Algebraic and coalgebraic modal logic: From Boolean algebras to semi-primal varieties

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Semi-primal algebras

Definition (Primal algebra)

An algebra $\mathbf{D}$ is *primal* if every operation $f\colon D^k \to D$ with $k \geq 1$ is term-definable in $\mathbf{D}$.

Foster 1953 [8] ; Foster, Pixley 1964 [9] ; Pixley 1971 [18]

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Theorem

For a finite algebra $\mathbf{D}$, t.f.a.e.:

1. $\mathbf{D}$ is semi-primal.
2. The variety $\mathbb{HSP}(\mathbf{D})$ is arithmetical (*i.e.*, congruence-distributive and -permutable) and all subalgebras of $\mathbf{D}$ are simple and rigid.

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3. The ternary discriminator is term-definable in $\mathbf{D}$ and all subalgebras of $\mathbf{D}$ are rigid.

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Semi-primal lattice-expansions

Proposition

For a finite algebra $\mathbf{D}$ with bounded lattice reduct, t.f.a.e.:

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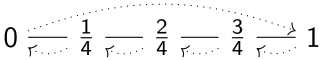
For the second part, note that we can define $T_0(x) = \tau_e(x \backslash 0)$ where e is the monoid unit of **D**.

Semi-primal chains: Examples

- The Post chains $\mathbf{P}_n = \langle \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \wedge, \vee, ', 0, 1 \rangle$.

$\mathbf{P}_4$:



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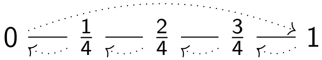

- The finite MV-chains

$$\text{Ł}_n = \langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle.$$

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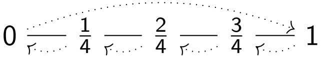

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$$\mathbf{M}_n = \langle \{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, \neg, 0, 1, (\tau_d \mid d \in M_n) \rangle.$$

Semi-primal chains: Examples

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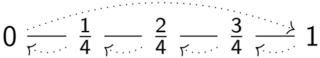

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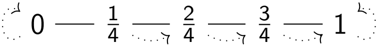
- The finite Cornish chains $\mathbf{C}_n = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, \neg, f, 0, 1 \rangle$.[1]

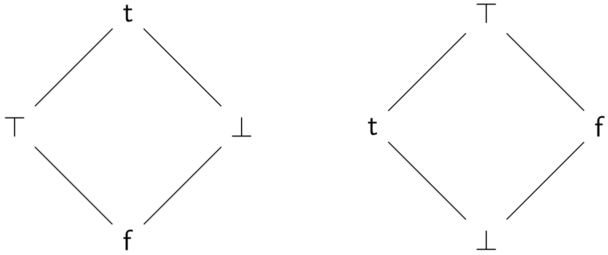
$\mathbf{C}_4$:



[1] Davey, Gair 2017 [5]

Semi-primal lattices: Examples (1)

$$\mathbf{FOUR} = \langle \{\mathsf{t}, \mathsf{f}, \top, \bot\}, \wedge, \vee, \otimes, \oplus, \neg, \supset, \mathsf{t}, \mathsf{f} \rangle.$$



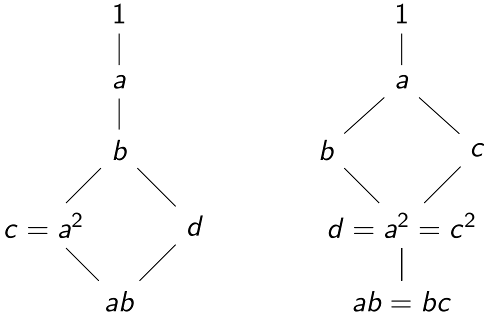
$\leq_t$ $\leq_k$

Figure: The truth-order $\leq_t$ and the knowledge-order $\leq_k$.

U. Rivieccio's PhD thesis

Semi-primal lattices: Examples (2)

- Residuated lattices, *e.g.*,

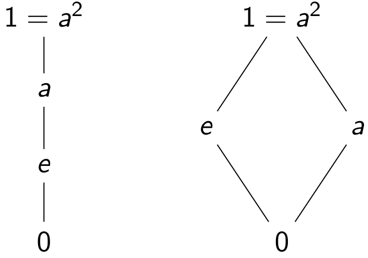


$\mathbf{R}^{6,2}_{1,11}$ $\mathbf{R}^{6,3}_{1,9}$

Notation from list of finite residuated lattices of size up to 6 by N. Galatos and P. Jipsen

Semi-primal lattices: Examples (3)

- De Morgan monoids (with unit e) / Relevant algebras (without e)



$\mathbf{C}_4^{01}$ $\mathbf{D}_4^{01}$

Moraschini, Raftery, Wannenburg 2019 [17]

Semi-primal duality

Let **D** be semi-primal bounded lattice-expansion, $\mathcal{A} := \mathbb{HSP}(\mathbf{D}) = \mathbb{ISP}(\mathbf{D})$.

There is a dual equivalence

$$\mathsf{Stone}_{\mathbf{D}} \underset{\Sigma'}{\overset{\Pi'}{\rightleftarrows}} \mathcal{A}$$

Keimel, Werner 1974 [12] ; Clark, Davey 1998 [4]

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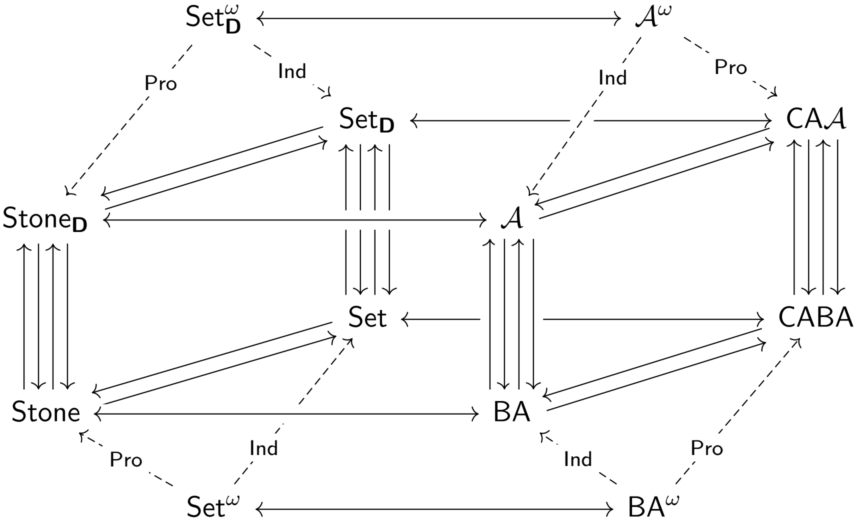
Definition (The category $\mathsf{Stone}_{\mathbf{D}}$)

The category $\mathsf{Stone}_{\mathbf{D}}$ has objects $(X, \mathbf{v})$ where $X \in \mathsf{Stone}$ and $\mathbf{v} \colon X \to \mathbb{S}(\mathbf{D})$ is continuous w.r.t. the upset topology on $\mathbb{S}(\mathbf{D})$.
A morphism $f \colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$ is a continuous map $X_1 \to X_2$ with $\mathbf{v}_2(f(x)) \leq \mathbf{v}_1(x)$ for all $x \in X_1$.

