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ALGEBRAIC AND COALGEBRAIC MODAL LOGIC: FROM BOOLEAN ALGEBRAS TO SEMI-PRIMAL VARIETIES

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Dedicated to my family

Abstract

We study many-valued variants of modal logic and, more generally, coalgebraic logic, under the assumption that the underlying algebra of truth-degrees is a semi-primal bounded lattice-expansion. Throwing light on the category-theoretical relation between the variety generated by such an algebra and the variety of Boolean algebras, we describe multiple adjunctions between these varieties. In particular, we show that the Boolean skeleton functor has two adjoints, both defined by taking certain Boolean powers, and we identify properties of these adjunctions which fully characterize semi-primality of an algebra. Making use of these relations, we show how to lift endofunctors encoding classical coalgebraic logics in order to obtain many-valued counterparts of these logics. We show that one-step completeness, expressivity and finite axiomatizability are preserved under this lifting, and we show that for classical modal logic and similar cases, an axiomatization of the lifted many-valued logic can be directly obtained from an axiomatization of the original logic. Lastly, we develop the theory of natural dualities for varieties generated by finite positive MV-chains and apply this to the algebraic study of the negation-free fragment of bimodal finite Łukasiewicz logic.

Acknowledgements

> *For this reason at the very beginning of this discussion I advisedly pointed to* experience *as furnishing the proof that the social impulses increase the capacity for happiness; and this makes the deductive derivation unnecessary.*
>
> – Moritz Schlick,
> (1930, while part of the Vienna Circle)[1]

First and foremost, I want to thank my supervisor Bruno Teheux for his constant support - both mathematical and mental - throughout all of my doctoral studies, for always taking the time to discuss (life and) mathematics, for giving me the freedom to pursue the things I did want to do and also for pushing me to do the things I did not want to do.

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[1] [Sch30b, p.190]; Dt.: *Deswegen habe ich zu Beginn dieser Erwägung mit vollem Bewußtsein gerade auf die* Erfahrung *hingewiesen als einen Beweis für die Förderung der Glücksfähigkeit durch die sozialen Triebe, welcher die deduktive Ableitung entbehrlich macht.* [Sch30a, p.139]

Further thanks go to the organizers of the *Barcelona Seminar on Non-Classical Logics* for having me present my research during this research visit. In a similar vein, I am grateful to the organizers of the *LLAMA Seminar* in Amsterdam, in particular Tobias Kappé, for inviting me to speak there in March 2024 and to Igor Sedlár for organizing the *Prague Workshop on Kleene Algebra and Many-Valued Logic* in November 2023 and having me as speaker there as well.

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Contents

Introduction

> *With "many-valued" systems of propositional logic a new domain of investigation has, in recent years, come into being; a domain which opens up surprising and unsuspected vistas. History, however, need only report about this new logic in the future.*
>
> – Jan Łukasiewicz
> (1935)[1]

> *The formulation and analysis of logics for various categories of coalgebras is the subject of current research. The assessment of the impact of these investigations on the evolution of modal logic is a task for the historians of the future.*
>
> – Robert Goldblatt
> (2003)[2]

Most people know that logic is important to do mathematics, fewer seem to know that the converse is nowadays equally true. In this thesis, we use tools from diverse mathematical areas such as *algebra*, *category theory* and *topology* to investigate the combination of two *non-classical logics*, namely *many-valued logic* and *(coalgebraic) modal logic*. Before we go into more detail about the content and structure of the thesis (in the second and third section of this introduction, respectively), in the following section we briefly recall some of the history of mathematical logic as a whole and of these two non-classical logics in particular.

Historical overview

We proceed to give a very short and rather selective summary of some important historical developments in mathematical logic related to the main topics of this thesis.

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[1] German original in [Łuk35, p.127]; English translation from [Łuk70, p.217]

[2] [Gol03, p.85]

But first, a few words of caution. In the author's opinion, history of science (or rather, the public perception and inexpert reproduction thereof) can have an 'unhealthy' tendency to attribute ground-breaking developments or scientific paradigm changes to but a handful of individuals (oftentimes even a single person). This contributes towards the (in the author's opinion) unrealistic idea that these 'selected few' have completely invented or revolutionized certain areas out of 'thin air' and, therefore, have an absolute 'ownership' of them. This, in return, sometimes even leads to a form of 'mystification' of these individuals as 'geniuses'. This can be harmful in at least two ways. First, it can give people too much 'authority', also outside of their own realms of research and expertise. Second, when it mixes up with the societal prejudices and discrimination of the time (where, sadly, the present is not yet exempt), there can be a bias towards who is even 'allowed' to be a so-called 'genius'[3]. Naturally, mathematical claims should never be judged by *who* is making them.

Unfortunately, usually for the sake of brevity, the author must himself be found guilty of reproducing the above-mentioned mistake throughout the thesis, arguably most severely in this historical overview. Nonetheless, the reader is encouraged to keep in mind that, more often than not, science is a collective endeavor and does not happen in a 'vacuum', that is, scientists are always embedded into the (scientific and general) societies and developments of their respective times[4].

The rise of mathematical logic

Formal logic, the science of valid arguments in terms of their *patterns* rather than their *contents*, exists at least since antiquity. For example, Aristotle included it in his *Organon* (*i.e.*, 'toolkit'), describing the *syllogisms* (which may nowadays be seen as 'ancient predicate logic'), and the Stoic philosopher Chrysippus analyzed sentences in terms of their *logical compositions* (which may nowadays be seen as 'ancient propositional logic'). For a long time, logic has been considered a part of philosophy rather than of mathematics (for the history of 'pre-mathematical' logic see, *e.g.*, [KK62, Chapters I-V]).

While Leibniz already highly anticipated (and to some degree developed) ideas of a *logical calculus* to be performed within some *characteristica universalis* (*i.e.*, 'universal language') during the late 17th century [KK62, Sections V.2-V.3], the 'birth' of *symbolic logic* or *mathematical logic* is commonly dated to 1847, with Boole's *Mathematical Analysis of Logic* [Boo47]

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[3] *e.g.*, compare how often you heard someone call a man a genius compared to a woman.

[4] I want to mention here that the above comments are of course *not* intended to *diminish* the importance of any particular work or the people involved in it.

and De Morgan's *Formal Logic* [DM47]. Boole in particular, also in his second book on the topic *The Laws of Thought* [Boo54], recognized the close resemblance between the logical operations of propositions and the *algebraic* manipulations of numbers, and in essence already described what is nowadays commonly known as *Boolean algebra* (also see the quote at the beginning of Chapter 3). The algebraic approach was further developed and popularized throughout the latter half of the 19th century, for example in Venn's *Symbolic Logic* [Ven81] (including the famous diagrams named after him) and Schröder's three-volume lectures series *Algebra der Logik* (1890-1905, beginning with [Sch90]), which also incorporated previous advances in the topic due to Peirce, like the addition of symbolic quantifiers.

Together with (but independently of) Peirce, it is usually Frege with his *Begriffsschrift* [Fre79], who is credited with first providing a system of predicate logic, which he based on the mathematical concepts of *functions and variables.* With some delay, Frege's work became highly influential when questions about the *logical foundations of mathematics* began to arise after the turn of the century. For example, Russell (whose famous paradox is also a reason why these questions came up in the first place) and Whitehead adopted his *logicism* (that is, the idea that mathematics can fundamentally be reduced to logic) when writing their three-volume *Principia Mathematica* (1910-1913, beginning with [RW10]). The *formalists*, on the other hand, aimed to reduce mathematics to purely *syntactical manipulations.* Most prominently among them, Hilbert was led to develop *proof theory* and his well-known program including the call for a complete, consistent and decidable formalism for mathematics in its entirety [Hil22, Hil28].

The formalist (and, to some degree, also the logicist) program was effectively shown to be impossible to carry out due to Gödel's *incompleteness theorems* [Gö31]. Nevertheless, these 'metamathematical' considerations are at the base of most important developments of contemporary research areas in mathematical logic, such as Tarski's *semantic theory of truth* [Tar44] in model theory, Cohen's *forcing* technique [Coh63] in set theory, Gentzen's *sequent calculus* [Gen35] in proof theory and Turing's general theory of *computability* [Tur37].

Throughout the 20th century, besides the development of what is nowadays considered *classical logic* sketched above, the topic of *non-classical logics* has evolved into an interesting research area of its own. For example, Brouwer opposed both the logicist and the formalist programs and instead advocated for the idea of *intuitionism*, a 'constructivist' philosophy challenging the law of excluded middle. Formalized by Heyting [Hey30], intuitionistic logic, together with many other non-classical logics, gained increased popularity throughout recent years, further driven by applications in theoretical computer science and beyond. In the following two subsections, we give separate brief historical overviews of the two most important types of non-classical logics related to this thesis, namely many-valued logic and modal logic. For more information, for the former we refer the reader to [Pri08a, 7.12, 11.8] together with the references therein and for the latter we refer the reader to [Gol03].

Beyond bivalence: Many-valued logic

While classical logic is based on the principle of *bivalence*, meaning the assumption that there exist exactly two truth-values *true* and *false*, this principle has been challenged early on in Eastern philosophy, for example, in the Buddhist Nāgārjuna's 'four-valued' *Catuṣkoṭi* (see,*e.g.*, [Gan04, Pri10]).

In mathematical logic, systems of *many-valued* propositional logics arose from the 1920s onwards with the finitely-valued systems of Post [Pos21] and Łukasiewicz [Łuk20], who later on generalized his system to have infinitely many truth-degrees together with Tarski [ŁT30]. Another type of finitely-valued systems has been studied by Gödel [Gö32] in relationship to intuitionistic logic and by Dummett [Dum59] in its infinite version. Important three-valued systems have, for example, been studied by Kleene [Kle38], Bochvar [Boc38] and Priest [Pri79], where the third truth-value is essentially intended to describe different forms of *unknown*. While these systems are all linearly ordered, the four-valued logic on 'how a computer should think' named after Belnap and Dunn [Bel77, Dun76] includes additional truth-values *both* and *none* and is not linearly ordered.

Algebraic advances in this domain were, for example, made by Rosenbloom [Ros42] for Post logic and Moisil [Moi40] who aimed to algebraically study Łukasiewicz logic. This was ultimately achieved via Chang's *MV-algebras* [Cha58]. Nowadays, a common algebraic framework for various many-valued logics (and, more generally, *substructural logics*) is provided by *residuated lattices* (see, *e.g.*, [GJKO07]).

Residuated lattices (more specifically, BL-algebras) also play a major role in the algebraic study of Zadeh's [Zad75] and later on Hájek's [Há98] treatment of *fuzzy logic*, an area studied by the former in relation to his fuzzy set theory [Zad65], where elements of a set can have a *degree* of membership in the real unit interval $[0, 1]$. Similarly, in fuzzy logic, *vagueness* is taken into account, allowing for arbitrary truth-degrees inside this interval.

Many-valued logic in general (and fuzzy logic in particular) have nowadays found far-reaching applications in mathematics, philosophy and most notably theoretical computer science, ranging from hardware design of many-valued circuits to the analysis of neural networks in artificial intelligence.

Thus, it is likely that they will remain active research areas in the near future.

Bare necessities: Modal logic

Discussions about *modalities* like *possibility* or *necessity* can, again, be traced back at least until the times of Aristotle in the West and Nāgārjuna in the East (see, *e.g.*, [Sch21]) and are well-known to have appeared throughout the entire history of philosophy. The mathematical evolution of modal logic began in the early 20th century, around the same time as that of many-valued logic[5].

Dissatisfied with the material implication of classical logic used in the *Principia Mathematica*, Lewis [Lew18] began to study his famous systems of modal logic (S1) - (S5). After the development of *universal algebra* in its modern form by Birkhoff [Bir35], the two topics became heavily intertwined. This can, for example, be seen in the work of McKinsey and Tarski, who in [MT44] studied *closure algebras* in relation to the system (S4) (nowadays called *topological semantics*) and utilized this in [MT48] to algebraically show that intuitionistic logic can be translated into this system, as anticipated by Gödel [Gö33]. Shortly thereafter, Jónsson and Tarski [JT51] generalized closure algebras to arbitrary *Boolean algebras with operators* and established their *topological duality*, extending Stone's [Sto36] famous duality for Boolean algebras. The significance of these results in modal logic, however, remained overlooked for some time thereafter.

Attention shifted towards modal validity with respect to *relational semantics* of *possible worlds*, when these concepts were used in their modern form by Kripke [Kri59, Kri63]. Nevertheless, algebraic semantics remained important, in particular because they can be used in their *interplay* with Kripke semantics via *complex algebras* and *ultrafilter extensions*. This was exploited by Lemmon [Lem66], using it to prove completeness via the *canonical model* method. Algebraic semantics perhaps also regained some of their former attraction with the discovery of modal logics which are incomplete in terms of Kripke semantics, for example by Thomason [Tho72] in the case of a tense logic.

These considerations also inspired a closer investigation of the relation between modal logic and first-order (and monadic second-order) logic, leading for example to the syntactical description of a collection of *canonical* formulas by Sahlqvist [Sah75], the characterization of first-order properties of frames

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[5] In fact, Łukasiewicz' three-valued logic was originally also intended to deal with the classical, inherently modal, problem of *future contingents* and *determinism*

which are also *modally definable* by Goldblatt and Thomason [GT75] and the identification of modal logic (on the level of models) as the fragment of first-order logic invariant under *bisimulation* by van Benthem [vB84].

Like many-valued logic, the development of modal logic was also highly influenced by advances in *computer science*. This can, for example, be seen in the work of Hennessy and Milner showing that modal equivalence and bisimilarity coincide for image-finite *labelled transition systems* (seen as computational *processes*) [HM80, HM85], the work of Pratt [Pra76] reasoning about *programs* in the modal language of *propositional dynamic logic* and the work of Ladner [Lad77] about the *computational complexity* of systems of modal logic.

Since the turn of the 21st century, computer science has had another important influence on the development of modal logic through the integration of *coalgebraic* methods into the area. This is the point where we let our short history end and let the present enter the stage.

Introduction to the thesis

In order to get the 'best of both worlds', enabling us to reason about modalities in settings of vagueness, research on *many-valued modal logics* has been active in recent years (see, *e.g.*, [Fit91, DG07, Pri08b, Mar09, CR10, BEGR11, HT13, RJJ17, VEG17, MM18, RV21], to name a few). In this thesis, we study this topic by means of *algebra* and *coalgebra*. In particular, we study it from the perspective of *coalgebraic logic*, the generalization of modal logic introduced by Moss in 1999 [Mos99].

People commonly distinguish between three approaches to coalgebraic logic. The *relation lifting approach* was introduced by Moss himself, and the *predicate lifting* approach was initiated by Pattinson in [Pat03a]. A unifying framework for both of these is found in the *abstract approach* or *algebra-coalgebra approach* developed by Kurz *et al.* [KKP04, BK05, KR12]. For standard modal logic, the development of all three approaches has resulted in interesting insights, generalizations and novel proof techniques (for a general overview of coalgebraic logic and a large collection of literature we refer the reader to [KP11] and the bibliography therein). Thus, it is all the more surprising that very little research on *many-valued* coalgebraic logic exists thus far. Among the few examples are [BKPV13] following the relation lifting approach and [BD16, LL23] following the predicate lifting approach. To the best of the authors knowledge, this thesis (together with the papers [KP23, KPT24a] it is based on) takes the first steps towards many-valued coalgebraic logic following the *abstract approach*.

In the Boolean two-valued setting, an *abstract coalgebraic logic* for an endofunctor $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ is a pair (L, δ) consisting of an endofunctor $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ (essentially determining syntax) together with a natural transformation δ which, over the usual dual adjunction between Set and BA, is used to relate T-coalgebras to L-algebras (essentially determining semantics). Important properties of coalgebraic logics, like *one-step completeness* [Pat03a, KKP04] and *expressivity* [Pat04, Kli07, Sch08, JS09] then directly correspond to purely category-theoretical properties of δ.

To retrieve standard modal logic as an example of an abstract coalgebraic logic, we set $\mathsf{T} = \mathcal{P}$ being the covariant powerset functor (whose coalgebras are Kripke frames), $\mathsf{L} = \mathsf{L}_\Box$ being the functor which sends a Boolean algebra $\mathbf{B}$ to the free Boolean algebra generated by the set of formal expressions $\{\Box b \mid b \in B\}$ modulo the familiar equations $\Box(b_1 \wedge b_2) \approx \Box b_1 \wedge \Box b_2$ and $\Box 1 \approx 1$ (whose algebras are modal algebras), and define the natural transformation δ such that it correspond to taking a Kripke frame to its complex modal algebra. This is not only an example of an abstract coalgebraic logic, but even of what we call a *concrete coalgebraic logic* in this thesis, meaning that $\mathsf{L}_\Box$ is defined in terms of a *presentation by operations and equations* [BK06, KP10, KR12], which essentially corresponds to an axiomatization of the corresponding variety of modal algebras. In this thesis, to move from the classical to the many-valued setting, we replace the (universal algebraic) variety $\mathsf{BA} = \mathbb{HSP}(\mathbf{2})$ generated by the two-element algebra $\mathbf{2}$ by varieties $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$ generated by other finite algebras $\mathbf{D}$ of *truth-degrees.*

More specifically, we discuss coalgebraic logics (L', δ') for which $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ is an endofunctor on the variety generated by a *semi-primal algebra* $\mathbf{D}$. Semi-primal algebras were introduced by Foster and Pixley [FP64a] to generalize *primal algebras*, which themselves were introduced by Foster [Fos53a] as immediate generalizations of the two-element Boolean algebra. Although usually not explicitly mentioned, modal extensions of semi-primal algebras have been studied before in a number of papers on finitely-valued modal logic. Standard examples of semi-primal algebras of truth-degrees in many-valued logic include the finite MV-chains, whose modal extensions have been studied in [HT13], and the Łukasiewicz-Moisil chains, whose extensions by tense operators have been studied in [DG07]. Extending its signature by certain unary operations, every algebra based on a finite bounded lattice can be rendered semi-primal. For the semi-primal algebras thus obtained from finite Heyting algebras, modal extensions have been studied in [Mar09]. Modal extensions of finite $\mathsf{FL}_{\mathrm{ew}}$-algebras in general have been studied in [BEGR11], and the recent paper [LL23] on coalgebraic many-valued logic also assumes the algebra of truth-degrees to be a $\mathsf{FL}_{\mathrm{ew}}$-algebra, either extended by these unary operations (rendering it semi-primal) or, even stronger, extended by the Baaz-delta [Baa96] and all constants (even rendering it primal). In this thesis, we offer a uniform framework for all of the above (this was strongly inspired by Maruyama [Mar12]) and explore it coalgebraically, notably also *through its relationship to Boolean coalgebraic logic.*

That is, we describe a way to systematically *lift* a $\mathbf{2}$-valued coalgebraic logic (L, δ) to obtain a $\mathbf{D}$-valued coalgebraic logic (L', δ') with $\mathsf{L}' \colon \mathcal{A} \to \mathcal{A}$. We show that important properties at the level of abstract coalgebraic logics like one-step completeness and expressivity of (L', δ') follow directly from the corresponding properties of (L, δ). Furthermore, on the level of concrete coalgebraic logics, we show that L' has a presentation by operations and equations given that L has one and, in certain cases, we demonstrate how one can directly obtain such a presentation from the original one. For example, this allows us to generalize algebraic completeness results for finitely-valued modal logics (over crisp Kripke frames) as in [Mar09, HT13] and expressivity results as in [MM18]. We emphasize again that, in the theory developed here, these results are all *direct consequences* of the corresponding well-established results of classical modal logic. This is some of the material of Chapter 4.

In order to achieve this, in Chapter 2 we study the *category-theoretical relationship* between the variety $\mathcal{A}$ generated by a semi-primal lattice-extension and BA itself (*i.e.*, on the 'propositional level'). This also leads to results interesting on their own. A classical theorem of Hu [Hu69, Hu71] states that a category of algebras is equivalent to BA if and only if it is generated by a primal algebra. We show that, also in the more general semi-primal case, the relationship between $\mathcal{A}$ and BA is still very 'close'. In fact, (for lattice-based algebras) we generalize Hu's Theorem by showing that a variety is generated by a semi-primal algebra if and only if there is a certain *constellation of adjunctions* involving the *Boolean skeleton* and *Boolean power* functors. These functors are also used to describe the more general *subalgebra adjunctions*, which are ultimately used to lift coalgebraic logics as sketched in the previous paragraph.

All of these studies also heavily rely on *duality theoretical* methods. There is a well-known topological duality [KW74, CD98] between $\mathcal{A}$ and a category $\mathsf{Stone}_{\mathbf{D}}$ of *structured Stone spaces.* In fact, we not only show how to lift coalgebraic logics, but also how to lift algebra-coalgebra dualities like Jónsson-Tarski duality [JT51, KKV03] or Došen duality [Doš89, BBdG22] to this 'semi-primal level' (the semi-primal Jónsson-Tarski duality thus obtained coincides with the one established directly in [Mar12]).

It is also illustrated throughout the thesis that the discrete counterpart of $\mathsf{Stone}_{\mathbf{D}}$, which we denote by $\mathsf{Set}_{\mathbf{D}}$, provides an adequate framework for the *relational semantics* of $\mathbf{D}$-valued coalgebraic logics. Indeed, in Chapter 3 we introduce and study these many-valued modal logics not only over usual

Kripke models, but over $\mathbf{D}$-*models*, which are structures (X, v, R, Val) with 'preconditions' $v\colon X \to \mathbb{S}(\mathbf{D})$ in the form of a subalgebra of $\mathbf{D}$ at every state. The valuation $\mathsf{Val}\colon \mathsf{Prop} \times X \to \mathbf{D}$ needs to 'respect' these valuations in the sense that $\mathsf{Val}(x, p) \in v(x)$ always holds (these semantics have previously appeared only in the case of finite MV-chains in [HT13, Teh16]). The corresponding $\mathbf{D}$-*frames* arise as coalgebras for $\mathcal{P}'\colon \mathsf{Set}_{\mathbf{D}} \to \mathsf{Set}_{\mathbf{D}}$, a natural lifting of the covariant powerset functor. In fact, the lifted coalgebraic logics (L', δ') described above naturally are logics for functors over $\mathsf{Set}_{\mathbf{D}}$ rather than Set, although the latter (*i.e.*, the 'usual' semantics) can also easily be accounted for via the former.

Similar considerations also motivate the work in Chapter 5, where there is a shift in attention towards *positive modal logic* introduced by Dunn in 1995 [Dun95]. We study many-valued positive modal logic, narrowing down our focus on *finite positive* MV-*chains* $\mathbf{PŁ}_n$ (the quasi-variety of positive MV-algebras was recently introduced in [AJKV22]). First, we establish *natural dualities* [CD98] for the varieties PMV_n they generate and use this, for example, to study their relationship to the variety DL of distributive lattices using *distributive skeletons* and *Priestley powers* similar to what is done in Chapter 2. Based on this, we introduce $\mathbf{PŁ}_n$-valued positive modal logic and study it by algebraic means. Again, we do this not only over Kripke frames, but over what we call Pos_n-frames, which are somewhat derived from the dualities and have a 'richer structure' including the 'preconditions' and a *partial order*, similar to the semantics of positive modal logic on ordered frames by Celani and Jansana [CJ97, CJ99]. We leave it for future work to fully connect the work of Chapter 5 to the previous chapters by means of a coalgebraic analysis, potentially not only for positive MV-chains but for arbitrary (adequately defined) *lattice-semi-primal* algebras.

Structure and contributions of the thesis

In the following, we give a general overview of the structure of the thesis, as well as some of its main novel research contributions. The structure of a specific chapter or section is usually also summarized at its beginning. In addition, each chapter ends with an individual conclusion, in which some open questions for further research related to the specific chapter are discussed. A more general conclusion is also found at the end of the thesis.

Chapter 1

In Chapter 1, which is mostly preliminary, we introduce the concept of primality and its variations, and we provide some examples of algebras which fall under their scope. More specifically, in Section 1.1, we recall the definitions, equivalent characterizations and important properties of algebras and varieties which are primal (Subsection 1.1.1), quasi- and semi-primal (Subsection 1.1.2) and lattice-primal (Subsection 1.1.3). In Section 1.2, we focus on examples of (mostly lattice-based) algebras which are primal (Subsection 1.2.1) or semi-primal (Subsection 1.2.2). We also give some characterizations of variations of primality for bounded residuated lattices (Subsection 1.2.3) and end the chapter by recalling Murskiĭ's Theorem about the 'proportion' of semi-primal algebras among all finite algebras (Subsection 1.2.4).

This chapter mostly serves introductory purposes. Nevertheless, the author hopes that it has merit both as an overview of primality and its variations and as a large assembly of examples of semi-primal bounded lattice expansions with a focus on their occurrences in logic. Furthermore, the systematic study of variations of primality for finite bounded residuated lattices of Subsection 1.2.3 to this extent has, to the best of the authors knowledge, not been done prior.

Chapter 2

Chapter 2, based on the paper [KPT24b], centers around the category-theoretical study of semi-primal varieties in terms of their relationship to the variety of Boolean algebras. In Section 2.1, we recall a well-known topological duality for semi-primal varieties and provide a novel proof thereof. In Section 2.2, we describe multiple adjunctions which connect this duality to Stone duality. We first give descriptions of these adjunctions on the topological side (Subsection 2.2.1), before explaining their actions on the algebraic side, in particular via Boolean skeletons (Subsection 2.2.2) and Boolean powers (Subsection 2.2.3). These are also used in the description of the more general subalgebra adjunctions (Subsection 2.2.4). At the end of the section, we show how the existence of these adjunctions fully characterizes semi-primality (Subsection 2.2.5). In Section 2.3, our focus shifts towards the discrete version of the semi-primal duality. In particular, we give two descriptions of the category of algebras involved in this duality (Subsection 2.3.1) and connect the discrete and topological dualities via an analogue of the Stone-Čech compactification (Subsection 2.3.2).

While the topological duality from Section 2.1 is well-known, we provide a new proof of this duality, obtaining it as extension of the finite duality via the Pro- and Ind-completion (Theorem 2.1.10). We use the same method in the proof of the discrete duality (Corollary 2.3.8) and give a new characterization of the corresponding category of algebras in terms of their Boolean skeletons (Theorem 2.3.10). The (according to the author's judgement) most remarkable results of Chapter 2 are found in Section 2.2, where it is shown that the Boolean skeleton functor has two adjoints defined by taking certain Boolean powers (Theorems 2.2.12 and 2.2.17), and that the existence of this constellation of adjunctions fully characterizes semi-primality of a lattice-based algebra (Theorem 2.2.20). The latter is a generalization of Hu's Theorem (Theorem 1.1.4), which characterizes primal algebras purely by its category-theoretical relation to the variety of Boolean algebras, to semi-primal algebras.

Chapter 3

In Chapter 3, we introduce many-valued modal logic and study this logic in the case where the algebra of truth-degrees is semi-primal. More specifically, after giving a general overview of many-valued modal logic in Section 3.1, we introduce the special case of semi-primal modal logic in Section 3.2. There, we introduce the richer relational semantics on frames and models with 'local preconditions' (Subsection 3.2.1) and prove that the logics have the Hennessy-Milner property (Subsection 3.2.2). In Section 3.3, we develop the algebraic framework to study these logics, proving algebraic completeness with respect to the corresponding modal algebras (Subsection 3.3.1). We also give an algebraic proof of a variation of the Goldblatt-Thomason Theorem (Subsection 3.3.2).

One purpose of this chapter is to set-up and motivate the research of Chapter 4. Nonetheless the chapter contains some novel results in its unified treatment of many-valued modal logic with both modalities $\Box$ and $\Diamond$ over a semi-primal algebra of truth-degrees. Most notably, these include the algebraic study of these logics in Subsection 3.3.1, where the corresponding varieties of modal algebras are axiomatized in general (Theorem 3.3.7) and more specifically for various examples, like in the case of bounded residuated lattices (Example 3.3.11).

Chapter 4

In Chapter 4, based on the papers [KP23, KPT24a], we extend the studies of the previous chapter to the level of coalgebraic logics. In Section 4.1, we set-up the chapter with an introduction to coalgebraic logic, in particular we define and give examples of algebras and coalgebras for a functor (Subsection 4.1.1) as well as abstract and concrete coalgebraic logics (Subsection 4.1.2), for which we also explain the properties of one-step completeness and expressivity (Subsection 4.1.3). In Section 4.2, we begin our study of many-valued coalgebraic logics, first specializing on primal algebras of truth-degrees. We explain how to lift classical abstract coalgebraic logics to primal varieties (Subsection 4.2.1) and obtain axiomatizations of these lifted logics in some concrete cases (Subsection 4.2.2). In Section 4.3, we extend this study from primal to semi-primal varieties. We show how to obtain semi-primal algebra-coalgebra dualities from classical ones (Subsection 4.3.1) before providing similar techniques for abstract coalgebraic logics (Subsection 4.3.2). Similarly to the primal case, we also explain how to lift axiomatizations in concrete cases (Subsection 4.3.3) and end the section by reviewing definability and the Goldblatt-Thomason Theorem from this coalgebraic perspective (Subsection 4.3.4).

To the best of the author's knowledge, this work is the first instance of many-valued coalgebraic logics being investigated via the 'abstract' algebra-coalgebra approach, and the author hopes that it succeeds in highlighting the benefits of this approach. The main results of this chapter concern the systematic method to lift algebra-coalgebra dualities (Theorem 4.3.4) and, arguably more importantly, coalgebraic logics (Definition 4.3.20) from the classical to the semi-primal level. We show that a many-valued coalgebraic logic thus obtained inherits one-step completeness (Theorem 4.3.21) and expressivity (Theorem 4.3.22) from the classical coalgebraic logic it stems from. Similarly, we exhibit close connections between these logics in the context of definability (Subsection 4.3.4). We also discuss how to obtain axiomatizations for these lifted coalgebraic logics and provide some examples, including the lifting of classical modal logic (Subsection 4.3.3).

Chapter 5

In Chapter 5, based on the papers [Poi23, Poi24], we study positive modal logic in the many-valued setting, narrowing down our focus from the previous chapters to the case of finite MV-chains. In Section 5.1, we apply the theory of natural dualities in order to obtain optimal natural dualities for varieties PMV_n generated by finite positive MV-chains $\mathbf{PŁ}_n$. After giving a brief introduction to natural duality theory (Subsection 5.1.1) and introducing finite positive MV-chains (Subsection 5.1.2), we establish said dualities (Subsection 5.1.3). We explore these dualities further in Section 5.2, where we give an explicit description of the dual category in the three-valued case (Subsection 5.2.1), study the relationship to Priestley duality in terms of distributive skeletons and Priestley powers (Subsection 5.2.2), give characterizations of the algebraically and existentially closed PMV_n algebras (Subsection 5.2.3) and describe the discrete dual category in the case of prime-indexed positive MV-chains $\mathbf{PŁ}_p$ (Subsection 5.2.4). Building on this work, in Section 5.3 we deal with positive modal logic over finite MV-chains. After giving a short introduction to classical positive modal logic (Subsection 5.3.1), we introduce semantics for the $\mathbf{PŁ}_n$-valued 'versions' of this logic based on ordered relational frames and models with local preconditions (Subsections 5.3.2). We prove algebraic completeness with respect to the corresponding modal PMV_n-algebras (Subsection 5.3.3) and utilize this in a specific example illustrating how the richer semantics are 'better-behaved' than the usual relational semantics in the context of definability and canonicity (Subsection 5.3.4).

The first main result of this chapter is the 'final' natural duality for PMV_n (Theorem 5.1.17), which leads to many interesting insights into the structure of these varieties in Section 5.2. Most notably among them is (according to the author's judgement) the relationship between this duality and Priestley duality in terms of an adjunction between the distributive skeleton and Priestley power functors (Theorems 5.2.4 and 5.2.8). This also provides an important tool to obtain the main results of the latter part of this chapter, in which we study positive modal logic over $\mathbf{PŁ}_n$ based on richer relational semantics (Subsection 5.3.2), proving algebraic completeness (Theorem 5.3.14). To the best of the author's knowledge, this (together with the paper [Poi24] it is based on) is the first instance of a many-valued positive modal logic in the literature.

Chapter 1

Variations of primality

> *Recently there has emerged a different tendency, namely, to view Boolean algebras structurally, as organic systems, rather than algorithmically.*
>
> – Marshall H. Stone
> (1938)[1]

In this chapter, we give an overview of primality and most of its variations which have previously been studied in universal algebra. We put special emphasis on semi-primal algebras and the varieties they generate, since they play a large role in later chapters of this thesis. We also provide a plethora of examples of quasi-primal, semi-primal and primal algebras, with a particular focus on semi-primal bounded lattice-based algebras which arise in the context of logic.

The chapter is structured as follows. In Section 1.1, we recall the basic definitions and equivalent characterizations of primal algebras (Subsection 1.1.1), quasi- and semi-primal algebras (Subsection 1.1.2) and lattice-primal algebras (Subsection 1.1.3). In Section 1.2, we give examples of lattice-based primal algebras (Subsection 1.2.1), chain-based semi-primal algebras (Subsection 1.2.2), and we explore and give examples of quasi-primality and semi-primality among residuated lattices (Subsection 1.2.3). At the end of the chapter we briefly recall Murskiĭ's Theorem (Subsection 1.2.4).

The study of primality and its variations is a classical topic in universal algebra, and most of the material covered in Section 1.1 has been established during the 'golden age' of this study during the 1950s - 1970s (see, *e.g.*, [Fos53a, FP64a, Hu69, Pix71]). General overviews of this topic are, for example, provided in [Qua79a, KP01, Wer78].

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[1] [Sto38, p.1]

Similarly, many of the examples collected in Section 1.2 are well-known. As a notable exception we mention that, to the best of the author's knowledge, the study of variations of primality for residuated lattices from Subsection 1.2.3 has not been previously considered to this extent. This subsection in particular is, like the entire chapter in general, partially based on [KPT24b, Section 2] co-authored by the author of this thesis.

While we almost exclusively consider lattice-based algebras, further examples in more generality may, for instance, be found in [Bur92, Wer78, MY68].

Throughout this thesis, we assume that the reader is familiar with the basic concepts and terminology of universal algebra and lattice theory (*e.g.*, algebras, terms, subalgebras, products, congruences, varieties, Boolean algebras, distributive lattices, ...), some introductory textbooks are [Grä79, BS81, Ber11] for universal algebra and [DP02, Grä03] for lattice theory.

1.1 Definitions and characterizations

In this section, we recall basic definitions and equivalent characterizations of primal algebras (Subsection 1.1.1), semi-primal and quasi-primal algebras (Subsection 1.1.2) and lattice-primal algebras (Subsection 1.1.3).

1.1.1 Primality

Let $\mathbf{2} = \langle \{0,1\}, \wedge, \vee, \neg, 0, 1 \rangle$ denote the two-element Boolean algebra. A very well-known fact about this algebra is that every k-ary operation $f\colon \{0,1\}^k \to \{0,1\}$ is term-definable in $\mathbf{2}$ (or, in the context of computer science, that every logic gate can be obtained as a network of the basic Boolean gates). As a straightforward generalization of this property, Foster introduced the concept of primality in his generalized 'Boolean' theory of universal algebras in 1953 [Fos53a, Fos53b] (therein, primal algebras are called *functionally strictly complete algebras*; to the best of the authors knowledge, Foster coined the terminology 'primal' two years later in [Fos55]).

Definition 1.1.1 (Primal algebra)**.** A finite non-trivial algebra $\mathbf{P}$ with carrier set P is called *primal* if every operation $f\colon P^k \to P$ (with $k \geq 1$) is term-definable in $\mathbf{P}$.

Note that this definition does not require a primal algebra $\mathbf{P}$ to contain constants (*i.e.*, nullary operations) for all members of P in its signature.

Possibly explaining the terminology 'primal', the fields $\mathbb{Z}/p\mathbb{Z}$ of prime-order p (with constants 0 and 1 included in the language) are primal. Other examples of primal algebras which are more interesting from a logical point of view are collected in Subsection 1.2.1.

The variety BA of *Boolean algebras* is generated by the two-element Boolean algebra in the sense that $\mathsf{BA} = \mathbb{HSP}(\mathbf{2})$. Therefore, it is an example of a primal variety defined as follows.

Definition 1.1.2 (Primal variety)**.** A variety $\mathcal{A}$ is called *primal* if there is a primal algebra $\mathbf{P} \in \mathcal{A}$ such that $\mathcal{A} = \mathbb{HSP}(\mathbf{P})$.

Many 'good' properties of the algebra **2** and the variety BA directly transfer to every primal algebra and variety, respectively. We proceed to describe some of these properties. In the following, recall that the *quasi-variety* generated by a finite algebra $\mathbf{M}$ is given by $\mathbb{ISP}(\mathbf{M})$, and that a variety is called *arithmetical* if it is congruence-distributive and congruence-permutable. For notation, given a variety $\mathcal{A}$, we use $\mathcal{A}^\omega$ to denote the full subcategory of finite members of $\mathcal{A}$. Given a class of algebras $\mathcal{B}$, we use $\mathbb{P}_\omega(\mathcal{B})$ to denote the class of all finite products of algebras in $\mathcal{B}$.

Proposition 1.1.3. *Let* $\mathbf{P}$ *be a primal algebra and let* $\mathcal{A} = \mathbb{HSP}(\mathbf{P})$ *be the primal variety it generates.*

(1) The algebra $\mathbf{P}$ *is simple, has no proper subalgebras, and the only homomorphism* $\mathbf{P} \to \mathbf{P}$ *is the identity* $\mathsf{id}_\mathbf{P}$ ([Fos53b, Theorem 7.1])*.*

(2) The variety $\mathcal{A}$ *is arithmetical* ([FP64b, Theorem 2.8])*.*

(3) The variety $\mathcal{A}$ *coincides with the quasi-variety generated by* $\mathbf{P}$*, that is,* $\mathcal{A} = \mathbb{ISP}(\mathbf{P})$ ([Fos53b, Theorem 9.2])*.*

(4) Every finite member of $\mathcal{A}$ *is isomorphic to a direct product of* $\mathbf{P}$*, that is,* $\mathcal{A}^\omega = \mathbb{IP}_\omega(\mathbf{P})$ ([Fos53b, Theorem 9.3])*.*

An algebra $\mathbf{A}$ for which the only automorphism $\mathbf{A} \to \mathbf{A}$ is $\mathsf{id}_\mathbf{A}$ is sometimes referred to as *rigid*. Thus, primal algebras are rigid due to part (1) of the above proposition.

A straightforward consequence of Theorem 1.1.9 of the next subsection is that conditions (1) and (2) of Proposition 1.1.3 are sufficient for $\mathbf{P}$ being primal.

From an algebraic point of view, Proposition 1.1.3 shows that primal varieties are similar to the variety BA. This has been verified from a category-theoretical point of view by Hu in his famous theorem(s) from [Hu69, Hu71], which can be subsumed as follows. To denote equivalence of two categories C and D, we use $\mathsf{C} \simeq \mathsf{D}$.

Theorem 1.1.4 (Hu's Theorem). *Let $\mathcal{A}$ be a variety. Then $\mathcal{A}$ is primal if and only if $\mathcal{A} \simeq \mathsf{BA}$.*

Hu's original proof of this theorem is partially based on *Stone duality* [Sto36, Sto37], the famous dual equivalence between BA and the category Stone of *Stone spaces*, that is, compact, Hausdorff and zero-dimensional topological spaces (Stone duality is discussed in more detail at the beginning of Section 2.1). More specifically, in [Hu69] Hu showed that every primal variety is dually equivalent to Stone, and together with Stone duality this clearly yields sufficiency in Theorem 1.1.4 as $\mathcal{A} \simeq \mathsf{Stone}^{op} \simeq \mathsf{BA}$.

While elegant, this duality-theoretical proof does not include a concrete description of the two functors establishing the dual equivalence $\mathcal{A} \simeq \mathsf{BA}$. Later on, in Section 2.2, we show that such a concrete description can be obtained via *Boolean skeletons* and *Boolean powers* (see 2.2.13).

Lastly, let us mention here that Hu's Theorem has been re-investigated via Lawvere theories by Porst in [Por00].

1.1.2 Semi-primality and quasi-primality

In this subsection, we recall basic facts about *semi-primal* algebras and the varieties they generate. The notion of semi-primality is on high importance later on (in particular, in Chapters 2 to 4).

To set up the definition of semi-primal algebras, we first recall from Proposition 1.1.3(1) that a primal algebra $\mathbf{P}$ can not have any proper subalgebra $\mathbf{S} \subsetneq \mathbf{P}$. Indeed, this is easy to see as follows. Suppose that $\mathbf{S} \subsetneq \mathbf{P}$ is a proper subalgebra of an algebra $\mathbf{P}$, and choose some elements $p \in P \backslash S$ and $s \in S$. Then, no unary operation $f \colon P \to P$ with $f(s) = p$ can be term-definable in $\mathbf{P}$, since every term of $\mathbf{P}$ necessarily preserves the subalgebra $\mathbf{S}$ in the sense that $f(S) \subseteq S$.

Therefore, if one wants to 'allow proper subalgebras', the notion of primality has to be weakened accordingly. The appropriate weakening, semi-primality, was inroduced by Foster and Pixley in 1964 [FP64a, FP64b].

Let $\mathbf{D}$ be an algebra[2] and let $f \colon D^k \to D$ be a k-ary operation. We say that f *preserves subalgebras* if, for every subalgebra $\mathbf{S} \subseteq \mathbf{D}$, it holds that $f(S^k) \subseteq S$ (operations which preserve subalgebras are also called *conservative* in parts of the literature). Of course, since subalgebras are closed under arbitrary terms, every operation which is term-definable in $\mathbf{D}$ necessarily preserves subalgebras. In semi-primal algebras, sufficiency also holds.

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[2] Throughout this thesis, we will often use the letter $\mathbf{D}$ to denote certain algebras, to reflect the fact that we want to think of them as algebras of *truth-**d**egrees* later on.

Definition 1.1.5 (Semi-primal algebra)**.** A non-trivial algebra $\mathbf{D}$ is called *semi-primal* if every operation $f\colon D^k \to D$ (with $k \geq 1$) which preserves subalgebras is term-definable in $\mathbf{D}$.

Semi-primal algebras are sometimes also referred to as *subalgebra-primal* in the literature (see, *e.g.*, [KP01]). However, throughout this thesis we will keep Foster and Pixley's original terminology. Similar to Definition 1.1.1, we also refer to the corresponding varieties as semi-primal.

Definition 1.1.6 (Semi-primal variety)**.** A variety $\mathcal{A}$ is called *semi-primal* if there is a semi-primal algebra $\mathbf{D} \in \mathcal{A}$ such that $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$.

It is obvious by definition that every subalgebra of a semi-primal algebra is again semi-primal, and that a semi-primal algebra is primal if and only if it has no proper subalgebras.

A further weakening of semi-primality is given by the notion of *quasi-primality*, introduced by Pixley in 1970 [Pix70] (therein, quasi-primal algebras are called *simple algebraic algebras*; Pixley coined the term 'quasi-primal' one year later in [Pix71]). Besides subalgebras, quasi-primality also takes isomorphisms between subalgebras into consideration. Given an algebra $\mathbf{D}$, an *internal isomorphism* of $\mathbf{D}$ is an isomorphism $\varphi\colon \mathbf{S}_1 \to \mathbf{S}_2$ between (not necessarily proper or distinct) subalgebras of $\mathbf{D}$. We say that an operation $f\colon D^k \to D$ *preserves internal isomorphisms* if, for every internal isomorphism φ as above, it satisfies $\varphi(f(d_1, \dots, d_k)) = f(\varphi(d_1), \dots, \varphi(d_k))$.

Definition 1.1.7 (Quasi-primal algebra)**.** An non-trivial algebra $\mathbf{D}$ is called *quasi-primal* if every operation $f\colon D^k \to D$ (with $k \geq 1$) which preserves subalgebras and internal isomorphisms of $\mathbf{D}$ is term-definable in $\mathbf{D}$.

As usual by now, we will also refer to varieties as *quasi-primal* if they are generated by some quasi-primal algebra.

If $\mathbf{D}$ is semi-primal, the only internal isomorphisms between non-trivial subalgebras of $\mathbf{D}$ are the identities $\mathsf{id}_{\mathbf{S}}\colon \mathbf{S} \to \mathbf{S}$ of non-trivial subalgebras $\mathbf{S} \subseteq \mathbf{D}$. Indeed, suppose towards contradiction that there is an internal isomorphism $\mathbf{S}_1 \to \mathbf{S}_2$ between two non-trivial subalgebras of a semi-primal algebra $\mathbf{D}$. Then there exists some $s \in S_1$ with $\varphi(s) \neq s$. Choosing any $s' \in S_1$ with $s' \neq s$, the binary operation $f\colon D^2 \to D$ defined by $f(s, s') = s'$ and $f(x, y) = x$ otherwise preserves subalgebras and, therefore, can be defined by a term-function of $\mathbf{D}$. But this implies

$$\varphi(s') = \varphi\big(f(s, s')\big) = f\big(\varphi(s), \varphi(s')\big) = \varphi(s)$$

and therefore $s = s'$, a contradiction.

With the above fact established, it is easy to see that a quasi-primal algebra $\mathbf{D}$ is semi-primal if and only if it has no internal isomorphisms between non-trivial subalgebras except for the identities $\mathsf{id}_{\mathbf{S}}$ of subalgebras $\mathbf{S} \subseteq \mathbf{D}$.

Similar to Proposition 1.1.3, we collect some well-known properties of quasi-primal algebras and varieties in the following. All of these were shown by Pixley [Pix70], as generalizations of the corresponding facts for semi-primal algebras due to Foster and Pixley [FP64a, FP64b].

Proposition 1.1.8. *Let* $\mathbf{D}$ *be a quasi-primal algebra and* $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$ *be the quasi-primal variety it generates.*

(1) The algebra $\mathbf{D}$ *is simple* ([Pix70, Theorem 3.1]).

(2) The variety $\mathcal{A}$ *is arithmetical* ([Pix70, Theorem 3.4]).

(3) The variety $\mathcal{A}$ *coincides with the quasi-variety generated by* $\mathbf{D}$*, that is,* $\mathcal{A} = \mathbb{ISP}(\mathbf{D})$ ([Pix70, Theorem 4.1(a)]).

(4) Every finite member of $\mathcal{A}$ *is isomorphic to a direct product of subalgebras of* $\mathbf{D}$*, that is,* $\mathcal{A}^{\omega} = \mathbb{IP}_{\omega}\mathbb{S}(\mathbf{D})$ ([Pix70, Theorem 4.1(b)]).

The corresponding facts for semi-primal algebras and varieties were already shown in [FP64a, Theorem 3.2] for (1), [FP64b, Theorem 2.8] for (2), [FP64a, Theorem 4.1] for (3) and [FP64a, Lemma 7.3] for (4).

In the following, we recall various equivalent characterizations of quasi-primality and their specializations to semi-primality. We begin with a well-known characterization of quasi-primal algebras in terms of the varieties they generate.

Theorem 1.1.9. *Let* $\mathbf{D}$ *be a finite non-trivial algebra and* $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$ *be the variety it generates.*

(1) $\mathbf{D}$ *is quasi-primal if and only if* $\mathcal{A}$ *is arithmetical and every non-trivial subalgebra of* $\mathbf{D}$ *is simple* ([Pix70, Theorem 5.1]).

(2) $\mathbf{D}$ *is semi-primal if and only if* $\mathcal{A}$ *is arithmetical, every non-trivial subalgebra of* $\mathbf{D}$ *is simple and the only non-trivial internal isomorphisms of* $\mathbf{D}$ *are the identities of subalgebras of* $\mathbf{D}$ ([FP64b, Theorem 3.1]).

In particular, part (2) of the above theorem implies that every subalgebra of a semi-primal algebra is rigid.

The next characterization of quasi-primality, due to Pixley [Pix71], is given via the existence of a certain discriminator term. A *discriminator* *algebra* is an algebra $\mathbf{D}$ in which the *ternary discriminator*, that is, the ternary operation $t\colon D^3 \to D$ defined by

$$t(x,y,z) = \begin{cases} z & \text{if } x = y \\ x & \text{if } x \neq y \end{cases}$$

is term-definable in $\mathbf{D}$.

Theorem 1.1.10 ([Pix71, Theorem 3.2])**.** *Let* $\mathbf{D}$ *be a finite non-trivial algebra.*

(1) $\mathbf{D}$ *is quasi-primal if and only if it is a finite discriminator algebra.*

(2) $\mathbf{D}$ *is semi-primal if and only if it is a finite discriminator algebra and the only non-trivial internal isomorphisms of* $\mathbf{D}$ *are the identities of subalgebras of* $\mathbf{D}$*.*

It is also possible to characterize quasi-primal algebras via the existence of a majority term. Recall that a *majority term* (or *ternary near-unanimity term*) is a ternary term $m(x,y,z)$ which satisfies the equations

$$m(x,x,y) = m(x,y,x) = m(y,x,x) = x.$$

In particular, every lattice $\langle L, \wedge, \vee\rangle$ has a majority-term given by the *median*

$$m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

In the presence of a majority term, the famous *Baker-Pixley Theorem* [BP75, Theorem 2.1] yields equivalent characterizations of quasi- and semi-primality. Let $\mathbf{D}$ be a finite algebra with a majority term. Then the Baker-Pixley says that an operation $f\colon D^k \to D$ is term-definable in $\mathbf{D}$ if and only if it every subalgebra $\mathbf{S}$ of $\mathbf{D}^2$ is closed under f (where f is defined on $\mathbf{D}^2$ component-wise as usual).

In the following, recall that the *diagonal* of a set A is given by $\Delta_A := \{(a,a) \mid a \in A\} \subseteq A^2$. In particular, if $\mathbf{A}$ is an algebra then $\Delta_{\mathbf{A}}$ always is a subalgebra of $\mathbf{A}^2$.

Theorem 1.1.11 ([BP75, Theorem 7.2])**.** *Let* $\mathbf{D}$ *be a finite non-trivial algebra.*

(1) $\mathbf{D}$ *is quasi-primal if and only if it has a majority term and every subalgebra of* $\mathbf{D}^2$ *is either a product of subalgebras or the graph of an internal isomorphism of* $\mathbf{D}$*.*

(2) $\mathbf{D}$ *is semi-primal if and only if it has a majority term and every subalgebra of* $\mathbf{D}^2$ *is a product of subalgebras or the diagonal* $\Delta_{\mathbf{S}}$ *of a subalgebra* $\mathbf{S} \subseteq \mathbf{D}$.

Lastly, we recall an equivalent characterization of semi-primality for algebras which are based on a bounded lattice. In the following theorem, the equivalence between (i) and (ii) may be seen as a special instance of the results of Foster's [Fos67], in which a similar characterization is given for all semi-primal algebras for which the intersection of all subalgebras contains at least two elements.

Theorem 1.1.12. *Let* $\mathbf{D}$ *be a finite algebra based on a bounded lattice* $\mathbf{D}^\flat = \langle D, \wedge, \vee, 0, 1 \rangle$. *Then the following are equivalent.*

(i) $\mathbf{D}$ *is semi-primal.*

(ii) For every $d \in D$, *the unary operation* $T_d \colon D \to D$ *defined by*

$$T_d(x) = \begin{cases} 1 & \textit{if } x = d \\ 0 & \textit{if } x \neq d \end{cases}$$

is term-definable in $\mathbf{D}$.

(iii) The unary operation T_0 *from (ii) and, for every* $d \in D \backslash \{0\}$, *the unary operation* $\tau_d \colon D \to D$ *defined by*

$$\tau_d(x) = \begin{cases} 1 & \textit{if } x \geq d \\ 0 & \textit{if } x \ngeq d \end{cases}$$

is term-definable in $\mathbf{D}$.

(iv) The unary operation T_0 *from (ii) and, for every* $d \in D \backslash \{1\}$, *the unary operation* $\eta_d \colon D \to D$ *defined by*

$$\eta_d(x) = \begin{cases} 0 & \textit{if } x \leq d \\ 1 & \textit{if } x \nleq d \end{cases}$$

is term-definable in $\mathbf{D}$.

Proof. Clearly (i) implies (ii), since all T_d preserve subalgebras (since 0 and 1 are contained in every subalgebra of $\mathbf{D}$) and, therefore, are term-definable by semi-primality.

Conversely, assuming (ii), we show that **D** is semi-primal using the characterization of Theorem 1.1.10. First we show that the ternary discriminator is term-definable in **D**. Consider the term

$$c(x,y) = \bigvee_{\ell \in L} \big((T_\ell(x) \wedge T_\ell(y)\big),$$

which satisfies

$$c(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and $d(x,y) := T_0(c(x,y))$ (note that this is the discrete metric on D). Now we can define the ternary discriminator via

$$t(x,y,z) = (d(x,y) \wedge x) \vee (c(x,y) \wedge z).$$

Next we show that the only internal isomorphisms of **D** are the identities of subalgebras of **D**. Let $\varphi : \mathbf{S}_1 \to \mathbf{S}_2$ be an internal isomorphism of **D** and let $s \in S_1$ be arbitrary. Then

$$1 = T_{\varphi(s)}\big(\varphi(s)\big) = \varphi\big(T_{\varphi(s)}(s)\big).$$

Since $\varphi(0) = 0$, we necessarily have $T_{\varphi(s)}(s) = 1$, which is equivalent to $\varphi(s) = s$. Therefore, (i) holds by Theorem 1.1.10.

(ii)⇒ (iii): If the T_d are term-definable in **D**, we can also define

$$\tau_d(x) = \bigvee_{d' \geq d} T_{d'}(x).$$

(iii) ⇒ (ii): If T_0 and the τ_d are term-definable in **D**, we can also define

$$T_d(x) = \tau_d(x) \wedge \bigwedge_{d' > d} T_0\big(\tau_{d'}(x)\big).$$

The equivalence (ii)⇔(iv) works similarly with

$$T_d(x) = T_0\big(\eta_d(x)\big) \wedge \bigwedge_{d' < d} \eta_{d'}(x),$$

finishing the proof. □

In particular, this implies that *every* finite bounded lattice **L** may be turned into a semi-primal algebra by adding all unary operations T_ℓ to its signature. Note that, as opposed to adding a constant for every member of **L**, adding all operations T_ℓ does not create new subalgebras.

The unary operation T_1 is also commonly referred to as *Baaz delta* due to [Baa96], where extensions of Gödel logic by this unary operation (therein denoted Δ) were studied.

Later on, in Subsections 1.2.2-1.2.3, we exclusively consider semi-primal algebras which are based on bounded lattices. The unary operations T_d and τ_d play an important role in this study throughout this thesis.

In the following subsection, we discuss another direction in which one may generalize primality in an ordered setting.

1.1.3 Lattice-primality

We now give a brief overview of an *ordered* version of primality, in particular a *lattice-ordered* version. The concept of order-primality was, to the best of the authors knowledge, first studied by Schweigert [Sch74] (in this German paper, order-primal algebras are called '*ordnungspolynomvollständig*', *i.e.*, order-polynomially complete).

Definition 1.1.13 (Order-primal & lattice-primal algebra)**.** A finite ordered algebra $\langle \mathbf{D}, \leq \rangle$ is called *order-primal* if, for every operation $f\colon D^k \to D$ (with $k \geq 1$), f is term-definable in $\mathbf{D}$ if and only if f preserves the order $\leq$. In particular, an order-primal algebra is called *lattice-primal* if the order $\leq$ is a lattice-order on $\mathbf{D}$.

Note that this definition implies that all primitive operations of an order-primal algebra need to be order-preserving. Also note that, in a lattice-primal algebra $\langle \mathbf{D}, \leq \rangle$, the corresponding lattice-operations $\wedge$ and $\vee$ are necessarily term-definable.

As usual, we also refer to a variety generated by an order- or lattice-primal algebra as *order-* or *lattice-primal*, respectively. Universal algebraic structure theory for order-primal algebras (and varieties) with arbitrary orders $\leq$ is rather complicated and non-uniform in the sense that it can heavily depend on specific properties of the corresponding order (see, *e.g.*, [DQS90, DRR05]). However, lattice-primal algebras are arguably almost as 'well-behaved' as primal algebras. This is witnessed by the following theorem similar to Hu's Theorem (see Theorem 1.1.4). Recall that the variety DL of *bounded distributive lattices* is the variety generated by the two-element bounded distributive lattice $\mathbf{2} = \langle \{0,1\}, \wedge, \vee, 0, 1 \rangle$, which is well-known to be lattice-primal (this can be easily seen using the Baker-Pixley Theorem). The 'lattice-ordered' version of Hu's Theorem is due to Quackenbush [Qua79b] (also see [DR82, DW83]).

Theorem 1.1.14 ([Qua79b, Theorem 1])**.** *Let $\mathcal{A}$ be a variety of algebras. Then $\mathcal{A}$ is lattice-primal if and only if $\mathcal{A} \simeq \mathsf{DL}$.*

Later on, in Chapter 5, we indicate how a concrete algebraic description of the functors involved in the categorical equivalence of the above theorem can be obtained via *distributive skeletons* and *Priestley powers*.

1.2 Examples

In this section, we collect various examples of quasi-primal and, in particular, semi-primal algebras. Our focus lies on algebras which have a bounded lattice-reduct, and which are significant in many-valued logic.

1.2.1 Post chains and other primal algebras

We begin with examples of primal algebras which have been previously studied in the context of logic. Note that, by definition of primality (see Defintion 1.1.1), it is obvious that, for every $n \in \mathbb{N}$, there is only one primal algebra of size n up to term-equivalence. Nevertheless, from an applied perspective, the choice of primitive operations might matter both from a technological point of view (*e.g.*, in designing many-valued circuits) and a philosophical one (*e.g.*, to interpret the elements and operations of the algebra in question). For these reasons, we describe distinct (albeit term-equivalent) primal algebras in this subsection.

Our first example is connected to one of the earliest instances of a many-valued logic, introduced by Post in 1921 [Pos21].

Definition 1.2.1 (Post chain)**.** Let $n \geq 1$ be a natural number. The $(n+1)$-*element Post chain* is given by

$$\mathbf{P}_n = \langle \{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, ', 0, 1 \rangle,$$

where $\langle \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \wedge, \vee, 0, 1 \rangle$ is the usual bounded lattice-order on this chain and the unary *Post negation* $'$ is defined by $0' = 1$ and $(\frac{i}{n})' = \frac{i-1}{n}$ for $i = 1, \dots, n$.

It was essentially already shown in Posts original work that, for every $n \geq 1$, the algebra $\mathbf{P}_n$ is primal [Pos21, Section 11] (also see [Fos53a, Theorem 35]). A finite axiomatization of the variety $\mathsf{Post}_n := \mathbb{HSP}(\mathbf{P}_n)$ was found by Rosenbloom in 1942 [Ros42].

As a side note, similar to the 2-element case, all primitive operations of $\mathbf{P}_n$ can also be defined via a single binary operation

$$\tfrac{i}{n} | \tfrac{j}{n} = \tfrac{k}{n} \text{ where } k = \mathsf{max}\{i+1, j+1\} \text{ with addition modulo } n+1,$$

which can be seen as a generalized Sheffer-stroke. This insight is due to Webb [Web36].

Our next example of a primal algebra arises in the context of the famous four-valued Belnap-Dunn logic [Bel77, Dun76] and, more generally, of *bilattice logics* as discussed by Arieli and Avron [AA96]. The basic idea is the following. Consider the four-element set $\{\mathsf{t}, \mathsf{f}, \top, \bot\}$ with the intended meanings of t and f being *true* and *false*, and of $\top$ and $\bot$ being *both* (or *contradiction*) and *none* (or *unknown*), respectively. This set can be (lattice)-ordered both by the *truth-order* $\leq_t$ and the *knowledge-order* $\leq_k$. These orders are depicted in Figure 1.1.

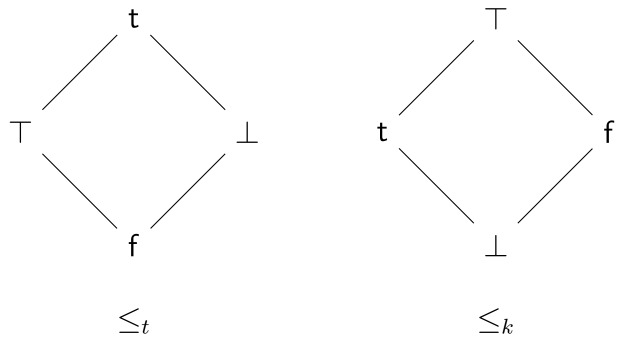


Figure 1.1: The truth-order $\leq_t$ and the knowledge-order $\leq_k$.

Definition 1.2.2. The *four-element bounded implicative bilattice* is given by

$$\mathbf{FOUR} = \langle \{\mathsf{t}, \mathsf{f}, \top, \bot\}, \wedge, \vee, \otimes, \oplus, \neg, \supset, \mathsf{t}, \mathsf{f}, \top, \bot \rangle.$$

Here, $\langle \{\mathsf{t}, \mathsf{f}, \top, \bot\}, \wedge, \vee, \mathsf{t}, \mathsf{f} \rangle$ is the bounded lattice corresponding to $\leq_t$ and $\langle \{\mathsf{t}, \mathsf{f}, \top, \bot\}, \otimes, \oplus, \bot, \top \rangle$ is the bounded lattice corresponding to $\leq_k$. The binary operation $\supset$ is defined by $x \supset y = y$ if $x \in \{\mathsf{t}, \top\}$ and $x \supset y = \mathsf{t}$ otherwise, and the unary operation $\neg$ exchanges t and t while fixing $\top$ and $\bot$.

The variety $\mathbb{HSP}(\mathbf{FOUR})$ generated by **FOUR** is the variety of *bounded classical implicative bilattices* and is axiomatized in Rivieccio's doctoral dissertation [Riv10, Definition 3.3.1]. Therein, it is also shown that the constant-free reduct of **FOUR** is quasi-primal [Riv10, Proposition 4.2.2], which immediately implies that **FOUR** itself is primal.

Lastly, we mention a recent example of a primal algebra in a logical context due to Bucciarelli, Ledda, Paoli and Salbira [SBLP23]. They define the variety $n\mathsf{BA}$ of (pure) *Boolean-like algebras of dimension* n as the variety generated by the algebra

$$\mathbf{n} = \langle \{e_1, \dots, e_n\}, q, e_1, \dots, e_n \rangle,$$

where the $n+1$-ary operation q is a *generalized if-then-else* operation which satisfies the equation

$$q(e_i, x_1, \dots, x_n) = x_i \text{ for every } i = 1, \dots, n.$$

In the same paper, it is shown that $\mathbf{n}$ is primal [SBLP23, Lemma 4].

In the next subsection, we shift our focus to semi-primal algebras.

1.2.2 Finite MV-chains and other semi-primal algebras

An algebraic counterpart of Łukasiewicz(-Tarski) infinite-valued logic [ŁT30] is provided by MV-*algebras*, introduced by Chang in 1958 [Cha58]. The variety MV of MV-algebras is generated by the *standard* MV-*algebra*

$$\mathbf{Ł} = \langle [0,1], \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle,$$

based on the real unit interval with its usual bounded lattice structure and additional operations

$$x \odot y = \mathsf{max}\{0, x+y-1\},\ \ x \oplus y = \mathsf{min}\{1, x+y\},\ \ \neg x = 1-x.$$

A detailed overview of MV-algebras and their relationship to many-valued logic may be found in the book [CDM00] (or [Mun11], for more advanced topics). In this paper, we focus on finite MV-chains, that is, the following finite subalgebras of the standard MV-algebra.

Definition 1.2.3 (Finite MV-chain)**.** Let $n \geq 1$ be a natural number. The $(n+1)$-*element* MV-*chain* is given by

$$\mathbf{Ł}_n = \langle \{0, \tfrac{1}{n}, \dots \tfrac{n-1}{n}, 1\}, \wedge, \vee, \odot, \oplus, \neg, 0, 1 \rangle,$$

considered as a subalgebra of the standard MV-algebra.

As usual, we use MV_n to denote the variety $\mathbb{HSP}(\mathbf{Ł}_n)$ generated by $\mathbf{Ł}_n$ and call their members MV_n-*algebras*.These varieties, first axiomatized by Grigolia in 1977 [Gri77], provide appropriate algebraic counterparts of Łukasiewicz *finitely-valued logics.* Historically the first, Łukasiewicz three-valued logic [Łuk20] (in Polish, see [Łuk70] for an english translation) with $\mathbf{Ł}_2$ as algebra of truth-degrees, is popular in the broader field of many-valued logics and their applications to computer science and philosophy.

For every $n \geq 1$, the finite MV-chain $\mathbf{Ł}_n$ is semi-primal. A complete proof of this fact is provided by Niederkorn in [Nie01, Proposition 2.1], however, it was already known much earlier (see, *e.g.*, [Dzi85]).

It is well-known that the subalgebras of $\mathbf{Ł}_n$ are exactly given by

$$\mathbf{Ł}_k \cong \langle \{0, \tfrac{\ell}{n}, \dots, \tfrac{(k-1)\ell}{n}, 1\}, \wedge, \vee, \odot, \oplus, \neg, 0, 1 \rangle,$$

where $n = k \cdot \ell$. Thus, the lattice $\mathbb{S}(\mathbf{Ł}_n)$ of subalgebras of $\mathbf{Ł}_n$ is isomorphic to the bounded lattice of divisors of n.

