structure on $\partial_\infty \mathbb{ADS}^{n,1}$.
We can distinguish three types of geodesics

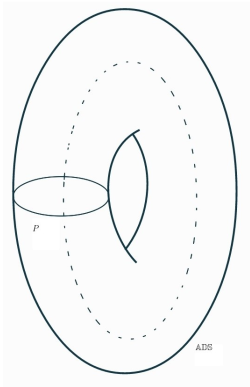


Figure 3.1: The three dimensional anti-de Sitter space is homeomorphic to a solid torus, and its ideal boundary is conformally equivalent to a $\mathbb{RP}^1 \times \mathbb{RP}^1$. A space like plane P intersects the ideal boundary of $\mathbb{ADS}^{2,1}$ at a geometric circle.

- Timelike geodesics: they are geodesics that correspond to a projective line, which is a closed non-trivial loop entirely contained in $\mathbb{ADS}^{n,1}$ that has length equal to π.
- Spacelike geodesic: they are geodesics that correspond to a projective line contained in $\mathbb{ADS}^{n,1} \cup \partial_\infty \mathbb{ADS}^{n,1}$ that intersects $\partial_\infty \mathbb{ADS}^{n,1}$ transversely at two points, they have infinite length.
- Light like geodesics: they are geodesics that correspond to a projective line contained in $\mathbb{ADS}^{n,1} \cup \partial_\infty \mathbb{ADS}^{n,1}$ which are tangent to $\partial_\infty \mathbb{ADS}^{n,1}$

3.1.2 Totally geodesic spaces

Let W be a k-dimensional linear subspace of $\mathbb{R}^{n,2}$. Since in this thesis we will focus on the three dimensional space $\mathbb{ADS}^{2,1}$, we will assume that $n = 2$ and $k = 3$, in that case we say that $W \cap \mathbb{H}^{2,1}$ is a plane (See Figure 3.1.2).
Let P be a plane of $\mathbb{H}^{2,1}$

- We say that P is space-like if any two points in P are connected by space-like geodesic which is in P.
- We say that P is time-like if it contains a time like geodesic.
- We say that P is light-like otherwise.

3.1.3 The $\mathrm{PSL}(2, \mathbb{R})$ model

In this thesis we will focus on $\mathbb{ADS}^{2,1}$, that is the three dimensional anti-de Sitter space. This space has a Lie group structure which is diffeomorphic to $\mathrm{PSL}(2, \mathbb{R})$ (that is the identity component of the isometry group of $\mathbb{H}^2$), and also it happens that in this case the isometry group of $\mathbb{ADS}^{2,1}$ that preserve the orientation and time orientation $SO_0(2,2)$ is isomorphic to $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$.

Let $\mathcal{M}(2,\mathbb{R})$, the space of 2×2 matrices with real coefficient, one can notice that the space $(\mathcal{M}(2,\mathbb{R}), -det)$ is isometric to $\mathbb{R}^{2,2}$.

$$\begin{aligned} \mathbb{R}^4 &\to \mathcal{M}_{2,2}(\mathbb{R}) \\ (x_1, x_2, x_3, x_4) &\mapsto \begin{pmatrix} x_1 - x_3 & x_4 - x_2 \\ x_2 + x_4 & x_1 + x_3 \end{pmatrix}. \end{aligned}$$

Under this isomorphism the space $\mathbb{H}^{2,1}$ is identified with the lie group $SL(2,\mathbb{R})$.
Note that the group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ acts on $\mathcal{M}(2,\mathbb{R})$ by left and right multiplication

$$(A,B).M := AMB^{-1}$$

We refer to [BS20, Section 3.1] to see why the action of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ preserves the quadratic form $-det$, and that the isometry group that preserve orientation of $SL(2,\mathbb{R})$ endowed with the quadratic form $-det$ is identified with $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) / \{(Id, Id), (-Id, -Id)\}$.
We refer to the same reference [BS20, Section 3.1] to see why $\kappa(X,Y) = 4tr(XY)$ is the killing form of $\mathcal{SL}(2,\mathbb{R})$ the Lie algebra of $SL(2,\mathbb{R})$.
Under these identifications, it follows that $\mathbb{ADS}^{2,1}$ is identified with $\mathrm{PSL}(2,\mathbb{R})$ and the isometry group of $\mathbb{ADS}^{2,1}$ that preserves orientation is identified with $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.
Note that with this Lie group model of $\mathbb{ADS}^{2,1}$ one can define the ideal boundary of anti-de Sitter space as

$$\partial_\infty \mathbb{ADS}^{2,1} = \{[X] \in \mathbb{P}(\mathcal{M}(2,\mathbb{R})) |\ det(X) = 0\}.$$

One can see that $\mathbb{ADS}^{2,1}$ is diffeomorphic to $\mathbb{D} \times \mathbb{RP}^1$ and that $\partial_\infty \mathbb{ADS}^{2,1}$ is identified with $\mathbb{RP}^1 \times \mathbb{RP}^1$ via the following map:

$$\begin{aligned} \partial_\infty \mathbb{ADS}^{2,1} &\to \mathbb{RP}^1 \times \mathbb{RP}^1 \\ [X] &\mapsto (\mathrm{Im}(X), \mathrm{Ker}(X)) \end{aligned}$$

Using the Lie group model of $\mathbb{ADS}^{2,1}$ we obtain a description of each geodesic type. Let's see the case of the one-parameter groups for the Lie group structure of $\mathrm{PSL}(2,\mathbb{R})$. For proofs and details we refer to [BS20, Section 3.5]

- Timelike geodesics are, up to conjugacy, of the form

$$t \mapsto \begin{pmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{pmatrix}$$

- Timelike geodesics are, up to conjugacy, of the form

$$t \mapsto \begin{pmatrix} cosh(t) & sinh(t) \\ sinh(t) & cosh(t) \end{pmatrix}$$

- And finally lightlike geodesics are up to conjugacy of the form

$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

It follows that any (unparameterized) timelike geodesic of $\mathbb{ADS}^{2,1}$ is of the form

$$L_{p,q} = \{X \in PSL(2,\mathbb{R}), X.p = q\}$$

for some $p, q \in \mathbb{H}^2$.
Reciprocally given any $p, q \in \mathbb{H}^2$, it follows that $L_{p,q}$ is a (unparameterized) timelike geodsic, for proofs see [BS20, Proposition 3.5.2].
A space-like (resp time-like, light-like) plane of $\mathbb{ADS}^{2,1}$ is the projectivization of a space-like (resp time-like, light-like) of $\mathbb{H}^{2,1}$.

3.1.4 Time orientation

For more details, we refer to [BS20, Section 2.3].
Let $\tilde{\mathbb{ADS}}$ be the Poincaré model of anti-de Sitter geometry introduced in [BS10]. That model is isometric to $\mathbb{D}^n \times \mathbb{R}$ equipped with the metric

$$\frac{4}{(1-r^2)^2}(dx_1^2 + ... + dx_n^2) - (\frac{1+r^2}{1-r^2})dt^2.$$

One can notice that the vector field $\frac{\partial}{\partial t}$ is a timelike non vanishing vector field of $\tilde{\mathbb{ADS}}^{n,1}$, which shows that $\tilde{\mathbb{ADS}}^{n,1}$ is time orientable. The choice of any time-orientation is preserved under the action of deck transformation of the covering $\tilde{\mathbb{ADS}}^{n,1} \to \mathbb{ADS}^{n,1}$, this shows that $\mathbb{ADS}^{n,1}$ is time orientable.

3.2 Convexity in $\mathbb{ADS}^{2,1}$

In this section we will define and investigate the notion of convexity in the three dimensional anti-de Sitter space. Later we will define and give some key theorems about globally hyperbolic manifolds, that are the quotient of a special category of convex subset of $\mathbb{ADS}^{2,1}$ by the product of two Fuchsian representation. These convex subsets have boundary at infinity equal to a quasi-circle (we will define this notion later). After, we will give some key theorems and questions about a non necessarily invariant such convex subsets (these objects includes lifts of globally hyperbolic manifolds).
As for geodesics, we define space-like, time-like and light-like curves.

Definition 3.2.1. *We say that a differentiable curve in $\mathbb{ADS}^{2,1}$ is space-like (resp time-like, light-like) if its tangent vector at any point is space-like (resp time-like, light-like) and we say that the curve is causal if its tangent vector at any point is light-like or time-like.*

Then we define what an achronal or acausal subset of $\mathbb{ADS}^{2,1}$ is.

Definition 3.2.2. *A subset $\Omega \subset \mathbb{ADS}^{2,1} \cup \partial_\infty \mathbb{ADS}^{2,1}$ is said to be achronal (resp. acausal) if there is no pair of points in Ω that can be joined by a timelike (resp spacelike) geodesic.*

Let's consider the projective model of $\mathbb{ADS}^{2,1}$. We say that a set $\Omega \subset \mathbb{RP}^3$ is convex, if it is contained and convex in some affine chart, that is we can join any two points of Ω by a projective segment. We say that a set $\Omega \subset \mathbb{RP}^3$ is properly convex if it's closure is

convex.
Note that $\mathbb{ADS}^{2,1}$ is not a convex subset of $\mathbb{RP}^3$, however we still define a convex subset of $\mathbb{ADS}^{2,1}$ to be a convex subset of $\mathbb{RP}^3$, which is equivalent to saying that $\Omega \subset \mathbb{ADS}^{2,1}$ is a convex subset of $\mathbb{ADS}^{2,1}$ if every two points of Ω can be joined by a Lorentzian geodesic segment. We say that projective plane P is a support plane for the convex subset Ω at the point p if P intersect $\partial\Omega$ at the point p and if the intersection between P and the interior of Ω is empty, Or in the case when Ω is contained in P. (see [BS20, Section 4] or [BDMS21a, Section 6.4]).

Next we will define the general notion of globally hyperbolic manifold. We will focus on $\mathbb{ADS}^{2,1}$ globally hyperblic manifolds, we will talk more about these objects later.

Definition 3.2.3. *Let (M, g) be a Lorentzian manifold and let X be an achronal subset, we define the domain of dependance of X to be*

$$\mathcal{D}(X) = \{p \in M, \textit{every inextensible causal curve through } p \textit{ meets } X\}$$

We say that X is a Cauchy surface of M if $\mathcal{D}(X) = M$. A Lorentzian manifold (M, g) is called globally hyperbolic spacetime (or manifold) if it admits a Cauchy surface.

Definition 3.2.4. *Let X be an achronal domain in $\mathbb{ADS}^{2,1} \cup \partial_\infty \mathbb{ADS}^{2,1}$, we define $\Omega(X) \subset \mathbb{ADS}^{2,1}$, the invisible domain of X, to be the points that are connected to X by no causal path.*

A globally hyperbolic spacetime has a well known topology as the following theorem shows, for proof see for example [Bee17], [Ger70], [BS03], [BS05].

Theorem 3.2.5. *Let (M, g) be a globally hyperbolic spacetime. Then*

- *Every two Cauchy surfaces of M are diffeomorphic*
- *There exists a submersion $\phi : M \to \mathbb{R}$ such that its fibers are Cauchy surfaces of M.*
- *Denote by Σ any Cauchy surface of M, then M is diffeomorphic to $\Sigma \times \mathbb{R}$*

3.2.1 Achronal/acausal meridians and quasicircles in $\mathbb{ADS}^{2,1}$

Let C be a continuous curve of $\partial_\infty \mathbb{ADS}^{2,1}$. The curve C is said to be achronal (resp acausal), if for any point p that belongs to the curve C, we can find a neighborhood U of p in the three dimensional anti-de Sitter space, such that $U \cap C$ lies outside the subset of U that can be reached from p via timelike paths (respectively timelike and lightlike paths). A curve C is said to be achronal meridian (resp acausal meridian) if it is achronal (resp acausal), and it bounds a disk in the three dimensional anti-de Sitter space.
Mess gave the following characterization of acausal meridians. Recall that $\partial_\infty \mathbb{ADS}^{2,1}$ is identified with $\mathbb{RP}^1 \times \mathbb{RP}^1$.