Keimel, Werner 1974 [12] ; Clark, Davey 1998 [4]

Dualities via categorical completions



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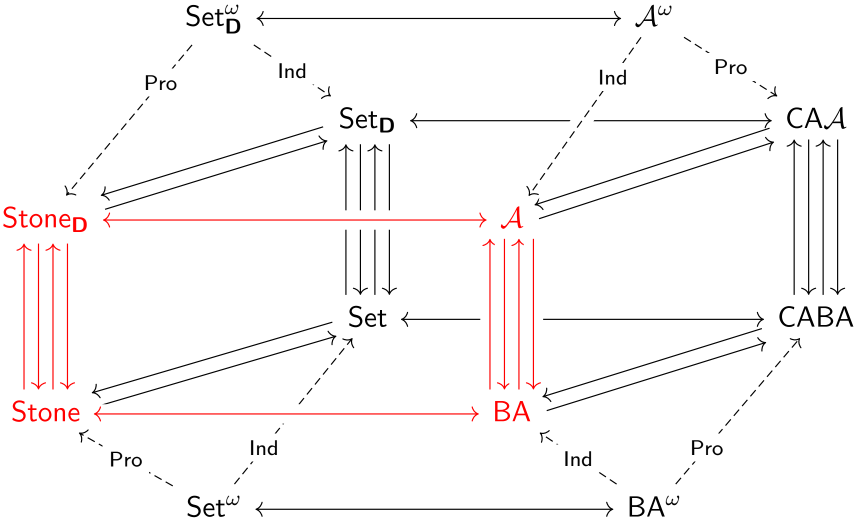
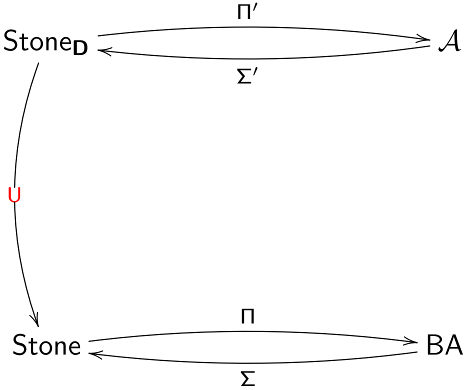


The adjoint functors on the dual side

$$\mathsf{Stone}_{\mathbf{D}} \underset{\Sigma'}{\overset{\Pi'}{\rightleftarrows}} \mathcal{A}$$

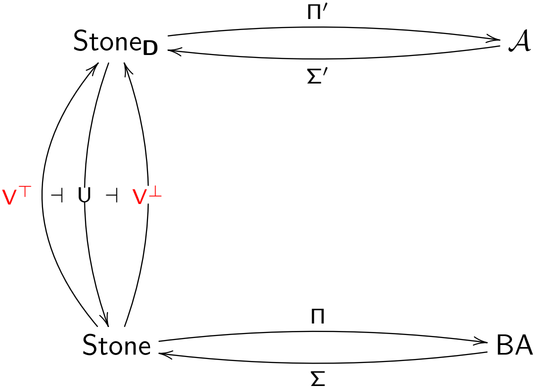
$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

The adjoint functors on the dual side



Let U be the the forgetful functor.

The adjoint functors on the dual side



Let U be the forgetful functor.
$\mathsf{V}^{\top}(X) = (X, \mathbf{v}^{\mathbf{D}})$ where $\mathbf{v}^{\mathbf{D}}(x) = \mathbf{D}$ for all $x \in X$.
$\mathsf{V}^{\perp}(X) = (X, \mathbf{v}^{\mathbf{E}})$ where $\mathbf{v}^{\mathbf{E}}(x) = \mathbf{E} := \langle 0, 1 \rangle$ for all $x \in X$.

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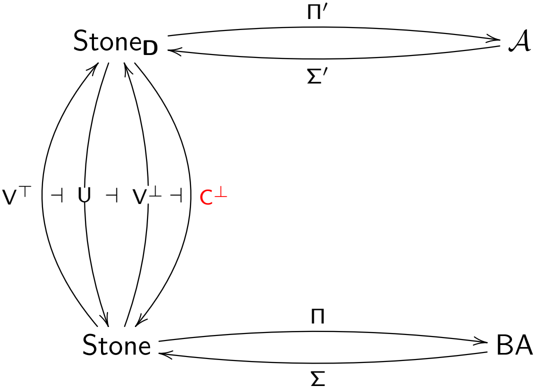

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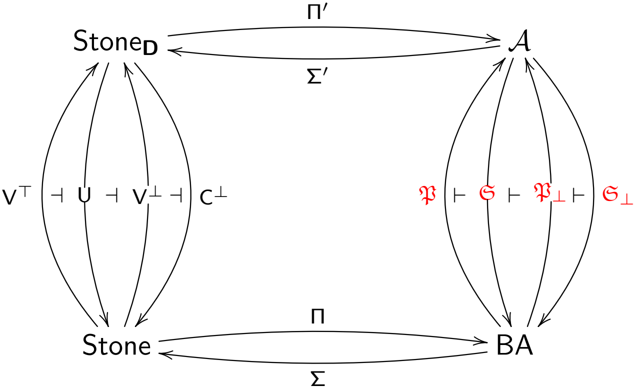
$C^\perp(X, \mathbf{v}) = \{x \in X \mid \mathbf{v}(x) = \mathbf{E}\}$.

The adjoint functors on the algebraic side



These functors all have algebraic duals.

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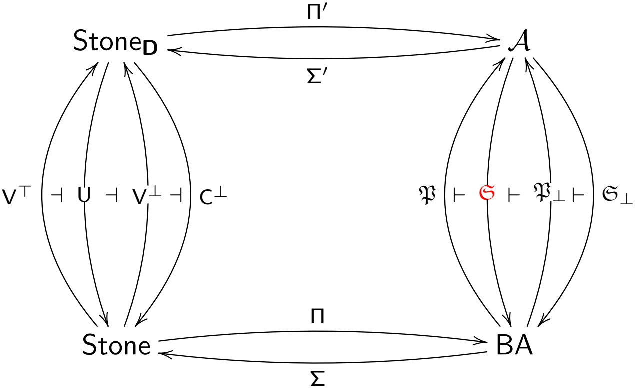

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The dual of the forgetful functor U is the Boolean skeleton functor $\mathfrak{S}$.

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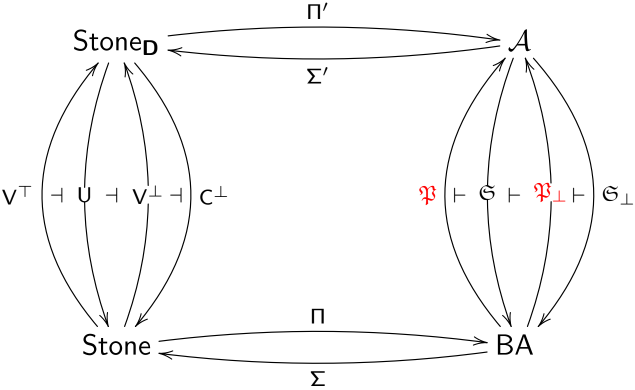

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The duals of $V^{\top}$ and $V^{\perp}$ are certain Boolean power functors $\mathfrak{P}$ and $\mathfrak{P}_{\perp}$.

Boolean skeletons

For every $d \in D$, the unary operation $T_d = \chi_{\{d\}}$ is term-definable in $\mathbf{D}$.

Definition (Boolean skeleton)

The *Boolean skeleton* of $\mathbf{A} \in \mathcal{A}$ is given by

$$\mathfrak{S}(\boldsymbol{A}) = (\mathfrak{S}(A), \wedge, \vee, T_0, 0, 1),$$

where $\mathfrak{S}(A) = \{a \in A \mid T_1(a) = a\}$.

Maruyama 2012 [16]

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Theorem

For every $\mathbf{A} \in \mathcal{A}$, there is a homeomorphism given by restriction $u \mapsto u|_{\mathfrak{S}(\mathbf{A})}$

$$\mathcal{A}(\mathbf{A}, \mathbf{D}) \cong \mathrm{BA}(\mathfrak{S}(\mathbf{A}), \mathbf{2})$$

This defines a natural isomorphism $\mathrm{U}\Sigma' \Rightarrow \Sigma\mathfrak{S}$.

Maruyama 2012 [16]

Boolean powers

Definition (Boolean power)

Let $\mathbf{M}$ be a finite algebra and $\mathbf{B} \in \mathrm{BA}$. The *Boolean power* $\mathbf{M}[\mathbf{B}]$ consists of all $\xi \colon M \to B$ which satisfy $\xi(m_1) \wedge \xi(m_2) = 0$ for $m_1 \neq m_2$ and $\bigvee_{m \in M} \xi(m) = 1$. If $\circ$ is some (for simplicity assume binary) operation of $\mathbf{M}$, define

$$(\xi \circ \xi')(m) = \bigvee_{m_1 \circ m_2 = m} \xi(m_1) \wedge \xi(m_2).$$

This turns $\mathbf{M}[\mathbf{B}]$ into a member of $\mathbb{HSP}(\mathbf{M})$.