Therefore, for all $n \geq 1$, the algebra $\mathbf{Ł}_n$ is semi-primal but not primal. However, it is easy to see that it becomes primal when one adds a constant for the element $\frac{1}{n}$, as every other non-zero element can be obtained as a $\oplus$-sum of this element. This (nowadays obvious) fact was shown by Słupecki already in 1936 for the three-valued case [Słu36].

Our next examples of semi-primal algebras are closely related to the previous ones. Indeed, they were introduced by Moisil [Moi40] under the name 'Łukasiewiczian algebras' in the hope to provide algebraic semantics for Łukasiewicz finitely-valued logic.

Definition 1.2.4 (Łukasiewicz-Moisil chain)**.** Let $n \geq 1$ be a natural number. The $(n+1)$*-element Łukasiewicz-Moisil chain* is given by

$$\mathbf{M}_n = \langle \{0, \tfrac{1}{n}, \dots \tfrac{n-1}{n}, 1\}, \wedge, \vee, \neg, 0, 1, (\tau_{\frac{i}{n}})_{i=1}^{n} \rangle,$$

where $\neg x = 1 - x$ is the MV-negation and $\tau_{\frac{i}{n}}$ is the characteristic function of the subset $\{\frac{j}{n} \mid j \geq i\}$ as in Theorem 1.1.12.

As usual, we use LM_n to denote the corresponding variety $\mathbb{HSP}(\mathbf{M}_n)$ of n*-valued Łukasiewicz-Moisil algebras.* For an axiomatization of this variety and further information about Łukasiewicz-Moisil algebras in general we refer to the book [BFGR91].

In this context, the unary operations $\tau_{\frac{i}{n}}$ are often called *Chrysippian endomorphisms*[3]. Note that they coincide with the operations τ_d from Theorem 1.1.12(iii). This theorem also shows that, for every $n \geq 1$, the algebra $\mathbf{M}_n$ is semi-primal, once we note that T_0 is term-definable in $\mathbf{M}_n$ via $\tau_1(\neg x)$.

The subalgebras of $\mathbf{M}_n$ are in one-to-one correspondence with subsets $S \subseteq \{\frac{i}{n} \mid i \geq \lceil \frac{n}{2} \rceil\}$, the corresponding subuniverse is then given by $S \cup \neg S$ where $\neg S = \{\neg s \mid s \in S\}$.

---

[3] After the stoic philosopher Chrysippus (279 - 206 BC).

As mentioned above, the algebras $\mathbf{M}_n$ were intended to provide algebraic semantics for Łukasiewicz n-valued logics. However, as noted in [Cig70, p.2] (attributed to personal communication with A. Rose), for $n \geq 4$, this fails. In this case, MV_n can be identified with the proper subclass of LM_n consisting of *proper* LM_n-algebras, identified by Cignoli in [Cig82]. The relationship between MV_n-algebras and LM_n has been further studied by Iorgulescu in a series of papers starting with [Ior98]. The logic associated with LM_n is nowadays usually called *Moisil logic*.

We conclude this subsection with another way to turn the $(n+1)$-element chain into a semi-primal algebra. It is due to Davey and Gair [DG17], who showed that the following *Cornish algebras* are semi-primal.

Definition 1.2.5 (Cornish chain)**.** Let $n \geq 1$ be a natural number. The *n-th semi-primal Cornish chain* is given by

$$\mathbf{CO}_n = \big(\{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, \neg, f, 0, 1\big),$$

where $\neg x = 1 - x$, $f(0) = 0, f(1) = 1$ and $f(\frac{i}{n}) = \frac{i+1}{n}$ for $1 \leq i \leq n-1$.

For all $n \geq 1$, the algebra $\mathbf{CO}_n$ is semi-primal. The proof of this fact can be found in [DG17, Example 5.15]. The only proper subalgebra of $\mathbf{CO}_n$ is given by the subuniverse $\{0, 1\}$.

Cornish algebras [Cor86] can be seen as generalized *Ockham algebras*, which provide an algebraic counterpart to a logic with a De Morgan negation, but without excluded middle or double negation.

In the next subsection, we give examples of semi-primal algebras based on lattices which are not necessarily linearly ordered.

1.2.3 Residuated Lattices and pseudo-logics

In this subsection, we mainly investigate variations of primality in the context of *residuated lattices*. Due to the additional operations these structures carry, simpler equivalent criteria for variations of primality may be obtained in this context. In particular, we characterize and give examples of semi-primal $\mathsf{FL}_{\mathrm{ew}}$-*algebras*. These algebras play an important role in many-valued logic since, oftentimes, algebras of truth-degrees are chosen to be $\mathsf{FL}_{\mathrm{ew}}$-algebras.

In a similar vein, towards the end of this subsection we recall a sufficient condition for quasi-primality of *pseudo-logics* from [DSW91, CD98]. Pseudo-logics may be seen as generalizations of bounded residuated lattices, where only a weak form of *negation* is contained as primitive operation (while residuated lattices, in particular $\mathsf{FL}_{\mathrm{ew}}$-algebras, put more emphasis on a canonical choice of *implication*).

Classes of *residuated lattices* (introduced by Dilworth and Ward in the late 1930s [DW38, Dil39]) provide algebraic counterparts for many substructural logics. For general surveys of residuated lattices and substructural logics, we refer to the book [GJKO07] (also see [JT02]). In this thesis, we only consider *bounded* residuated lattices.

Definition 1.2.6 (Bounded residuated lattice). A *bounded residuated lattice* is an algebra

$$\mathbf{R} = \langle R, \wedge, \vee, 0, 1, \odot, /, \backslash, e \rangle,$$

such that $\langle R, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice, $\langle R, \odot, e \rangle$ is a monoid and the *residuation laws*

$$x \odot y \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x \backslash z$$

hold in $\mathbf{R}$. The bounded residuated lattice $\mathbf{R}$ is called *integral* if, in addition, it satisfies $e = 1$.

In the following, we show that quasi-primality of a bounded residuated lattice is equivalent to term-definability of τ_e (as defined in Theorem 1.1.12) for its neutral element e. This is similar to [Jip03, pp.44-45], where the analogous result is shown for $\tau_e \wedge e$ instead.

Proposition 1.2.7. *Let* $\mathbf{R} = \langle R, \wedge, \vee, 0, 1, \odot, /, \backslash, e \rangle$ *be a finite bounded residuated lattice. Then* $\mathbf{R}$ *is quasi-primal if and only if*

$$\tau_e(x) = \begin{cases} 1 & \textit{if } x \geq e \\ 0 & \textit{if } x \ngeq e \end{cases}$$

is term-definable in $\mathbf{R}$.

Proof. We use the equivalent characterization of quasi-primality from Theorem 1.1.10 via the ternary discriminator.

First suppose that $\mathbf{R}$ is quasi-primal, then the ternary discriminator t is term-definable in $\mathbf{R}$. Then τ_e is term-definable via

$$\tau_e(x) = \neg t(e, x \wedge e, 0).$$

Here, we use the usual definition $\neg e = x \backslash 0$. Indeed, if $x \geq e$, then we have

$$\neg t(e, x \wedge e, 0) = \neg t(e, e, 0) = \neg 0 = 1,$$

where the last equation holds due to $y \leq \neg 0 \Leftrightarrow 0 \odot y \leq 0$ and $0 \odot y = 0$ (since $0 \odot y \leq 0 \Leftrightarrow 0 \leq 0/y$ which is always true).

Conversely, assume that τ_e is term-definable in $\mathbf{R}$. Similar to [Jip03, p. 45], we define $x \leftrightarrow y := x\backslash y \wedge y\backslash x$ and

$$t(x,y,z) = \big(z \wedge \tau_e(x \leftrightarrow y)\big) \vee \big(x \wedge \neg(\tau_e(x \leftrightarrow y)\big)$$

defines the ternary discriminator because

$$x = y \Leftrightarrow (x \leftrightarrow y) \geq e \Leftrightarrow \tau_e(x \leftrightarrow y) = 1$$

and $\neg 1 = 0$. □

In particular, for bounded *integral* residuated lattices, quasi-primality corresponds to term-definability of T_1 (as defined in Theorem 1.1.12). In particular, the following yields a sufficient condition for quasi-primality in this case.

Corollary 1.2.8. *Let* $\mathbf{R}$ *be a finite bounded integral residuated lattice. If every element of* $\mathbf{R}\backslash\{0,1\}$ *is non-idempotent, then* $\mathbf{R}$ *is quasi-primal.*

Proof. Suppose that every element $a \in \mathbf{R}$ satisfies $a \odot a \neq a$. Then, since $\mathbf{R}$ is integral, it follows from $a \odot a \leq a \odot 1 = a$ that $a \odot a < a$. Therefore, since $\mathbf{R}$ is finite, for every $a \neq 1$, there exists some n_a with $a^{n_a} = 0$. Set $n = \mathsf{max}\{n_a \mid a \in \mathbf{R}\backslash\{1\}\}$. Then $T_1(x) = x^n$ is term-definable and $\mathbf{R}$ is quasi-primal by Proposition 1.2.7. □

We can also give a slightly easier characterization of semi-primality for bounded integral lattices as follows. They are exactly the ones for which all τ_d are term-definable. Note that this is a weaker condition than that of Theorem 1.1.12, where we additionally require T_0 to be term-definable.

Proposition 1.2.9. *Let* $\mathbf{R}$ *be a finite bounded residuated lattice. Then* $\mathbf{R}$ *is semi-primal if and only if, for every* $r \in \mathbf{R}\backslash\{0\}$*, the unary operation*

$$\tau_r(x) = \begin{cases} 1 & \textit{if } x \geq r \\ 0 & \textit{if } x \ngeq r \end{cases}$$

is term-definable in $\mathbf{R}$.

Proof. By Theorem 1.1.12, we know that $\mathbf{R}$ is semi-primal if and only if T_0 and all τ_r are term-definable in $\mathbf{R}$. Assume that all τ_r are term-definable in $\mathbf{R}$. As before, set $\neg x = x\backslash 0$. Then it is easy to see that $T_0 = \tau_e(\neg x)$ holds. Indeed, by the residuation law we have

$$e \leq x\backslash 0 \Leftrightarrow x \odot e \leq 0 \Leftrightarrow x = 0,$$

which finishes the proof. □

It also follows immediately from Proposition 1.2.7 that a bounded residuated lattice $\mathbf{R}^c$ expanded with τ_e and a constant $\hat{r}$ for every $d \in R$ is primal. In this case, it is also easy to see that we can define

$$\tau_r(x) = \tau_e(\hat{r} \backslash x)$$

for every $r \in R$. In particular, if the residuated lattice is integral, this corresponds to the common expansion of the language in many-valued modal logic by truth-constants and T_1 (often denoted Δ), rendering the algebra of truth-degrees primal.

Before we list a number of concrete examples, we show that, in the case of $\mathsf{FL}_{\mathrm{ew}}$-algebras, the converse of Corollary 1.2.8 holds if $\mathbf{R}$ is linear. First we introduce $\mathsf{FL}_{\mathrm{ew}}$-algebras. Recall that a residuated lattice is *commutative* if its monoid operation $\odot$ is commutative. In this case, the residuation operations $/$ and $\backslash$ coincide, and is simply denoted by $\rightarrow$. A $\mathsf{FL}_{\mathrm{ew}}$-algebra is a bounded integral commutative residuated lattice, explicitly defined as follows.

Definition 1.2.10 ($\mathsf{FL}_{\mathrm{ew}}$-algebra)**.** A $\mathsf{FL}_{\mathrm{ew}}$-*algebra* is an algebra

$$\mathbf{R} = \langle R, \wedge, \vee, 0, 1, \odot, \rightarrow \rangle$$

such that $\langle R, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice, $\langle R, \odot, 1 \rangle$ is a commutative monoid and the residuation law

$$x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z$$

holds in $\mathbf{R}$.

Examples of famous subvarieties of the variety of $\mathsf{FL}_{\mathrm{ew}}$-algebras are the variety of MV-algebras, the variety of Heyting algebras, the variety of Gödel algebras and the variey of Boolean algebras.

In [Kow04, Theorem 3.10], Kowalski showed that a $\mathsf{FL}_{\mathrm{ew}}$-algebra $\mathbf{R}$ is quasi-primal if and only if there is some $n \geq 0$ such that

$$x \vee \neg(x^n) = 1 \tag{1.1}$$

is satisfied in $\mathbf{R}$, where, as usual, we define $\neg x$ as $x \rightarrow 0$ and x^n refers to the n-th power with respect to $\odot$ (in fact, Kowalski even showed that every variety of $\mathsf{FL}_{\mathrm{ew}}$-algebras satisfying the above equation is a discriminator variety).

With this, we can prove the following partial converse of Corollary 1.2.8.

Corollary 1.2.11. *Let* $\mathbf{R}$ *be a finite* $\mathsf{FL}_{\mathrm{ew}}$*-algebra based on a chain. Then* $\mathbf{R}$ *is quasi-primal if and only if no element of* $\mathbf{R} \backslash \{0, 1\}$ *is idempotent.*

Proof. We already showed that the condition is sufficient in Corollary 1.2.8. For necessity, assume that $a \in \mathbf{R}\backslash\{0,1\}$ is idempotent. Then $\neg a < a$, since for all $b \geq a$ we have $a \odot b \geq a \odot a = a$. Therefore, for all $n \geq 1$, we have $a \vee \neg(a^n) = a \vee \neg a = a \neq 1$. Therefore, $\mathbf{R}$ does not satisfy Equation (1.1), *i.e.*, it is not quasi-primal. $\square$

We now give some concrete examples of semi-primal bounded residuated lattices. We begin with some semi-primal $\mathsf{FL}_{\mathrm{ew}}$-chains. Due to Corollary 1.2.11, we only need to consider $\mathsf{FL}_{\mathrm{ew}}$-algebras without idempotent elements $a \notin \{0,1\}$. In [GJ17], Galatos and Jipsen provide a list of all finite residuated lattices of size up to 6. In the following, we use the notation used therein.

The only quasi-primal $\mathsf{FL}_{\mathrm{ew}}$-chain with three elements is $\mathbf{Ł}_2$, and, therefore, semi-primal. There are two quasi-primal $\mathsf{FL}_{\mathrm{ew}}$-chains with 4 elements, $\mathbf{R}^{4,1}_{1,5}$ and $\mathbf{R}^{4,1}_{1,6}$, and there are six quasi-primal $\mathsf{FL}_{\mathrm{ew}}$-chains with 5 elements $\mathbf{R}^{5,1}_{1,17}, \ldots, \mathbf{R}^{5,1}_{1,22}$. All of them are depicted in Figure 1.2.

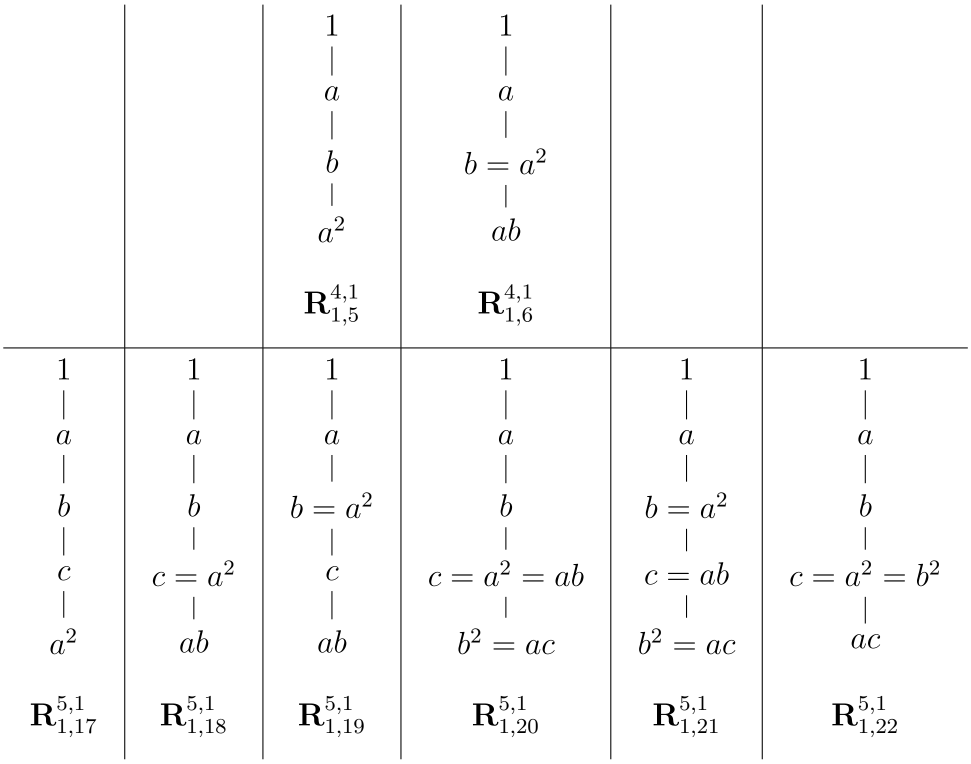
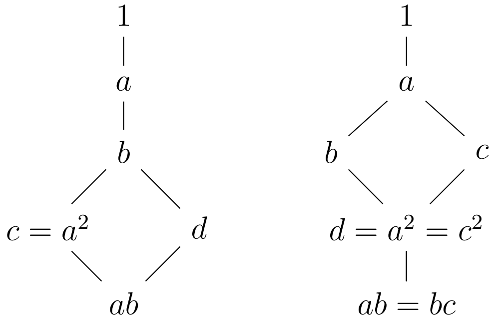


Figure 1.2: The quasi-primal $\mathsf{FL}_{\mathrm{ew}}$-chains of order four or five.

The algebra $\mathbf{R}^{4,1}_{1,5}$ is semi-primal because its only proper subalgebra is given by $\{0, a, 1\}$ (note that $b \to 0 = a$ implies that a is in the subalgebra generated by b). The algebra $\mathbf{R}^{4,1}_{1,6}$ coincides with $\mathbf{Ł}_3$ and, therefore, is also semi-primal.

Among the quasi-primal $\mathsf{FL}_{\mathrm{ew}}$-chains of order 5, all except $\mathbf{R}^{5,1}_{1,17}$ are semi-primal. For a detailed proof of this fact, see [KPT24b, Appendix A]. In that same appendix, to also provide examples of $\mathsf{FL}_{\mathrm{ew}}$-algebras which are not based on a chain, it is also shown that the algebras $\mathbf{R}^{6,2}_{2,11}$ and $\mathbf{R}^{6,3}_{1,9}$ of order 6 depicted in Figure 1.3 are semi-primal.



$\mathbf{R}^{6,2}_{1,11}$ $\mathbf{R}^{6,3}_{1,9}$

Figure 1.3: Two semi-primal $\mathsf{FL}_{\mathrm{ew}}$-algebras of order six.

On the negative side, Corollary 1.2.11 implies that there are finite *Heyting algebras*, *i.e.*, $\mathsf{FL}_{\mathrm{ew}}$-algebras where $\odot = \wedge$, which are not quasi-primal. In particular, it shows that the finite Heyting chains (also known as finite *Gödel chains*) are not quasi-primal. However, there are various expansions of Heyting algebras which are quasi-primal. Among them are the finite *bi-Heyting algebras* (introduced in [Rau74] under the name 'semi-Boolean algebras'), which arise in the context of *bi-intuitionistic logic.* A *bi-Heyting algebra* is a Heyting algebra with a binary 'co-implication' $\leftarrow$, which satisfies the dual residuation law

$$x \leftarrow y \leq z \Leftrightarrow x \leq y \vee z.$$

It is easy to see that $\big((1 \leftarrow x) \rightarrow 0\big)$ defines the unary operation T_1 in every bi-Heyting algebra with a unique atom and $\big(1 \leftarrow (x \rightarrow 0)\big)$ defines η_0 in every bi-Heyting algebra with a unique co-atom. Therefore, by Proposition 1.2.7, every finite bi-Heyting algebra with a unique atom is quasi-primal. While the three-element bi-Heyting chain is semi-primal as well (because it only has one proper subalgebra, which is rigid), every bi-Heyting chain of length ≥ 4 only becomes semi-primal after adding all unary terms τ_d to satisfy the condition of Proposition 1.2.9.

To also give examples of semi-primal algebras based on bounded residuated lattices which are *non-integral* (*i.e.*, satisfying $1 \neq e$), we consider the *De Morgan monoids* $\mathbf{C}_4$ and $\mathbf{D}_4$ depicted in Figure 1.4.

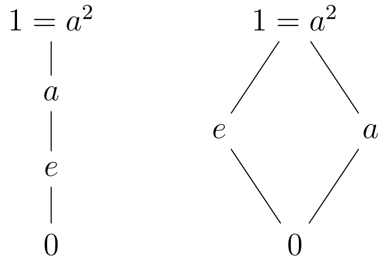


Figure 1.4: The De Morgan monoids $\mathbf{C}_4$ and $\mathbf{D}_4$.

They are bounded commutative residuated lattices with an additional involution $\sim$ which, in both examples, is defined by ${\sim}e = a$ and ${\sim}0 = 1$. Our names for these algebras are the same as in the paper [MRW19] by Moraschini, Raftery and Wannenburg, where it is shown that these algebras generate minimal subvarieties of the variety of all De Morgan monoids [MRW19, Theorem 6.1]. That paper also shows that the respective e-free reducts $\mathbf{C}_4^-$ and $\mathbf{D}_4^-$ generate minimal proper extensions of BA inside the variety of all *relevant algebras* [MRW19, Theorem 7.8]. Furthermore, the first part of the following claim is also shown therein [MRW19, p.2793].

Proposition 1.2.12. *The De Morgan monoids* $\mathbf{C}_4$ *and* $\mathbf{D}_4$ *are primal. Their e-free reducts* $\mathbf{C}_4^-$ *and* $\mathbf{D}_4^-$ *are semi-primal relevant algebras.*

Proof. Starting with $\mathbf{C}_4$, we directly verify that it satisfies characterization (ii) from Theorem 1.1.12. First we define T_1 and, therefore, $T_0(x) = T_1({\sim}x)$. As in [DSW91], we do this by, for all $\ell \in \{0, e, a\}$, defining unary terms u_ℓ satisfying $u_\ell(1) = 1$ and $u_\ell(\ell) = 0$. For instance, we can define such terms by

$$u_0(x) = x, \;\; u_e(x) = {\sim}\big(({\sim}x)^2\big) \text{ and } u_a(x) = {\sim}\big(({\sim}x) \odot 1\big).$$

Through these terms we can clearly define $T_1(x) = u_0(x) \wedge u_e(x) \wedge u_a(x)$. Lastly, we need to define τ_ℓ for $\ell \in \{e, a\}$. Again, it suffices to find terms τ_ℓ^* which satisfy

$$\tau_\ell^*(x) = \begin{cases} 1 & \text{if } x \geq \ell \\ \neq 1 & \text{if } x \ngeq \ell, \end{cases}$$

since then we get $\tau_\ell = T_1(\tau_\ell^*)$. Our desired terms are given by

$$\tau_e^*(x) = \big(({\sim}x)^2 \odot x\big) \vee x^2 \text{ and } \tau_a^*(x) = x^2.$$

This concludes the proof for $\mathbf{C}_4$. The proof for $\mathbf{D}_4$ is completely analogous, except that we use $\tau^*_e(x) = \big((\sim x)^2 \odot x\big) \vee x$ instead. Thus, we showed that these two algebras are semi-primal, and since they don't have any proper subalgebras they are primal. Since we never relied on the constant e in the above, the second part of the statement follows. Note that, for both algebras, if we exclude e from the signature, then $\{0, 1\}$ becomes a proper subalgebra. □

We end this subsection with a discussion of quasi-primal *pseudo-logics*, as defined in [CD98, p.121].

Definition 1.2.13 (Pseudo-logic)**.** A *pseudo-logic* is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, ', 0, 1\rangle,$$

where $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice and the unary operation $'$ (called the *pseudo-negation*) satisfies $0' = 1$ and $1' = 0$.

In particular, every bounded integral residuated lattice defines a pseudo-logic with its canonical negation $x' = x\backslash 0$.

In [DSW91], Davey Schumann and Werner showed that a finite pseudo-logic $\mathbf{L}$ is quasi-primal if the following two properties are satisfied:

1. There is no $a \in L\backslash\{0\}$ with $a' = 1$,
2. For all $a \in L$ there exists an $n \geq 1$ with $a \wedge a^{(2n)} = 0$ (where $a^{(k)}$ denotes the k-fold iteration of $'$ on a).

Together with the characterization of semi-primality from Theorem 1.1.10, we can use this to find examples semi-primal pseudo-logics. For this, we only need to assure that the conditions mentioned above are satisfied and that there are no non-trivial internal isomorphisms. For example, the pseudo-logics depicted in Figure 1.5 are semi-primal (in this figure, the pseudo-negation $'$ is indicated by dotted arrows).

While semi-primal algebras may seem rare, in the next subsection we recall a theorem which states that, in some sense, almost all algebras in a fixed signature are semi-primal.

1.2.4 Murskiĭ's Theorem

In the following we recall Murskiĭ's surprising theorem about the ratio of semi-primal algebras of a fixed signature compared to that of all algebras of that signature under increasing order. The original result was proved in 1975 [Mur75] (in Russian), the version we recall here is found in [Ber11, Section 6.2].

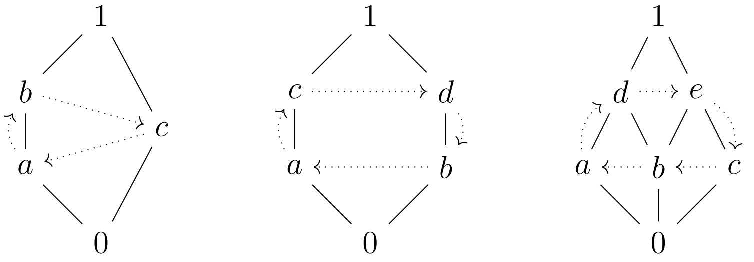


Figure 1.5: Three examples of semi-primal pseudo-logics.

Theorem 1.2.14 (Murskiĭ's Theorem). *Let σ be an algebraic language which contains at least one operation symbol which is at least binary. Let $A_{\sigma,n}$ be the number of algebras of type σ and size n and let $SP_{\sigma,n}$ be the number of such algebras which are semi-primal. Then*

$$\lim_{n\to\infty} \frac{SP_{\sigma,n}}{A_{\sigma,n}} = 1.$$

In fact, assuming that σ contains at least two operations, at least one of which is at least binary, then the same holds even for the ratio of *primal* algebras amongst all algebras (see, *e.g.*, [Ber11, Theorem 6.17]). For a signature σ which only contains one single operation which is at least binary, the ratio of primal algebras converges towards $\frac{1}{e} \approx 0.368$ (this is attributed to unpublished notes of R. O. Davies in [Qua79a]).

In the light of Theorem 1.2.14, one may justify the claim that *almost all* finite lattice-based algebras are (semi-)primal.

1.3 Conclusion of Chapter 1

In this short preliminary chapter, we gave an overview of some variations of primality and a number of examples of (semi-)primal bounded lattice reducts. In the next chapter, we study semi-primal varieties from a category-theoretical point of view and in Chapter 4 we put this to use in our study of many-valued coalgebraic logic with a semi-primal algebra of truth-degrees. In (the conclusion of) Chapter 5, we approach an appropriate notion of *lattice-semi-primal* algebra (combining lattice-primality and semi-primality) as well. Our main examples for such algebras are the *positive* MV-*chains* $\mathbf{PŁ}_n$, that is, the finite MV-chains $\mathbf{Ł}_n$ without negation.

In Subsection 1.2.3, we gave some criteria to recognize quasi-primal and semi-primal residuated lattices (possibly with additional structure). We leave a more *exhaustive* study and a *complete characterization* of variations of primality for residuated lattices open for future research.

Another interesting question for future research is that of analogues of primality for *infinite algebras*. A direct adaptation of Definition 1.1.1, as for instance investigated in [vN14], necessarily requires algebras with infinite signatures. In the author's opinion, it would be interesting to find some analogues of (quasi/semi/*etc.*)-primality which work for infinite algebras with finite signatures. In particular, motivated by the examples $\mathbf{Ł}_n$, is there a generalization of semi-primality which applies to the standard MV-chain $[0, 1]_{\mathbf{Ł}}$? For example, while the characterization of Theorem 1.1.12(ii) does not hold anymore in this algebra, it is still possible to *separate values* via terms, in the sense that for all $r_1, r_2 \in [0, 1]$, there exists a unary MV-term $T_{r_1,r_2}(x)$ with $r_1 \mapsto 1$ and $r_2 \mapsto 0$.

Chapter 2

Perspectives on semi-primal varieties

> *'Nevertheless, this might be of use in all sorts of later investigations in algebraic topology and elsewhere. We could add a couple of footnotes to the present paper.'*
> *So much for the initial proposal that there should be a new subject - now called category theory.*
>
> – Saunders Mac Lane
> (2002, quoting a letter to Samuel Eilenberg from 1942)[1]

In this chapter, we study varieties generated by semi-primal bounded lattice expansions by means of category theory. We discuss a well-known topological duality for such semi-primal varieties, and we explore the relationship between Stone duality and this topological duality. In particular, we show that there are various adjunctions between BA and any lattice-based semi-primal variety $\mathcal{A}$, which can be described in terms of the Boolean skeleton and the Boolean power constructions. Later on, in Chapter 4, we make use of the subalgebra adjunctions discussed here (Subsection 2.2.4) in order to lift algebra/coalgebra dualities as well as coalgebraic logics from the classical to the semi-primal level.

From now on, for the entirety of Chapter 2 and throughout many parts of the chapters that follow, we fix the following assumption and notation.

Assumption 2.0.1 (Main Assumption)**.** The algebra $\mathbf{D}$ is a semi-primal algebra with a reduct $\mathbf{D}^\flat = \langle D, \wedge, \vee, 0, 1\rangle$ which is a bounded lattice. We use $\mathcal{A} := \mathbb{HSP}(\mathbf{D})$ to denote the variety generated by $\mathbf{D}$. Furthermore, we use $\mathbf{E} := \langle 0, 1\rangle$ to denote the unique smallest subalgebra of $\mathbf{D}$.

---

[1] [ML02, p.130]

Note that all examples of semi-primal algebras considered in Section 1.2 are lattice-based and, therefore, satisfy this assumption.

The chapter is structured as follows. In Section 2.1, we present and provide a new proof of a well-known dual equivalence between the semi-primal variety generated by $\mathbf{D}$ and a category $\mathsf{Stone}_{\mathbf{D}}$ of structured Stone spaces. In Section 2.2, we study various adjunctions between $\mathsf{Stone}_{\mathbf{D}}$ and Stone, as well as their algebraic duals. We show that the dual of the forgetful functor $\mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$ corresponds to the *Boolean skeleton* functor (Subsection 2.2.2), which has two adjoints taking certain *Boolean powers*, and we show that this situation completely characterizes semi-primality of a lattice-based algebra $\mathbf{D}$ (Subsections 2.2.3 and 2.2.5). We also describe the *subalgebra adjunctions* which we will use in Chapter 4 to lift coalgebraic logics (Subsection 2.2.4). Lastly, in Section 2.3, we discuss canonical extensions of algebras in semi-primal varieties and we show that they are fully characterized by their Boolean skeletons (Subsection 2.3.1). We also relate the topological and the discrete semi-primal dualities via an adaptation of the Stone-Čech compactification (Subsection 2.3.2).

The topological duality for semi-primal varieties discussed in Section 2.1 goes back to Keimel and Werner [KW74], who presented it in terms of sheaves. It has also been re-phrased in terms of natural dualities in [DW83, CD98]. However, we give a proof of this duality for lattice-based algebras which does neither rely on sheaves nor on the theory of natural dualities. Instead, the proof based on techniques from [Joh82] proceeds by lifting the dual equivalence between the corresponding categories of finite objects to the level of the Ind- and Pro-completions. We also employ these techniques to establish the discrete version of this duality in Section 2.3, in order to study canonical extensions in this setting. From the point of view of natural dualities, canonical extensions have also been studied in a more general setting in [DP12, DHP17] (however, we do not rely on the results therein). Categories similar to $\mathsf{Stone}_{\mathbf{D}}$ and $\mathsf{Set}_{\mathbf{D}}$ (see Definitions 2.1.2 and 2.3.1) have been studied before in the context of *fuzzy sets*, initially by Goguen [Gog67, Gog74] and later on, for example, in [Bar86, Wal04].

This chapter may be seen as an extended version of [KPT24b, Sections 3-5] co-authored by the author of this thesis.

From now on, we assume familiarity with basic concepts and terminology of category theory (*e.g.*, morphism, adjunction, natural transformation, limits and colimits, ...). A standard reference is the introductory textbook [ML97], we additionally refer the reader to [Joh82] for Pro- and Ind-completions and [AHS06] for information about topological functors.

2.1 Semi-primal topological duality

In this section, we introduce a well-known topological duality for the variety $\mathcal{A}$, originally due to Keimel and Werner [KW74] (also see [CD98, Theorem 3.3.14] for a description via natural dualities). We introduce the category $\mathsf{Stone}_{\mathbf{D}}$ dual to $\mathcal{A}$ and give a self-contained proof of this duality. For this, we use techniques described in [Joh82], based on the categorical completions under filtered colimits (*i.e.*, the Ind-completion) and cofiltered limits (*i.e.*, the Pro-completion), respectively.

To set the scene, recall that one of the nicest 'features' of the variety of Boolean algebras BA is the famous *Stone duality* [Sto36] . Categorically speaking, it asserts that there is a dual equivalence between BA and the category Stone of *Stone spaces* (that is, compact, Hausdorff and zero-dimensional topological spaces) with continuous maps. It is established by the following contravariant functors $\Sigma\colon \mathsf{BA} \to \mathsf{Stone}$ and $\Pi\colon \mathsf{Stone} \to \mathsf{BA}$.

$$\begin{tikzcd}
\mathsf{Stone} \arrow[r, bend left, "\Pi"] & \mathsf{BA} \arrow[l, bend left, "\Sigma"]
\end{tikzcd}$$

Given a Boolean algebra $\mathbf{B}$, the corresponding Stone space $\Sigma(\mathbf{B})$ is based on the set

$$\Sigma(\mathbf{B}) = \{u \subseteq B \mid u \text{ is an ultrafilter of } \mathbf{B}\},$$

that is, the collection of ultrafilters of $\mathbf{B}$ (recall that an *ultrafilter* is a maximal proper subset which is upwards closed and closed under finite meets). The topology on $\Sigma(\mathbf{B})$ is generated by the basis consisting of

$$U_b = \{u \in \Sigma(\mathbf{B}) \mid b \in u\} \text{ for all } b \in \mathbf{B}.$$

Conversely, given a Stone space X, the Boolean algebra $\Pi(X)$ consists of the clopen (*i.e.*, closed and open) subsets of X with the usual set-theoretical Boolean operations. Both functors Σ and Π are defined on morphisms via preimages.

Equivalently up to natural isomorphism, Σ and Π are given by the hom-functors

$$\Sigma \cong \mathsf{BA}(-, \mathbf{2}) \text{ and } \Pi \cong \mathsf{Stone}(-, 2),$$

where in the latter equation 2 denotes the two-element discrete space. Stone duality can now be subsumed as follows.

Theorem 2.1.1 (Stone duality). *The functors Σ and Π defined above establish a dual equivalence between* BA *and* Stone.

We now discuss a similar topological duality for $\mathcal{A}$. First we introduce the category of topological structures which will turn out to be dual to $\mathcal{A}$. In the following, we always consider the subalgebra lattice $\mathbb{S}(\mathbf{D})$ as a complete lattice in the usual sense, that is, ordered by inclusion where $\mathbf{S}_1 \wedge \mathbf{S}_2$ is the intersection $\mathbf{S}_1 \cap \mathbf{S}_2$ and $\mathbf{S}_1 \vee \mathbf{S}_2$ is the subalgebra generated by the union $S_1 \cup S_2$.

Definition 2.1.2 (The category $\mathsf{Stone}_{\mathbf{D}}$). The category $\mathsf{Stone}_{\mathbf{D}}$ has objects $(X, \mathbf{v})$ where $X \in \mathsf{Stone}$ is a Stone space and

$$\mathbf{v}\colon X \to \mathbb{S}(\mathbf{D})$$

assigns to every point $x \in X$ a subalgebra $\mathbf{v}(x) \subseteq \mathbf{D}$, such that for every subalgebra $\mathbf{S} \subseteq \mathbf{D}$, the preimage $\mathbf{v}^{-1}(\mathbf{S}{\downarrow})$ is closed. A morphism $f\colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$ in $\mathsf{Stone}_{\mathbf{D}}$ is a continuous map $X_1 \to X_2$ which satisfies

$$\mathbf{v}_2\big(f(x)\big) \leq \mathbf{v}_1(x).$$

for every $x \in X_1$.

In the framework of natural dualities [CD98], the category $\mathcal{X}$ dual to $\mathcal{A}$ is defined slightly differently, using Stone spaces with closed unary relations (*i.e.*, subsets) instead. In the following remark, we show that this category is isomorphic to $\mathsf{Stone}_{\mathbf{D}}$.

Remark 2.1.3 (Relationship to natural dualities). Let $\mathcal{X}$ be the category with objects $\big(X, \{R^{\mathbf{S}} \mid \mathbf{S} \in \mathbb{S}(\mathbf{D})\}\big)$, where $X \in \mathsf{Stone}$ and $R^{\mathbf{S}}$ is a closed subset of X for each subalgebra $\mathbf{S} \subseteq \mathbf{D}$, satisfying the two conditions

- $R^{\mathbf{D}} = X$,
- $R^{\mathbf{S_1}} \cap R^{\mathbf{S_2}} = R^{\mathbf{S_1} \cap \mathbf{S_2}}$ for all $\mathbf{S_1}, \mathbf{S_2} \in \mathbb{S}(\mathbf{D})$.

A morphism $f\colon \big(X_1, \{R_1^{\mathbf{S}} \mid \mathbf{S} \in \mathbb{S}(\mathbf{D})\}\big) \to \big(X_2, \{R_2^{\mathbf{S}} \mid \mathbf{S} \in \mathbb{S}(\mathbf{D})\}\big)$ in $\mathcal{X}$ is a continuous relation-preserving map $X_1 \to X_2$, that is, it satisfies $x \in R_1^{\mathbf{S}} \Rightarrow f(x) \in R_2^{\mathbf{S}}$ for all $x \in X_1$ and $\mathbf{S} \in \mathbb{S}(\mathbf{D})$.

The categories $\mathcal{X}$ and $\mathsf{Stone}_{\mathbf{D}}$ are isomorphic, as witnessed by the following mutually inverse functors Φ and Ψ. The functor $\Phi\colon \mathcal{X} \to \mathsf{Stone}_{\mathbf{D}}$ is given on objects by $\big(X, \{R^{\mathbf{S}} \mid \mathbf{S} \in \mathbb{S}(\mathbf{D})\}\big) \mapsto (X, \mathbf{v}_R)$, where

$$\mathbf{v}_R(x) = \bigwedge\{\mathbf{S} \mid x \in R^{\mathbf{S}}\}.$$

The functor $\Psi\colon \mathsf{Stone}_{\mathbf{D}} \to \mathcal{X}$ is given on objects by $(X, \mathbf{v}) \mapsto \big(X, \{R_{\mathbf{v}}^{\mathbf{S}} \mid \mathbf{S} \in \mathbb{S}(\mathbf{D})\}\big)$, where

$$R_{\mathbf{v}}^{\mathbf{S}} = v^{-1}(\mathbf{S}{\downarrow}).$$

Both Φ and Ψ map every morphism to itself. To check that Φ and Ψ are mutually inverse, we note that $\Psi\Phi(X, \{R^{\mathbf{S}} \mid \mathbf{S} \in \mathbb{S}(\mathbf{D})\}) = (X, \{R^{\mathbf{S}}_{\mathbf{v}_R} \mid \mathbf{S} \in \mathbb{S}(\mathbf{D})\})$ by definition satisfies

$$x \in R^{\mathbf{S}}_{\mathbf{v}_R} = \mathbf{v}_R^{-1}(\mathbf{S}{\downarrow}) \Leftrightarrow \mathbf{v}_R(x) = \bigwedge\{\mathbf{T} \mid x \in R^{\mathbf{T}}\} \leq \mathbf{S} \Leftrightarrow x \in R^{\mathbf{S}},$$

where the implication $\Rightarrow$ of the last equivalence is due to the fact that the $R^{\mathbf{S}}$ are compatible with intersections.

Conversely, we note that $\Phi\Psi(X, \mathbf{v}) = (X, \mathbf{v}_{R_{\mathbf{v}}})$ by definition satisfies

$$\mathbf{v}_{R_{\mathbf{v}}}(x) = \bigwedge\{\mathbf{S} \mid x \in R^{\mathbf{S}}_{\mathbf{v}}\} = \bigwedge\{\mathbf{S} \mid x \in \mathbf{v}^{-1}(\mathbf{S}{\downarrow})\} = \mathbf{v}(x)$$

as desired. This finishes the proof. ■

In the following, we describe two contravariant functors $\Sigma' \colon \mathcal{A} \to \mathsf{Stone}_{\mathbf{D}}$ and $\Pi' \colon \mathsf{Stone}_{\mathbf{D}} \to \mathcal{A}$ which, as we shall see later, establish a dual equivalence which generalizes Stone duality as described above.

$$\begin{tikzcd}
\mathsf{Stone}_{\mathbf{D}} \arrow[r, bend left, "\Pi'"] & \mathcal{A} \arrow[l, bend left, "\Sigma'"]
\end{tikzcd}$$

For an algebra $\mathbf{A} \in \mathcal{A}$, we define $\Sigma'(\mathbf{A}) \in \mathsf{Stone}_{\mathbf{D}}$ by

$$\Sigma'(\mathbf{A}) = \big(\mathcal{A}(\mathbf{A}, \mathbf{D}), \mathbf{im}\big)$$

where $\mathbf{im}$ takes a homomorphism $u \colon \mathbf{A} \to \mathbf{D}$ to its image $\mathbf{im}(u) = u(\mathbf{A}) \in \mathbb{S}(\mathbf{D})$. A clopen subbase for the topology on $\mathcal{A}(\mathbf{A}, \mathbf{D})$ is given by the collection of sets of the form

$$[a : d] = \{u \in \mathcal{A}(\mathbf{A}, \mathbf{D}) \mid u(a) = d\} \text{ for all } a \in A, d \in D.$$

On the homomorphism $h \in \mathcal{A}(\mathbf{A}_1, \mathbf{A}_2)$, the functor Σ' acts via composition

$$\begin{aligned} \Sigma' h \colon \mathcal{A}(\mathbf{A}_2, \mathbf{D}) &\to \mathcal{A}(\mathbf{A}_1, \mathbf{D}) \\ u &\mapsto u \circ h. \end{aligned}$$

Note that this is a morphism in $\mathsf{Stone}_{\mathbf{D}}$ since $\mathbf{im}(u \circ h) \leq \mathbf{im}(u)$.

Before we define the functor Π', we need to find a canonical way to consider D as a member of $\mathsf{Stone}_{\mathbf{D}}$. For this, simply endow D with the discrete topology and

$$\langle \cdot \rangle \colon D \to \mathbb{S}(\mathbf{D})$$

mapping every element $d \in D$ to the subalgebra $\langle d \rangle \leq \mathbf{D}$ it generates. Now, as expected, we can define the functor Π' on objects $(X, \mathbf{v}) \in \mathsf{Stone}_{\mathbf{D}}$ via

$$\Pi'(X, \mathbf{v}) = \mathsf{Stone}_{\mathbf{D}}\big((X, \mathbf{v}), (D, \langle \cdot \rangle)\big)$$

with the algebraic operations defined pointwise. This means that the carrier-set of $\Pi'(X, \mathbf{v})$ is the set of continuous maps $\alpha \colon X \to D$ which respect $\mathbf{v}$ in the sense that they satisfy

$$\alpha(x) \in \mathbf{v}(x)$$

for all $x \in X$. Again, on morphisms $f \colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$ the functor is defined via composition

$$\begin{aligned} \Pi' f \colon \mathsf{Stone}_{\mathbf{D}}\big((X_2, \mathbf{v}_2), (D, \langle \cdot \rangle)\big) &\to \mathsf{Stone}_{\mathbf{D}}\big((X_1, \mathbf{v}_1), (D, \langle \cdot \rangle)\big) \\ \alpha &\mapsto \alpha \circ f. \end{aligned}$$

This is well-defined due to the condition on morphisms in $\mathsf{Stone}_{\mathbf{D}}$, which implies

$$(\alpha \circ f)(x) = \alpha\big(f(x)\big) \in \mathbf{v}_2\big(f(x)\big) \leq \mathbf{v}_1(x)$$

for all $x \in X_1$. It is also clearly a homomorphism since the operations are defined pointwise.

The dual equivalence between $\mathsf{Stone}_{\mathbf{D}}$ and $\mathcal{A}$ has first been described by Keimel and Werner in [KW74], who proved it via the theory of sheaves. This duality also fits the general framework of *natural dualities*, as shown in the book by Clark and Davey [CD98, Theorem 3.3.14]). In the following, we present and prove this duality theorem in a self-contained way which particularly suits our purpose. Our proof relies on categorical constructions from [Joh82] and, to the best of the authors knowledge, first appeared in [KPT24b].

Theorem 2.1.4 (Semi-primal topological duality)**.** *The functors Σ' and Π' defined above establish a dual equivalence between $\mathcal{A}$ and* $\mathsf{Stone}_{\mathbf{D}}$*.*

In particular, in the case where $\mathbf{D} = \mathbf{2}$ is the two-element Boolean algebra, we recover Stone duality as described in Theorem 2.1.1 (if $\mathbf{D}$ is primal, then $\mathsf{Stone}_{\mathbf{D}}$ is isomorphic to Stone since $\mathbb{S}(\mathbf{D}) = \{\mathbf{D}\}$ is a one-element poset). In this case, we proceed to denote the functors involved by Σ and Π (instead of Σ' and Π').

In the remainder of this section, we work to provide an alternative proof of Theorem 2.1.4. The idea is to directly prove the duality on the finite level, and then lift it to the infinite level using certain categorical completions. In the following, recall that a category I is *filtered* if it satisfies the following three properties:

- I is not empty.
- For all $i, j \in I$, there is some $k \in I$ for which morphisms $i \to k$ and $j \to k$ exist.

- For all parallel morphisms $f, g\colon i \to j$ there is some $k \in I$ and a morphism $h\colon j \to k$ such that $h \circ g = h \circ g$.

Note that this generalizes the notion of *directed poset*, since a poset is filtered as a category if and only if it is directed. Of course, we say that a category I is *cofiltered* if I^{op} is filtered. As usual, we call the limit of a small diagram $I \to \mathsf{C}$ *cofiltered* if I is filtered, and similarly we define *filtered colimits*. It was shown in [AN82] that filtered and directed colimits are 'essentially the same'. Colimits over directed diagrams are also (misleadingly) referred to as *inductive* (or *direct*) limits, which explains the name Ind-completion. Similarly, limits over codirected diagrams are sometimes referred to as *projective* (or *inverse*) limits, which explains the name Pro-completion.

Definition 2.1.5 (Ind-&Pro-completion)**.** Let C be a finitely complete and cocomplete category. The Ind-*completion* of C, denoted $\mathsf{Ind}(\mathsf{C})$, is the completion of C under filtered colimits. Dually, the Pro-*completion* of C, denoted $\mathsf{Pro}(\mathsf{C})$ is the completion of C under cofiltered limits.

More material about these completions can be found in Johnstone's book [Joh82, Chapter VI] (in particular, a more rigorous definition of the Ind-completion is given in VI.1.2 therein). This book also contains some notes about the history and development of these topics.

As an example, is is easy to see that $\mathsf{Ind}(\mathsf{Set}^{\omega}) \simeq \mathsf{Set}$, since every set can be written as a directed union (*i.e.*, a filtered colimit) of its finite subsets. As another example, it is also well-known that $\mathsf{Ind}(\mathsf{BA}^{\omega}) \simeq \mathsf{BA}$ which, together with Stone duality, also implies $\mathsf{Pro}(\mathsf{Set}^{\omega}) \simeq \mathsf{Stone}$ (because by definition it holds that $\mathsf{Pro}(\mathsf{C}) \simeq \mathsf{Ind}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$). More generally, $\mathsf{Ind}(\mathcal{V}^{\omega}) \simeq \mathcal{V}$ holds for every locally finite variety of algebras $\mathcal{V}$ (see, *e.g.*, [Joh82, VI.2.2]).

Lemma 2.1.6 ([Joh82, Lemma VI.3.1])**.** *Let* C *and* D *be essentially small, finitely complete and cocomplete categories which are dually equivalent. Then* $\mathsf{Ind}(\mathsf{C})$ *is dually equivalent to* $\mathsf{Pro}(\mathsf{D})$ *as well.*

Dualities arising this way are sometimes (*e.g.*, in [Joh82]) called *Stone-type dualities*.

Our argument to prove Theorem 2.1.4 now has the following outline. The role of C will be played by $\mathcal{A}^{\omega}$. Since $\mathcal{A}$ is locally finite (see, *e.g.*, [CD98, Lemma 1.3.2]), as mentioned above already, we know that $\mathsf{Ind}(\mathcal{A}^{\omega}) \simeq \mathcal{A}$ holds. The role of D will be played by $\mathsf{Stone}^{\omega}_{\mathbf{D}}$. Since the topology doesn't matter here (because it is always discrete), we will denote this category by $\mathsf{Set}^{\omega}_{\mathbf{D}}$ instead. To get the finite dual equivalence, we first make the following observation.

Lemma 2.1.7. *Let $\mathbf{S}_1, \dots, \mathbf{S}_n$ be subalgebras of $\mathbf{D}$. Then the set of homomorphisms $\mathcal{A}(\prod_{i\leq n} \mathbf{S}_i, \mathbf{D})$ consists exactly of the projections followed by inclusions*

$$\mathsf{pr}_i \colon \prod_{i\leq n} \mathbf{S_i} \to \mathbf{S_i} \hookrightarrow \mathbf{D}$$

in each component $i \leq n$.

Proof. Our proof is similar to the one of [CGL18, Theorem 2.5]. Start with a homomorphism $u\colon \prod_{i\leq n} \mathbf{S}_i \to \mathbf{D}$. Since, due to Proposition 1.1.8(2), the variety $\mathcal{A}$ is congruence-distributive, it has the Fraser-Horn property, meaning that the congruence $\theta := \mathsf{ker}(u)$ is a product of congruences θ_i on $\mathbf{S}_i$. By the isomorphism theorem we find

$$(\prod_{i\leq n} \mathbf{S}_i)/\theta \cong \prod_{i\leq n} (\mathbf{S}_i/\theta_i) \cong \mathsf{im}(u).$$

Since $\mathsf{im}(u)$ is a subalgebra of $\mathbf{D}$ and thus simple (Proposition 1.1.8(1)), at most one factor of $\prod_{i\leq n}(\mathbf{S}_i/\theta_i)$ can be non-trivial. Since $\mathsf{im}(u)$ contains at least two elements (that is, 0 and 1), precisely one factor, say $\mathbf{S}_j/\theta_j$, is non-trivial. Since $\mathbf{S}_j$ is itself semi-primal, it is simple, so $\mathbf{S}_j/\theta_j \cong \mathbf{S}_j$. Therefore, u induces an internal isomorphism $\mathbf{S}_j \cong \mathsf{im}(u)$, but since $\mathbf{S}_j$ is semi-primal this can only be the identity on $\mathbf{S}_j$, thus u coincides with the projection pr_j. □

With this at hand, it is easy to establish the semi-primal duality on the finite level. In the special case $\mathbf{D} = \mathbf{2}$, this corresponds to the duality between the category BA^ω of finite Boolean algebras and the category Set^ω of finite sets.

Theorem 2.1.8 (Finite semi-primal duality)**.** *The (restrictions of the) functors Π' and Σ' establish a dual equivalence between the categories $\mathsf{Set}^\omega_\mathbf{D}$ and $\mathcal{A}^\omega$.*

Proof. For $(X, \mathbf{v}) \in \mathsf{Set}^\omega_\mathbf{D}$ we have

$$\Sigma'\Pi'(X, \mathbf{v}) = \Big(\mathcal{A}(\prod_{x\in X} \mathbf{v}(x), \mathbf{D}), \mathbf{im}\Big).$$

By Lemma 2.1.7, this is equal to $(\{\mathsf{pr}_x \mid x \in X\}, \mathbf{im})$, which is clearly isomorphic to $(X, \mathbf{v})$.

On the other hand, starting with a finite algebra $\mathbf{A} \in \mathcal{A}^\omega$, we know by Proposition 1.1.8(4) that it is isomorphic to a finite product of subalgebras $\mathbf{A} = \prod_{i\leq n} \mathbf{S}_i$. Now, again due to Lemma 2.1.7, we get $\Sigma'(\mathbf{A}) = (\{\mathsf{pr}_i \mid i \leq n\}, \mathbf{im})$, and thus

$$\Pi'\Sigma'(\mathbf{A}) \cong \prod_{i\leq n} \mathbf{im}(\mathsf{pr}_i) \cong \prod_{i\leq n} \mathbf{S}_i$$

as desired. To see that Π' and Σ' form a dual adjunction we note that for $\mathbf{A} = \prod_{i \leq n} \mathbf{S}_i \in \mathcal{A}^\omega$ and $(X, \mathbf{v}) \in \mathsf{Set}^\omega_\mathbf{D}$, we have

$$\mathcal{A}^\omega\big(\Pi'(X, \mathbf{v}), \mathbf{A}\big) \cong \prod_{i \leq n} \mathcal{A}^\omega\big(\Pi'(X, \mathbf{v}), \mathbf{S}_i\big)$$

and

$$\begin{aligned} \mathsf{Set}^\omega_\mathbf{D}\big(\Sigma'(\mathbf{A}), (X, \mathbf{v})\big) &\cong \mathsf{Set}^\omega_\mathbf{D}\big(\coprod_{i \leq n}(\{\mathsf{pr}_i\}, \mathbf{im}), (X, \mathbf{v})\big) \\ &\cong \prod_{i \leq n} \mathsf{Set}^\omega_\mathbf{D}\big((\{\mathsf{pr}_i\}, \mathbf{im}), (X, \mathbf{v})\big), \end{aligned}$$

where the coproduct in $\mathsf{Set}^\omega_\mathbf{D}$ is the obvious disjoint union. So we only need to show that

$$\mathcal{A}^\omega\big(\Pi'(X, \mathbf{v}), \mathbf{S_i}\big) \cong \mathsf{Set}^\omega_\mathbf{D}\big((\{\mathsf{pr}_i\}, \mathbf{im}), (X, \mathbf{v})\big)$$

holds. But this is obvious since the elements of the left-hand side are exactly the projections with image contained in $\mathbf{S}_i$, which are in bijective correspondence with the points of X with $\mathbf{v}(x) \leq \mathbf{S}_i$, that is, with elements of the right-hand side. □

Now, in order to successfully apply Lemma 2.1.6, it remains to show that $\mathsf{Pro}(\mathsf{Set}^\omega_\mathbf{D}) \simeq \mathsf{Stone}_\mathbf{D}$. In order to do this, we first show that $\mathsf{Stone}_\mathbf{D}$ is complete. For categories of fuzzy sets which are similar to $\mathsf{Stone}_\mathbf{D}$, this has already been shown by Goguen [Gog74, Theorem 14].

Proposition 2.1.9 ($\mathsf{Stone}_\mathbf{D}$ is complete)**.** *The category* $\mathsf{Stone}_\mathbf{D}$ *has all small limits. In particular, it has all cofiltered limits.*

Proof. First we show that $\mathsf{Stone}_\mathbf{D}$ has all products. For an indexed family $(X_i, \mathbf{v}_i)_{i \in I}$, we claim that the product is given by

$$\prod_{i \in I}(X_i, \mathbf{v}_i) = (\prod_{i \in I} X_i, \bigvee \mathbf{v}_i),$$

where $\bigvee \mathbf{v}_i(x) = \bigvee(\mathbf{v}_i(x_i))$ for all $x \in \prod X_i$. It follows from

$$(\bigvee \mathbf{v}_i)^{-1}(\mathbf{S}{\downarrow}) = \prod \mathbf{v}_i^{-1}(\mathbf{S}{\downarrow})$$

that this defines an object of $\mathsf{Stone}_\mathbf{D}$. Note that the canonical projections are morphisms in $\mathsf{Stone}_\mathbf{D}$ because of

$$\mathbf{v}_i\big(\mathsf{pr}_i(x)\big) = \mathbf{v}_i(x_i) \leq \bigvee_{j \in I} \mathbf{v}_j(p_j) = (\bigvee \mathbf{v}_j)(x).$$

If $(\gamma_i \colon (Y, \mathbf{w}) \to (X_i, \mathbf{v}_i) \mid i \in I)$ is another cone, there is a unique continuous map $f \colon Y \to \prod X_i$ with $\mathsf{pr}_i \circ f = \gamma_i$. This map is a morphism in $\mathsf{Stone}_{\mathbf{D}}$ since

$$\big(\bigvee \mathbf{v}_i\big)\big(f(y)\big) = \bigvee \mathbf{v}_i\big(\mathsf{pr}_i(f(y))\big) = \bigvee \mathbf{v}_i\big(\gamma_i(f(y))\big) \leq \mathbf{w}(y),$$

where the last inequality follows from $\mathbf{v}_i(\gamma_i)(y) \leq \mathbf{w}(y)$, which is true since γ_i is a morphism in $\mathsf{Stone}_{\mathbf{D}}$ for every $i \in I$.

The equaliser of $f_1, f_2 \colon (X, \mathbf{v}_1) \to (Y, \mathbf{w})$ is simply given by $(E, \mathbf{v}|_E)$ where $E \subseteq X$ is the corresponding equalizer in Stone.

Thus, since we showed that $\mathsf{Stone}_{\mathbf{D}}$ has all products and equalisers, we conclude that $\mathsf{Stone}_{\mathbf{D}}$ is complete. □

We are now ready to prove that $\mathsf{Pro}(\mathsf{Set}^{\omega}_{\mathbf{D}}) \simeq \mathsf{Stone}_{\mathbf{D}}$ which, as discussed above, implies Theorem 2.1.4.

Theorem 2.1.10. $\mathsf{Pro}(\mathsf{Set}^{\omega}_{\mathbf{D}})$ *is categorically equivalent to* $\mathsf{Stone}_{\mathbf{D}}$.

Proof. As a consequence of Proposition 2.1.9, the category $\mathsf{Stone}_{\mathbf{D}}$ has all cofiltered limits, so the natural inclusion functor $\iota \colon \mathsf{Set}^{\omega}_{\mathbf{D}} \hookrightarrow \mathsf{Stone}_{\mathbf{D}}$ has a unique *cofinitary* (that is, cofiltered limit preserving) extension

$$\hat{\iota} \colon \mathsf{Pro}(\mathsf{Set}^{\omega}_{\mathbf{D}}) \hookrightarrow \mathsf{Stone}_{\mathbf{D}}.$$

Since ι is fully faithful, in order to conclude that the functor $\hat{\iota}$ is fully faithful as well, it suffices to show that ι takes all objects to finitely copresentable objects in $\mathsf{Stone}_{\mathbf{D}}$ (this is due to the analogue of [Joh82, Theorem VI.1.8] for the Pro-completion). So we need to show that any $(C, \mathbf{w}) \in \mathsf{Stone}_{\mathbf{D}}$, where C is a finite discrete space, is finitely copresentable. In other words, we need to show that, whenever $(X, \mathbf{v}) \cong \lim_{i \in I}(X_i, \mathbf{v}_i)$ is a cofiltered limit of a diagram $(f_{ij} \colon (X_j, \mathbf{v}_j) \to (X_i, \mathbf{v}_i) \mid i \leq j)$ in $\mathsf{Stone}_{\mathbf{D}}$ with limit morphisms $p_i \colon (X, \mathbf{v}) \to (X_i, \mathbf{v}_i)$, any morphism $f \colon (X, \mathbf{v}) \to (C, \mathbf{w})$ factors essentially uniquely through one of the p_i. For this we can employ an argument similar to the one in the proof of [RZ10, Lemma 1.1.16(b)]. On the underlying level of Stone, where finite discrete spaces are finitely copresentable, the continuous map f factors essentially uniquely through some p_i, say via the continuous map $g_i \colon X_i \to C$. However, g_i is not necessarily a morphism in $\mathsf{Stone}_{\mathbf{D}}$. Consider $J = \{j \geq i\}$, and for each $j \in J$ define $g_j = f_{ij} \circ g_i$. Define the continuous maps $\mu \colon X \to \mathbb{S}(\mathbf{D})^2$ and $\mu_j \colon X_j \to \mathbb{S}(\mathbf{D})^2$ for all $j \in J$ by

$$\mu(x) = \big(\mathbf{w}(f(x)), \mathbf{v}(x)\big) \text{ and } \mu_j(x) = \big(\mathbf{w}(g_j(x)), \mathbf{v}_j(x)\big).$$

Since $\mu(X) = \lim_{j \in J} \mu_j(X_j) = \bigcap_{j \geq i} \mu_j(X_j)$ is contained in the finite set $\mathbb{S}(\mathbf{D})^2$ and J is directed, there is some $k \in J$ for which

$$\mu(X) = \mu_k(X_k)$$

holds. But now, since f is a morphism in $\mathsf{Stone}_{\mathbf{D}}$, we have that $\mu(X) \subseteq \{(\mathbf{S}, \mathbf{T}) \mid \mathbf{S} \leq \mathbf{T}\}$, and thus the same holds for $\mu_k(X_k)$. Thus, g_k is a morphism in $\mathsf{Stone}_{\mathbf{D}}$ which has the desired properties.

To finish the proof, we show that $\hat{\iota}$ is essentially surjective, in other words, we show that every element $(X, \mathbf{v})$ of $\mathsf{Stone}_{\mathbf{D}}$ is isomorphic to a cofiltered limit of elements of $\mathsf{Set}^{\omega}_{\mathbf{D}}$. We do this in a manner similar to [RZ10, Theorem 1.1.12], where a direct proof of $\mathsf{Pro}(\mathsf{Set}^{\omega}) \simeq \mathsf{Stone}$ is provided. Let $\mathcal{R}$ consist of all finite partitions of X into clopen sets. Together with the order $R \leq R'$ if and only if R' refines R, this forms a codirected set and in [RZ10, Theorem 1.1.12] it is shown that $X \cong \lim_{R\in\mathcal{R}} R$. We now turn every $R \in \mathcal{R}$ into a member of $\mathsf{Set}^{\omega}_{\mathbf{D}}$ by endowing it with an appropriate $\mathbf{v}_R \colon R \to \mathbb{S}(\mathbf{D})$ and show that $(X, \mathbf{v}) = \lim_{R\in\mathcal{R}}(R, \mathbf{v}_R)$. For $R \in \mathcal{R}$, say $R = \{\Omega_1, \dots, \Omega_k\}$, we define

$$\mathbf{v}_R^{-1}(\mathbf{S}{\downarrow}) = \{\Omega_i \mid \Omega_i \cap \mathbf{v}^{-1}(\mathbf{S}{\downarrow}) \neq \varnothing\}.$$

The map $p_R \colon X \to R$ defined by $p_R(x) = \Omega_i \Leftrightarrow x \in \Omega_i$ is a morphism in $\mathsf{Stone}_{\mathbf{D}}$, since $\mathbf{v}(x) = \mathbf{S}$ and $x \in \Omega_i$ implies $\mathbf{v}_R(p_R(x)) \in \mathbf{v}_R^{-1}(\mathbf{S}{\downarrow})$. It is easy to see that this defines a cone over the diagram $(R, \mathbf{v}_R)_{R\in\mathcal{R}}$, so there is a unique $f \colon (X, \mathbf{v}) \to \lim_{R\in\mathcal{R}}(R, \mathbf{v}_R)$ in $\mathsf{Stone}_{\mathbf{D}}$. As in Stone, the map f is a homeomorphism. To complete the proof, it suffices to show that f^{-1} is a morhpism in $\mathsf{Stone}_{\mathbf{D}}$ as well. Say $\lim_{R\in\mathcal{R}}(R, \mathbf{v}_R) = (Y, \mathbf{w})$ and let $\pi_R \colon (Y, \mathbf{w}) \to (R, \mathbf{v}_R)$ denote the limit morphisms. Assuming $\mathbf{w}(y) = \mathbf{S}$ we want to show $f^{-1}(y) \in \mathbf{v}^{-1}(\mathbf{S}{\downarrow})$. Let $\Omega \subseteq X$ be an arbitrary clopen set containing $f^{-1}(y)$. Then $R = \{\Omega, X\backslash\Omega\} \in \mathcal{R}$ and

$$\Omega = p_R\big(f^{-1}(y)\big) = \pi_R(y) \in \mathbf{v}_R^{-1}(\mathbf{S}{\downarrow}).$$

By definition, this means that $\Omega \cap \mathbf{v}^{-1}(\mathbf{S}{\downarrow}) \neq \varnothing$. Since this holds for every Ω containing $f^{-1}(y)$, this implies that $f^{-1}(y)$ is in the closure $\overline{\mathbf{v}^{-1}(\mathbf{S}{\downarrow})}$. However, this closure coincides with $\mathbf{v}^{-1}(\mathbf{S}{\downarrow})$, since by definition of $\mathsf{Stone}_{\mathbf{D}}$ this is a closed set to begin with. □

Using similar techniques, Cignoli, Dubuc and Mundici [CDM04] proved that there is a dual equivalence between the category $\mathsf{Ind}(\mathsf{MV}^{\omega})$ of locally finite MV-algebras and a category similar to $\mathsf{Stone}_{\mathbf{D}}$, for which $\mathbb{S}(\mathbf{D})$ is replaced by the lattice of *supernatural numbers* with the Scott topology.