Lemma 3.2.6. *[Mes07] An acausal meridian in $\partial_\infty \mathbb{ADS}^{2,1} = \mathbb{RP}^1 \times \mathbb{RP}^1$ is the graph of a an orientation preserving homeomorphism $f : \mathbb{RP}^1 \to \mathbb{RP}^1$*

We say that an acausal meridian is normalized if it is the graph of an orientation preserving homeomorphism $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ that satisfies $f(0) = 0, f(1) = 1$ and $f(\infty) = \infty$, or in other words if it goes thought the points $\{(0,0),(1,1),(\infty,\infty)\}$.
We call an acausal meridian quasi-circle if it is the graph of a quasi-symmetric map, and we say that it is k quasi-circle if it is the graph of a k quasi-symmetric map.
As in the case of quasi-circles in $\partial_\infty\mathbb{H}^3$, quasi-circle in $\partial_\infty\mathbb{ADS}^{2,1}$ satisfy the following compactness statement (which is a direct consequence of the compactness statement about quasi-symmetric maps)

Lemma 3.2.7. *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of k quasi-circles in $\mathbb{ADS}^{2,1}$, then $(C_n)_{n\in\mathbb{N}}$ has a subsequence that converge to either a k-quasi-circle C_∞ or to $\{p\} \times S^1 \cup S^1 \times \{q\}$ for some $p, q \in S^1$.*

3.2.2 Globally hyperbolic manifolds

Let S be a closed hyperbolic surface, and let $\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ be Fuchsian representation. Since $Isom_0(\mathbb{ADS}^{2,1})$ is identified with $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, we can define the representation $\rho := (\rho_1, \rho_2) : \pi_1(S) \to Isom_0(\mathbb{ADS}^{2,1})$. However this representation does not act properly discontinuously on $\mathbb{ADS}^{2,1}$ (see [Mes07]). But Mess has shown the existence of a maximal (maximal in the sense of inclusion) convex subset $\Omega_\rho \subset \mathbb{ADS}^{2,1}$ such that ρ acts properly discontinuously on Ω_ρ. The quotient Ω_ρ/ρ is an $\mathbb{ADS}^{2,1}$ globally hyperbolic manifold (see [Mes07]). Before we state Mess theorem we will talk more about the maximal convex subset Ω_ρ.
We need to define the notion of proper achronal subset

Definition 3.2.8. *We say that a subset $X \subset \mathbb{ADS}^{2,1}$ is proper achronal if there is a space like plane P such that $X \subset (\mathbb{ADS}^{2,1} \subset \partial_\infty\mathbb{ADS}^{2,1}) \setminus \overline{P}$ and if X is achronal as a subset of $(\mathbb{ADS}^{2,1} \subset \partial_\infty\mathbb{ADS}^{2,1}) \setminus \overline{P}$.*

Proposition 3.2.9. *[BS20, Proposition 4.6.1] Let C be a proper achronal meridian in $\partial_\infty\mathbb{ADS}^{2,1}$, then $\Omega(C)$ is convex and if C is different from the boundary of a spacelike plane than $\Omega(C)$ is a proper convex set.*

In particular, we get that a proper achronal meridian is contained in an affine chart of $\mathbb{RP}^3$ which complement is a space-like plane.

Definition 3.2.10. *Let C be a proper achronal meridian in $\partial_\infty\mathbb{ADS}^{2,1}$, we define* CH(C) *to be the convex hull (in $\mathbb{RP}^3$) of C, which can be taken in an affine chart containing C.*

Note that by Proposition 3.2.9 the convex hull of C is contained in $\mathbb{ADS}^{2,1} \cup \partial_\infty\mathbb{ADS}^{2,1}$. Since $\Omega(C)$ is a convex subset of $\mathbb{ADS}^{2,1}$, then for any convex subset K of $\mathbb{ADS}^{2,1} \cup \partial_\infty\mathbb{ADS}^{2,1}$, such that $\partial_\infty\mathbb{ADS}^{2,1} \cap K := C$, we have that $\mathrm{CH(C)} \subset \mathrm{K} \subset \overline{\Omega(\mathrm{C})}$.
In the case when C is the boundary of a space-like plane P, the convex hull of C (which is $P \cup \partial_\infty P$) is a totally geodesic copy of $\mathbb{H}^2 \cup \partial_\infty\mathbb{H}^2$ in $\mathbb{ADS}^{2,1} \cup \partial_\infty\mathbb{ADS}^{2,1}$. In the case when C is not the boundary of a space-like plane, both of CH(C) and $\overline{\Omega(C)}$ are homeomorphic to the closed ball $\overline{B}^2$. Then in that case the boundary $\partial\mathrm{CH(C)} \cap \mathbb{ADS}^{2,1}$ (resp $\partial\Omega(C) \cap \mathbb{ADS}^{2,1}$) consists of the disjoint union of two topological discs $\partial^\pm\mathrm{CH(C)}$ (resp $\partial^\pm\Omega(C)$).
The Lorentzian metric of $\mathbb{ADS}^{2,1}$ induces a path metric on $\partial\mathrm{CH(C)}$, each of $\partial^\pm\mathrm{CH(C)}$

with this induced path metric is isometric to $\mathbb{H}^2$, and each of $\partial^{\pm}\mathrm{CH}(\mathrm{C})$ is a pleated surface. This gives two measured laminations on $\mathbb{H}^2$ which are the bending laminations of $\partial^{\pm}\mathrm{CH}(\mathrm{C})$), for more details and proves we refer to [Mes07] or to [BS20, Part II].
Let S be a closed hyperbolic surface and let $\rho_l, \rho_r : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ be two Fuchsian representations. There exists a unique quasi-symetric map $f : \partial_\infty\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2$ which is ρ_1, ρ_2 equivariant. We denote by ρ the representation $\rho := (\rho_1, \rho_2)$. Let C_ρ be the meridian in $\partial_\infty\mathbb{ADS}^{2,1}$, which is the graph of f. Mess has prove that $\Omega(C_\rho)$ is the maximal convex subset on which ρ acts properly discontinuously.

Theorem 3.2.11. *[Mes07] The set $\Omega(C_\rho)$ is the maximal convex subset in which ρ acts properly discontinuously, and the quotient of $\Omega(C_\rho)$ by ρ is a globally hyperbolic manifold diffeomorphic to $S \times \mathbb{R}$*

A maximal globally hyperbolic manifold plays the role of a complete Riemannian manifold, in the following sense

Definition 3.2.12. *Let (M, g) be an anti-de Sitter globally hyperbolic manifold, we say that it is maximal if any isometric embedding of (M, g) into another globally anti-de Sitter globally hyperbolic manifold (M', g') that sends a Cauchy surface of (M, g) into a Cauchy surface of (M', g'), is surjective.*

As a consequence we get the following corollary, see Corollary [BS20, Corollaary 5.1.5]

Corollary 3.2.13. *An anti-de Sitter globally hyperbolic manifold M is maximal if and only if $\widetilde{M}$ is isometric to the invisible domain of a proper achronal meridian in $\mathbb{ADS}^{2,1}$.*

In this thesis we will consider only anti-de Sitter Globally hyperbolic manifolds that are diffeomerphic to $S \times \mathbb{R}$ where S is a closed surface of genus bigger or equal to 2. In what follows we will show that these manifolds share many properties similar to the quasi-Fuchsian manifolds.
Define $\mathcal{GHM}(S)$ to be the deformation space of maximal globally hyperbolic spacetimes diffeomorphic to $S \times \mathbb{R}$, that is

$$\mathcal{GHM}(S) = \{g \text{ Globally hyperbolic metric on } S \times \mathbb{R}\} / \mathit{Diff}_0(S \times \mathbb{R})$$

Note that from the discussion above, every globally hyperbolic manifold is the quotient of a maximal covex subset Ω_ρ by a representation ρ of the form $\rho := (\rho_1, \rho_2) : \pi_1 \to \mathrm{PSL}(2,\mathbb{R})$, where ρ_1, ρ_2 are Fuchsian representations. This gives a natural identification between $\mathcal{GHM}(S)$ and $\mathcal{T}(S) \times \mathcal{T}(S)$, for proofs and more details we refer to [BS20, Section 5.5].

3.2.3 The convex core of globally hyperbolic spacetimes

Let (M, g) (or just M) be a maximal globally hyperbolic manifold which is isometric to $\Omega(C_\rho)/\rho$, where $\rho = (\rho_1, \rho_2) : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, and ρ_1, ρ_2 are Fuchsian representations. We begin by giving the definition of the convex core of M

Definition 3.2.14. *We call the convex core of M, the quotient of the convex hull of C_ρ by ρ and we denote it by $C(M)$. That is $C(M) = \mathrm{CH}(\mathrm{C}_\rho)/\rho$.*

If C_ρ is the boundary of a space-like plane, then $C(M)$ is one totally geodesic surface homeomorphic to S, in that case we say that M is Fuchsian.
As in the case of quasi-Fuchsian manifolds $C(M)$ has the same homotopy type as M. If C_ρ is not the boundary of a space-like plane, then $\partial C(M)$ the boundary of $C(M)$ consists of the disjoint union of two connected components $\partial^\pm C(M)$, each of them is homeomorphic to S.
Mess [Mes07] (or also see [BS20, Section 7.5]) has proved that the Lorentzian metric of a maximal globally hyperbolic manifold induces a hyperbolic path metric on $\partial C(M) \cap \mathbb{ADS}^{2,1}$, this gives two hyperbolic metrics h^+ and h^- on S. Mess gave the following conjecture

Conjecture 3.2.15. *[Mes07] Let $h^+, h^- \in \mathcal{T}(S)$ be two hyperbolic metrics, then there exists a unique maximal globally hyperbolic manifold $M \in \mathcal{MGH}(S)$ such that h^+ (resp h^-) is the induced metric on $\partial^+ C(M)$ (resp $\partial^- C(M)$).*

In [Dia13] the author has shown the existence part of that conjecture

Theorem 3.2.16. *[Dia13] Let $h^+, h^- \in \mathcal{T}(S)$ be two hyperbolic metrics, then there exists a maximal globally hyperbolic manifold $M \in \mathcal{MGH}(S)$ such that h^+ (resp h^-) is the induced metric on $\partial^+ C(M)$ (resp $\partial^- C(M)$).*

The uniqueness part still open question.

Also as in the case of quasi-Fuchsian manifolds each component $\partial^\pm C(M)$ is a pleated surface, and its bending lamination defines a measured lamination on S. Mess made the following conjecture

Conjecture 3.2.17. *Let L^+ and L^- be two measured lamiantions that fill S, then there exists a unique maximal globally hyperbolic manifold M such that L^+ (resp L^-) is the bending lamination of $\partial^+ C(M)$ (resp $\partial^- C(M)$).*

The existence part was confirmed by Bonsate and Schlenker in [BS12]

Theorem 3.2.18. *[BS12, Theorem 1.4] Let $L^+, L^- \in \mathcal{ML}(S)$ be two measured laminations that fill S, then there exists a maximal globally hyperbolic spacetime $M \in \mathcal{MGH}(S)$ such that L^+ (resp L^-) is the bending lamination of $\partial^+ C(M)$ (resp $\partial^- C(M)$).*

However the uniqueness part is still unsolved. In [BS12] the authors showed a partiel statement about uniqueness.

Theorem 3.2.19. *[BS12, Theorem 1.6] Let L^+ and L^- be two measured laminations that fill S, there exists $\epsilon > 0$ such that for any $0 < t < \epsilon$, there exists a unique maximal globally hyperbolic spacetime M such that tL^+ (resp tL^-) is the bending lamination of $\partial^+ C(M)$ (resp $\partial^- C(M)$).*

Unlike the quasi-Fuchsian case, even if $L^\pm$ are discrete measured laminations we don't know yet if the maximal globally hyperbolic manifold that realise them as bending laminations of the convex core is unique or not.

3.2.4 Foliations of $M \in \mathcal{MGH}(S)$

Let $M \in \mathcal{MGH}(S)$, assume that S is a smoothly embedded surface in M. We define B to be the shape operator of S in the same way as we define the Riemannian shape operator. We say that S has a constant mean curvature if $tr(B)$ is constant at any point of S, we denote then this mean curvature by H (the sign of H will depend on the orientation that we chose). Then the following theorem holds

Theorem 3.2.20. *[BBZ07] Every maximal globally hyperbolic anti-de Sitter manifold with compact Cauchy surface is uniquely foliated by closed CMC surfaces, where the mean curvature H varies in $(-\infty, +\infty)$.*

Moreover, for any H, there is a unique S_H, a smoothly embedded surface in M, that has a constant mean curvature equal to H.
We also have a theorem in the setting of globally hyperbolic manifolds, which is similar to the main theorem of [Lab91, Theorem 2]. Assume again that S is a smoothly embedded surface in a globally hyperbolic manifold $M \in \mathcal{MGH}(S)$, we say that S is a k-surface if its induced metric has a constant Gaussian curvature equal to k, or equivalently the determinant of B at any point of S is constant and equal $-1-k$.

Theorem 3.2.21. *[BBZ08] Let M be a maximal globally hyperbolic anti-de Sitter manifold with compact Cauchy surface. Then each connected component of $M \setminus C(M)$ is uniquely foliated by closed constant Gaussian curvature surfaces, where the Gaussian curvature K varies in $(0, +\infty)$.*

Assume that S is smoothly embedded in $M \in \mathcal{MGH}(S)$, and let B be its shape operator, which points in the direction of the differential of the Gauss map towards the future. We say that S is convex if $\det(B) > 0$ at any point on S, and we say that it is concave if $\det(B) < 0$. Tamburelli in [Tam18] has shown a Lorentzian analogue of the main theorem of [Lab92a].