Foster 1953 [8]

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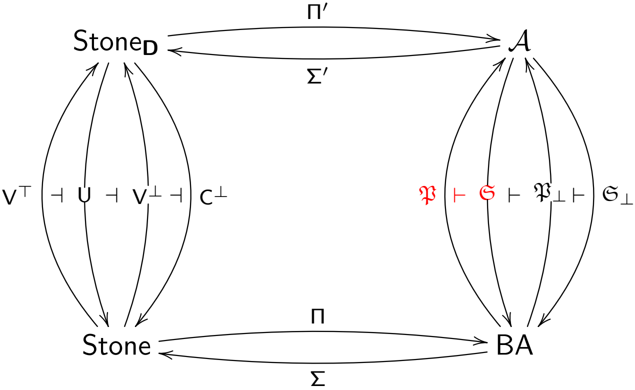
Theorem

The Boolean power functor $\mathfrak{P}(\mathbf{B}) = \mathbf{D}[\mathbf{B}]$ *is right-adjoint to the Boolean skeleton functor* $\mathfrak{S}$.

Foster 1953 [8]

The adjoint functors on the algebraic side (ctd.)



So we have an algebraic description of the adjunction $\mathfrak{P} \vdash \mathfrak{S}$.

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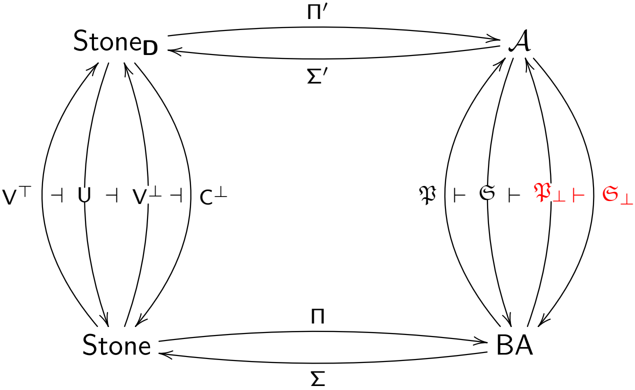

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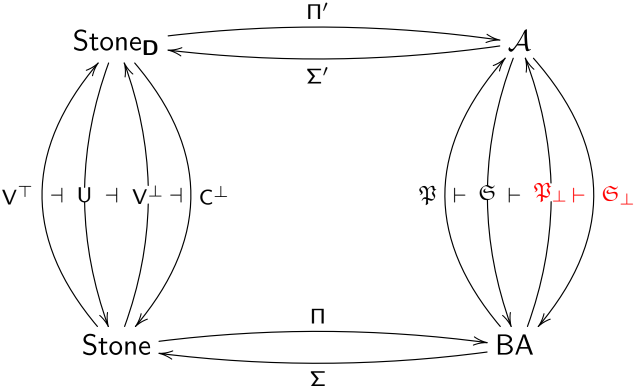

So we have an algebraic description of the adjunction $\mathfrak{P} \vdash \mathfrak{S}$.
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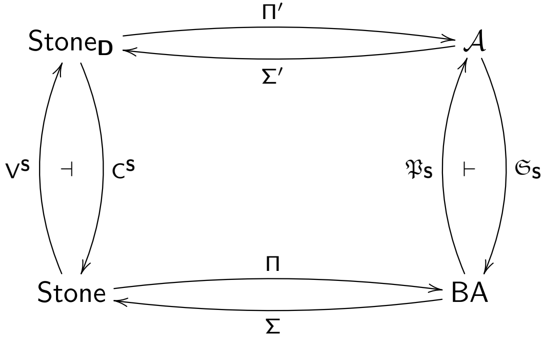
In fact, it is a specific instance of the more general subalgebra adjunctions.

The subalgebra adjunctions



For every $\mathbf{S} \in \mathbb{S}(\mathbf{D})$ there is an adjunction $\mathsf{V}^{\mathbf{S}} \dashv \mathsf{C}^{\mathbf{S}}$.

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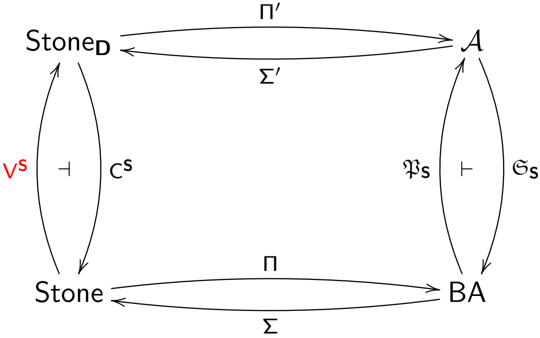

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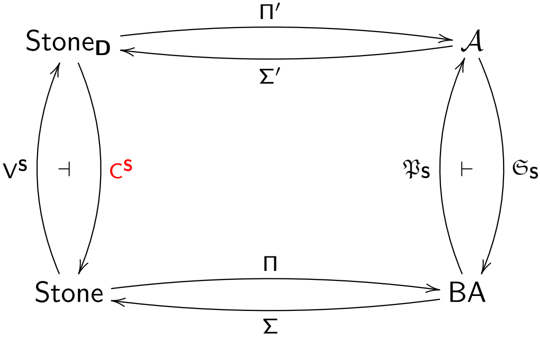

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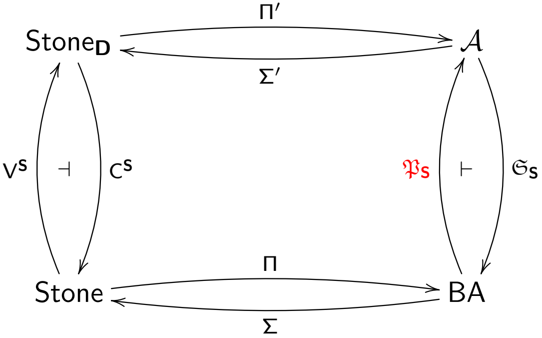
The subalgebra adjunctions



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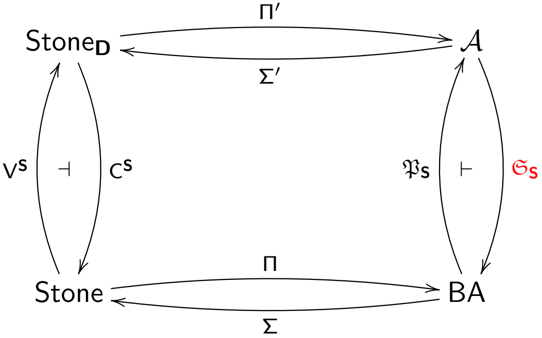
$\mathfrak{P}_{\mathbf{S}}$ takes the Boolean power $\mathfrak{P}_{\mathbf{S}}(\mathbf{B}) = \mathbf{S}[\mathbf{B}]$.

The subalgebra adjunctions



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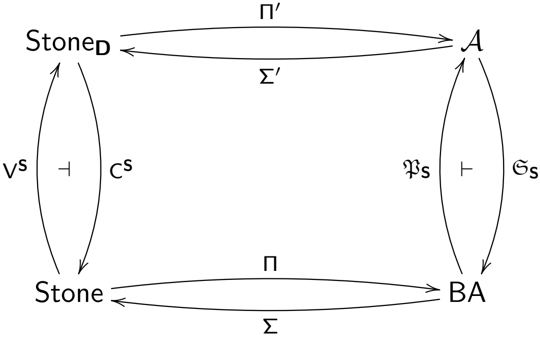
$\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) = \{x \in X \mid \mathbf{v}(x) \subseteq \mathbf{S}\}$.
$\mathfrak{P}_{\mathbf{S}}$ takes the Boolean power $\mathfrak{P}_{\mathbf{S}}(\mathbf{B}) = \mathbf{S}[\mathbf{B}]$.
$\mathfrak{S}_{\mathbf{S}}$ takes the Boolean skeleton of a *quotient*.