In Section 2.3, we investigate the other dual equivalence which can be obtained from the finite dual equivalence of Theorem 2.1.8. More specifically, there we describe $\mathsf{Ind}(\mathsf{Set}^{\omega}_{\mathbf{D}})$ and its dual, the category of profinite algebras $\mathsf{Pro}(\mathcal{A}^{\omega})$. This may be seen as a 'semi-primal version' of the duality between $\mathsf{Ind}(\mathsf{Set}^{\omega}) \simeq \mathsf{Set}$ and $\mathsf{Pro}(\mathsf{BA}^{\omega}) \simeq \mathsf{CABA}$, the category of complete and atomic Boolean algebras.

Before that, in the following section we investigate the relationship between $\mathsf{Stone}_{\mathbf{D}}$ and Stone in terms of various adjunctions. Dually, and perhaps more interestingly, we investigate the relationship between $\mathcal{A}$ and BA. We will also see that this relationship fully characterizes semi-primality of $\mathbf{D}$.

2.2 A collection of adjunctions

In this section, we explore the relationship between Stone duality and the semi-primal topological duality from the previous section. First, we explore the connection between $\mathsf{Stone}_{\mathbf{D}}$ and Stone, which can be expressed in terms of a chain of four adjoint functors (similar to one in [Wal04]). Then, we look at the duals of these functors on the algebraic side, aiming to provide purely algebraic descriptions of them in order to gain insight into the structure of $\mathcal{A}$ relative to that of BA. The entire situation is summarized in Figure 2.1, which we will have fully described at the end of this section (note that left-adjoints on the topological side correspond to right-adjoints on the algebraic side and vice-versa, since the functors Π', Σ' and Σ, Π which establish the two dualities are all contravariant).

$$\begin{tikzcd}
\mathsf{Stone}_{\mathbf{D}} \arrow[r, shift left, "\Pi'"] \arrow[d, bend right=15, "\mathsf{U}"'] \arrow[d, bend left=60, "\mathsf{C}^{\mathbf{E}}"] \arrow[d, phantom, bend right=45, "\dashv"] \arrow[d, phantom, "\dashv"] \arrow[d, phantom, bend left=45, "\dashv"] & \mathcal{A} \arrow[l, shift left, "\Sigma'"] \arrow[d, bend right=15, "\mathfrak{S}"'] \arrow[d, bend left=60, "\mathfrak{S}_{\mathbf{E}}"] \arrow[d, phantom, bend right=45, "\vdash"] \arrow[d, phantom, "\vdash"] \arrow[d, phantom, bend left=45, "\vdash"] \\
\mathsf{Stone} \arrow[u, bend left=60, "\mathsf{V}^{\mathbf{D}}"] \arrow[u, bend right=15, "\mathsf{V}^{\mathbf{E}}"'] \arrow[r, shift left, "\Pi"] & \mathsf{BA} \arrow[l, shift left, "\Sigma"] \arrow[u, bend left=60, "\mathfrak{P}"] \arrow[u, bend right=15, "\mathfrak{P}_{\mathbf{E}}"']
\end{tikzcd}$$

Figure 2.1: Chain of adjunctions on the topological and the algebraic side.

This section is structured as follows. In Subsection 2.2.1, we describe the four functors $\mathsf{V}^{\mathbf{D}}$, U, $\mathsf{V}^{\mathbf{E}}$, $\mathsf{C}^{\mathbf{E}}$ appearing on the topological side of Figure 2.1. In Subsection 2.2.2, we give a purely algebraic description of $\mathfrak{S}$ and in Subsection 2.2.3, we give a purely algebraic description of $\mathfrak{P}$. In Subsection 2.2.4, we put these into the more general context of the subalgebra adjunctions.

Lastly, in Subsection 2.2.5, we show that the existence of these adjunctions fully characterizes semi-primality of a lattice-based algebra.

2.2.1 Four functors on the topological side

Our starting point is the obvious *forgetful functor* $\mathsf{U}\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$, sending every object $(X, \mathbf{v})$ to its underlying Stone space X, and sending every morphism to itself. This functor has both a left-adjoint and a right-adjoint $\mathsf{V}^{\mathbf{D}} \dashv \mathsf{U} \dashv \mathsf{V}^{\mathbf{E}}$. The two functors $\mathsf{V}^{\mathbf{D}}, \mathsf{V}^{\mathbf{E}}\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$ are defined similarly. On objects they are given by

$$\mathsf{V}^{\mathbf{D}}(X) = (X, \mathbf{v}^{\mathbf{D}}) \text{ where } \forall x \in X : \mathbf{v}^{\mathbf{D}}(x) = \mathbf{D},$$
$$\mathsf{V}^{\mathbf{E}}(X) = (X, \mathbf{v}^{\mathbf{E}}) \text{ where } \forall x \in X : \mathbf{v}^{\mathbf{E}}(x) = \mathbf{E},$$

and both functors are given by the identity on morphisms (this is well-defined since every continuous map $X_1 \to X_2$ is a well-defined $\mathsf{Stone}_{\mathbf{D}}$-morphism $(X_1, \mathbf{v}^{\mathbf{D}}) \to (X_2, \mathbf{v}^{\mathbf{D}})$ and similarly for $\mathbf{E}$). Recall that we use $\mathbf{E}$ to denote the smallest subalgebra of $\mathbf{D}$, which is unique and well-defined since $\mathbf{D}$ is based on a bounded lattice (Assumption 2.0.1).

To see that $\mathsf{V}^{\mathbf{D}} \dashv \mathsf{U}$ holds, note that by definition we have

$$f \in \mathsf{Stone}_{\mathbf{D}}\big((X, \mathbf{v}^{\mathbf{D}}), (Y, \mathbf{w})\big) \Leftrightarrow f \in \mathsf{Stone}(X, Y) \wedge \forall x : \mathbf{w}(f(x)) \leq \mathbf{v}^{\mathbf{D}}(x),$$

and $\mathbf{w}\big(f(x)\big) \leq \mathbf{v}^{\mathbf{D}}(x) = \mathbf{D}$ is trivially satisfied for every $f \in \mathsf{Stone}(X, Y)$.

Similarly we see $\mathsf{U} \dashv \mathsf{V}^{\mathbf{E}}$, since every $f \in \mathsf{Stone}(X, Y)$ automatically satisfies $\mathbf{v}^{\mathbf{E}}(f(x)) \leq \mathbf{w}(x)$ and, therefore, $f \in \mathsf{Stone}_{\mathbf{D}}\big((X, \mathbf{w}), (Y, \mathbf{v}^{\mathbf{E}})\big)$.

The functor $\mathsf{V}^{\mathbf{E}}$ also has a right-adjoint $\mathsf{C}^{\mathbf{E}}\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$ defined by

$$\mathsf{C}^{\mathbf{E}}(X, \mathbf{v}) = \{x \in X \mid \mathbf{v}(x) = \mathbf{E}\}$$

on objects. On morphisms $f\colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$ it acts via restriction $f \mapsto f|_{\mathsf{C}^{\mathbf{E}}(X, \mathbf{v})}$, which is well-defined since $f \in \mathsf{Stone}_{\mathbf{D}}\big((X_1, \mathbf{v}_1), (X_2, \mathbf{v}_2)\big)$ and $x \in \mathsf{C}^{\mathbf{E}}(X_1, \mathbf{v}_1)$ means

$$\mathbf{v}_2(f(x)) \leq \mathbf{v}_1(x) = \mathbf{E},$$

which is equivalent to $f(x) \in \mathsf{C}^{\mathbf{E}}(X_2, \mathbf{v}_2)$. Again, $\mathsf{V}^{\mathbf{E}} \dashv \mathsf{C}^{\mathbf{E}}$ is easy to see since

$$\begin{aligned} f \in \mathsf{Stone}_{\mathbf{D}}\big((X, \mathbf{v}^{\mathbf{E}}), (Y, \mathbf{w})\big) &\Leftrightarrow \forall x : \mathbf{w}\big(f(x)\big) \leq \mathbf{E} \\ &\Leftrightarrow f \in \mathsf{Stone}\big(X, \mathsf{C}^{\mathbf{E}}(Y, \mathbf{w})\big) \end{aligned}$$

holds. Later on, we show that both the adjunctions $\mathsf{V}^{\mathbf{D}} \dashv \mathsf{U}$ and $\mathsf{V}^{\mathbf{E}} \dashv \mathsf{C}^{\mathbf{E}}$ are particular instances of *subalgebra adjunctions*, which exist for every subalgebra $\mathbf{S} \subseteq \mathbf{D}$ (see Subsection 2.2.4).

The functor $\mathsf{V}^{\mathbf{D}}$ preserves almost all limits, however, there is one important exception. The terminal object (that is, the limit of the empty diagram) in $\mathsf{Stone}_{\mathbf{D}}$ is given by $(\{*\}, \mathbf{v}^{\mathbf{E}})$, implying that $\mathsf{V}^{\mathbf{D}}$ does not preserve it. Therefore, contrary to a claim made in [Wal04], no further left-adjoint of $\mathsf{V}^{\mathbf{D}}$ exists.

It is obvious that both the unit $\mathsf{id}_{\mathsf{Stone}} \Rightarrow \mathsf{U} \circ \mathsf{V}^{\mathbf{D}}$ of the adjunction $\mathsf{V}^{\mathbf{D}} \dashv \mathsf{U}$ and the counit $\mathsf{U} \circ \mathsf{V}^{\mathbf{E}} \Rightarrow \mathsf{id}_{\mathsf{Stone}}$ of the adjunction $\mathsf{U} \dashv \mathsf{V}^{\mathbf{E}}$ are natural isomorphisms (since they both are the identity map on every component $X \in \mathsf{Stone}$). We hold on to this fact, as it will be interesting on the algebraic side later on (see Corollaries 2.2.14 and 2.2.19).

Proposition 2.2.1. *The category* Stone *is categorically equivalent to both*

- *a coreflective subcategory of* $\mathsf{Stone}_{\mathbf{D}}$*, witnessed by the fully faithful functor* $\mathsf{V}^{\mathbf{D}}$ *and*
- *a reflective and coreflective subcategory of* $\mathsf{Stone}_{\mathbf{D}}$*, witnessed by the fully faithful functor* $\mathsf{V}^{\mathbf{E}}$*.*

All functors which we described in this subsection can be carried through the dualities, resulting in a a corresponding chain of adjunctions between $\mathcal{A}$ and BA. For example, the dual of U is given by $\Pi\mathsf{U}\Sigma' \colon \mathcal{A} \to \mathsf{BA}$. In the next subsection we show that this functor can be understood algebraically as the Boolean skeleton. Throughout the subsections that follow, we give similar algebraic descriptions of all functors between $\mathcal{A}$ and BA which appear in Figure 2.1.

2.2.2 The Boolean skeleton functor

For MV-algebras, in particular for MV_n-algebras (recall that this corresponds to the case where $\mathbf{D} = \mathbf{Ł}_n$), the Boolean skeleton is a well-known and useful tool (see, for example, [CDM00]). An appropriate generalization of this concept to arbitrary semi-primal algebras based on a bounded lattice was given by Maruyama in [Mar12] (therein, the Boolean skeleton is referred to as the *Boolean core*; other authors also refer to it as *Boolean center*).

Due to Theorem 1.1.12 and [Mar12, Lemma 3.11] (which shows that the following really defines a Boolean algebra), the next definition is justified.

Definition 2.2.2 (Boolean skeleton)**.** Let $\mathbf{A} \in \mathcal{A}$. The *Boolean skeleton* of $\mathbf{A}$ is the following Boolean algebra

$$\mathfrak{S}(\mathbf{A}) = \langle \mathfrak{S}(A), \wedge, \vee, T_0, 0, 1 \rangle,$$

whose carrier set is

$$\mathfrak{S}(A) = \{a \in A \mid T_1(a) = a\}.$$

The lattice operations $\wedge, \vee$ and the bounds $0, 1$ are inherited from $\mathbf{A}$. The unary operations T_0 and T_1 correspond to the ones from Theorem 1.1.12 (which also shows they are term-definable in $\mathbf{D}$), interpreted in $\mathbf{A}$.

For example, for each $\mathbf{A} \in \mathcal{A}, a \in \mathbf{A}$ and $d \in \mathbf{D}$, we have $T_d(a) \in \mathfrak{S}(\mathbf{A})$. This is due to the fact that the equation $T_1(T_d(x)) \approx T_d(x)$ holds in $\mathbf{D}$, and therefore also in $\mathbf{A}$.

Remark 2.2.3. For $\mathbf{A} \in \mathcal{A}$, suppose that $A' \subseteq A$ is a subset such that $\langle A', \wedge, \vee, T_0, 0, 1\rangle$ forms a Boolean algebra. Then, for all $a' \in A'$, we have $T_1(a') = T_1(T_0(T_0(a'))) = T_0(T_0(a')) = a'$ and thus $a' \in \mathfrak{S}(A)$ (the second equation always holds since $\mathcal{A} \models T_1(T_0(x)) \approx T_0(x)$, which is easily checked in $\mathbf{D}$). Therefore, $\mathfrak{S}(A)$ is the largest such subset. ■

To extend the construction of the Boolean skeleton to a functor $\mathfrak{S}\colon \mathcal{A} \to \mathsf{BA}$, on homomorphisms $h \in \mathcal{A}(\mathbf{A}_1, \mathbf{A}_2)$ we define $\mathfrak{S}f$ to be the restriction $h|_{\mathfrak{S}(\mathbf{A}_1)}$. This is well-defined since

$$a \in \mathfrak{S}(\mathbf{A}_1) \Leftrightarrow T_1(a) = a \Rightarrow T_1\big(h(a)\big) = h\big(T_1(a)\big) = h(a) \Leftrightarrow h(a) \in \mathfrak{S}(\mathbf{A}_2),$$

where we used that every homomorphism h preserves terms, in particular T_1. We call the resulting functor the *Boolean skeleton functor*. The following is, arguably, the most important property of this functor.

Proposition 2.2.4. *For all $\mathbf{A} \in \mathcal{A}$, there is a homeomorphism between $\mathsf{U}\Sigma'(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{D})$ and $\Sigma\mathfrak{S}(\mathbf{A}) = \mathsf{BA}(\mathfrak{S}(\mathbf{A}), \mathbf{2})$ given by $u \mapsto u|_{\mathfrak{S}(\mathbf{A})}$.*

Proof. First we show that the map is a bijection. For injectivity, suppose that u_1 and u_2 satisfy $u_1|_{\mathfrak{S}(\mathbf{A})} = u_2|_{\mathfrak{S}(\mathbf{A})}$. Take an arbitrary element $a \in \mathbf{A}$ and say $u_1(a) = d \in \mathbf{D}$. Using the fact that $T_d(a) \in \mathfrak{S}(\mathbf{A})$, we get

$$1 = T_d\big(u_1(a)\big) = u_1\big(T_d(a)\big) = u_2\big(T_d(a)\big) = T_d\big(u_2(a)\big),$$

which implies $u_2(a) = d$ and, since a was arbitrary, that $u_1 = u_2$.

For surjectivity, let $u \in \mathsf{BA}(\mathfrak{S}(\mathbf{A}), \mathbf{2})$ be arbitrary. Due to [Mar12, Lemma 3.12], we get a well-defined homomorphism $\bar{u}\colon \mathbf{A} \to \mathbf{D}$ by setting

$$\bar{u}(a) = d \Leftrightarrow u\big(T_d(a)\big) = 1.$$

Since for $a \in \mathfrak{S}(\mathbf{A})$ we have

$$u\big(T_1(a)\big) = 1 \Leftrightarrow u(a) = 1 \text{ and}$$
$$u\big(T_0(a)\big) = 1 \Leftrightarrow T_0\big(u(a)\big) = 1 \Leftrightarrow u(a) = 0,$$

we conclude that $\bar{u}|_{\mathfrak{S}(\mathbf{A})} = u$.

We now have a bijection between two Stone spaces, so to show that this is a homeomorphism it suffices to show this bijection is continuous. But this is easy to see, since the preimage of an open subbase element $[a : i] \subseteq \mathsf{BA}(\mathfrak{S}(\mathbf{A}), \mathbf{2})$ is the corresponding open subbase element $[a : i] \subseteq \mathcal{A}(\mathbf{A}, \mathbf{D})$ (recall that the subbases were described in the paragraph after Remark 2.1.3). This finishes the proof. □

The fact that the Boolean skeleton functor $\mathfrak{S}\colon \mathcal{A} \to \mathsf{BA}$ is (up to natural isomorphism) the dual of the forgetful functor $\mathsf{U}\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$ is an easy consequence of this proposition.

Corollary 2.2.5 ($\mathfrak{S}$ is dual of U)**.** *There is a natural isomorphism between the functor $\mathfrak{S}$ and the dual $\Pi\mathsf{U}\Sigma'$ of the forgetful functor U.*

Proof. By Proposition 2.2.4, for every $\mathbf{A} \in \mathcal{A}$, setting

$$\begin{aligned} \phi_{\mathbf{A}}\colon \mathsf{U}\Sigma'(\mathbf{A}) &\to \Sigma\mathfrak{S}(\mathbf{A}) \\ u &\mapsto u|_{\mathfrak{S}(\mathbf{A})} \end{aligned}$$

defines an isomorphism. We show that it even defines a natural isomorphism $\phi\colon \mathsf{U}\Sigma' \Rightarrow \Sigma\mathfrak{S}$. For this, we need to show that, for any homomorphism $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ the diagram

$$\begin{tikzcd}
\mathsf{U}\Sigma'(\mathbf{A}_2) \arrow[r, "\phi_{\mathbf{A}_2}"] \arrow[d, "\mathsf{U}\Sigma' h"'] & \Sigma\mathfrak{S}(\mathbf{A}_2) \arrow[d, "\Sigma\mathfrak{S}h"] \\
\mathsf{U}\Sigma'(\mathbf{A}_1) \arrow[r, "\phi_{\mathbf{A}_1}"'] & \Sigma\mathfrak{S}(\mathbf{A}_1)
\end{tikzcd}$$

commutes (recall that Σ and Σ' are contravariant). For $u \in \mathsf{U}\Sigma'(\mathbf{A}_2) = \mathcal{A}(\mathbf{A}_2, \mathbf{D})$ we have

$$(\phi_{\mathbf{A}_1} \circ \mathsf{U}\Sigma' h)(u) = \phi_{\mathbf{A}_1}(u \circ h) = (u \circ h)|_{\mathfrak{S}(\mathbf{A}_1)}$$

on the one hand and

$$(\Sigma\mathfrak{S}h \circ \phi_{\mathbf{A}_2})(u) = \Sigma\mathfrak{S}h(u|_{\mathfrak{S}(\mathbf{A}_2)}) = u|_{\mathfrak{S}(\mathbf{A}_2)} \circ h|_{\mathfrak{S}(\mathbf{A}_1)}$$

on the other hand, and clearly these two coincide. Therefore, we showed that ϕ is a natural isomorphism.

Applying Π to ϕ and using the fact that $\Pi\Sigma$ is naturally isomorphic to $\mathsf{id}_{\mathsf{BA}}$, we find the desired natural isomorphism $\Pi\phi\colon \mathfrak{S} \Rightarrow \Pi\mathsf{U}\Sigma'$ as well. $\square$

One immediate consequence of the above is that the Boolean skeleton functor $\mathfrak{S}\colon \mathcal{A} \to \mathsf{BA}$ is faithful (since it is dual to the forgetful functor $\mathsf{U}\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$, which is easily seen to be faithful).

Another immediate consequence is that $\mathfrak{S}$ has both adjoints and, therefore, preserves all limits and colimits.

In the next subsection, we give an explicit algebraic description the right-adjoint of the Boolean skeleton functor. As it turns out, this corresponds to another well-known construction in universal algebra.

2.2.3 The Boolean power functor

In this subsection, we give an algebraic description of a functor $\mathfrak{P}\colon \mathsf{BA} \to \mathcal{A}$ which is naturally isomorphic to the dual $\Pi'\mathsf{V}^{\mathbf{D}}\Sigma$ of the functor $\mathsf{V}^{\mathbf{D}}$. Since we already know that $\mathsf{V}^{\mathbf{D}}$ is left-adjoint to the forgetful functor U, and that the Boolean skeleton functor $\mathfrak{S}$ is naturally isomorphic to the dual of U, this immediately yields that $\mathfrak{P}$ is right-adjoint to $\mathfrak{S}$ (also see Figure 2.1).

The functor $\mathfrak{P}$ turns out to be an instance of the well-known construction of the *Boolean power*, which was already introduced for arbitrary finite algebras in Foster's original paper which also introduced primal algebras [Fos53a] (therein called *Boolean extension*). Boolean powers are special instances of Boolean products (see, *e.g.*, [BS81, Chapter IV]), but for our purposes it is more convenient to work with the following equivalent definition found, for example, in [Bur75].

Definition 2.2.6 (Boolean power)**.** Given a Boolean algebra $\mathbf{B} \in \mathsf{BA}$ and a finite algebra $\mathbf{M}$, the *Boolean power* $\mathbf{M}[\mathbf{B}]$ is defined on the carrier set

$$M[B] \subseteq B^M$$

consisting of all maps $\xi\colon M \to B$ which satisfy the two conditions

- If ℓ and ℓ' are distinct elements of M, then $\xi(\ell) \wedge \xi(\ell') = 0$,
- $\bigvee\{\xi(\ell) \mid \ell \in M\} = 1$.

If $o^{\mathbf{M}}\colon M^k \to M$ is a k-ary operation of $\mathbf{M}$, we define a corresponding operation $o^{\mathbf{M}[\mathbf{B}]}\colon M[B] \to M[B]$ by

$$o^{\mathbf{M}[\mathbf{B}]}(\xi_1, \dots, \xi_k)(\ell) = \bigvee_{o^{\mathbf{M}}(\ell_1,\dots,\ell_k)=\ell} \big(\xi_1(\ell_1) \wedge \cdots \wedge \xi_k(\ell_k)\big).$$

The resulting algebra $\mathbf{M}[\mathbf{B}] = \langle M[B], \{o^{\mathbf{M}[\mathbf{B}]} \mid o \text{ in the signature of } \mathbf{M}\}\rangle$ is a member of the variety $\mathbb{HSP}(\mathbf{M})$ generated by $\mathbf{M}$ (since it satisfies the same equations as $\mathbf{M}$).

Oftentimes (*e.g.*, in [BS81]) the Boolean power $\mathbf{M}[\mathbf{B}]$ is equivalently defined as the collection of all continuous maps $\Sigma(\mathbf{B}) \to M$ from the Stone-dual of $\mathbf{B}$ to M (considered as a discrete space) with component-wise operations. However, one advantage of the formulation in Definition 2.2.6 is that it is more constructive, for example, it can still be carried out in choice-free settings (like that of [BH20]).

There is a straightforward way to extend the construction of the Boolean power to a functor as follows.

Definition 2.2.7 (Boolean power functor)**.** For a finite algebra $\mathbf{M}$, we define the *Boolean power functor* $\mathfrak{P}_{\mathbf{M}} \colon \mathsf{BA} \to \mathbb{HSP}(\mathbf{M})$ as follows. On objects $\mathbf{B} \in \mathsf{BA}$ we define

$$\mathfrak{P}_{\mathbf{M}}(\mathbf{B}) = \mathbf{M}[\mathbf{B}]$$

and for a Boolean homomorphism $h \colon \mathbf{B}_1 \to \mathbf{B}_2$, the homomorphism

$$\mathfrak{P}_{\mathbf{M}} h \colon \mathbf{M}[\mathbf{B}_1] \to \mathbf{M}[\mathbf{B}_2]$$

is defined via composition $\xi \mapsto h \circ \xi$ (note that this is a homomorphism because operations in $\mathbf{M}[\mathbf{B}_1]$ are defined by Boolean expressions, which commute with h).

We will often (in particular, always in this subsection) use the shorthand notation $\mathfrak{P}$ for $\mathfrak{P}_{\mathbf{D}} \colon \mathsf{BA} \to \mathcal{A}$. In the remainder of this subsection we aim to show that $\mathfrak{P}$ is indeed right-adjoint to the Boolean skeleton functor $\mathfrak{S}$. To this end, we shall make use of the following well-known properties of the Boolean power.

Lemma 2.2.8 ([Bur75, Proposition 2.1])**.** *The functor* $\mathfrak{P}_{\mathbf{M}}$ *has the following properties.*

(1) $\mathfrak{P}_{\mathbf{M}}(\mathbf{2}) \cong \mathbf{M}$.

(2) $\mathfrak{P}_{\mathbf{M}}$ *preserves products.*

In particular, $\mathfrak{P}_{\mathbf{M}}(\mathbf{2}^{\kappa}) \cong \mathbf{M}^{\kappa}$ *holds for all index sets* κ.

Next, we describe the interplay between the functors $\mathfrak{S}$ and $\mathfrak{P}$. Here, the unary terms T_d from Proposition 1.1.12 play an important role once again.

Proposition 2.2.9. *For every* $\mathbf{A} \in \mathcal{A}$*, there is an embedding* $\mathcal{T}_{(\cdot)}\colon \mathbf{A} \hookrightarrow \mathfrak{P}(\mathfrak{S}(\mathbf{A}))$ *given by* $a \mapsto \mathcal{T}_a$ *where*

$$\mathcal{T}_a(d) = T_d(a).$$

The restriction to $\mathfrak{S}(\mathbf{A})$ *yields an isomorphism* $\mathfrak{S}(\mathbf{A}) \cong \mathfrak{S}\big(\mathfrak{P}(\mathfrak{S}(\mathbf{A}))\big)$.

Proof. The map is well-defined, that is, $\mathcal{T}_a$ is in $\mathfrak{P}(\mathfrak{S}(\mathbf{A}))$, since the equations $T_d(x) \wedge T_{d'}(x) \approx 0$ (for distinct d, d') and $\bigvee\{T_d(x) \mid d \in D\} \approx 1$ hold in $\mathbf{D}$.

We now fix an embedding $\mathbf{A} \hookrightarrow \mathbf{D}^I$. It is easy to see that $\mathcal{T}_{(\cdot)}$ is injective since, for distinct $a, a' \in \mathbf{A}$, there is some component $i \in I$ with $a(i) = d \neq a'(i)$, thus $\mathcal{T}_a(d) \neq \mathcal{T}_{a'}(d)$. To conclude that $\mathcal{T}_{(\cdot)}$ is an embedding we need to show that it is a homomorphism, that is, we want to show that for any k-ary operation $o\colon D^k \to D$ of $\mathbf{D}$ we have

$$\mathcal{T}_{o^{\mathbf{A}}(a_1,\ldots,a_k)} = o^{\mathbf{D}[\mathfrak{S}(\mathbf{A})]}(\mathcal{T}_{a_1}, \ldots, \mathcal{T}_{a_k}).$$

By definition, the i-th component of the left-hand side is given by

$$\mathcal{T}_{o^{A}(a_1,\ldots,a_k)}(d)(i) = T_d\big(o^{\mathbf{D}}(a_1(i), \ldots, a_k(i))\big) = \begin{cases} 1 & \text{if } o^{\mathbf{D}}(a_1(i), ..., a_k(i)) = d \\ 0 & \text{otherwise.} \end{cases}$$

The right-hand side is given by

$$o^{\mathbf{D}[\mathfrak{S}(\mathbf{A})]}(\mathcal{T}_{a_1}, \ldots \mathcal{T}_{a_k})(d) = \bigvee_{o^{\mathbf{D}}(d_1,\ldots,d_k)=d} \big(\mathcal{T}_{a_1}(d_1) \wedge \cdots \wedge \mathcal{T}_{a_k}(d_k)\big).$$

In its i-th component, this again corresponds to

$$\bigvee_{o^{\mathbf{D}}(d_1,\ldots,d_k)=d} \big(T_{d_1}(a_1(i)) \wedge \cdots \wedge T_{d_k}(a_k(i))\big) = \begin{cases} 1 & \text{if } o^{\mathbf{D}}\big(a_1(i), ..., a_k(i)\big) = d \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we showed that $\mathcal{T}_{(\cdot)}$ is an embedding, which concludes the proof of the first statement.

For the second statement note that, since $\mathfrak{S}$ preserves injectivity of homomorphisms, it suffices to show that the restriction of $\mathcal{T}_{(\cdot)}$ to $\mathfrak{S}(\mathbf{A})$ is a surjection onto $\mathfrak{S}\big(\mathfrak{P}(\mathfrak{S}(\mathbf{A}))\big)$. So consider an element $\xi \in \mathfrak{S}\big(\mathfrak{P}(\mathfrak{S}(\mathbf{A}))\big)$, that is $\xi \in \mathfrak{P}(\mathfrak{S}(\mathbf{A}))$ and $T_1^{\mathbf{D}[\mathfrak{S}(\mathbf{A})]}(\xi) = \xi$. The latter, by definition, means

$$\begin{aligned} &T_1^{\mathbf{D}[\mathfrak{S}(\mathbf{A})]}(\xi)(1) = \xi(1), \\ &T_1^{\mathbf{D}[\mathfrak{S}(\mathbf{A})]}(\xi)(0) = \bigvee\{\xi(d) \mid d \in D, d \neq 1\} = \xi(0) \text{ and} \\ &T_1^{\mathbf{D}[\mathfrak{S}(\mathbf{A})]}(\xi)(d) = \bigvee \varnothing = 0 = \xi(d) \text{ for all } d \in D \backslash \{0, 1\}. \end{aligned}$$

We claim that $\xi = \mathcal{T}_{\xi(1)}$. Indeed, we know that $\xi(1) \in \mathfrak{S}(\mathbf{A})$, so $\xi(1) = T_1(\xi(1))$. Furthermore, in the component $i \in I$, we have $\xi(0)(i) = 1$ if and only if $\xi(1)(i) = 0$, so $T_0(\xi(1)) = T_1(\xi(0)) = \xi(0)$, since $\xi(0) \in \mathfrak{S}(\mathbf{A})$. Finally, for $d \notin \{0, 1\}$, we have $T_d(\xi(1)) = 0$, since for all $i \in I$ we have $\xi(1)(i) \in \{0, 1\}$. This concludes the proof. □

Since $\mathfrak{S}$ is dual to the essentially surjective functor U, we know that for every $\mathbf{B} \in \mathsf{BA}$ there exists some $\mathbf{A} \in \mathcal{A}$ such that there is an isomorphism $\mathbf{B} \cong \mathfrak{S}(\mathbf{A})$. Therefore, the following is a direct consequence of the second part of Proposition 2.2.9.

Corollary 2.2.10. *Any Boolean algebra* $\mathbf{B} \in \mathsf{BA}$ *is isomorphic to* $\mathfrak{S}(\mathfrak{P}(\mathbf{B}))$.

Another immediate consequence of Proposition 2.2.9 is the following.

Corollary 2.2.11. *For every Boolean algebra* $\mathbf{B} \in \mathsf{BA}$*, the algebra* $\mathfrak{P}(\mathbf{B})$ *is the largest algebra in* $\mathcal{A}$ *which has* $\mathbf{B}$ *as Boolean skeleton. That is, for every algebra* $\mathbf{A} \in \mathcal{A}$ *with* $\mathfrak{S}(\mathbf{A}) \cong \mathbf{B}$*, there exists an embedding* $\mathbf{A} \hookrightarrow \mathfrak{P}(\mathbf{B})$.

We now have everything at hand to prove the main theorem of this subsection.

Theorem 2.2.12 (Boolean skeleton/power adjunction)**.** *The Boolean power functor* $\mathfrak{P}\colon \mathsf{BA} \to \mathcal{A}$ *is naturally isomorphic to the dual of* $\mathsf{V}^{\mathbf{D}}$ *and, therefore,* $\mathfrak{S} \dashv \mathfrak{P}$.

Proof. First we prove the statement on the finite level. In other words, we want to show that

$$\Sigma'\mathfrak{P}(\mathbf{B}) \cong \mathsf{V}^{\mathbf{D}}\Sigma(\mathbf{B})$$

holds (in $\mathsf{Stone}_{\mathbf{D}}$) for every finite Boolean algebra $\mathbf{B} \in \mathsf{BA}^{\omega}$. More explicitly, after spelling out the definition of the functors involved, we want to show

$$\big(\mathcal{A}(\mathfrak{P}(\mathbf{B}), \mathbf{D}), \mathbf{im}\big) \cong \big(\mathsf{BA}(\mathbf{B}, \mathbf{2}), \mathbf{v}^{\mathbf{D}}\big) \tag{2.1}$$

for every finite Boolean algebra $\mathbf{B}$. First, since $\mathbf{B}$ is finite, there is some positive integer k such that $\mathbf{B} \cong \mathbf{2}^k$. We combine the following isomorphisms in Stone. Due to Lemma 2.1.7, we know

$$\mathcal{A}\big(\mathfrak{P}(\mathbf{B}), \mathbf{D}\big) \cong \mathsf{BA}\big(\mathfrak{S}(\mathfrak{P}(\mathbf{B})), \mathbf{2}\big),$$

and due to Corollary 2.2.10, we know

$$\mathfrak{S}\big(\mathfrak{P}(\mathbf{B})\big) \cong \mathbf{B}.$$

Putting these together, we get

$$\mathcal{A}(\mathfrak{P}(\mathbf{B}), \mathbf{D}) \cong \mathsf{BA}(\mathbf{B}, \mathbf{2}).$$

In fact, this even yields an isomorphism in $\mathsf{Stone}_{\mathbf{D}}$, as desired in Equation (2.1), because

$$\big(\mathcal{A}(\mathfrak{P}(\mathbf{B}), \mathbf{D}), \mathbf{im}\big) \cong \big(\mathcal{A}(\mathbf{D}^k, \mathbf{D}), \mathbf{im}\big) \cong \big(\mathcal{A}(\mathbf{D}^k, \mathbf{D}), \mathbf{v}^{\mathbf{D}}\big),$$

where the last equation holds due to Lemma 2.1.7.

So we know that the restriction of $\mathfrak{P}$ to the category of finite Boolean algebras $\mathfrak{P}^\omega \colon \mathsf{BA}^\omega \to \mathcal{A}$ is dual to the restriction $(\mathsf{V}^{\mathbf{D}})^\omega$ of $\mathsf{V}^{\mathbf{D}}$ to the category $\mathsf{Set}^\omega_{\mathbf{D}}$. There is a unique (up to natural isomorphism) finitary (*i.e.*, filtered colimit preserving) extension of $\mathfrak{P}^\omega$ to $\mathsf{Ind}(\mathsf{BA}^\omega) \simeq \mathsf{BA}$, and this extension is naturally isomorphic to the dual of $\mathsf{V}^{\mathbf{D}}$ (since $\mathsf{V}^{\mathbf{D}}$ preserves all limits except for the terminal object, it is the unique cofinitary extension of $(\mathsf{V}^{\mathbf{D}})^\omega$). To show that $\mathfrak{P}$ coincides with this extension, it suffices to show that $\mathfrak{P}$ is finitary as well. Since $\mathfrak{P}$ preserves monomorphisms (it is easy to see by definition that if $h \in \mathsf{BA}(\mathbf{B}_1, \mathbf{B}_2)$ is injective, then $\mathfrak{P}h$ is injective as well), we can apply [AMSW19, Theorem 3.4], which states that $\mathfrak{P}$ is finitary if and only if the following holds.

Fact. *For every Boolean algebra* $\mathbf{B} \in \mathsf{BA}$ *and every finite subalgebra* $\mathbf{A} \hookrightarrow \mathfrak{P}(\mathbf{B})$*, the inclusion factors through the image of the inclusion of some finite subalgebra* $\mathbf{B}' \hookrightarrow \mathbf{B}$ *under* $\mathfrak{P}$*.*

To see this, write $\mathbf{A} \cong \prod_{i \leq n} \mathbf{S}_i$ as product of finite subalgebras of $\mathbf{D}$. Then, by Corollary 2.2.10, we know that $\mathfrak{S}(\mathbf{A}) \cong \mathbf{2}^n$ embeds into $\mathbf{B}$. Now, by Lemma 2.2.8, we have $\mathfrak{P}(\mathbf{2}^n) \cong \mathbf{D}^n$ and the natural inclusion $\prod_{i \leq n} \mathbf{S}_i \hookrightarrow \mathbf{D}^n$ yields our factorization

$$\begin{tikzcd}
\mathbf{A} \arrow[r, hook] \arrow[dr, hook] & \mathfrak{P}(\mathbf{B}) \\
& \mathfrak{P}(\mathbf{2}^n) \arrow[u, hook]
\end{tikzcd}$$

as desired. This concludes the proof. □

In particular, if $\mathbf{D}$ is primal we have $\mathsf{V}^{\mathbf{D}} = \mathsf{V}^{\mathbf{E}}$, which means that U and $\mathsf{V}^{\mathbf{D}}$ actually form an equivalence of categories. On the algebraic side, we thus get an explicit categorical equivalence witnessing Hu's theorem (recall Theorem 1.1.4).

Corollary 2.2.13 (Hu's Theorem, explicitly)**.** *If* $\mathbf{D}$ *is primal, then* $\mathfrak{S}$ *and* $\mathfrak{P}$ *establish a categorical equivalence between* $\mathcal{A}$ *and* BA*.*

We also immediately get the algebraic analogue of the first item of Proposition 2.2.1.

Corollary 2.2.14. *The functor* $\mathfrak{P}$ *is fully faithful and identifies* BA *with a reflective subcategory of* $\mathcal{A}$.

By now we found detailed descriptions of most of the functors appearing in Figure 2.1. However, we are still missing an algebraic understanding of the adjunction $\mathfrak{S}_{\mathbf{E}} \dashv \mathfrak{P}_{\mathbf{E}}$. This gap is filled in the next subsection. As we will see, it is an instance of the more general *subalgebra adjunctions* and, as the notation already suggests, closely related to $\mathfrak{S} \dashv \mathfrak{P}$.

2.2.4 The subalgebra adjunctions

In this subsection, we describe the subalgebra adjunctions which generalize both the adjunction $\mathsf{V}^{\mathbf{D}} \dashv \mathsf{U}$ and $\mathsf{V}^{\mathbf{E}} \dashv \mathsf{C}^{\mathbf{E}}$ from Subsection 2.2.1.

We begin with the description on the topological side, which is fairly easy. For every subalgebra $\mathbf{S} \subseteq \mathbf{D}$, there is an adjunction

$$\begin{tikzcd} \mathsf{Stone} \arrow[r, bend left, "\mathsf{V}^{\mathbf{S}}"] \arrow[r, phantom, "\perp"] & \mathsf{Stone}_{\mathbf{D}} \arrow[l, bend left, "\mathsf{C}^{\mathbf{S}}"] \end{tikzcd} \tag{2.2}$$

given by the functors defined as follows .

The functor $\mathsf{V}^{\mathbf{S}} \colon \mathsf{Stone} \to \mathsf{Stone}_{\mathbf{D}}$ is given on objects by

$$\mathsf{V}^{\mathbf{S}}(X) = (X, \mathbf{v}^{\mathbf{S}}) \text{ where } \forall x \in X : \mathbf{v}^{\mathbf{S}}(x) = \mathbf{S}.$$

To every continuous map $f \colon X_1 \to X_2$ between Stone spaces, the functor $\mathsf{V}^{\mathbf{S}}$ assigns itself (as a well-defined morphism $(X_1, \mathbf{v}^{\mathbf{S}}) \to (X_2, \mathbf{v}^{\mathbf{S}})$ in $\mathsf{Stone}_{\mathbf{D}}$).

The functor $\mathbf{C}^{\mathbf{S}} \colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$ is given on objects by

$$\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) = \{x \in X \mid \mathbf{v}(x) \leq \mathbf{S}\},$$

which is well-defined since $\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) = \mathbf{v}^{-1}(\mathbf{S}{\downarrow})$ is a closed subspace of X (see Definition 2.1.2). On morphisms, the functor $\mathsf{C}^{\mathbf{S}}$ acts via restriction, that is, given a morphism $f \colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$, we set $\mathsf{C}^{\mathbf{S}} f = f|_{\mathsf{C}^{\mathbf{S}}(X_1)} \colon \mathsf{C}^{\mathbf{S}}(X_1) \to \mathsf{C}^{\mathbf{S}}(X_2)$. This is well-defined since

$$\begin{aligned} x \in \mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1) &\Leftrightarrow \mathbf{v}_1(x) \leq \mathbf{S} \\ &\Leftrightarrow \mathbf{v}_2\big(f(x)\big) \leq \mathbf{v}_1(x) \leq \mathbf{S} \\ &\Leftrightarrow f(x) \in \mathsf{C}^{\mathbf{S}}(X_2, \mathbf{v}_2). \end{aligned}$$

Comparing this with Subsection 2.2.1, one can easily verify that $V^{\mathbf{S}} \dashv \mathsf{C}^{\mathbf{S}}$. Indeed, the adjunction $V^{\mathbf{S}} \dashv \mathsf{C}^{\mathbf{S}}$ generalizes the following adjunctions in Figure 2.1:

- $\mathsf{V}^{\mathbf{D}} \dashv \mathsf{U}$ arises in the case where $\mathbf{S} = \mathbf{D}$ is the largest subalgebra,
- $\mathsf{V}^{\mathbf{E}} \dashv \mathsf{C}^{\mathbf{E}}$ arises in the case where $\mathbf{S} = \mathbf{E}$ is the smallest subalgebra.

What is special about these two extremal cases is the additional adjunction $\mathsf{U} \dashv \mathsf{V}^{\mathbf{D}}$, which 'glues' the two adjunctions into the chain described in Subsection 2.2.1.

In order to better understand the subalgebra adjunction corresponding to the subalgebra $\mathbf{S} \subseteq \mathbf{D}$, we dissect it into two parts as follows.

$$\begin{tikzcd}
\mathsf{Stone} \arrow[r, bend left, "\mathsf{V}^{\mathbf{S}}"] \arrow[r, phantom, "\perp"] & \mathsf{Stone}_{\mathbf{S}} \arrow[l, bend left, "\mathsf{U}"] \arrow[r, bend left, "\iota^{\mathbf{S}}"] \arrow[r, phantom, "\perp"] & \mathsf{Stone}_{\mathbf{D}} \arrow[l, bend left, "(\mathsf{C}^{\mathbf{S}}, -)"]
\end{tikzcd}$$

Here, $\iota^{\mathbf{S}}$ is the natural inclusion and the functor $(\mathsf{C}^{\mathbf{S}}, -)$ is defined by

$$(X, \mathbf{v}) \mapsto \big(\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}), \mathbf{v}|_{\mathsf{C}^{\mathbf{S}}(X)}\big)$$

on objects and, exactly like $\mathsf{C}^{\mathbf{S}}$, acts via restriction on morphisms. Slightly abusing notation, we re-use $\mathsf{U}\colon \mathsf{Stone}_{\mathbf{S}} \to \mathsf{Stone}$ to denote the forgetful functor. It is easy to see that this really is a decomposition of the adjunction from Equation (2.2), that is,

$$\mathsf{V}^{\mathbf{S}} = \iota^{\mathbf{S}} \circ \mathsf{V}^{\mathbf{S}} \text{ and } \mathsf{C}^{\mathbf{S}} = \mathsf{U} \circ (\mathsf{C}^{\mathbf{S}}, -).$$

As before, we want to carry everything over to the algebraic side, where the dissection takes place through the subvariety

$$\mathcal{A}_{\mathbf{S}} := \mathbb{HSP}(\mathbf{S})$$

of $\mathcal{A}$ generated by $\mathbf{S}$. For an overview, we illustrate the entire situation in Figure 2.2. Recall that $\mathbf{S} \leq \mathbf{D}$ is itself semi-primal (see Subsection 1.1.2), so the semi-primal topological duality given by the functors we denote $\Sigma'_{\mathbf{S}}$ and $\Pi'_{\mathbf{S}}$ as well as the adjunction $\mathfrak{S} \dashv \mathfrak{P}_{\mathbf{S}}$ make sense in this context. Again, $\iota_{\mathbf{S}}$ denotes the natural inclusion, this time on the algebraic side. Although it may seem obvious, it is not immediate that $\iota_{\mathbf{S}}$ really is the dual of $\iota^{\mathbf{S}}$. To prove it, we make use of the unary term

$$\chi_S(x) := \bigvee_{s \in S} T_s(x),$$

which will play an important role for the remainder of the subsection. On $\mathbf{D}$, this simply corresponds to the characteristic function of $S \subseteq D$. It is, furthermore, characteristic for the subvariety $\mathcal{A}_{\mathbf{S}}$ in the following sense.

$$\begin{tikzcd}
\mathsf{Stone}_{\mathbf{D}} \arrow[r, bend left, "\Pi'"] \arrow[d, bend left, "(\mathsf{C}^{\mathbf{S}}{,}-)"] \arrow[d, phantom, "\dashv"] & \mathcal{A} \arrow[l, bend left, "\Sigma'"] \arrow[d, bend left, "\mathsf{Q}_{\mathbf{S}}"] \arrow[d, phantom, "\vdash"] \\
\mathsf{Stone}_{\mathbf{S}} \arrow[r, bend left, "\Pi'_{\mathbf{S}}"] \arrow[u, bend left, "\iota^{\mathbf{S}}"] \arrow[d, bend left, "\mathsf{U}"] \arrow[d, phantom, "\dashv"] & \mathcal{A}_{\mathbf{S}} \arrow[l, bend left, "\Sigma'_{\mathbf{S}}"] \arrow[u, bend left, "\iota_{\mathbf{S}}"] \arrow[d, bend left, "\mathfrak{S}"] \arrow[d, phantom, "\vdash"] \\
\mathsf{Stone} \arrow[r, bend left, "\Pi"] \arrow[u, bend left, "\mathsf{V}^{\mathbf{S}}"] & \mathsf{BA} \arrow[l, bend left, "\Sigma"] \arrow[u, bend left, "\mathfrak{P}_{\mathbf{S}}"]
\end{tikzcd}$$

Figure 2.2: Dissecting the subalgebra adjunction of $\mathbf{S} \subseteq \mathbf{D}$.

Lemma 2.2.15. *An algebra in* $\mathcal{A}$ *is a member of* $\mathcal{A}_{\mathbf{S}}$ *if and only if it satisfies the equation* $\chi_S(x) \approx 1$.

Proof. Clearly, every member of $\mathcal{A}_{\mathbf{S}}$ satisfies the equation since $\mathbf{S}$ satisfies it. For the other direction, let $\mathbf{A} \in \mathcal{A}$ satisfy $\chi_S(a) = 1$ for all $a \in A$. We know that $\mathbf{A}$ can be embedded into some $\mathbf{D}^I$, and for each $a \in \mathbf{A}$ and $i \in I$, we have $\chi_{\mathbf{S}}(\mathsf{pr}_i(a)) = 1$, which implies that $\mathsf{pr}_i(a) \in \mathbf{S}$. Therefore, $\mathbf{A}$ can even be embedded into $\mathbf{S}^I$. $\square$

Now take $\mathbf{A} \in \mathcal{A}_{\mathbf{S}}$ and let $u \in \mathcal{A}(\iota_{\mathbf{S}}(\mathbf{A}), \mathbf{D})$ be a homomorphism. Since u preserves equations, for every $a \in A$ we get

$$\chi_S(a) = 1 \Rightarrow \chi_S\big(u(a)\big) = 1,$$

which implies $u \in \mathcal{A}(\mathbf{A}, \mathbf{S})$. So we showed $\mathcal{A}(\mathbf{A}, \mathbf{D}) = \mathcal{A}_{\mathbf{S}}(\mathbf{A}, \mathbf{S})$ for $\mathbf{A} \in \mathcal{A}_{\mathbf{S}}$, which immediately implies the following.

Corollary 2.2.16. *The inclusion functor* $\iota_{\mathbf{S}} \colon \mathcal{A}_{\mathbf{S}} \to \mathcal{A}$ *is (up to natural isomorphism) the dual of the inclusion functor* $\iota^{\mathbf{S}} \colon \mathsf{Stone}_{\mathbf{S}} \to \mathsf{Stone}_{\mathbf{D}}$.

To completely understand Figure 2.2, we only need to describe the functor $\mathsf{Q}_{\mathbf{S}} \colon \mathcal{A} \to \mathcal{A}_{\mathbf{S}}$. In the following, let $\eta \colon \mathsf{id}_{\mathcal{A}} \Rightarrow \iota_{\mathbf{S}} \circ \mathsf{Q}_{\mathbf{S}}$ denote the unit of the adjunction $\mathsf{Q}_{\mathbf{S}} \dashv \iota_{\mathbf{S}}$. By duality, for any $\mathbf{A} \in \mathcal{A}$ the algebra $\mathsf{Q}_{\mathbf{S}}(\mathbf{A})$ is universal for $\mathcal{A}_{\mathbf{S}}$ in the following sense.

Fact. *For every* $\mathbf{B} \in \mathcal{A}_{\mathbf{S}}$ *and every homomorphism* $h \colon \mathbf{A} \to \mathbf{B}$*, there is a* *unique* $\hat{h}\colon \mathsf{Q}_{\mathbf{S}}(\mathbf{A}) \to \mathbf{B}$ *such that* $\hat{h} \circ \eta_{\mathbf{A}} = h$.

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "\alpha_{\mathbf{A}}"] \arrow[dr, "h"'] & \mathsf{Q}_{\mathbf{S}}(\mathbf{A}) \arrow[d, dashed, "\exists \hat{h}"] \\
& \mathbf{B}
\end{tikzcd}$$

Therefore, the functor $\mathsf{Q}_{\mathbf{S}}$ may be understood as a quotient (in fact, as the largest quotient contained in $\mathcal{A}_{\mathbf{S}}$). There is a well-known connection between quotients and equations which goes back to Banaschewski and Herrlich [BH76] (also see [ACMU21, Section 3] or [MU19, Remark 3.4]). Not surprisingly, the equation corresponding to the quotient $\mathsf{Q}_{\mathbf{S}}$ is given by $\chi_S(x) \approx 1$, which is an easy consequence of the above discussion together with Lemma 2.2.15. Thus, we can summarize the results of this subsection as follows.

Theorem 2.2.17 (Subalgebra adjunctions, algebraically)**.** *For every subalgebra* $\mathbf{S} \subseteq \mathbf{D}$, *there is an adjunction*

$$\begin{tikzcd}
\mathsf{BA} \arrow[r, bend left, "\mathfrak{P}_{\mathbf{S}}"] \arrow[r, phantom, "\top"] & \mathcal{A} \arrow[l, bend left, "\mathfrak{S}_{\mathbf{S}}"]
\end{tikzcd}$$

which can be dissected as

$$\begin{tikzcd}
\mathsf{BA} \arrow[r, bend left, "\mathfrak{P}_{\mathbf{S}}"] \arrow[r, phantom, "\top"] & \mathcal{A}_{\mathbf{S}} \arrow[l, bend left, "\mathfrak{S}"] \arrow[r, bend left, "\iota_{\mathbf{S}}"] \arrow[r, phantom, "\top"] & \mathcal{A} \arrow[l, bend left, "\mathsf{Q}_{\mathbf{S}}"]
\end{tikzcd}$$

where $\iota_{\mathbf{S}}$ *is the natural inclusion functor of the subvariety* $\mathbb{HSP}(\mathbf{S}) \hookrightarrow \mathcal{A}$ *and* $\mathsf{Q}_{\mathbf{S}}$ *is the quotient functor corresponding to the equation* $\chi_S(x) \approx 1$.

In particular, in the case where $\mathbf{S} = \mathbf{E}$ is the smallest subalgebra of $\mathbf{D}$, we recover the adjunction between $\mathfrak{P}_{\mathbf{E}}$ and $\mathfrak{S}_{\mathbf{E}}$ from Figure 2.1.

Corollary 2.2.18. *The functor* $\mathfrak{P}_{\mathbf{E}}\colon \mathsf{BA} \to \mathcal{A}$ *is, up to categorical equivalence, an inclusion. The functor* $\mathfrak{S}_{\mathbf{E}}\colon \mathcal{A} \to \mathsf{BA}$ *is, up to categorical equivalence, the quotient by the equation*

$$\chi_E(x) \approx 1,$$

where $\mathbf{E} = \langle 0, 1 \rangle$ *is the smallest subalgebra of* $\mathbf{D}$.

Proof. Being the smallest subalgebra of a semi-primal algebra, $\mathbf{E}$ is primal. Therefore, by Corollary 2.2.13, the adjunction $\mathfrak{S} \dashv \mathfrak{P}_{\mathbf{E}}$ is an equivalence of categories. The statement now follows from Theorem 2.2.17. ☐

Clearly, Corollary 2.2.14 holds not only for $\mathfrak{P}$, but for all functors $\mathfrak{P}_{\mathbf{S}}$. Among them, $\mathfrak{P}_{\mathbf{E}}$ is special in the sense that it also has a right-adjoint. This yields the following algebraic version of the second item of Proposition 2.2.1.

Corollary 2.2.19. *The functor* $\mathfrak{P}_{\mathbf{E}}$ *is fully faithful and identifies* BA *with a reflective and coreflective subcategory of* $\mathcal{A}$*.*

Later on, in Subsections 4.3.1 and 4.3.2, we make use of the subalgebra adjunctions in order to lift endofunctors $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ to ones $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$. This allows us, for example, to lift classical coalgebraic logics to many-valued ones.

In this subsection, we showed that if a finite lattice-based algebra $\mathbf{D}$ is semi-primal, then there is an adjunction $\mathfrak{P}_{\mathbf{E}} \dashv \mathfrak{S} \dashv \mathfrak{P}_{\mathbf{D}}$, where $\mathbf{E}$ is the smallest subalgebra of $\mathbf{D}$. In the next subsection, we show that, conversely, the existence of an adjunction resembling this one fully characterizes semi-primality of a finite lattice-based algebra $\mathbf{D}$.

2.2.5 Characterizing semi-primality via adjunctions

The aim of this subsection is to find sufficient conditions for semi-primality of the algebra $\mathbf{M}$ in terms of $\mathfrak{P}_{\mathbf{M}}$ and its adjoint. We will then show that, in particular, these conditions are consequences of $\mathsf{U}\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$ and its dual $\mathfrak{S}\colon \mathcal{A} \to \mathsf{BA}$ from Figure 2.1 being (essentially) *topological functors*.

Recall from Definition 2.2.7 that the Boolean power functor $\mathfrak{P}_{\mathbf{M}}\colon \mathsf{BA} \to \mathbb{HSP}(\mathbf{M})$ can be defined for arbitrary finite algebras $\mathbf{M}$. Of course, if $\mathbf{S} \subseteq \mathbf{M}$ is a subalgebra, then $\mathfrak{P}_{\mathbf{S}}$ can also be seen as a functor $\mathsf{BA} \to \mathbb{HSP}(\mathbf{M})$. In the following, we will not distinguish between these two functors in our notation, since its type will always be clear from the context. The functor $\mathfrak{P}_{\mathbf{M}}$ is faithful (unless $\mathbf{M}$ is trivial), but it is usually not full. In fact, it is easy to see that $\mathfrak{P}_{\mathbf{M}}$ can only be full if $\mathbf{M}$ does not have any non-trivial automorphisms.

In the main theorem of this subsection, we show that if $\mathfrak{P}_{\mathbf{M}}$ is full and has a left-adjoint resembling $\mathfrak{S}$, then a lattice-based algebra $\mathbf{M}$ is semi-primal. Thus, this can be seen as a generalization of Hu's Theorem (Theorem 1.1.4) to a characterization of semi-primal (lattice-based) algebras via its category-theoretical relationship to BA.

Theorem 2.2.20 (Semi-primality via adjunctions)**.** *Let* $\mathbf{M}$ *be a finite lattice-based algebra. Then* $\mathbf{M}$ *is semi-primal if and only if* $\mathfrak{P}_{\mathbf{M}}$ *is full and there is a faithful functor* $\mathfrak{s}\colon \mathbb{HSP}(\mathbf{M}) \to \mathsf{BA}$ *which satisfies*

$$\mathfrak{P}_{\mathbf{E}} \dashv \mathfrak{s} \dashv \mathfrak{P}_{\mathbf{M}}$$

(where, as before, $\mathbf{E} = \langle 0, 1\rangle$ *is the smallest subalgebra of* $\mathbf{M}$*).*

Proof. If $\mathbf{M}$ is semi-primal, then $\mathfrak{P}_{\mathbf{M}}$ is full since it is dual to the full functor $\mathsf{V}^{\mathbf{D}}$, the functor $\mathfrak{s} = \mathfrak{S}$ is faithful since it is dual to the faithful functor U and $\mathfrak{P}_{\mathbf{E}} \dashv \mathfrak{S} \dashv \mathfrak{P}_{\mathbf{M}}$ was shown in the last two subsections.

Now for the converse, assume that $\mathfrak{P}_{\mathbf{M}}$ is full and there is a faithful functor $\mathfrak{s}\colon \mathbb{HSP}(\mathbf{M}) \to \mathsf{BA}$ with $\mathfrak{P}_{\mathbf{E}} \dashv \mathfrak{s} \dashv \mathfrak{P}_{\mathbf{M}}$. For abbreviation, we write $\mathcal{V}$ for $\mathbb{HSP}(\mathbf{M})$. We will make use of the following properties of $\mathfrak{s}$:

(i) The unit $\eta\colon \mathsf{id}_{\mathcal{V}} \Rightarrow \mathfrak{P}_{\mathbf{M}} \circ \mathfrak{s}$ is a monomorphism in each component.

(ii) The functor $\mathfrak{s}$ preserves monomorphisms and finite products.

Condition (i) follows from $\mathfrak{s}$ being faithful and (ii) follows from $\mathfrak{s}$ being a right-adjoint.

Our first goal is to prove the equivalence

$$\mathfrak{s}(\mathbf{A}) \cong \mathbf{2} \Leftrightarrow \exists \mathbf{S} \in \mathbb{S}(\mathbf{M}) : \mathbf{A} \cong \mathbf{S}. \tag{2.3}$$

If $\mathfrak{s}(\mathbf{A}) \cong \mathbf{2}$, use that by (i) there is an embedding $\mathbf{A} \hookrightarrow \mathfrak{P}_{\mathbf{M}}(\mathfrak{s}(\mathbf{A}))$. Since $\mathfrak{P}_{\mathbf{M}}(\mathfrak{s}(\mathbf{A})) \cong \mathbf{M}$, it follows that $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{M}$. Conversely, first note that $\mathfrak{s}(\mathbf{M}) \cong \mathbf{2}$ since, using that $\mathfrak{P}_{\mathbf{M}}$ is full and $\mathfrak{s} \dashv \mathfrak{P}_{\mathbf{M}}$, we have

$$1 = |\mathsf{BA}(\mathbf{2},\mathbf{2})| = |\mathcal{V}(\mathbf{M},\mathbf{M})| = |\mathcal{V}\big(\mathbf{M},\mathfrak{P}_{\mathbf{M}}(\mathbf{2})\big)| = |\mathsf{BA}\big(\mathfrak{s}(\mathbf{M}),\mathbf{2}\big)|,$$

which is only possible for $\mathfrak{s}(\mathbf{M}) \cong \mathbf{2}$. Now if $\mathbf{A} \cong \mathbf{S} \in \mathbb{S}(\mathbf{M})$, then due to (ii), the natural embedding $\mathbf{S} \hookrightarrow \mathbf{M}$ induces an embedding $\mathfrak{s}(\mathbf{S}) \hookrightarrow \mathfrak{s}(\mathbf{M})$. Therefore $\mathfrak{s}(\mathbf{S}) \cong \mathbf{2}$ since $\mathfrak{s}(\mathbf{M}) \cong \mathbf{2}$ does not have any proper subalgebras.

Next we show that $\mathbf{M}$ does not have any non-trivial internal isomorphisms. For every subalgebra $\mathbf{S} \in \mathbb{S}(\mathbf{M})$, there is a bijection between the set of Boolean homomorphisms $\mathfrak{s}(\mathbf{S}) \to \mathbf{2}$ and the set of homomorphisms $\mathbf{S} \to \mathfrak{P}_{\mathbf{M}}(\mathbf{2})$. Due to Equation 2.3, we have $\mathfrak{s}(\mathbf{S}) \cong \mathbf{2}$, so the former only has one element. Since $\mathfrak{P}_{\mathbf{M}}(\mathbf{2}) \cong \mathbf{M}$, this means that there is only one homomorphism $\mathbf{S} \to \mathbf{M}$, namely the identity on $\mathbf{S}$. Every non-trivial internal isomorphism with domain $\mathbf{S}$ would define another such homomorphism, resulting in a contradiction.

We now show that $\mathbf{M}$ is semi-primal, using the characterization of semi-primality from part (2) of Theorem 1.1.11. That is, we want to show that $\mathbf{M}$ has a majority term and every subalgebra of $\mathbf{M}^2$ is either a product of subalgebras or the diagonal of a subalgebra of $\mathbf{M}$. Since $\mathbf{M}$ is based on a lattice, a majority term is given by the median. Let $\mathbf{A} \leq \mathbf{M}^2$ be a subalgebra and let $\iota\colon \mathbf{A} \hookrightarrow \mathbf{M}^2$ be its natural embedding. Due to (ii) above, this embedding induces an embedding $\mathfrak{s}(\mathbf{A}) \hookrightarrow \mathfrak{s}(\mathbf{M}^2)$ into $\mathfrak{s}(\mathbf{M}^2) \cong \mathbf{2}^2$.

Therefore, either $\mathfrak{s}(\mathbf{A}) \cong \mathbf{2}^2$ or $\mathfrak{s}(\mathbf{A}) \cong \mathbf{2}$. Let $p_1 \colon \mathbf{A} \to \mathbf{M}$ and $p_2 \colon \mathbf{A} \to \mathbf{M}$ be ι followed by the respective projections $\mathbf{M}^2 \to \mathbf{M}$.

First assume that p_1 and p_2 coincide. Then clearly $\mathbf{A}$ embeds into $\mathbf{M}$, and therefore it is isomorphic to some subalgebra $\mathbf{S}$ of $\mathbf{M}$. Since $\mathbf{M}$ has no non-trivial internal isomorphisms, $\mathbf{A}$ needs to coincide with the diagonal of $\mathbf{S}$.

If p_1 and p_2 are distinct then, using that $\mathfrak{s}$ is faithful, the morphisms $\mathfrak{s}p_1 \colon \mathfrak{s}(\mathbf{A}) \to \mathbf{2}$ and $\mathfrak{s}p_2 \colon \mathfrak{s}(\mathbf{A}) \to \mathbf{2}$ are distinct as well. This implies that $\mathfrak{s}(\mathbf{A}) \cong \mathbf{2}^2$. Using the adjunction $\mathfrak{P}_E \dashv \mathfrak{s}$ we get

$$4 = |\mathsf{BA}\big(\mathbf{2}^2, \mathfrak{s}(\mathbf{A})\big)| = |\mathcal{V}(\mathbf{E}^2, \mathbf{A})| \text{ and } 4 = |\mathsf{BA}\big(\mathbf{2}^2, \mathfrak{s}(\mathbf{M}^2)\big)| = |\mathcal{V}(\mathbf{E}^2, \mathbf{M}^2)|.$$

So there are exactly four distinct homomorphisms $\mathbf{E}^2 \to \mathbf{A}$ and, since ι is a monomorphism, their compositions with ι are also four distinct homomorphisms $\mathbf{E}^2 \to \mathbf{M}^2$. Therefore, every of the former homomorphisms arises in such a way. In particular, the natural embedding $\mathbf{E}^2 \hookrightarrow \mathbf{M}^2$ arises in this way, which implies $(0,1) \in \mathbf{A}$ and $(1,0) \in \mathbf{A}$. As noted in [DSW91], this leads to $\mathbf{A} = p_1(\mathbf{A}) \times p_2(\mathbf{A})$, since whenever $(a,b), (c,d) \in \mathbf{A}$ we also have

$$(a,d) = \big((a,b) \wedge (1,0)\big) \vee \big((c,d) \wedge (0,1)\big) \in \mathbf{A}.$$

This concludes the proof. □

In the remainder of this subsection, we show how the above theorem relates to the theory of *topological functors* (see, *e.g.*, [AHS06, Chapter VI.21] or [Bor94, Chapter 7]).