Theorem 3.2.22. *[Tam18] Let S be a closed hyperbolic surface, and let g^+ and g^- be two Riemannian metrics on S that have curvature strictly smaller than -1, there exists $M \in \mathcal{MGH}(S)$, that contains a convex surface S^+ and a concave surface S^- such that the induced metric on S^+ is isotopic to g^+ and the induced metric on S^- is isotopic to g^-.*

However, we still don't know if the maximal globally hyperbolic spacetime M in Theorem 3.2.22 is unique or not (that would be an analogue of the main theorem of [Sch06b]). In [FS16] the authors suggested many possible parameterization of $\mathcal{MGH}(S)$. One can also ask if there is an analogue of [Mes23, Theorem C] in the setting of globally hyperbolic manifolds

Question 3.2.23. *Let S be a closed hyperbolic surface. Let $h \in \mathcal{T}(S)$ and let $L \in \mathcal{ML}(S)$. Is there a unique $M \in \mathcal{MGH}(S)$ such that the induced metric on $\partial^+ C(M)$ is isotopic to h and the bending lamination of $\partial^- C(M)$ is equal to L ?*

3.2.5 Mess diagram

Let $L \in \mathcal{ML}(S)$, we denote by E_l^L (resp E_r^L) the left (resp the right) earthquake that has shearing lamination equal to L. Recall that an earthquake has a natural action on $\mathcal{T}(S)$, that is it defines a map

$$E_l^L, E_r^L : \mathcal{T}(S) \to \mathcal{T}(S)$$

For more details how this map is defined we refer to [Mar16b, Section 7.2.2]. Mess has related the geometry of the convex cores of $\mathcal{MGH}(S)$ and the earthquake maps.
Recall that a globally hyperbolic manifold is the quotient of an invisible domain by the representation $\rho = (\rho_l, \rho_r)$. Also recall that $\mathcal{T}(S)$ can be identified with the Fuchsian representations from $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$, up to conjugation.

Theorem 3.2.24. *[Mes07] Let $M \in \mathcal{MGH}(S)$ be a globally hyperbolic manifold that is identified with (ρ_l, ρ_r) (that is ρ_l, ρ_r are the left and right representations that define M). Let $h^+ \in \mathcal{T}(S)$ (resp $h^- \in \mathcal{T}(S)$) be the induced path metric on $\partial^+ C(M)$ (resp $\partial^- C(M)$) and let $L^+ \in \mathcal{ML}(S)$ (resp $L^- \in \mathcal{ML}(S)$) be the bending lamination of $\partial^+ C(M)$ (resp $\partial^- C(M)$).*
Then

$$\rho_l = E_l^{L^+}(h^+), \rho_r = E_r^{L^+}(h^+)$$
$$\rho_l = E_l^{L^+}(h^+), \rho_r = E_r^{L^+}(h^+)$$

Then if we reformulate Theorem 3.2.18 using Theorem 3.2.24 we obtain the following theorems.

Theorem 3.2.25. *Let $\lambda, \mu \in \mathcal{ML}(S)$ be two measured laminations which fill S. Then $E_\lambda^l \circ E_\mu^l : \mathcal{T}(S) \to \mathcal{T}(S)$ has a fixed point in $\mathcal{T}(S)$.*

3.3 Globally hyperbolic manifolds from universal point of view

Recall that for any maximal globally hyperbolic manifold $M \in \mathcal{MGH}(S)$, there exists a left and right fuchsian representations $\rho_l, \rho_r : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ such that M is isometric to $\Omega(C_\rho)/\rho$, where $\rho = (\rho_l, \rho_r)$, C_ρ is the meridian which is the graph of the unique ρ_l, ρ_r equivariant quasi-symmetric map, and $\Omega(C_\rho)$ is the invisible domain (which is convex) of C_ρ.
In this section we will study the invisible domains of quasi-circles, or more generally convex subsets of $\mathbb{ADS}^{2,1}$ that their closures intersect the boundary at infinity of $\mathbb{ADS}^{2,1}$ at a quasi-circle, these objects are lifts of globally hyperbolic manifolds.
Here is a universal version of Theorem 3.2.20

Theorem 3.3.1. *[Tam19, Theorem 3.1] Let $C \subset \partial_\infty \mathbb{ADS}^{2,1}$ be a meridian which is not the boundary of a space-like plane. Then $\Omega(C)$ is foliated by constant mean curvature surfaces S_H, such that each surface S_H has a constant mean curvature equal to H, and its boundary at infinity is equal to C, and H vary in $(-\infty, +\infty)$.*

The next theorem shows that each connected component of the invisible domain of a quasi-circle minus the convex hull of the quasi-circle is foliated by K-surfaces that intersect the ideal boundary of $\mathbb{ADS}^{2,1}$ at the quasi-circle. This a universal version of Theorem 3.2.21

Theorem 3.3.2. *[BS18] Let $C \subset \partial_\infty \mathbb{ADS}^{2,1}$ be an acausal meridian and let $k \in (-\infty, -1)$. Then:*

- *There are exactly two properly embedded K-surfaces $S_k^{\pm}(C)$ in $\mathbb{ADS}^{2,1}$ spanning C. These are each homeomorphic to disks, are disjoint, and bound a closed properly convex region $CK(C)$ in $\mathbb{ADS}^{2,1}$ which contains a neighborhood of the convex hull $CH(C)$.*
- *Further, the K-surfaces spanning C, for $K \in (-\infty, -1)$, form a foliation of $E(C)\backslash CH(C)$.*
- *Moreover, C is a quasicircle if and only if any spanning K-surface has bounded principal curvatures.*

Then as in the hyperbolic space there is two isometries $V_k^{\pm} : \mathbb{H}_k^2 \to S_k^{\pm}$, these isometries extends to a homeomorphism from $\partial_\infty \mathbb{H}^2$ to C.

Proposition 3.3.3. *[BDMS21a, Propossition 7.2] For any quasi-circle C of the $\partial_\infty \mathbb{ADS}^{2,1}$, the Lorentzian metric if $\mathbb{ADS}^{2,1}$ induces a complete space-like metric on each of $S_K^{\pm}(C)$ for any $K < -1$. And any isometry $V_k^{\pm} : \mathbb{H}_k^2 \to S_k^{\pm}$ has a unique continous extension to a homeomorphism $V_k : \mathbb{H}_k^2 \cup \partial_\infty \mathbb{H}_k^2 \to S_k^{\pm} \cup C$.*
That also holds when $k = -1$, in that case S_{-1}^{pm} are the boundary of the convex hull of C.

Note that as in the hyperbolic space we can define the gluing map as follows

$$\Phi_{C,K} = \partial(V_K^-)^{-1} \circ V_K^+$$

Then the following theorem, which is seen as a universal analogue of Theorem 3.2.22 when the metrics have constant Gaussian curvature, holds

Theorem 3.3.4. *[BDMS21a, Theorem E] Let $k \in (-\infty, -1)$. For any normalized quasi-symmetric homeomorphism f there is a normalized quasi-circle C of $\partial_\infty \mathbb{ADS}^{2,1}$ such that f is the gluing map between the past and future k surfaces that spain C .*

The next Theorem is a universal version of Theorem 3.2.19.

Theorem 3.3.5. *[BDMS21a, Theorem D] For any normalized quasi-symmetric homeomorphism f, there is a normalized quasi-circle C of $\partial_\infty \mathbb{ADS}^{2,1}$ such that f is the gluing map of the boundaries of the convex hull of C.*

The uniqueness of the quasi-circle still unkown in both of Theorem 3.3.4 and Theorem 3.3.5. The authors in [MS21] gave an universal version of Theorem 3.2.18

Theorem 3.3.6. *[MS21, Theorem B] Let L^+ and L^- be two bounded measured laminations in $\mathbb{H}^2$ that strongly fill. There exists a parameterized quasi-circle $u : \mathbb{RP}^1 \to \partial_\infty \mathbb{ADS}^{2,1}$ such that the pull back of upper (resp lower) bending laminations of the boundary of the convex hull of $Im(u)$ by u is L^+ (resp L^-).*

However we still can ask the uniqueness in Theorem 3.3.6, and also the theorem does not give a full characterization of the pair of measured laminations that can be the bending laminations of the convex hull of such a quasi-cirlce.

3.3.1 The left and right projections

Let S be a space-like surface in $\mathbb{ADS}^{2,1}$. Recall that any time like geodesic in $\mathbb{ADS}^{2,1}$ (we take the $\mathrm{PSL}(2,\mathbb{R})$ model) is of the form $L_{x,x'} = \{A \in PSL(2,\mathbb{R}), Ax = x'\}$. Then the space-like surface S defines a map

$$\begin{aligned}\pi : S \to & \mathbb{H}^2 \times \mathbb{H}^2 \\ s \mapsto & (\pi_l(s), \pi_r(s))\end{aligned}$$

where $L_{(\pi_l(s),\pi_r(s))}$ is the unique time-like geodesic that goes thought s. When the intrinsic curvature of S does not vanish (especially when S is convex) the maps π_l and π_r are local diffeomorphism. Krasnov and Schlenker in [KS07b] have computed the pull-back of the hyperbolic metric of $\mathbb{H}^2$ by each of Π_l and Π_r.

Theorem 3.3.7. *[KS07b] The following equalities hold,*

$$\begin{aligned}\Pi_l^*(h_{-1})(v,w) = I\left((E + J_I B)v, (E + J_I B)w\right) \\ \Pi_r^*(h_{-1})(v,w) = I\left((E - J_I B)v, (E - J_I B)w\right)\end{aligned}$$

where E denotes the identity operator, and J_I is the complex structure over TS induced by I. (Note that while both J_I and B depend on the choice of an orientation on S, the product is independent of the orientation).

Also the left and right projections extends to the boundary at infinity.

Lemma 3.3.8. *[BS10, Lemma 3.18 and Remark 3.19] Let S be a properly embedded space-like convex surface of $\mathbb{ADS}^{2,1}$ such that $\partial_\infty S$ is an acausal meridian C. In that case, each of the maps Π_l and Π_r is a diffeomorphism onto the hyperbolic plane, and it has a continuous extension $\pi_l, \pi_r : C \to \partial_\infty \mathbb{H}^2$ that satisfies $\pi_l(x, f(x)) = x$ and $\pi_r(x, f(x)) = f(x)$.*

In the case when $k = -1$ (that is in the case of $\partial^{\pm} C(M)$), the maps Π_l, Π_r are not well defined, because $\partial^{\pm} C(M)$ are not smoothly embedded in M. Howeever we know that $\partial^{\pm} C(M)$ are pleated surfaces, in particluar they are C^1 outside the bending locus. The points where $\partial^{\pm} C(M)$ is C^1 are dense.
It follows that $\partial^+ CH(C)$ is isometric to an open subset $\mathcal{U}^+$ (resp. $\mathcal{U}^-$) of the hyperbolic plane that is bounded by a (possibly empty) set of disjoint geodesics. An isometry $V^+_{C,-1} : \mathcal{U}^+ \to \partial^+ CH(C)$ (resp. $V^-_{C,-1} : \mathcal{U}^- \to \partial^-$CH(C)) is given by a bending map, with bending determined by a measured geodesic lamination λ^+ of $\mathcal{U}^+$ (resp. λ^- of $\mathcal{U}^-$). The projection maps Π_l^+ and Π_r^+ (resp. Π_l^- and Π_r^-) for $\partial^+ CH(C)$ (resp. for $\partial^- CH(C)$) are well defined only at the dense set of points along which $\partial^+ CH(C)$ (resp. $\partial^- CH(C)$) is C^1, this is the complement of the leaves of λ^+ (resp. of λ^-) which have positive measure. Mess observed the following relationship between the bending measure and earthquakes., and later Benedetti-Bonsante have extended that to the universal case

Theorem 3.3.9. *[RB+09] The compositions $\Pi_l^+ \circ V^+_{C,-1} : \mathcal{U}^+ \to \mathbb{H}^2$, $\Pi_r^+ \circ V^+_{C,-1} : \mathcal{U}^+ \to \mathcal{H}^2$, $\Pi_l^- \circ V^-_{C,-1} : \mathcal{U}^- \to \mathbb{H}^2$, and $\Pi_r^- \circ V^-_{C,-1} : \mathcal{U}^- \to \mathcal{H}^2$ are surjective earthquake maps, with $\Pi_l^+ \circ V_+^{C,-1} : U_+ \to H^2$ (resp. $\Pi_r^- \circ V_-^{C,-1} : U_- \to H^2$) shearing to the left along λ_+ (resp. along λ_-) and $\Pi_l^- \circ V^-_{C,-1} : U^- \to \mathbb{H}^2$ (resp. $\Pi_r^+ \circ V^+_{C,-1} : U^+ \to \mathbb{H}^2$) shearing to the right along λ_- (resp. along λ_+).*

3.3.2 The width of the convex hull of a meridian

The width of the convex hull of a meridian is the time distance between the two boundaries.

Definition 3.3.10. *Let $C \subset \partial\mathbb{ADS}^{2,1}$ be an achronal meridian. The width $w(C)$ of C is the supremum of the time distance between a point of $\partial^- CH(C)$ and $\partial^+ CH(C)$.*

It happens that there is a strong relation between the width of the convex hull of a quasi-circle C and the quasi-symmetric constant of C.