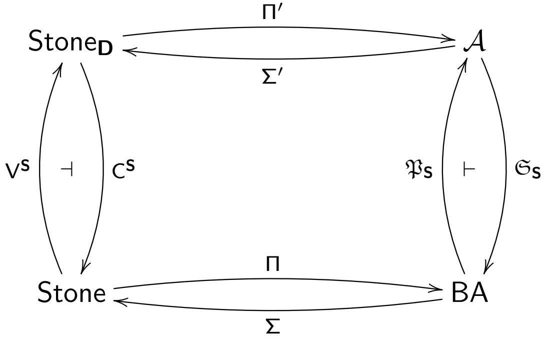
The subalgebra adjunctions



Any $(X, \mathbf{v}) \in \mathrm{Stone}_{\mathbf{D}}$ can be recovered from all $\mathrm{V}^{\mathbf{S}}\mathrm{C}^{\mathbf{S}}(X, \mathbf{v})$ via the *coend*

$$(X, \mathbf{v}) \cong \int^{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathrm{V}^{\mathbf{S}}\mathrm{C}^{\mathbf{S}}(X, \mathbf{v}).$$

The subalgebra adjunctions



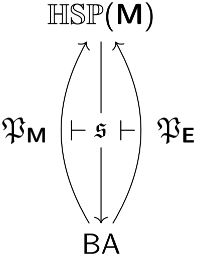
Dually, any $\mathbf{A} \in \mathcal{A}$ can be recovered from all $\mathfrak{P}_{\mathbf{S}}\mathfrak{S}_{\mathbf{S}}(\mathbf{A})$ via the *end*

$$\mathbf{A} \cong \int_{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}).$$

Theorem

Let $\mathbf{M}$ be a bounded lattice-based algebra with smallest subalgebra $\mathbf{E} = \langle 0, 1 \rangle$. Then $\mathbf{M}$ is semi-primal if and only if there exists a topological adjunction

$$\mathfrak{P}_{\mathbf{M}} \vdash \mathfrak{s} \vdash \mathfrak{P}_{\mathbf{E}}.$$



Characterizing semi-primality

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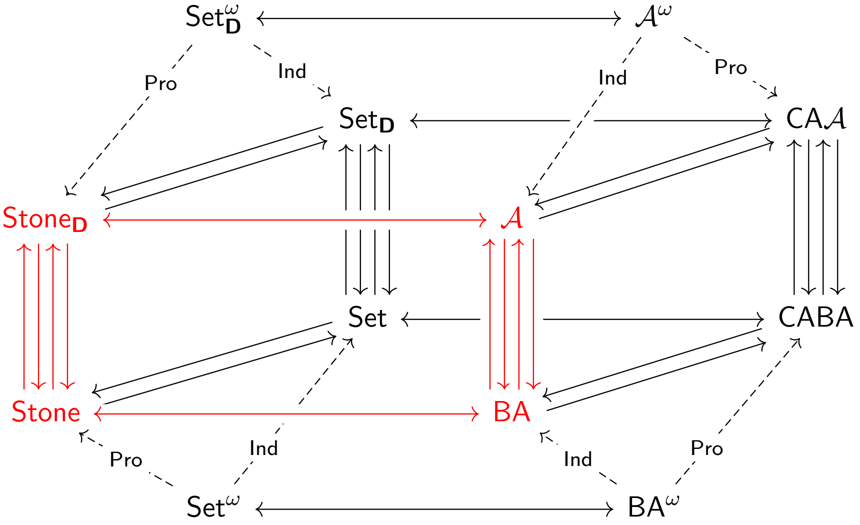
In particular, if **M** has no proper subalgebras this adjunction 'collapses' to a categorical equivalence and we recover Hu's Theorem.[2]

Corollary (Hu)

A variety is generated by a primal algebra if and only if it is categorically equivalent to BA.

[2] Hu 1971 [11]

Dualities via categorical completions (ctd.)



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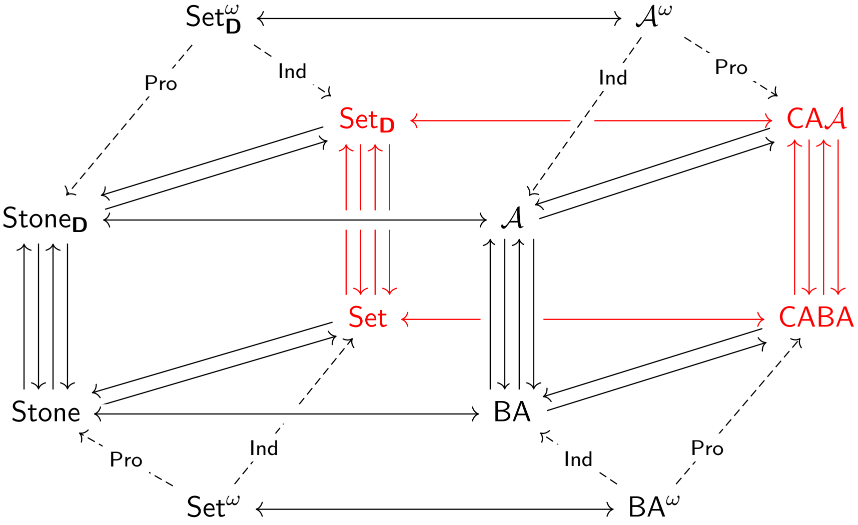

Discrete semi-primal duality

$$\mathsf{Set}_{\mathbf{D}} \underset{\Sigma'_{\mathrm{dis}}}{\overset{\Pi'_{\mathrm{dis}}}{\rightleftarrows}} \mathsf{CA}\mathcal{A}$$

Definition (The category $\mathsf{Set}_{\mathbf{D}}$)

The category $\mathsf{Set}_{\mathbf{D}}$ has objects (X, v) where $X \in \mathsf{Set}$ and $v \colon X \to \mathbb{S}(\mathbf{D})$.
A morphism $f \colon (X_1, v_1) \to (X_2, v_2)$ is a map $f \colon X_1 \to X_2$ with $v_2(f(x)) \leq v_1(x)$ for all $x \in X$.

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Theorem

An algebra $\mathbf{A} \in \mathcal{A}$ is a member of $\mathsf{CA}\mathcal{A}$ if and only if its Boolean skeleton $\mathfrak{S}(\mathbf{A})$ is a member of CABA.

Table of Contents

Many-valued modal logic

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- We define $\mathfrak{M}, w \Vdash \varphi$ iff $\mathrm{Val}(w, \varphi) = 1$.
- Recover classical modal logic if $\mathbf{D} = \mathbf{2} \in \mathrm{BA}$.

Examples from many-valued modal logic (1)

Let $\mathbf{D}$ be the $(n+1)$-element finite MV-chain

$$\text{Ł}_n = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle.$$

Hansoul, Teheux 2013 [10] ; Bou, Esteva, Godo, Rodríguez 2011 [1]

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The algebraic counterpart of the corresponding modal logic:

Definition

A *modal MV_n-algebra* is an algebra $(\mathbf{A}, \Box)$ with $\mathbf{A} \in MV_n = \mathbb{HSP}(\mathbf{Ł}_n)$,

- $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$,
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Examples from many-valued modal logic (2)

$$\mathbf{H} = \langle H, \wedge, \vee, \rightarrow 0, 1, (T_d \mid d \in H)\rangle,$$

where $\langle H, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a finite Heyting algebra expanded by unary

$$\mathrm{T}_d(x) = \begin{cases} 1 & \text{if } x = d, \\ 0 & \text{if } x \neq d. \end{cases}$$

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Maruyama 2009 [15]

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Examples from many-valued modal logic (3)

Let **D** be given by the $(n+1)$-element Łukasiewicz-Moisil chain

$$\mathbf{M}_n = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, \neg, 0, 1, (\tau_d \mid d \in M_n)\rangle.$$

where $\neg$ is the MV-negation and $\tau_d = \chi_{\{x \geq d\}}$ similar to before.

Diaconescu, Georgescu 2007 [6]

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The algebraic counterpart of the corresponding tense logic:

Definition

A *tense* $\text{Ł}M_n$*-algebra* is an algebra $(\mathbf{A}, G, H)$ with $\mathbf{A} \in \text{ŁM}_n = \mathbb{HSP}(\mathbf{M}_n)$,

- $G(x \wedge y) = Gx \wedge Gy$ and $G1 = 1$,
- $H(x \wedge y) = Hx \wedge Hy$ and $H1 = 1$,
- $x \leq GPx$ and $x \leq HFx$,
- $G\tau_d(x) = \tau_d(Gx)$ for all $d \in M_n \backslash \{0\}$,
- $H\tau_d(x) = \tau_d(Hx)$ for all $d \in M_n \backslash \{0\}$.

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Algebras and Coalgebras

Let C be a category and let $\mathsf{F}\colon \mathsf{C} \to \mathsf{C}$ be an endofunctor.