Intuitively speaking, topological functors behave similarly to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$ from the category Top of all topological spaces to Set. This functor has the following well-known property. Let X be a set, let $(X_i, \tau_i)_{i \in I}$ be an (arbitrary) collection of topological spaces and, for every $i \in I$, let $g_i \colon X \to X_i$ be a map. Then there is a unique coarsest topology on X which renders all of the g_i continuous, namely the *initial topology.* Also recall that the forgetful functor $\mathsf{Top} \to \mathsf{Set}$ is faithful and has both adjoints, its left-adjoint taking the discrete topology and its right-adjoint taking the trivial topology.

To generalize this, let $\mathsf{F} \colon \mathsf{C} \to \mathsf{D}$ be a functor. A F-*structured source* is a collection of morphisms $(f_i \colon d \to \mathsf{F}(c_i))_{i \in I}$ in D. A *lift* of this F-source is a collection of morphisms $(\bar{f}_i \colon c \to c_i)_{i \in I}$ such that $\mathsf{F}(c) = d$ and $\mathsf{F}(\bar{f}_i) = f_i$ for all $i \in I$. Furthermore, this lift is *initial* if it is universal in the sense that for all other lifts $(\bar{f}'_i \colon c' \to c_i)_{i \in I}$ there exists a unique morphism $g \colon c' \to c$ with $\bar{f}'_i \circ g = \bar{f}_i$ for all $i \in I$.

Definition 2.2.21 (Topological functor)**.** Let C and D be categories. We call a functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$

(1) *topological* if it is faithful and every F-structured source has an initial lift and

(2) *essentially topological* if it is topological up to categorical equivalences of C and D.

The need for this distinction arises because certain properties of topological functors, *e.g.*, *amnesticity* [AHS06, Definition 3.27], are not preserved under categorical equivalence (this issue is addressed in [nLa22]).

The following is our key observation for the last part of this subsection.

Proposition 2.2.22 (U is topological)**.** *The forgetful functor* $\mathsf{U}\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}$ *is topological and the Boolean skeleton functor* $\mathfrak{S}\colon \mathcal{A} \to \mathsf{BA}$ *is essentially topological.*

Proof. We only need to show that U is topological, which immediately implies that $\mathfrak{S}$ is essentially topological due to [AHS06, Theorem 21.9] together with the fact that $\mathfrak{S}$ is naturally isomorphic to the dual of U (see Corollary 2.2.5).

It is obvious that U is faithful since it is the identity on morphisms. Now let $X \in \mathsf{Stone}$ be a Stone space and let $(f_i\colon X \to \mathsf{U}(X_i, \mathbf{v}_i))_{i\in I}$ be a U-structured source (*i.e.*, a collection of continuous maps) indexed by a class I. We define $\mathbf{v}\colon X \to \mathbb{S}(\mathbf{D})$ by

$$\mathbf{v}(x) = \bigvee_{i\in I} \mathbf{v}_i\big(f_i(x)\big),$$

which is well-defined, since $\mathbb{S}(\mathbf{D})$ is finite. The fact that $(X, \mathbf{v})$ is a member of $\mathsf{Stone}_{\mathbf{D}}$ follows from the fact that $\mathbf{v}^{-1}(\mathbf{S}{\downarrow}) = \bigcap_{i\in I} f_i^{-1}(\mathbf{v}_i^{-1}(\mathbf{S}{\downarrow}))$ is closed. Every f_i is now also a morphism in $\mathsf{Stone}_{\mathbf{D}}$, which defines a lift of the U-structured source.

We now show that the source thus defined is initial. Assume there are $\mathsf{Stone}_{\mathbf{D}}$-morphisms $(g_i\colon (Y, \mathbf{w}) \to (X_i, \mathbf{v}_i))_{i\in I}$ and a continuous map $g\colon Y \to X$ with $f_i \circ g = g_i$ for all $i \in I$. All we need to show is that g defines a $\mathsf{Stone}_{\mathbf{D}}$-morphism $(Y, \mathbf{w}) \to (X, \mathbf{v})$. To see this simply note that

$$\mathbf{v}\big(g(y)\big) = \bigvee_{i\in I} \mathbf{v}_i\big(f_i(g(y))\big) = \bigvee_{i\in I} \mathbf{v}_i\big(g_i(y)\big) \leq \mathbf{w}(y),$$

which concludes the proof. □

We can now easily show the following characterization of semi-primality of a lattice-based algebra $\mathbf{M}$ via the existence of a topological functor from $\mathbb{HSP}(\mathbf{M})$ to BA which has the Boolean power functors with respect to $\mathbf{M}$ and the smallest subalgebra $\mathbf{E} \subseteq \mathbf{M}$ as right- and left-adjoint, respectively.

Corollary 2.2.23 (Semi-primality via topological functor). *Let* $\mathbf{M}$ *be a finite lattice-based algebra. Then* $\mathbf{M}$ *is semi-primal if and only if there is an essentially topological functor* $\mathfrak{s} \colon \mathbb{HSP}(\mathbf{M}) \to \mathsf{BA}$ *which satisfies*

$$\mathfrak{P}_{\mathbf{E}} \dashv \mathfrak{s} \dashv \mathfrak{P}_{\mathbf{M}},$$

where $\mathbf{E} = \langle 0, 1 \rangle$ *is the smallest subalgebra of* $\mathbf{M}$.

Proof. In Proposition 2.2.22, we showed that if $\mathbf{M}$ is semi-primal, then the Boolean skeleton functor $\mathfrak{S}$ is essentially topological.

Conversely, if such an essentially topological $\mathfrak{s}$ exists, it is faithful by definition and both its adjoints $\mathfrak{P}_{\mathbf{M}}$ and $\mathfrak{P}_{\mathbf{E}}$ are full by [AHS06, Proposition 21.12]. Therefore, due to Theorem 2.2.20, $\mathbf{M}$ is semi-primal. □

This can be seen as a generalization of Hu's Theorem (Theorem 1.1.4), in the following sense. Hu's Theorem states that there is a categorical equivalence defined by $\mathfrak{P}_{\mathbf{M}} \colon \mathsf{BA} \to \mathbb{HSP}(\mathbf{M})$ if and only if $\mathbf{M}$ is primal. Corollary 2.2.23 states that $\mathfrak{P}_{\mathbf{M}}$ and $\mathfrak{P}_{\mathbf{E}}$ are adjoints of a topological functor if and only if $\mathbf{M}$ is semi-primal. In particular, if $\mathbf{M}$ has no proper subalgebras, this topological adjunction turns into an equivalence since the two adjoints $\mathfrak{P}_{\mathbf{M}} = \mathfrak{P}_{\mathbf{E}}$ collapse.

To end this section, we use the fact that U is topological to characterize the regular monomorphisms in $\mathsf{Stone}_{\mathbf{D}}$ and thus, by duality, the regular epimorphisms in $\mathcal{A}$.

Corollary 2.2.24 (Regular monos of $\mathsf{Stone}_{\mathbf{D}}$). *A morphism* $f \colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$ *in* $\mathsf{Stone}_{\mathbf{D}}$ *is a regular monomorphism if and only if it is injective and* $\mathbf{v}_1(x) = \mathbf{v}_2(f(x))$ *for all* $x \in X_1$.

Proof. It follows from [AHS06, Proposition 21.13] that a morphism f in $\mathsf{Stone}_{\mathbf{D}}$ is a regular monomorphism if and only if $f \colon X_1 \to X_2$ is a monomorphism (*i.e.*, injective) in Stone and f is initial, meaning that for all $(Y, \mathbf{w}) \in \mathsf{Stone}_{\mathbf{D}}$ and continuous maps $g \colon Y \to X_1$, if $f \circ g$ is a morphism $(Y, \mathbf{w}) \to (X_2, \mathbf{v}_2)$, then g is a morphism $(Y, \mathbf{w}) \to (X_1, \mathbf{v}_1)$. We now show that this is the case if and only if $\mathbf{v}_1(x) = \mathbf{v}_2(f(x))$ for all $x \in X_1$.

Assuming $\mathbf{v}_1(x) = \mathbf{v}_2(f(x))$ for all x, let $g \colon Y \to X_1$ be a continuous map such that $f \circ g$ is a morphism $(Y, \mathbf{w}) \to (X_2, \mathbf{v}_2)$ in $\mathsf{Stone}_{\mathbf{D}}$. Then we have

$$\mathbf{v}_1\big(g(y)\big) = \mathbf{v_2}\big(f(g(x))\big) \leq \mathbf{w}(y)$$

for all $y \in Y$, thus g is a morphism $(Y, \mathbf{w}) \to (X_1, \mathbf{v}_1)$.

Conversely, if there exists some $x_0 \in X_1$ such that $\mathbf{v}_2(f(x_0)) < \mathbf{v}_1(x_0)$, then define $(Y, \mathbf{w})$ by $Y = X_1$ and $\mathbf{w}(x_0) = \mathbf{v}_2(x_0)$ and $\mathbf{w}(x) = \mathbf{v}_1(x)$ for all $x \neq x_0$. Let $g\colon X_1 \to X_1$ be the identity. By construction we have that $f \circ g$ is a morphism as desired, but g is not since $\mathbf{v}_1(g(x_0)) = \mathbf{v}_1((x_0) > \mathbf{v}_2(f(x_0)) = \mathbf{w}(x_0)$. Therefore, f is not initial. $\square$

In this section, we obtained algebraic descriptions of all the functors between $\mathcal{A}$ and BA appearing on the right-hand side of Figure 2.1, thereby gaining insight into semi-primal algebras and their relationship to Boolean algebras. Furthermore, we now showed which properties of the Boolean skeleton functor $\mathfrak{S}$ completely characterize semi-primality. In the next section, we investigate how canonical extensions of algebras in $\mathcal{A}$ behave and can be characterized by their Boolean skeletons. In particular, we show that the algebras that can be obtained as canonical extensions of algebras in $\mathcal{A}$ are precisely the ones whose Boolean skeletons are canonical extensions of Boolean algebras, that is, the algebras whose Boolean skeletons are complete and atomic.

2.3 Discrete duality and canonical extensions

In this section, we describe a semi-primal discrete duality similar to the well-known discrete duality between Set and CABA, the latter denoting the category of *complete atomic Boolean algebras* with complete homomorphisms (on the object level, this duality goes back to [Tar35]). It can be obtained from the finite duality given in Theorem 2.1.8 in a similar way to the one of Section 2.1, except that now we lift it to the level of $\mathsf{Ind}(\mathsf{Set}^{\omega}_{\mathbf{D}})$ and $\mathsf{Pro}(\mathcal{A}^{\omega})$. The members of the latter category are known to be precisely the *canonical extensions* [GJ04] of members of $\mathcal{A}$ (see [DP12]), and we provide two new characterizations of this category (Corollary 2.3.8 and Theorem 2.3.10). Lastly, we show that, as in the primal case of BA, the topological duality from Section 2.1 can be connected to its discrete version via an analogue of the *Stone-Čech compactification* (Proposition 2.3.11).

2.3.1 Semi-primal discrete duality

Our first goal is to identify $\mathsf{Ind}(\mathsf{Set}^{\omega}_{\mathbf{D}})$. Although it may not be surprising, it still takes some work to prove that it can be identified with the following category.

Definition 2.3.1 (The category $\mathsf{Set}_{\mathbf{D}}$)**.** The category $\mathsf{Set}_{\mathbf{D}}$ has objects of the form (X, v) where $X \in \mathsf{Set}$ and $v\colon X \to \mathbb{S}(\mathbf{D})$ is an arbitrary map. A morphism $f\colon (X_1, v_1) \to (X_2, v_2)$ is a map $X_1 \to X_2$ which satisfies

$$v_2\big(f(x)\big) \leq v_1(x)$$

for all $x \in X_1$.

In the context of fuzzy sets, Goguen [Gog67, Gog74] initiated the study of categories similar to $\mathsf{Set}_{\mathbf{D}}$. This research was continued, *e.g.*, in [Bar86, Wal04]. Sticking to the notation of [Gog74], for a complete lattice $\mathcal{V}$, the category $\mathsf{Set}(\mathcal{V})$ of $\mathcal{V}$-fuzzy sets has objects (X, A) where $A\colon X \to \mathcal{V}$. Morphisms $(X_1, A_1) \to (X_2, A_2)$ are maps $f\colon X_1 \to X_2$ which satisfy $A_2(f(x)) \geq A_1(x)$ for all $x \in X_1$. For the purpose of fuzzy set theory (as introduced by Zadeh [Zad65]), people were mainly interested in the case where $\mathcal{V} = [0, 1]$. However, we retrieve $\mathsf{Set}_{\mathbf{D}}$ in the case where $\mathcal{V}$ is the order-dual of $\mathbb{S}(\mathbf{D})$.

Since we are interested in the Ind-completion of $\mathsf{Set}^{\omega}_{\mathbf{D}}$, we will first discuss (filtered) colimits in the category $\mathsf{Set}_{\mathbf{D}}$. We show that, on the object level they coincide with filtered colimits in Set, which can be described via certain quotients of disjoint unions. The additional structure is then given by 'minimizing over the equivalence classes' as described in the following.

Lemma 2.3.2. *The category* $\mathsf{Set}_{\mathbf{D}}$ *is cocomplete. The colimit* $\mathrm{colim}_{i\in I}(X_i, v_i)$ *of a filtered diagram* $\big(f_{ij}\colon (X_i, v_i) \to (X_j, v_j) \mid i \leq j\big)$ *is realized by*

$$\big((\coprod_{i\in I} X_i)/{\sim}, \bar{v}\big)$$

defined as follows. For $x_i \in X_i$ *and* $x_j \in X_j$*,*

$$x_i \sim x_j \iff \exists k \geq i, j : f_{ik}(x_i) = f_{jk}(x_j)$$

and

$$\bar{v}([x_i]) = \bigwedge_{x_i \sim x_j \in X_j} v_j(x_j),$$

where $[x_i]$ *denotes the equivalence class of* x_i *with respect to* $\sim$*.*

Proof. The proof that $\mathsf{Set}_{\mathbf{D}}$ is cocomplete is completely analogous to the one in [Wal04, Propositions 5 & 8]. For filtered colimits, on the underlying level of Set we know that $X := \coprod_{i\in I}(X_i)/{\sim}$ with the canonical inclusions $\rho_i\colon X_i \to X$ is the colimit of the diagram. To see that all the ρ_i are morphisms in $\mathsf{Set}_{\mathbf{D}}$ we note that

$$\bar{v}\big(\rho_i(x_i)\big) = \bigwedge_{x_i \sim x_j \in X_j} v_j(x_j) \leq v_i(x_i).$$

Given another cocone $\gamma_i \colon (X_i, v_i) \to (Y, w)$, the unique map $g \colon X \to Y$ is a morphism in $\mathsf{Set}_{\mathbf{D}}$ since, for $x_i \in X_i$ and $x_i \sim x_j \in X_j$ we have $w\big(g(\rho_j(x_j))\big) = w(\gamma_j(x_j)) \leq v_j(x_j)$ and thus

$$w\big(g([x_i])\big) \leq \bigwedge_{x_i \sim x_j \in X_j} v_j(x_j) = \bar{v}([x_i]),$$

which concludes the proof. □

Since we are interested in an Ind-completion and, therefore, in finitely presentable objects, we make note of the following observation.

Lemma 2.3.3. *Let* $\mathsf{F} \colon \mathsf{C} \to \mathsf{D}$ *be a functor between categories* C *and* D *which both admit filtered colimits. If* F *has a right-adjoint* G *which preserves filtered colimits, then* F *preserves finitely presentable objects.*

Proof. Let $c \in \mathsf{C}$ be finitely presentable. We want to show that $\mathsf{F}(c)$ is finitely presentable in D. Let $\mathrm{colim}_I d_i$ be a filtered colimit in D. Then

$$\mathsf{D}\big(\mathsf{F}(c), \mathrm{colim}_I d_i\big) \cong \mathrm{colim}_I \mathsf{C}\big(c, \mathsf{G}(d_i)\big) \cong \mathrm{colim}_I \mathsf{D}\big(\mathsf{F}(c), d_i\big),$$

where the first isomorphism comes from the fact that G preserves filtered colimits and c is finitely presentable. □

It is easy to see that all adjunctions between $\mathsf{Stone}_{\mathbf{D}}$ and Stone described in Section 2.2 have their analogously defined discrete counterparts between $\mathsf{Set}_{\mathbf{D}}$ and Set. With the previous lemma, this allows us to show that all objects of $\mathsf{Set}_{\mathbf{D}}$ whose underlying sets are finite, are finitely presentable in $\mathsf{Set}_{\mathbf{D}}$.

Corollary 2.3.4. *If X is a finite set, then (X, v) is finitely presentable in* $\mathsf{Set}_{\mathbf{D}}$ *for every $v \colon X \to \mathbb{S}(\mathbf{D})$.*

Proof. Let $X = \{x_1, \dots, x_n\}$ and let $v(x_i) = \mathbf{S}_i$. Then we can clearly identify

$$(X, v) \cong \coprod_{1 \leq i \leq n} (\{x_i\}, v^{\mathbf{S}_i}),$$

where $v_i^{\mathbf{S}}(x_i) = \mathbf{S}_i$ as before. Since filtered colimits commute with finite limits in Set, it now suffices to show that all $(\{x_i\}, v^{\mathbf{S}_i})$ are finitely presentable. Just like in Subsection 2.2.4, we can define the adjunction $\mathsf{V}^{\mathbf{S}} \dashv \mathsf{C}^{\mathbf{S}}$ between $\mathsf{Set}_{\mathbf{D}}$ and Set for every subalgebra $\mathbf{S} \leq \mathbf{D}$. By Lemma 2.3.3 it now suffices to show that $\mathsf{C}^{\mathbf{S}}$ preserves filtered colimits. So let $(X, \bar{v})$ be a filtered colimit as in Lemma 2.3.2. We know that $\mathsf{C}^{\mathbf{S}}(X) = \{[x_i] \mid \exists x_i \sim x_j \in X_j, v_j(x_j) \leq \mathbf{S}\}$. Therefore, for all $[x_i] \in \mathsf{C}^{\mathbf{S}}$, we can choose representatives with $x_i \in \mathsf{C}^{\mathbf{S}}(X_i, v_i)$. This yields an isomorphism between $\mathsf{C}^{\mathbf{S}}(X)$ and $\mathrm{colim}\mathsf{C}^{\mathbf{S}}(X_i, v_i)$. □

Now we have all the necessary tools at our disposal to easily show that $\mathsf{Set}_{\mathbf{D}}$ is indeed categorically equivalent to $\mathsf{Ind}(\mathsf{Set}^{\omega}_{\mathbf{D}})$.

Theorem 2.3.5. $\mathsf{Ind}(\mathsf{Set}^{\omega}_{\mathbf{D}})$ *is categorically equivalent to* $\mathsf{Set}_{\mathbf{D}}$.

Proof. Since $\mathsf{Set}_{\mathbf{D}}$ is cocomplete, the inclusion $\iota\colon \mathsf{Set}^{\omega}_{\mathbf{D}} \to \mathsf{Set}_{\mathbf{D}}$ has a unique finitary extension $\hat{\iota}\colon \mathsf{Ind}(\mathsf{Set}^{\omega}_{\mathbf{D}}) \to \mathsf{Set}_{\mathbf{D}}$. Since ι is fully faithful and, by Corollary 2.3.4, maps all objects to finitely presentable objects in $\mathsf{Set}_{\mathbf{D}}$, this extension is also fully faithful. To see that it is essentially surjective note that, just like in Set, every object $(X, v) \in \mathsf{Set}_{\mathbf{D}}$ is the directed colimit of its finite subsets $F \subseteq X$ considered as $(F, v|_F)$. □

We now take a closer look at the category $\mathsf{Pro}(\mathcal{A}^{\omega})$, which is dually equivalent to $\mathsf{Set}_{\mathbf{D}}$ by Lemma 2.1.6 together with Theorem 2.1.8. It is well-known that the profinite algebras in $\mathsf{Pro}(\mathcal{A}^{\omega})$ can be identified with the *canonical extensions* [GJ04] of algebras in $\mathcal{A}$. In [DP12] a description of these canonical extensions as topological algebras can be found. However, as in the case of complete atomic Boolean algebras $\mathsf{CABA} \simeq \mathsf{Pro}(\mathsf{BA}^{\omega})$, this need not be the only description. In the following, we apply some results of Section 2.2 to find two simple alternative descriptions. The first one is in terms of (arbitrary) products of subalgebras of $\mathbf{D}$ with complete homomorphisms.

Definition 2.3.6. Let $\hat{\mathcal{A}}$ be the category with algebras from $\mathbb{IPS}(\mathbf{D})$ as objects and complete (with respect to the lattice-operations) homomorphisms as morphisms.

We can essentially repeat our proof of the finite duality from Theorem 2.1.8, once we prove the following result analogous to Lemma 2.1.7.

Proposition 2.3.7. *Let* $\mathbf{A} = \prod_{i \in I} \mathbf{S}_i \in \hat{\mathcal{A}}$. *Then the complete homomorphisms* $\mathbf{A} \to \mathbf{D}$ *are precisely the projections (followed by inclusions) in each component.*

Proof. By Proposition 2.2.4, there is a natural bijection between $\mathcal{A}(\mathbf{A}, \mathbf{D})$ and $\mathsf{BA}(\mathfrak{S}(\mathbf{A}), \mathbf{2})$ given by $u \mapsto u|_{\mathfrak{S}(\mathbf{A})}$. In particular, if u is complete, then so is its restriction. Since $\mathfrak{S}(\mathbf{A}) = \mathbf{2}^I$, the only complete homomorphisms $\mathfrak{S}(\mathbf{A}) \to \mathbf{2}$ are the projections, and they are the restrictions of the respective projections $\mathbf{A} \to \mathbf{D}$. □

We now get our first description of $\mathsf{Pro}(\mathcal{A}^{\omega})$ as follows.

Corollary 2.3.8. $\mathsf{Pro}(\mathcal{A}^{\omega})$ *is categorically equivalent to* $\hat{\mathcal{A}}$.

Proof. By Theorem 2.3.5, it suffices to show that $\mathsf{Set}_{\mathbf{D}}$ is dually equivalent to $\hat{\mathcal{A}}$. This is done completely analogous to the proof of Theorem 2.1.8. □

Our second description of $\mathsf{Pro}(\mathcal{A}^\omega)$ is in terms of the Boolean skeleton.

Definition 2.3.9 (The category $\mathsf{CA}\mathcal{A}$)**.** The category $\mathsf{CA}\mathcal{A}$ has as objects algebras $\mathbf{A} \in \mathcal{A}$ which have a complete lattice-reduct and which satisfy $\mathfrak{S}(\mathbf{A}) \in \mathsf{CABA}$. The morphisms in $\mathsf{CA}\mathcal{A}$ are the complete homomorphisms.

In light of this definition, the following shows that algebras in $\mathsf{Pro}(\mathcal{A}^\omega)$ can be recognised by their Boolean skeletons.

Theorem 2.3.10. $\mathsf{Pro}(\mathcal{A}^\omega)$ *is categorically equivalent to* $\mathsf{CA}\mathcal{A}$*.*

Proof. Using Corollary 2.3.8, we show that $\mathsf{CA}\mathcal{A}$ is categorically equivalent to $\hat{\mathcal{A}}$. Clearly there is a fully faithful inclusion functor $\hat{\mathcal{A}} \hookrightarrow \mathsf{CA}\mathcal{A}$. So it suffices to show that this functor is essentially surjective. In other words, we want to show that every object of $\mathsf{CA}\mathcal{A}$ is isomorphic to a product of subalgebras of $\mathbf{D}$.

So consider an arbitrary $\mathbf{A} \in \mathsf{CA}\mathcal{A}$. Since the adjunction $\mathfrak{S} \dashv \mathfrak{P}$ restricts to CABA and $\mathsf{CA}\mathcal{A}$, we can use Corollary 2.2.11 to get a *complete* embedding $\eta_{\mathbf{A}} \colon \mathbf{A} \hookrightarrow \mathfrak{P}(\mathfrak{S}(\mathbf{A}))$. Since $\mathfrak{S}(\mathbf{A})$ is in CABA it is isomorphic to $\mathbf{2}^I$ for some index set I. Thus we have $\mathfrak{P}(\mathfrak{S}(\mathbf{A})) \cong \mathfrak{P}(\mathbf{2}^{\mathbf{I}}) \cong \mathbf{D}^I$ by Lemma 2.2.8. We show that $\mathbf{A}$ is isomorphic to the direct product of subalgebras $\prod_{i\in I} \mathsf{pr}_i(\eta_{\mathbf{A}}(A))$. For this it suffices to show that the injective homomorphism $\eta_{\mathbf{A}}$ maps onto it. So let α be an element of this product. For each $i \in I$, choose $a_i \in \mathbf{A}$ such that $\mathsf{pr}_i(\eta_{\mathbf{A}}(a_i)) = \alpha(i)$. Since $\mathbf{2}^I \cong \mathfrak{S}(\mathbf{A}) \subseteq \mathbf{A}$ holds, all atoms $b_i \in \mathbf{2}^I$ (defined by $b_i(j) = 1$ iff $j = i$) can be considered as members of $\mathbf{A}$. Now define

$$a = \bigvee \{a_i \wedge b_i \mid i \in I\}.$$

Since $\mathbf{A}$ is complete, we have $a \in \mathbf{A}$. Furthermore, since $\eta_{\mathbf{A}}$ is a complete homomorphism, we have $\eta_{\mathbf{A}}(a) = \alpha$ (because $\mathsf{pr}_i(\eta_{\mathbf{A}}(a)) = \eta_{\mathbf{A}}(a_i) = \alpha(i)$), which finishes the proof. □

2.3.2 Stone-Čech compactification

With the results from this section up to this point, it is clear that the chains of adjunctions from Section 2.2 (see Figure 2.1) have their discrete counterparts, analogously defined, not only between $\mathsf{Set}_{\mathbf{D}}$ and Set, but also between $\mathsf{CA}\mathcal{A}$ and CABA. To make a connection between Figure 2.1 and its discrete counterpart, we finish this section by connecting the respective dualities as indicated in Figure 2.3.

Here, $(-)^\flat \colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Set}_{\mathbf{D}}$ is the forgetful functor with respect to topology and $\iota_c \colon \mathsf{CA}\mathcal{A} \to \mathcal{A}$ is the obvious inclusion functor (note that both these

$$\begin{tikzcd}
\mathsf{Stone}_{\mathbf{D}} \arrow[r, shift left] \arrow[d, bend left=40, "(-)^{\flat}"] \arrow[d, phantom, "\dashv"] & \mathcal{A} \arrow[l, shift left] \arrow[d, bend left=40, "(-)^{\delta}"] \arrow[d, phantom, "\vdash"] \\
\mathsf{Set}_{\mathbf{D}} \arrow[u, bend left=40, "\beta_{\mathbf{D}}"] \arrow[r, shift left] & \mathsf{CA}\mathcal{A} \arrow[l, shift left] \arrow[u, bend left=40, "\iota_c"]
\end{tikzcd}$$

Figure 2.3: Compactification and canonical extension.

functors are not full). The functor $(-)^{\delta}\colon \mathcal{A} \to \mathsf{CA}\mathcal{A}$ takes an algebra to its *canonical extension*. In the primal case $\mathbf{D} = \mathbf{2}$, it is well-known that $\beta_{\mathbf{2}} =: \beta$ is the *Stone-Čech compactification* (see, *e.g.*, [Joh82, Section IV.2]). This has been generalized to the *Bohr compactification* in a (much broader) framework which includes ours in [DHP17]. However, since things are particularly simple in our setting, we directly show how to define $\beta_{\mathbf{D}}$.

Given $(X, v) \in \mathsf{Set}_{\mathbf{D}}$, there is a natural way to extend v to the Stone-Čech compactification $\beta(X)$ of X. Indeed, since $v\colon X \to \mathbb{S}(\mathbf{D})$ can be thought of as a continuous map between discrete spaces, by the universal property of β it has a unique continuous extension $\tilde{\mathbf{v}}\colon \beta(X) \to \mathbb{S}(\mathbf{D})$. Here, $\tilde{\mathbf{v}}^{-1}(\mathbf{S}{\downarrow})$ is given by the topological closure of $v^{-1}(\mathbf{S}{\downarrow})$ in $\beta(X)$. Thus, for every morphism $f\colon (X_1, v_1) \to (X_2, v_2)$ in $\mathsf{Set}_{\mathbf{D}}$, the continuous map βf defines a morphism $(\beta(X_1), \tilde{\mathbf{v}}_{\mathbf{1}}) \to (\beta(X_2), \tilde{\mathbf{v}}_{\mathbf{2}})$ in $\mathsf{Stone}_{\mathbf{D}}$. This is due to the observation that whenever $x \in \tilde{\mathbf{v}}_{\mathbf{1}}^{-\mathbf{1}}(\mathbf{S}{\downarrow}) = \overline{v_1^{-1}(\mathbf{S}{\downarrow})}$, by continuity of βf and the morphism property of f, we have $\beta f(x) \in \overline{v_2^{-1}(\mathbf{S}{\downarrow})} = \tilde{\mathbf{v}}_{\mathbf{2}}^{-1}(\mathbf{S}{\downarrow})$.

Proposition 2.3.11. *The functor* $\beta_{\mathbf{D}}\colon \mathsf{Set}_{\mathbf{D}} \to \mathsf{Stone}_{\mathbf{D}}$ *defined on objects by*

$$\beta_{\mathbf{D}}(X, v) = (\beta(X), \tilde{\mathbf{v}}),$$

and by $f \mapsto \beta f$ *on morphisms is the dual of the canonical extension functor* $(-)^{\delta}\colon \mathcal{A} \to \mathsf{CA}\mathcal{A}$.

Proof. It suffices to show that $\beta_{\mathbf{D}}$ satisfies the following universal property. Given $(Y, \mathbf{w}) \in \mathsf{Stone}_{\mathbf{D}}$, every $\mathsf{Set}_{\mathbf{D}}$-morphism $f\colon (X, v) \to (Y, \mathbf{w})$ extends uniquely to a $\mathsf{Stone}_{\mathbf{D}}$-morphism $\tilde{f}\colon (\beta(X), \tilde{\mathbf{v}}) \to (Y, \mathbf{w})$. On the levels of Set and Stone, we get a unique continuous extension $\tilde{f}$. To show it is a $\mathsf{Stone}_{\mathbf{D}}$-morphism, similarly to before, note that if $x \in \overline{v^{-1}(S{\downarrow})}$, then by continuity

$$\tilde{f}(x) \in \overline{f\big(v^{-1}(\mathbf{S}{\downarrow})\big)} \subseteq \overline{\mathbf{w}^{-1}(\mathbf{S}{\downarrow})}.$$

Since $\mathbf{w}^{-1}(\mathbf{S}{\downarrow})$ is closed it equals its own closure. □

This nicely wraps up this chapter by connecting all of its main sections. In the following concluding section, we hint at some potential directions for future research along similar lines.

2.4 Conclusion of Chapter 2

We explored semi-primality by means of category theory, showing how a variety generated by a semi-primal lattice expansion relates to the variety of Boolean algebras. Various adjunctions provide insight into the many similarities between these varieties. Furthermore, we investigated the corresponding discrete duality and characterized canonical extensions via their Boolean skeletons.

A schematic summary of the results of this chapter is given in Figure 2.4[2], which also emphasizes once more how close BA and $\mathcal{A}$ really are. Building on the results of this chapter, in Chapter 3, we investigate the role of semi-primality in the context of *many-valued modal logic* and in Chapter 4 in the context of *coalgebraic logics*. In particular, in the latter the subalgebra adjunctions from Subsection 2.2.4 are used in order to obtain many-valued coalgebraic logics from their classical counterparts.

$$\begin{tikzcd}
& \mathsf{Set}_{\mathbf{D}}^{\omega} \arrow[rrrr, leftrightarrow] \arrow[ddl, dashed, "\mathsf{Pro}"'] \arrow[dr, dashed, "\mathsf{Ind}"] & & & & \mathcal{A}^{\omega} \arrow[ddl, dashed, "\mathsf{Ind}"'] \arrow[dr, dashed, "\mathsf{Pro}"] & \\
& & \mathsf{Set}_{\mathbf{D}} \arrow[rrrr, leftrightarrow] \arrow[dll, shift right=1] \arrow[dd, shift left=1] \arrow[dd, shift left=3] & & & & \mathsf{CA}\mathcal{A} \arrow[dll, shift right=1] \arrow[dd, shift left=1] \arrow[dd, shift left=3] \\
\mathsf{Stone}_{\mathbf{D}} \arrow[urr, shift right=1] \arrow[rrrr, leftrightarrow] \arrow[dd, shift left=1] \arrow[dd, shift left=3] & & & & \mathcal{A} \arrow[urr, shift right=1] \arrow[dd, shift left=1] \arrow[dd, shift left=3] & & \\
& & \mathsf{Set} \arrow[rrrr, leftrightarrow] \arrow[dll, shift right=1] \arrow[uu, shift left=1] \arrow[uu, shift left=3] & & & & \mathsf{CABA} \arrow[dll, shift right=1] \arrow[uu, shift left=1] \arrow[uu, shift left=3] \\
\mathsf{Stone} \arrow[urr, shift right=1] \arrow[rrrr, leftrightarrow] \arrow[uu, shift left=1] \arrow[uu, shift left=3] & & & & \mathsf{BA} \arrow[urr, shift right=1] \arrow[uu, shift left=1] \arrow[uu, shift left=3] & & \\
& \mathsf{Set}^{\omega} \arrow[ul, dashed, "\mathsf{Pro}"] \arrow[uur, dashed, "\mathsf{Ind}"'] \arrow[rrrr, leftrightarrow] & & & & \mathsf{BA}^{\omega} \arrow[ul, dashed, "\mathsf{Ind}"] \arrow[uur, dashed, "\mathsf{Pro}"'] &
\end{tikzcd}$$

Figure 2.4: Summary of the results of Chapter 2.

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[2] I like to refer to the diagram in Figure 2.4 as *semi-primal bi-temple*.

To end this chapter, we indicate some questions and directions for future related research. One purpose of this research was to set an example in exploring concepts in universal algebra through a category theoretical lens. In the future, one could investigate other variants of primality in a similar manner. For example, a similar study of *quasi-primality* (Definition 1.1.7) would be a logical next step. There also are some more 'intermediate' variations of primality (with varying restrictions on internal isomorphisms) as follows.

Definition 2.4.1. A finite algebra $\mathbf{M}$ is called

(1) *Demi-primal* if it is quasi-primal and has no proper subalgebras (see [Qua71]).

(2) *Demi-semi-primal* if it is quasi-primal and every internal isomorphism of $\mathbf{M}$ can be extended to an automorphism of $\mathbf{M}$ (see [Qua71]).

(3) *Infra-primal* if it is demi-semi primal and every internal isomorphism is an automorphism on its domain (see [Fos69]).

The 'hierarchy' between these variations of primality is depicted in Figure 2.5.

$$\begin{tikzcd}
& \text{quasi-primal} & \\
& \text{demi-semi-primal} \arrow[u, Rightarrow] & \\
& & \text{infra-primal} \arrow[ul, Rightarrow] \\
\text{demi-primal} \arrow[uur, Rightarrow] & & \text{semi-primal} \arrow[u, Rightarrow] \\
& \text{primal} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Figure 2.5: Hierarchy of sub-variants of quasi-primality.

While they coincide in the semi-primal case, for quasi-primal varieties the dualities obtained by the theory of natural dualities [CD98] and in terms of sheaves [KW74] differ. In the following, we recall the latter.

Definition 2.4.2 (The category $\mathsf{Stone}_{\mathsf{Iso}(\mathbf{Q})}$)**.** Let $\mathbf{Q}$ be a quasi-primal algebra. We define a category $\mathsf{Stone}_{\mathsf{Iso}(\mathbf{Q})}$ as follows. Objects are $(X, \mathbf{v})$ precisely as in the case of $\mathsf{Stone}_{\mathbf{D}}$, that is, $\mathbf{v}\colon X\colon \mathbb{S}(\mathbf{Q})$. A morphism $(X, \mathbf{v}) \to (Y, \mathbf{w})$ is a map $f\colon X \to Y$ and a choice of homomorphism $f_x\colon \mathbf{w}(f(x)) \to \mathbf{v}(x)$ for every $x \in X$.

This generalizes Definition 2.1.2, which is recovered if there are no non-trivial internal isomorphisms (*i.e.*, if the algebra is semi-primal). In [KW74] it is shown that the variety $\mathcal{A} := \mathbb{HSP}(\mathbf{Q})$ is dually equivalent to $\mathsf{Stone}_{\mathsf{Iso}(\mathbf{Q})}$.

Of course, there still is an obvious forgetful functor $\mathsf{U}\colon \mathsf{Stone}_{\mathsf{Iso}(\mathbf{Q})} \to \mathsf{Stone}$ which, however, is usually (*i.e.*, in the presence of internal isomorphisms) not faithful and does not have a left-adjoint anymore (because it does not preserve equalizers).

Nevertheless, the subalgebra adjunctions can be generalized differently. The functor $\mathsf{V}^{\mathbf{S}}\colon \mathsf{Stone} \to \mathsf{Stone}_{\mathsf{Iso}(\mathbf{Q})}$ is essentially defined as in the semi-primal case, taking an object X to $(X, \mathbf{v}^{\mathbf{S}})$ and a morphism $f\colon X \to Y$ to itself with $\mathsf{id}_{\mathbf{S}}$ in every 'component' $x \in X$. The right-adjoint of this functor is $\mathsf{C}^{\mathbf{S}}\colon \mathsf{Stone}_{\mathsf{Iso}(\mathbf{Q})} \to \mathsf{Stone}$ which is defined on objects by

$$\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) = \coprod_{x \in X} \mathcal{A}\big(\mathbf{v}(x), \mathbf{S}\big)$$

and it sends a morphism $f_x\colon \mathbf{w}(f(x)) \to \mathbf{v}(x)$ to

$$\begin{aligned} \mathsf{C}^{\mathbf{S}}f\colon \coprod_{x \in X} \mathcal{A}\big(\mathbf{v}(x), \mathbf{S}\big) &\to \coprod_{y \in Y} \mathcal{A}\big(\mathbf{w}(y), \mathbf{S}\big) \\ \big(\varphi\colon \mathbf{v}(x) \to \mathbf{S}\big) &\mapsto \big(\varphi \circ f_x\colon \mathbf{w}(f(x)) \to \mathbf{S}\big). \end{aligned}$$

Let $\mathcal{A}^T$ be the variety generated by $\mathbf{Q}^T$, the '*semi-primalification*' of $\mathbf{Q}$, which is obtained by adding all T_q to the signature of $\mathbf{Q}$ as primitives. Then there is an adjunction between $\mathsf{Stone}_{\mathbf{Q}^T}$ and $\mathsf{Stone}_{\mathsf{Iso}(\mathbf{Q})}$ defined similarly to the one above and the 'quasi-primal' subalgebra adjunctions are obtained from this adjunction and the 'semi-primal' subalgebra adjunctions. We leave the details and further ramifications of these claims for future research.

Another related albeit fairly unexplored area is yet another variant of primality, which might be considered 'orthogonal' to semi-primality in some sense (replacing subalgebras by quotients). A finite algebra $\mathbf{H}$ is *hemi-primal* [Fos70] if every operation which preserves congruences is term-definable in $\mathbf{H}$. To the best of the authors knowledge, no duality for varieties generated by hemi-primal algebras exist as of yet. Thus, we pose this as an open problem, which could perhaps be solved by establishing first a finite duality, and extending it to the entire variety similarly to what was done in Section 2.1.

Lastly, we mention another category-theoretical approach to universal algebra, which has not been discussed in this paper, namely via Lawvere theories. For example, primality and Hu's Theorem has been analyzed in this context in [Por00]. In future work, one could also try to gain more insight into variants of primality in this context.

Chapter 3

Many-valued modal logic over semi-primal varieties

> *There is not only a close analogy between the operations of the mind in general and its operations in the particular science of Algebra, but there is to a considerable extent an exact agreement in the laws by which the two classes of operations are conducted.*
>
> – George Boole
> (1854)[1]

In this chapter, we begin our study of many-valued modal logic under the assumption that the algebra **D** of truth-degrees is semi-primal. In particular, we introduce (crisp) **D**-frames, which can be seen as Kripke frames with local preconditions on their valuations. We show that the corresponding modal logic has the Hennessy-Milner property. Furthermore, using an algebraic approach, we provide an algebraic completeness result and an appropriate version of the Goldblatt-Thomason theorem for **D**-frames.

Many of the results directly proved here are also consequences of more general results about many-valued coalgebraic logics from the next chapter. Therefore, this chapter can be seen as set-up for (or special case of) the study conducted in the next chapter. Nevertheless, we do not rely on coalgebraic methods in this chapter and present more 'standard' proofs of many results, which are interesting in their own right.

With the exception of Section 3.1, throughout this chapter we still always work under the main assumption from the previous chapter (*i.e.*, Assumption 2.0.1).

The chapter is structured as follows. In Section 3.1, we give a general

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[1] [Boo54, Chapter 1]

introduction to many-valued modal logic on crisp frames. In Section 3.2, we focus on the case where the algebra of truth-degrees is semi-primal. We introduce the more general semantics over **D**-frames and **D**-models (Subsection 3.2.1) and show that their logics all have the Hennessy-Milner property (Subsection 3.2.2). In Section 3.3, we focus on the algebraic semantics of these logics. We provide an algebraic completeness theorem (Subsection 3.3.1) and a many-valued analogue of the Goldblatt-Thomason theorem (Subsection 3.3.2).

Nowadays, there exists a plethora of literature about many-valued modal logic (see, *e.g.*, [Fit91, Pri08b, CR10, BEGR11, VEG17, MM18]). This chapter is mostly influenced by prior work in this area conducted by Hansoul and Teheux [HT13, Teh16] and Maruyama [Mar09, Mar12]. The papers [HT13, Teh16] subsume the only previous instance of many-valued modal logic where **D**-frames and **D**-models are considered, namely in the case where $\mathbf{D} = \mathbf{Ł}_n$ is a finite MV-chain. In particular, our study of definability in Subsection 3.3.2 is a direct generalization of [Teh16]. The papers [Mar09, Mar12] provide inspiration for the algebraic semantics we consider in Section 3.3. In particular, the paper [Mar12] is the first instance where arbitrary semi-primal algebras of truth-degrees are considered, and the 'Kripke condition' proved therein constitutes the heart of the Truth Lemma used herein (see Lemma 3.3.6). However, while Maruyama only considers the $\Box$-fragment of our logic, we consider the full logic containing both the $\Box$- and $\Diamond$-modality. In the absence of a De Morgan involution in **D**, we make use of Dunn's axioms for *positive modal logic* [Dun95] (also see Subsection 5.3.1) to describe the interplay of these two modalities.

From now on, we assume familiarity with basic concepts and terminology of modal logic (*e.g.*, Kripke frame/model, truth and validity, bounded morphism, bisimiliarity, modal definability, . . .). For an introduction to and good overview of modal logic, we refer the reader to the book [BdRV01] (the book [BvBW07] contains a lot of additional material).

3.1 Introduction to many-valued modal logic

In this section, we recall the basic definitions in many-valued modal logic over crisp frames, as well as some prior results in this area. From Section 3.2 onwards, we exclusively consider many-valued modal logic with a semi-primal lattice-expansion as algebra of truth-degrees.

According to Priest [Pri08b], the first studies of many-valued (more specifically, three-valued) modal logics were conducted by Segerberg [Seg67]. Some other early instances are [Tho78, Mor79, Ost88]. There appears to be a broad consensus that the topic was re-introduced and popularized by Fitting's papers [Fit91, Fit92], where many-valued modal logics over finite Heyting algebras (and, for the first time, frames with many-valued accessibility relations) were studied. A good introduction to many-valued modal logic in general is provided in [BEGR11].

For now we only assume that $\mathbf{D}$ is an arbitrary algebra with a complete bounded lattice reduct. We will consider the following three modal languages defined by $\mathbf{D}$. Throughout this chapter, we fix a countable set Prop of *propositional variables.*

Definition 3.1.1 (Modal languages/formulas over $\mathbf{D}$)**.** The *full modal language over* $\mathbf{D}$, denoted $\mathcal{L}_{\mathbf{D}}^{\Box\Diamond}$ consists of the signature of $\mathbf{D}$ and two unary operation symbols $\{\Box, \Diamond\}$. The set $\mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$ of *modal* $\mathbf{D}$*-formulas modal* $\mathbf{D}$*-formulas* is inductively defined by

$$\varphi ::= p \in \mathsf{Prop} \mid o(\varphi_1, \dots, \varphi_n) \mid \Box\varphi \mid \Diamond\varphi,$$

where o ranges over all primitive operations of $\mathbf{D}$. The $\Box$*-modal language over* $\mathbf{D}$, denoted $\mathcal{L}_{\mathbf{D}}^{\Box}$, is the $\Diamond$-free reduct of $\mathcal{L}_{\mathbf{D}}^{\Box\Diamond}$. Similarly, the $\Diamond$*-modal language over* $\mathbf{D}$, denoted $\mathcal{L}_{\mathbf{D}}^{\Diamond}$, is the $\Box$-free reduct of $\mathcal{L}_{\mathbf{D}}^{\Box\Diamond}$. The corresponding subsets of formulas are the sets of $\Box$*-modal* $\mathbf{D}$*-formulas* $\mathsf{Form}_{\mathbf{D}}^{\Box}$ and $\Diamond$*-modal* $\mathbf{D}$*-formulas* $\mathsf{Form}_{\mathbf{D}}^{\Diamond}$.

Since it is usually clear from the context, we often omit the subscripts in the above definition. Note that $\top$, $\bot$, $\varphi_1 \wedge \varphi_2$ and $\varphi_1 \vee \varphi_2$ always are well-defined $\mathbf{D}$-modal formulas since we assume that $\mathbf{D}$ is based on a bounded lattice. In the case where $\mathbf{D}$ is even based on a $\mathsf{FL}_{\mathrm{ew}}$-algebra, we can also build formulas $\varphi_1 \to \varphi_2$ and $\varphi_1 \odot \varphi_2$.

We interpret formulas via *Kripke semantics*. Recall that a *Kripke frame* (or *crisp frame* or simply *frame*) is a relational structure (X, R), where X is a set of *worlds* or *states* and $R \subseteq X^2$ is its binary *accessibility relation*. To define validity of modal formulas, we endow these frames with valuations taking values in $\mathbf{D}$ as follows.

Definition 3.1.2 ($\mathbf{D}$-valued model)**.** A $\mathbf{D}$*-valued model* is a triple (X, R, Val) where (X, R) is a Kripke frame and

$$\mathsf{Val}\colon X \times \mathsf{Prop} \to \mathbf{D}$$

is a map, which we call the *valuation* of the model.

If (X, R, Val) is a $\mathbf{D}$-valued model, then the valuation can be extended to all formulas of $\mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$, by inductively defining

$$\mathsf{Val}\big(x, o(\varphi_1, \dots, \varphi_n)\big) = o^{\mathbf{D}}\big(\mathsf{Val}(x, \varphi_1), \dots \mathsf{Val}(x, \varphi_n)\big)$$

for all primitive operations o of the algebra $\mathbf{D}$ and

$$\mathsf{Val}(x, \Box\varphi) = \bigwedge_{xRx'} \mathsf{Val}(x', \varphi) \text{ and } \mathsf{Val}(x, \Diamond\varphi) = \bigvee_{xRx'} \mathsf{Val}(x', \varphi),$$

where we stipulate $\bigwedge \varnothing = 1$ and $\bigvee \varnothing = 0$.

Local truth is now defined as follows. Given a modal $\mathbf{D}$-formula $\varphi \in \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$ and a $\mathbf{D}$-valued model $\mathfrak{M} = (X, R, \mathsf{Val})$, we say that φ is *true* at state $x \in X$ in $\mathfrak{M}$ if and only if

$$\mathsf{Val}(x, \varphi) = 1.$$

We denote this by $\mathfrak{M}, x \Vdash \varphi$. As usual, we write $\mathfrak{M} \Vdash \varphi$ if φ is true at *every* state of $\mathfrak{M}$.

Validity in a frame $\mathfrak{F} = (X, R)$ is also defined in the usual way. That is, we say that φ is *valid* at a state $x \in X$ in $\mathfrak{F}$ if and only if it is true at x in every $\mathbf{D}$-valued model based on $\mathfrak{F}$. We denote this by $\mathfrak{F}, x \models \varphi$. We write $\mathfrak{F} \models \varphi$ if this formula is is valid at every state $x \in X$ and say that φ is *valid* in $\mathfrak{F}$. Now we define the minimal $\mathbf{D}$-valued modal logic (on crisp frames) as follows.

Definition 3.1.3 ($\mathbf{D}$-valued modal logics)**.** The *full* $\mathbf{D}$*-valued modal logic* $\Lambda_{\mathbf{D}}^{\Box\Diamond}$ on crisp frames is the set of all modal $\mathbf{D}$-formulas which are valid in every frame. The $\mathbf{D}$*-valued* $\Box$*-modal logic* is $\Lambda_{\mathbf{D}}^{\Box} := \Lambda_{\mathbf{D}}^{\Box\Diamond} \cap \mathsf{Form}_{\mathbf{D}}^{\Box}$ and the $\mathbf{D}$*-valued* $\Diamond$*-modal logic* is $\Lambda_{\mathbf{D}}^{\Diamond} := \Lambda_{\mathbf{D}}^{\Diamond\Box} \cap \mathsf{Form}_{\mathbf{D}}^{\Diamond}$.

As our careful distinctions indicate, $\Box$ and $\Diamond$ are not necessarily interdefinable in the many-valued setting. However, it is possible in the usual way if there is a *De Morgan involution* $\neg\colon \mathbf{D} \to \mathbf{D}$ which is term-definable in $\mathbf{D}$, in which case the above distinction becomes redundant. Here, by an *involution* we mean a bijection f which satisfies $f^2 = \mathsf{id}$ and we say it is a *De Morgan involution* if it satisfies the De Morgan laws $f(x \wedge y) = f(x) \vee f(y)$ and $f(x \vee y) = f(x) \wedge f(y)$. For example, this is the case in the standard MV-algebra $\mathbf{Ł}$ and all its subalgebras (see Subsection 1.2.2), or the four-element implicative bilattice **FOUR** (see Definition 1.2.2). The modal logics $\Lambda_{\mathbf{Ł}}^{\Box}$ and $\Lambda_{\mathbf{Ł}_n}^{\Box}$ were described by Hansoul and Teheux in [HT13]. A modal logic similar to $\Lambda_{\mathbf{FOUR}}^{\Box}$ was studied by Rivieccio, Jung and Jansana [Riv10, JR13, RJJ17]. An example of an algebra which does *not* have a term-definable involution is the standard Gödel chain $\mathbf{G}$ based on $[0, 1]$. Here, the logic $\Lambda_{\mathbf{G}}^{\Box\Diamond}$ can not be immediately obtained from $\Lambda_{\mathbf{G}}^{\Box}$ and $\Lambda_{\mathbf{G}}^{\Diamond}$. In fact, it has only recently been axiomatized by Rodriguez and Vidal in [RV21] (the asymmetry between Gödel modal logics has also been studied in [CR10]).

We end this subsection by putting the framework considered here into the broader context of many-valued modal logics. There are two more possible generalizations to be made.

First, while for us only the top-element 1 of $\mathbf{D}$ is used in the definition of modal satisfaction, one could consider other sets of *designated truth-values* $A \subseteq D$ and define

$$(X, R, \mathsf{Val}) \Vdash^{A} \varphi \Leftrightarrow \mathsf{Val}(w, \varphi) \in A.$$

For example, in the case of **FOUR** it is more customary to have both $\top$ and $\mathtt{t}$ as designated truth-values. However, in case where $\mathbf{D}$ is semi-primal, the relation $\Vdash^{A}$ can always be recovered from the relation $\Vdash$ via

$$\mathfrak{M}, x \Vdash^{A} \varphi \Leftrightarrow \mathfrak{M}, x \Vdash \chi_{A}(\varphi),$$

where $\chi_A \colon \mathbf{D} \to \mathbf{D}$ is the characteristic function of A (with $\{0,1\}$ considered as subset of $\mathbf{D}$), which is term-definable in $\mathbf{D}$.

The second generalization is to consider $\mathbf{D}$-*labelled frames* as well, that is, to have $\mathbf{D}$-valued accessibility relations $R \colon X^2 \to \mathbf{D}$ instead of crisp ones. While it does not seem clear how to interpret $\Box$ and $\Diamond$ in this case over arbitrary complete bounded lattices, in the case where $\mathbf{D}$ is a $\mathsf{FL}_{\mathsf{ew}}$-algebra, valuations are usually extended via the rules

$$\mathsf{Val}(x, \Box\varphi) = \bigwedge_{x' \in X} R(x, x') \to \mathsf{Val}(x', \varphi)$$

and

$$\mathsf{Val}(x, \Diamond\varphi) = \bigvee_{x' \in X} R(x, x') \odot \mathsf{Val}(x', \varphi)$$

for the modalities.

For a good overview of many-valued modal logic over finite $\mathsf{FL}_{\mathsf{ew}}$-algebras in this generality, we refer the reader to [BEGR11]. In particular, the $\Box$-fragment of this logic is axiomatized therein for the case where $\mathbf{D} = \mathbf{Ł}_n$ is a finite MV-chain [BEGR11, Subsection 5.1] (also see [BCR22] for an algebraic treatment). Since this axiomatization only depends on the unary terms τ_d, analogous results can be obtained for any semi-primal $\mathsf{FL}_{\mathsf{ew}}$-algebra.

3.2 Semi-primal algebras of truth-degrees

From now on, we exclusively study $\mathbf{D}$-valued modal logics for the case where the algebra $\mathbf{D}$ of truth-degrees is semi-primal, that is, we work under Assumption 2.0.1 again.

3.2.1 Frames and models with preconditions

Motivated by results from Chapter 2, we will not only consider Set-based relational structures, but also ones which are based on $\mathsf{Set}_{\mathbf{D}}$ (recall Definition 2.3.1), which we call (crisp) $\mathbf{D}$-frames. One can think of those as frames with local preconditions on possible valuations. Usual frames then correspond to $\mathbf{D}$-frames with 'trivial' preconditions.

We begin with the definition of $\mathbf{D}$-frames. In the case $\mathbf{D} = \mathbf{Ł}_n$, these frames have been introduced in [HT13, Definition 7.2] (and have also been considered in [Teh16]).

Definition 3.2.1 ($\mathbf{D}$-frame)**.** A *crisp* $\mathbf{D}$*-frame* or *Kripke* $\mathbf{D}$*-frame* or simply $\mathbf{D}$*-frame* is a triple (X, v, R) such that

- $(X, v) \in \mathsf{Set}_{\mathbf{D}}$, that is, $v\colon X \to \mathbb{S}(\mathbf{D})$,
- (X, R) is a frame,

and these two structures satisfy the *compatibility condition*

$$xRx' \Rightarrow v(x') \leq v(x)$$

for all $x, x' \in X$.

The logical significance of v and the compatibility condition from the above definition become clear at the level of models.

Definition 3.2.2 ($\mathbf{D}$-model)**.** A $\mathbf{D}$*-model* is a quadruple (X, v, R, Val) where (X, v, R) is a $\mathbf{D}$-frame and (X, R, Val) is a $\mathbf{D}$-valued model which satisfies

$$\mathsf{Val}(x, p) \in v(x)$$

for all states $x \in X$ and propositional variables $p \in \mathsf{Prop}$.

These valuations can be inductively extended to all formulas $\varphi \in \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$ as in the case of $\mathbf{D}$-valued models (see the discussion after Definition 3.1.2). The extended valuation

$$\mathsf{Val}\colon X \times \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond} \to \mathbf{D}$$

still always satisfies $\mathsf{Val}(x, \varphi) \in v(x)$, which can be inductively seen as follows. On formulas of the form $\varphi = o(\varphi_1, \dots, \varphi_n)$ with a primitive n-ary operation o of $\mathbf{D}$, assuming that $\mathsf{Val}(x, \varphi_i) \in v(w)$ for all $i = 1, \dots, n$, we have

$$\mathsf{Val}(x, \varphi) = o^{\mathbf{D}}(\mathsf{Val}(x, \varphi_1), \dots, \mathsf{Val}(x, \varphi_n)) \in v(x),$$

because $v(x) \in \mathbb{S}(\mathbf{D})$ is a subalgebra, thus closed under $o^{\mathbf{D}}$. For the $\Box$-modality, we have

$$\mathsf{Val}(x, \Box\varphi) = \bigwedge_{xRx'} \mathsf{Val}(x', \varphi) \in v(x),$$

since, due to the compatibility condition on **D**-frames, we have

$$\mathsf{Val}(x', \varphi) \in v(x') \leq v(x) \text{ whenever } xRx'.$$

Therefore, $\mathsf{Val}(x, \Box\varphi)$ is either $1 \in v(x)$, or a meet of elements of $v(x)$ (in fact, a finite one since **D** is finite) and thus contained in $v(x)$, because it is a subalgebra of **D**. Thus, we can think of **D**-frames as frames which have preconditions, in the sense that they only allow certain valuations.

Besides being more general, the following two reasons justify working with **D**-frames and **D**-models. First, in light of Section 2.3, we know that canonical extensions of algebras in $\mathcal{A}$ can be naturally identified with members of $\mathsf{Set}_{\mathbf{D}}$. Therefore, the *canonical model* also naturally fits into this environment. Indeed, as we see later in Subsection 3.3.1, the canonical model really is a **D**-model in a natural way. Secondly, the definition of **D**-frame and **D**-model is 'correct' from a coalgebraic perspective, as we show later on. For example, we will show that **D**-frames are exactly the coalgebras for a naturally defined lifting $\mathcal{P}' \colon \mathsf{Set}_{\mathbf{D}} \to \mathsf{Set}_{\mathbf{D}}$ of the powerset functor $\mathcal{P} \colon \mathsf{Set} \to \mathsf{Set}$ to the category $\mathsf{Set}_{\mathbf{D}}$ (see Example 4.3.18).

The validity relation on **D**-frames is defined as expected, that is, we say that a formula φ is *valid* in a **D**-frame if and only if it is true in every **D**-model based on it. Note that validity in a frame (X, R) coincides with validity in the corresponding **D**-frame $(X, v^{\mathbf{D}}, R)$, where $v^{\mathbf{D}}$ is constant $v(x) = \mathbf{D}$ (as in the definition of $\mathsf{V}^{\mathbf{D}}$ in Subsection 2.2.1).

As an example, we consider the case $\mathbf{D} = \mathbf{Ł}_n$ (with $n \geq 2$) and the formula

$$\varphi = \Diamond(p \vee \neg p).$$

This formula is valid in every $\mathbf{Ł}_n$-frame (X, v, R) which satisfies

$$\forall x \exists x' : \big(xRx' \wedge v(x') = \mathbf{Ł}_1\big),$$

that is, every state has some crisp successor (identifying $\mathbf{Ł}_1$ with $\mathbf{2}$). Meanwhile, frames without preconditions (X, R) never satisfy this formula, since any constant valuation $\mathsf{Val}(x, p) = \ell \in \mathbf{Ł}_n \backslash \{0, 1\}$ yields a counter-model.

This example illustrates that, in general, the validity relation defined here can not be reduced to the one defined in the previous subsection. Indeed, the generalization to **D**-frames opens up some new questions, for example about modal definability. Nevertheless, it is easy to see that the set $\Lambda^{\Box\Diamond}$ of formulas valid in *all* frames (recall Definition 3.1.3), coincides with the set of formulas valid in all **D**-frames.

Proposition 3.2.3. *A modal* **D***-formula* φ *is in* $\Lambda^{\Box\Diamond}$ *if and only if it is valid in all* **D***-frames.*

Proof. Suppose that φ is valid in every **D**-frame. Then, in particular, it is valid in every frame, since a frame (W, R) can be identified with the corresponding **D**-frame $(X, v^{\mathbf{D}}, R)$ as described above.

Conversely, suppose that φ is not valid in every **D**-frame. Then there exists a **D**-model (X, R, v, Val) with $\mathsf{Val}(x, \varphi) \neq 1$ for some $x \in X$. But then the same is true for the model (X, R, Val), so φ is not valid in all frames. □

To end this subsection, we introduce the appropriate notion of morphism between **D**-frames. First, recall that a *bounded morphism* or *p-morphism* between frames $(X_1, R_1) \to (X_2, R_2)$ is a map $f\colon X_1 \to X_2$ which satisfies the two conditions

- if $x_1 R_1 x_1'$, then $f(x_1) R_2 f(x_1')$ and
- if $f(x_1) R_2 x_2'$, there exists x_1' with $x_1 R x_1'$ and $f(x_1') = x_2'$.

A bounded morphism between **D**-frames now simply has to be both a bounded morphism and a morphism in $\mathsf{Set}_{\mathbf{D}}$ at the same time.

Definition 3.2.4 (Bounded **D**-morphism)**.** For two **D**-frames (X_1, v_1, R_1) and (X_2, v_2, R_2), a *bounded* **D***-morphism* $(X_1, v_1, R_1) \to (X_2, v_2, R_2)$ is a map $f\colon X_1 \to X_2$ which is a bounded morphism $(X_1, R_1) \to (X_2, R_2)$ and a $\mathsf{Set}_{\mathbf{D}}$-morphism $(X_1, v_1) \to (X_2, v_2)$, that is,

$$v_2\big(f(x)\big) \leq v_1(x)$$

holds for all $x \in X_1$.

A bounded **D**-morphism between **D**-models

$$f\colon (X_1, v_1, R_1, \mathsf{Val}_1) \to (X_2, v_2, R_2, \mathsf{Val}_2)$$

additionally satisfies

$$\mathsf{Val}_1(x, p) = \mathsf{Val}_2\big(f(x), p\big)$$

for all $x \in X_1$ and $p \in \mathsf{Prop}$.

We show that this definition makes sense from a logical perspective, that is, bounded **D**-morphisms preserve truth in **D**-models.

Proposition 3.2.5. *Let f be a bounded $\mathbf{D}$-morphism between $\mathbf{D}$-models $\mathfrak{M}_1 = (X_1, v_1, R_1, \mathsf{Val}_1)$ and $\mathfrak{M}_2 = (X_2, v_2, R_2, \mathsf{Val}_2)$, and let $\varphi \in \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$ be a modal $\mathbf{D}$-formula. Then we have*

$$\mathsf{Val}_1(x, \varphi) = \mathsf{Val}_2\big(f(x), \varphi\big)$$

for all $x \in X_1$. In particular, this implies

$$\mathfrak{M}_1, x \models \varphi \Leftrightarrow \mathfrak{M}_2, f(x) \models \varphi$$

for all $x \in X_1$.

Proof. By induction on the formula φ. If $\varphi = p \in \mathsf{Prop}$ is a propositional variable, then the statement holds by definition. If $\varphi = o(\varphi_1, \dots, \varphi_n)$, the statement follows by induction. Now consider the case $\varphi = \Box\psi$. Then, by definition we have

$$\mathsf{Val}_1(x, \Box\psi) = \bigwedge_{xR_1x_1'} \mathsf{Val}_1(x_1', \psi) =: d_1$$

and

$$\mathsf{Val}_2\big(f(x), \Box\psi\big) = \bigwedge_{f(x)R_2x_2'} \mathsf{Val}_2(x_2', \psi) =: d_2.$$

By the first condition on bounded morphisms and the inductive assumption, we get

$$d_2 \leq \bigwedge_{xR_1x_1'} \mathsf{Val}_2\big(f(x_1'), \psi\big) = \bigwedge_{xR_1x_1'} \mathsf{Val}(x_1', \psi) = d_1.$$

Conversely, by the second condition on bounded morphisms and the inductive assumption we get

$$d_2 \geq \bigwedge_{xR_1x_1'} \mathsf{Val}_2\big(f(x_1'), \psi\big) = d_1$$

thus d_1 and d_2 coincide. The case for $\varphi = \Diamond\psi$ is similar. ☐

In the next subsection, we investigate modal equivalence and bisimilarity in the setting described here.

3.2.2 A many-valued Hennessy-Milner property

In this subsection, we show that image-finite $\mathbf{D}$-models have the Hennessy-Milner property [HM80, HM85], meaning that two states are modally equivalent if and only if they are bisimilar. The proof is inspired by [MM18], where expressivity for chain-based many-valued modal logics is studied. However, we do not require the assumption that $\mathbf{D}$ is linearly ordered here. We start with the basic definitions in the many-valued case, for an overview of the Hennessy-Milner property for classical modal logic we refer the reader to [BdRV01, Section 2.2].

Definition 3.2.6 (Modal equivalence)**.** Let $\mathfrak{M}_1 = (X_1, v_1, R_1, \mathsf{Val}_1)$ and $\mathfrak{M}_2 = (X_2, v_2, R_2, \mathsf{Val}_2)$ be two $\mathbf{D}$-models. We say that $x_1 \in X_1$ and $x_2 \in X_2$ are *fully modally equivalent* if

$$\mathsf{Val}_1(x_1, \varphi) = \mathsf{Val}_2(x_2, \varphi)$$

holds for all $\varphi \in \mathsf{Form}^{\Box\Diamond}_{\mathbf{D}}$. They are $\Box$-*modally equivalent* if the above holds for all $\varphi \in \mathsf{Form}^{\Box}_{\mathbf{D}}$ and $\Diamond$-*modally equivalent* if it holds for all $\varphi \in \mathsf{Form}^{\Diamond}_{\mathbf{D}}$.