Proposition 3.3.11. *Let $C \subset \partial_\infty\mathbb{ADS}^{2,1}$ be an acausal meridian. Then C is a quasi-circle if and only if $w(C) < \frac{\pi}{2}$. Furthermore, if C_n is a sequence of quasi-circles whose optimal quasi-symmetric constant diverges to infinity, then there exist isometries $\phi_n \in Isom_{\mathbb{ADS}^{2,1}}$ such that $\phi_n(C_n)$ converges to the rhombus C^*, so that in particular $w(C_n) \to \frac{\pi}{2}$.*

3.4 Globally hyperbolic convex subsets

Let Ω be a convex subset of $\mathbb{ADS}^{2,1} \cup \partial_\infty\mathbb{ADS}^{2,1}$ which is homeomorphic to the sphere with a spacelike boundary and which intersects $\partial_\infty\mathbb{ADS}^{2,1}$ at a quasicircle. Note that the lift of any globally hyperbolic manifold (not maximal) lifts to a such convex subset. In the next definition we give the notion of **globally hyperbolic convex subset**.

Definition 3.4.1. *Let Ω be a convex subset of $\mathbb{ADS}^{2,1} \cup \partial_\infty\mathbb{ADS}^{2,1}$ such that*

- *Ω is homeomorphic to the ball.*
- *$\partial_\infty\Omega$ is a quasi-circle.*
- *$\partial\Omega \cap \mathbb{ADS}^{2,1}$ is the disjoint union of two smooth spacelike disks $\partial^\pm\Omega$.*
- *The induced metric on each of $\partial^\pm\Omega$ has curvature in the interval $(-\frac{1}{\epsilon}, -1-\epsilon)$, for some $\epsilon > 0$.*

We call a convex Ω with the preceding properties by a ***globally hyperbolic convex subset****.*

Note that the intersection of the boundary of Ω with $\mathbb{ADS}^{2,1}$ consists of the disjoint union of two topological disks, each of them is space-like ans has a convex induced metric. That is $\partial\Omega \cap \mathbb{ADS}^{2,1} = \partial^\pm\Omega$, where each of $\partial^\pm\Omega$ is a space-like topological disk, and the induced metric on it has curvature that belongs to the interval $\left[-\frac{1}{\epsilon}, -1\right]$.
Note that in the case when C is a quasi-circle, then $S_K^+ \cup S_K^- \cup C$ forms the boundary of a globally hyperbolic convex subset Ω. In what follows, we generalise Theorem 3.3.4 to a more general case, without assuming that the induced metric on the boundary of Ω has constant curvature (See Figure 3.4).

Question 3.4.2. *Let h^+ and h^- be two complete, conformal Riemannian metrics on the disc $\mathbb{D}$ that have Gaussian curvature belonging to the interval $\left[-\frac{1}{\epsilon}, -1-\epsilon\right]$ for some $\epsilon > 0$. Let f be a quasi-symmetric map. Is there a unique (up to isometry) globally hyperbolic convex subset Ω such that the induced metric on $\partial^+\Omega$ (resp $\partial^-\Omega$) is isometric to $(\mathbb{D}, h^+)$ (resp $(\mathbb{D}, h^-)$) and the gluing map equal to f.*

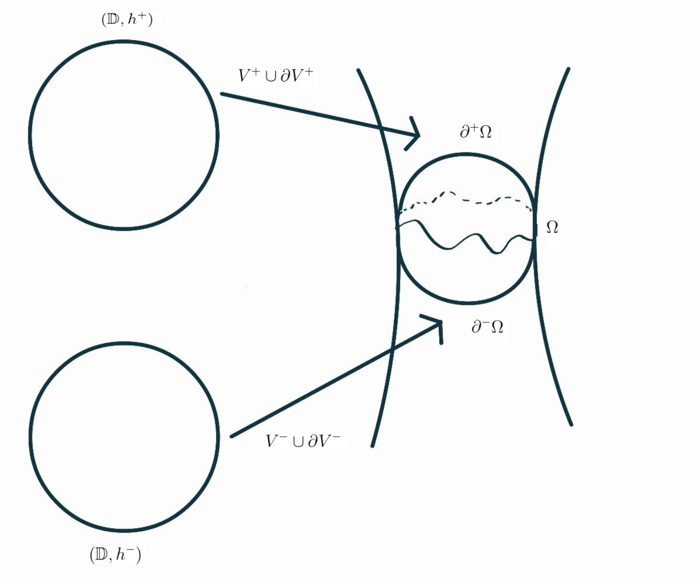


Figure 3.2: The statement of Question 3.4.2 is a universal version of Theorem 3.2.22.

We give the following theorem

Theorem 3.4.3. *Let h^+ and h^- be two complete, conformal metrics on the disc $\mathbb{D}$ that have curvatures in an interval of the form $(-\frac{1}{\epsilon}, -1-\epsilon)$, for some $\epsilon > 0$. Assume moreover that any derivative of h^+ or h^- of order p is bounded by some positive number M_p. Let f be a normalized quasi-symmetric map. Then there exists a globally hyperbolic convex subset Ω such that the induced metric on $\partial^+\Omega$ is isometric to $(\mathbb{D}, h^+)$, the induced metric on $\partial^-\Omega$ is isometric to $(\mathbb{D}, h^-)$, and the gluing map is equal to f.*

In this section we will talk more about the geometry of the boundary of Ω. In the rest of the section we will consider globally hyperbolic manifolds that the induced metrics on their boundary are isometric to metrics on $\mathbb{D}$ that have bounded derivatives at any order with respect to the hyperbolic metric as in definition 0.4.1.
We will follows the proofs and arguments of [BDMS21a, Section 7] that the authors gave when the induced metrics on the boundary of Ω have constant curvatures, and generalise them to the case when the induced metrics have bounded derivatives at any order with respect to the hyperbolic metric.

We need the following lemma

Lemma 3.4.4. *[BDMS21a, Lemma 7.9] Let S_n be a sequence of properly embedded space-like disks spanning a sequence of k-quasi-circles C_n. Then if C_n converge to the union $\{p\} \times \mathbb{RP}^1 \cup \mathbb{RP}^1 \times \{q\}$ of a line of the left ruling and a line of the right ruling, then S_n converge to the lightlike plane with boundary at infinity $\{p\} \times \mathbb{RP}^1 \cup \mathbb{RP}^1 \times \{q\}$. If C_n converges to a k-quasi-circle C, then up to a subsequence, S_n converges to a locally convex properly embedded surface spanning the curve C.*

Then we use similar arguments to [BDMS21a, Proposition 7.10] and [BDMS21a, Lemma 7.11] to show the following proposition:

Proposition 3.4.5. *Let $k > 1$ and $\epsilon > 0$. Let $S \subset \mathbb{AdS}^{2,1}$ be a properly embedded convex spacelike disk spanning a k-quasicircle C. Assume that S is isometric to $(\mathbb{D}, h)$ where h is a conformal complete metric that has curvature in $\left(-\frac{1}{\epsilon}, -1-\epsilon\right)$. Assume that there is a sequence $(M_p)_{p\in\mathbb{N}}$ of positive numbers such that each derivative of h of order p is bounded by M_p. Then there exists D that depends on k, ϵ, and $(M_p)_{p\in\mathbb{N}}$ such that the principal curvatures of S are in the interval $\left(\frac{1}{D}, D\right)$.*

Proof. We argue by contradiction. Assume the existence of a sequence of properly embedded convex spacelike disks S_n spanning k-quasicircles C_n, satisfying the hypothesis of Proposition 3.4.5. Assume there exists a sequence of points $p_n \in S_n$ such that one of the principal curvatures at p_n goes to infinity. Note that the product of the principal curvatures is bounded (because the curvatures belong to $(-\frac{1}{\epsilon}, \epsilon)$). If one principal curvature goes to 0, then the other goes to ∞. So assume without loss of generality that the largest principal curvature at p_n goes to ∞.
Up to normalization by isometries, we can assume that p_n are equal to a fixed point p, and $T_{p_n}S_n$ are equal to a fixed space-like tangent plane. By Proposition 3.4.4 and since all S_n are tangent to the same space-like tangent plan, the surfaces S_n converge in the Hausdorff sense to a locally convex properly embedded surface spanning a k-quasi-circle C.
Let $V_n : (\mathbb{D}, h_n) \to S_n$ be isometric embeddings. Let x_n be a sequence of points in the disc $\mathbb{D}$ such that $V_n(x_n) = p$. Let $x_0 \in \mathbb{D}$ be a fixed point, and let g_n be a sequence of elements of $\mathrm{PSL}(2,\mathbb{R})$ such that $g_n(x_n) = x_0$.
We denote $h'_n := g_n^*(h_n)$. Note that the metrics h'_n have uniformly bounded derivatives. That is the bounds depend only on the order of the derivatives and not on n or the points, because they are pullbacks of the metrics h_n by hyperbolic isometries (where h_n has uniformly bounded derivatives on the disc at any order).
Then, up to extracting a subsequence, h'_n converges smoothly (C^∞ on compact subsets) to a metric h'_∞. By [Sch96, Theorem 5.6], either $\phi_n : (\mathbb{D}, h'_n) \to S_n$ converge smoothly to an isometric embedding $\phi_\infty : (\mathbb{D}, h_\infty) \to S_\infty$, or there is a complete geodesic γ in $(\mathbb{D}, h_\infty)$ which is sent by ϕ_∞ to a spacelike geodesic $\Gamma \in \mathbb{AdS}^{2,1}$. Moreover, in the latter case, the integral of the mean curvatures of ϕ_n goes to ∞ at any point in a neighborhood of γ. According to [Sch96, Lemma 5.4], the length of the third fundamental form for ϕ_n of any geodesic segment transverse to γ also goes to ∞. This implies that the limit surface S_∞ must be contained in the union of the past-directed lightlike half-planes bounded by Γ. However, this contradicts the fact that the ideal boundary of S_∞ is a quasi-circle (Proposition 3.4.4).
Therefore, the isometries $\phi_n : (\mathbb{D}, h'_n) \to S_n$ must converge, up to extracting a subsequence, to an isometric embedding $\psi_\infty : (\mathbb{D}, h'_\infty) \to S_\infty$. This contradiction confirms that the principal curvatures of S_n at p cannot go to ∞. □

From that we deduce that the composition of the left and right projections with the isometries that parameterize the surfaces $\partial^\pm\Omega$ From that, we will deduce that the gluing maps are well defined. Before that, we need to show the following lemma

Lemma 3.4.6. *Let Ω be a globally hyperbolic convex subset spanning a k-quasi-circle at infinity, and let the induced metric on its spacelike boundaries have curvatures in $(-\frac{1}{\epsilon}, -1-\epsilon)$. Let $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm\Omega$ be isometries. Assume that each derivative of $h^\pm$ at order p is bounded by M_p on $\mathbb{D}$. Let $\Pi^\pm_l$, $\Pi^\pm_r$ be the left and right projections.*

Then there exists $A > 0$ depending only on ϵ, k, and $(M_p)_{p\in\mathbb{N}}$ such that $\Pi_l^\pm \circ V^\pm$ and $\Pi_r^\pm \circ V^\pm$ are A-quasi-isometries.

Proof. From Proposition 3.4.5, there exists $D > 0$ depending only on k, the quasi-symmetric constant of $\partial_\infty\Omega$, ϵ, and $(M_p)_{p\in\mathbb{N}}$ such that the principal curvatures of $\partial^\pm\Omega$ are in the interval $\left[\frac{1}{D}, D\right]$. By Theorem 3.3.7, we obtain that the projection maps $\Pi_l^\pm$, $\Pi_r^\pm$ are A-bilipschitz for some constant A that depends on D (which in turn depends on ϵ, k, and $(M_p)_{p\in\mathbb{N}}$).
Therefore, the maps $\Pi_l^\pm \circ V^\pm : (\mathbb{D}, h^\pm) \to \mathbb{H}^2$ are A-bilipschitz diffeomorphisms. The same argument applies to $\Pi_r^\pm$. $\square$

Note that from the proof of Lemma 3.4.6, the maps $\Pi_l^\pm \circ V^\pm$ extend to a homeomorphism $\partial(\Pi_l^\pm \circ V^\pm) : \partial\mathbb{D} \to \partial_\infty\mathbb{H}^2$.
We define $\partial V^\pm := \pi_l^{-1} \circ \partial(\Pi_l^\pm \circ V^\pm)$, where π_l is the extension of Π_l to the boundary at infinity.
For a prof when $K = -1$ (that is when Ω is the convex hull of a quasi-circle) we refer to [BDMS21a, Section 7].

Definition 3.4.7. *Let Ω be a globally hyperbolic convex subset. Let $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm\Omega$ be isometries.*
We define the gluing map to be $\Phi_\Omega = (\partial V^-)^{-1} \circ \partial V^+$.
Note that the isometries $V^\pm$ and the gluing map Φ_Ω are defined up to composition by Möbius maps. Also note that Möbius maps do not necessarily preserve the metrics $h^\pm$. However, we still have a uniquely defined normalized gluing map.
Furthermore, if $\partial_\infty\Omega$ passes through $0, 1, \infty$, there exist unique metrics $h^\pm$ and isometries $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm\Omega$ such that $\partial V^\pm(p) = (p, p)$ for any $p = 0, 1, \infty$.