$\alpha\colon \mathsf{F}(A) \to A$	$\gamma\colon X \to \mathsf{F}(X)$
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Morphisms:

$$\begin{array}{ccc} \mathsf{F}(A_1) & \xrightarrow{\alpha_1} & A_1 \\ {\scriptstyle \mathsf{F}h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathsf{F}(A_2) & \xrightarrow[\alpha_2]{} & A_2 \end{array} \qquad\qquad \begin{array}{ccc} X_1 & \xrightarrow{\gamma_1} & \mathsf{F}(X_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{F}f} \\ X_2 & \xrightarrow[\gamma_2]{} & \mathsf{F}(X_2) \end{array}$$

Gives rise to categories Alg(F) and Coalg(F).

Jónsson-Tarski duality, coalgebraically

$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

Start with Stone duality $\Pi\colon \mathsf{Stone} \to \mathsf{BA}$ (takes clopens) and $\Sigma\colon \mathsf{BA} \to \mathsf{Stone}$ (takes ultrafilters).

Kupke, Kurz, Venema 2003 [14]

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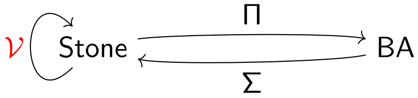

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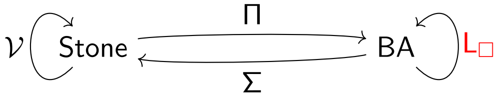

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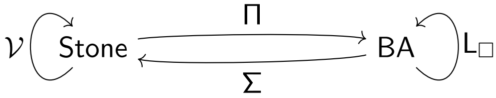

$\delta \colon \mathsf{L}_\Box \Pi \Rightarrow \Pi \mathcal{V}$

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Jónsson-Tarski duality: There is a natural isomorphism $\mathsf{L}_\Box \Pi \cong \Pi \mathcal{V}$.

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Classical modal logic, coalgebraically

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Kupke, Kurz, Pattinson 2004 [13]

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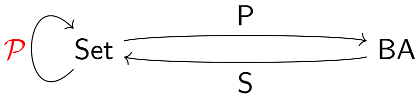

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$$\mathcal{P} \circlearrowright \mathsf{Set} \underset{\mathsf{S}}{\overset{\mathsf{P}}{\rightleftarrows}} \mathsf{BA} \circlearrowleft \color{red}{\mathsf{L}_\Box}$$

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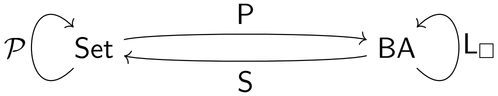
Kupke, Kurz, Pattinson 2004 [13]

Classical modal logic, coalgebraically



$\delta: L_\Box P \Rightarrow P\mathcal{P}$

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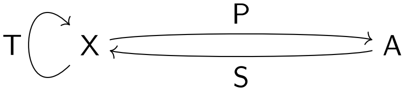
Sending a Kripke frame to its *complex algebra* can be realized by a *natural transformation* $L_\Box P \Rightarrow P\mathcal{P}$.

Kupke, Kurz, Pattinson 2004 [13]

Abstract and concrete coalgebraic logics



Definition (Coalgebraic logic)

Let X be a concrete category, let A be a variety of algebras, let P and S establish a dual adjunction and let $T\colon X \to X$ be an endofunctor.

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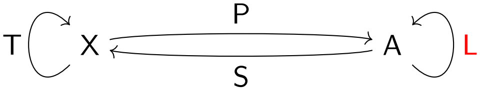

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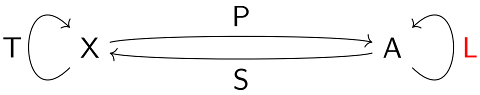
Abstract and concrete coalgebraic logics



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2. A *concrete coalgebraic logic* for T is a triple (L, δ, E) consisting of an abstract coalgebraic logic (L, δ) and a presentation E of L by operations and equations.

One-step completeness and expressivity

Definition (One-step completeness)

An abstract coalgebraic logic (L, δ) for T is *one-step complete* if δ is a monomorphism, *i.e.*, every component of δ is injective.

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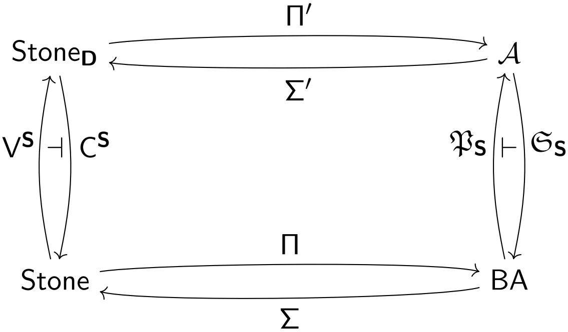
For example, the abstract coalgebraic logic $(\mathsf{L}_\Box, \delta)$ for $\mathcal{P}_{\text{fin}}$ is expressive. This is also known as the *Hennessy-Milner property*.

Lifting algebra-coalgebra dualities



$$(X, \mathbf{v}) \cong \int^{\mathbf{S}} \mathsf{V}^{\mathbf{S}} \mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) \text{ and } \mathbf{A} \cong \int_{\mathbf{S}} \mathfrak{P}_{\mathbf{S}} \mathfrak{S}_{\mathbf{S}}(\mathbf{A}).$$

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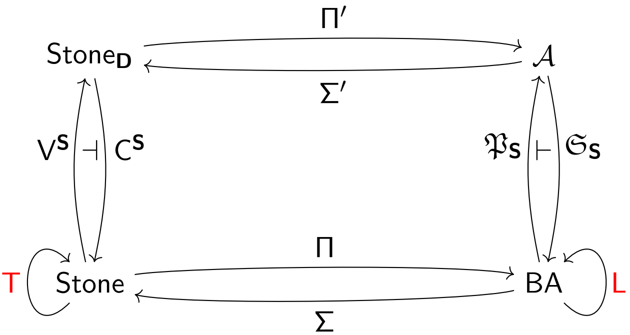

Suppose T and L are duals of each other.

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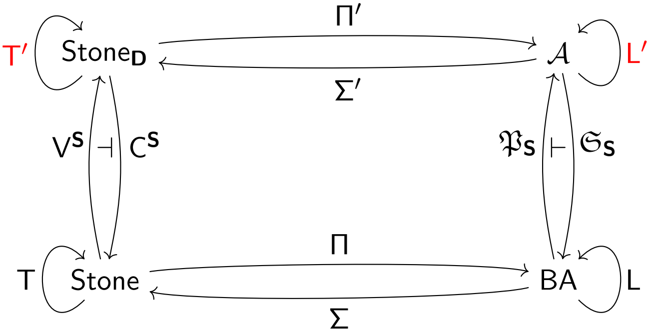
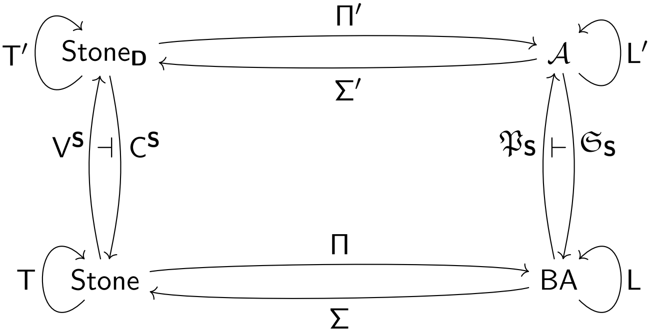


Suppose T and L are duals of each other. Define

$$\mathsf{T}'(X, \mathbf{v}) \cong \int^{\mathbf{S}} \mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) \text{ and } \mathsf{L}'(\mathbf{A}) \cong \int_{\mathbf{S}} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}).$$

Lifting algebra-coalgebra dualities



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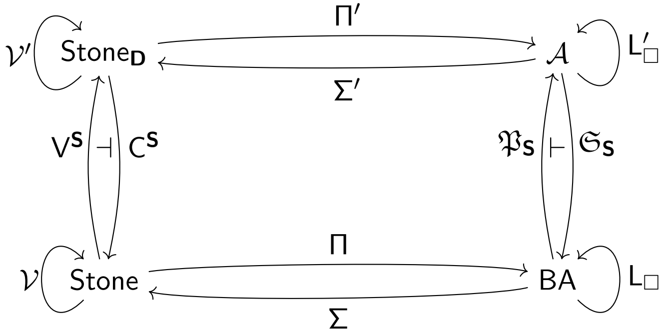
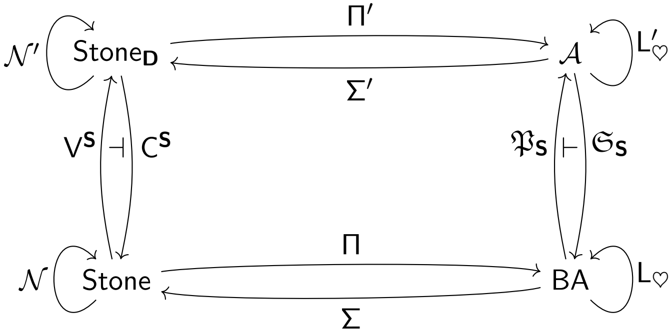
Then T′ and L′ are duals of each other as well.