Bisimulations are also defined analogously to the classical case.

Definition 3.2.7 (Bisimulation)**.** Let $\mathfrak{M}_1 = (X_1, v_1, R_1, \mathsf{Val}_1)$ and $\mathfrak{M}_2 = (X_2, v_2, R_2, \mathsf{Val}_2)$ be two $\mathbf{D}$-models. A non-empty binary relation $B \subseteq X_1 \times X_2$ is a *bisimulation* if the following three conditions hold:

- If $x_1 B x_2$, then $\mathsf{Val}_1(x_1, p) = \mathsf{Val}_2(x_2, p)$ for all $p \in \mathsf{Prop}$.
- (*Back condition*) If $x_1 B x_2$ and $x_2 R_2 x_2'$, then there exists some $x_1' B x_2'$ with $x_1 R_1 x_1'$.
- (*Forth condition*) If $x_1 B x_2$ and $x_1 R_1 x_1'$, then there exists some $x_1' B x_2'$ with $x_2 R_2 x_2'$.

We call two states $x_1 \in X_1$ and $x_2 \in X_2$ *bisimilar* if there exists a bisimulation with $x_1 B x_2$.

For example, the graph of every bounded $\mathbf{D}$-morphism is a bisimulation. In general, bisimilarity is stronger than modal equivalence.

Proposition 3.2.8. *Let* $\mathfrak{M}_1 = (X_1, v_1, R_1, \mathsf{Val}_1)$ *and* $\mathfrak{M}_2 = (X_2, v_2, R_2, \mathsf{Val}_2)$ *be two* $\mathbf{D}$*-models. If* x_1 *and* x_2 *are bisimilar, then they are fully modally equivalent.*

Proof. Let $B \subseteq X_1 \times X_2$ be a bisimulation with $x_1 B x_2$. Then, by definition we know that $\mathsf{Val}_1(x_1, p) = \mathsf{Val}(x_2, p)$ for all $p \in \mathsf{Prop}$. This can be inductively extended to all modal $\mathbf{D}$-formulas $\varphi \in \mathsf{Form}^{\Box\Diamond}_{\mathbf{D}}$, completely analogous to the proof of Proposition 3.2.5. $\Box$

As in the classical case, the converse does not hold in general. However, the *Hennessy-Milner property* asserts that modal equivalence and bisimiliarity coincide on image-finite frames, where we call a **D**-frame (X, v, R) or frame (X, R) *image-finite* if, for all $x \in X$, the set of successors

$$R[x] = \{x' \in X \mid xRx'\}$$

is finite (a **D**-model or model is image-finite if its underlying **D**-frame or frame is image-finite, respectively).

Our proof of this property is similar to the one of [MM18, Theorem 1], with the exception that therein only chain-based algebras and full modal equivalence are considered. The proof is based on the fact that we can distinguish between distinct truth-values by certain formulas. More specifically, we make use of the term-definable unary operations

$$T_d(x) = \begin{cases} 1 & \text{if } x = d \\ 0 & \text{if } x \neq d \end{cases}$$

from Theorem 1.1.12 again. Since they are term-definable in **D**, it makes sense to consider modal **D**-formulas of the form $T_d(\varphi)$.

Theorem 3.2.9 (Hennessy-Milner property)**.** *Let* $\mathfrak{M}_1 = (X_1, v_1, R_1, \mathsf{Val}_1)$ *and* $\mathfrak{M}_2 = (X_2, v_2, R_2, \mathsf{Val}_2)$ *be two image-finite* **D***-models, and let* $x_1 \in X_1$ *and* $x_2 \in X_2$.

(1) If x_1 *and* x_2 *are* $\Box$*-modally equivalent, then they are bisimilar.*

(2) If x_1 *and* x_2 *are* $\Diamond$*-modally equivalent, then they are bisimilar.*

(3) If x_1 *and* x_2 *are fully modally equivalent, then they are bisimilar.*

Proof. To prove (1), we show that $\Box$-modal equivalence is itself a bisimulation. Assume towards contradiction that the forth-condition does not hold (the other case where the back-condition does not hold is analogous). Then there is $x_1' \in X_1$ with $x_1 R_1 x_1'$ and no element of the finite set of successors

$$R_2[x_2] = \{y_1', \dots, y_n'\}$$

is modally equivalent to x_1'. If this set is empty, consider the formula $\varphi = \Box 0$. Then $\mathsf{Val}_2(x_2, \varphi) = 1$ while $\mathsf{Val}(x_1, \varphi) = 0$ (since x_1 has at least one successor x_1'), contradicting modal equivalence between the two. Otherwise, proceed as follows. For all $i = 1, \dots, n$, let $\varphi_i \in \mathsf{Form}_{\mathbf{D}}^{\Box}$ be a modal formula which witnesses that x_1' and y_n' are not $\Box$-modally equivalent, say

$$\mathsf{Val}_2(y_i', \varphi_i) =: d_i \neq e_i := \mathsf{Val}_1(x_1', \varphi_i)$$

for all $i = 1, \dots, n$. Now consider the formula

$$\varphi = \Box\big(T_{d_1}(\varphi_1) \vee \dots \vee T_{d_n}(\varphi_n)\big).$$

Since $x_1 R_1 x_1'$ holds, we know that

$$\mathsf{Val}_1(x_1, \varphi) \leq \mathsf{Val}(x_1', \varphi) = T_{d_1}(e_1) \vee \dots \vee T_{d_n}(e_n) = 0.$$

On the other hand, we have

$$\mathsf{Val}_2(x_2, \varphi) = \bigwedge_{i=1,\dots,n} \big(T_{d_1}(\mathsf{Val}_2(y_i, \varphi_1)) \vee \dots \vee T_{d_n}(\mathsf{Val}_2(y_i, \varphi_n))\big) = 1,$$

again contradicting modal equivalence of x_1 and x_2. This finishes the proof of statement (1).

The argument for statement (2) is similar, except that we use $\varphi = \Diamond 1$ in the case where $R_2[x_2] = \varnothing$ and

$$\varphi = \Diamond\big(T_0(T_{d_1}(\varphi_1)) \wedge \dots \wedge T_0(T_{d_n}(\varphi_n))\big)$$

otherwise. Since statement (3) is an immediate consequence of both (1) and (2), this finishes the proof. □

Extending the work of [MM18], expressivity of many-valued modal logics is also studied in [BD16] from a coalgebraic perspective (via the predicate lifting approach).

In the next section, we begin the algebraic study of modal **D**-valued logics.

3.3 Algebraic framework

In this section, we study many-valued modal logics over semi-primal algebras algebraically. The main results are an algebraic completeness theorem (Theorem 3.3.7) and an analogue of the Goldblatt-Thomason Theorem [GT75] in our setting (Theorem 3.3.24).

3.3.1 Canonical model and completeness

In this section, we introduce the algebraic counterparts to the many-valued modal logics over semi-primal algebras introduced in the previous sections. Recall that $\mathcal{A}$ denotes the variety $\mathbb{HSP}(\mathbf{D})$ (also see Assumption 2.0.1).

Recall that, for all $d \in \mathbf{D}$, the unary operations

$$\tau_d(x) = \begin{cases} 1 & \text{if } x \geq d \\ 0 & \text{if } x \ngeq d \end{cases} \qquad \text{and} \qquad \eta_d(x) = \begin{cases} 0 & \text{if } x \leq d \\ 1 & \text{if } x \nleq d \end{cases}$$

are term-definable in **D** (see Theorem 1.1.12). Therefore, the following defines a variety of algebras.

Definition 3.3.1 (Modal $\mathcal{A}$-algebra)**.** A *modal* $\mathcal{A}$*-algebra* or $\mathcal{A}^{\Box\Diamond}$*-algebra* is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$, where $\mathbf{A} \in \mathcal{A}$ and $\Box, \Diamond \colon A \to A$ are unary operations satisfying the equations

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3) $\tau_d(\Box x) = \Box \tau_d(x)$ for all $d \neq 0$,

(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$,

(D1) $\Diamond 0 = 0$,

(D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,

(D3) $\eta_d(\Diamond x) = \Diamond \eta_d(x)$ for all $d \neq 1$,

(P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

We denote the variety of modal $\mathcal{A}$-algebras by $\mathcal{A}^{\Box\Diamond}$.

Note that the first two 'box-axioms' (B1)-(B2) and, equivalently, the first two 'diamond-axioms' (D1)-(D2), are the usual axioms defining modal algebras in the classical case. Since we do not assume that **D** has a De Morgan involution, we need to consider these two operators separately. To make sure they correspond to the same relation, the *positivity-axioms* (P1)-(P2) are added. These axioms are the ones used in Dunn's *positive modal logic* [Dun95] as well (also see Subsection 5.3.1). We now show that, as noted in Section 3.1, if **D** has a De Morgan involution then the operators $\Box$ and $\Diamond$ are inter-definable. In the following, recall that a *De Morgan involution* on **D** is a self-inverse bijection $\neg \colon D \to D$ which satisfies the De Morgan laws $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$.

Proposition 3.3.2. *Suppose that* **D** *has a De Morgan involution* $\neg$ *and let* $\langle \mathbf{A}, \Box, \Diamond \rangle$ *be a* $\mathcal{A}^{\Box\Diamond}$*-algebra. Then*

$$\Diamond a = \neg \Box \neg a$$

holds for all $a \in A$.

Proof. First we show that **D** satisfies the quasi-equations

$$\bigwedge_{d \in D} \tau_d(x) = T_0 \eta_{\neg d}(y) \leftrightarrow (y = \neg x).$$

The direction '$\leftarrow$' is easy to check directly by the definitions involved. We show direction '$\rightarrow$' by contrapositive. Suppose that $y \neq \neg x$. If $y \not\leq \neg x$, take $d = x$ and find

$$\tau_x(x) = 1 \text{ but } T_0\big(\eta_{\neg x}(y)\big) = T_0(1) = 0.$$

If $\neg x \not\leq y$ (equivalently $\neg y \not\leq \neg x$) holds instead, take $d = \neg y$ and find

$$\tau_{\neg y}(x) = 0 \text{ but } T_0(\eta_{\neg\neg y}(y)) = T_0(0) = 1.$$

Therefore, $\mathbf{D}$ satisfies the above quasi-equations, which implies that $\mathbf{A}$ satisfies them as well.

To finish the proof, given $a \in A$, by the above it suffices to show

$$\tau_d(\Box\neg a) = T_0\big(\eta_{\neg d}(\Diamond a)\big)$$

for all $d \in D$. Since $\tau_d(\Box\neg a), \eta_{\neg d}(\Diamond a)$ are both in the Boolean skeleton $\mathfrak{S}(\mathbf{A})$ (recall Definition 2.2.2), this is equivalent to

$$\tau_d(\Box\neg a) \wedge \eta_{\neg d}(\Diamond a) = 0 \text{ and } \tau_d(\Box\neg a) \vee \eta_{\neg d}(\Diamond a) = 1.$$

The first equation follows from

$$\begin{aligned} \tau_d(\Box\neg a) \wedge \eta_{\neg d}(\Diamond a) &= \Box\tau_d(\neg a) \wedge \Diamond\eta_{\neg d}(a) && \text{(B3),(D3)} \\ &\leq \Diamond\big(\tau_d(\neg a) \wedge \eta_{\neg d}(a)\big) && \text{(P2)} \\ &= \Diamond 0 = 0, && \text{(D1)} \end{aligned}$$

and the second one from

$$\begin{aligned} \tau_d(\Box\neg a) \vee \eta_{\neg d}(\Diamond a) &= \Box\tau_d(\neg a) \vee \Diamond\eta_{\neg d}(a) && \text{(B3),(D3)} \\ &\geq \Box\big(\tau_d(\neg a) \vee \eta_{\neg d}(a)\big) && \text{(P1)} \\ &= \Box 1 = 1, && \text{(B1)} \end{aligned}$$

finishing the proof. □

We now show how $\mathbf{D}$-frames and $\mathcal{A}^{\Box\Diamond}$-algebras are related. The following, defined similarly to [Mar12, Definition 3.16], generalizes the *ultrafilter extension* of a modal algebra from the classical case (see, *e.g.*, [BdRV01, Section 5.3]).

Definition 3.3.3 (Canonical $\mathbf{D}$-frame of an $\mathcal{A}^{\Box\Diamond}$-algebra)**.** Let $\langle \mathbf{A}, \Box, \Diamond \rangle$ be an $\mathcal{A}^{\Box\Diamond}$-algebra. The *canonical* $\mathbf{D}$*-frame of* $\langle \mathbf{A}, \Box, \Diamond \rangle$ is given by

$$\langle \mathbf{A}, \Box, \Diamond \rangle^+ := \big(\mathsf{S}'(\mathbf{A}), R_\Box\big),$$

where $\mathsf{S}'(\mathbf{A}) = (\mathcal{A}(\mathbf{A}, \mathbf{D}), \mathsf{im})$ is the functor described explicitly at the beginning of Subsection 4.3.2 and

$$u_1 R_\Box u_2 \Leftrightarrow \forall a \in \mathbf{A}\colon u_1(\Box a) \leq u_2(a).$$

As shown in [Mar12, Lemma 3.17], this really is a $\mathbf{D}$-frame.

In this definition, we used the operator $\Box$ to define $R_\Box$. In the following, we show that this coincides with the natural $\Diamond$-induced relation $u_1 R_\Diamond u_2 \Leftrightarrow u_2(a) \leq u_1(\Diamond a)$ for all $a \in \mathbf{A}$.

Lemma 3.3.4. *Let $\langle \mathbf{A}, \Box, \Diamond \rangle$ be an $\mathcal{A}^{\Box\Diamond}$-algebra and let $u_1, u_2 \in \mathcal{A}(\mathbf{A}, \mathbf{D})$. Then*

$$u_1 R_\Box u_2 \Leftrightarrow \forall a \in \mathbf{A} \colon u_2(a) \leq u_1(\Diamond a)$$

holds in $\langle \mathbf{A}, \Box, \Diamond \rangle^+$.

Proof. For the direction '$\Rightarrow$', suppose towards contradiction that $u_1 R_\Box u_2$ holds but there is some $a \in \mathbf{A}$ with $u_2(a) \not\leq u_1(\Diamond a)$. Let $d = u_1(\Diamond(a))$. On the one hand, by (P1) we have

$$u_1\big(\Box T_0 \eta_d(a) \vee \Diamond \eta_d(a)\big) \geq u_1\big(\Box(T_0\eta_d(a) \vee \eta_d(a))\big) = u_1(\Box 1) = 1.$$

On the other hand, we have $u_1(\Diamond \eta_d(a)) = \eta_d(u_1(\Diamond a)) = 0$ and

$$u_1\big(\Box T_0 \eta_d(a)\big) \leq u_2\big(T_0 \eta_d(a)\big) = 0$$

by definition of d and $u_1 R_\Box u_2$. Therefore, we also have

$$u_1\big(\Box T_0\eta_d(a) \vee \Diamond\eta_d(a)\big) = u_1\big(\Box T_0 \eta_d(a)\big) \vee u_1\big(\Diamond \eta_d(a)\big) = 0,$$

a contradiction.

Similarly, for the other direction '$\Leftarrow$', suppose towards contradiction that $u_1(\Diamond a) \geq u_2(a)$ holds for all $a \in A$, but there exists some a with $u_1(\Box a) \not\leq u_2(a)$. Let $d = u_1(\Box a)$. On the one hand, by (P2) we have

$$u_1\big(\Box \tau_d(a) \wedge \Diamond T_0 \tau_d(a)\big) \leq u_1\big(\Diamond(\tau_d(a) \wedge T_0\tau_d(a))\big) = u_1(\Diamond 0) = 0.$$

On the other hand, we have $u_1(\Box \tau_d(a)) = \tau_d(u_1(\Box a)) = 1$ and

$$u_1\big(\Diamond T_0 \tau_d(a)\big) \geq u_2\big(T_0\tau_d(a)\big) = 1.$$

Therefore, we also have

$$u_1\big(\Box\tau_d(a) \wedge \Diamond T_0 \tau_d(a)\big) = u_1\big(\Box \tau_d(a)\big) \wedge u_1\big(\Diamond T_0 \tau_d(a)\big) = 1,$$

a contradiction. This finishes the proof. $\Box$

In particular, we now define the *canonical* **D**-*model* as follows. Let $\mathbf{F} := \mathsf{Free}_{\mathcal{A}^{\Box\Diamond}}(\mathsf{Prop})$ be the free $\mathcal{A}^{\Box\Diamond}$-algebra generated by the countable set Prop.

Definition 3.3.5 (Canonical $\mathbf{D}$-model)**.** The *canonical* $\mathbf{D}$*-model* is the $\mathbf{D}$-model

$$\mathfrak{M}^c = (\mathbf{F}^+, \mathsf{Val}^c)$$

where

$$\mathsf{Val}^c(u, p) = u\big([p]\big)$$

for all $u \in \mathcal{A}(\mathbf{F}, \mathbf{D})$ and $p \in \mathsf{Prop}$. Here, $[p]$ denotes the equivalence class of p in $\mathbf{F}$.

Clearly this is a $\mathbf{D}$-model, since $\mathsf{Val}^c(u, p) = u([p]) \in \mathsf{im}(u)$. To obtain an algebraic completeness theorem, the following is crucial.

Lemma 3.3.6 (Truth Lemma)**.** *For every* $\varphi \in \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$*, we have*

$$\mathsf{Val}^c(u, \varphi) = u\big([\varphi]\big)$$

in the canonical $\mathbf{D}$*-model.*

Proof. By induction on φ, where the cases for the propositional connectives of $\mathbf{D}$ are obvious. Suppose $\varphi = \Box\psi$. By definition and inductive hypothesis we have

$$\mathsf{Val}^c(u, \Box\psi) = \bigwedge_{uR_\Box u'} \mathsf{Val}^c(u', \psi) = \bigwedge_{uR_\Box u'} u'\big([\psi]\big).$$

Due to Maruyama's *Kripke condition* [Mar12, Proposition 3.14], we get

$$\bigwedge_{uR_\Box u'} u'\big([\psi]\big) = u\big(\Box[\psi]\big) = u\big([\Box\psi]\big)$$

as desired.

The case $\varphi = \Diamond\psi$ works similarly once we prove the following analogue of Maruyama's Kripke condition for the $\Diamond$-modality.

Fact. *Let* $\langle \mathbf{A}, \Box, \Diamond \rangle$ *be an* $\mathcal{A}^{\Box\Diamond}$*-algebra and let* $u \in \mathcal{A}(\mathbf{A}, \mathbf{D})$*. Then*

$$u(\Diamond a) = \bigvee_{uR_\Box u'} u'(a)$$

holds for all $a \in \mathbf{A}$*.*

The proof of this fact is similar to the one of [Mar12, Proposition 3.14]. Due to Lemma 3.3.4, we know that

$$u(\Diamond a) \geq \bigvee_{uR_\Box u'} u'(a)$$

holds. Suppose towards contradiction that this inequality is strict and set $d := \bigvee_{uR_\Box u'} u'(a)$. Define the subset $J \subseteq \mathfrak{S}(\mathbf{A})$ by

$$b \in J \Leftrightarrow u(\Diamond b) = 0.$$

This is an ideal since $b_1 \leq b_2$ and $b_2 \in J$ imply $u(\Diamond b_1) \leq u(\Diamond b_2) = 0$ and $b_1, b_2 \in J$ implies $u(\Diamond(b_1 \vee b_2)) = u(\Diamond b_1) \vee u(\Diamond b_2) = 0$. This ideal does not contain $\eta_d(a)$ since

$$u\big(\Diamond \eta_d(a)\big) = \eta_d\big(u(\Diamond a)\big) = 1.$$

Therefore, there exists a prime ideal which contains J and which does not contain $\eta_d(a)$. Let $P\colon \mathfrak{S}(\mathbf{A}) \to \mathbf{2}$ be its characteristic function. By Proposition 2.2.4, there is a unique homomorphism $p\colon \mathbf{A} \to \mathbf{D}$ which extends P. Now we have $uR_\Box p$ since for $e = u(\Diamond a')$, we have

$$\eta_e\big(p(a')\big) = p\big(\eta_e(a')\big) = u\big(\Diamond \eta_e(a')\big) = \eta_e\big(u(\Diamond a')\big) = 0,$$

which implies $p(a') \leq e = u(\Diamond a')$ (recall Lemma 3.3.4). However, by construction we also have $p(\eta_d(a)) = 1$, which implies $p(a) \not\leq d = \bigvee_{uR_\Box u'} u'(a)$, contradicting $uR_\Box p$. $\Box$

Therefore, we get an algebraic completeness result as follows. Later on, we also prove a similar, albeit more general version of this theorem via coalgebra (see Sections 4.3.2 and 4.3.3).

Theorem 3.3.7 (Algebraic completeness). *A modal* $\mathbf{D}$*-formula* $\varphi(p_1, \dots, p_n)$ *is in* $\Lambda_{\mathbf{D}}^{\Box\Diamond}$ *if and only if the equation* $\varphi(x_1, \dots, x_n) \approx 1$ *holds in* $\mathcal{A}^{\Box\Diamond}$.

Proof. Suppose that the equation $\varphi = 1$ does not hold in $\mathcal{A}^{\Box\Diamond}$. Then, in particular, $[\varphi] = [1]$ does not hold in $\mathbf{F}$. This means that there is some $d \in \mathbf{D}$ with $\tau_d[\varphi] \neq \tau_d[1] = 1$. Thus, there exists a Boolean homomorphism $U\colon \mathfrak{S}(\mathbf{F}) \to \mathbf{2}$ with $U([1]) = 1$ and $u([\varphi]) = 0$. This homomorphism can be extended to a $\mathcal{A}$-homomorphism $u\colon \mathbf{F} \to \mathbf{D}$ which extends u. This homomorphism satisfies $u([1]) = 1$ and $u([\varphi]) \neq 1$. This means that $\mathsf{Val}^{\mathrm{c}}(u, [\varphi]) \neq 1$, witnessing that $\varphi \notin \Lambda_{\mathbf{D}}^{\Box\Diamond}$ in the canonical $\mathbf{D}$-model. $\Box$

Now we can also easily get similar results for $\Lambda_{\mathbf{D}}^{\Box}$ and $\Lambda_{\mathbf{D}}^{\Diamond}$. The corresponding varieties of algebras are the following.

Definition 3.3.8 ($\mathcal{A}^{\Box}$-&$\mathcal{A}^{\Diamond}$-algebra). A $\Box$*-modal* $\mathcal{A}$*-algebra* or $\mathcal{A}^{\Box}$*-algebra* is an algebra $\langle \mathbf{A}, \Box \rangle$ where $\mathbf{A} \in \mathcal{A}$ and $\Box\colon A \to A$ satisfies the equations

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$, (B3) $\Box\tau_d(x) = \tau_d(\Box x)$ for all $d \in D\backslash\{0\}$.

Similarly, a *$\Diamond$-modal $\mathcal{A}$-algebra* or *$\mathcal{A}^\Diamond$-algebra* is an algebra $\langle \mathbf{A}, \Diamond\rangle$ where $\mathbf{A} \in \mathcal{A}$ and $\Diamond\colon A \to A$ satisfies the equations

(D1) $\Diamond 1 = 1$,

(D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,

(D3) $\Diamond\eta_d(x) = \eta_d(\Diamond x)$ for all $d \in D\backslash\{1\}$.

We denote the variety of $\Box$-modal $\mathcal{A}$-algebras and of $\Diamond$-modal $\mathcal{A}$-algebras by $\mathcal{A}^\Box$ and $\mathcal{A}^\Diamond$, respectively.

With the proofs already established in this subsection, it is straightforward to get the following analogues of Theorem 3.3.7.

Corollary 3.3.9 (Algebraic completeness for $\Lambda^\Box$, $\Lambda^\Diamond$). *Let $\varphi(p_1, \ldots, p_n) \in \mathsf{Form}^\Box_\mathbf{D}$ and $\varphi'(p'_1, \ldots, p'_m) \in \mathsf{Form}^\Diamond_\mathbf{D}$.*

(1) φ is in $\Lambda^\Box_\mathbf{D}$ if and only if the equation $\varphi(x_1, \ldots, x_n) \approx 1$ holds in $\mathcal{A}^\Box$.

(2) φ' is in $\Lambda^\Diamond_\mathbf{D}$ if and only if the equation $\varphi'(x_1, \ldots, x_m) \approx 1$ holds in $\mathcal{A}^\Diamond$.

To end this section, we give some examples on how to get more ‘explicit’ axiomatizations of the varieties of modal algebras from Definitions 3.3.1 and 3.3.8. We start with the case where $\mathbf{D} = \text{Ł}_n$ is a finite MV-chain, and show how to obtain the following combined result from [BEGR11, HT13].

Example 3.3.10 (Alternative axiomatization of $\mathsf{MV}^\Box_n$). An algebra $\langle \mathbf{A}, \Box\rangle$ with $\mathbf{A} \in \mathsf{MV}_n$ and unary operation $\Box$ is in $\mathsf{MV}^\Box_n$ if and only if it satisfies the equations

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3') $\Box(x \oplus x) = \Box x \oplus \Box x$,

(B4') $\Box(x \odot x) = \Box x \odot \Box x$.

Proof. It is shown in [Ost88] that every $\tau_\ell(x)$ can be obtained exclusively as combination of terms $x \oplus x$ and $x \odot x$. Therefore, the axioms (B3')-(B4') imply the axiom (B3) from Definition 3.3.8. Conversely, note that the equations

$$\tau_\ell(x \oplus x) = \tau_{\lceil \frac{\ell}{2} \rceil}(x) \text{ and } \tau_\ell(x \odot x) = \tau_{\lceil \frac{\ell+1}{2} \rceil},$$

are satisfied in Ł_n, where for any rational $q \in \mathbb{Q}$ we define $\lceil q \rceil$ to be the smallest element of Ł_n which is above q. Therefore, using (B3), for every $\ell \in \text{Ł}_n\backslash\{0\}$, we can compute

$$\tau_\ell(\Box x \oplus \Box x) = \tau_{\lceil \frac{\ell}{2} \rceil}(\Box x) = \Box\tau_{\lceil \frac{\ell}{2} \rceil}(x) = \Box\tau_\ell(x \oplus x) = \tau_\ell\big(\Box(x \oplus x)\big).$$

But this implies $\Box x \oplus \Box x = \Box(x \oplus x)$ since $\mathbf{Ł}_n$ satisfies the quasi-equation

$$\bigwedge_{\ell \neq 0} \big(\tau_\ell(x) \approx \tau_\ell(y)\big) \to x \approx y.$$

The case for $\Box(x \odot x)$ is similar. $\Box$

Next, we consider the example where $\mathbf{R}$ is a bounded residuated lattice endowed with the unary operation τ_e (where e is the neutral element with respec to the monoid operation $\odot$) and added truth constants, which is primal due to Proposition 1.2.9.

Example 3.3.11. Let $\mathbf{R}$ be a finite bounded residuated lattice which is quasi-primal or expanded with τ_e, and which is expanded with a constant $\hat{r}$ for every $r \in R$. Let $\mathbf{A} \in \mathcal{A}$.

(1) An algebra $\langle \mathbf{A}, \Box \rangle$ is in $\mathcal{A}^\Box$ if and only if it satisfies the equations

(B1) $\Box 1 = 1$,
(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,
(B3') $\tau_e(\Box x) = \Box \tau_e(x)$,
(B4') $\Box(\hat{r} \backslash x) = \hat{r} \backslash \Box x$ for all $r \neq 0$.

(2) If $\mathbf{R}$ is a $\mathsf{FL}_{\mathrm{ew}}$-algebra, then $\langle \mathbf{A}, \Box \rangle$ is in $\mathcal{A}^\Box$ if and only if it satisfies the equations

(B1) $\Box 1 = 1$,
(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,
(B3') $T_1(\Box x) = \Box T_1(x)$,
(B4') $\Box(\hat{r} \to x) = \hat{r} \to \Box x$ all $r \neq 0$.

(3) If $\mathbf{R}$ is a quasi-primal $\mathsf{FL}_{\mathrm{ew}}$-chain expanded by constants, then $\langle \mathbf{A}, \Box \rangle$ is in $\mathcal{A}^\Box$ if and only if it satisfies the equations

(B1) $\Box 1 = 1$,
(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,
(B3') $\Box(x \odot x) = \Box x \odot \Box x$,
(B4') $\Box(\hat{r} \to x) = \hat{r} \to \Box x$ all $r \neq 0$.

(4) An algebra $\langle \mathbf{A}, \Diamond \rangle$ is in $\mathcal{A}^\Diamond$ if and only if it satisfies the equations

(D1) $\Diamond 0 = 0$,
(D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,
(D3') $\Diamond \tau_e\big(\tau_e(x \backslash \hat{r}) \backslash 0\big) = \tau_e\big(\tau_e(\Diamond x \backslash \hat{r}) \backslash 0\big)$ for all $r \neq 1$.

Proof. As noted after Proposition 1.2.9, we have that $\tau_r(x) = \tau_e(\hat{r}\backslash x)$ because

$$e \leq \hat{r}\backslash x \Leftrightarrow \hat{r} \odot e \leq x \Leftrightarrow d \leq x.$$

Therefore, equations (B3')-(B4') imply (B3) from Definition 3.3.8. Conversely, if $\Box$ preserves all τ_r, then it preserves τ_e in particular. Furthermore, we can then compute

$$\begin{aligned} \tau_s\big(\Box(\hat{r}\backslash x)\big) &= \Box\tau_s(\hat{r}\backslash x) \\ &= \Box\tau_{r\odot s}(x) \\ &= \tau_{r\odot s}(\Box x) = \tau_s(\hat{r}\backslash \Box x) \end{aligned}$$

for all s, which implies (B4') as desired. Here we used $\tau_s(\hat{r}\backslash x) = \tau_{s\odot r}(x)$, which holds because in $\mathbf{R}$ because of the residuation law

$$s \leq r\backslash x \Leftrightarrow r \odot s \leq x.$$

Statement (2) is an immediate specialization of statement (1) since in a $\mathsf{FL}_{\mathrm{ew}}$-algebra it holds that $e = 1$ and $a\backslash b = a \to b$. For statement (3), recall that in a finite $\mathsf{FL}_{\mathrm{ew}}$-chain is quasi-primal if and only if no elements other than $\{0, 1\}$ are idempotent. Therefore, it is easy to see that $T_1(x) = x^n$ for $n = |R|$. Therefore, if $\Box(x \odot x) = \Box x \odot \Box x$ then $T_1(\Box x) = \Box T_1(x)$. Conversely, suppose that (B3) from Definition 3.3.8 holds. Note that, since $\mathbf{R}$ is chain-based, for every $s \in R$, there exists a unique (minimal) s' such that

$$s \leq x \odot x \Leftrightarrow s' \leq x.$$

Therefore we can compute

$$\begin{aligned} \tau_s\big(\Box(x \odot x)\big) &= \Box\tau_s(x \odot x) \\ &= \Box\tau_{s'}(x) \\ &= \tau_{s'}(\Box x) = \tau_s(\Box x \odot \Box x) \end{aligned}$$

for all $s \in R$, which implies (B3').

For the proof of part of statement (4) we simply verify that in $\mathbf{R}$ we have that

$$\eta_r(x) = \tau_e\big(\tau_e(x\backslash\hat{r})\backslash 0\big).$$

This is confirmed by the chain of equivalences

$$\begin{aligned} \tau_e\big(\tau_e(x\backslash r)\backslash 0\big) = 1 &\Leftrightarrow e \leq \tau_e(x\backslash r)\backslash 0 \\ &\Leftrightarrow \tau_e(x\backslash r) \leq 0 \\ &\Leftrightarrow e \not\leq x\backslash r \Leftrightarrow x \not\leq r. \end{aligned}$$

Therefore, (D3') in the statement is equivalent to (D3) in Definition 3.3.8. $\Box$

The axiomatizations of $\mathcal{A}^{\Box}$ obtained here are similar to the ones obtained in [BEGR11, Subsection 4.4] for FL_{ew}-algebras with truth constants and a unique co-atom.

While the presentation of $\mathcal{A}^{\Diamond}$ given in statement (4) above is not the 'nicest', in the case where $\mathbf{D}$ is a *bi-Heyting algebra* we can find a better one. Recall from 1.2.3 that a *bi-Heyting algebra* $\mathbf{B}$ is a Heyting algebra with a co-implication satisfying

$$x \leftarrow y \leq z \Leftrightarrow x \leq y \vee z.$$

Since they are expansions of FL_{ew}-algebras, a finite bi-Heyting algebra $\mathbf{B}$ is quasi-primal if and only if T_1 is term-definable in $\mathbf{B}$. In particular, this holds if $\mathbf{B}$ has a unique atom, as noted in Subsection 1.2.3. As before, adding truth-constants then renders the algebra primal.

Example 3.3.12. Let $\mathbf{D} = \mathbf{B}$ be a finite bi-Heyting algebra expanded by a truth-constant $\hat{b}$ for every $b \in B$.

(1) If $\mathbf{B}$ is quasi-primal (or $\mathbf{B}$ is expanded by T_1), then an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ is in $\mathcal{A}^{\Box\Diamond}$ if and only if it satisfies the equations

(B1) $\Box 1 = 1$, (D1) $\Diamond 0 = 0$,
(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$, (D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,
(B3') $\Box T_1(x) = T_1(\Box x)$, (D3') $\Diamond \eta_0(x) = \eta_0(\Diamond x)$,
(B4') $\Box(\hat{b} \rightarrow x) = \hat{b} \rightarrow \Box x$ all $b \neq 0$, (D4') $\Diamond(x \leftarrow \hat{b}) = \Diamond x \leftarrow \hat{b}$ all $b \neq 1$,
(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$, (P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

(2) If $\mathbf{B}$ has a unique atom and a unique co-atom, then $\langle \mathbf{A}, \Box, \Diamond \rangle$ is in $\mathcal{A}^{\Box\Diamond}$ if and only if it satisfies the equations

(B1) $\Box 1 = 1$, (D1) $\Diamond 0 = 0$,
(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$, (D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,
(B3') $\Box\big(\neg(1 \leftarrow x)\big) = \neg(1 \leftarrow \Box x)$, (D3') $\Diamond\big(1 \leftarrow (\neg x)\big) = 1 \leftarrow (\neg \Diamond x)$,
(B4') $\Box(\hat{b} \rightarrow x) = \hat{b} \rightarrow \Box x$ all $b \neq 0$, (D4') $\Diamond(x \leftarrow \hat{b}) = \Diamond x \leftarrow \hat{b}$ all $b \neq 1$,
(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$, (P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

Proof. Statement (2) is a special case of statement (1) since if $\mathbf{B}$ has a unique atom and co-atom it holds in $\mathbf{B}$ that

$$T_1(x) = \neg(1 \leftarrow x) \text{ and } \eta_0(x) = 1 \leftarrow (\neg x),$$

where we abbreviate $\neg x = x \to 0$ as usual. The proof of statement (1) is similar to that of Example 3.3.11(2), noting that

$$\tau_b(x) = T_1(\hat{b} \to x) \text{ and } \eta_b(x) = \eta_0(x \leftarrow \hat{b}),$$

which shows that (B3')-(B4') and (D3')-(D4') imply (B3), (D3) from Definition 3.3.1. For the converse, use that

$$\tau_d(\hat{b} \to x) = \tau_{d\wedge b}(x) \text{ and } \eta_d(x \leftarrow \hat{b}) = \eta_{d\vee b}(x)$$

in an argument similar to the proof of Example 3.3.11(2). □

In our next example, we consider the primal four-element bilattice **FOUR** from Definition 1.2.2 as algebra of truth-degrees.

Example 3.3.13. Let $\mathbf{D} = \mathbf{FOUR}$ be the four-element bounded implicative bilattice. An algebra $\langle \mathbf{A}, \Box \rangle$ is in $\mathcal{A}^\Box$ if and only if it satisfies (B1)-(B2) from Definition 3.3.8 and the additional equations

(B3') $\Box(t \to x) = \mathsf{t} \to \Box x$,

(B4') $\Box\big(\mathsf{t} \to (\neg x \supset x)\big) = \mathsf{t} \to (\neg\Box x \supset \Box x)$,

(B5') $\Box(x * \mathsf{t}) = \Box x * \mathsf{t}$ and $\Box(x * x) = \Box x * \Box x$,

where

$$x \to y = (x \supset y) \wedge (\neg y \supset \neg x) \text{ and } x * y = \neg(y \to \neg x).$$

Proof. To show that (B3')-(B5') imply (B3) from Definition 3.3.8, one can verify that

$$\tau_{\mathsf{t}}(x) = (\mathsf{t} \to x) \wedge (x * x), \quad \tau_\top(x) = \tau_{\mathsf{t}}(x * \mathsf{t}) \text{ and } \quad \tau_\bot(x) = \mathsf{t} \to (\neg x \supset x).$$

Conversely, assuming (B3) immediately yields (B4') by the above. To get (B3'), note that

$$\tau_{\mathsf{t}}(\mathsf{t} \to x) = \tau_{\mathsf{t}}(x) \qquad \tau_\top(\mathsf{t} \to x) = \tau_{\mathsf{t}}(x) \text{ and } \qquad \tau_\bot(\mathsf{t} \to x) = \tau_\bot(x)$$

hold in **FOUR** and to get (B5'), note that

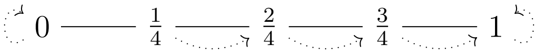
$$\tau_{\mathsf{t}}(x * x) = \tau_{\mathsf{t}}(x) \qquad \tau_\top(x * x) = \tau_\top(x) \text{ and } \qquad \tau_\bot(x * x) = \tau_{\mathsf{t}}(x)$$

holds in **FOUR**, finishing the proof. □

For a more 'exotic' example, we also give another axiomatization of $\mathcal{A}^\Box$ where $\mathbf{D} = \mathbf{CO}_n$ is the semi-primal Cornish chain from Definition 1.2.5. Recall that

$$\mathbf{CO}_n = \big(\{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, \wedge, \vee, \neg, f, 0, 1\big),$$

is the $(n+1)$-element chain with the MV-negation $\neg$ and unary operation f which fixes 0 and 1 and sends $\frac{k}{n}$ to $\frac{k+1}{n}$ otherwise. For example, this operation on $\mathbf{CO}_4$ is depicted below (indicated by dotted lines).



Example 3.3.14. Let $\mathbf{D} = \mathbf{CO}_n$ be the n-th semi-primal Cornish chain. An algebra $\langle \mathbf{A}, \Box \rangle$ is in $\mathcal{A}^\Box$ if and only if it satisfies the equations

(B1) $\Box 1 = 1$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$,

(B3') $\Box\big(\neg f^{n-1}(\neg x)\big) = \neg f^{n-1}(\neg \Box x)$,

(B4') $\Box f(x) = f(\Box x)$.

Proof. First, one can easily check in $\mathbf{CO}_n$ that

$$T_1(x) = \neg f^{n-1}(\neg x)$$

and

$$\tau_{\frac{k}{n}}(x) = T_1\big(f^{n-k}(x)\big) \text{ for all } k \in \{1, \dots, n-1\}.$$

Therefore, if (B3') and (B4') hold then (B3) from Definition 3.3.8 holds. Conversely, if (B3) holds then (B3') holds by the first equation above and to verify (B4') we first note

$$\tau_{\frac{k}{n}}(f(x)) = \begin{cases} \tau_{\frac{k}{n}}(x) & \text{if } k = 1 \\ \tau_{\frac{k-1}{n}}(x) & \text{if } k \in \{2, \dots, n\}, \end{cases}$$

which can again be easily checked in $\mathbf{CO}_n$. Now we can use this to show

$$\begin{aligned} \tau_{\frac{1}{n}}\big(\Box f(x)\big) &= \Box \tau_{\frac{1}{n}}\big(f(x)\big) \\ &= \Box \tau_{\frac{1}{n}}(x) \\ &= \tau_{\frac{1}{n}}(\Box x) = \tau_{\frac{1}{n}}\big(f(\Box x)\big) \end{aligned}$$

and

$$\begin{aligned} \tau_{\frac{k}{n}}\big(\Box f(x)\big) &= \Box \tau_{\frac{k}{n}}\big(f(x)\big) \\ &= \Box \tau_{\frac{k-1}{n}}(x) \\ &= \tau_{\frac{k-1}{n}}(\Box x) = \tau_{\frac{k}{n}}\big(f(\Box x)\big) \end{aligned}$$

for all $k \in \{2, \dots, n\}$, which implies (B4'). $\Box$

Most of the proofs in the above examples follow the same pattern, which we now present in a general form.

Proposition 3.3.15. *Let* $t_1(x), \ldots, t_m(x)$ *be a collection of unary* $\mathbf{D}$*-terms.*

(1) Suppose that for all $d \in D\backslash\{0\}$*, the unary term* τ_d *can be obtained as a combination of the terms* $t_i(x)$ *and for all* $d \in D\backslash\{0\}$ *and* $i \in \{1, \ldots, m\}$*, there is some* $d[i] \in D$ *with*

$$\tau_d\big(t_i(x)\big) = \tau_{d[i]}(x).$$

Then $\langle \mathbf{A}, \Box \rangle$ *with* $\mathbf{A} \in \mathcal{A}$ *and unary operation* $\Box$ *is in* $\mathcal{A}^{\Box}$ *if and only if it satisfies* (B1), (B2) *and*

$$\Box t_i(x) = t_i(\Box x) \text{ for all } i = 1, \ldots, m.$$

(2) Suppose that for all $d \in D\backslash\{1\}$*, the unary term* η_d *can be obtained as a combination of the terms* $t_i(x)$ *and for all* $d \in D\backslash\{1\}$ *and* $i \in \{1, \ldots, m\}$*, there is some* $d[i] \in D$ *with*

$$\eta_d\big(t_i(x)\big) = \eta_{d[i]}(x).$$

Then $\langle \mathbf{A}, \Diamond \rangle$ *with* $\mathbf{A} \in \mathcal{A}$ *and unary operation* $\Diamond$ *is in* $\mathcal{A}^{\Diamond}$ *if and only if it satisfies* (D1), (D2) *and*

$$\Diamond t_i(x) = t_i(\Diamond x) \text{ for all } i = 1, \ldots, m.$$

Proof. We only show how to prove (1), since (2) is completely analogous. Since all τ_d can be obtained as combinations of t_i, it is clear that algebras satisfying $\Box t_i = t_i(\Box x)$ also satisfy $\Box \tau_d(x) = \tau_d(\Box x)$. Conversely, assuming that (B3) from Definition 3.3.8 holds, for every i and d we find

$$\tau_d\big(\Box t_i(x)\big) = \Box \tau_d(t_i(x)) = \Box \tau_{d[i]}(x) = \tau_{d[i]}(\Box x) = \tau_d\big(t_i(\Box x)\big),$$

in particular this also holds if $d[i] = 0$ since $\tau_0 = 1$ is preserved by $\Box$ due to (B1). Since the above equation holds for all $d \neq 0$ and $\mathbf{D}$ satisfies the quasi-equation

$$\bigwedge_{d \neq 0} \big(\tau_d(x) \approx \tau_d(y)\big) \to x \approx y,$$

we get $t_i(\Box x) = \Box(t_i x)$ as desired. $\Box$

In the next subsection, we investigate many-valued modal definability in our setting via the algebraic approach as well.

3.3.2 A many-valued Goldblatt-Thomason Theorem

In this subsection, we begin our study of modal definability for **D**-frames. Our main result is an analogue of the *Goldblatt-Thomason Theorem* [GT75, Theorem 8]. In its original form, this theorem states that a first-order definable class of Kripke frames is definable by a set of modal formulas if and only if it reflects ultrafilter extensions and is closed under disjoint unions, generated subframes and bounded morphic images. We aim to show that a similar result holds for modal definability on **D**-frames, which we now define. The proofs in this chapter are similar to the ones in [Teh16], where the case $\mathbf{D} = \mathbf{Ł}_n$ was already considered.

Definition 3.3.16 (Definability for **D**-frames)**.** Let $\mathcal{C}$ be a class of **D**-frames. We say that $\mathcal{C}$ is *definable* if there is a set of formulas $\Phi \subseteq \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$ such that

$$\mathfrak{F} \in \mathcal{C} \Leftrightarrow \mathfrak{F} \models \Phi$$

holds for every **D**-frame $\mathfrak{F}$.

As discussed in Subsection 3.2.1, definability for **D**-frames differs from definability of frames by modal **D**-formulas. For example, in the case $\mathbf{D} = \mathbf{Ł}_n$ with $n \geq 2$, the formula $\Diamond(p \vee \neg p)$ defines the class of **D**-frames (X, v, R) which satisfy

$$\forall x : \exists x' : xRx' \wedge v(x') = \mathbf{Ł}_1.$$

Meanwhile, there are no frames (X, R) which satisfy the formula $\Diamond(p \vee \neg p)$.

For **D**-frames, the appropriate notions of disjoint union, generated subframe and bounded morphic image are as follows.

Definition 3.3.17 (Constructions on **D**-frames)**.** Let $\mathfrak{F} = (X, v, R)$, $\mathfrak{F}' = (X', v', R')$ be **D**-frames and let $\{\mathfrak{F}_i = (X_i, v_i, R_i)\}_{i\in I}$ be an indexed family of **D**-frames.

(1) The *disjoint union* of $\{\mathfrak{F}_i = (X_i, v_i, R_i)\}_{i\in I}$ is the **D**-frame

$$\biguplus \mathfrak{F}_i = \Big(\biguplus X_i, \biguplus v_i, \bigcup R_i\Big)$$

where $\biguplus X_i = \bigcup \{i\} \times X_i$ is the usual disjoint union and $\big(\biguplus v_i\big)(j, x_j) = v_j(x_j)$ for all $j \in I$ and $x_j \in X_j$.

(2) We say that $\mathfrak{F}$ is a *generated* **D***-subframe* of $\mathfrak{F}'$ if

- (X, R) is a *generated subframe* of (X', R'), that is, $X \subseteq X'$ and $x \in X \Rightarrow R'[x] \subseteq X$ and

- for all $x \in X$ it holds that $v(x) = v'(x)$.

(3) We say that $\mathfrak{F}$ is a *bounded* **D**-*morphic image* of $\mathfrak{F}'$ if there exists a surjective bounded **D**-morphism $\mathfrak{F}' \twoheadrightarrow \mathfrak{F}$.

As expected, validity on **D**-frames is preserved under these operations.

Proposition 3.3.18. *Let $\mathfrak{F} = (X, v, R)$, $\mathfrak{F}' = (X', v', R')$ be* **D***-frames and let $\{\mathfrak{F}_i = (X_i, v_i, R_i)\}_{i \in I}$ be an indexed family of* **D***-frames. Let $\varphi \in \mathsf{Form}^{\Box\Diamond}_{\mathbf{D}}$.*

(1) If $\mathfrak{F}_j \models \varphi$ for every $j \in I$, then $\biguplus \mathfrak{F}_i \models \varphi$.

(2) If $\mathfrak{F}' \models \varphi$ and $\mathfrak{F}$ is a generated **D***-subframe of $\mathfrak{F}'$, then $\mathfrak{F} \models \varphi$.*

(3) If $\mathfrak{F}' \models \varphi$ and $\mathfrak{F}$ is a bounded **D***-morphic image of $\mathfrak{F}'$, then $\mathfrak{F} \models \varphi$.*

Proof. For statement (1), suppose that $(\biguplus \mathfrak{F}_i, \mathsf{Val})$ is an arbitrary **D**-model based on $\biguplus \mathfrak{F}_i$ and let $(j, x_j) \in \biguplus X_i$ be an arbitrary state. Then the restriction $\mathsf{Val}|_{\{j\} \times X_j}$ yields a well-defined **D**-model based on $\mathfrak{F}_j$. Since we have $\mathfrak{F}_j \models \varphi$, we now find

$$\mathsf{Val}\big((j, x_j), \varphi\big) = \mathsf{Val}_j(x_j, \varphi) = 1$$

as desired.

For statement (2), let (X, R, Val) be a **D**-model based on $\mathfrak{F}$. Then $\mathsf{Val}' \colon X' \times \mathsf{Prop} \to \mathbf{D}$ given by

$$\mathsf{Val}'(x', p) = \begin{cases} \mathsf{Val}(x', p) & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

yields a well-defined **D**-model based on $\mathfrak{F}'$, since $v(x') = v'(x')$ for all $x' \in X$. Since (X, R) is a generated subframe of (X', R') and $\mathfrak{F}' \models \varphi$, it is now easy to see (by induction on the complexity of the formula φ) that

$$\mathsf{Val}(x, \varphi) = \mathsf{Val}'(x, \varphi) = 1$$

holds as desired. This finishes the proof, since statement (3) is an easy consequence of Proposition 3.2.5. □

Remark 3.3.19. Note that, in the definition of generated **D**-subframe (Definition 3.3.17(2)), we require the inclusion $X \hookrightarrow X'$ to be a *regular* monomorphism in $\mathsf{Set}_{\mathbf{D}}$ (recall Corollary 2.2.24). The above proof illustrates that, for preservation of validity, we really require $v(x) = v'(x)$ rather than $v'(x) \leq v(x)$ (which would only mean that the inclusion is a monomorphism in $\mathsf{Set}_{\mathbf{D}}$, but not necessarily regular). ■

We already explained how to obtain **D**-frames from $\mathcal{A}^{\Box\Diamond}$-algebras in Definition 3.3.3. Conversely, we now show how to obtain $\mathcal{A}^{\Box\Diamond}$-algebras from **D**-frames.

Definition 3.3.20 (Complex algebra of a **D**-frame)**.** Let $\mathfrak{F} = (X, v, R)$ be a **D**-frame. The *complex algebra* of $\mathfrak{F}$ is the $\mathcal{A}^{\Box\Diamond}$-algebra

$$\mathfrak{F}_+ := \langle \mathsf{P}'(X, v), \Box_R, \Diamond_R \rangle$$

where $\mathsf{P}'(X, v) = \prod_{x \in X} v(x)$ is the functor described explicitly at the beginning of Subsection 4.3.2 and the unary operations $\Box_R$ and $\Diamond_R$ are defined by

$$(\Box_R f)(x) = \bigwedge_{xRx'} f(x') \text{ and } (\Diamond_R f)(x) = \bigvee_{xRx'} f(x')$$

for all $f \in \mathsf{P}'(X, v), x \in X$.

The fact that $\Box_R f$ and $\Diamond_R f$ are well-defined members of $\mathsf{P}'(X, v)$ is due to the compatibility condition of **D**-frames (see Definition 3.2.1). It is also easy to directly verify that the above construction results in a $\mathcal{A}^{\Box\Diamond}$-algebra. We are now ready to define canonical extensions of **D**-frames and **D**-models.

Definition 3.3.21 (Canonical extension)**.** Let $\mathfrak{F}$ be a **D**-frame and let $\mathfrak{M} = (\mathfrak{F}, \mathsf{Val})$ be a **D**-model based on $\mathfrak{F}$.

(1) The *canonical extension* of $\mathfrak{F}$ is given by

$$\mathfrak{Ce}(\mathfrak{F}) = (\mathfrak{F}_+)^+,$$

that is, the canonical **D**-frame of the complex algebra of $\mathfrak{F}$.

(2) The *canonical extension* of $\mathfrak{M}$ is given by

$$\mathfrak{Ce}(\mathfrak{M}) = \big(\mathfrak{Ce}(\mathfrak{F}), \mathsf{Val}^e\big),$$

where

$$\mathsf{Val}^e(\alpha, p) = \alpha\big(\mathsf{Val}(-, p)\big)$$

for every $\alpha \in \mathsf{S}'\mathsf{P}'(X, v)$ and $p \in \mathsf{Prop}$.

Note that the canonical model is well-defined, since $\alpha \in \mathsf{S}'\mathsf{P}'(X, v) = \mathcal{A}(\prod v(x), \mathbf{D})$ and $\mathsf{Val}(-, p)$ is an element of $\prod_{x \in X} v(x)$. Furthermore, we have $\mathsf{Val}^e(\alpha, p) = \alpha(\mathsf{Val}(-, p)) \in \mathsf{im}(\alpha)$ as needed.

We now show that validity is reflected under canonical extensions.

Proposition 3.3.22. *Let $\mathfrak{F}$ be a* $\mathbf{D}$*-frame. Then*

$$\mathfrak{Ce}(\mathfrak{F}) \models \varphi \Rightarrow \mathfrak{F} \models \varphi$$

holds for all $\varphi \in \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$.

Proof. Let $\mathfrak{M} = (X, v, R, \mathsf{Val})$ be a $\mathbf{D}$-model based on $\mathfrak{F} = (X, v, R)$. We show that there is a *canonical embedding* $\iota\colon \mathfrak{M} \hookrightarrow \mathfrak{Ce}(\mathfrak{M})$ given by evaluations $x \mapsto \mathsf{ev}_x$ where $\mathsf{ev}_x(\alpha) = \alpha(x)$.

This map ι is injective, since if $x_1 \neq x_2$, we can always consider the characteristic function χ of $\{x_1\}$, which satisfies $\mathsf{ev}_{x_1}(\chi) = 1$ and $\mathsf{ev}_{x_2}(\chi) = 0$, witnessing $\mathsf{ev}_{x_1} \neq \mathsf{ev}_{x_2}$.

We now show that $x_1 R x_2$ holds if and only if $\mathsf{ev}_{x_1} R_{\Box_R} \mathsf{ev}_{x_2}$ holds. If $x_1 R x_2$, then we have

$$\mathsf{ev}_{x_1}(\Box_R f) = (\Box_R f)(x_1) = \bigwedge_{x_1 R x'} f(x') \leq f(x_2) = \mathsf{ev}_{x_2}(f),$$

which implies $\mathsf{ev}_{x_1} R_{\Box_R} \mathsf{ev}_{x_2}$ (see Definition 3.3.3). Conversely, if $\neg(x_1 R x_2)$, then let χ be the characteristic function of $R[x_1]$. Then we have

$$\mathsf{ev}_{x_1}(\Box_R \chi) = (\Box_R \chi)(x) = \bigwedge_{x_1 R x'} \chi(x') = 1 \text{ and } \mathsf{ev}_{x_2}(\chi) = \chi(x_2) = 0,$$

witnessing $\neg(\mathsf{ev}_{x_1} R_{\Box_R} \mathsf{ev}_{x_2})$.

Next, we show that $v(x) = \mathsf{im}(\mathsf{ev}_x)$ for all $x \in W$. If $s \in v(x)$, then we can define $\alpha(x) = s$ and $\alpha(x') = 0$ otherwise to find $s = \mathsf{ev}_x(\alpha)$, thus $s \in \mathsf{im}(ev_x)$. Conversely, if $s \in \mathsf{im}(\mathsf{ev}_x)$, then there is some $\alpha \in \prod v(x)$ with $\mathsf{ev}_x = \alpha(x) = s$, which implies $s \in v(x)$.

Lastly, we note that, for every $p \in \mathsf{Prop}$ and $x \in X$, we have

$$\mathsf{Val}(x, p) = \mathsf{ev}_x\big(\mathsf{Val}(-, p)\big) = \mathsf{Val}^e(\mathsf{ev}_x, p) = \mathsf{Val}\big(\iota(x), p\big).$$

Similarly to the proof of the Truth Lemma 3.3.6, this can be extended to all $\varphi \in \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$ due to Maruyama's Kripke condition [Mar12, Proposition 3.14] for the $\Box$-modality and its analogue for the $\Diamond$-modality (see the proof of Lemma 3.3.6). $\Box$

There is one more technical lemma needed in order to prove our many-valued version of the Goldblatt-Thomason Theorem. Before we state it, we explain how to consider $\mathbf{D}$-frames as first-order structures. The corresponding language $\mathcal{L}(\mathsf{Krip}_{\mathbf{D}})$ consists of one binary relation symbol R and a unary relation symbol S for every $S \in \mathbb{S}(\mathbf{D})$. Then a $\mathbf{D}$-frame $\mathfrak{F} = (W, v, R)$ can be understood as a $\mathcal{L}(\mathsf{Krip}_{\mathbf{D}})$-structure, setting $R^{\mathfrak{F}} = R$ and

$$S^{\mathfrak{F}} = v^{-1}(\mathbf{S}{\downarrow}) \text{ for all } \mathbf{S} \in \mathbb{S}(\mathbf{D}).$$

We now show that any first-order definable class of $\mathbf{D}$-frames (equivalently, a class closed under ultrapowers) closed under bounded morphic images is also closed under taking canonical extensions.

Lemma 3.3.23. *Let $\mathfrak{F} = (X, v, R)$ be a $\mathbf{D}$-frame. Then $\mathfrak{Ce}(\mathfrak{F})$ is a bounded $\mathbf{D}$-morphic image of an ultrapower of $\mathfrak{F}$.*

Proof. First, expand the language $\mathcal{L}(\mathsf{Krip}_{\mathbf{D}})$ by a unary predicate P_Y for every $Y \subseteq X$ and interpret these unary predicates in the obvious way setting $P_Y^{\mathfrak{F}} = Y$. The resulting structure has an ω-saturated ultrapower $\mathfrak{F}_\omega$ by [CK90, Theorem 6.1.8]. For $\xi \in \mathfrak{F}_\omega$, consider the set

$$U_\xi = \{Y \subseteq X \mid \xi \in P_Y^{\mathfrak{F}_\omega}\}.$$

This defines an ultrafilter on the Boolean algebra $\mathbf{2}^X = \mathfrak{S}(\mathsf{P}'(X, v))$. Thus, by (the discrete version of) Proposition 2.2.4, it can be uniquely identified with a homomorphism $u_\xi \colon \mathsf{P}'(X, v) \to \mathbf{D}$, that is, a member of $\mathfrak{Ce}(\mathfrak{F})$.

The assignment $\xi \mapsto U_\xi$ yields a surjective bounded morphism of frames due to [Gol89, Theorem 3.6.1], and therefore the same is true for the assignment $\xi \mapsto u_\xi$. So all that is left to show is that it is also a morphism in $\mathsf{Set}_{\mathbf{D}}$.

That is, we want to show that if $\xi \in S^{\mathfrak{F}_\omega}$, then $\mathsf{im}(u_\xi) \leq \mathbf{S}$. For $\alpha \in \mathsf{P}'(X, v)$, we have the following equivalences

$$u_\xi(\alpha) \in \mathbf{S} \Leftrightarrow \exists s \in \mathbf{S} \colon T_s(\alpha) \in U_\xi \Leftrightarrow \exists s \in \mathbf{S} \colon \xi \in P_{T_s(\alpha)}^{\mathfrak{F}_\omega},$$

where in the last statement note that $T_s(\alpha) \in 2^X$ means that $P_{T_s(\alpha)}$ makes sense in this context. We know that α satisfies $\alpha(x) \in v(x)$ for all $x \in X$. In our extended first-order language, this can now be expressed as

$$\mathfrak{F} \models \forall x : \big(S(x) \to \bigvee_{s \in S} P_{T_S(\alpha)}(x)\big).$$

Since $\mathfrak{F}$ satisfies this formula, the same is true for its elementary extension $\mathfrak{F}^\omega$. Thus, if $\xi \in S^{\mathfrak{F}_\omega}$, then there is some $s \in \mathbf{S}$ such that $\xi \in P_{T_s(\alpha)}^{\mathfrak{F}_\omega}$. Therefore, by the above equivalence, we have $u_\xi(\alpha) \in \mathbf{S}$. Since α was arbitrary, we thus get $\mathsf{im}(u_\xi) \leq \mathbf{S}$ as desired. □

With this, we can now state the main theorem of this subsection, giving a structural characterization of elementary classes of $\mathbf{D}$-frames which are modally definable (recall Definition 3.3.16).

Theorem 3.3.24 (Goldblatt-Thomason for $\mathbf{D}$-frames)**.** *Let $\mathcal{C}$ be a first-order definable class of $\mathbf{D}$-frames. Then $\mathcal{C}$ is modally definable if and only if it reflects canonical extensions and is closed under disjoint unions, generated $\mathbf{D}$-subframes and bounded $\mathbf{D}$-morphic images.*

Proof. We already know that the conditions are necessary for modal definability by Propositions 3.3.18 and 3.3.22. Now we show that they are sufficient as well.

So assume that $\mathcal{C}$ is a class of $\mathbf{D}$-frames with the stated closure properties. Note that, due to Lemma 3.3.23, the class $\mathcal{C}$ is also closed under canonical extensions. We show that the set of formulas $\Phi = \{\varphi \mid \forall \mathfrak{G} \in \mathcal{C}\colon \mathfrak{G} \models \varphi\}$ defines the class $\mathcal{C}$.

In other words, we want to show the implication

$$\mathfrak{F} \models \Phi \Rightarrow \mathfrak{F} \in \mathcal{C}$$

holds for all $\mathbf{D}$-frames $\mathfrak{F}$. Since $\mathfrak{F} \models \Phi$. The complex algebra $\mathfrak{F}_+$ is a member of the variety generated by $\mathcal{C}_+ := \{\mathfrak{G}_+ \mid \mathfrak{G} \in \mathcal{C}\}$. Therefore, by Birkhoff's Theorem, it is a homomorphic image of a subalgebra of a product of algebras in $\mathcal{C}_+$, say we have

$$\prod_{i \in I} (\mathfrak{G}_i)_+ \hookleftarrow \mathbf{A} \twoheadrightarrow \mathfrak{F}_+$$

for $\{\mathfrak{G}_i \mid i \in I\} \subseteq \mathcal{C}$. By duality, we get

$$\biguplus_{i \in I} \mathfrak{Ce}(\mathfrak{G}_i) \twoheadrightarrow \mathbf{A}^+ \hookleftarrow \mathfrak{Ce}(\mathfrak{F})$$

in the category of $\mathbf{D}$-frames. Now, since all $\mathfrak{G}_i$ are in $\mathcal{C}$, the same is true for all $\mathfrak{Ce}(\mathfrak{G}_i)$, their disjoint union and its bounded $\mathbf{D}$-morphic image $\mathbf{A}^+$ and the generated $\mathbf{D}$-subframe $\mathfrak{Ce}(\mathfrak{F})$ of $\mathbf{A}^+$. Since $\mathcal{C}$ reflects canonical extensions, finally we get $\mathfrak{F} \in \mathcal{C}$, finishing the proof. $\square$

Just like in [Teh16, Theorem 4.7], this immediately yields a corresponding definability result for frames without preconditions. It is not surprising in the light of the recent paper [BCN23], where it is shown that most finitely-valued modal logics can define exactly the same classes of frames which classical modal logic can define. In the following, *definable* refers to definable by the validity relation of $\mathbf{D}$-valued modal logic defined in Section 3.1.

Corollary 3.3.25 (**D**-valued Goldblatt-Thomason). *Let $\mathcal{C}$ be a first-order definable class of Kripke frames. Then $\mathcal{C}$ is definable by a set of formulas $\Phi \subseteq \mathsf{Form}_{\mathbf{D}}^{\Box\Diamond}$ if and only if it reflects ultrafilter extensions and is closed under disjoint unions, generated subframes and bounded morphic images.*

Proof. As described in Subsection 3.2.1, every frame (X, R) can be identified with its corresponding **D**-frame $(X, v^{\mathbf{D}}, R)$. Thus, necessity follows from Propositions 3.3.18 and 3.3.22 as before.

To see that the conditions are also sufficient, suppose that $\mathcal{C} \subseteq \mathsf{Krip}$ has these closure properties. Then consider the class of **D**-frames

$$\mathcal{C}' = \{(X, v, R) \mid (X, R) \in \mathcal{C}\} \subseteq \mathsf{Krip}_{\mathbf{D}},$$

that is, the class of **D**-frames whose underlying frames are in $\mathcal{C}$. By Theorem 3.3.24, the class $\mathcal{C}'$ is definable by a set of modal formulas Φ. It is easy to see that this same set of modal formulas also defines the class $\mathcal{C}$. ☐

Using the variety $\mathcal{A}^{\Box}$ or $\mathcal{A}^{\Diamond}$ instead, we can prove similar theorems for definability by sets of formulas $\Phi \subseteq \mathsf{Form}_{\mathbf{D}}^{\Box}$ or $\Phi' \subseteq \mathsf{Form}_{\mathbf{D}}^{\Diamond}$, respectively. Note that, due to Lemma 3.3.4, both the '$\Box$-canonical extension' and the '$\Diamond$-canonical extension' one has to use there coincide with the canonical extension from Definition 3.3.21. Therefore, we get the following result, even in the absence of a De Morgan involution in **D**.

Corollary 3.3.26 (Definability by unimodal fragments). *Let $\mathcal{C}$ be a first-order definable class of* **D***-frames. Then the following are equivalent.*

(i) $\mathcal{C}$ is modally definable.

(ii) $\mathcal{C}$ is modally definable by some $\Phi \subseteq \mathsf{Form}_{\mathbf{D}}^{\Box}$.

(iii) $\mathcal{C}$ is modally definable by some $\Phi' \subseteq \mathsf{Form}_{\mathbf{D}}^{\Diamond}$.