The gluing maps are always quasi-symmetric

Proposition 3.4.8. *Let Ω be a globally hyperbolic convex subset. The gluing map is quasi-symmetric.*

Proof. It follows from the fact that $\Phi_\Omega = \partial((\Pi_l^- \circ V^-)^{-1} \circ (\Pi_l^+ \circ V^+))$, and each of $\Pi_l^- \circ V^-$ and $\Pi_l^+ \circ V^+$ is a bilipchitz diffeomorphism (then quasi-isometric). $\square$

Proposition 3.4.9. *Let $k > 1$ and $\epsilon > 0$. Let Ω be a globally hyperbolic convex subset. Assume that Ω spans a k quasi-circle and assume the existence of the isometries $V^\pm$: $(\mathbb{D}, h^\pm) \to \partial^\pm\Omega$ such that the curvatures of $h^\pm$ are in the interval $(-\frac{1}{\epsilon}, -1-\epsilon)$. Then $V^\pm$ extends to a homeomorphism $\overline{V}^\pm : \mathbb{RP}^1 \cup \mathbb{D} \to \partial^\pm\Omega \cup \partial_\infty\Omega$.*

We denote the gluing map of the convex Ω by Φ_Ω.
We will need the following lemma later.

Lemma 3.4.10. *[BDMS21a, Lemma 4.9] For any constant $A > 1$ and for any $x \in \mathbb{H}^2$, there exists a compact region Q of $\mathbb{H}^2$ such that if f is a normalized A-quasi-isometry of $\mathbb{H}^2$, then $f(x) \in Q$.*

Now we proceed the proof that the correspondence between globally hyperbolic convex subsets and the gluing maps is proper and continuous.

Proposition 3.4.11. *Assume that all the metrics on the disc in the statement are complete and conformal, and they have curvatures in an interval of the form $(-\frac{1}{\epsilon}, -1-\epsilon)$, for some $\epsilon > 0$. Let Ω_n be a sequence of globally hyperbolic convex subsets such that:*

- *There are isometries $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm\Omega_n$.*
- *The gluing maps Φ_{Ω_n} are normalized and uniformly quasi-symmetric and they converge in the C^0 topology to a k quasi-symmetric map f.*
- *The metrics $h_n^\pm$ converge C^∞ uniformly on compact subsets to some metrics $h^\pm$ on $\mathbb{D}$.*
- *Any derivative of h_n or $h^\pm$ of order p is bounded by $M_p > 0$ on the disc $\mathbb{D}$ (M_p does not depend on n).*

Then the globally hyperbolic convex subsets Ω_n converge to a globally hyperbolic convex subset Ω in the Hausdorff sense, and the isometries $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm\Omega_n$ converge to isometries $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm\Omega$, and Φ_Ω, the gluing map of Ω, is equal to f.

We will split the proof of Proposition 3.4.11 into lemmas and propositions.

3.4.1 Properness of the gluing maps

In this section we prove the properness of the gluing maps.

Proposition 3.4.12. *Let $\epsilon > 0$ and $(M_p)_{p\in\mathbb{N}}$ a sequence of positive numbers. For any $k > 1$ there exists $k' > 1$ such that for any globally hyperbolic convex subset Ω in which the induced metrics on $\partial^\pm\Omega$ have curvatures in $(-\frac{1}{\epsilon}, -1-\epsilon)$ and any derivative of the metric of order p is bounded by M_p. We assume that the gluing map is normalized. If the gluing map Φ_Ω is k-quasi-symmetric then $\partial_\infty\Omega$ is a k' quasi-circle.*

Before proceeding with the proof, let us give the following lemma.

Lemma 3.4.13. *[BDMS21a, Lemma 8.4] Let P be a totally geodesic spacelike plane. We denote by P^+ (resp P^-) the union of all future-oriented (resp past-oriented) timelike geodesic segments of length $\frac{\pi}{2}$ starting orthogonally from P.*

- *let $S \subset \mathbb{ADS}^{2,1}$ be a spacelike past convex surface, and let P be a spacelike totally geodesic plane. In the neighborhood of all points $x \in S \cap P$, the intersection $S \cap P^+$ is locally convex in the induced metric on S.*
- *Let $S \subset \mathbb{ADS}^{2,1}$ be a spacelike future convex surface, and let P be a spacelike totally geodesic plane. In the neighborhood of all points $x \in S \cap P$, the intersection $S \cap P^-$ is locally convex in the induced metric on S.*

Now we give a proof of Proposition 3.4.12, note that the proof is similar to the proof of [BDMS21a, Proposition 8.3].

Proof. We argue by contradiction. Suppose that Ω_n is a sequence of normalized globally hyperbolic convex subsets such that k'_n, the optimal quasi-symmetric constant of their ideal boundary $\partial_\infty\Omega_n$, converges to ∞. We will prove that k_n, the quasi-symmetric constants of their gluing maps Φ_{Ω_n}, must also go to ∞. Let $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm\Omega_n$ be isometries such that the normalized gluing map $\Phi_{\Omega_n} = (\partial V_n^-)^{-1} \circ (\partial V_n^+)$. By applying an isometry of $\mathbb{ADS}^{2,1}$ to Ω_n, we can assume that $\partial V_n^+(p) = (p,p)$ for any $p = 0, 1, \infty$. Since we assumed that the gluing map is normalized, it follows that $\partial V_n^-(p) = (p,p)$ for any $p = 0, 1, \infty$. Given $k'_n \to \infty$, Proposition 3.3.11 implies that the width $w(C_n)$ of $C_n := \partial_\infty\Omega_n$ goes to $\frac{\pi}{2}$. After adjusting by isometries, we can assume that C_n converges to a rhombus $C_\diamond$ as in [BDMS21a, Example 6.7] (note that even after applying these isometries, we can still assume that C_n and $C_\diamond$ are normalized).

Let's work in the projective model of $\mathbb{ADS}^{2,1}$ with the coordinates so that in the affine chart $x_4 = 1$, $\partial_\infty\mathbb{ADS}^{2,1}$ is the hyperboloid $x_1^2 + x_2^2 = x_3^2 + 1$, with $\mathbb{ADS}^{2,1}$ seen as the region $x_1^2 + x_2^2 < x_3^2 + 1$. We may then arrange that the vertices of $C_\diamond$ are the points $(\pm\sqrt{2}, 0, -1)$ and $(0, \pm\sqrt{2}, 1)$. Since $\partial^+\Omega_n$ is in the future of $\mathrm{CH(C_n)}$ but contained in the invisible domain $E(C_n)$, and since both $\overline{E(C_n)}$ and $\mathrm{CH(C_n)}$ converge to $\mathrm{CH(C_\diamond)}$, we have that $\partial^+\Omega_n$ converge to S_∞^+, where S_∞^+ is the union of the two future faces of $\mathrm{CH(C_\diamond)}$. Similarly, $\partial^-\Omega_n$ converge to S_∞^-, where S_∞^- is the union of the two past faces of $\mathrm{CH(C_\diamond)}$.

Let P be the spacelike totally geodesic plane equal to the intersection with $\mathbb{ADS}^{2,1}$ of the plane $z = 0$ in $\mathbb{R}^3$. Let $a = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$, $b = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$, $c = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$, and $d = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$. Thus, a, b, c, d are the intersection points of ∂P with $C_\diamond$ occurring in the cyclic order a, b, c, d.

For all $n \in \mathbb{N}$, let a_n, b_n, c_n, d_n be the intersection points of C_n with P, such that $a_n \to a$, $b_n \to b$, $c_n \to c$, $d_n \to d$. Define $a_n^\pm, b_n^\pm, c_n^\pm, d_n^\pm$ by the equalities $\partial V_n^\pm(a_n) = a_n^\pm$, $\partial V_n^\pm(b_n) = b_n^\pm$, $\partial V_n^\pm(c_n) = c_n^\pm$, $\partial V_n^\pm(d_n) = d_n^\pm$. Note that $a_n^-, b_n^-, c_n^-, d_n^-$ are the images of $a_n^+, b_n^+, c_n^+, d_n^+$ under Φ_{Ω_n}.

We will show that the cross-ratio of $a_n^+, b_n^+, c_n^+, d_n^+$ tends to 0, while the cross-ratio of $a_n^-, b_n^-, c_n^-, d_n^-$ tends to infinity.

Let P^+ be the future of P as in Lemma 3.4.13. Let Q be the timelike plane defined by the equation $x_2 = 0$. Then the path $Q \cap S_\infty^+ \cap P^+$ from $Q \cap \overline{ab}$ to $Q \cap \overline{cd}$ along the piecewise lightlike geodesic $Q \cap S_\infty^+$ has length zero in the $\mathbb{ADS}^{2,1}$ metric. This implies that the lengths of the paths $Q \cap S_n^+ \cap P^+$ converge to zero.

Note that $S_n^+ \cap P^+$ has a locally convex boundary with respect to the induced metric of S_n^+. Then the set $U_n^+ := (V_n^+)^{-1}(S_n^+ \cap P^+)$ is a region of $(\mathbb{D}, h_n^+)$ that has a locally convex boundary, so it is globally convex. Since a_n, b_n, c_n, d_n are the intersection points of C_n and P^+ by Proposition 3.4.9, the set U_n^+ contains the points $\partial V_n^+(a_n) = a_n^+$, $\partial V_n^+(b_n) = b_n^+$, $\partial V_n^+(c_n) = c_n^+$, $\partial V_n^+(d_n) = d_n^+$ in its ideal boundary. Also, since U_n^+ is convex, it contains the geodesics (with respect to the metric h_n^+) $\overline{a_n^+ b_n^+}$ and $\overline{c_n^+ d_n^+}$, which means that $S_n^+ \cap P^+$ contains the geodesics $\gamma(\overline{a_n^+ b_n^+}) = V_n^+(\overline{a_n^+ b_n^+})$ and $\gamma(\overline{c_n^+ d_n^+}) = V_n^+(\overline{c_n^+ d_n^+})$.

Also note that the path $P \cap S_n^+ \cap P_+$ crosses the two geodesics $\gamma(\overline{a_n^+ b_n^+})$ and $\gamma(\overline{c_n^+ d_n^+})$. Recall that the length of $P \cap S_n^+ \cap P_+$ converges to 0, so the distance between the two geodesics $\gamma(\overline{a_n^+ b_n^+})$ and $\gamma(\overline{c_n^+ d_n^+})$ goes to zero. It follows that the cross-ratio of $(a_n^+, b_n^+, c_n^+, d_n^+)$ goes to 0.

We apply a similar argument on the surfaces S_n^-. We denote by T the timelike plane defined by the equation $x_1 = 0$. The path $T \cap S_\infty^- \cap P^-$ has length zero. By similar arguments as above (applying Lemma 3.4.13 on $P^- \cap S_\infty^-$ and considering the path $T \cap$

$S_\infty^- \cap P^-$), the cross-ratio $(d_n^-, a_n^-, b_n^-, c_n^-)$ converges to 0. This implies that the cross-ratio of $(a_n^-, b_n^-, c_n^-, d_n^-)$ converges to ∞. We deduce that the quasi-symmetric constants of the gluing maps Φ_{Ω_n} diverge.

$\square$

3.4.2 Continuity of the gluing maps

The proof that we give for the next proposition is similar to the prof of [BDMS21a, Proposition 8.2]

Proposition 3.4.14. *Let Ω_n be a sequence of globally hyperbolic convex subsets. Assume the existence of isometries $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm \Omega_n$ such that the metrics $h_n^\pm$ converge C^∞ on compact subsets to $h^\pm$, and all the metrics have curvatures in $(-\frac{1}{\epsilon}, -1-\epsilon)$ for some $\epsilon > 0$. Assume there exists a sequence of positive numbers $(M_p)_{p\in\mathbb{N}}$ such that any derivative of h_n or h of order p is bounded by M_p. Also assume that all the gluing maps Φ_{Ω_n} are normalized and uniformly quasi-symmetric (they are all k-quasi-symmetric for the same k), and that Φ_{Ω_n} converge to a quasi-symmetric map f in the C^0 topology. Assume further that Ω_n converge (which is always possible, up to extracting a subsequence, after applying isometries on Ω_n) in the Hausdorff topology to a hyperbolic convex subset Ω.*
Then $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm \Omega_n$ converge to isometries $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm \Omega$, and the gluing map Φ_Ω of Ω is equal to f.