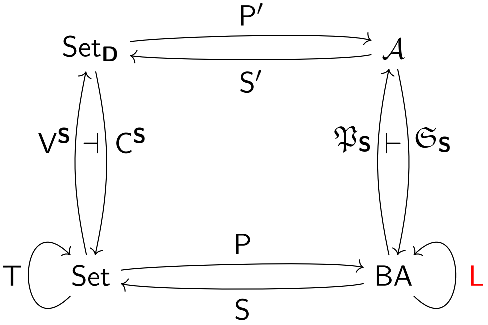


For example, this can be used to obtain Maruyama's [16] 'semi-primal version' of Jónsson-Tarski duality as lifting of the 'original' Jónsson-Tarski duality.

Lifting algebra-coalgebra dualities



It can also be used to obtain a ‘semi-primal version’ of Došen duality from the ‘original’ one, described as algebra-coalgebra duality by Bezhanishvilis, de Groot [2].



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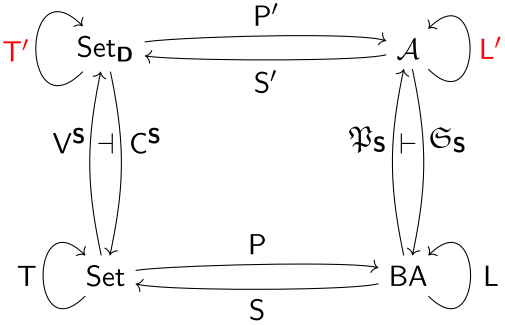
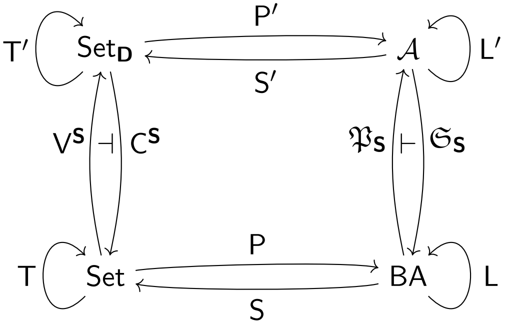


$\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$

Start with an abstract coalgebraic logic (L, δ) for T.
Similarly to before, we can lift T and L to T′ and L′.

Lifting abstract coalgebraic logics



$\color{red}{\delta'}\colon \mathsf{L'P'} \Rightarrow \mathsf{P'T'}$

$\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$

Start with an abstract coalgebraic logic (L, δ) for T.
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How to obtain δ' from δ

$$\mathsf{L}'\mathsf{P}'(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}\mathsf{P}'(X,v) \xrightarrow{\text{limit}} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}\mathsf{P}'(X,v)$$

$$\mathsf{P}'\mathsf{T}'(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X,v) \xrightarrow{\text{limit}} \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X,v)$$

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 & & \downarrow \cong \\
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\end{array}$$

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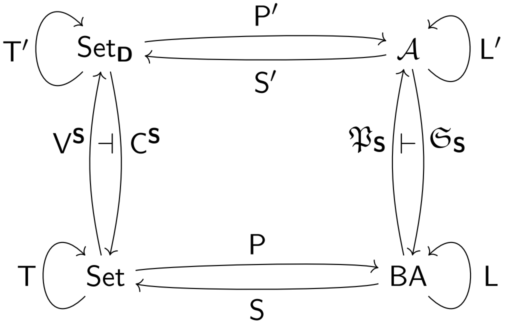

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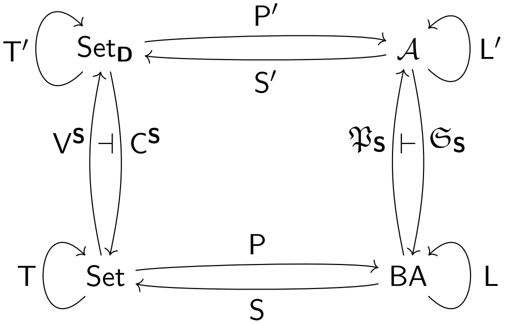
Lifting abstract coalgebraic logics



$\delta' \colon \mathsf{L'P'} \Rightarrow \mathsf{P'T'}$

$\delta \colon \mathsf{LP} \Rightarrow \mathsf{PT}$

Start with an abstract coalgebraic logic (L, δ) for T.
Similarly to before, we can lift T and L to T′ and L′.
Furthermore, we can define an appropriate δ' from δ.
Thus we obtain a many-valued abstract coalgebraic logic (L', δ') for T′.

One-step completeness and expressivity

Theorem

Let (L', δ') be the lifting of (L, δ) as defined on the previous slides.

1. If (L, δ) is one-step complete, then (L', δ') is one-step complete.

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Corollary

If (L, δ) is one-step complete/expressive, then so is $(\mathsf{L}', \delta^\top)$.

$$\mathsf{T} \circlearrowright \mathsf{Set} \underset{\mathsf{C}^{\mathbf{D}}}{\overset{\mathsf{V}^{\mathbf{D}}}{\rightleftarrows}} \mathsf{Set}_{\mathbf{D}} \underset{\mathsf{S}'}{\overset{\mathsf{P}'}{\rightleftarrows}} \mathcal{A} \circlearrowleft \mathsf{L}' \qquad \delta^\top = \delta' \mathsf{V}^\top$$

Lifting concrete coalgebraic logics (1)

$$\tau_d(x) = \begin{cases} 1 & \text{if } x \geq d \\ 0 & \text{if } x \not\geq d. \end{cases}$$

Theorem

Let $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ have a presentation by one unary operation $\Box$ and equations which all hold in $\mathbf{D}$ if $\Box$ is replaced by any τ_d, including the equation $\Box(x \wedge y) = \Box x \wedge \Box y$.

Then L' has a presentation by one unary operation $\Box'$ and the following equations.

- $\Box'$ satisfies all equations which the original $\Box$ satisfies,
- $\Box'\tau_d(x) = \tau_d(\Box' x)$ for all $d \in D \backslash \{0\}$.

Lifting concrete coalgebraic logics (2)

$$\eta_d(x) = \begin{cases} 0 & \text{if } x \leq d \\ 1 & \text{if } x \not\leq d. \end{cases}$$

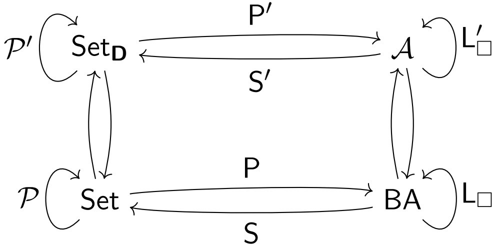
Theorem

Let $\mathrm{L}\colon \mathrm{BA} \to \mathrm{BA}$ have a presentation by one unary operation $\Diamond$ and equations which all hold in $\mathbf{D}$ if $\Diamond$ is replaced by any η_d, including the equation $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$.
Then L' has a presentation by one unary operation $\Diamond'$ and the following equations.

- $\Diamond'$ satisfies all equations which the original $\Diamond$ satisfies,
- $\Diamond' \eta_d(x) = \eta_d(\Diamond' x)$ for all $d \in D \backslash \{1\}$.

Many-valued modal logic as lifting of classical modal logic

The functor $\mathsf{L}_\Box$ has a presentation by $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

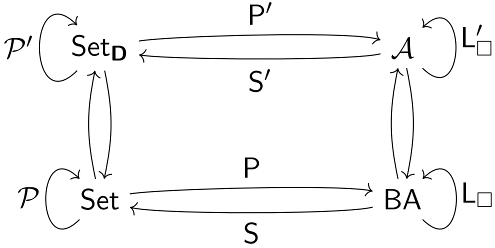


Many-valued modal logic as lifting of classical modal logic

The functor $\mathsf{L}_\Box$ has a presentation by $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

Therefore, the functor $\mathsf{L}'_\Box$ has a presentation by

$\Box'(x \wedge y) = \Box' x \wedge \Box' y$, $\Box' 1 = 1$ and $\Box' \tau_d(x) = \tau_d(\Box' x)$ for all $d \in D \setminus \{0\}$.