Furthermore, the analogous equivalence holds for first-order definable classes of Kripke frames (see Corollary 3.3.25). Later on, in Subsection 4.3.4, we also study modal definability from a coalgebraic perspective. In particular, there we give a coalgebraic generalization of the above corollary (see Corollary 4.3.34).

We now conclude Chapter 3, discussing some additional topics for future research related to these topics.

3.4 Conclusion of Chapter 3

We discussed many-valued modal logic (with both modalities $\Box$ and $\Diamond$) with a semi-primal lattice-expansion as algebra of truth-degrees. We considered semantics over Kripke frames and 'richer' semantics over (crisp) $\mathbf{D}$-frames. We studied these logics primarily algebraically and proved results about expressivity, completeness and definability.

In the next chapter, we (re-)investigate and generalize the topics from this chapter using methods from coalgebraic logic. In particular, we show how the logics considered here arise as natural 'liftings' of classical modal logic to the semi-primal level.

The semantics on $\mathbf{D}$-frames appear to be more 'appropriate' when it comes to the study of definability as in Subsection 3.3.2 and, similarly, axiomatic extensions of the minimal $\mathbf{D}$-valued modal logic. Similarly to [HT13], a theory of canonicity and Sahlqvist formulas for the semantics over $\mathbf{D}$-frames can be developed.

The general tendency in many-valued modal logic is still to exclusively consider semantics on Set-based relational structures (either with a crisp or a many-valued accessibility relation). However, it would be interesting to study some 'richer semantics' for other classes of many-valued modal logics as well. As a positive example, we mention the recent paper [FGMR24], which considers semantics for Gödel modal logic based on the duality between finite Gödel algebras and *finite forests*. In the context of this chapter, an obvious question is what would be an appropriate analogue of '$\mathbf{D}$-frame' for modal logic over the standard MV-chain $\mathbf{Ł}$ based on the standard unit interval $[0, 1]$. Note that, in this case, a 'local constraint' should not be an *arbitrary subalgebra* of $\mathbf{Ł}$, but rather a *complete* one. Otherwise, for example with $v(x) = \mathbf{Ł} \cap \mathbb{Q}$, it would be possible to violate $\mathsf{Val}(x, \Box p) \in v(x)$ by approaching an irrational number r as meet of rational numbers.

Moving beyond semi-primal algebras, we note that for *every* finite lattice-based algebras of truth-degrees $\mathbf{D}$ there is a natural duality [CD98] defined by the NU Strong Duality Theorem (also see Subsection 5.1.1). More specifically, the variety $\mathcal{A} = \mathbb{HSP}(\mathbf{D})$ is dually equivalent to a category $\mathcal{X}$ of structured Stone spaces determined by an *alter ego* $\widetilde{\mathbf{D}}$ of $\mathbf{D}$. For $\mathbf{X} \in \mathcal{X}$ a valuation $\mathsf{Val}\colon \mathbf{X} \times \mathsf{Prop} \to \widetilde{\mathbf{D}}$ needs to be a $\mathcal{X}$-morphism in the first component. It might suffice to find the correct general *compatibility conditions* between the accessibility relation R on X and the additional structure on $\mathcal{X}$ in order to ensure that the same is true for the extended valuation to all modal formulas. In Chapter 5 we conduct similar research in the case where $\mathbf{D} = \mathbf{PŁ}_n$ is the $(n + 1)$-element *positive* MV-*chain*.

Lastly, results like Corollary 3.3.26 suggest that $\Box$ and $\Diamond$ are also inter-definable in the absence of a De Morgan involution. In fact, it follows from the results of Section 4.3 that the varieties $\mathcal{A}^{\Box\Diamond}$ and $\mathcal{A}^{\Box}$ are *categorically equivalent*. Moving beyond semi-primal algebras of truth-degrees, we ask the general question for necessary and sufficient conditions on $\mathbf{D}$ for this to be the case.

Chapter 4

Many-valued coalgebraic logic over semi-primal varieties

> *It is all very well to aim for a more "abstract" and a "cleaner" approach to semantics, but if the plan is to be any good, the operational aspects cannot be completely ignored.*
>
> – Dana S. Scott
> (1970)[1]

In this chapter, we study many-valued coalgebraic logics over semi-primal algebras. Following the general theme of the previous chapters, we show that these logics are very 'well-behaved' and closely related to their classical counterparts. More specifically, using the subalgebra adjunctions from Subsection 2.2.4, we show how to lift classical coalgebraic logics (and algebra-coalgebra dualities like Jónsson-Tarski duality) to the many-valued level, and we show that important properties like completeness, expressivity and finite axiomatizability are preserved under this lifting. We also show how to specifically obtain axiomatizations of the many-valued coalgebraic logics directly from their classical counterparts using the unary terms τ_d and η_d similarly to the last chapter.

Again, for the entirety of this chapter, we work under Assumption 2.0.1. In Section 4.2, we even assume that $\mathbf{D}$ is primal.

The chapter is structured as follows. In Section 4.1, we give an introduction to the abstract approach to coalgebraic logics from [KKP04]. In particular, we introduce algebras and coalgebras for an endofunctor (Subsection 4.1.1), abstract and concrete coalgebraic logics (Subsection 4.1.2) and one-step completeness and expressivity (Subsection 4.1.3). In Section 4.2, we

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[1] [Sco70, p.2]

explain how to lift abstract (Subsection 4.2.1) and concrete (Subsection 4.2.2) coalgebraic logics to primal varieties. In Section 4.3, we generalize these results further to semi-primal varieties (Subsections 4.3.2 and 4.3.3) after we show how to similarly lift algebra-coalgebra dualities to the semi-primal level (Subsection 4.3.1). We also investigate the relationship between various notions of coalgebraic definability (Subsection 4.3.4).

There is a common distinction between three distinct (albeit interrelated) approaches to coalgebraic logic. The topic was introduced by Moss in 1999 [Mos99], who used the *relation lifting approach*. Soon after, the *predicate lifting* approach was initiated by Pattinson in [Pat03a]. A unifying framework for both of these is found in the *abstract (or algebraic) approach* developed by Kurz *et al.* [KKP04, BK05, KR12]. For classical modal logic, all three approaches were fruitfully followed, leading to interesting insights, generalizations and novel proof techniques. Thus, it is all the more surprising that very little research on many-valued coalgebraic logic exists thus far. Examples are [BKPV13] following the relation lifting approach and [BD16, LL23] following the predicate lifting approach. To the best of the authors knowledge, the papers [KP23, KPT24a], co-authored by the author of this thesis, took the first steps towards many-valued coalgebraic logic following the abstract approach. This chapter may be seen as an extended version of these two papers.

Most of the preliminaries about coalgebraic logic required for this chapter are presented in Section 4.1. Nevertheless, some familiarity with basic concepts and terminology of this area might be helpful. A good overview of the various approaches to coalgebraic logic is provided in [KP11], which also contains a lot of references. In addition, we refer the reader to [KKV03, BCM22, BBdG22] for algebra-coalgebra dualities, to [KKP04, BK05, KR12] for algebraic semantics of coalgebraic logics, to [Gum99, Rut00, Pat03b] for the general theory of coalgebras as state-based systems, to [Pat04, Kli07, Sch08, JS09] for expressivity of coalgebraic logics and to [BK06, KP10, KR12] for presentations of functors by operations and equations.

4.1 Introduction to coalgebraic logic

Coalgebraic (modal) logic was introduced by Moss in 1999 [Mos99] and soon thereafter developed into an active research area (for an overview see, *e.g.*, [KP11] and the references therein). In this section, we give an overview of this topic, as set-up for the later sections of this chapter.

Note that, as mentioned at the beginning of this chapter, our approach to coalgebraic modal logic is the ‘abstract’ one introduced in [KKP04] and further developed (among others) in [BK05, KR12].

The section is structured as follows. In Subsection 4.1.1, we introduce coalgebras and algebras for endofunctors, and discuss important examples like Kripke- and neighborhood frames and modal- and neighborhood algebras. In Subsection 4.1.2, we define abstract and concrete coalgebraic logics and show how classical modal logic fits this framework. Lastly, in Subsection 4.1.3, we introduce two important properties of coalgebraic logics, namely (one-step) completeness and expressivity.

4.1.1 Coalgebras and algebras

We start with the definition of coalgebra for an endofunctor and some examples related to modal logic.

Definition 4.1.1 (Coalgebra for an endofunctor)**.** Given a category C and an endofunctor $\mathsf{T}\colon \mathsf{C} \to \mathsf{C}$, a T-*coalgebra* is a C-morphism $\gamma\colon X \to \mathsf{T}(X)$, where $X \in \mathsf{C}$. Given another T-coalgebra $\gamma'\colon X' \to \mathsf{T}(X')$, a T-*coalgebra morphism* $\gamma \to \gamma'$ is a C-morphism $f\colon X \to X'$ for which the square

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "f"'] & \mathsf{T}(X) \arrow[d, "\mathsf{T}f"] \\ X' \arrow[r, "\gamma'"] & \mathsf{T}(X') \end{tikzcd}$$

commutes. We denote the category of T-coalgebras with coalgebra morphisms by $\mathsf{Coalg}(\mathsf{T})$.

Coalgebras offer a convenient category-theoretical framework to describe various transition-systems (*e.g.*, both deterministic or non-deterministic automata). A general theory of *universal coalgebra* similar to that of universal algebra can be found in [Rut00]. The following well-known example relates this to classical modal logic.

Example 4.1.2 (Kripke frames as coalgebras)**.** The category $\mathsf{Coalg}(\mathcal{P})$ of coalgebras for the covariant powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ is isomorphic to the category Krip of Kripke frames with bounded morphisms.

Indeed, a $\mathcal{P}$-coalgebra is a map of the form $\gamma\colon X \to \mathcal{P}(X)$. This can be identified (and vice versa) with a relational structure (X, R_γ) via

$$x_1 R_\gamma x_2 \Leftrightarrow x_2 \in \gamma(x_1)$$

or, in other words, $\gamma(x) = R[x]$ for all $x \in X$. A coalgebra morphism between two $\mathcal{P}$-coalgebras is map f such that the square

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "f"'] & \mathcal{P}(X) \arrow[d, "\mathcal{P}f"] \\ X' \arrow[r, "\gamma'"] & \mathcal{P}(X') \end{tikzcd}$$

commutes. Spelling out the definitions, this means

$$R_{\gamma'}[f(x)] = f\big(R_{\gamma}[x]\big)$$

for all $x \in X$. Now the inclusion '$\supseteq$' is equivalent to

$$x_1 R_{\gamma} x_2 \Rightarrow f(x_1) R_{\gamma'} f(x_2)$$

and the converse inclusion '$\subseteq$' is equivalent to

$$f(x_1) R_{\gamma'} x_2' \Rightarrow \text{ there is some } x_2 \in R[x_1] \text{ with } f(x_2) = x_2'.$$

Therefore, f is a $\mathcal{P}$-coalgebra morphism $\gamma \to \gamma'$ if and only if it is a bounded morphism $(X, R_{\gamma}) \to (X', R_{\gamma'})$. ■

The following closely related example was described in detail in [KKV03].

Example 4.1.3 (Descriptive general frames as coalgebras)**.** We consider the coalgebras for the *Vietoris functor* $\mathcal{V} \colon \mathsf{Stone} \to \mathsf{Stone}$. It takes a Stone space X to its *Vietoris space* $\mathcal{V}(X)$ defined as follows [Vie22]. The carrier set of $\mathcal{V}(X)$ is the collection $\mathsf{K}(X)$ of closed subsets of X. The topology is generated by the subbasis consisting of all sets of the form

$$[U] = \{K \in \mathsf{K}(X) \mid K \subseteq U\} \text{ and } \langle U \rangle = \{K \in \mathsf{K}(X) \mid K \cap U \neq \varnothing\},$$

where $U \subseteq X$ is an open subset.

Recall that a *descriptive general frame* is a relational structure (X, R) based on a Stone space X, such that $R[x] \subseteq X$ is always closed and

$$R^{-1}[C] := \{x \in X \mid \exists x' \in C : xRx'\}$$

is clopen whenever $C \subseteq X$ is clopen.

The category $\mathsf{Coalg}(\mathcal{V})$ of coalgebras for the Vietoris functor $\mathcal{V} \colon \mathsf{Stone} \to \mathsf{Stone}$ is isomorphic to the category of descriptive general frames with continuous bounded morphisms [KKV03, Theorem 3.13]. ■

Dealing with non-normal logics, Kripke semantics are usually replaced by the more general *neighborhood semantics* (see, *e.g.*, the book [Pac17]). These can also be described and studied coalgebraically [HKP09].

Example 4.1.4 (Neighborhood frames as coalgebras). A *neighborhood frame* is a pair (W, N) where W is a set and $N\colon W \to \mathcal{P}\mathcal{P}(W)$ sends w to the collection of its *neighborhoods* $N(w)$. A *neighborhood morphism* $f\colon (W_1, N_1) \to (W_2, N_2)$ is a map $f\colon W_1 \to W_2$ which satisfies

$$Y \in N_2\big(f(x)\big) \Leftrightarrow f^{-1}(Y) \in N_1(x)$$

for all $x \in W_1$ and $Y \subseteq W_2$.

The category of neighborhood frames is isomorphic to the category of coalgebras for the *neighborhood functor* $\mathcal{N} = \wp \circ \wp$, where $\wp\colon \mathsf{Set} \to \mathsf{Set}$ is the contravariant powerset functor [HKP09, Subsection 2.3]. ■

Similar to Example 4.1.3, there is also a notion of *descriptive neighborhood frame* introduced by Došen [Doš89]. Their coalgebraic description is due to [BBdG22] as follows.

Example 4.1.5 (Descriptive neighborhood frames as coalgebras). Descriptive neighborhood frames can be identified with coalgebras for the following endofunctor $\mathcal{D}\colon \mathsf{Stone} \to \mathsf{Stone}$. This functor takes a Stone space X to the $\mathcal{D}$-*hyperspace* $\mathcal{D}(X) = \mathcal{P}(\mathsf{Clp}(X))$, where $\mathsf{Clp}(X)$ denotes the collection of clopen subsets of X. The topology on $\mathcal{D}(X)$ is generated by a subbasis consisting of all sets of the form

$$[C] = \{N \subseteq \mathsf{Clp}(X) \mid C \in N\} \text{ and } \langle C\rangle = \{N \subseteq \mathsf{Clp}(X) \mid X\backslash C \notin N\},$$

where $C \subseteq X$ is a clopen subset.

It is shown in [BBdG22, Theorem 5.4] that the category of descriptive neighborhood frames is isomorphic to the category of coalgebras for the functor $\mathcal{D}\colon \mathsf{Stone} \to \mathsf{Stone}$. ■

We now define and give some examples of algebras for an endofunctor as well. Not surprisingly, they are defined very similarly to coalgebras for an endofunctor.

Definition 4.1.6 (Algebra for an endofunctor). Given a category C and an endofunctor $\mathsf{L}\colon \mathsf{C} \to \mathsf{C}$, a L-*algebra* is a C-morphism $\alpha\colon \mathsf{L}(A) \to A$, where $A \in \mathsf{C}$. Given another L-algebra $\alpha'\colon \mathsf{L}(A') \to A'$, a L-*algebra morphism* $\alpha \to \alpha'$ is a C-morphism $h\colon A \to A'$ for which the square

$$\begin{tikzcd}
\mathsf{L}(A) \arrow[r, "\alpha"] \arrow[d, "\mathsf{L}h"'] & A \arrow[d, "h"] \\
\mathsf{L}(A') \arrow[r, "\alpha'"] & A'
\end{tikzcd}$$

commutes. We denote the category of L-algebras with algebra morphisms by $\mathsf{Alg}(\mathsf{L})$.

In this thesis, we exclusively consider algebras over endofunctors $\mathsf{L}\colon \mathsf{V} \to \mathsf{V}$ where V is a variety of algebras. In the context of coalgebraic logic, these functors should be thought of as 'adding modal operators'. Before we make this more precise, we give two examples.

Example 4.1.7 (Modal algebras as algebras for a functor)**.** For the first example, recall that a (classical) $\Box$-*modal algebra* is a pair $(\mathbf{B}, \Box)$, where $\mathbf{B}$ is a Boolean algebra and $\Box\colon B \to B$ preserves the top-element 1 and finite meets. It was shown in [KKV03, Proposition 3.17] that the category of modal algebras with $\Box$-preserving homomorphisms is equivalent to the category $\mathsf{Alg}(\mathsf{L})$ for the functor $\mathsf{L}_\Box\colon \mathsf{BA} \to \mathsf{BA}$ which assigns to a Boolean algebra $\mathbf{B}$ the free Boolean algebra generated by the underlying meet-semilattice of $\mathbf{B}$.

Equivalently, as shown in [KKP04], the functor $\mathsf{L}_\Box$ can be described in terms of a *presentation by operations and equations* as follows. For a Boolean algebra $\mathbf{B}$, the Boolean algebra $\mathsf{L}_\Box(\mathbf{B})$ is the free Boolean algebra generated by the set of formal expressions $\{\Box b \mid b \in \mathbf{B}\}$, modulo the equations $\Box 1 \approx 1$ and $\Box(b_1 \wedge b_2) \approx \Box b_1 \wedge \Box b_2$. Given a Boolean homomorphism $h\colon \mathbf{B}_1 \to \mathbf{B}_2$, the homomorphism $\mathsf{L}_\Box h\colon \mathsf{L}_\Box(\mathbf{B}_1) \to \mathsf{L}^\Box(\mathsf{Alg}(B_2))$ is the unique homomorphism which extends the map $\{\Box b_1 \mid b_1 \in \mathbf{B}_1\} \to \mathsf{L}^\Box(\mathbf{B}_2)$ defined by $\Box b_1 = [\Box h(b_1)]$. Now, a $\mathsf{L}_\Box$ algebra is by definition a homomorphism $\alpha\colon \mathsf{L}_\Box(\mathbf{B}) \to \mathbf{B}$. By definition of $\mathsf{L}_\Box$, such homomorphisms are in one-to-one correspondence with maps $\{\Box b \mid b \in \mathbf{B}\} \to \mathbf{B}$ which respect the equations defining $\mathsf{L}_\Box$.

Thus, the variety $\mathsf{BA}^\Box$ of ($\Box$-)modal algebras is isomorphic to the category $\mathsf{Alg}(\mathsf{L}^\Box)$ of algebras for the functor $\mathsf{L}_\Box$ described above. ■

Of course, similarly the variety of $\Diamond$-modal algebras (*i.e.*, modal algebras defined in terms of $\Diamond$ instead of $\Box$) is isomorphic to the category $\mathsf{Alg}(\mathsf{L}_\Diamond)$, where $\mathsf{L}_\Diamond$ takes a Boolean algebra $\mathbf{B}$ to the free Boolean algebra generated by the set of formal expressions $\{\Diamond b \mid b \in \mathbf{B}\}$, modulo the equations $\Diamond 0 \approx 0$ and $\Diamond(b_1 \vee b_2) \approx \Diamond b_1 \vee \Diamond b_2$.

While modal algebras are used in classical modal logic with Kripke semantics, in the case of neighborhood semantics one uses the following more general variety of algebras.

Example 4.1.8 (Neighborhood algebras as algebras for a functor)**.** For our next example, we consider *neighborhood algebras.* Recall that a *neighborhood algebra* is a pair $(\mathbf{B}, \triangle)$, where $\mathbf{B}$ is a Boolean algebra and $\triangle\colon B \to B$ is an arbitrary unary operation. As noted in [BBdG22, Proposition 2.6], we can again describe the variety of neighborhood algebras as category of algebras for a functor $\mathsf{L}_\triangle \colon \mathsf{BA} \to \mathsf{BA}$. The functor $\mathsf{L}_\triangle$ sends a Boolean algebra to the free Boolean algebra generated by the underlying set of $\mathbf{B}$ or, to help our intuition, the Boolean algebra generated by the set of formal expressions $\{\triangle b \mid b \in \mathbf{B}\}$. The $\mathsf{L}_\triangle$-algebras $\alpha \colon \mathsf{L}_\triangle(\mathbf{B}) \to \mathbf{B}$ can then be identified with arbitrary maps $\{\triangle b \mid b \in \mathbf{B}\} \to \mathbf{B}$.

Thus, the variety NA of neighborhood algebras is isomorphic to the category $\mathsf{Alg}(\mathsf{L}_\triangle)$ of algebras for the functor $\mathsf{L}_\triangle \colon \mathsf{BA} \to \mathsf{BA}$. ■

All of the functors $\mathsf{L}_\Box$, $\mathsf{L}_\Diamond$ and $\mathsf{L}_\triangle$ are defined in terms of a *presentation by operations and equations* [BK06, KP10, KR12]. Following [BK06, Definition 6], a *finitary presentation* of an endofunctor $\mathsf{L} \colon \mathsf{V} \to \mathsf{V}$ on a variety V consists of a collection of *operation symbols* $\mathcal{O}$ together with their arities $\mathsf{ar} \colon \mathcal{O} \to \mathbb{N}$ and a collection of *rank*-1 *equations* $\mathcal{E}$. Here, an equation $t_1 \approx t_2$ between terms formed from a set of variables, operations of $\mathcal{V}$ and operations in $\mathcal{O}$ is called *rank*-1 if every variable occurring in t_1 or t_2 is in the scope of precisely one operation from $\mathcal{O}$.

For example, the functor $\mathsf{L}_\Box$ from Example 4.1.7 has a presentation by one operation symbol $\mathcal{O} = \{\Box\}$ with is unary (*i.e.*, $\mathsf{ar}(\Box) = 1$) and the set of rank-1 equations $\mathcal{E} = \{\Box 1 \approx 1, \Box(b_1 \wedge b_2) \approx \Box b_1 \wedge \Box b_2\}$ (note that $\Box 1 \approx 1$ is a rank-1 equation because 1 is not a variable).

Similarly, the functor $\mathsf{L}_\triangle$ from Example 4.1.8 has a presentation by one unary operation symbol $\mathcal{O} = \{\triangle\}$ and no equations, that is, $\mathcal{E} = \varnothing$.

It was shown in [KR12, Theorem 4.7] that an endofunctor on a variety has a finitary presentation by operations and equations if and only if it preserves *sifted colimits* (for an introduction to sifted colimits and their role in universal algebra we refer the reader to [ARVL10]).

In the following subsection, we define coalgebraic logics, which put coalgebras and algebras for an endofunctor in correspondence.

4.1.2 Abstract and concrete coalgebraic logics

In this subsection, we introduce and give some examples of abstract and concrete coalgebraic logics. An abstract coalgebraic logic for T-coalgebras consists of an endofunctor $\mathsf{L} \colon \mathsf{V} \to \mathsf{V}$ defined on a variety V (which essentially determines syntax) and a natural transformation δ which is used to take T-coalgebras to L-algebras (essentially determining semantics). All of this happens 'on top of' a dual adjunction between the base categories of T and L. Before we give the general definition, we discuss two examples.

Both examples are built on the dual adjunction between Set and BA determined by the contravariant functors $\mathsf{S} \colon \mathsf{BA} \to \mathsf{Set}$ taking a Boolean algebra to its set of ultrafilters and $\mathsf{P}\colon \mathsf{Set} \to \mathsf{BA}$ taking a set to its powerset algebra. Our first example is due to [KKP04]

Example 4.1.9 (Classical modal logic, coalgebraically)**.** We know by Example 4.1.2 that coalgebras for the covariant powerset functor $\mathcal{P}$ are Kripke frames and by Example 4.1.7 that algebras for the functor $\mathsf{L}_\Box$ are modal algebras. We now define a natural transformation $\delta\colon \mathsf{L}_\Box\mathsf{P} \Rightarrow \mathsf{P}\mathcal{P}$ as follows. For a set W, the component $\delta_W\colon \mathsf{L}_\Box\mathsf{P}(W) \to \mathsf{P}\mathcal{P}(W)$ of δ is given by

$$\Box Y \mapsto \{Z \subseteq W \mid Z \subseteq Y\} \text{ for } Y \subseteq W.$$

Let $\gamma_R\colon W \to \mathcal{P}(W)$ be a $\mathcal{P}$-coalgebra, which we identify with the Kripke frame (W, R), where $\gamma_R(w) = R[w] = \{w' \in W \mid wRw'\}$. We apply P to γ_R to obtain a homomorphism $\mathsf{PT}(W) \to \mathsf{P}(W)$. This can be composed with δ_W to obtain a $\mathsf{L}_\Box$-algebra

$$\mathsf{P}\gamma_R \circ \delta_W\colon \mathsf{L}_\Box\mathsf{P}(W) \to \mathsf{P}(W).$$

Untangling the definitions shows that the operator corresponding to this modal algebra is defined on a subset $Y \subseteq W$ by

$$\Box Y = \{w \in W \mid \gamma_R(w) \subseteq Y\} = \{w \in W \mid wRw' \Rightarrow w' \in Y\}.$$

This is commonly known as the *complex algebra* of the frame (W, R) (see, *e.g.*, [BdRV01, Section 5.2]. ■

So, what is needed to relate coalgebras and algebras is a natural transformation of a certain type. The following example illustrates this once more in the context of neighborhood semantics.

Example 4.1.10 (Neighborhood semantics, coalgebraically)**.** We know by Example 4.1.4 that coalgebras for the neighborhood functor $\mathcal{N}$ are neighborhood frames and by Example 4.1.8 that algebras for the functor $\mathsf{L}_\triangle$ are neighborhood algebras. We now define a natural transformation $\delta\colon \mathsf{L}_\triangle\mathsf{P} \Rightarrow \mathsf{P}\mathcal{N}$ as follows. For a set $W \in \mathsf{Set}$, the component $\delta_W\colon \mathsf{L}_\triangle\mathsf{P}(W) \to \mathsf{P}\mathcal{N}(W)$ of δ is given by

$$\Delta Y \mapsto \{N \subseteq \mathcal{P}(W) \mid Y \in N\} \text{ for } Y \subseteq W.$$

Given a coalgebra $\gamma_N\colon W \to \mathcal{N}(W)$, we can again use P and δ to define a $\mathsf{L}_\triangle$-algebra

$$\mathsf{P}\gamma \circ \delta_W\colon \mathsf{L}_\triangle\mathsf{P}(W) \to \mathsf{P}(W).$$

Untangling the definitions, we see that the operation of this neighborhood algebra takes $Y \subseteq W$ to

$$\triangle W = \{w \in W \mid Y \in \gamma(w)\}.$$

This is the assignment of a neighborhood algebra to a neighborhood frame described in [Doš89, Section 2]. ■

With these two examples at hand, we are ready to give a general definition of abstract coalgebraic logic.

Definition 4.1.11 (Abstract coalgebraic logic)**.** Let C be a concrete category and let V be a variety of algebras. Let the functors $\mathsf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{V}$ and $\mathsf{S}\colon \mathsf{V} \to \mathsf{C}^{\mathrm{op}}$ form a dual adjunction $\mathsf{S} \dashv \mathsf{P}$ (we will always identify them with contravariant functors between C and V). Let $\mathsf{T}\colon \mathsf{C} \to \mathsf{C}$ be an endofunctor. An *abstract coalgebraic logic for* T is a pair (L, δ) consisting of an endofunctor $\mathsf{L}\colon \mathsf{V} \to \mathsf{V}$ and a natural transformation $\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$, called the *one-step semantics*.

$$\begin{tikzcd} \mathsf{C} \arrow[loop left, "\mathsf{T}"] \arrow[r, bend left, "\mathsf{P}"] & \mathsf{V} \arrow[l, bend left, "\mathsf{S}"] \arrow[loop right, "\mathsf{L}"] \end{tikzcd}$$

If $\mathsf{C} = \mathsf{Set}$, $\mathsf{V} = \mathsf{BA}$ and P and S is the dual adjunction from the above two examples, we call the coalgebraic logic *classical*.

As we saw in the previous two examples, in an abstract coalgebraic logic (L, δ) the natural transformation δ is used to relate T-coalgebras to L-algebras as follows. Starting with a T-coalgebra $\gamma\colon X \to \mathsf{T}(X)$, we can first apply the (contravariant) functor P to obtain $\mathsf{P}\gamma\colon \mathsf{PT}(X) \to \mathsf{P}(X)$. Now the component $\delta_X\colon \mathsf{LP}(X) \to \mathsf{PT}(X)$ of δ is precisely what is needed to obtain an L-algebra

$$\mathsf{P}\gamma \circ \delta_X\colon \mathsf{LP}(X) \to \mathsf{P}(X).$$

Now suppose that the functor L has a presentation by operations and equations. Then $\mathsf{Alg}(\mathsf{L})$ forms a variety, whose equational logic can be thought of as a modal logic. We obtain the corresponding semantics on coalgebras as follows. Given a coalgebra $\gamma\colon X \to \mathsf{T}(X)$, we use δ to associate the corresponding L-algebra to it. The initial L-algebra $\mathsf{L}(I) \to I$ exists and the unique morphism

$$\begin{tikzcd} \mathsf{L}(I) \arrow[r] \arrow[d, "\mathsf{L}[[\cdot]]"'] & I \arrow[d, "{[[\cdot]]}"] \\ \mathsf{LP}(X) \arrow[r] & \mathsf{P}(X) \end{tikzcd}$$

yields the *interpretation* of formulas $[[\cdot]]\colon I \to \mathsf{P}(X)$. Via the dual adjunction, this corresponds to $\mathsf{th}\colon X \to \mathsf{S}(I)$, which assigns a point of X to its *theory*. In this case, we call the coalgebraic logic concrete.

Definition 4.1.12 (Concrete coalgebraic logic)**.** A *concrete coalgebraic logic* is an abstract coalgebraic logic (L, δ) together with a presentation $(\mathcal{O}, \mathcal{E})$ of L by operations and equations.

The logics considered in Examples 4.1.9 and 4.1.10 are therefore both examples of concrete coalgebraic logics.

In the next subsection, we recall how completeness and the Hennessy-Milner property are reflected in coalgebraic logics.

4.1.3 One-step completeness and expressivity

In this subsection, we give an overview of the two most important properties an abstract coalgebraic logic might have, namely (one-step) completeness and expressivity. In particular, we show how these correspond to properties of the one-step semantics δ and its adjoint-transpose $\delta^{\dagger}$.

We start with the definition of one-step completeness. This concept was introduced for predicate liftings by Pattinson in [Pat03a]. In the framework of abstract coalgebraic logics, following [KKP04], this property is easy to state.

Definition 4.1.13 (One-step completeness)**.** An abstract coalgebraic logic (L, δ) is called *one-step complete* if δ is a monomorphism.

As shown in [KKP04, Proposition 5.6], for classical concrete coalgebraic logics, one-step completeness always implies completeness for T-coalgebras in the sense that every formula which is true in all T-coalgebras (with the semantics described in the discussion before Definition 4.1.12) can be derived in the consequence relation induced by the variety $\mathsf{Alg}(\mathsf{L})$. Since we want to prove the analogous result for the semi-primal case later on, we give a sketch of the proof here. The way we present the proof here is very similar to [KP10, Theorem 6.15] and makes use of the *initial algebra sequence*.

Theorem 4.1.14 (Classical one-step complete ⇒ complete)**.** *Let* (L, δ) *be a classical concrete coalgebraic logic for* $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$. *Then one-step completeness implies completeness.*

Proof. Recall that $\mathbf{2}$ is the initial object in BA and $\mathbb{1} := \{*\}$ is the terminal object in Set. We now construct the *initial algebra sequence* as follows. Start with e_0 being the unique homomorphism $\mathbf{2} \to \mathsf{L}(\mathbf{2})$ and inductively define $e_{n+1} = \mathsf{L}e_n$.

$$\mathbf{2} \xrightarrow{e_0} \mathsf{L}(\mathbf{2}) \xrightarrow{e_1} \mathsf{L}^2(\mathbf{2}) \xrightarrow{e_2} \dots$$

The members of $\mathsf{L}^n(\mathbf{2})$ are the (equivalence classes of) formulas of depth n. Dually, we define the terminal coalgebra sequence, starting with p_0 being the unique map $\mathsf{T}(\mathbb{1}) \to \mathbb{1}$ and $p_{n+1} = Tp_n$. We transport this sequence to BA via the contravariant functor P and obtain a similar sequence.

$$\mathsf{P}(\mathbb{1}) \xrightarrow{\mathsf{P}p_0} \mathsf{PT}(\mathbb{1}) \xrightarrow{\mathsf{P}p_1} \mathsf{PT}^2(\mathbb{1}) \xrightarrow{\mathsf{P}p_2} \dots$$

We now connect these two sequences via a sequence of homomorphisms $[[\cdot]]_n \colon \mathsf{L}^n(\mathbf{2}) \to \mathsf{PT}^n(\mathbb{1})$. Here, $[[\cdot]]_0 \colon \mathbf{2} \to \mathsf{P}(\mathbb{1})$ is the unique such homomorphism and $[[\cdot]]_{n+1} = \delta_{\mathsf{T}^n(\mathbb{1})} \circ \mathsf{L}[[\cdot]]_n$. Since L and δ preserve monomorphisms, every $[[\cdot]]$ is a monomorphism.

$$\begin{tikzcd}
\mathsf{P}(\mathbb{1}) \arrow[r, "\mathsf{P}p_0"] & \mathsf{PT}(\mathbb{1}) \arrow[r, "\mathsf{P}p_1"] & \mathsf{PT}^2(\mathbb{1}) \arrow[r, "\mathsf{P}p_2"] & \dots \\
\mathbf{2} \arrow[u, "{[[\cdot]]_0}"] \arrow[r, "e_0"] & \mathsf{L}(\mathbf{2}) \arrow[u, "{[[\cdot]]_1}"] \arrow[r, "e_1"] & \mathsf{L}^2(\mathbf{2}) \arrow[u, "{[[\cdot]]_2}"] \arrow[r, "e_2"] & \dots
\end{tikzcd}$$

Now, to show completeness, suppose that $\varphi \neq \psi$ holds in the initial L-algebra. Then there is some n such that these formulas are different in $\mathsf{L}^n(\mathbf{2})$. Let i be a one-sided inverse of p_0. Then $\gamma = \mathsf{T}^n i \colon \mathsf{T}^{n+1}(\mathbb{1}) \to \mathsf{T}^n(\mathbb{1})$ is a coalgebra. Since $[[\cdot]]_n$ is injective, we have $[[\varphi]]_n \neq [[\psi]]_n$ and $[[\cdot]]_n$ coincides with the interpretation map $[[\cdot]] \colon I \to \mathsf{PT}^n(\mathbb{1})$. Thus, the coalgebra γ provides a counter-example to $\varphi = \psi$ as desired. $\square$

Similarly to how completeness of (L, δ) is related to δ being a monomorphism, *expressivity* [Pat04, Kli07, Sch08, JS09] can be related to the adjoint-transpose of δ being a monomorphism. We use [JS09, Theorem 4] as definition of expressivity. Recall that, in the setting of Definition 4.1.11, the *adjoint-transpose* $\delta^\dagger \colon \mathsf{TS} \Rightarrow \mathsf{SL}$ is the natural transformation obtained from δ as composition

$$\mathsf{TS} \xrightarrow{\varepsilon \mathsf{TS}} \mathsf{SPTS} \xrightarrow{\mathsf{S}\delta\mathsf{S}} \mathsf{SLPS} \xrightarrow{\mathsf{SL}\eta} \mathsf{SL}$$

where ε and η are the unit and counit of the adjunction.

Definition 4.1.15 (Expressivity)**.** Let (L, δ) be an abstract coalgebraic logic for $\mathsf{T} \colon \mathsf{C} \to \mathsf{C}$ satisfying the following conditions.

1. The category $\mathsf{Alg}(\mathsf{L})$ has an initial object.
2. The category C has $(\mathcal{M}, \mathcal{E})$-factorizations with $\mathcal{M}$ being a collection of monomorphisms and $\mathcal{E}$ being a collection of epimorphisms.

3. The functor T preserves members of $\mathcal{M}$.

We say that (L, δ) is *expressive* if every component of the adjoint-transpose $\delta^\dagger$ of δ is in $\mathcal{M}$.

Assuming the coalgebraic logic is concrete, expressivity is also known as the *Hennessy-Milner property*, stating that for every T-coalgebra $\gamma\colon X \to \mathsf{T}(X)$, two states $x, y \in X$ have the same theory if and only if they are behaviourally equivalent (bisimilar). Here, we call $x_1 \in X_1$ and $x_2 \in X_2$ *behaviourally equivalent* if there exist coalgebra morphisms $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ (into the same coalgebra) with $f_1(x_1) = f_2(x_2)$.

The logic of Example 4.1.9 is not expressive, but becomes expressive if restricted to *image-finite* Kripke frames (that is, the powerset functor is replaced by the finite powerset functor $\mathcal{P}_\omega$). Similar results for the logic defined in Example 4.1.10 and the appropriate definition of image-finite neighborhood frames can be found in [HKP09].

In the following sections, we develop methods to lift classical coalgebraic logics to the semi-primal level and show that one-step completeness and expressivity are preserved under this lifting. Before we deal with the general semi-primal case, we illustrate the results in the simpler primal case in the next section.

4.2 Lifting coalgebraic logics to primal varieties

In this section, we show how to lift abstract and concrete classical coalgebraic logics to the primal level. All of the results here (with the exception of Theorems 4.2.7 and 4.2.10) are generalized from primal to semi-primal varieties in Section 4.3. Nevertheless, taking care of the primal case provides a valuable set-up for this more general case. This section is based on [KP23], co-authored by the author of this thesis.

For this entire section, we strengthen Assumption 2.0.1 to the following.

Assumption 4.2.1. The algebra $\mathbf{D}$ is a primal algebra with a reduct $\mathbf{D}^\flat = \langle D, \wedge, \vee, 0, 1\rangle$ which is a bounded lattice. We use $\mathcal{A} := \mathbb{HSP}(\mathbf{D})$ to denote the variety generated by $\mathbf{D}$.

Note that the assumption that $\mathbf{D}$ comes equipped with a lattice structure can essentially be made without loss of generality, since every possible lattice-order on D is term-definable in a primal algebra $\mathbf{D}$.

4.2.1 Lifting abstract coalgebraic logics: Primal case

To lift abstract coalgebraic logics, we will consider various functors between Set, BA and $\mathcal{A}$ from previous chapters. The entire constellation is summarized in Figure 4.1. Here, the Boolean skeleton functor $\mathfrak{S}\colon \mathcal{A} \to \mathsf{BA}$ and the Boolean power functor $\mathfrak{P}\colon \mathsf{BA} \to \mathcal{A}$ establish a categorical equivalence as in Corollary 2.2.13.

$$\begin{tikzcd}
& \mathcal{A} \arrow[dl, "\mathsf{S}'"'] \arrow[dr, "\mathfrak{S}"] & \\
\mathsf{Set} \arrow[ur, bend left, "\mathsf{P}'"] \arrow[rr, shift left, "\mathsf{P}"] & & \mathsf{BA} \arrow[ul, bend right, "\mathfrak{P}"'] \arrow[ll, shift left, "\mathsf{S}"]
\end{tikzcd}$$

Figure 4.1: Functors between Set, BA and a primal variety $\mathcal{A}$.

The functors $\mathsf{P}\colon \mathsf{Set} \to \mathsf{BA}$ and $\mathsf{S}\colon \mathsf{BA} \to \mathsf{Set}$ are the ones already defined in Subsection 4.1.2 and P' and S' are defined similarly. That is, P' assigns the algebra $\mathsf{P}'(X) = \mathbf{D}^X$ to a set X and sends a map $f\colon X \to X'$ to the homomorphism $\mathsf{P}'f\colon \mathbf{D}^{X'} \to \mathbf{D}^X$ defined by composition $\alpha \mapsto \alpha \circ f$. The functor S' assigns the set of homomorphisms $\mathsf{S}'(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{D})$ to an algebra $\mathbf{A} \in \mathcal{A}$ and sends a homomorphism $h\colon \mathbf{A} \to \mathbf{A}'$ to the map $\mathsf{S}'h\colon \mathcal{A}(\mathbf{A}', \mathbf{D}) \to \mathcal{A}(\mathbf{A}, \mathbf{D})$ defined by composition $u \mapsto u \circ h$. Like in the case where $\mathbf{D} = \mathbf{2}$, the functors P' and S' establish a dual adjunction between Set and $\mathcal{A}$ (this also follows that P' and S' arise as compositions of functors involved in Figure 2.3). The corresponding natural transformations $\eta'\colon 1_{\mathcal{A}} \Rightarrow \mathsf{P}'\mathsf{S}'$ and $\varepsilon'\colon 1_{\mathsf{Set}} \Rightarrow \mathsf{S}'\mathsf{P}'$ are given by evaluation. In the following, we collect some useful properties which we need later on.

Lemma 4.2.2. *The functors* P,S,P',S', $\mathfrak{P}, \mathfrak{S}$ *and the natural transformations* $\varepsilon, \eta, \varepsilon', \eta'$ *satisfy the following properties.*

(1) $\phi_{\mathbf{A}}\colon \mathcal{A}(\mathbf{A}, \mathbf{D}) \to \mathsf{BA}\big(\mathfrak{S}(\mathbf{A}), \mathbf{2}\big)$ *given by restriction* $u \mapsto u|_{\mathfrak{S}(A)}$ *defines a natural isomorphism* $\mathsf{S}' \cong \mathsf{S}\mathfrak{S}$. *There also exists a natural isomorphism* $\mathsf{S} \cong \mathsf{S}'\mathfrak{P}$.

(2) $\psi_X\colon \mathbf{2}^X \to \mathfrak{S}(\mathbf{D}^X)$, *which identifies* $\mathbf{2}^X$ *with a subset of* $\mathbf{D}^X$ *in the obvious way defines a natural isomorphism* $\mathsf{P} \cong \mathfrak{S}\mathsf{P}'$. *There also exists a natural isomorphism* $\mathsf{P}' \cong \mathfrak{P}\mathsf{P}$.

(3) $\varepsilon = \mathsf{S}\psi \circ \phi\mathsf{P}' \circ \varepsilon'$ *and* $\mathfrak{S}\eta' = \psi\mathsf{S}' \circ \mathsf{P}\phi \circ \eta\mathfrak{S}$.

Proof. In both (1) and (2), the second statement is an immediate consequence of the first one because $\mathfrak{P}$ and $\mathfrak{S}$ form a categorical equivalence. The first part of (1) is (the discrete version of) Proposition 2.2.4.

For the first part of (2), note that ψ_X is well-defined since $\beta \in \mathbf{2}^X$ satisfies $T_1(\beta(x)) = \beta(x)$ in every component $x \in X$. Since the Boolean operations are defined component-wise, it is a homomorphism, and it is clearly injective. It is also surjective, since whenever an element $\alpha \in \mathbf{D}^X$ has a component with $\alpha(x) \notin \{0, 1\}$, we have $T_1(\alpha(x)) \neq \alpha(x)$. Naturality is straightforward by definition.

For (3), we need to show that the following diagrams commute for all $X \in \mathsf{Set}$ and $\mathbf{A} \in \mathcal{A}$.

$$\begin{tikzcd}
X \arrow[r, "\varepsilon_X"] \arrow[d, "\varepsilon'_X"'] & \mathsf{BA}(\mathbf{2}^X, \mathbf{2}) \\
\mathcal{A}(\mathbf{D}^X, \mathbf{D}) \arrow[r, "\phi_{\mathbf{D}^X}"'] & \mathsf{BA}(\mathfrak{S}(\mathbf{D}^X), \mathbf{2}) \arrow[u, "\mathsf{S}\psi_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{S}(\mathbf{A}) \arrow[r, "\mathfrak{S}\eta'_{\mathbf{A}}"] \arrow[d, "\eta_{\mathfrak{S}(\mathbf{A})}"'] & \mathfrak{S}(\mathbf{D}^{\mathcal{A}(\mathbf{A},\mathbf{D})}) \\
\mathbf{2}^{\mathsf{BA}(\mathfrak{S}(\mathbf{A}),\mathbf{2})} \arrow[r, "\mathsf{P}\phi_{\mathbf{A}}"'] & \mathbf{2}^{\mathcal{A}(\mathbf{A},\mathbf{D})} \arrow[u, "\psi_{\mathcal{A}(\mathbf{A},\mathbf{D})}"']
\end{tikzcd}$$

For the diagram on the left, given $x \in X$, we compute

$$\mathsf{S}\psi_X \circ \phi_{\mathbf{D}^X} \circ \varepsilon'_X(x) = \mathsf{S}\psi_X \circ \phi_{\mathbf{D}^X}(\mathsf{ev}_x) = \mathsf{S}\psi_X(\mathsf{ev}_x|_{\mathfrak{S}(\mathbf{D}^X)}) = \mathsf{ev}_x|_{\mathfrak{S}(\mathbf{D}^X)} \circ \psi_X,$$

which, on $\beta \in \mathbf{2}^X$, is given by

$$\mathsf{ev}_x|_{\mathfrak{S}(\mathbf{D}^X)} \circ \psi_X(\beta) = \mathsf{ev}_x|_{\mathfrak{S}(\mathbf{D}^X)}(\beta) = \beta(x).$$

This coincides with $\varepsilon_X(x)(\beta) = \mathsf{ev}_x(\beta) = \beta(x)$ as desired.

For the diagram on the right, given $b \in \mathfrak{S}(\mathbf{A})$, similarly we compute

$$\psi_{\mathcal{A}(\mathbf{A},\mathbf{D})} \circ \mathsf{P}\phi_{\mathbf{A}} \circ \eta_{\mathfrak{S}(\mathbf{A})}(b) = \psi_{\mathcal{A}(\mathbf{A},\mathbf{D})} \circ \mathsf{P}\phi_{\mathbf{A}}(\mathsf{ev}_b) = \psi_{\mathcal{A}(\mathbf{A},\mathbf{D})}(\mathsf{ev}_b \circ \phi_{\mathbf{A}}),$$

which is given on $u \in \mathcal{A}(\mathbf{A}, \mathbf{D})$ by

$$\psi_{\mathcal{A}(\mathbf{A},\mathbf{D})}(\mathsf{ev}_b \circ \phi_{\mathbf{A}})(u) = \psi_{\mathcal{A}(\mathbf{A},\mathbf{D})}\big(\mathsf{ev}_b(u|_{\mathfrak{S}(\mathbf{A})})\big) = u(b).$$

This coincides with $\mathfrak{S}\eta'_{\mathbf{A}}(b)(u) = \eta'_{\mathbf{A}}|_{\mathfrak{S}(\mathbf{A})}(b)(u) = u(b)$ as desired. □

Now, for an endofunctor $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$, suppose we are given a classical abstract coalgebraic logic (L, δ) with $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ and $\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$. We now lift this to an abstract coalgebraic logic (L', δ') with endofunctor $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ and one-step semantics $\delta'\colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}$ as follows.

Definition 4.2.3 (Lifting of a coalgebraic logic, primal)**.** Let (L, δ) be a classical abstract coalgebraic logic for $\mathsf{T} \colon \mathsf{Set} \to \mathsf{Set}$. Then

$$\mathsf{L}' = \mathfrak{P}\mathsf{L}\mathfrak{S} \text{ and } \delta' = \mathfrak{P}\delta$$

defines an abstract coalgebraic logic (L', δ') for T, which we call the *lifting of* (L, δ) *to* $\mathcal{A}$.

This is well-defined since, by Lemma 4.2.2(2), the natural transformation $\mathfrak{P}\delta \colon \mathfrak{P}\mathsf{LP} \to \mathfrak{P}\mathsf{PT}$ can be identified up to natural isomorphism with one from $\mathfrak{P}\mathsf{LP} \cong \mathfrak{P}\mathsf{L}\mathfrak{S}\mathsf{P}' = \mathsf{L}'\mathsf{P}'$ to $\mathfrak{P}\mathsf{PT} \cong \mathsf{P}'\mathsf{T}$. The entire situation is summarized in Figure 4.2.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[loop above, "\mathsf{L}'"] \arrow[dl, bend right=15, "\mathsf{S}'"'] \arrow[dr, bend left=15, "\mathfrak{S}"] & \\
\mathsf{Set} \arrow[loop left, "\mathsf{T}"] \arrow[ur, bend left=25, "\mathsf{P}'"] \arrow[rr, bend left=5, "\mathsf{P}"] & & \mathsf{BA} \arrow[loop right, "\mathsf{L}"] \arrow[ul, bend right=25, "\mathfrak{P}"'] \arrow[ll, bend left=5, "\mathsf{S}"]
\end{tikzcd}$$

Figure 4.2: Classical coalgebraic logic and its lifting to the primal level.

We now show that all important properties of coalgebraic logics discussed in Section 4.1 are preserved under this lifting, in particular one-step completeness (recall Definition 4.1.13) and expressivity (recall Definition 4.1.15).

Theorem 4.2.4 (Inherited properties, primal)**.** *Let* (L', δ') *be the lifting of a coalgebraic logic* (L, δ) *to* $\mathcal{A}$.

(1) If L *has a finitary presentation by operations and equations, then* L' *has one as well.*

(2) If (L, δ) *is one-step complete, then so is* (L', δ').

(3) If (L, δ) *is expressive, then so is* (L', δ').

Proof. (1): Recall that an endofunctor on a variety has a finitary presentation if and only if it preserves sifted colimits [KR12, Theorem 4.7]. Of course, if L preserves sifted colimits then, by definition, so does L'.

(2): If δ is a component-wise monomorphism, then so is δ', since $\mathfrak{P}$ preserves monomorphisms.

(3): We show that $(\delta')^\dagger = \delta^\dagger\mathfrak{S}$ holds up to natural isomorphism, from which the statement follows since it implies that if $\delta^\dagger$ is a component-wise monomorphism, then so is $(\delta')^\dagger$. So we want to show that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{TS'} \arrow[r, "\varepsilon'\mathsf{TS'}"] \arrow[d] \arrow[dr, phantom, "D_1"] & \mathsf{S'P'TS'} \arrow[r, "\mathsf{S'}\delta'\mathsf{S'}"] \arrow[d] \arrow[dr, phantom, "D_2"] & \mathsf{S'L'P'S'} \arrow[r, "\mathsf{S'L'}\eta'"] \arrow[d] \arrow[dr, phantom, "D_3"] & \mathsf{S'L'} \arrow[d] \\
\mathsf{TS}\mathfrak{S} \arrow[r, "\varepsilon\mathsf{TS}\mathfrak{S}"'] & \mathsf{SPTS}\mathfrak{S} \arrow[r, "\mathsf{S}\delta\mathsf{S}\mathfrak{S}"'] & \mathsf{SLPS}\mathfrak{S} \arrow[r, "\mathsf{SL}\eta\mathfrak{S}"'] & \mathsf{SL}\mathfrak{S}
\end{tikzcd}$$

Here, by definition, the top edge of the diagram is the adjoint-transpose $(\delta')^\dagger$ and the bottom edge is $\delta^\dagger\mathfrak{S}$. All vertical arrows are natural isomorphisms obtained via ϕ and ψ from Lemma 4.2.2. The diagram D_2 commutes by definition of δ', using that $\mathsf{S'}\delta' = \mathsf{S'}\mathfrak{P}\delta$ and $\mathsf{S'}\mathfrak{P} \cong \mathsf{S}$ by Lemma 4.2.2(1). To finish the proof we show that D_1 and D_3 commute as well.

To see that D_1 commutes, we apply the first equation of Lemma 4.2.2(3) to compute

$$\begin{aligned}
\mathsf{SPT}\Phi \circ \mathsf{S}\Psi\mathsf{TS'} \circ \Phi\mathsf{P'TS'} \circ \varepsilon'\mathsf{TS'} &= \\
\mathsf{SPT}\Phi \circ (\mathsf{S}\Psi \circ \Phi\mathsf{P'} \circ \varepsilon')\mathsf{TS'} &= \\
\mathsf{SPT}\Phi \circ \varepsilon\mathsf{TS'}&,
\end{aligned}$$

which coincides with $\varepsilon\mathsf{TS}\mathfrak{S} \circ \mathsf{T}\Phi$.

Similarly, to see that D_3 commutes we apply the second equation of Lemma 4.2.2(3) to compute

$$\begin{aligned}
\mathsf{SL}\eta\mathfrak{S} \circ \mathsf{SLP}\Phi \circ \mathsf{SL}\Psi\mathsf{S'} \circ \Phi\mathsf{L'P'S'} &= \\
\mathsf{SL}(\Psi\mathsf{S'} \circ \mathsf{P}\Phi \circ \eta\mathfrak{S}) \circ \Phi\mathsf{L'P'S'} &= \\
\mathsf{SL}\mathfrak{S}\eta' \circ \Phi\mathsf{L'P'S'}&,
\end{aligned}$$

which coincides with $\Phi\mathsf{L'} \circ \mathsf{S'L'}\eta'$. Thus, the entire diagram commutes, finishing the proof. □

If (L, δ) is a concrete coalgebraic logic, then part (1) of the Theorem 4.2.4 implies that $(\mathsf{L'}, \delta')$ can also be turned into a concrete coalgebraic. In this case, it can be shown analogously to Theorem 4.1.14 that one-step completeness implies completeness (also see Corollary 4.3.24 in the semi-primal case). Therefore, together with part (2) of Theorem 4.2.4 the lifting of a complete classical concrete coalgebraic logic is always complete with respect to T-coalgebras as well.

So we showed that the lifting (L', δ') of a coalgebraic logic (L, δ) inherits desirable properties from the original logic, which is satisfactory from a theoretical point of view. From a more practical point of view, one important question still needs to be answered, namely that of finding a concrete presentation of L' and determining its relationship to a presentation of L. Indeed, Theorem 4.2.4(1) only states that the *existence* of such a presentation is preserved, without any explicit way of obtaining it from the original one. In the following section, we give some partial solutions to this problem.

4.2.2 Lifting concrete coalgebraic logics: Primal case

In this subsection, we aim to relate presentations of $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ to presentations of the corresponding lifted functor $\mathsf{L}' = \mathfrak{P}\mathsf{L}\mathfrak{S}\colon \mathcal{A} \to \mathcal{A}$ (recall Definition 4.2.3). In the light of Chapter 3, we utilize the unary terms τ_d and η_d again.

In the following, we use D^+ to denote $D\backslash\{0\}$ and D^- to denote $D\backslash\{1\}$. Given an element $e \in D$, both the map $\tau_{(\cdot)}(e)\colon D^+ \to 2$ defined by $d \mapsto \tau_d(e)$ and the map $\eta_{(\cdot)}(e)\colon D^- \to 2$ defined similarly fully determine the element e via

$$e = \bigvee\{d \mid \tau_d(e) = 1\}$$

or

$$e = \bigwedge\{d \mid \eta_d(e) = 0\},$$

respectively. In the following, we characterize all maps of these forms by their lattice-theoretic properties.

Lemma 4.2.5. *Let $\mathcal{T}\colon D^+ \to 2$ be a map which, for all $d_1, d_2 \in D^+$, satisfies*

$$\mathcal{T}(d_1 \vee d_2) = \mathcal{T}(d_1) \wedge \mathcal{T}(d_2). \tag{4.1}$$

Then $\mathcal{T} = \tau_{(\cdot)}(e)$ for $e = \bigvee\{d \mid \mathcal{T}(d) = 1\}$.

Proof. The case $e = 0$ can only occur if $\mathcal{T}(d) = 0$ for all $d \in D^+$, which implies $\mathcal{T}(d) = 0 = \tau_d(0)$ for all $d \in D$. Now assume that $e \neq 0$. First we show that $\mathcal{T}(e) = 1$. Since e is a finite join we can apply Equation 4.1 to find

$$\mathcal{T}(e) = \mathcal{T}\big(\bigvee\{d \mid \mathcal{T}(d) = 1\}\big) = \bigwedge\{\mathcal{T}(d) \mid \mathcal{T}(d) = 1\} = 1.$$

Furthermore, since Equation 4.1 implies that $\mathcal{T}$ is order-reversing, we have $\mathcal{T}(c) = 1$ for all $c \leq e$ as well. Now let $c \not\leq e$. Then we have $\mathcal{T}(c) = 0$, since otherwise $\mathcal{T}(c) = 1$ leads to the contradiction

$$e = \bigvee\{d \mid \mathcal{T}(d) = 1\} \geq e \vee c > e.$$

Altogether, we have shown that $\mathcal{T}(d) = 1$ if and only if $e \geq d$, so $\mathcal{T}(d) = \tau_d(e)$. □

Of course, the following can be shown completely analogously by order-duality.

Lemma 4.2.6. *Let $\mathcal{H}\colon D^- \to 2$ be a map which, for all $d_1, d_2 \in D^+$, satisfies*

$$\mathcal{H}(d_1 \wedge d_2) = \mathcal{H}(d_1) \vee \mathcal{H}(d_2). \tag{4.2}$$

Then $\mathcal{H} = \eta_{(\cdot)}(e)$ for $e = \bigwedge\{d \mid \mathcal{H}(d) = 0\}$.

Suppose that $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ has a presentation by one unary operation, denoted $\Box$, and equations which are satisfied by the terms τ_d, in the sense that all the equations obtained by replacing $\Box$ by any τ_d hold in $\mathbf{D}$. In particular, this is true for the equations $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$ from Example 4.1.7. Under these circumstances, we can find a presentation of the corresponding lifted functor $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ as follows. The idea is to 'approach' a presentation of L' by introducing a modal operator for every $d \in D^+$, intended to correspond to $\tau_d\Box$ for the 'lifted' $\Box'$. However, only if these modal operators are 'consistent' in the sense of Lemma 4.2.5, we can replace them by a single operator again.

Theorem 4.2.7 (Lifting presentations, primal)**.** *Let $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ have a presentation by one unary operation $\Box$ and equations which are satisfied (in $\mathbf{D}$) by all τ_d, $d \in D^+$. Let $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ be the lifting of L.*

(1) The functor L' can be presented by unary operations $\Box'_d$ for every $d \in D^+$ and the following equations.

  - *The equations for $\Box$, where $\Box$ is replaced by $\Box'_1$.*
  - *$\Box'_1\tau_d(x) = \Box'_d x$ for all $d \in D^+$.*
  - *$T_1(\Box'_d x) = \Box'_d x$ for all $d \in D^+$.*

(2) If, in the variety $\mathsf{Alg}(\mathsf{L}')$ axiomatized by the presentation of (1), the equation

$$\Box_{d_1 \vee d_2} x = \Box_{d_1} x \wedge \Box_{d_2} x \tag{4.3}$$

holds for all $d_1, d_2 \in D^+$, then L' can also be presented by one unary operation $\Box'$ and the following equations.

- *The equations for $\Box$, where $\Box$ is replaced by $\Box'$.*
- $\Box'\tau_d(x) = \tau_d(\Box' x)$ *for all* $d \in D^+$*.*

Proof. (1): Let $\mathsf{L}^+ \colon \mathcal{A} \to \mathcal{A}$ be the functor presented by the operations $\Box_d$ and equations as in the statement. We want to show that L' is naturally isomorphic to L^+. Since both these functors are finitary (because they preserve sifted colimits, in particular they preserve filtered colimits), it suffices to show that their restrictions to finite algebras are naturally isomorphic. The restrictions of P and S to the categories Set^ω of finite sets and BA^ω of finite Boolean algebras form a dual equivalence. Similarly, the restrictions of P' and S' form a dual equivalence between Set^ω and $\mathcal{A}^\omega$ (see Theorem 2.1.8). Therefore, it suffices to show

$$\mathsf{S}'\mathsf{L}^+\mathsf{P}' \cong \mathsf{SLP},$$

since, due to Lemma 4.2.2, for the right-hand side we have further natural isomorphisms $\mathsf{SLP} \cong \mathsf{S}'\mathfrak{P}\mathsf{L}\mathfrak{S}\mathsf{P}' = \mathsf{S}'\mathsf{L}'\mathsf{P}'$.

Spelling this out, we want to find a bijection between the sets of homomorphisms $\mathcal{A}(\mathsf{L}^+(\mathbf{D}^X), \mathbf{D})$ and $\mathsf{BA}(\mathsf{L}(\mathbf{2}^X), \mathbf{2})$ which is natural in $X \in \mathsf{Set}$. By definition of L^+, the set $\mathcal{A}(\mathsf{L}^+(\mathbf{D}^X), \mathbf{D})$ can be naturally identified with the collection of all maps (whose domain is simply a set of formal expressions)

$$f \colon \{\Box_d a \mid d \in D^+, a \in D^X\} \to D, \text{ where } f \text{ respects the equations of } \mathsf{L}^+.$$

Similarly, the set $\mathsf{BA}(\mathsf{L}(\mathbf{2}^X), \mathbf{2})$ can be naturally identified with the collection of all maps

$$g \colon \{\Box b \mid b \in 2^X\} \to 2, \text{ where } g \text{ respects the equations of } \mathsf{L}.$$

Given f as above, we assign to it g_f defined by

$$g_f(\Box b) = f(\Box_1 b).$$

This is well-defined, since $T_1(f(\Box_1 b)) = f(\Box_1 b)$ implies $f(\Box_1 b) \in 2$, and g_f respects the equations of L, because f does for $\Box$ replaced by $\Box_1$.

Conversely, given g as above, we assign to it f_g defined by

$$f_g(\Box_d a) = g\big(\Box \tau_d(a)\big).$$

Since the equations of L are satisfied by τ_d and respected by g, they are also respected by f_g. The remaining equations of L^+ are respected by f_g, since, for all $d \in D^+$ we can directly verify

$$f_g\big(\Box_1 \tau_d(a)\big) = g\big(\Box T_1(\tau_d(a))\big) = g\big(\Box \tau_d(a)\big) = f_g(\Box_d a),$$

where we used $T_1(\tau_d(a)) = \tau_d(a)$ since $\tau_d(a) \in 2^X$ and

$$T_1\big(f_g(\Box_d a)\big) = T_1\big(g(\Box\tau_d(a))\big) = g\big(\Box\tau_d(a)\big) = f_g(\Box_d a),$$

where we used $T_1(g(\Box\tau_d a)) = g(\Box\tau_d a)$ since $g(\Box\tau_d a) \in 2$.

Now we show that these two assignments are mutually inverse. For this we compute

$$f_{g_f}(\Box_d a) = g_f\big(\Box\tau_d(a)\big) = f\big(\Box_1\tau_d(a)\big) = f(\Box_d a),$$

where in the last equation we used that f respects the corresponding equation of L^+ and

$$g_{f_g}(\Box b) = f_g(\Box_1 b) = g\big(\Box T_1(b)\big) = g(\Box b),$$

where in the last equation we used $b \in 2^X$ again.

For naturality, we need to show that, given a map $m\colon X_1 \to X_2$, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{A}(\mathsf{L}^+(\mathbf{D}^{X_1}), \mathbf{D}) \arrow[r, "g_{(\cdot)}"] \arrow[d, "\mathsf{S}'\mathsf{L}^+\mathsf{P}'m"'] & \mathsf{BA}(\mathsf{L}(\mathbf{2}^{X_1}), \mathbf{2}) \arrow[d, "\mathsf{SLP}m"] \\
\mathcal{A}(\mathsf{L}^+(\mathbf{D}^{X_2}), \mathbf{D}) \arrow[r, "g_{(\cdot)}"'] & \mathsf{BA}(\mathsf{L}(\mathbf{2}^{X_2}), \mathbf{2})
\end{tikzcd}$$

Let $f\colon \{\Box_d a \mid d \in D^+, a \in D^{X_1}\} \to D$ be given as before. On the one hand, for $\alpha \in D^{X_2}$ and $\beta \in 2^{X_2}$ we have $\mathsf{S}'\mathsf{L}^+\mathsf{P}'m(f)(\Box_d\alpha) = f(\Box_d(\alpha \circ m))$ and therefore $g_{\mathsf{S}'\mathsf{L}^+\mathsf{P}'m(f)}(\Box\beta) = f(\Box_1(\beta \circ m))$. On the other hand, $\mathsf{SLP}m(g_f)(\Box\beta) = g_f(\Box(\beta \circ m)) = f(\Box_1(\beta \circ m))$. Thus, the diagram commutes.

(2): Let $\mathsf{L}^\star\colon \mathcal{A} \to \mathcal{A}$ be defined by one unary operation $\Box'$ and equations as in the statement and let L^+ be defined as in the proof of (1). For the same reason as before, it suffices to show

$$\mathsf{S}'\mathsf{L}^\star\mathsf{P}' \cong \mathsf{S}'\mathsf{L}^+\mathsf{P}'.$$

Again, $\mathsf{S}'\mathsf{L}^+\mathsf{P}'(X) = \mathcal{A}(\mathsf{L}^+(\mathbf{D}^X), \mathbf{D})$ is essentially the collection of maps

$$f\colon \{\Box_d a \mid d \in D^+, a \in D^X\} \to D, \text{ where } f \text{ respects the equations of } \mathsf{L}^+,$$

and $\mathsf{S}'\mathsf{L}^\star\mathsf{P}'(X)$ is essentially the collection of maps

$$h\colon \{\Box a \mid a \in D^X\} \to D, \text{ where } h \text{ respects the equations of } \mathsf{L}^\star.$$

Given h as above, we send it to

$$f_h(\Box_d a) = h\big(\Box' \tau_d(a)\big).$$

Checking that this is well-defined is routine by now, the only non-trivial part being

$$T_1\big(f_h(\Box_d(a)\big) = T_1\big(h(\Box' \tau_d(a))\big) = h\big(\Box' T_1(\tau_d(a))\big) = f_h(\Box_d a),$$

which uses the fact that h respects the corresponding equation $\Box' T_1(x) = T_1(\Box' x)$ of $\mathsf{L}^\star$.