Proof. Up to normalizing by isometries of $\mathbb{ADS}^{2,1}$, we can assume that $\partial V_n^+(p) = (p, p)$ for any $p = 0, 1, \infty$. Note that after this normalization Ω_n converge in the Hausdorff topology to some convex subset Ω that has a quasi-circle as ideal boundary. Since we assumed that the gluing maps Φ_{Ω_n} are normalized, we will also have that $\partial V_n^-(p) = (p, p)$ for any $p = 0, 1, \infty$.
We will show that each of the isometries $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm \Omega_n$ converges to an isometry $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm \Omega$. To do that, we will show that there exists $x_0 \in \mathbb{D}$ such that $V_n^\pm(x_0)$ is in a compact subset of $\mathbb{ADS}^{2,1}$, and their convergence will follow from [Sch96, Theorem 5.4].
We denote by $\Pi_{n,l}^\pm$ (resp $\Pi_{n,r}^\pm$) the left (resp the right) projection from $\partial^\pm \Omega_n$. Let $x_0 \in \mathbb{D}$ be a fixed point. From Lemma 3.4.6, there is A that depends only of ϵ, k and the sequence $(M_p)_{p\in\mathbb{N}}$ such that each of $\Pi_{n,l}^\pm \circ V_n^\pm$ and $\Pi_{n,r}^\pm \circ V_n^\pm$ is an A quasi-isometry from $\mathbb{H}^2$ to $\mathbb{H}^2$. Also, since $\partial V_n^\pm(p) = (p, p)$, for any $p = 0, 1, \infty$, the A quasi-isometries $\Pi_{n,l}^\pm \circ V_n^\pm$ and $\Pi_{n,r}^\pm \circ V_n^\pm$ are normalized. Then by Lemma 3.4.10 there are compact subsets K_l and K_r of $\mathbb{H}^2$ such that the images $\Pi_{n,l}^\pm \circ V_n^\pm(x_0)$ and $\Pi_{n,r}^\pm \circ V_n^\pm(x_0)$ belong to K_l and K_r respectively. Hence $V_n^\pm(x_0)$ lie on a subset of light like geodesics $L_{x,y}$ where x vary in K_l, a compact subset of $\mathbb{H}^2$, and y vary in K_r, also a compact subset of $\mathbb{H}^2$. This implies that each of the sequences $V_n^\pm(x_0)$ lies on a compact subset of $\mathbb{ADS}^{2,1}$ (recall that Ω_n converge in the Hausdorff sense). Then the isometries $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm \Omega_n$ converge to the isometry $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm \Omega$. Note that the gluing map Φ_{Ω_n} satisfies

$$\Phi_{\Omega_n} = \partial((\Pi_{n,l}^- \circ V_n^-)^{-1} \circ (\Pi_{n,r}^+ \circ V_n^+))$$

and note that all maps $(\Pi_{n,l}^- \circ V_n^-)^{-1} \circ (\Pi_{n,r}^+ \circ V_n^+)$ are normalized uniformly quasi-isometries. Since $(\Pi_{n,l}^- \circ V_n^-)^{-1} \circ (\Pi_{n,r}^+ \circ V_n^+)$ converges to $(\Pi_l^- \circ V^-)^{-1} \circ (\Pi_r^+ \circ V^+)$, then also Φ_{Ω_n} converge to $\partial((\Pi_l^- \circ V^-)^{-1} \circ (\Pi_r^+ \circ V^+)) = (\partial V^-)^{-1} \circ \partial V^+ = \Phi_\Omega$. Since we have also assumed that Φ_{Ω_n} converges to f, we deduce $\Phi_\Omega = f$. $\square$

3.5 The induced metrics on the boundary of globally hyperbolic convex subsets

In this section, we will give the proof of Theorem 0.4.2 provided in the introduction. First, we will approximate any complete, conformal metric h by invariant metrics h_n. Note that the main difficulty is that h_n must have bounded derivatives at any order, where the bounds depend only on the order and not on n. Once this approximation is done, we can use an approximation of Ω, the globally hyperbolic convex subset that satisfies our theorem, by Ω_n, which are lifts of globally hyperbolic manifolds.

3.5.1 Approximation

We begin by approximating the metrics

Proposition 3.5.1. *Let S_n be a sequence of closed surfaces having the genus g_n converging to ∞. Let $\rho_n : \pi_1(S_n) \to \mathrm{PSL}(2, \mathbb{R})$ be a sequence of Fuchsian representations that have injectivity radius going to ∞. Let h be a complete conformal Riemannian metric on the disc $\mathbb{D}$ that has a curvature in $\left(-\frac{1}{\epsilon}, -1-\epsilon\right)$, for some $\epsilon > 0$. also assume that there is a sequence of positive real numbers $(M_p)_{p\in\mathbb{N}}$ such that each derivative of h of order p is bounded by M_p. Then there exists a sequence h_n of complete, conformal Riemannian metrics that have curvature in $\left(-\frac{1}{\epsilon'}, -1-\epsilon'\right)$, for some $\epsilon' > 0$, each h_n is ρ_n-invariant, and h_n converge C^∞ uniformly on compact subsets of $\mathbb{D}$ to h. Moreover, there is a sequence of positive real numbers $(M'_p)p \in \mathbb{N}$ such that each derivative of h_n of order p is bounded by M'_p.*

To do that, we will use curvatures. We will approximate K_h, the curvature of h, by smooth functions K_n invariant under ρ_n. Then, we will show that each K_n is the curvature of a metric h_n which is invariant under ρ_n. Finally, we will show that the metric h_n converges to h smoothly on compact subsets of $\mathbb{D}$. Note that for us it is important to have bounded derivatives with bounds that do not depend on n.

Lemma 3.5.2. *Let $K : \mathbb{D} \to \left(-\frac{1}{\epsilon}, -1-\epsilon\right)$ be a smooth function such that any derivative of it at order p is bounded by some $M_p > 0$ uniformly on $\mathbb{D}$. Let $\rho_n : \pi_1(S_n) \to \mathrm{PSL}(2, \mathbb{R})$ be a sequence of Fuchsian representations that have injectivity radius growing to ∞. Then there exists a sequence of smooth functions $K_n : \mathbb{D} \to \left(-\frac{1}{\epsilon'}, -1-\epsilon'\right)$, such that each K_n is ρ_n-equivariant, K_n converges C^∞ on compact subsets to K, and each derivative of order p of K_n is uniformly bounded on the disc $\mathbb{D}$ by some M'_p, where M'_p does not depend on n.*

Proof. Let D_n be a fundamental domain of ρ_n. Let $0 < r_n$ be such that $B(o, 2r_n) \subset D_n$, where $B(o, r)$ is the hyperbolic ball centered at o, the center of $\mathbb{D}$. Since the injectivity radius of ρ_n go to ∞, we can assume that r_n go to ∞.
Let $\phi : \mathbb{R} \to [0, 1]$ be a smooth function such that $\phi \mid_{[0,1]}= 1$ and $\phi(x) = 0, \forall x \in \mathbb{R} \setminus [0, 2]$ (see Urysohn smooth lemma). We define the function k'_n to be

$$K'_n = \begin{cases} K \ \text{on } B(o, r_n) \\ A_n \ \text{on } B(o, 2r_n) \setminus B(o, r_n) \\ -\frac{1}{\epsilon} \ \text{on } D_n \setminus B(o, 2r_n) \end{cases}$$

where $A_n = \phi(\frac{d_{\mathbb{H}^2}(o,x)}{r_n})K + (1-\phi(\frac{d_{\mathbb{H}^2}(o,x)}{r_n}))(-\frac{1}{\epsilon})$.
Note that that K_n' has values in the interval $\left[-\frac{1}{\epsilon}, -1-\epsilon\right]$. Also note that the derivatives of K_n' depend only on K and ϕ, $d_{\mathbb{H}^2}(x,o)$ and $\frac{1}{r_n}$ which can be assumed arbitrary big. Then we deduce that any derivative of K_n' of order α is bounded (with respect to the hyperbolic metric) by a constant M_α' that does not depend on n (it depends on the function ϕ and on the bound of the derivative of h of order α) .
We define the map K_n on the disc $\mathbb{D}$ by extending K_n' by reflections (that is K_n is the unique ρ_n invariant map that extends K_n' to the disc). Note that K_n is smooth since K_n' is constant on a neighborhood of the boundary of the fundamental domain D_n. It is also clear that K_n converge uniformly on compact subsets to K (because r_n is growing to ∞). □

We will use the next two theorems

Theorem 3.5.3. *[Duc91] Let $k : \mathbb{D} \to \mathbb{R}_-$ be a C^∞ function, then there exists a unique complete metric h on $\mathbb{D}$ which is conformal to dz^2 and has curvature equal to k.*

and

Theorem 3.5.4. *[KW74] Let $k : S \to \mathbb{R}_-$ be a C^∞ function, and let $[g]$ be a conformal class on S. Then there exists a unique complete metric h on S conformal to g that has curvature equal to k.*

Lemma 3.5.5. *Let h_n be a sequence of complete metrics on $\mathbb{D}$, and let h be also a complete metric on $\mathbb{D}$, assume that all the metrics are conformal to dz^2, moreover assume that k_h the curvature of h belongs to $\left[-\frac{1}{\epsilon}, -1-\epsilon\right]$ for some $\epsilon > 0$.*
For each n we denote by k_n the curvature of h_n. If k_n converge uniformly C^∞ on compact subsets to k, then h_n converge, up to extracting, uniformly C^∞ on compact subsets to h. Moreover, if there is a sequence $(M_p)_{p\in\mathbb{N}}$ such that any derivative of K_n of order p is bounded by M_p, then there is a sequence of positive real numbers M_p' such that any derivative of h_n of order p is bounded by M_p'.

Proof. Let f be a smooth function on a closed hyperbolic disc B.
In this proof we will use euclidean norms, and we will use the fact that the hyperbolic metric and the euclidean metric are bi-Lipchitz on compact subsets. We denote

$$\|f\|_{k,p} := (\sum_{\alpha\leq k} \int_B |\partial^\alpha f|^p \, dx)^{\frac{1}{p}}$$

and we denote

$$\|f\|_{C^{k,\alpha}(B)} := \sum_{|\beta|\leq k} \left\|\partial^\beta f\right\|_\infty + \sum_{|\beta|=k} \left[\partial^\beta f\right]_\alpha$$

where $\left[\partial^\beta f\right]_\alpha := \sup_{0<R<1, z\in B} R^{-\alpha} \sup\left\{|u(x)-u(y)| \, ; x,y \in B(z,R)\cap B\right\}$ (here, by $B(z,R)$, we mean the euclidean ball).
Recall that at a point p of the disc $\mathbb{D}$, the hyperbolic metric h_{-1} is equal to $h_{-1} := \frac{dz^2}{(1-\|p\|^2)^2}$.
In particular, the hyperbolic metric and the euclidean metric are bi-Lipschitz to each other on compact subsets of $\mathbb{D}$ with a bi-Lipschitz constant that depends on that compact subset.

We will show that if K, the curvature of a complete conformal metric h on $\mathbb{D}$, belongs to an interval of the form $(-\frac{1}{\epsilon}, -1-\epsilon)$ and every derivative of K or order p is bounded by $M_p > 0$, than all the partial derivatives of the metric at any order are bounded by constants that depend only on M_p and ϵ.
Fix $r > 0$. Denote by $B(x, r)$ the hyperbolic ball of center x and radius r. The metric $h = e^{2u}h_{-1}$, where u is a smooth function and h_{-1} is the hyperbolic metric. Note that (see [Tro91]) the function u is bounded on the disc because the curvature of h is negative and bounded.
Up to apply a hyperbolic isometry that sends the ball $B(x, r)$ to the hyperbolic ball $B(o, r)$, we can always remain in the hyperbolic disc $B = B(o, r)$ which is centered at o and have hyperbolic radius r. Let g be a hyperbolic isometry, then we get $g^*(h) = e^{u^*}h_{-1}$, where $u^* = u \circ g$, in particular u^* still bounded on $\mathbb{D}$.
Since the hyperbolic metric and the euclidean metric are bi-Lipchitz with a bi-Lipchitz constant that depends only on r, we find that all the derivatives of K restcited to B are bounded with respect to the euclidean metric at any order. We will show that this implies that the derivatives of u^* are bounded by constants that depend only on the bounds of the derivatives of K. Again, since the euclidean metric and the hyperbolic metric are bi-Lipchitz with a constant that depends only on r (that we can control), then the derivatives of u^* will be bounded by the hyperbolic metrics and the bounds depend only on r and the hyperbolic bounds of the derivatives of K. When we apply an isometry to go back to the ball $B(x, r)$, we find that all the derivatives of u are bounded on $B(x, r)$ with respect to the hyperbolic metric, and the bounds depend only on ϵ and the bounds of K.
Now we restrict our self to the hyperbolic ball $B(0, r)$. Recall the equation $e^{2u^*}K^* + 1 = -\Delta u^*$, where $h^* = e^{2u^*}h_1$ and Δu^* is the hyperbolic Laplacian and K^* is the curvature of h^*. By [Nic20, Theorem 10.3.1], for any $k \in \mathbb{N}$ and $p > 1$ there exists a constant $C_{k,p}$ such that the following inequality holds

$$\|u^*\|_{k+2,p} \leq C_{k,p}(\|\Delta u^*\|_{k,p} + \|u^*\|_p)$$

Note that since μ is uniformly bounded with bounds that depend only ϵ, and since k^* and all its derivatives are uniformly bounded at any order, it follows that by applying induction on the equality $e^{2u^*}K^* + 1 = -\Delta u^*$ we get the existence of constants $M'_{k,p}$ such that $\|u^*\|_{k,p}$ is bounded for any k, p by $M'_{k,p}$.
By Morrey inequality there exist constant $C'_{k,p}$ such that

$$\|u^*\|_{C^{k-\gamma,\gamma-\frac{2}{p}}} \leq C'_{k,p} \|u^*\|_{k,p}$$

where $\gamma = \left\lfloor \frac{2}{p} \right\rfloor + 1$.
Then there exists $M''_{k,\alpha}$ that depend only on M_p and ϵ such that $\|u^*\|_{C^{k,\alpha}(B(x,r))}$ is bounded by $M''_{k,\alpha}$, this implies in particular that all the derivatives of u^* are bounded at any order by constants that depends only on ϵ and $(M_p)_{p\in\mathbb{N}}$.
In particular the lemma will follows because the metric h_n will converge uniformly C^∞ on compact subsets up to extracting a subsequence (because their derivatives are uniformly bounded at any order with respect the hyperbolic metric, this follows from the arguments above). Since the curvatures of h_n converge to the curvature of h, we obtain that h_n converge (up to extracting) to h.