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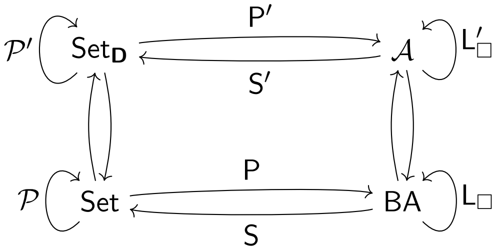
$(\mathsf{L}_\Box, \delta)$ is (one-step) complete $\Rightarrow$ $(\mathsf{L}'_\Box, \delta')$ is (one-step) complete.



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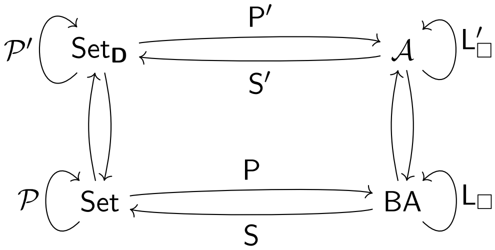
Replacing $\mathcal{P}$ by $\mathcal{P}_{\text{fin}}$: $(\mathsf{L}_\Box, \delta)$ is expressive $\Rightarrow$ $(\mathsf{L}'_\Box, \delta')$ is expressive.



Lifted semantics

Definition ($\mathbf{Set}_{\mathbf{D}}$-frame & $\mathbf{Set}_{\mathbf{D}}$-model)

A $\mathsf{Set}_{\mathbf{D}}$-*frame* is a structure (W, v, R) with $v \colon X \to \mathbb{S}(\mathbf{D})$ and binary relation $R \subseteq W^2$ satisfying

$$wRw' \Rightarrow v(w') \subseteq v(w)$$

for all $w, w' \in W$.

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for all $w, w' \in W$. A $\mathsf{Set}_{\mathbf{D}}$-*model* adds a $\mathsf{Set}_{\mathbf{D}}$-*valuation* $\mathsf{Val}\colon W \times \mathsf{Prop} \to \mathbf{D}$ which satisfies

$$\mathsf{Val}(w, p) \in v(w)$$

for all $w \in W$.

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For example, if $\mathbf{D} = \text{Ł}_2$ is the three-element MV-chain, the formula

$$\Diamond(p \vee \neg p)$$

is valid on a $\mathsf{Set}_{\mathbf{D}}$-frame if and only if $\forall w \exists w'\colon wRw' \wedge v(w') = \mathbf{2}$, while it is not satisfied in any frame.

Alternative axiomatizations: Some case studies (1)

If $\mathbf{D} = \mathbf{Ł}_n$ is a finite MV-chain, then $\mathrm{L}'_\Box$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\Box(x \oplus x) = \Box x \oplus \Box x$,

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If $\mathbf{D}$ is a finite bounded residuated lattice with τ_e (monoid unit e) and truth-constants, then $\mathrm{L}'_\Box$ has a presentation by

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In particular, if $\mathbf{D}$ is a finite $\mathrm{FL_{ew}}$-algebra with truth-constants where only $0, 1$ are idempotent, then $\mathrm{L}'_\Box$ has a presentation by

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(B4) $\Box(r \rightarrow x) = r \rightarrow \Box x$ all $r \neq 0$.

Alternative axiomatizations: Some case studies (2)

If **D** is a finite bi-Heyting algebra with truth-constants and with a unique atom and coatom, then $\mathrm{L}'_{\Box}$ has a presentation by

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\Box\big(\neg(1 \leftarrow x)\big) = \neg(1 \leftarrow \Box x)$,

(B4) $\Box(b \rightarrow x) = b \rightarrow \Box x$ all $b \neq 0$,

(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$,

(D1) $\Diamond 0 = 0$,

(D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,

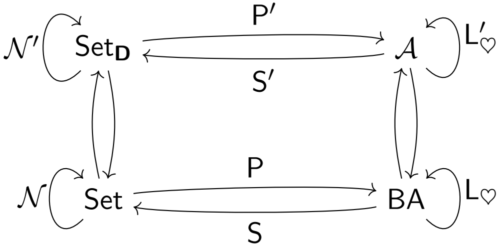
(D3) $\Diamond\big(1 \leftarrow (\neg x)\big) = 1 \leftarrow (\neg \Diamond x)$,

(D4) $\Diamond(x \leftarrow b) = \Diamond x \leftarrow b$ all $b \neq 1$,

(P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

Many-valued modal logic for crisp neighborhoods

The neighborhood functor $\mathcal{N}$ is the contravariant powerset functor composed with itself. The functor $\mathsf{L}_\heartsuit$ has a presentation by one unary operation $\heartsuit$ and *no* equations.



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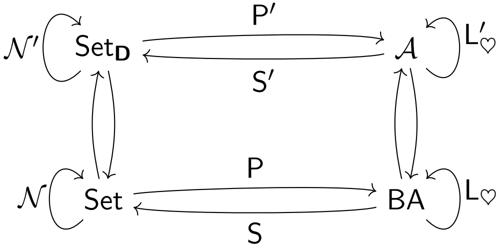

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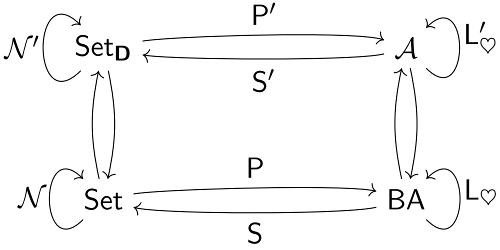

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We don't know a concrete presentation for $\mathsf{L}'_\heartsuit$ yet, unless **D** is primal.

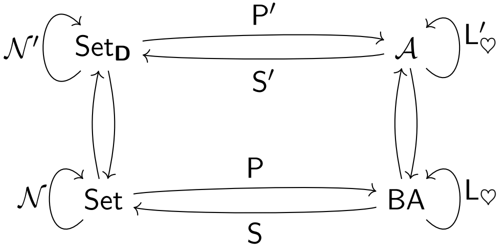

Table of Contents

Finite positive MV-chains

Definition (finite positive MV-chain)

Let $n \geq 1$ be a natural number. The $(n+1)$*-element positive* MV*-chain* is given by

$$\mathbf{P}\textbf{Ł}_n = \langle \{0, \frac{1}{n}, \ldots \frac{n-1}{n}, 1\}, \wedge, \vee, \odot, \oplus, 0, 1 \rangle,$$

understood as a reduct of $\textbf{Ł}_n$.

We write PMV_n for the variety $\mathbb{HSP}(\mathbf{P}\textbf{Ł}_n)$ generated by $\mathbf{P}\textbf{Ł}_n$.

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3. The unary terms $\tau_d = \chi_{\{x \geq d\}}$ are still term-definable in $\mathbf{P\L}_n$.

Natural duality for PMV_n

Theorem

There is a (natural) duality between PMV_n and a category $\mathcal{X}_n$ of *Priestley spaces with additional subrelations of the order*.
An optimal *dualising structure* is determined by a subset $\mathcal{S}_n \subseteq \mathbb{S}(\leq)$.

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Example (Dual category for PMV_2)

In the case of the three-element chain $\mathbf{P\text{Ł}}_2$, we have $\mathcal{S}_2 = \{\lhd, \leq\}$ with

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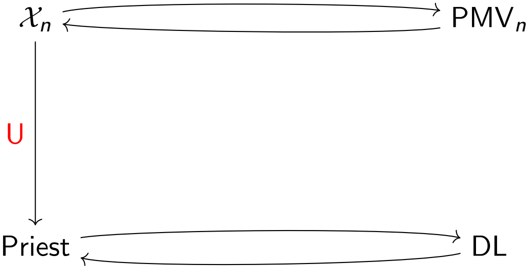
$$\lhd = \{(a, b) \in \mathbf{PŁ}_2^2 \mid a = 0 \text{ or } b = 1\}.$$

The members of $\mathcal{X}_2$ are of the form $(X, \leq^X, \lhd^X)$ and need to satisfy an additional separation property:
If $x \ntriangleleft^X y$ but $x \leq^X y$, then there exist a clopen upset U and a clopen downset D with the following properties

- $x \notin D$ and $y \notin U$,
- For all $z, z' \in X$, if $z \lhd^X z'$ then $z \in D$ or $z' \in U$.