Conversely, given f as above, we assign to it

$$h_f(\Box' a) = \bigvee \{c \mid f(\Box_c a) = 1\}.$$

First, given $d \in D^+$, using that $\tau_c \circ \tau_d = \tau_d$ holds for all $c \in D^+$, we note

$$\begin{aligned} h_f\big(\Box' \tau_d(a)\big) &= \bigvee \{c \mid f\big(\Box_c \tau_d(a)\big) = 1\} \\ &= \bigvee \{c \mid f\big(\Box_1 \tau_c(\tau_d(a))\big) = 1\} \\ &= \bigvee \{c \mid f(\Box_d a) = 1\}. \end{aligned}$$

Since, on the right-hand side, the formula $f(\Box_d a) = 1$ is independent of c, this join is either equal to $\bigvee \varnothing = 0$ if $f(\Box_d a) = 0$ or $\bigvee D^+ = 1$ if $f(\Box_d a) = 1$. On the other hand, by assumption we can apply Lemma 4.2.5, which yields

$$\tau_d\big(h_f(\Box' a)\big) = \tau_d\Big(\bigvee \{c \mid f(\Box_c a) = 1\}\Big) = f(\Box_d a)$$

as well.

The two assignments $f \mapsto h_f$ and $h \mapsto f_h$ thus defined are mutually inverse since

$$f_{h_f}(\Box_d a) = h_f\big(\Box' \tau_d(a)\big) = \bigvee \{c \mid f\big(\Box_c \tau_d(a)\big) = 1\} = f(\Box_d a)$$

holds again by Lemma4.2.5 and

$$h_{f_h}(\Box' a) = \bigvee \{c \mid h\big(\Box' \tau_c(a)\big) = 1\} = \bigvee \{c \mid \tau_c\big(h(\Box' a)\big) = 1\} = h(\Box' a).$$

Analogous to (1), it is straightforward to show that the isomorphism thus defined is natural. $\Box$

Since the above proof is already rather technical, we only presented it for the case of one unary operation, but there is a straightforward generalization of Theorem 4.2.7 to presentations of L by one operation which is not necessarily unary (the operations $\Box_d$ and $\Box'$ will simply have the same arity).

In particular, part (2) of this theorem applies if the 'original' operation $\Box$ preserves meets, as shown in the following.

Corollary 4.2.8. *Let* L *be as in Theorem 4.2.7, such that* $\Box(x\wedge y) = \Box x \wedge \Box y$ *holds in the variety* $\mathsf{Alg}(\mathsf{L})$*. Then the lifting* L' *can be presented by one unary operation* $\Box'$ *and the following equations.*

- *The equations for* $\Box$*, where* $\Box$ *is replaced by* $\Box'$*.*
- $\Box'\tau_d(x) = \tau_d(\Box' x)$ *for all* $d \in D^+$*.*

Proof. We verify Equation 4.3 from Theorem 4.2.7(2) by

$$\begin{aligned}\Box_{d_1\vee d_2} x &= \Box_1 \tau_{d_1\vee d_2}(x)\\ &= \Box_1\big(\tau_{d_1}(x) \wedge \tau_{d_2}(x)\big)\\ &= \Box_1\tau_{d_1}(x) \wedge \Box_1\tau_{d_2}(x) = \Box_{d_1} x \wedge \Box_{d_2} x,\end{aligned}$$

and the statement immediately follows from that theorem. ☐

If (L,δ) is a concrete coalgebraic logic for $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$, where $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ is endowed with a presentation which satisfies the condition of Theorem 4.2.7, it is now easy to describe the lifting (L',δ') as a concrete coalgebraic logic as well. The only missing piece is an explicit description of the natural transformation $\delta'\colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}$. As seen in the proof of Theorem 4.2.7, for a set X the component $\delta'_X\colon \mathsf{L}'(\mathbf{D}^X) \to \mathbf{D}^{\mathsf{T}(X)}$ is defined on $Z \subseteq X$ by

$$\delta'_X(\Box_d a)(Z) = \delta_X(\Box\tau_d(a))(Z).$$

Given that the additional condition of part (2) of Theorem 4.2.7 is also satisfied, δ' can be described as

$$\delta'_X(\Box a)(Z) = \bigvee\{d \mid \delta\big(\Box\tau_d(a)\big)(Z) = 1\}.$$

In the following, we show that the machinery developed in this subsection works well with respect to classical modal logic as considered in Example 4.1.9.

Example 4.2.9 (Lifting classical modal logic to the primal level)**.** Let $(\mathsf{L}_\Box,\delta)$ be the coalgebraic logic for $\mathcal{P}$ which corresponds to classical modal logic as in Example 4.1.9, in particular $\mathsf{L}_\Box\colon \mathsf{BA} \to \mathsf{BA}$ is presented by a unary operation $\Box$ and the equations $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$.

Let $(\mathsf{L}'_\Box,\delta')$ be the lifting of $(\mathsf{L}_\Box,\delta)$ to $\mathcal{A}$. By Corollary 4.2.8, we know that L' has a presentation by a unary operation $\Box'$ and equations

$$\Box'(x\wedge y) = \Box' x \wedge \Box' y,\ \ \Box' 1 = 1 \text{ and } \tau_d(\Box' x) = \Box'\tau_d(x) \text{ for all } d \in D^+.$$

Note that the variety $\mathsf{Alg}(\mathsf{L}')$ is exactly the variety $\mathcal{A}^\Box$ from Definition 3.3.8.

The natural transformation δ' has components $\delta'_X \colon \mathsf{L}'(\mathbf{D}^X) \to \mathbf{D}^{\mathcal{P}(X)}$, defined by

$$\delta'_X(\Box' a)(Z) = \bigvee \{d \mid \delta_X\big(\Box \tau_d(a)\big)(Z) = 1\}.$$

Now, since $\delta_X(\Box \tau_d(a))(Z) = 1 \Leftrightarrow \forall z \in Z : \tau_d(a(z)) = 1 \Leftrightarrow \forall z \in Z : a(z) \geq d$, we can rewrite this as

$$\begin{aligned} \bigvee \{d \mid \delta_X\big(\Box \tau_d(a)\big)(Z) = 1\} &= \bigvee \{d \mid \bigwedge_{y \in Y} a(z) \geq d\} \\ &= \bigvee \{d \mid \tau_d\big(\bigwedge_{z \in Z} a(z)\big) = 1\} \\ &= \bigwedge_{z \in Z} a(z). \end{aligned}$$

Thus, this corresponds to the usual semantics of a many-valued box over Kripke frames defined via meet (recall Section 3.1).

Since we know that $(\mathsf{L}_\Box, \delta)$ is one-step complete (and thus complete), by Theorem 4.2.4(2) (and Theorem 4.1.14), the logic $(\mathsf{L}'_\Box, \delta')$ is one-step complete (and thus complete) as well. This corresponds to Corollary 3.3.9(1).

Furthermore, from Theorem 4.2.4(3) we conclude that, replacing $\mathcal{P}$ by the finite-powerset functor $\mathcal{P}_\omega$, the logic (L', δ') is expressive for image-finite frames. This corresponds to the Hennessy-Milner property we proved directly in Theorem 3.2.9(2). ■

The applicability of Theorem 4.2.7 does depend on the specific choice of a presentation of L. For instance, we already noted that the functor $\mathsf{L}_\Box$ is naturally isomorphic to $\mathsf{L}_\Diamond$ which is presented by one unary operation $\Diamond$ and equations $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ and $\Diamond 0 = 0$. If $\mathbf{D}$ is not linear, it is easy to check that $\tau_d(x \vee y) = \tau_d(x) \vee \tau_d(y)$ does not hold in general (simply choose incomparable elements x and y and set $d = x \vee y$). Therefore, the presentation of $\mathsf{L}_\Diamond$ can not be lifted via Theorem 4.2.7. However, with Lemma 4.2.6, the following order-dual version of Theorem 4.2.7 can be applied in this case.

Theorem 4.2.10. *Let* $\mathsf{L} \colon \mathsf{BA} \to \mathsf{BA}$ *have a presentation by one unary operation* $\Diamond$ *and equations which are satisfied (in* $\mathbf{D}$*) by all* η_d*,* $d \in D^-$*. Let* $\mathsf{L}' \colon \mathcal{A} \to \mathcal{A}$ *be the lifting of* L*.*

(1) *The functor* L' *can be presented by unary operations* $\Diamond'_d$ *for every* $d \in D^-$ *and the following equations.*
  - *The equations for* $\Diamond$*, where* $\Diamond$ *is replaced by* $\Diamond'_0$*.*
  - $\Diamond'_0 \eta_d(x) = \Diamond'_d x$ *for all* $d \in D^-$*.*

- $T_1(\Diamond'_d x) = \Diamond'_d x$ *for all* $d \in D^-$.

(2) *If, in the variety* $\mathsf{Alg}(\mathsf{L}')$ *axiomatized by the presentation of (1), the equation*
$$\Diamond_{d_1 \wedge d_2} x = \Diamond_{d_1} x \vee \Diamond_{d_2} x \tag{4.4}$$
holds for all $d_1, d_2 \in D^-$, *then* L' *can also be presented by one unary operation* $\Diamond'$ *and the following equations.*

- *The equations for* $\Diamond$, *where* $\Diamond$ *is replaced by* $\Diamond'$.
- $\Diamond' \eta_d(x) = \eta_d(\Diamond' x)$ *for all* $d \in D^-$.

Analogous to Corollary 4.2.8, Equation 4.4 of Theorem 4.2.10 can be deduced if $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ holds in $\mathsf{Alg}(\mathsf{L})$. Thus, we can also concretely present the lifting $(\mathsf{L}'_\Diamond, \delta')$ of classical modal logic, presenting $\mathsf{L}'_\Diamond$ by one unary operation $\Diamond'$ satisfying

$$\Diamond'(x \vee y) = \Diamond' x \vee \Diamond' y, \ \ \Diamond' 1 = 1 \text{ and } \eta_d(\Diamond' x) = \Diamond' \eta_d(x) \text{ for all } d \in D^-.$$

Thus we have $\mathsf{Alg}(\mathsf{L}'_\Diamond) \cong \mathcal{A}^\Diamond$.

The semantics of $\Diamond'$ are (as usual for many-valued diamonds over Kripke frames) defined by joins, that is, for $a \in \mathbf{D}^X$ and $Y \in \mathcal{P}(X)$ we have

$$\delta'_X(\Diamond' a)(Y) = \bigvee_{y \in Y} a(y).$$

Thus we get completeness and (with $\mathcal{P}_\omega$ instead of $\mathcal{P}$) expressivity for $(\mathsf{L}'_\Diamond, \delta')$ (which were proved directly in Corollary 3.3.9(2) and Theorem 3.2.9(2), respectively) by Theorem 4.2.4 and the respective properties of $(\mathsf{L}_\Diamond, \delta)$ again.

We finish this subsection with an example to illustrate a situation where part (1) of Theorem 4.2.7 can be applied, but part (2) can not.

Example 4.2.11 (Lifting classical non-normal modal logic to primal level)**.** Let $\mathsf{T} = \mathcal{N}$ be the neighborhood functor and let $(\mathsf{L}_\triangle, \delta)$ be the corresponding concrete coalgebraic logic from Example 4.1.10, that is, $\mathsf{L}_\triangle$ has a presentation by one unary operation $\triangle$ and no equations.

Recall that the natural transformation δ has components $\delta_X \colon \mathsf{L}_\triangle(\mathbf{2}^X) \to \mathbf{2}^{\mathcal{N}(X)}$ defined by
$$\delta_X(\Box b)(N) = N(b),$$
in other words, $\delta_X(\Box b)(N) = 1$ if and only if the subset $b \in 2^X$ is an element of the collection of neighborhoods N.

Since the presentation of L doesn't include any equations, it trivially satisfies the conditions of Theorem 4.2.7. Therefore, the lifting $(\mathsf{L}'_\triangle, \delta')$ of the above logic to $\mathcal{A}$ can be described as follows. The functor $\mathsf{L}' \colon \mathcal{A} \to \mathcal{A}$ has a presentation by unary operations $\triangle'_d$ for all $d \in D^+$ with equations

$$\triangle_1 \tau_d(x) = \triangle_d x \text{ and } T_1(\triangle_d x) = \triangle_d x \text{ for all } d \in D^+.$$

The semantics δ' can be described by

$$\delta'_X(\triangle_d a)(N) = \delta_X\big(\triangle \tau_d(a)\big)(N) = N\big(\tau_d(a)\big),$$

which means that $\delta'_X(\triangle_d a) = 1$ if and only if the subset $\{x \in X \mid a(x) \geq d\}$ is an element of the collection of neighborhoods N. Therefore, it can easily be shown by counter-example that $\triangle_{d_1 \vee d_2} x = \triangle_{d_1} x \wedge \triangle_{d_2} x$ does not hold in $\mathsf{Alg}(\mathsf{L}'_\triangle)$, which means that the above presentation can not be simplified to the one using a single unary operation via Theorem 4.2.7(2). At this point, the question whether or not the presentation can be simplified to one which only uses one unary operation in a different way remains open.

Since $(\mathsf{L}_\triangle, \delta)$ is one-step complete and expressive for image-finite neighborhood frames, we can again immediately conclude that the same is true for $(\mathsf{L}'_\triangle, \delta')$ by Theorem 4.2.4. ■

Note that if we replace the functor $\mathcal{N}$ in the above example by its subfunctor which only allows collections of neighborhoods which are closed under finite intersections and supersets (that is, collections of neighborhoods which are filters or empty), we know that there is a corresponding concrete coalgebraic logic (L, δ) such that the presentation of L contains the equation $\Box(x \wedge y) = \Box x \wedge \Box y$. Thus, Corollary 4.2.8 applies in this case again.

In the following section, we aim to obtain similar results for the more general case where $\mathbf{D}$ can be semi-primal but not primal. In this case, the base category $\mathsf{Set}_{\mathbf{D}}$ also enters the picture, and we lift both functors T and L in ‘parallel’.

4.3 Lifting coalgebraic logics to semi-primal varieties

In this section, we show how to lift classical algebra-coalgebra dualities and classical coalgebraic logics to the level of $\mathsf{Stone}_{\mathbf{D}}$ and $\mathsf{Set}_{\mathbf{D}}$, respectively. From now on, we again work under Assumption 2.0.1. The section may be seen as an extended version of [KPT24a], co-authored by the author of this thesis.

This section is structured as follows. In Subsection 4.3.1, we show how to lift algebra-coalgebra dualities from the classical case to their semi-primal analogues (Theorem 4.3.4). In particular, we exemplify this with Jónsson-Tarski duality (Theorem 4.3.11) and Došen duality (Theorem 4.3.14). In

Subsection 4.3.2, we similarly show how to lift classical abstract coalgebraic logics to the semi-primal level. We show that one-step completeness and expressivity stay preserved under this process (Theorems 4.3.21 and 4.3.22, respectively). Lastly, in Subsection 4.3.3, we show how to obtain axiomatizations of these lifted logics in particular cases and we discuss the case of lifting classical modal logic (Example 4.3.27) and non-normal modal logic (Example 4.3.28).

4.3.1 Lifting algebra-coalgebra dualities

In this subsection, we describe a canonical way to lift endofunctors on Stone to ones one $\mathsf{Stone}_{\mathbf{D}}$ and, dually, to lift endofunctors on BA to ones on $\mathcal{A}$. In particular, if $\mathsf{T}\colon \mathsf{Stone} \to \mathsf{Stone}$ and $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ are dual (in the sense that $\mathsf{L} \cong \Pi\mathsf{T}\Sigma$), then their respective liftings $\mathsf{T}'\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}_{\mathbf{D}}$ and $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ are dual as well (in the sense that $\mathsf{L}' \cong \Pi'\mathsf{T}'\Sigma'$). For example, the 'semi-primal version' of Jónsson-Tarski duality which Maruyama established directly in [Mar12] can also be obtained by lifting the classical Jónsson-Tarski duality in this systematic way we describe here (Theorem 4.3.11). Other dualities that can be lifted to the 'semi-primal level' include Došen duality [Doš89] as framed in [BBdG22] by algebras and coalgebras (Theorem 4.3.14).

In order to lift endofunctors defined on Stone or BA, we will make use of the *subalgebra adjunctions* from Subsection 2.2.4. An overview of the functors involved in this process is given in Figure 4.3.

$$\begin{tikzcd}
\mathsf{Stone}_{\mathbf{D}} \arrow[loop left, "\mathsf{T}'"] \arrow[r, bend left, "\Pi'"] \arrow[d, bend left, "\mathsf{C}^{\mathbf{S}}"] & \mathcal{A} \arrow[loop right, "\mathsf{L}'"] \arrow[l, bend left, "\Sigma'"] \arrow[d, bend left, "\mathfrak{S}_{\mathbf{S}}"] \\
\mathsf{Stone} \arrow[loop left, "\mathsf{T}"] \arrow[r, bend left, "\Pi"] \arrow[u, bend left, "\mathsf{V}^{\mathbf{S}}"] \arrow[u, phantom, "\dashv"] & \mathsf{BA} \arrow[loop right, "\mathsf{L}"] \arrow[l, bend left, "\Sigma"] \arrow[u, bend left, "\mathfrak{P}_{\mathbf{S}}"] \arrow[u, phantom, "\vdash"]
\end{tikzcd}$$

Figure 4.3: Lifting algebra-coalgebra dualities via subalgebra adjunctions.

In the following, we show that the subalgebra adjunctions can be used to reconstruct (up to isomorphism) an object $(X, \mathbf{v}) \in \mathsf{Stone}_{\mathbf{D}}$ from the information carried by all $\mathsf{V}^{\mathbf{S}}\mathsf{C}^{\mathbf{S}}(X, \mathbf{v})$ as a *coend* (for the general theory of ends and coends see, *e.g.*, [ML97, pp. 222-227]).

More specifically, considering $\mathbb{S}(\mathbf{D})$ as a posetal category ordered by inclusion, we define a coend diagram $\mathbb{S}(\mathbf{D})^{\mathrm{op}} \times \mathbb{S}(\mathbf{D}) \to \mathsf{Stone}_{\mathbf{D}}$ corresponding to $(X, \mathbf{v})$ as follows. A pair of subalgebras $(\mathbf{S}, \mathbf{T})$ gets assigned to $\mathsf{V}^{\mathbf{S}}\mathsf{C}^{\mathbf{T}}(X, \mathbf{v})$ and if $\mathbf{S}_1 \leq \mathbf{S}_2$ and $\mathbf{T}_1 \leq \mathbf{T}_2$, then the inclusion $\mathsf{C}^{\mathbf{T}_1}(X, \mathbf{v}) \hookrightarrow \mathsf{C}^{\mathbf{T}_2}(X, \mathbf{v})$ yields a well-defined morphism $\mathsf{V}^{\mathbf{S}_2}\mathsf{C}^{\mathbf{T}_1}(X, \mathbf{v}) \hookrightarrow \mathsf{V}^{\mathbf{S}_1}\mathsf{C}^{\mathbf{T}_2}(X, \mathbf{v})$ in $\mathsf{Stone}_{\mathbf{D}}$.

Proposition 4.3.1. *Let* $(X, \mathbf{v}) \in \mathsf{Stone}_{\mathbf{D}}$. *Then* $(X, \mathbf{v})$ *is isomorphic to the coend of the diagram defined above, that is,*

$$(X, \mathbf{v}) \cong \int^{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathsf{V}^{\mathbf{S}}\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}).$$

Proof. The inclusion maps $\iota_{\mathbf{S}} \colon \mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) \hookrightarrow X$ are morphisms $\mathsf{V}^{\mathbf{S}}\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) \hookrightarrow (X, \mathbf{v})$ because if $x \in \mathsf{C}^{\mathbf{S}}(X)$, then $\mathbf{v}(x) \leq \mathbf{S}$ and thus $\mathbf{v}(\iota(x)) = \mathbf{v}(x) \leq \mathbf{S} = \mathbf{v}^{\mathbf{S}}(x)$. Since the diagram only contains inclusion maps, clearly this defines a cowedge.

Now assume that $c_{\mathbf{S}} \colon \mathsf{V}^{\mathbf{S}}\mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$ is another cowedge. Then the underlying map of $c_{\mathbf{D}} \colon X_1 \to X_2$ yields a well-defined morphism from $(X_1, \mathbf{v}_1)$ to $(X_2, \mathbf{v}_2)$. To see this, take $x \in X_1$ and note that $\mathbf{v}_2(c_{\mathbf{D}}(x)) \leq \mathbf{S}$ whenever $x \in \mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1)$, because the diagram

$$\begin{tikzcd}
\mathsf{V}^{\mathbf{D}}\mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1) \arrow[r, hook] \arrow[d, hook] & \mathsf{V}^{\mathbf{S}}\mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1) \arrow[d, "c_{\mathbf{S}}"] \\
\mathsf{V}^{\mathbf{D}}\mathsf{C}^{\mathbf{D}}(X_1, \mathbf{v}_1) \arrow[r, "c_{\mathbf{D}}"'] & (X_2, \mathbf{v}_2)
\end{tikzcd}$$

commutes. Therefore, we have $\mathbf{v}_2(c_{\mathbf{D}}(x)) \leq \bigwedge\{\mathbf{S} \mid x \in \mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1)\} = \mathbf{v}_1(x)$, which finishes the proof. □

By the dualities between $\mathsf{C}^{\mathbf{S}}$ and $\mathfrak{S}_{\mathbf{S}}$ and $\mathsf{V}^{\mathbf{S}}$ and $\mathfrak{P}_{\mathbf{S}}$ established in Subsection 2.2.4, we immediately get the following statement about dually recovering algebras $\mathbf{A} \in \mathcal{A}$ as a certain *ends*.

Corollary 4.3.2. *Let* $\mathbf{A} \in \mathcal{A}$. *Then* $\mathbf{A}$ *is isomorphic to the end of the diagram dual to the coend diagram corresponding to* $\Sigma'(\mathbf{A})$, *that is,*

$$\mathbf{A} \cong \int_{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}).$$

The presentations of Proposition 4.3.1 and Corollary 4.3.2 yield canonical ways to lift functors from Stone to $\mathsf{Stone}_{\mathbf{D}}$ and from BA to $\mathcal{A}$ as follows.

Definition 4.3.3 (Lifting of Stone- or $\mathcal{A}$-endofunctor)**.** Let $\mathsf{T}\colon \mathsf{Stone} \to \mathsf{Stone}$ and $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ be endofunctors.

(1) The *lifting of* T *to* $\mathsf{Stone}_{\mathbf{D}}$ is the functor $\mathsf{T}'\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}_{\mathbf{D}}$ defined on objects by

$$\mathsf{T}'(X, \mathbf{v}) = \int^{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X, \mathbf{v}).$$

(2) The *lifting of* L *to* $\mathcal{A}$ is the functor $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ defined on objects by

$$\mathsf{L}'(\mathbf{A}) = \int_{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}).$$

The definitions of T' and L' on morphisms are discussed in the next paragraph.

Let $f\colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$ be a $\mathsf{Stone}_{\mathbf{D}}$-morphism. Then we can define $\mathsf{T}'f$ by the universal property of the coend, once we define a cowedge $\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1) \to \mathsf{T}'(X_2, \mathbf{v}_2)$ as follows.

$$\begin{tikzcd}
\mathsf{V}^{\mathbf{S}_2}\mathsf{T}\mathsf{C}^{\mathbf{S}_2}(X_1, \mathbf{v}_1) \arrow[r] & \mathsf{V}^{\mathbf{S}_2}\mathsf{T}\mathsf{C}^{\mathbf{S}_2}(X_2, \mathbf{v}_2) \arrow[dr] & \\
\mathsf{V}^{\mathbf{S}_2}\mathsf{T}\mathsf{C}^{\mathbf{S}_1}(X_1, \mathbf{v}_1) \arrow[r] \arrow[u] \arrow[d] & \mathsf{V}^{\mathbf{S}_2}\mathsf{T}\mathsf{C}^{\mathbf{S}_1}(X_2, \mathbf{v}_2) \arrow[u] \arrow[d] & \mathsf{T}'(X_2, \mathbf{v}_2) \\
\mathsf{V}^{\mathbf{S}_1}\mathsf{T}\mathsf{C}^{\mathbf{S}_1}(X_1, \mathbf{v}_1) \arrow[r] & \mathsf{V}^{\mathbf{S}_1}\mathsf{T}\mathsf{C}^{\mathbf{S}_1}(X_2, \mathbf{v}_2) \arrow[ur] &
\end{tikzcd}$$

Here, we have $\mathbf{S}_1 \leq \mathbf{S}_2$, all vertical arrows arise from inclusion mappings and all horizontal arrows are defined by application of the corresponding functors to f. The triangle on the right commutes because $\mathsf{T}'(X_2, \mathbf{v}_2)$ is a cowedge and the two smaller rectangles commute by functoriality of T.

We define L' on morphisms in a similar manner by duality. Since the definitions of T' and L' are completely dual, the following is obvious.

Theorem 4.3.4 (Lifting algebra-coalgebra dualities)**.** *If* $\mathsf{T}\colon \mathsf{Stone} \to \mathsf{Stone}$ *and* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *are dual (that is,* $\mathsf{L} \cong \Pi\mathsf{T}\Sigma$*), then the corresponding liftings* $\mathsf{T}'\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}_{\mathbf{D}}$ *and* $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$ *are dual as well (that is,* $\mathsf{L}' \cong \Pi'\mathsf{T}'\Sigma'$*).*

For example, the 'semi-primal version' of Jónsson-Tarski duality due to Maruyama [Mar12] can be obtained from the (usual) Jónsson-Tarski duality by this method. To illustrate this, we first show that there is an easier description of T', given that T *preserves mono- and epimorphisms.* If T preserves monomorphisms, then it also preserves finite intersections in the sense of [Trn69, Proposition 2.1] (the proof therein still works for Stone instead of Set, which is easy to check). It also preserves restrictions of morphisms $f\colon X \to Y$ to $X_0 \subseteq X$ in the sense that $\mathsf{T}(f|_{X_0}) = (\mathsf{T}f)|_{\mathsf{T}X_0}$ if we identify $\mathsf{T}X_0$ with a subset of X. If T also preserves epimorphisms, then T preserves images in the sense that $\mathsf{T}(f(X)) = \mathsf{T}f(\mathsf{TX})$ for every $f\colon X \to Y$ (since Stone is a regular category in which all epimorphisms are regular).

Proposition 4.3.5. *Let* $\mathsf{T}\colon \mathsf{Stone} \to \mathsf{Stone}$ *preserve mono- and epimorphisms. Then the following functor* $\hat{\mathsf{T}}\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}_{\mathbf{D}}$ *is naturally isomorphic to the lifting* T'*. On objects* $(X, \mathbf{v}) \in \mathsf{Stone}_{\mathbf{D}}$*, the functor* $\hat{\mathsf{T}}$ *is defined by*

$$\hat{\mathsf{T}}(X, \mathbf{v}) = (\mathsf{T}(X), \hat{\mathbf{v}}),$$

where, for $Z \in \mathsf{T}(X)$*, considering* $\mathsf{TC}^{\mathbf{S}}(X, \mathbf{v})$ *as subspace of* $\mathsf{T}(X)$*,*

$$\hat{\mathbf{v}}(Z) = \bigwedge\{\mathbf{S} \mid Z \in \mathsf{TC}^{\mathbf{S}}(X, \mathbf{v})\}.$$

On morphisms, $\hat{\mathsf{T}}$ *acts precisely like* T*.*

Proof. First note that $\hat{T}$ is well-defined on objects because $\mathsf{C}^{\mathbf{S}}\hat{\mathsf{T}}(X, \mathbf{v}) \cong \mathsf{TC}^{\mathbf{S}}(X, \mathbf{v})$ and T preserves intersections as discussed in the paragraph before the proposition. It is also well-defined on morphisms since, given a $\mathsf{Stone}_{\mathbf{D}}$-morphism $f\colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$, we have

$$(\mathsf{T}f)\big(\mathsf{TC}^{\mathbf{S}}(X_1, \mathbf{v}_1)\big) = \mathsf{T}f\big(\mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v})\big) \subseteq \mathsf{TC}^{\mathbf{S}}(X_2, \mathbf{v}_2),$$

where in the first step we used that T preserves images and restrictions as described above and in the second step we used that T preserves inclusions up to isomorphisms and $f(\mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1)) \subseteq \mathsf{C}^{\mathbf{S}}(X_2, \mathbf{v}_2)$, which holds because f is a morphism in $\mathsf{Stone}_{\mathbf{D}}$.

Next we show that $\hat{\mathsf{T}}(X, \mathbf{v})$ with the inclusion maps $i_{\mathbf{S}}\colon \mathsf{V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X, \mathbf{v}) \hookrightarrow (\mathsf{T}(X), \hat{\mathbf{v}})$ is a cowedge over the diagram defining T'. Since T preserves monomorphisms, we only need to make sure that the $i_{\mathbf{S}}$ really are $\mathsf{Stone}_{\mathbf{D}}$-morphisms. But this is easy to see, since for $Z \in \mathsf{TC}^{\mathbf{S}}(X, \mathbf{v})$ we have

$$\hat{\mathbf{v}}(Z) = \bigwedge\{\mathbf{R} \in \mathbb{S}(\mathbf{D}) \mid Z \in \mathsf{TC}^{\mathbf{R}}(X, \mathbf{v})\} \leq \mathbf{S}.$$

For universality, simply note that for every other cowedge given by a collection $c_{\mathbf{S}}\colon \mathsf{V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X, \mathbf{v}) \to (Y, \mathbf{w})$, the underlying map of $c_{\mathbf{D}}\colon \mathsf{T}(X) \to Y$

also defines a morphism $\hat{\mathsf{T}}(X, \mathbf{v}) \to (Y, \mathbf{w})$, and this morphism witnesses the universal property (due to an argument similar to the one in the proof of Proposition 4.3.1).

The diagram after Definition 4.3.3 was used to define T' on morphisms $f\colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v}_2)$. Since the diagram

$$\begin{tikzcd}
\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X_1, \mathbf{v}_1) \arrow[r, "\mathsf{T}f|_{\mathsf{C}^{\mathbf{S}}(X_1,\mathbf{v}_1)}"] \arrow[d, hook] & \mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X_2, \mathbf{v}_2) \arrow[d, hook] \\
\hat{\mathsf{T}}(X_1, \mathbf{v}_1) \arrow[r, "\mathsf{T}f"'] & \hat{\mathsf{T}}(X_2, \mathbf{v}_2)
\end{tikzcd}$$

commutes for every $\mathbf{S} \in \mathbb{S}(\mathbf{D})$, the morphisms $\mathsf{T}'f$ and $\hat{\mathsf{T}}f$ coincide by the uniqueness clause of the universal property of the coend. $\square$

From now on, we will not distinguish between $\hat{\mathsf{T}}$ and T' in our notation if T preserves mono- and epimorphisms.

In particular, Proposition 4.3.5 applies in the (for us) important special case where T is the dual of a functor $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ appearing in a concrete coalgebraic logic, that is, the functor L has a presentation by operations and equations. We prove this fact in the following.

Corollary 4.3.6. *If* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *has a presentation by operations and equations, then it preserves mono- and epimorphisms. Therefore, for* $\mathsf{T} = \Sigma\mathsf{L}\Pi$ *the dual of* L*, the lifting* T' *can be obtained as in Proposition 4.3.5.*

Proof. The functor L has a presentation if and only if it preserves sifted colimits [KR12]. Since every filtered colimit is sifted, L is finitary and preserves monomorphisms due to [KP10, Lemma 6.14].

If $e\colon \mathbf{B} \to \mathbf{C}$ is a (necessarily regular) epimorphism between Boolean algebras, then $\mathbf{C}$ is isomorphic to a quotient of $\mathbf{B}$ by a congruence (namely, the kernel of e). Such a quotient is a reflexive coequalizer, which is preserved by L. Therefore, $\mathsf{L}e\colon \mathsf{L}(\mathbf{B}) \to \mathsf{L}(\mathbf{C})$ is a coequalizer in BA, which implies that $\mathsf{L}e$ is an epimorphism again. $\square$

Our next goal is to show that if $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ has a finitary presentation by operations and equations, the same is true for its lifting L' (where for now, we only focus on the *existence* of such a presentation). For this, we need the following lemma.

Lemma 4.3.7. *Let* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *be a functor and let* $\mathsf{T} = \Sigma\mathsf{L}\Pi$ *be its dual.*

(1) If L *is finitary (*i.e.*, preserves filtered colimits), then so is* L'*.*

(2) If L *preserves mono- and epimorphisms, then* T' *preserves all equalizers which* TU *preserves.*

Proof. (1): The functor $\mathfrak{S}_{\mathbf{S}}$ preserves all colimits because it is a left-adjoint, L is finitary by assumption and it is shown as in the proof of [KPT24b, Theorem 4.11] that $\mathfrak{P}_{\mathbf{S}}$ is finitary. Since, in addition, filtered colimits commute with finite limits, this implies that L' is finitary as well.

(2): Let $f, g\colon (X_1, \mathbf{v}_1) \to (X_2, \mathbf{v_2})$ be two $\mathsf{Stone}_{\mathbf{D}}$-morphisms. It is easy to check that the equalizer of f and g is given by $(E, \mathbf{w})$ where $E \subseteq X_1$ is the equalizer of f and g in Stone and $\mathbf{w}$ is the restriction of $\mathbf{v}_1$ to E. Now, assuming that $\mathsf{T}(E)$ is the equalizer of $\mathsf{T}f$ and $\mathsf{T}g$ in Stone, we show that $(\mathsf{T}(E), \hat{\mathbf{w}})$ (in the notation of Proposition 4.3.5) is the corresponding equalizer in $\mathsf{Stone}_{\mathbf{D}}$. By definition, for $Z \in \mathsf{T}(E)$ we have

$$\begin{aligned}\hat{\mathbf{w}}(Z) &= \bigwedge\{\mathbf{S} \mid Z \in \mathsf{TC}^{\mathbf{S}}(E, \mathbf{w})\}\\ &= \bigwedge\{\mathbf{S} \mid Z \in \mathsf{TC}^{\mathbf{S}}(X, \mathbf{v}_1) \cap \mathsf{T}(E)\}\\ &= \bigwedge\{\mathbf{S} \mid Z \in \mathsf{TC}^{\mathbf{S}}(X, \mathbf{v}_1)\} = \hat{\mathbf{v}}_1(Z),\end{aligned}$$

where we used that T preserves finite intersections and that $\mathsf{C}^{\mathbf{S}}(E, \mathbf{w}) = \mathsf{C}^{\mathbf{S}}(X, \mathbf{v}_1) \cap E$ since $\mathbf{w}$ is the restriction of $\mathbf{v}_1$. Thus, $\hat{\mathbf{w}}$ is the restriction of $\hat{\mathbf{v}}_1$ to $\mathsf{T}(E)$, finishing the proof. □

We can now easily conclude the following.

Corollary 4.3.8. *If* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *has a finitary presentation by operations and equations, then the same is true for the lifting* $\mathsf{L}'\colon \mathcal{A} \to \mathcal{A}$*.*

Proof. By [KR12, Theorem 4.7], we know that L' has a presentation if and only if it preserves sifted colimits. By [ARVL10, Theorem 7.7], we know that L' preserves sifted colimits if and only if it preserves filtered colimits and reflexive coequalizers. Since L has a presentation, it preserves filtered colimits and reflexive coequalizers. By part (1) of Lemma 4.3.7, we know that L' preserves filtered colimits and by part (2), we know that L' preserves reflexive coequalizers since T' preserves coreflexive equalizers (because TU preserves them as well). □

Note that we showed that a presentation of L' necessarily *exists*, however our proof does not indicate what such a presentation actually '*looks like*'.

In Subsection 4.3.3, we describe some circumstances under which one can directly obtain a presentation of L' from a given presentation of L (similarly to Subsection 4.2.2).

To end this section, we discuss two examples of classical algebra-coalgebra dualities which can be lifted in this way, namely Jónsson-Tarski duality [JT51] and Došen duality [Doš89].

The following coalgebraic version of Jónsson-Tarski duality [JT51] between descriptive general frames and modal algebras is due to [KKV03]. Recall that the Vietoris functor $\mathcal{V}\colon \mathsf{Stone} \to \mathsf{Stone}$ was defined in Example 4.1.3 and the functor $\mathsf{L}_\Box\colon \mathsf{BA} \to \mathsf{BA}$ was defined in Example 4.1.7.

Theorem 4.3.9 (Jónsson-Tarski duality, coalgebraically [KKV03])**.** *The two functors $\mathcal{V}$ and $\mathsf{L}_\Box$ are* Stone*-duals of each other, that is, $\mathcal{V} \cong \Sigma\mathsf{L}_\Box\Pi$.*

Since $\mathcal{V}$ satisfies the conditions of Proposition 4.3.5, we can give a more explicit description of its lifting $\mathcal{V}'\colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}_{\mathbf{D}}$ as follows.

Example 4.3.10 (Lifting of Vietoris functor)**.** The category of coalgebras $\mathsf{Coalg}(\mathcal{V}')$ is isomorphic to (also see Example 4.3.18) the category of *descriptive general* **D***-frames*, which are triples $(X, \mathbf{v}, R)$ where

- $(X, \mathbf{v}) \in \mathsf{Stone}_{\mathbf{D}}$.
- (X, R) is a descriptive general frame.
- If $x_1 R x_2$ then $\mathbf{v}(x_2) \leq \mathbf{v}(x_1)$.

Note that the last item again is the compatibility condition for **D**-frames from Definition 3.2.1. A morphism of descriptive general **D**-frames $(X_1, \mathbf{v}_1, R_1) \to (X_2, \mathbf{v}_2, R_2)$ is a map $f\colon X_1 \to X_2$ which is both a $\mathsf{Stone}_{\mathbf{D}}$-morphism and a bounded morphism.

By Proposition 4.3.5, it is easy to see that the lifting $\mathcal{V}'$ of $\mathcal{V}$ coincides with the functor described by Maruyama in [Mar12, Section 4] (also see Example 4.3.18 for the similar case of the powerset functor). ■

Since $\mathsf{L}_\Box$ has a presentation by operations and equations, due to Corollary 4.3.8 we know that the same is true for $\mathsf{L}'_\Box$. A concrete presentation of this functor is provided later on in Subsection 4.3.3. Next, note that Theorem 4.3.4 immediately yields the following.

Theorem 4.3.11 (Lifting Jónsson-Tarski duality)**.** *The two functors $\mathcal{V}'$ and $\mathsf{L}'_\Box$ are $\mathsf{Stone}_{\mathbf{D}}$-duals of each other, that is, $\mathcal{V}' \cong \Sigma'\mathsf{L}'_\Box\Pi'$.*

Our second duality concerns Došen duality [Doš89] between descriptive neighborhood frames and neighborhood algebras. As a algebra-coalgebra duality it is due to [BBdG22]. Recall that the functor $\mathcal{D}\colon \mathsf{Stone} \to \mathsf{Stone}$ was defined in Example 4.1.5 and $\mathsf{L}_\triangle\colon \mathsf{BA} \to \mathsf{BA}$ was defined in Example 4.1.8.

Theorem 4.3.12 (Došen duality, coalgebraically [BBdG22])**.** *The two functors* $\mathcal{D}$ *and* $\mathsf{L}_\triangle$ *are* Stone*-duals of each other, that is,* $\mathcal{D} \cong \Sigma \mathsf{L}_\triangle \Pi$.

Again, we use Proposition 4.3.5 to find a more concrete description of the lifting $\mathcal{D}' \colon \mathsf{Stone}_{\mathbf{D}} \to \mathsf{Stone}_{\mathbf{D}}$ (also see the Example 4.3.19 for the similar case of the neighborhood functor).

Example 4.3.13 (Lifting of descriptive neighborhood functor)**.** The coalgebras for the functor $\mathcal{D}'$ are *descriptive neighborhood* $\mathbf{D}$*-frames*, that is, structures of the form $(X, \mathbf{v}, N)$ where

- $(X, \mathbf{v}) \in \mathsf{Stone}_{\mathbf{D}}$.
- (X, N) is a descriptive neighborhood frame.
- If $x \in \mathsf{C}^{\mathbf{S}}(X, \mathbf{v})$, then there is a collection of clopens

  $$N^{\mathbf{S}}(x) \subseteq \mathcal{P}\big(\mathsf{C}^{\mathbf{S}}(X, \mathbf{v})\big)$$

  such that for all $Y \subseteq X$

  $$Y \in N(x) \Leftrightarrow Y \cap \mathsf{C}^{\mathbf{S}}(X, \mathbf{v}) \in N^{\mathbf{S}}(x).$$

  holds.

Similar to Example 4.3.10, a morphism of descriptive neighborhood $\mathbf{D}$-frames is a map which is both a $\mathsf{Stone}_{\mathbf{D}}$-morphism and a $\mathcal{D}$-coalgebra morphism. ■

As before, we know due to Corollary 4.3.8 that $\mathsf{L}'_\triangle$ has a presentation by operations and equations. However, outside of the primal case (recall Theorem 4.2.7), we do not know such a concrete presentation as of yet.

The combination of Theorems 4.3.12 and 4.3.4 immediately yields the following.

Theorem 4.3.14 (Lifting Došen duality)**.** *The two functors* $\mathcal{D}'$ *and* $\mathsf{L}'_\triangle$ *are* $\mathsf{Stone}_{\mathbf{D}}$*-duals of each other, that is,* $\mathcal{D}' \cong \Sigma' \mathsf{L}'_\triangle \Pi'$.

In the following subsection, similarly to this subsection (and generalizing Subsection 4.2.1), we describe how to lift classical abstract coalgebraic logics to the semi-primal level.

4.3.2 Lifting abstract coalgebraic logics: Semi-primal case

In this subsection, we discuss how to lift classical abstract coalgebraic logics (L, δ) (recall Definition 4.1.11) to many-valued abstract coalgebraic logics (L', δ'), where L' is an endofunctor on the variety $\mathcal{A}$. Compared to the primal case, where this is more easily achived via the dual equivalence due to Hu's Theorem (as demonstrated in Subsection 4.2.1), here we follow a strategy similar to the previous subsection.

We will discuss how to lift Set-endofunctors T to endofunctors T' on $\mathsf{Set}_{\mathbf{D}}$ (recall Definition 2.3.1). In order to lift abstract coalgebraic logics, we also explain how to lift $\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$ to a natural transformation $\delta'\colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}'$.

Analogous to the case of Stone and $\mathsf{Stone}_{\mathbf{D}}$ and as mentioned in Section 2.3, the subalgebra adjunctions also exist between Set and $\mathsf{Set}_{\mathbf{D}}$. Slightly abusing notation, we keep denoting the functors involved by $\mathsf{V}^{\mathbf{S}}\colon \mathsf{Set} \to \mathsf{Set}_{\mathbf{D}}$ and $\mathsf{C}^{\mathbf{S}}\colon \mathsf{Set}_{\mathbf{D}} \to \mathsf{Set}$ (see Figure 4.4).

$$\begin{tikzcd}
\mathsf{Set}_{\mathbf{D}} \arrow[loop left, "\mathsf{T}'"] \arrow[rr, bend left, "\mathsf{P}'"] \arrow[d, bend left, "\mathsf{C}^{\mathbf{S}}"] \arrow[d, phantom, "\dashv"] & & \mathcal{A} \arrow[loop right, "\mathsf{L}'"] \arrow[ll, bend left, "\mathsf{S}'"] \arrow[d, bend left, "\mathfrak{S}_{\mathbf{S}}"] \arrow[d, phantom, "\vdash"] & \delta'\colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}' \\
\mathsf{Set} \arrow[loop left, "\mathsf{T}"] \arrow[u, bend left, "\mathsf{V}^{\mathbf{S}}"] \arrow[rr, bend left, "\mathsf{P}"] & & \mathsf{BA} \arrow[loop right, "\mathsf{L}"] \arrow[ll, bend left, "\mathsf{S}"] \arrow[u, bend left, "\mathfrak{P}_{\mathbf{S}}"] & \delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}
\end{tikzcd}$$

Figure 4.4: Lifting coalgebraic logics via subalgebra adjunctions.

The functors $\mathsf{S}'\colon \mathcal{A} \to \mathsf{Set}_{\mathbf{D}}$ and $\mathsf{P}'\colon \mathsf{Set}_{\mathbf{D}} \to \mathcal{A}$ are defined similarly to Σ' and Π' from Section 2.1 and form a dual adjunction between $\mathsf{Set}_{\mathbf{D}}$ and $\mathcal{A}$. In the light of Figure 2.3, we can define S' as $(-)^{\flat} \circ \Sigma'$ and P' as $\iota_c \circ \Pi'$.

Explicitly, this means the functor $\mathsf{S}'\colon \mathcal{A} \to \mathsf{Set}_{\mathbf{D}}$ is defined on objects $\mathbf{A} \in \mathcal{A}$ by

$$\mathsf{S}'(\mathbf{A}) = \big(\mathcal{A}(\mathbf{A}, \mathbf{D}), \mathsf{im}\big),$$

where every homomorphism $u\colon \mathbf{A} \to \mathbf{D}$ gets assigned its image $\mathsf{im}(u) = u(A) \leq \mathbf{D}$. To a homomorphism $h\colon \mathbf{A} \to \mathbf{A}'$ the functor S' assigns the

$\mathsf{Set}_{\mathbf{D}}$-morphism $\mathsf{S}'h\colon \mathcal{A}(\mathbf{A}', \mathbf{D}) \to \mathcal{A}(\mathbf{A}, \mathbf{D})$ given by $u \mapsto u \circ h$.

The functor $\mathsf{P}'\colon \mathsf{Set}_{\mathbf{D}} \to \mathcal{A}$ is defined on objects by

$$\mathsf{P}'(X, v) = \prod_{x \in X} v(x).$$

To a morphism $f\colon (X_1, v_1) \to (X_2, v_2)$, the functor P' assigns the homomorphism $\alpha \mapsto \alpha \circ f$. These functors define a dual adjunction. The corresponding natural transformations $\eta'\colon \mathsf{id}_{\mathcal{A}} \Rightarrow \mathsf{P}'\mathsf{S}'$ and $\varepsilon'\colon \mathsf{id}_{\mathsf{Set}_{\mathbf{D}}} \Rightarrow \mathsf{S}'\mathsf{P}'$ are given by evaluations, that is, for for all $\mathbf{A} \in \mathcal{A}$ and $(X, v) \in \mathsf{Set}_{\mathbf{D}}$ we have

$$\begin{aligned} \eta'_{\mathbf{A}}\colon \mathbf{A} &\to \prod_{u \in \mathcal{A}(\mathbf{A},\mathbf{D})} \mathsf{im}(u) & \qquad \varepsilon'_{(X,v)}\colon X &\to \mathcal{A}\big(\prod_{x \in X} v(x), \mathbf{D}\big) \\ a &\mapsto \mathsf{ev}_a & x &\mapsto \mathsf{ev}_x \end{aligned}$$

where $\mathsf{ev}_a(u) = u(a)$ and $\mathsf{ev}_x(\alpha) = \alpha(x)$.

Of course, compared to the previous section, there is no longer a full dual equivalence between the left- and right-hand sides of Figure 4.4 (except when restricted to the finite level). Fortunately, the following useful relationships still hold (this is similar to Lemma 4.2.2 in the primal case).

Lemma 4.3.15. *The functors involved in Figure 4.4 and the natural transformations $\varepsilon, \eta, \varepsilon', \eta'$ satisfy the following properties.*

(1) For all $\mathbf{S} \in \mathbb{S}(\mathbf{D})$, there is a natural isomorphism $\Theta^{\mathbf{S}}\colon \mathsf{P}'\mathsf{V}^{\mathbf{S}} \Rightarrow \mathfrak{P}_{\mathbf{S}}\mathsf{P}$, with components

$$\Theta^{\mathbf{S}}_X(\alpha)(s) = T_s(\alpha),$$

where $\alpha \in \mathsf{P}'\mathsf{V}^{\mathbf{S}}(X) \cong \mathbf{S}^X$ and T_s are the unary terms defined in Theorem 1.1.12.

(2) For all $\mathbf{S} \in \mathbb{S}(\mathbf{D})$, there is a natural isomorphism $\Psi^{\mathbf{S}}\colon \mathsf{P}\mathsf{C}^{\mathbf{S}} \Rightarrow \mathfrak{S}_{\mathbf{S}}\mathsf{P}'$ given by the identification of $\mathbf{2}^{\mathsf{C}^{\mathbf{S}}(X,v)}$ with

$$\mathfrak{S}_{\mathbf{S}}\big(\prod_X v(x)\big) \cong \mathfrak{S}\big(\prod_{\mathsf{C}^{\mathbf{S}}(X,v)} v(x)\big) \cong \prod_{\mathsf{C}^{\mathbf{S}}(X,v)} \mathfrak{S}\big(v(x)\big) \cong \mathbf{2}^{\mathsf{C}^{\mathbf{S}}(X,v)}.$$

In particular, for $\mathbf{S} = \mathbf{D}$, there is a natural isomorphism $\Psi\colon \mathsf{PU} \Rightarrow \mathfrak{S}\mathsf{P}'$.

(3) There is a natural isomorphism $\Phi\colon \mathsf{U}\mathsf{S}' \Rightarrow \mathsf{S}\mathfrak{S}$ given by restriction

$$\begin{aligned} \Phi_{\mathbf{A}}\colon \mathcal{A}(\mathbf{A}, \mathbf{D}) &\to \mathsf{BA}\big(\mathfrak{S}(\mathbf{A}), \mathbf{2}\big) \\ u &\mapsto u|_{\mathfrak{S}(\mathbf{A})}. \end{aligned}$$

(4) For Ψ and Φ from (2) and (3), the identities $\varepsilon\mathsf{U} = \mathsf{S}\Psi \circ \Phi\mathsf{P}' \circ \mathsf{U}\varepsilon'$ and $\mathfrak{S}\eta' = \Psi\mathsf{S}' \circ \mathsf{P}\Phi \circ \eta\mathfrak{S}$ hold.

Proof. For part (1), showing that Θ_X is a homomorphism is analogous to Proposition 2.2.9. It is injective because $\alpha_1 \neq \alpha_2$ means there is some $x \in X$ such that $\alpha_1(x) \neq \alpha(x)$. Then, for $s = \alpha(x)$ we have $T_s(\alpha_1) \neq T_s(\alpha_2)$.

To see that Θ_X is surjective, let $\xi \colon S \to 2^X$ be in $\mathfrak{P}_{\mathbf{S}}\mathsf{P}(X)$. By definition of the Boolean power this means that, in every component $x \in X$, there is a unique $s_x \in \mathbf{S}$ with $\xi(s_x)(x) = 1$. Thus $\alpha(x) = s_x$ is in the preimage of ξ.

For naturality, we need to show that the following diagram commutes for any morphism $f \colon Y \to X$.

$$\begin{tikzcd}
\mathsf{P}'\mathsf{V}^{\mathbf{S}}(X) \arrow[r, "\Theta_X"] \arrow[d, "\mathsf{P}'\mathsf{V}^{\mathbf{S}}f"'] & \mathfrak{P}_{\mathbf{S}}\mathsf{P}(X) \arrow[d, "\mathfrak{P}_{\mathbf{S}}\mathsf{P}f"] \\
\mathsf{P}'\mathsf{V}^{\mathbf{S}}(Y) \arrow[r, "\Theta_Y"'] & \mathfrak{P}_{\mathbf{S}}\mathsf{P}(Y)
\end{tikzcd}$$

Given $\alpha \in \mathsf{P}'\mathsf{V}^{\mathbf{S}}(X) \cong \mathbf{S}^X$, on the one hand we have $\mathfrak{P}_{\mathbf{S}}\mathsf{P}f(\theta_X(\alpha)) = \mathsf{P}f \circ \Theta_X(\alpha)$, which sends $s \in S$ to $\mathsf{P}f(T_s(\alpha)) = T_s(\alpha \circ f)$. On the other hand we have $\Theta_Y(\mathsf{P}'\mathsf{V}^{\mathbf{S}}f(\alpha)) = \Theta_Y(\alpha \circ f)$ sends s to $T_s(\alpha \circ f)$ as well. This finishes the proof of part (1).

The equations in (2) follow from the results of Section 2.3, the fact that $\mathfrak{S}$ preserves limits and $\mathfrak{S}(\mathbf{S}) \cong \mathbf{2}$ holds for all $\mathbf{S} \in \mathbb{S}(\mathbf{D})$. Naturality is easy to check by definitions.

The proof of (3) is completely analogous to that of Proposition 2.2.4 and the proof of (4) is completely analogous to that of Lemma 4.2.2(3). □

Exactly like in Definition 4.3.3, we can use the subalgebra adjunctions to lift an endofunctor $\mathsf{T} \colon \mathsf{Set} \to \mathsf{Set}$ to one $\mathsf{T}' \colon \mathsf{Set}_{\mathbf{D}} \to \mathsf{Set}_{\mathbf{D}}$.

Definition 4.3.16 (Lifting Set-endofunctor)**.** Let $\mathsf{T} \colon \mathsf{Set} \to \mathsf{Set}$ be a functor. The *lifting of* T *to* $\mathsf{Set}_{\mathbf{D}}$ is the functor $\mathsf{T}' \colon \mathsf{Set}_{\mathbf{D}} \to \mathsf{Set}_{\mathbf{D}}$ defined on objects by

$$\mathsf{T}'(X, v) = \int^{\mathbf{S} \in \mathbb{S}(\mathbf{D})} \mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X, v)$$

and on morphisms as discussed in the paragraph after Definition 4.3.3.

The canonical lifting T' of T can again be described more concretely if the functor T preserves mono- and epimorphisms. In particular, this is true for Set-endofunctors which are *standard* (that is, inclusion-preserving), and up to what it does on the empty set, *every* Set-endofunctor is naturally isomorphic to one which is standard [Trn69]. The proof of Proposition 4.3.5 can be adapted to obtain the following.

Proposition 4.3.17. *Let* $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ *preserve mono- and epimorphisms. Then, up to natural isomorphism,* T' *is defined on objects by*

$$\mathsf{T}'(X, v) = \big(\mathsf{T}(X), \hat{v}\big),$$

where, for $Z \in \mathsf{T}(X)$*, considering* $\mathsf{TC}^{\mathbf{S}}(X, v)$ *as subspace of* $\mathsf{T}(X)$*,*

$$\hat{v}(Z) = \bigwedge\{\mathbf{S} \mid Z \in \mathsf{TC}^{\mathbf{S}}(X, v)\}.$$

On morphisms, T' *acts precisely like* T*.*

An obvious question related to modal logic is what the lifting of the covariant powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ and its corresponding coalgebras look like. As a matter of fact, they are exactly the $\mathbf{D}$-frames from Definition 3.2.1.

Example 4.3.18 (Lifting of powerset functor)**.** Since $\mathcal{P}$ is standard, by Proposition 4.3.17 its lifting is given on objects by $\mathcal{P}'(X, v) = (\mathcal{P}(X), \hat{v})$, where, for $Y \subseteq X$ we have

$$\hat{v}(Y) = \bigwedge\{\mathbf{S} \mid Y \in \mathcal{P}\mathsf{C}^{\mathbf{S}}(X, v)\} = \bigvee\{v(y) \mid y \in Y\},$$

where the second equation holds because both terms describe the smallest subalgebra $\mathbf{S}$ which satisfies $v(y) \leq \mathbf{S}$ for all $y \in Y$.

We show that the category of $\mathbf{D}$-frames with bounded $\mathbf{D}$-morphisms is isomorphic to $\mathsf{Coalg}(\mathcal{P}')$. Given a $\mathcal{P}'$-coalgebra $\gamma\colon (X, v) \to \mathcal{P}'(X, v)$, we define the corresponding Kripke $\mathbf{D}$-frame (X, v, R_γ) by $x_1 R_\gamma x_2 \Leftrightarrow x_2 \in \gamma(x_1)$. The compatibility condition is satisfied because γ being a $\mathsf{Set}_{\mathbf{D}}$-morphism implies

$$\hat{v}\big(\gamma(x)\big) = \bigvee\{v(x') \mid x' \in \gamma(x)\} \leq v(x).$$

Conversely, to every $\mathbf{D}$-frame (X, v, R) we can associate the $\mathcal{P}'$-coalgebra $\gamma_R\colon (X, v) \to \mathcal{P}'(X, v)$ given by $\gamma_R(x) = R[x]$. Again, the compatibility condition is what is needed to assure that this is a morphism in $\mathsf{Set}_{\mathbf{D}}$. ■

In the following, we give a concrete description of the category $\mathsf{Coalg}(\mathcal{N}')$ of coalgebras for the lifting $\mathcal{N}'$ of the neighborhood functor $\mathcal{N}$ (recall Example 4.1.4) as well.

Example 4.3.19 (Lifting of neighborhood functor)**.** Since $\mathcal{N}$ is standard, by Proposition 4.3.17 its lifting satisfies

$$\mathsf{C}^{\mathbf{S}}\big(\mathcal{N}'(X,v)\big) = \mathcal{N}\iota\big(\mathcal{N}(\mathsf{C}^{\mathbf{S}}(X,v))\big),$$

where $\iota\colon \mathsf{C}^{\mathbf{S}}(X,v) \hookrightarrow X$ is the natural inclusion. The map $\mathcal{N}\iota$ sends $N^{\mathbf{S}} \subseteq \mathfrak{P}(\mathsf{C}^{\mathbf{S}}(X,v))$ to $N \subseteq \mathfrak{P}(X)$ defined by

$$Y \in N \Leftrightarrow Y \cap \mathsf{C}^{\mathbf{S}}(X,v) \in N^{\mathbf{S}}.$$

The coalgebras $\gamma\colon (X,v) \to \mathcal{N}'(X,v)$ can therefore be identified with *neighborhood* $\mathbf{D}$*-frames*, which are triples (X,v,N) such that (X,N) is a neighborhood frame, $(X,v) \in \mathsf{Set}_{\mathbf{D}}$ and whenever $x \in \mathsf{C}^{\mathbf{S}}(X,\mathbf{v})$, there is a collection of subsets $N^{\mathbf{S}}(x) \subseteq \mathcal{P}(\mathsf{C}^{\mathbf{S}}(X,\mathbf{v}))$ such that for all $Y \subseteq X$

$$Y \in N(x) \Leftrightarrow Y \cap \mathsf{C}^{\mathbf{S}}(X,\mathbf{v}) \in N^{\mathbf{S}}(x)$$

holds.

Morphisms in $\mathsf{Coalg}(\mathcal{N}')$ are maps which simultaneously are morphisms between neighborhood frames and morphisms in $\mathsf{Set}_{\mathbf{D}}$. ■

We now describe how to lift a classical abstract coalgebraic logic (L,δ) for $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ to a many-valued one (L',δ') for $\mathsf{T}'\colon \mathsf{Set}_{\mathbf{D}} \to \mathsf{Set}_{\mathbf{D}}$. Since by now we have a way to lift T (by Definition 4.3.16) and L (by Definition 4.3.3), we only need to build from the classical one-step semantics $\delta\colon \mathsf{LP} \Rightarrow \mathsf{PT}$ the many-valued one-step semantics, that is, a natural transformation $\delta'\colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}'$.

By definition we have

$$\mathsf{L}'\mathsf{P}'(X,v) = \int_{\mathbf{S}\in\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}\mathsf{P}'(X,v)$$

and by Lemma 4.3.15(2), there is a natural isomorphism

$$\mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}\mathsf{P}' \cong \mathfrak{P}_{\mathbf{S}}\mathsf{LPC}^{\mathbf{S}}.$$

Furthermore, we have

$$\mathsf{P}'\mathsf{T}'(X,v) = \mathsf{P}'\Big(\int^{\mathbf{S}\in\mathbb{S}(\mathbf{D})} \mathsf{V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X,v)\Big) \cong \int_{\mathbf{S}\in\mathbb{S}(\mathbf{D})} \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X,v)$$

since P' is right-adjoint as a functor $\mathsf{Set}_{\mathbf{D}}^{\mathrm{op}} \to \mathcal{A}$ and due to Lemma 4.3.15(1) we know $\mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}} \cong \mathfrak{P}_{\mathbf{S}}\mathsf{PTC}^{\mathbf{S}}$. Because δ is natural, for every $(X,v) \in \mathsf{Set}_{\mathbf{D}}$ we can define a wedge

$$\mathsf{L}'\mathsf{P}'(X,v) \longrightarrow \mathfrak{P}_{\mathbf{S}}\mathsf{LPC}^{\mathbf{S}}(X,v) \longrightarrow \mathfrak{P}_{\mathbf{S}}\mathsf{PTC}^{\mathbf{S}}(X,v) \longrightarrow \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{TC}^{\mathbf{S}}(X,v)$$

where the first arrow is the corresponding limit morphism up to the first natural isomorphism mentioned above, the second arrow is $\mathfrak{P}_{\mathbf{S}}\delta_{\mathsf{C}^{\mathbf{S}}(X,v)}$ and the last arrow is the second natural isomorphism mentioned above. Thus the universal property of the end $\mathsf{P}'\mathsf{T}'(X,v)$ yields a morphism

$$\delta'_{(X,v)} \colon \mathsf{L}'\mathsf{P}'(X,v) \to \mathsf{P}'\mathsf{T}'(X,v),$$

which in fact defines a natural transformation $\mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}'$, by naturality of δ and of all isomorphisms involved in the definition of δ'. The procedure to obtain δ' from δ is summarized again in Figure 4.5. Now we have everything at hand to define the lifting of a classical abstract coalgebraic logic to the semi-primal level.

$$\begin{tikzcd}
\mathsf{L}'\mathsf{P}'(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}\mathsf{P}'(X,v) \arrow[r, "\text{limit}"] \arrow[ddd, dashed, "\exists!\delta'_{(X,v)}"'] \arrow[dddr, "\text{wedge}"'] & \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}\mathsf{P}'(X,v) \arrow[d, "\cong"] \\
 & \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathsf{P}\mathsf{C}^{\mathbf{S}}(X,v) \arrow[d, "\mathfrak{P}_{\mathbf{S}}\delta\mathsf{C}^{\mathbf{S}}"] \\
 & \mathfrak{P}_{\mathbf{S}}\mathsf{P}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X,v) \arrow[d, "\cong"] \\
\mathsf{P}'\mathsf{T}'(X,v) = \int_{\mathbb{S}(\mathbf{D})} \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X,v) \arrow[r, "\text{limit}"] & \mathsf{P}'\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}(X,v)
\end{tikzcd}$$

Figure 4.5: How to obtain δ' from δ.

Definition 4.3.20 (Lifting of a coalgebraic logic)**.** Let (L,δ) be a classical abstract coalgebraic logic for $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$. The *lifting of* (L,δ) *to* $\mathcal{A}$ is the abstract coalgebraic logic (L',δ') for T', where L' and T' are the liftings of L and T to $\mathcal{A}$ and $\mathsf{Set}_{\mathbf{D}}$, respectively, and δ' is the natural transformation defined in the previous paragraph.

This definition is a direct generalization of Definition 4.2.3 from primal algebras to semi-primal algebras $\mathbf{D}$.

In the remainder of this subsection, we show that under the assumption that L preserves mono- and epimorphisms (in particular, if the coalgebraic logic is concrete), one-step completeness and expressivity of a classical abstract coalgebraic logic are preserved under this lifting (generalizing Theorem 4.2.4 from the primal case). Afterwards, in Subsection 4.3.3, we deal with concrete coalgebraic logics, in particular we discuss how to lift axiomatizations (*i.e.*, presentations of functors) as well.

First, we deal with the preservation of one-step completeness (Definition 4.1.13) under the lifting of Definition 4.3.20.