$\square$

Proposition 3.5.6. *Let S_n be a sequence of closed surfaces having genus g_n converging to ∞. Let $\rho_n : \pi_1(S_n) \to \mathrm{PSL}(2,\mathbb{R})$ be a sequence of Fuchsian representations that have injectivity radius converging to ∞. Let h be a complete, conformal Riemannian metric on the disc $\mathbb{D}$ that has curvature in $(-\frac{1}{\epsilon}, -1-\epsilon)$ for some $\epsilon > 0$, and any derivative of h of order p is bounded by some M_p. Then there exists a sequence h_n of complete, conformal Riemannian metrics that have curvature in $(-\frac{1}{\epsilon'}, -1-\epsilon')$, for some $\epsilon' > 0$, each h_n is ρ_n-invariant, and h_n converges C^∞ on compact subsets of $\mathbb{D}$ to h, and any derivative of h_n of order p is bounded by some M'_p, where M'_p does not depend on n.*

Proof. By Lemma 3.5.2 we construct $K_n : \mathbb{D} \to (-\frac{1}{\epsilon'}, -1-\epsilon')$, a sequence of ρ_n invariant functions that converge C^∞ on compact subsets to K_h, where K_h is the curvature of h (recall that K_n have uniformly bounded derevatives). By Theorem 3.5.4, for any n there is a unique complete conformal metric h_n on the disc which has curvature equal to h_n, and which is ρ_n invariant. Since K_n the curvatures of the metrics h_n converge smoothly to K_h, Lemma 3.5.5 implies that the metrics h_n converge, up to extracting a subsequence, to h. The derivatives of h_n at any order all uniformly bounded (independently on n) by the last remark in the proof of Proposition 3.5.5. ☐

3.6 The proof of the main theorem

In this subsection we will prove the main theorem. Before proceeding with the proof we need to recall the following statements.
The next theorem, is the group action invariant version of our main theorem.

Theorem 3.6.1. *[Tam18] Given two metrics g^+ and g^- with curvature $\kappa < -1$ on a closed, oriented surface S of genus $g \geq 2$, there exists an $\mathbb{ADS}^{2,1}$ manifold N with smooth, space-like, strictly convex boundary such that the induced metrics on the two connected components of ∂N are isotopic to g^+ and g^-.*

The next proposition shows that equivariant quasi-symmetric maps are dense in the set of quasi-symmetric maps.

Proposition 3.6.2. *[BDMS21a, Proposition 9.1] Let f be a normalized quasi-symmetric map. There is a sequence of equivariant normalized uniformly quasi-symmetric maps, $\rho_n^+, \rho_n^- : \pi_1(S_n) \to \mathrm{PSL}(2,\mathbb{R})$, that converge to f. Here, S_n is a sequence of closed surfaces with genus g_n going to ∞, and ρ_n^+, ρ_n^- are a sequence of Fuchsian representations whose injectivity radius go to ∞.*

Now we proceed the proof of our main theorem

Theorem 3.6.3. *Let $h^\pm$ be two complete conformal Riemannian metrics on the disk $\mathbb{D}$ that have curvature in $(-\frac{1}{\epsilon}, -1-\epsilon)$ for some $\epsilon > 0$, and each derivative of $h^\pm$ of order p is bounded by $M_p > 0$ (with respect to the hyperbolic metric). Let f be a normalized quasi-symmetric map. There exists a normalized globally hyperbolic convex subset Ω, and normalized isometries $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm\Omega$ such that $f = \Phi_\Omega$.*

Proof. By Proposition 3.6.2 there are quasi-Fuchsian representations $\rho_n^+, \rho_n^- : \pi_1(S_n) \to \mathrm{PSL}(2,\mathbb{R})$, and f_n a sequence of ρ_n^+, ρ_n^- equivariant quasi-symmetric maps that converge in the C^0 topology to f. Also by Proposition 3.5.1 there is a ρ_n^+ (resp ρ_n^-) metrics h_n^+ (resp

h_n^-) that converge C^∞ on compact subsets of $\mathbb{D}$ to h^+ (resp h^-). Also by Proposition 3.5.1, the metrics $h_n^\pm$ have bounded uniformly derivatives (that is the bounds depend only on the order of the derevative and do not depend on n or the point on the $\mathbb{D}$).
By Theorem 3.6.1, there is exists a globally hyperbolic manifold M_n diffeomorphic to $S_n \times [0,1]$ such that the induced metric on $S_n \times \{1\}$ is homotopic to h_n^+/ρ^+n and the induced metric on $S_n \times \{0\}$ is homotopic h_n^-/ρ_n^+.
The globally hyperbolic manifold M_n lifts to a globally hyperbolic convex subset Ω_n, up to normalizing, we have isometries $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm\Omega_n$ such that $\partial V_n^\pm(p) = (p,p)$ for $p = 0, 1, \infty$. Since there is a unique ρ_n^+, ρ_n^- equivariant quasi-symmetric map, we have that $\Phi_{\Omega_n} = f_n$.
By Proposition 3.6.2, the maps f_n are uniformly quasi-symmetric. Then there is $k' > 1$ such that $\partial_\infty\Omega_n$ is a k' quasi-circle for any n. This implies that Ω_n converge in the Hausdorff topology to a globally hyperbolic convex subset Ω . Then by Proposition 3.4.14, the isometries $V_n^\pm : (\mathbb{D}, h_n^\pm) \to \partial^\pm\Omega_n$ converge to an isometry $V^\pm : (\mathbb{D}, h^\pm) \to \partial^\pm\Omega$ and the symmetric map $\Phi_\Omega = f$. □

Chapter 4

Possible development

4.1 Smooth grafting

Let $\mathcal{P}(S)$ be the set of complex projective structures on the surface S. If S is a closed surface of genus g then $\mathcal{P}(S)$ is a complex manifolds of dimension $12g-12$. Recall that the projection $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ that associates to a complex projective surface the underlying complex structure is holomorphic.
In [BMS13], the authors defined the map

$$SGr_s : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$$

where $s > 0$ (see Figure 4.1).

The map is defined as follows, given $h_1, h_2 \in \mathcal{T}(S)$, by Corollary 1.18.4 we can assume that (up to isotopy) $h_2 = h_1(b, b)$ where b is an operator as defined in Proposition 1.18.2. Consider the metric $I_s = cosh^2(\frac{1}{2}s)h_1$ and the operator $B_s = -tanh(\frac{1}{2}s)b$. One can check that

$$\nabla \cdot B_s = 0, \quad K_s = -1 + \det(B_s)$$

Where ∇ is the Levi-Civita connection for I_s (which is equal to the Levi-Civita connection for h), and K_s is the curvature of I_s (which is constant and equal to $-\frac{1}{\cosh^2(\frac{s}{2})}$).

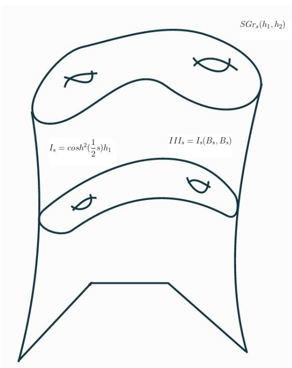
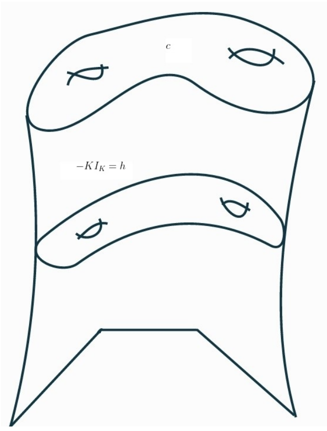


Figure 4.1: We consider the hyperbolic end E such that there is an embedded colsed surface S that has first fundamental form equal to I_s and shape operator B_s, the map $SGr_s(h_1, h_2)$ is the projective structure at infinity given in that hyperbolic end.



Let $\tilde{S}$ be a universal cover of S, denote by $\tilde{I}_s$ and $\tilde{B}_s$ the lifts of I_s and B_s respectively. There exists a convex equivariant immersion

$$\sigma_S : \tilde{S} \to \mathbb{H}3$$

such that the pull-back of the metric of $\mathbb{H}^3$ is $\tilde{I}_s$ and the pull back of the shape operator is $\tilde{B}_s$. The map σ_s is uniquely determined up to elements of $\mathrm{PSL}_2(\mathbb{C})$, once we state that the orientation on $\tilde{S}$ at $\sigma_s(\tilde{p})$ coincides with the orientation induced by the normal vector $\tilde{n}_s(\tilde{p})$ pointing towards the concave part.
We define the map

$$dev_s : \tilde{S} \to \mathbb{CP}^1$$

as follow. Let $\tilde{p} \in \tilde{S}$, we define $dev_s(\tilde{p}) \in \mathbb{CP}^1$ to be the endpoint of the geodesic that starts from $dev_s(\tilde{p})$ and has velocity $\tilde{n}_s(\tilde{p})$.
We define $SGr_s(h_1, h_2)$ to be the $\mathbb{CP}^1$ structure on S constructed above. This map can be seen as an extension for the classical grafting map in the following sense

Proposition 4.1.1. *[BMS13, Proposition 6.2] Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic metrics converging to a hyperbolic metric h on S, and let $(h_n^\star)_{n\in\mathbb{N}}$ be a sequence of hyperbolic metrics converging projectively to $[\lambda]$ in the Thurston boundary of T. If θ_n is a sequence of positive numbers such that $\theta_n \ell h_n^\star \to \iota(\lambda, \bullet)$, then $SG_r'(\theta_n)(h_n, h_n^\star)$ converges to $Gr_{\lambda/2}(h)$.*

The map SGr_s induce the following map.
Let $h_1 \in \mathcal{T}(S)$, define ther map

$$\begin{aligned} sgr_{s,h_1} : \mathcal{T}(S) &\to \mathcal{T}(S) \\ h &\mapsto \pi(SGr_s(h, h_2)) \end{aligned}$$

One can ask if the map sgr_{s,h_1} is injective ? This is equivalent to the following question

Question 4.1.2. *Is there a unique hyperbolic end with conformal structure at infinity c, and containing an embedded surface of constant curvature k with induced metric proportional to h? (see Figure 4.1)*

4.1.1 Grafting and smooth grafting from universal point of view

The grafting map $Gr : \mathcal{P}(S) \to \mathcal{ML}(S) \times \mathcal{T}(S)$ was defined by Thurston, and he showed that it is a homeomorphism, and Bonahon has shown that it is a bitangentaible map. Later Scannell and Wolf have proved that for any measured lamination $L \in \mathcal{ML}(S)$, the map $gr_L = gr(., L) = \pi \circ Gr(., L)$ is a biholomorphism map from $\mathcal{T}(S)$ to itself. Then Dumas and Wolf have proved that for any hyperbolic metric $h \in \mathcal{T}(S)$ the map $gr_h = gr(h, .) = \pi \circ Gr(h, .)$ is a homeomorphism from $\mathcal{T}(S)$ to $\mathcal{ML}(S)$. In this section we will give the maps defined in [Sch20] and explain why they can be seen as a universal analogue to the maps Gr, gr_h, gt_L. Later we we will give a universal analogue to the smooth grafting map SGr.

Definition 4.1.3. *Let $\phi : \mathbb{D} \to \mathbb{H}^3$ be a locally convex immersion Since the immersion is locally convex we can define the Gauss map*

$$G : Im(\phi) \to \mathbb{CP}^1$$

that is the map that to a point $p \in Im(\phi)$ associate the endpoint at infinity to the geodesic ray starting from p in the direction of the normal vector pointing from p to the concave side. The Gauss map is a local homeomorphism..
Let $U_0 : D \to S$ be the Riemann uniformization map for the pull-back $(\varphi \circ G)^*(c_\infty)$ by the Gauss map of the conformal metric at infinity, and let $U_1 : D \to S$ be the Riemann uniformization map for the induced metric I on S. We call $\partial(U_1^{-1} \circ U_0) : \mathbb{RP}^1 \to \mathbb{RP}^1$ the induced metric gluing map for φ.