Distributive skeletons



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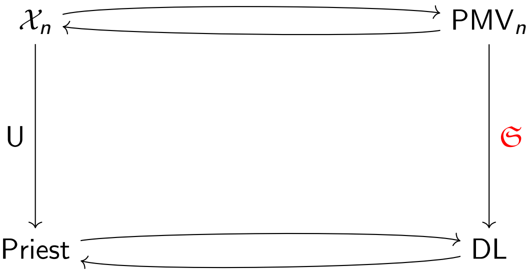

There again is a forgetful functor U.
Its dual is given by the *distributive skeleton* defined analogously to the Boolean skeleton.

$$\mathrm{PMV}_n(\mathbf{A}, \mathbf{PŁ}_n) \cong \mathrm{DL}(\mathfrak{S}(\mathbf{A}), \mathbf{2}) \text{ via restriction } p \mapsto p|_{\mathfrak{S}(\mathbf{A})}.$$

Introduction to positive modal logic

- *Positive modal logic*: The $\{\wedge, \vee, 0, 1, \Box, \Diamond\}$-reduct of standard modal logic. Algebraically, move to *modal distributive lattices* $(\mathbf{L}, \Box, \Diamond)$ with

(B1) $\Box 1 = 1$,
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- Shortcomings of 'usual' semantics over Set-frames. *E.g.*, the consequence pairs

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define the same class of frames but are not mutually inter-derivable anymore.

- Better-behaved semantics over Pos-*frames* $(X, \leq, R)$ adding a *partial order*. Now the above correspond to the distinct classes with reflexive

$$R_\Box := R \circ \leq \text{ and } R_\Diamond := R \circ \geq$$

Dunn 1995 [7] ; Celani, Jansana 1997 [3]

Positive modal logic over finite MV-chains: Semantics

Signature $\mathcal{L}^{\Box\Diamond}_{\mathsf{PMV}} = \{\wedge, \vee, \oplus, \odot, 0, 1, \Box, \Diamond\}$, inductively define formulas $\mathsf{Form}^{\Box\Diamond}_{\mathsf{PMV}}$ (with countable set of propositional variables Prop) as usual.

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Definition (Pos_n-frame & Pos_n-model)

A Pos_n*-frame* is a structure $(X, \leq, v, R)$ such that $v \colon X \to \mathbb{S}(\mathbf{P\L}_n)$ and the accessibility relation satisfies the *compatibility conditions*

- For all $x, y \in X$ it holds that

$$x \leq y \Rightarrow R[x] \leq_{\mathrm{EM}} R[y].$$

- Whenever $x, y \in X$ satisfy $y \in R[x]$, there exist $y', y'' \in R[x]$ with

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A Pos$_n$-*model* adds a valuation $\mathsf{Val} \colon X \times \mathsf{Prop} \to \mathbf{P}\text{Ł}_n$ satisfying

- If $x \leq y$, then $\mathsf{Val}(x, p) \leq \mathsf{Val}(y, p)$ for all $p \in \mathsf{Prop}$.
- $\mathsf{Val}(x, p) \in v(x)$ for all $x \in X$ and $p \in \mathsf{Prop}$.

Positive modal logic over finite MV-chains: Algebras

Definition (Modal PMV_n-algebras)

A *modal* PMV_n*-algebra* is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$, where $\mathbf{A} \in \mathrm{PMV}_n$ and $\Box, \Diamond \colon A \to A$ satisfy

B1 $\Box 1 = 1$,

B2 $\Box(x \wedge y) = \Box x \wedge \Box y$,

B3 $\tau_d(\Box x) = \Box \tau_d(x)$,

P1 $\Box(x \vee y) \leq \Box x \vee \Diamond y$,

D1 $\Diamond 0 = 0$,

D2 $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,

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We denote the variety of modal PMV_n-algebras by mPMV_n.

The axioms B3 and D3 can be equivalently replaced by

B$\oplus$ $\Box(x \oplus x) = \Box x \oplus \Box x$,

B$\odot$ $\Box(x \odot x) = \Box x \odot \Box x$,

D$\oplus$ $\Diamond(x \oplus x) = \Diamond x \oplus \Diamond x$,

D$\odot$ $\Diamond(x \odot x) = \Diamond x \odot \Diamond x$.

Algebraic completeness

Theorem

Let $\varphi, \psi \in \mathrm{Form}_{\mathrm{PMV}}^{\Box\Diamond}$ be modal PMV-formulas. Then the following are equivalent.

1. $\varphi \vdash \psi$ is valid on all Pos_n-frames.
2. $\varphi \vdash \psi$ is valid on all Set-frames.
3. $mPMV_n \models \varphi \leq \psi$.

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The additional 'richness' of the semantics over Pos_n-frames plays a role when it comes to axiomatic extensions, definability and canonicity.

A case study in canonicity

In modal logic over $\mathbf{Ł}_n$ (*i.e.*, with negation), the formulas

$$\Box(x \oplus x) \to \Box x \text{ and } \Diamond(x \oplus x) \to \Diamond x$$

both define the Set_n-frames (X, v, R) which satisfy $xRy \Rightarrow v(y) = \mathbf{PŁ}_1$. The former is canonical[3], so the latter is derivable from it.

[3] Hansoul, Teheux 2013 [10]

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In modal logic over $\mathbf{PŁ}_n$, this is not the case anymore. The semantics over Pos_n reflect this as for any Pos_n-frame $\mathfrak{F}$ we have:

1. The consequence pair $\Box(p \oplus p) \vdash \Box p$ is valid in $\mathfrak{F}$ if and only if $\mathfrak{F}$ satisfies

$$\forall x \forall y \colon \big(xRy \to \exists y' : (xRy' \wedge y' \leq y \wedge v(y') = \mathbf{PŁ}_1)\big).$$

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1. Perspectives on semi-primal varieties

A. Kurz, W.P., B. Teheux: *New perspectives on semi-primal varieties*. Journal of Pure and Applied Algebra **228**(4), 107525, 2024. doi: 10.1016/j.jpaa.2023.107525

2. Many-valued coalgebraic logic: From Boolean algebras to semi-primal varieties

A. Kurz, W.P., B. Teheux: *Many-valued coalgebraic logic over semi-primal varieties*. To appear in: Logical Methods in Computer Science (LMCS), 2024. https://arxiv.org/abs/2308.14581

A. Kurz, W.P.: *Many-Valued Coalgebraic Logic: From Boolean Algebras to Primal Varieties*. CALCO, 2023. doi: 10.4230/LIPIcs.CALCO.2023.17

3. Many-valued positive modal logic over finite MV-chains

W.P.: *Positive modal logic over finite MV-chains*. To appear in: Advances in Modal Logic (AiML), 2024.

W.P.: *Natural dualities for varieties generated by finite positive MV-chains*. To appear in: Algebra Universalis, 2024. https://arxiv.org/abs/2309.16998

Future Research

Investigate broader classes of algebras of truth-degrees, *e.g.*

- Quasi-primal = Finite discriminator algebras

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Investigate broader classes of logics, *e.g.*

- Many-valued modal logic with many-valued accessibility relation
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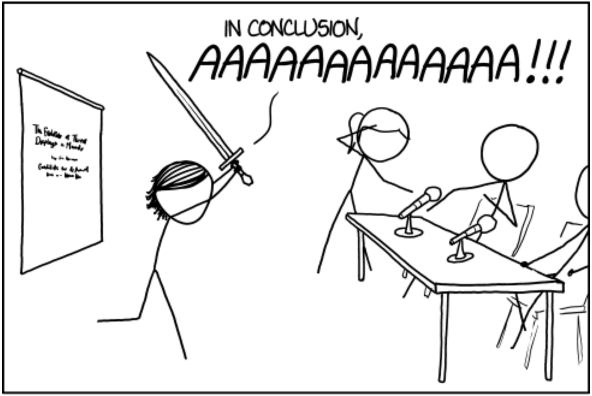
- Propositional Dynamic Logic, Linear Temporal Logic, *etc.*

Thanks for your attention!



THE BEST THESIS DEFENSE IS A GOOD THESIS OFFENSE.

https://xkcd.com/1403/

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