In [Sch20] the author formulate universal statements of the graftings map

Question 4.1.4. *Let $\sigma : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$ be a quasi-symmetric homeomorphism. Is there a unique locally convex pleated isometric immersion $\varphi : \mathbb{H}^2 \to \mathbb{H}^3$ such that the induced metric gluing map of φ is σ?*

Question 4.1.5. *[Sch21, Question 5.2] Let L be a bounded measured lamination on $\mathbb{H}^2$. Is there a unique immersed locally convex pleated surface $\Sigma \subset \mathbb{H}^3$ together with a map $u : \mathbb{H}^3 \to N^1\Sigma$ which is a parameterization of $N^1\Sigma$ such that*

- *$G \circ u$ is conformal, where G is the hyperbolic Gauss map of Σ,*
- *the pull-back by u of the measured bending lamination of Σ corresponds to l?*

Let L be a bounded measured lamination on $\mathbb{H}^2$. We denote by Σ the pleated surface obtained by pleating the hyperbolic disc (in $\mathbb{H}^3$) along the measured lamination L, and we denote by $G : N^1\Sigma \to \mathbb{CP}^1$ the Gauss map. Let $u : \mathbb{D} \to \Sigma$ be the Riemann uniformization map for the conformal structure which the pull back of the conformal structure of $\mathcal{CP}^1$ by G. And we denote by π_Σ the canonical projection $\pi : N^1\Sigma \to \Sigma$.
In [EM87] the authors have show the following lemma

Lemma 4.1.6. *There exists $\delta > 0$ such that, if L is bounded, then the map $\pi \circ u : D \to \Sigma$ is within distance δ from a quasi-conformal map, with quasi-conformal constant depending only on the bound on L.*

A direct consequence of Lemma 4.1.6 is that the map $\pi \circ u$ extends to a quasi-symmetric map $\partial\pi \circ u : \mathbb{RP}^1 \to \partial_\infty \Sigma$

Definition 4.1.7. *We denote by:*

- $GR_0(L) : \mathbb{RP}^1 \to \partial\infty\Sigma$ *the boundary extension of* $\pi \circ u$*,*
- $GR_1(L)$ *the measured lamination corresponding to* $(\pi \circ u)^*(L)$ *on* D*,*
- $GR(L) = (GR_0(L), GR_1(L))$*.*

Then the author in [Sch20],

Question 4.1.8. *Is the map* $GR_0 : \mathcal{ML}(\mathbb{H}^2) \to \mathcal{T}$ *a homeomorphism from bounded measured laminations to quasi-symmetric homeomorphisms?*

And

Question 4.1.9. *Is the map* $GR_1 : \mathcal{ML}(\mathbb{H}^2) \to \mathcal{ML}(\mathbb{H}^2)$ *a homeomorphism from bounded measured laminations to bounded measured laminations?*

Note the analogy between Question 4.1.8 and Question 4.1.4 which should be a universal analogue of the main theorem of [DW08]. And the analogy between Question 4.1.9 and Question 4.1.5 which should be a universal analogue of the main theorem of [SW02].

One can also can define and study the Smooth grafting as maps from $\mathcal{T}(\mathbb{H}^2) \times \mathcal{T}(\mathbb{H}^2)$, the key theorem that we will need is the following universal analogue of Theorem 1.18.3.

Theorem 4.1.10. *[BS10] Let* $f : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$ *be a quasi-symmetric map. There exists a unique quasi-conformal minimal Lagrangian map* $m : \mathbb{H}^2 \to \mathbb{H}^2$ *that its extension to ideal boundary is equal to* f*.*

Let's denote the hyperbolic metric on $\mathbb{H}^2$ by h, and we denote by ∇ its Levi-Civita connection. As in the case of closed surfaces, there is a unique bundle morphism $b : T\mathbb{H}^2 \to T\mathbb{H}^2$ such that

- $m^*h = h(b, b)$
- $det(b) = 1$
- b is self adjoint with respect to h
- b satisfies the Coddazi equation $d^\nabla b = 0$

Let f be a quasi-symmetric map, and let $s > 0$, then we define the map

$$sgr_s : \mathcal{T}(\mathbb{H}^2) \to \mathcal{T}(\mathbb{H}^2)$$

as follows,
Let $m : \mathbb{D} \to \mathbb{D}$ be the unique quasi-conformal minimal Lagrangian map that extends f, and let b be the bundle morphism defined above. Then there exists a unique immersion $\psi : \mathbb{D} \to \mathbb{H}^3$ such that the pull back of the metric of $\mathbb{H}^3$ by ψ is equal to $cos(\frac{1}{2}s)h$ and the pull back of the shape operator by ψ is equal to $-tanh(\frac{1}{2}s)b$.

Definition 4.1.11. *We define* sgr_s *to be the gluing map associated to the immersion* $\psi : \mathbb{D} \to \mathbb{H}^3$*.*

Question 4.1.12. *Is the map* $sgr_s : \mathcal{T}(\mathbb{H}^2) \to \mathcal{T}(\mathbb{H}^2)$ *a homeomorphism ?*

4.2 Land-slides flow

4.2.1 Land-slides flow

As mentioned in Section 3.2.5, the earthquake map has an interpretation related to the geometry of the boundary of the convex core of the anti-de Sitter manifolds. Using that point of view, the authors in [BMS13] extend the earthquake map to a map from $\mathcal{T}(S) \times \mathcal{T}(S)$ to $\mathcal{T}(S)$ and which converge (in a sense that we will explain) to the usual earthquake map.

The authors in [BMS13] have defined for any θ in $\mathbb{R}$ a map $L_\theta : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)\times\mathcal{T}(S)$ such that the left projection (resp. the right) of L_θ extends the left projection (resp. the right) earthquake map. For details in what follows, we refer to [BMS13, Section 3]. Let $h, h^* \in \mathcal{T}(S)$, without loss of generality we can assume that h and h^* are two normalized hyperbolic metrics, that is the identity map $(S, h) \to (S, h^*)$ is minimal Lagrangian, it follows the existence of a unique operator b as in Corollary 1.18.4 such that $h^*(,) = h(b, b)$.
Let $\theta \in \mathbb{R}$, we define

$$I_\theta = cos^2(\frac{\theta}{2})h,\ B_\theta = \tan(\frac{\theta}{2})b$$

There exists an (unique up to isometry) equivariant map

$$\phi : \tilde{S} \to \mathbb{ADS}^{2,1}$$

that its image, denoted by $\tilde{F}$, is a space-like surface, and the pull back of $\mathbb{ADS}^{2,1}$ metric by ϕ is equal to $cos^2(\frac{\theta}{2})\tilde{h}$ and the pull back of the shape operator of $\tilde{F}$ is equal to $\tan(\frac{\theta}{2})\tilde{b}$. We will require that $\tilde{n}$, the normal field that induces the right orientation on $\tilde{S}$ points toward the convex side, and that $\tilde{n}$ is a future directed vector field.
Since the induced metric on $Im(\phi) := \tilde{F}$ is complete, then the holonomy of Φ has two Fuchsian components ρ_l, ρ_r. Let (S, h_l) (resp (S, h_r)) is be the hyperbolic surface isometric to $\mathbb{H}^2/\rho_l$ (resp (S, h_l)). We define $L_\theta(h, h^*) = (h_l, h_r)$.
The authours have proved the following theorem, that shows that L_θ extends the earthquake map.

Theorem 4.2.1. *Let $h \in \mathcal{T}$, let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of hyperbolic metrics, and let $\lambda \neq 0$ be a measured lamination. Consider a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\to\infty} \theta_n l_{\theta_n} = i(\lambda, .)$ in the sense of convergence of the length spectra of simple closed curves. Then*

$$\lim_{n\to\infty} h_n^1 = E_{\frac{\lambda}{2}}(h), \lim_{n\to\infty} \theta_n l_{h_n^2} = i(\lambda, .)$$

where $(h_n^1, h_n^2) := L_{\theta_n}(h, h_n^*)$.

4.2.2 Extention of the Land-slides flow to universal Teichmüller space

Let $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ be a quasi-symmetric map. By Theorem 4.1.10, the map f has a unique minimal Lagrangian extension $m : \mathbb{D} \to \mathbb{D}$. We denote the usual hyperbolic metric on $\mathbb{D}$ by h. There exists an operator $b : T\mathbb{D} \to \mathbb{D}$ as in Theorem 4.1.10 such that $m^*h = h(b, b)$.

Let $f_1, f_2 : \mathbb{RP}^1 \to \mathbb{RP}^1$ be two quasi-symmetric maps. Let $\Psi := f_2 \circ f_1^{-1}$. Then Ψ is a quasi-symmetric map. Denote by m its unique minimal Lagrangian extention and let b be the operator mentioned in Theorem 4.1.10.
We define

$$B_\theta := cos(\frac{\theta}{2})Id + sin(\frac{\theta}{2})Jb$$

where J is the almost complex structure in $\mathbb{D}$.
Set

$$h_\theta := h(B_\theta, B_\theta), h_\theta^* := h(B_{\pi+\theta}, B_{\pi+\theta})$$

See [BMS13, Lemma 8.3] to see why h_θ and h_θ^* are hyperbolic metrics, and why the identity map $(\mathbb{D}, h_\theta) \to (\mathbb{D}, h_\theta^*)$ is minimal Lagrangian.
The Identity map from $(\mathbb{D}, h) \to (\mathbb{D}, h_\theta)$ is quasi-conformal and determines a quasi-symmetric map f_θ. In addition, the identity map $(\mathbb{D}, h) \to (\mathbb{D}, h_\theta^*)$ is quasi-conformal and defines a quasi-symmetric map f_θ^*.
We define $L_\theta(f, f^*) = (f_\theta, f_\theta^*)$. We refer to [BMS13, Section 8.2] to see why this map is a natural extension of L_θ to the universal case.

4.2.3 Weil-Petersson curves

Recall that there is a one-to-one correspondence between normalized quasi-circles of $\mathbb{CP}^1$ and normalized quasi-symmetric maps via the correspondence of conformal welding. The space of normalized quasi-symmetric maps forms the universal Teichmüller space $\mathcal{T}$ (see Section 1.10).
This motivated the authors (in particular, Takhtajan and Teo [TT06]) to define a metric on the universal Teichmüller space, called the Weil-Petersson metric. The Weil-Petersson metric turns $\mathcal{T}$ into a Hilbert manifold. The universal Teichmüller space, endowed with the topology compatible with the Weil-Petersson metric, has many connected components. One of these components, denoted by $\mathcal{T}_0$, is exactly the closure of the smooth maps. Any quasi-symmetric map that belongs to $\mathcal{T}_0$ is called a Weil-Petersson homeomorphism. Additionally, we call a quasi-circle a Weil-Petersson curve if its conformal welding map is a Weil-Petersson homeomorphism.
Equivalently, a quasi-symmetric homeomorphism is Weil-Petersson if and only if it has a quasi-conformal extension $\tilde{f}$ to the disc $\mathbb{D}$, and $\tilde{f}$ is L^2 with respect to the hyperbolic metric of the disc (see [Bis19] for more details). The Weil-Petersson class of quasi-circles has been a topic of interest and has been studied by many authors, see, for example, [TT06] and [Cui00]. In particular, Bishop in [Bis19] gave many characterizations for a Weil-Petersson curve (or equivalently a Weil-Petersson homeomorphism). In this subsection, we will ask some questions about characterizing a Weil-Petersson curve.
As mentioned in Section 2.7.5, we believe that if $\phi : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$ is a parameterized Jordan curve such that the pull-back of the bending laminations of $\mathrm{CH}(\mathrm{Im}(\phi))$, the convex hull of $\mathrm{Im}(\phi)$ in $\mathbb{H}^3$, strongly fill (see Definition 2.7.25) and are bounded by $c < \pi$, then ϕ is a parameterized quasi-circle. We can also ask if there is a similar condition on the bending laminations to deduce whether a curve is Weil-Petersson.

Question 4.2.2. *Let $\phi : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$ be a parameterized quasi-circle. Denote $C :=$*

$Im(\phi)$, and let CH(C) *denote the convex hull of C in $\mathbb{H}^3$. We denote by L^+ and L^- the pull-back of the bending laminations of ∂*CH(C) *by ϕ.*

- *Are there conditions on L^+ and L^- to deduce that C is a smooth curve?*
- *Are there conditions on L^+ and L^- to deduce that C is a Weil-Petersson curve?*

Recall that we have seen in Section 1.16 that, according to the work of [Thu97] and [Šar06], any homeomorphism f of S^1 that preserves the orientation has a unique extension to an earthquake map E, and this homeomorphism is quasi-symmetric if and only if the shearing lamination of E is bounded. One can ask under what condition on the shearing lamination of the earthquake E the map f is a Weil-Petersson homeomorphism.

Question 4.2.3. *Let $f : \partial_\infty\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2$ be a homeomorphism that preserves orientation. Let E be the unique left earthquake map whose extension to the boundary $\partial_\infty\mathbb{H}^2$ is equal to f, that is, $E\mid_{\partial_\infty\mathbb{H}^2}= f$.*
Under what condition on the shearing lamination of E can we deduce that f is a Weil-Petersson homeomorphism?

In Section 2.7 we have defined the notion of gluing maps (see Section 2.7.3). For any Jordan curve C, and for any $k \in (-1, 0)$ there exists a gluing map $\Phi_{C,k}$ that corresponds to the two k surface $S^\pm_k(C)$ that span the Jordan curve C at infinity. It is natural to ask more about the relation between these gluing maps and the Jordan curve C, in particular

Question 4.2.4. *Let $C \subset \partial_\infty\mathbb{H}^3$. Let $k \in (-1, 0)$, and let $\Phi_{C,k}$ be the gluing map that corresponds to the two k-surfaces spanning C. Is it true that $\Phi_{C,k}$ is a Weill-Petersson homeomorphism if and only if C is a Weill-Petersson curve ?*

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