Theorem 4.3.21 (Inheritance of one-step completeness). *Let* (L, δ) *be a classical abstract coalgebraic logic for a functor* $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ *such that* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *and* T *preserve mono- and epimorphisms. If* (L, δ) *is one-step complete, then so is its lifting* (L', δ')*.*

Proof. By Definition 4.1.13, we have to show that if δ is a component-wise monomorphism, then so is δ'. It suffices to show that $\mathfrak{S}\delta' = \delta\mathsf{U}$ holds up to natural isomorphism, since $\mathfrak{S}$ is faithful and thus reflects monomorphisms. For all $(X, v) \in \mathsf{Set}_{\mathbf{D}}$, by our definition of δ' (in the special case where $\mathbf{S} = \mathbf{D}$), the following diagram commutes.

$$\begin{tikzcd}
\mathsf{L}'\mathsf{P}'(X,v) \arrow[r, "f"] \arrow[d, "\delta'_{(X,v)}"'] & \mathfrak{P}\mathsf{L}\mathfrak{S}\mathsf{P}'(X,v) \arrow[r, "\cong"] & \mathfrak{P}\mathsf{L}\mathsf{P}\mathsf{U}(X,v) \arrow[d, "\mathfrak{P}\delta_X"] \\
\mathsf{P}'\mathsf{T}'(X,v) \arrow[r, "g"'] & \mathsf{P}'\mathsf{V}^{\mathbf{D}}\mathsf{T}\mathsf{U}(X,v) \arrow[r, "\cong"] & \mathfrak{P}\mathsf{P}\mathsf{T}\mathsf{U}(X,v)
\end{tikzcd}$$

Here, f and g are defined via the corresponding limit morphisms. We now apply $\mathfrak{S}$ to this diagram and use the fact that $\mathfrak{S}\mathfrak{P} \cong \mathsf{id}_{\mathsf{BA}}$ to get the following commutative diagram.

$$\begin{tikzcd}
\mathfrak{S}\mathsf{L}'\mathsf{P}'(X,v) \arrow[r, "\mathfrak{S}f"] \arrow[d, "\mathfrak{S}\delta'_{(X,v)}"'] & \mathfrak{S}\mathfrak{P}\mathsf{L}\mathfrak{S}\mathsf{P}'(X,v) \arrow[r, "\cong"] & \mathsf{L}\mathsf{P}\mathsf{U}(X,v) \arrow[d, "\delta_X"] \\
\mathfrak{S}\mathsf{P}'\mathsf{T}'(X,v) \arrow[r, "\mathfrak{S}g"'] & \mathfrak{S}\mathsf{P}'\mathsf{V}^{\mathbf{D}}\mathsf{T}\mathsf{U}(X,v) \arrow[r, "\cong"] & \mathsf{P}\mathsf{T}\mathsf{U}(X,v)
\end{tikzcd}$$

To conclude our proof, it remains to be shown that $\mathfrak{S}f$ and $\mathfrak{S}g$ are isomorphisms.

The fact that $\mathfrak{S}f$ commutes follows by duality from Proposition 4.3.5, where it is shown that the cowedge morphism corresponding to $\mathbf{S} = \mathbf{D}$ of the

(Stone) dual of L' is the identity on the underlying space, thus applying the forgetful functor (the Stone dual of $\mathfrak{S}$) yields an isomorphism.

Similarly, by Proposition 4.3.17 (we also use the notation used there), again we know that the cowedge morphism $\mathsf{V}^{\mathbf{D}}\mathsf{TU}(X, v) \to \mathsf{T}'(X, v)$ is the identity map as a (notably non-identity) morphism $(\mathsf{T}(X), v^{\mathbf{D}}) \to (\mathsf{T}(X), \hat{v})$. The homomorphism g is obtained by applying P' to this morphism, and it is easy to see that this is the natural inclusion $g\colon \prod_{Z \in \mathsf{T}(X)} \hat{v}(Z) \hookrightarrow \prod_{Z \in \mathsf{T}(X)} \mathbf{D}$. Applying $\mathfrak{S}$ to this natural inclusion clearly yields (up to identifying $\mathbf{2}$ with the subset $\{0, 1\} \subseteq D$) the identity $\mathbf{2}^{\mathsf{T}(X)} \to \mathbf{2}^{\mathsf{T}(X)}$. Thus $\mathfrak{S}g$ is also an isomorphism, which concludes the proof. ☐

Similarly, we show that expressivity (Definition 4.1.15) is preserved under the lifting to the semi-primal level as follows.

Theorem 4.3.22 (Inheritance of expressivity)**.** *Let* (L, δ) *be a classical abstract coalgebraic logic for a functor* $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ *such that* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *and* T *preserve mono- and epimorphisms. If* (L, δ) *is expressive, then so is its lifting* (L', δ').

Proof. In light of Definition 4.1.15, we note that by duality it can be seen that $\mathsf{Alg}(\mathsf{L}')$ has an initial object if $\mathsf{Alg}(\mathsf{L})$ has one (endow the terminal coalgebra of the dual of L with the 'bottom' evaluation $\mathbf{v}^{\mathbf{E}}$ which always assigns the smallest subalgebra $\mathbf{E} \subseteq \mathbf{D}$). Furthermore, $\mathsf{Set}_{\mathbf{D}}$ has epi-mono factorizations for essentially the same reason that Set does.

We need to show that if the adjoint-transpose $\delta^\dagger$ is a component-wise monomorphism, then so is the adjoint-transpose $(\delta')^\dagger$. This works similarly to the proof in the primal case (Theorem 4.2.4). It suffices to show that $\mathsf{U}(\delta')^\dagger = \delta^\dagger\mathfrak{S}$ holds up to natural isomorphism, since U is fully faithful and thus reflects monomorphisms. In other words, we want to show that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{UT'S'} \arrow[r, "\mathsf{U}\varepsilon'\mathsf{T'S'}"] \arrow[d] \arrow[dr, phantom, "D_1"] & \mathsf{US'P'T'S'} \arrow[r, "\mathsf{US'}\delta'\mathsf{S'}"] \arrow[d] \arrow[dr, phantom, "D_2"] & \mathsf{US'L'P'S'} \arrow[r, "\mathsf{US'L'}\eta'"] \arrow[d] \arrow[dr, phantom, "D_3"] & \mathsf{S'L'} \arrow[d] \\
\mathsf{TS}\mathfrak{S} \arrow[r, "\varepsilon\mathsf{TS}\mathfrak{S}"'] & \mathsf{SPTS}\mathfrak{S} \arrow[r, "\mathsf{S}\delta\mathsf{S}\mathfrak{S}"'] & \mathsf{SLPS}\mathfrak{S} \arrow[r, "\mathsf{SL}\eta\mathfrak{S}"'] & \mathsf{SL}\mathfrak{S}
\end{tikzcd}$$

Here, the upper edge of the (entire) diagram is $\mathsf{U}(\delta')^\dagger$ and the lower edge is $\delta^\dagger\mathfrak{S}$. All vertical arrows are natural isomorphisms obtained via the natural isomorphisms $\Psi\colon \mathsf{PU} \Rightarrow \mathfrak{S}\mathsf{P}'$, $\Phi\colon \mathsf{US}' \Rightarrow \mathsf{S}\mathfrak{S}$ from Lemma 4.3.15, the identity $\mathsf{UT}' = \mathsf{TU}$ (which clearly holds by Proposition 4.3.17) and natural isomorphism $\mathfrak{S}\mathsf{L}' \cong \mathsf{L}'\mathfrak{S}$ (which exists by the dual of Proposition 4.3.5).

The diagram D_1 commutes, because by applying the first equation of Lemma 4.3.15(4), we obtain

$$\begin{aligned} &\mathsf{SPT}\Phi \circ \mathsf{S}\Psi\mathsf{T}'\mathsf{S}' \circ \Phi\mathsf{P}'\mathsf{T}'\mathsf{S}' \circ \mathsf{U}\varepsilon'\mathsf{T}'\mathsf{S}' = \\ &\qquad \mathsf{SPT}\Phi \circ (\mathsf{S}\Psi \circ \Phi\mathsf{P}' \circ \mathsf{U}\varepsilon')\mathsf{T}'\mathsf{S}' = \\ &\qquad\qquad \mathsf{SPT}\Phi \circ \varepsilon\mathsf{UT}'\mathsf{S}', \end{aligned}$$

which coincides with $\varepsilon\mathsf{TS}\mathfrak{S} \circ \mathsf{T}\Phi$.

The diagram D_3 commutes for similar reasons since, applying the second equation of Lemma 4.3.15(4), we can compute

$$\begin{aligned} &\mathsf{SL}\eta\mathfrak{S} \circ \mathsf{SLP}\Phi \circ \mathsf{SL}\Psi\mathsf{S}' \circ \Phi\mathsf{L}'\mathsf{P}'\mathsf{S}' = \\ &\qquad \mathsf{SL}(\Psi\mathsf{S}' \circ \mathsf{P}\Phi \circ \eta\mathfrak{S}) \circ \Phi\mathsf{L}'\mathsf{P}'\mathsf{S}' = \\ &\qquad\qquad \mathsf{SL}\mathfrak{S}\eta' \circ \Phi\mathsf{L}'\mathsf{P}'\mathsf{S}', \end{aligned}$$

which coincides with $\Phi\mathsf{L}' \circ \mathsf{S}'\mathsf{L}'\eta'$.

Finally, to see that the diagram D_2 commutes, one uses $\mathfrak{S}\delta' = \delta\mathsf{U}$ (up to natural isomorphisms), as shown in the proof of Theorem 4.3.21. □

While (L', δ') is a coalgebraic logic for the lifting T' of T, it also directly yields a coalgebraic logic for T itself. Indeed, with the exception of [HT13], all results on many-valued modal logic interpret formulas over Kripke frames (*i.e.*, $\mathcal{P}$-coalgebras) rather than **D**-frames (*i.e.*, $\mathcal{P}'$-coalgebras). This is easily dealt with, since from (L', δ') we can always obtain a coalgebraic logic $(\mathsf{L}', \delta^{\mathbf{D}})$ for T by composing with the adjunction $\mathsf{V}^{\mathbf{D}} \dashv \mathsf{U}$. That is, we simply define $\delta^{\mathbf{D}} \colon \mathsf{L}'\mathsf{P}'\mathsf{V}^{\mathbf{D}} \to \mathsf{P}'\mathsf{V}^{\mathbf{D}}\mathsf{T}$ to be $\delta'\mathsf{V}^{\mathbf{D}}$ (which is well-defined because $\mathsf{T}'\mathsf{V}^{\mathbf{D}} = \mathsf{V}^{\mathbf{D}}\mathsf{T}$). In the case $\mathsf{T} = \mathcal{P}$, this essentially means that we identify a Kripke frame (W, R) with the corresponding **D**-frame $(W, v^{\mathbf{D}}, R)$ (which from a logical perspective means that models can have arbitrary valuations $\mathsf{Val} \colon W \times \mathsf{Prop} \to \mathbf{D}$). It is obvious by definition that one-step completeness of (L', δ') implies one-step completeness of $(\mathsf{L}', \delta^{\mathbf{D}})$. Furthermore, the fact that $(\delta^{\mathbf{D}})^\dagger = \mathsf{U}(\delta')^\dagger$ yields the analogous result for expressivity. Thus, together with Theorems 4.3.21 and 4.3.22, we showed the following.

Corollary 4.3.23. *Let* (L, δ) *be a classical abstract coalgebraic logic for* T *as in Theorem 4.3.21, let* (L', δ') *be its lifting and let* $\delta^{\mathbf{D}} = \delta'\mathsf{V}^{\mathbf{D}}$*. Then* $(\mathsf{L}', \delta^{\mathbf{D}})$ *is an abstract coalgebraic logic for* T*, which is one-step complete if* (L, δ) *is one-step complete and expressive if* (L, δ) *is expressive.*

In this subsection, we showed that both one-step completeness and expressivity of classical coalgebraic logics are preserved under the lifting to the many-valued level. On the level of abstract coalgebraic logics this is satisfying, but from a more 'practical' perspective these results only become interesting once we discuss concrete coalgebraic logics and provide an axiomatization of the lifted logic. This is the content of the next section, where we also explicitly show how our results apply to classical modal logic (Kripke semantics) and to neighborhood semantics.

4.3.3 Lifting concrete coalgebraic logics: Semi-primal case

In this subsection, we deal with liftings of classical concrete coalgebraic logics. Most notably, we generalize Corollary 4.2.8 to the semi-primal case, that is, we provide a method which allows us to find a presentation of $\mathsf{L}' \colon \mathcal{A} \to \mathcal{A}$, given a presentation of $\mathsf{L} \colon \mathsf{BA} \to \mathsf{BA}$ in certain cases. We also show how our tools may successfully be used in some sample applications.

But first, a note on completeness in the case of concrete coalgebraic logics. Assume that $\mathsf{L} \colon \mathsf{BA} \to \mathsf{BA}$ has a presentation by operations and equations. By Corollary 4.3.8, we know that its lifting $\mathsf{L}' \colon \mathcal{A} \to \mathcal{A}$ has a presentation by operations and equations as well. Furthermore, we know that one-step completeness of a classical abstract coalgebraic logic (L, δ) transfers to the lifted logic (L', δ') by Theorem 4.3.21. As discussed in Theorem 4.1.14, for classical concrete coalgebraic logics (L, δ), one-step completeness implies completeness with respect to the semantics determined by the initial algebra [KKP04, KP10]. The proof of this fact (as in Theorem 4.1.14) can be easily adapted to work for (L', δ') as well, after one notes that L' preserves monomorphisms (because L preserves monomorphisms, $\mathfrak{S}\mathsf{L}' \cong \mathsf{L}\mathfrak{S}$ and $\mathfrak{S}$ preserves and reflects monomorphisms). For semi-primal $\mathsf{FL}_{\mathrm{ew}}$-algebras, a similar result has been shown in [LL23].

Corollary 4.3.24 (Semi-primal one-step completeness ⇒ completeness). *Let* (L', δ') *be a concrete coalgebraic logic for* $\mathsf{T}' \colon \mathsf{Set} \to \mathsf{Set}$ *such that* L' *preserves monomorphisms. Then one-step completeness implies completeness. In particular, this holds if* (L', δ') *is a lifting of a classical coalgebraic logic as in Theorem 4.3.21.*

Proof. The proof is analogous to the proof of Theorem 4.1.14. The only difference is that the initial algebra sequence is now given by iterating L' on $\mathbf{E}$, the smallest subalgebra of $\mathbf{D}$. Since $\mathbf{E}$ contains all constants from the signature of $\mathbf{D}$, modal $\mathbf{D}$-formulas of depth n can then be identified with elements of $(\mathsf{L}')^n(\mathbf{E})$. □

This means that, as soon as we find a presentation of the lifted functor L' occurring in the lifting (L', δ') of the classical concrete logic (L, δ), we get the many-valued completeness result directly from the corresponding classical one. In the following, we show that it is sometimes possible to come up with a presentation of L' in a straightforward way. For this, we make use of the algebraic structure of $\mathbf{D}$.

First we note that Lemmas 4.2.5 and 4.2.6 still hold in the semi-primal case, having the exact same proof. Similarly to Corollary 4.2.8, we now show hot to axiomatize $\mathsf{Alg}(\mathsf{L}')$ given that L has a presentation by a unary meet-preserving operation.

Theorem 4.3.25. *Let* $\mathsf{L}\colon \mathsf{BA} \to \mathsf{BA}$ *have a presentation by one unary operation* $\Box$ *and equations which all hold in* $\mathbf{D}$ *if* $\Box$ *is replaced by any* $\tau_d, d \in D^+$, *including the equation* $\Box(x \wedge y) = \Box x \wedge \Box y$. *Then* L' *has a presentation by one unary operation* $\Box'$ *and the following equations.*

- $\Box'$ *satisfies all equations which the original* $\Box$ *satisfies,*
- $\Box' \tau_d(x) = \tau_d(\Box' x)$ *for all* $d \in D^+$.

Proof. Let $\mathsf{L}^\tau \colon \mathcal{A} \to \mathcal{A}$ be the endofunctor presented by the operation $\Box'$ and the equations from the statement. Since L^τ and L' are both finitary and the functors P', S' restrict to a dual equivalence between the full subcategories $\mathcal{A}^\omega$ and $\mathsf{Set}^\omega_{\mathbf{D}}$ consisting of the corresponding finite members, it suffices to show

$$\mathsf{S}'\mathsf{L}^\tau\mathsf{P}' \cong \mathsf{S}'\mathsf{L}'\mathsf{P}'$$

on the finite level, and from now on we only consider the restrictions of functors to this finite level. Let T' and T be the duals of L' and L, respectively. Since $\mathsf{S}'\mathsf{L}'\mathsf{P}' \cong \mathsf{T}'$ holds, we can equivalently show $\mathsf{S}'\mathsf{L}^\tau\mathsf{P}' \cong \mathsf{T}'$. By Definition 4.3.20 and Proposition 4.3.17, the functor T' is completely characterized by $\mathsf{C}^{\mathbf{S}}\mathsf{T}' \cong \mathsf{T}\mathsf{C}^{\mathbf{S}}$ for all $\mathbf{S} \in \mathbb{S}(\mathbf{D})$ (note that, in particular this includes $\mathsf{U}\mathsf{T}' \cong \mathsf{T}\mathsf{U}$). Thus, altogether it suffices to show

$$\mathsf{C}^{\mathbf{S}}\mathsf{S}'\mathsf{L}^\tau\mathsf{P}' \cong \mathsf{SLPC}^{\mathbf{S}} \text{ for all } \mathbf{S} \in \mathbb{S}(\mathbf{D}).$$

By definition of the functors involved, we want to find a bijection between the sets $\mathcal{A}(\mathsf{L}^\tau(\prod_X v(x)), \mathbf{S})$ and $\mathsf{BA}(\mathsf{L}(\mathbf{2}^{\mathsf{C}^{\mathbf{S}}(X,v)}), \mathbf{2})$ which is natural in $(X, v) \in \mathsf{Set}^\omega_{\mathbf{D}}$. By definition of L^τ, the former set is naturally isomorphic to the collection of all functions

$$f\colon \{\Box' a \mid a \in \prod_{x \in X} v(x)\} \to \mathbf{S} \text{ satisfying the equations of } \mathsf{L}^\tau.$$

Similarly, the latter set is naturally isomorphic to the collection of all functions

$$g\colon \{\Box b \mid b \in \mathbf{2}^{\mathsf{C}^{\mathbf{S}}(X,v)}\} \to \mathbf{2} \text{ satisfying the equations of } \mathsf{L}.$$

Given f as above, we assign to it g_f defined by $g_f(\Box b) = f(\Box' b^0)$, where $b^0(x) = b(x)$ for $x \in \mathsf{C}^{\mathbf{S}}(X,v)$ and $b^0(x) = 0$ otherwise. The map g_f is well-defined since $T_1(f(\Box' b^0)) = f(\Box' T_1(b^0)) = f(\Box' b^0)$, so $f(\Box' b^0) \in \{0,1\}$. Furthermore, g_f satisfies the equations of L because they are included in the equations of L^τ, which f satisfies.

Conversely, given g as above we assign to it f_g defined by

$$f_g(\Box' a) = \bigvee\{d \mid g\big(\Box \tau_d(a^\flat)\big) = 1\},$$

where $a^\flat$ is the restriction of a to $\mathsf{C}^{\mathbf{S}}(X,v)$. Due to

$$\begin{aligned} g\big(\Box \tau_{(d_1 \vee d_2)}(a^\flat)\big) &= g\big(\Box(\tau_{d_1}(a^\flat) \wedge \tau_{d_2}(a^\flat))\big) \\ &= g\big(\Box \tau_{d_1}(a^\flat)\big) \wedge g\big(\Box \tau_{d_2}(a^\flat)\big), \end{aligned}$$

the condition of Lemma 4.2.5 is satisfied here. Therefore, $\tau_d(f_g(\Box' a)) = g(\Box \tau_d(a^\flat))$. On the other hand, we use $\tau_c \circ \tau_d = \tau_d$ and compute

$$\begin{aligned} f_g\big(\Box' \tau_d(a)\big) &= \bigvee\{c \mid g\big(\Box \tau_c(\tau_d(a^\flat))\big) = 1\} \\ &= \bigvee\{c \mid g\big(\Box \tau_d(a^\flat)\big) = 1\} \\ &= g\big(\Box \tau_d(a^\flat)\big). \end{aligned}$$

Thus, we showed that f_g satisfies the equations $\Box'(\tau_d(x)) = \tau_d(\Box x)$ for all $d \in D^+$. The reason that f_g satisfies the remaining equations of L^τ, *i.e.*, the equations of L, is that these equations are satisfied by all τ_d and preserved by g. For example, we see that f_g preserves meets by computing

$$f_g\big(\Box'(a_1 \wedge a_2)\big) = \bigvee\{d \mid g\big(\Box \tau_d(a_1^\flat \wedge a_2^\flat)\big) = 1\}$$

and thus $\tau_d(f_g(\Box'(a_1 \wedge a_2))) = g(\Box \tau_d(a_1^\flat \wedge a_2^\flat))$, which is equal to $g(\Box \tau_d(a_1^\flat)) \wedge g(\Box \tau_d(a_2^\flat))$ because both τ_d and g preserve meets. On the other hand, we compute

$$\begin{aligned} \tau_d\big(f_g(\Box' a_1) \wedge f_g(\Box' a_2)\big) &= \tau_d\big(f_g(\Box' a_1)\big) \wedge \tau_d\big(f_g(\Box' a_2)\big) \\ &= g\big(\Box \tau_d(a_1^\flat)\big) \wedge g\big(\Box \tau_d(a_2^\flat)\big). \end{aligned}$$

Thus, we showed that $\tau_d(f_g(\Box'(a_1 \wedge a_2))) = \tau_d(f_g(\Box' a_1) \wedge f_g(\Box' a_2))$ holds for all $d \in D^+$, which implies $f_g(\Box'(a_1 \wedge a_2)) = f_g(\Box' a_1) \wedge f_g(\Box' a_2)$ as desired.

Naturality of the bijection $g \mapsto g_{(\cdot)}$ is easy to check by definition, so we are left to show that the two assignments $f \mapsto g_f$ and $g \mapsto f_g$ are mutually inverse. To show $g_{f_g} = g$, we simply compute

$$\begin{aligned} g_{f_g}(\Box b) &= f_g(\Box' b^0) = \bigvee\{d \mid g\big(\Box \tau_d((b^0)^\flat)\big) = 1\} \\ &= \bigvee\{d \mid g(\Box b) = 1\} \\ &= g(\Box b), \end{aligned}$$

where we used $(b^0)^\flat = b$ which is clear by the definitions and $\tau_d(b) = b$ because $b \in 2^{\mathsf{C}^{\mathbf{S}}(X,v)}$. Showing that $f_{g_f} = f$ is more involved. We first compute

$$\begin{aligned} f_{g_f}(\Box' a) &= \bigvee\{d \mid g_f\big(\Box \tau_d(a^\flat)\big) = 1\} \\ &= \bigvee\{d \mid f\big(\Box' \tau_d((a^\flat)^0)\big) = 1\} \\ &= \bigvee\{d \mid \tau_d\big(f(\Box'(a^\flat)^0\big) = 1\} = f\big(\Box'(a^\flat)^0\big). \end{aligned}$$

This means we have to show that $f(\Box' a) = f(\Box' \tilde{a})$ always holds for $\tilde{a}(x) = a(x)$ on $\mathsf{C}^{\mathbf{S}}(X,v)$ and $\tilde{a}(x) = 0$ on $X \backslash \mathsf{C}^{\mathbf{S}}(X,v)$. Clearly this holds if f is constant, so assume that f is not constant. It suffices to show $f(\Box' \alpha) = 1$ for

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in \mathsf{C}^{\mathbf{S}}(X,v) \\ 0 & \text{if } x \notin \mathsf{C}^{\mathbf{S}}(X,v), \end{cases}$$

because this implies $f(\Box' \tilde{a}) = f(\Box'(a \wedge \alpha)) = f(\Box' a) \wedge f(\Box' \alpha) = f(\Box' a)$. In order to show that $f(\Box' \alpha) = 1$, we show that $f(\Box' c_x) = 1$ holds for all $x \in X \backslash \mathsf{C}^{\mathbf{S}}(X,v)$, where

$$c_x(y) = \begin{cases} 1 & \text{if } y \neq x \\ 0 & \text{if } y = x, \end{cases}$$

which is sufficient because $\alpha = \bigwedge\{c_x \mid x \in X \backslash \mathsf{C}^{\mathbf{S}}(X,v)\}$ (note that this is a finite meet because X is finite). So let $x \in X \backslash \mathsf{C}^{\mathbf{S}}(X,v)$ and choose some $d \in D \backslash v(x)$. Let s be the minimal element of $\mathbf{S}$ strictly above d (which exists because $d \neq 1$). Let c_x^d be defined by

$$c_x^d(y) = \begin{cases} 1 & \text{if } y \neq x \\ d & \text{if } y = x, \end{cases}.$$

Now $\tau_d(f(\Box' c_x^d)) = f(\Box' \tau_d(c_x^d)) = f(\Box' 1) = 1$ (since otherwise $f(\Box' 1) = 0$ and the fact that f is order-preserving would imply that f is constant 0).

Since $f(\Box' c_x^d) \in \mathbf{S}$ and $f(\Box' c_x^d) \geq d$, due to our choice of s we have $f(\Box' c_x^d) \geq s$ as well. This implies

$$1 = \tau_s\big(f(\Box' c_x^d)\big) = f\big(\Box' \tau_s(c_x^d)\big) = f(\Box' c_x)$$

as desired, finishing the proof. $\Box$

With the presentation of this theorem, the corresponding natural transformation $\delta' \colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathsf{T}'$ can be obtained from δ component-wise via

$$\begin{aligned} \delta'_{(X,v)} \colon \mathsf{L}'\Big(\prod_{x \in X} v(x)\Big) &\to \prod_{Z \in \mathsf{T}X} \hat{v}(Z) \\ \Box' a &\mapsto \Big(Z \mapsto \bigvee \{d \mid \delta\big(\Box \tau_d(a)\big)(Z) = 1\}\Big). \end{aligned}$$

This means that in this case we have a full and explicit description of the lifted concrete coalgebraic logic (L', δ').

Similar to the primal case (Subsection 4.2.2), the applicability of Theorem 4.3.25 depends on the specific choice of presentation of L. For example, while it does apply to the functor $\mathsf{L}_\Box$ presented by $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \wedge \Box y$, it does not apply to the (naturally isomorphic) functor $\mathsf{L}_\Diamond$ presented by $\Diamond 0 = 0$ and $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$. However, not surprisingly, in this example the order-dual version of Theorem 4.3.25 applies. Let $D^- := D \backslash \{1\}$ and recall that for $d \in D^-$ we define

$$\eta_d(x) = \begin{cases} 0 & \text{if } x \leq d \\ 1 & \text{if } x \nleq d. \end{cases}$$

The following is proved completely analogous to Theorem 4.3.25.

Theorem 4.3.26. *Let $\mathsf{L} \colon \mathsf{BA} \to \mathsf{BA}$ preserve mono- and epimorphisms and have a presentation by one unary operation $\Diamond$ and equations which all hold in $\mathbf{D}$ if $\Diamond$ is replaced by any $\eta_d, d \in D^-$, including the equation $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$. Then L' has a presentation by one unary operation $\Diamond'$ and the following equations.*

- *$\Diamond'$ satisfies all equations which the original $\Diamond$ satisfies,*
- *$\Diamond' \eta_d(x) = \eta_d(\Diamond' x)$ for all $d \in D^-$.*

Of course, Theorems 4.3.25 and 4.3.26 do not exhaust all possible presentations a functor L may have, and as of yet we know no systematic method to directly lift presentations that don't fall within the scope of these theorems. Nevertheless, these theorems already cover some ground, most notably including classical modal logic. In the following, we show how our results apply, among others, to this case.

Example 4.3.27 (Lifting classical modal logic)**.** As in Example 4.1.9, let $\mathsf{T} = \mathcal{P}$ be the covariant powerset functor and $\mathsf{L}_\Box \colon \mathsf{BA} \to \mathsf{BA}$ be the functor presented by a unary operation $\Box$ and equations $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \wedge \Box y$. The natural transformation $\delta \colon \mathsf{LP} \Rightarrow \mathsf{PT}$ is given on $W \in \mathsf{Set}$ by

$$\Box Y \mapsto \{Z \subseteq W \mid Z \subseteq Y\} \text{ for } Y \subseteq W.$$

It is well-known that the classical coalgebraic logic (L, δ) thus defined is (one-step) complete for $\mathcal{P}$ and expressive if we replace $\mathcal{P}$ by the finite powerset functor $\mathcal{P}_\omega$.

Now let $\mathcal{P}'$ be the lifting of $\mathcal{P}$ to $\mathsf{Set}_\mathbf{D}$. As described in Example 4.3.18, the category of coalgebras $\mathsf{Coalg}(\mathcal{P}')$ is isomorphic to the category of $\mathbf{D}$-frames (W, v, R) where $(W, v) \in \mathsf{Set}_\mathbf{D}$ is compatible with the accessibility relation in the sense that $w_1 R w_2 \Rightarrow v(w_2) \leq v(w_1)$.

Let $(\mathsf{L}'_\Box, \delta')$ be the lifting of $(\mathsf{L}_\Box, \delta)$ to $\mathcal{A}$. By Theorem 4.3.25, we know that $\mathsf{L}'_\Box$ has a presentation by one unary operation $\Box'$ and equations

$$\Box' 1 = 1, \quad \Box'(x \wedge y) = \Box' x \wedge \Box' y, \quad \Box' \tau_d(x) = \tau_d(\Box' x), \text{ for all } d \in D^+.$$

The natural transformation $\delta' \colon \mathsf{L}'\mathsf{P}' \Rightarrow \mathsf{P}'\mathcal{P}'$ has components

$$\delta'_{(X,v)}(\Box' a)(Z) = \bigvee \{d \mid \delta_X\big(\Box \tau_d(a)\big)(Z) = 1\}.$$

Since we have the chain of equivalences

$$\begin{aligned} \delta_X\big(\Box \tau_d(a)\big)(Z) = 1 &\Leftrightarrow \forall z \in Z \colon \tau_d(a)(z) = 1 \\ &\Leftrightarrow \forall z \in Z \colon a(z) \geq d \\ &\Leftrightarrow \bigwedge_{z \in Z} a(z) \geq d \Leftrightarrow \tau_d\big(\bigwedge_{z \in Z} a(z)\big) = 1, \end{aligned}$$

we can simplify this to

$$\bigvee \{d \mid \delta_X\big(\Box \tau_d(a)\big)(Z) = 1\} = \bigvee \{d \mid \tau_d\big(\bigwedge_{z \in Z} a(z)\big) = 1\} = \bigwedge_{z \in Z} a(z).$$

Thus, δ' yields the conventional semantics of a many-valued box-modality via meets, as described in Section 3.1.

From Theorem 4.3.21 and Corollary 4.3.23 we get that (L', δ') and $(\mathsf{L}', \delta^\mathbf{D})$ are one-step complete since (L, δ) is one-step complete. By Corollary 4.3.24 this implies completeness for $\mathbf{D}$-frames and frames alike. Such a completeness results has been proven directly in the case where $\mathbf{D} = \mathbf{Ł}_n$ is a finite Łukasiewicz chain in [BEGR11, HT13] and, although only for $(\mathsf{L}', \delta^\mathbf{D})$, in the case where $\mathbf{D}$ is a Heyting algebra expanded with the unary operations T_d in [Mar09].

Replacing $\mathcal{P}$ by $\mathcal{P}_\omega$, from Theorem 4.3.22 we get that (L', δ') and $(\mathsf{L}', \delta^{\mathbf{D}})$ are expressive, that is, they satisfy the Hennessy-Milner property. In the case where $\mathbf{D} = \text{Ł}_n$ with semantics $\delta^{\mathbf{D}}$, this has been shown in [MM18] (see also [BD16] for a coalgebraic treatment via predicate liftings). For $\mathbf{D}$-frames and one-step semantics δ', this corresponds to Theorem 3.2.9.

Of course, it is also possible to use Theorem 4.3.26 instead to get the analogous completeness and expressivity result for the many-valued $\Diamond'$ satisfying $\Diamond' 0 = 0$, $\Diamond'(x \vee y) = \Diamond' x \vee \Diamond' y$ and $\Diamond' \eta_d(x) = \eta_d(\Diamond' x)$ for all $d \in D^-$. Here, as usual, a formula $\Diamond' \varphi$ is interpreted on $\mathbf{D}$-frames or frames as a join. Since $\mathsf{L}_\Box$ and $\mathsf{L}_\Diamond$ are naturally isomorphic, the same is true for $\mathsf{L}'_\Box$ and $\mathsf{L}'_\Diamond$. Indeed, this strongly suggests that $\Box'$ and $\Diamond'$ are inter-definable as well, even in the absence of a De Morgan negation. ■

As second example, we consider the lifting of non-normal modal logic over neighborhood frames from Example 4.1.10.

Example 4.3.28 (Lifting neighborhood semantics)**.** In this example, the theory of Subsection 4.3.2 of lifting abstract coalgebraic logics still applies but obtaining an axiomatization via Theorem 4.2.7 or 4.2.10 is not possible.

Let $\mathsf{L}_\triangle \colon \mathsf{BA} \to \mathsf{BA}$ be the functor from Example 4.1.8 which has a presentation by one unary operation $\triangle$ and no (that is, the empty set of) equations. Let δ be as in Example 4.1.10. The concrete coalgebraic logic $(\mathsf{L}_\triangle, \delta)$ for $\mathcal{N}$ is again complete, and expressive if we replace $\mathcal{N}$ by an appropriate $\mathcal{N}_\omega$.

Therefore, the lifting $(\mathsf{L}'_\triangle, \delta')$ of $(\mathsf{L}_\triangle, \delta)$ to $\mathcal{A}$ is a complete (or expressive) abstract coalgebraic logic for $\mathcal{N}'$ (or $\mathcal{N}'_\omega$, respectively).

Furthermore, by Theorem 4.3.8 we know that L' does have a finitary presentation by operations and equations. However, as of yet we do not know a concrete presentation of L' in the case where $\mathbf{D}$ is semi-primal but not primal (for the primal case, recall Theorem 4.2.7).

However, considering the *filter functor* $\mathcal{M} \colon \mathsf{Set} \to \mathsf{Set}$, that is, the subfunctor of $\mathcal{N}$ which only allows collections of neighborhoods which are (empty or non-empty) *filters* (*i.e.*, closed under finite intersections and supersets), Theorem 4.2.7 does apply again.

In this case, $\mathsf{L} \colon \mathsf{BA} \to \mathsf{BA}$ can be presented by one unary operation $\Box$ and the equation $\Box(x \wedge y) = \Box x \wedge \Box y$. Then the concrete coalgebraic logic (L, δ) for $\mathcal{M}$ is well-known to be complete. It is expressive if $\mathcal{M}$ is replaced by the functor $\mathcal{M}_\omega$ corresponding to (image-)finite filter frames [HKP09].

Let (L', δ') be the lifting of (L, δ) to $\mathcal{A}$. By Theorem 4.2.7, we get a presentation of L' by one unary operation $\Box'$ and equations

$$\Box'(x \wedge y) = \Box' x \wedge \Box' y, \qquad \Box' \tau_d(a) = \tau_d(\Box a) \text{ for all } d \in D^+.$$

The corresponding semantics $\delta'_{(X,v)} \colon \mathsf{L}'\mathsf{P}'(X,v) \to \mathsf{P}'\mathcal{M}'(X,v)$ are given by

$$\delta'_{(X,v)}(\Box' a)(N) = \bigvee \{d \mid \delta_X\big(\Box \tau_d(a)\big)(N) = 1\} = \bigvee \{d \mid \tau_d(a) \in N\}.$$

This can be interpreted as follows. Given a neighborhood model (W, N, Val) where $\mathsf{Val} \colon W \times \mathsf{Prop} \to \mathbf{D}$, for a formula $\varphi \in \mathbf{D}^W$ we have

$$\mathsf{Val}(w, \Box' \varphi) = \bigvee \{d \mid \tau_d(\varphi) \in N(w)\}.$$

By Theorem 4.3.21 and Corollary 4.3.24, we know that (L', δ') is complete for $\mathcal{M}'$- and $\mathcal{M}$-coalgebras. Replacing $\mathcal{M}$ by $\mathcal{M}_\omega$ we also get the corresponding expressivity results by Theorem 4.3.22. ■

4.3.4 Goldblatt-Thomason revisited

In this subsection, we re-investigate the results of Subsection 3.3.2 coalgebraically. In particular, we generalize the many-valued Goldblatt-Thomason Theorem for $\mathbf{D}$-frames (Theorem 3.3.24) and frames (Corollary 3.3.25) to definability for classes of T'-coalgebras and T-coalgebras (using the notation from the previous subsections). This subsection heavily relies on [KR07], where coalgebraic Goldblatt-Thomason theorems are discussed in some generality.

Throughout this subsection, we assume that (L, δ) is a classical *concrete* coalgebraic logic for a standard functor $\mathsf{T} \colon \mathsf{Set} \to \mathsf{Set}$. Furthermore, (L', δ') is the lifting to $\mathcal{A}$ for $\mathsf{T}' \colon \mathsf{Set}_{\mathbf{D}} \to \mathsf{Set}_{\mathbf{D}}$ and $(\mathsf{L}', \delta^{\mathbf{D}})$ is the corresponding many-valued coalgebraic logic for T (recall the discussion before Corollary 4.3.23). Furthermore, let $\mathsf{L}(I) \to I$ be the initial L-algebra and $\mathsf{L}'(I') \to I'$ be the initial L'-algebra.

Definition 4.3.29 (Definability for coalgebras)**.** Let $\mathcal{C} \subseteq \mathsf{Coalg}(\mathsf{T})$ be a class of T-coalgebras and let $\mathcal{C} \subseteq \mathsf{Coalg}(\mathsf{T}')$ be a class of T'-coalgebras.

(1) $\mathcal{C}$ is (L, δ)-*definable* if there is some $\Phi \subseteq I$ such that

$$\gamma \colon X \to \mathsf{T}(X) \in \mathcal{C} \Leftrightarrow [[\varphi]] = X \text{ for all } \varphi \in \Phi.$$

(2) $\mathcal{C}$ is $(\mathsf{L}', \delta^{\mathbf{D}})$-*definable* if there is some $\Phi' \subseteq I'$ such that

$$\gamma \colon X \to \mathsf{T}(X) \in \mathcal{C} \Leftrightarrow [[\varphi]](x) = 1 \text{ for all } \varphi \in \Phi', x \in X.$$

(3) Similarly, $\mathcal{C}'$ is (L', δ')-*definable* if there is some $\Phi' \subseteq I'$ such that

$$\gamma\colon (X, v) \to \mathsf{T}'(X, v) \in \mathcal{C}' \Leftrightarrow [[\varphi]](x) = 1 \text{ for all } \varphi \in \Phi', x \in X.$$

Note that, in the above, the definition of $[[\cdot]]$ depends on the coalgebraic logic in question (recall the discussion before Definition 4.1.12).

In the coalgebraic setting, we replace disjoint unions, generated subframes and bounded morphic images (recall Definition 3.3.17) by coproducts, subcoalgebras and coalgebra morphic images. For the Set-based case, these notions can, for example, be found in [Gum99, Section 4]. We generalize them to the $\mathsf{Set}_{\mathbf{D}}$-based case in order to fit our purposes.

Definition 4.3.30 (Constructions on T'-coalgebras)**.** Let the morphisms $\gamma\colon (X, v) \to \mathsf{T}'(X, v)$, $\gamma'\colon (X', v') \to \mathsf{T}'(X', v')$ be T'-coalgebras and let $\{\gamma_i\colon (X_i, v_i) \to \mathsf{T}'(X_i, v_i)\}_{i \in I}$ be an indexed family of T'-coalgebras.

(1) The *coproduct* of $\{\gamma_i\}_{i\in I}$ is given by the unique morphism

$$\coprod \gamma_i\colon \coprod (X_i, v_i) \to \mathsf{T}'\big(\coprod (X_i, v_i)\big)$$

obtained by the universal property of the coproduct (in $\mathsf{Set}_{\mathbf{D}}$) from

$$\begin{tikzcd}
(X_i, v_i) \arrow[r, "e_i"] \arrow[d, "\gamma_i"'] & \coprod (X_i, v_i) \arrow[d, dashed, "\coprod \gamma_i"] \\
\mathsf{T}'(X_i, v_i) \arrow[r, "\mathsf{T}' e_i"'] & \mathsf{T}'\big(\coprod (X_i, v_i)\big)
\end{tikzcd}$$

where e_i are the coproduct morphisms.

(2) We say that γ is a *subcoalgebra* of γ' if $X \subseteq X'$ and the embedding $\iota\colon (X, v) \hookrightarrow (X', v')$ is a regular monomorphism in $\mathsf{Set}_{\mathbf{D}}$ and a T'-coalgebra morphism $\gamma \to \gamma'$.

(3) We say that γ is a *coalgebra morphic image* of γ' if there is a surjective coalgebra morphism $f\colon \gamma' \to \gamma$ (*i.e.*, a coalgebra morphism which is an epimorphism $(X', v') \to (X, v)$ in $\mathsf{Set}_{\mathbf{D}}$).

We rely on the following theorem of [KR07] which, as mentioned in that paper, is general enough to cover both (L, δ) and (L', δ').

Theorem 4.3.31 ([KR07, Theorem 3.15]). *Assume there exists a natural transformation* $\lambda\colon \mathsf{TS} \Rightarrow \mathsf{SL}$ *such that, whenever* $\mathbf{B} = \lim_{i\in I} \mathbf{B}_i$ *is a cofiltered limit of finite Boolean algebras in* BA *with limit morphisms* $p_i\colon \mathbf{B} \to \mathbf{B}_i$*, the following diagram*

$$\begin{tikzcd}
\mathsf{TS}(\mathbf{B}) \arrow[r, "\lambda_{\mathbf{B}}"] & \mathsf{SL}(\mathbf{B}) \\
\mathsf{TS}(\mathbf{B}_i) \arrow[r, "\cong"'] \arrow[u, "\mathsf{TS}p_i"] & \mathsf{SL}(\mathbf{B}_i) \arrow[u, "\mathsf{SL}p_i"']
\end{tikzcd}$$

commutes. Then a class $\mathcal{C} \subseteq \mathsf{Coalg}(\mathsf{T})$ *is* (L, δ)*-definable if and only if* $\mathcal{C}$ *is closed under coproducts, subcoalgebras, quotients and ultrafilter extensions, and* $\mathcal{C}$ *reflects ultrafilter extensions.*

Here, the *ultrafilter extension* of a coalgebra $\gamma\colon X \to \mathsf{T}(X)$ is obtained by first using δ to transform it into a L-algebra as described in the discussion after Definition 4.1.11 and then using λ in a similar manner to turn this L-algebra back into a coalgebra, which we will denote by $\mathfrak{Ce}(\gamma)$. Similarly, the *canonical extension* $\mathfrak{Ce}(\gamma')$ of a T'-coalgebra γ' is defined if a natural transformation $\lambda'\colon \mathsf{T}'\mathsf{S}' \Rightarrow \mathsf{S}'\mathsf{L}'$ with the analogous property exists.

Furthermore, the analogue of Theorem 4.3.31 holds for (L', δ')-definability, given that such a λ' exists (since the theorem is proved in sufficient generality in [KR07]). In the following, we show that such a λ' exists whenever a λ as in Theorem 4.3.31 exists. The way to obtain λ' from λ is similar to obtaining δ' from δ (recall the discussion after Example 4.3.19). In order to carry this out, we need some more natural isomorphisms similar to the ones from Lemma 4.3.15.

Proposition 4.3.32. *Suppose that* λ *as in Theorem 4.3.31 exists. Then there exists a* $\lambda'\colon \mathsf{T}'\mathsf{S}' \Rightarrow \mathsf{S}'\mathsf{L}'$ *such that whenever* $\mathbf{A} = \lim_{i\in I} \mathbf{A}_i$ *is a cofiltered limit of finite algebra in* $\mathcal{A}$ *with limit morphisms* $p_i'\colon \mathbf{A} \to \mathbf{A}_i$*, the diagram*

$$\begin{tikzcd}
\mathsf{T}'\mathsf{S}'(\mathbf{A}) \arrow[r, "\lambda'_{\mathbf{A}}"] & \mathsf{S}'\mathsf{L}'(\mathbf{A}) \\
\mathsf{T}'\mathsf{S}'(\mathbf{A}_i) \arrow[r, "\cong"'] \arrow[u, "\mathsf{T}'\mathsf{S}'p_i'"] & \mathsf{S}'\mathsf{L}'(\mathbf{A}_i) \arrow[u, "\mathsf{S}'\mathsf{L}'p_i'"']
\end{tikzcd}$$

commutes.

Proof. First we sketch how to obtain λ' from λ, then we show that this λ' has the desired property.

By definition we have

$$\mathsf{T}'\mathsf{S}'(\mathbf{A}) = \int^{\mathbf{S}\in\mathbb{S}(\mathbf{D})} \mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}\mathsf{S}'(\mathbf{A})$$

and for every $\mathbf{S} \in \mathbb{S}(\mathbf{D})$ there is a natural isomorphism

$$\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}\mathsf{S}' \cong \mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{S}\mathfrak{S}_{\mathbf{S}},$$

defined as in the 'topological case' discussed in Subsection 2.2.4 (since S and S' completely coincide with Σ and Σ' up to 'forgetting topology'). Furthermore, we have

$$\mathsf{S}'\mathsf{L}'(\mathbf{A}) = \mathsf{S}'\big(\int_{\mathbf{S}\in\mathbb{S}(\mathbf{D})} \mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}(\mathbf{A})\big) \cong \int^{\mathbf{S}\in\mathbb{S}(\mathbf{D})} \mathsf{S}'\mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}(\mathbf{A})$$

because S' is left-adjoint as a functor $\mathcal{A} \to \mathsf{Set}_{\mathbf{D}}^{\mathrm{op}}$ and for the same reasons as above we again know that $\mathsf{S}'\mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}} \cong \mathsf{V}^{\mathbf{S}}\mathsf{S}\mathsf{L}\mathfrak{S}_{\mathbf{S}}$ essentially from Subsection 2.2.4. Because λ is natural, for every $\mathbf{A} \in \mathcal{A}$ we can define a cowedge

$$\mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{C}^{\mathbf{S}}\mathsf{S}'(\mathbf{A}) \longrightarrow \mathsf{V}^{\mathbf{S}}\mathsf{T}\mathsf{S}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}) \longrightarrow \mathsf{V}^{\mathbf{S}}\mathsf{S}\mathsf{L}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}) \longrightarrow \mathsf{S}'\mathsf{L}'(\mathbf{A})$$

where the first arrow is the first natural isomorphism mentioned above, the second arrow is $\mathsf{V}^{\mathbf{S}}\lambda_{\mathfrak{S}_{\mathbf{S}}(\mathbf{A})}$ and the third arrow is the corresponding coend morphism up to the other natural isomorphism mentioned above. The universal property of the coend $\mathsf{T}'\mathsf{S}'(\mathbf{A})$ yields a unique morphism

$$\lambda'_{\mathbf{A}} \colon \mathsf{T}'\mathsf{S}'(\mathbf{A}) \to \mathsf{S}'\mathsf{L}'(\mathbf{A}),$$

which in fact defines a natural transformation $\mathsf{T}'\mathsf{S}' \Rightarrow \mathsf{S}'\mathsf{L}'$, by naturality of λ and of all isomorphisms involved in the definition of λ'. The procedure to obtain λ' from λ is summarized again in Figure 4.6. Now we show that this λ' has the desired property. Suppose towards contradiction that it does not have this property. Then there is cofiltered limit $\mathbf{A} = \lim_{i\in I} \mathbf{A}_i$ as in the statement such that the property is falsified by some $p'_i \colon \mathbf{A} \to \mathbf{A}_i$. It suffices to show $\mathsf{U}\lambda'_{\mathbf{A}} = \lambda_{\mathfrak{S}(\mathbf{A})}$ up to natural isomorphism, because due to the fact that $\mathfrak{S}$ preserves limits and is injective on morphisms, we then have $\mathfrak{S}(\mathbf{A}) = \lim_{i\in I} \mathbf{A}_i$ and $\mathfrak{S}p'_i$ witnesses that λ does also not have the property, a contradiction. To see $\mathsf{U}\lambda'_{\mathbf{A}} = \lambda_{\mathfrak{S}(\mathbf{A})}$, apply U to the diagram in Figure 4.6 instatiated with $\mathbf{S} = \mathbf{D}$ and use $\mathsf{U}\mathsf{V}^{\mathbf{D}} \cong \mathsf{id}_{\mathsf{Set}}$ and $\mathsf{U}\mathsf{T}' \cong \mathsf{T}\mathsf{U}$. □

$$\begin{tikzcd}
\mathsf{T}'\mathsf{S}'(\mathbf{A}) = \int^{\mathbb{S}(\mathbf{D})} \mathsf{V}^{\mathsf{S}}\mathsf{T}\mathsf{C}^{\mathsf{S}}\mathsf{S}'(\mathbf{A}) \arrow[dddd, dashed, "\exists! \lambda'_{\mathbf{A}}"'] & & \mathsf{V}^{\mathsf{S}}\mathsf{T}\mathsf{C}^{\mathsf{S}}\mathsf{S}'(\mathbf{A}) \arrow[ll, "\text{colimit}"'] \arrow[d, "\cong"] \arrow[ddddll, "\text{cowedge}"'] \\
 & & \mathsf{V}^{\mathsf{S}}\mathsf{T}\mathsf{S}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}) \arrow[d, "\mathsf{V}^{\mathsf{S}}\lambda\mathfrak{S}_{\mathbf{S}}"] \\
 & & \mathsf{V}^{\mathsf{S}}\mathsf{S}\mathsf{L}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}) \arrow[dd, "\cong"] \\
 & & \\
\mathsf{S}'\mathsf{L}'(\mathbf{A}) = \int^{\mathbb{S}(\mathbf{D})} \mathsf{S}'\mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}(\mathbf{A}) \arrow[rr, "\text{colimit}"] & & \mathsf{S}'\mathfrak{P}_{\mathbf{S}}\mathsf{L}\mathfrak{S}_{\mathbf{S}}(\mathbf{A})
\end{tikzcd}$$

Figure 4.6: How to obtain λ' from λ.

Therefore, we know that whenever Theorem 4.3.31 can be applied to the classical coalgebraic logic (L, δ), its many-valued analogue can be applied to (L', δ'). To end this subsection, we relate this to the 'intermediate' $(\mathsf{L}, \delta^{\mathbf{D}})$-definability, that is, many-valued definability for classes of T-coalgebras. In particular, we aim to generalize Corollary 3.3.25. As it turns out, all we need for this is the following fact relating the canonical extension of a T'-coalgebra to the ultrafilter extension of its underlying T-coalgebra.

Lemma 4.3.33. *Let λ' be obtained from λ as described in the proof of Proposition 4.3.32 and let $\gamma' \in \mathsf{Coalg}(T')$. Then there is an isomorphism*

$$\mathsf{U}\big(\mathfrak{Ce}(\gamma')\big) \cong \mathfrak{Ue}\big(\mathsf{U}(\gamma')\big)$$

in the category $\mathsf{Coalg}(\mathsf{T})$ (where $\mathfrak{Ce}$ denotes the canonical extension obtained from δ' and λ' and $\mathfrak{Ue}$ denotes the ultrafilter extension obtained from δ and λ).

Proof. Say $\gamma' \colon (X, v) \to \mathsf{T}'(X, v)$ and abbreviate $\mathbf{X} := (X, v)$. The proof is subsumed by the diagram depicted in Figure 4.7. In this large diagram, following the top-most edge yields $\mathsf{U}\big(\mathfrak{Ce}(\gamma')\big)$ and following the bottom-most edge yields $\mathfrak{Ue}(\mathsf{U}\gamma')$. All vertical arrows are natural isomorphisms previously discussed. The diagrams D_1, D_2, D_4 and D_8 clearly commute because therein we always apply the same natural isomorphism vertically. The combined diagram of D_3 and D_6 corresponds to $\mathsf{U}\lambda' = \lambda\mathfrak{S}$ as discussed in the proof of

$$\begin{tikzcd}
\mathsf{US'P'}(\mathbf{X}) \arrow[r, "\mathsf{US'P'}\gamma'"] \arrow[d] \arrow[dr, phantom, "D_1"] & \mathsf{US'P'T'}(\mathbf{X}) \arrow[r, "\mathsf{US'}\delta'_{\mathbf{X}}"] \arrow[d] \arrow[dr, phantom, "D_2"] & \mathsf{US'L'P'}(\mathbf{X}) \arrow[r, "\mathsf{U}\lambda'_{\mathsf{P'}(\mathbf{X})}"] \arrow[d] \arrow[dr, phantom, "D_3"] & \mathsf{UT'S'P'}(\mathbf{X}) \arrow[d] \\
\mathsf{S}\mathfrak{S}\mathsf{P'}(\mathbf{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "D_4"] & \mathsf{S}\mathfrak{S}\mathsf{P'T'}(\mathbf{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "D_5"] & \mathsf{S}\mathfrak{S}\mathsf{L'P'}(\mathbf{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "D_6"] & \mathsf{TUS'P'}(\mathbf{X}) \arrow[d] \\
\mathsf{SPU}(\mathbf{X}) \arrow[r, "\mathsf{SPU}\gamma'"'] & \mathsf{SPUT'}(\mathbf{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "D_7"] & \mathsf{SL}\mathfrak{S}\mathsf{P'}(\mathbf{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "D_8"] & \mathsf{TS}\mathfrak{S}\mathsf{P'}(\mathbf{X}) \arrow[d] \\
 & \mathsf{SPTU}(\mathbf{X}) \arrow[r, "\mathsf{S}\delta_{\mathsf{U}(\mathbf{X})}"'] & \mathsf{SLPU}(\mathbf{X}) \arrow[r, "\lambda_{\mathsf{PU}(\mathbf{X})}"'] & \mathsf{TSPU}(\mathbf{X})
\end{tikzcd}$$

Figure 4.7: Canonical extension and ultrafilter extension.

Proposition 4.3.32 and the combined diagram of D_5 and D_7 corresponds to $\mathfrak{S}\delta' = \delta\mathsf{U}$ as discussed in the proof of Theorem 4.3.21. □

From this, we now get the following analogue of Corollary 3.3.25, showing that, for T-coalgebras, $\mathbf{D}$-valued definability coincides with classical definability.

Corollary 4.3.34. *Suppose that λ as in Theorem 4.3.31 exists and let $\mathcal{C} \subseteq \mathsf{Coalg}(\mathsf{T})$ be an elementary class of T-coalgebras. Then the following are equivalent.*

- *(i) $\mathcal{C}$ is (L, δ)-definable.*
- *(ii) $\mathcal{C}$ is $(\mathsf{L}', \delta^{\mathbf{D}})$-definable.*
- *(iii) $\mathcal{C}$ is closed under coproducts, subcoalgebras and coalgebra morphic images and reflects ultrafilter extensions.*

Proof. The equivalence (i)⇔(iii) is Theorem 4.3.31 and (ii)⇒(iii) is straightforward as in [KR07]. We prove the remaining implication (iii)⇒(ii).

We define a class $\mathcal{C}'$ of T'-coalgebras via

$$\mathcal{C}' = \{\gamma' \in \mathsf{Coalg}(\mathsf{T}') \mid \mathsf{U}\gamma' \in \mathcal{C}\}.$$

We show that $\mathcal{C}'$ is (L', δ')-definable, by showing that it is closed under coproducts, subcoalgebras, coalgebra morphic images and it reflects canonical extensions. Closure under coproducs follows from the corresponding closure property of $\mathcal{C}$ because

$$\mathsf{U}(\coprod \gamma_i') = \coprod \mathsf{U}(\gamma_i') \in \mathcal{C}$$

holds whenever $\{\gamma_i'\} \subseteq \mathcal{C}'$. Closure under subcoalgebras and coalgebra morphic images is (even more) straightforward by definition and the corresponding closure properties of $\mathcal{C}$.

Lastly, suppose that γ' is any T'-coalgebra which satisfies $\mathfrak{Ce}(\gamma') \in \mathcal{C}'$. Then, by definition of $\mathcal{C}'$ and Lemma 4.3.33 we find

$$\mathfrak{Ue}(\mathsf{U}\gamma') \cong \mathsf{U}\big(\mathfrak{Ce}(\gamma')\big) \in \mathcal{C}'$$

and, therefore, $\mathsf{U}\gamma' \in \mathcal{C}$ because $\mathcal{C}$ is closed under ultrafilter extensions. In return, this again yields $\gamma' \in \mathcal{C}'$ as desired.

Therefore, $\mathcal{C}'$ is (L', δ')-definable by some set $\Phi \subseteq I'$, and that same subset witnesses that $\mathcal{C}$ is $(\mathsf{L}', \delta^{\mathbf{D}})$-definable, finishing the proof. □

With this, we end the main part of Chapter 4. In the following concluding section, we give an outlook of potential future research related to this chapter.

4.4 Conclusion of Chapter 4

We provided a general method to lift classical algebra-coalgebra dualities and coalgebraic logics to the semi-primal level. More specifically, we showed how to get from a classical abstract coalgebraic logic (L, δ) to a $\mathbf{D}$-valued abstract coalgebraic logic (L', δ') in the sense that L' is defined on $\mathcal{A}$. We also showed that (L', δ') inherits many properties from (L, δ) with regards to completeness, expressivity and definability. Furthermore, L' has a presentation by operations and equations if L has one, and we partially answered how such a presentation of L' can be obtained directly from a presentation of L. In particular, we showed how all of this applies to the lifting of classical modal logic. In the following, we offer some more ideas for future related research.

In Subsection 4.3.1, we described how to lift classical algebra-coalgebra dualities building on Stone duality. Recently, in the two papers [BCM22, BBdG22], classical algebra-coalgebra dualities building on 'Tarski duality' between Set and CABA were studied as well. In particular, in [BCM22] the duality between Kripke frames and complete atomic modal algebras with $\Box$ preserving arbitrary meets (which the authors refer to as 'Thomason duality') was established as algebra-coalgebra duality between the covariant powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ and a functor which we call $\mathsf{cL}_\Box\colon \mathsf{CABA} \to \mathsf{CABA}$, which can also be defined similarly to $\mathsf{L}_\Box$ but using a proper class of equations (assuring preservation of meets of arbitrary cardinality), see [BBdG22, Example 3.12]. Looking at this in the context of Section 2.3 and Subsection 4.3.1, it seems likely that such dualities can also be lifted to algebra-coalgebra dualities building on the discrete semi-primal duality between $\mathsf{Set}_{\mathbf{D}}$ and $\mathsf{CA}\mathcal{A}$ (recall Definition 2.3.9), as indicated in Figure 4.8.

$$\begin{tikzcd}
\mathsf{Set}_{\mathbf{D}} \arrow[loop left, "\mathsf{T}'"] \arrow[r, bend left=10, "\mathsf{c}\Pi'"] \arrow[d, bend left, "\mathsf{C}^{\mathbf{S}}"] \arrow[d, phantom, "\dashv"] & \mathsf{CA}\mathcal{A} \arrow[loop right, "\mathsf{cL}'"] \arrow[l, bend left=10, "\mathsf{c}\Sigma'"] \arrow[d, bend left, "\mathfrak{S}_{\mathbf{S}}"] \arrow[d, phantom, "\vdash"] \\
\mathsf{Set} \arrow[loop left, "\mathsf{T}"] \arrow[r, bend left=10, "\mathsf{c}\Pi"] \arrow[u, bend left, "\mathsf{V}^{\mathbf{S}}"] & \mathsf{CABA} \arrow[loop right, "\mathsf{cL}"] \arrow[l, bend left=10, "\mathsf{c}\Sigma"] \arrow[u, bend left, "\mathfrak{P}_{\mathbf{S}}"]
\end{tikzcd}$$

Figure 4.8: Lifting discrete algebra-coalgebra dualities.

Here, the way to lift T is the same as discussed in Subsection 4.3.2 and the way to lift cL to cL' is dual to this (*i.e.*, via ends as in Definition 4.3.3). We conjecture that the category of algebras for the functor $\mathsf{cL}'_\Box$ is isomorphic to the category of *complete modal $\mathcal{A}$-algebras* $\langle \mathbf{A}, \Box\rangle$, where $\mathbf{A} \in \mathsf{CA}\mathcal{A}$ and $\Box$ satisfies the axioms

$$\Box 1 = 1, \qquad \Box\Big(\bigwedge_{i\in I} x_i\Big) = \bigwedge_{i\in I} \Box x_i, \qquad \Box\tau_d(x) = \tau_d(\Box x), \text{ for all } d \in D^+.$$

This also relates to *canonical extensions* of modal algebras.

In this chapter, we dealt with $\mathbf{D}$-valued coalgebraic logics, using the *abstract approach* to coalgebraic logic. In the future, it would also be interesting to stuy these coalgebraic logics using the *relation lifting approach* [Mos99] and the *predicate lifting approach* [Pat03a]. Little prior work exists on these approaches in a many-valued setting (few examples are [BKPV13] following the relation lifting approach and [BD16, LL23] following the predicate lifting approach).

The *relation lifting approach* is, roughly speaking, based on a technique to lift a binary relation $B \subseteq X \times Y$ to a relation $\widehat{B} \subseteq \mathsf{T}(X) \times \mathsf{T}(Y)$. The set of formulas of the T-coalgebraic logic is closed under $\nabla\alpha$ whenever $\alpha \in \mathsf{T}(\mathsf{Form})$ and the semantics of this *cover modality* ∇ are given on a coalgebra $\gamma \colon X \to \mathsf{T}(X)$ by

$$x \models \nabla\alpha \text{ if and only if } \gamma(x) \widehat{\models} \alpha.$$

A $\mathbf{D}$-valued relation lifting now has to lift a $\mathbf{D}$-valued relation $B \subseteq X \times Y \to D$ to one $\widehat{B} \subseteq \mathsf{T}(X) \times \mathsf{T}(Y) \to D$. In order to do this, dissect B into $|D^+|$-many binary relations $B_d \subseteq X \times Y$ with $xB_dy \Leftrightarrow B(x,y) \geq d$ and define

$$\widehat{B}(V,W) = \bigvee \{d \in D \mid V\widehat{B_d}W\}.$$

In other words, we define the many-valued relation lifting from the classical one in a way similarly to Subsection 4.3.3. Semantically we then define

$$\models(x, \nabla\alpha) = \widehat{\models}(\gamma(x), \alpha).$$

In the case where $\mathbf{D}$ is linearly ordered and $\mathsf{T} = \mathcal{P}$, from this we retrieve for $\Phi \subseteq \mathsf{Form}$ the definition

$$\models (x, \nabla\Phi) = \bigwedge_{x' \in \gamma(x)} \bigvee_{\varphi \in \Phi} \models (x', \varphi) \wedge \bigwedge_{\varphi \in \Phi} \bigvee_{x' \in \gamma(x)} \models (x', \varphi)$$

as expected. In [KKV12] a proof system for the classical cover modality is established, essentially as a concrete coalgebraic logic $\mathbf{M} \colon \mathsf{BA} \to \mathsf{BA}$. Adapting the proof of Theorem 4.2.7 in the case where $\mathbf{D}$ is linear and primal, one can show that the lifted functor $\mathbf{M}' \colon \mathcal{A} \to \mathcal{A}$ and the corresponding logic can be described by the axioms (∇1)-(∇3) from [KKV12] and the additional axiom

$$\tau_d(\nabla\alpha) = \nabla(\mathsf{T}\tau_d)(\alpha) \text{ for all } d \in D^+.$$

The details of this claim, as well as the generalization to non-linear and non-primal algebras is left for future work.

The *predicate lifting approach* to coalgebraic logic identifies two-valued modalities for $\mathsf{T} \colon \mathsf{Set} \to \mathsf{Set}$ with natural transformations $2^{(\cdot)} \Rightarrow 2^{(\cdot)} \circ \mathsf{T}$. By the Yoneda Lemma, these can be identified with maps $\mathsf{T}(2) \to 2$. For example, in the case where $\mathsf{T} = \mathcal{P}$ is the powerset functor, there are 16 such maps, and all of them can be obtained as Boolean combinations of '$\Box$' and '$\Diamond$' corresponding to the characteristic functions of $\{\varnothing, \{1\}\}$ and $\{\{0,1\}, \{1\}\}$, respectively. In the $\mathbf{D}$-valued case, predicate liftings are natural transformations $D^{(\cdot)} \Rightarrow D^{(\cdot)} \circ \mathsf{T}$. In the case where $|D| = 3$ and $\mathsf{T} = \mathcal{P}$, there are $3^{2^3} = 6561$ possible predicate liftings (that is, modalities). Is is still possible to obtain all of them as combinations (using operations from **D**) of '$\Box$' and '$\Diamond$' which are identified with $\bigwedge$ and $\bigvee$ as maps $\mathcal{P}(D) \to D$, respectively?

The relationship between the three approaches to coalgebraic logic in the classical case has been described in [KL12]. After carrying out the 'program' above, one could also paint the 'complete picture' for the semi-primal case.

The last open question we pose here is the following. Theorems 4.3.25 and 4.3.26 only deal with presentations of L by a single unary operation, though the proof allows a straightforward generalization to presentations by a single n-ary operation. The case of presentations by more than one operation and mutual interplay seems more involved. For example, is it possible to lift the presentation of the probabilistic logic for the distribution functor from [CP04] to the semi-primal level?

Chapter 5

Many-valued positive modal logic

> *One could say that denial is already related to the logical place that is determined by the proposition that is denied.*
> *The denying proposition determines a logical place* other *than does the proposition denied.*
>
> – Ludwig Wittgenstein (1921)[1]

In this chapter, we study *positive* modal logic over finite MV-chains, which may be seen as the negation-free and implication-free reduct of many-valued modal logic over finite MV-chains. To this end, we first establish natural dualities for the varieties PMV_n generated by the finite positive MV-chains $\mathbf{PŁ}_n$. We also show that there is an adjunction between PMV_n and DL given by the constructions of the *distributive skeletons* and *Priestley powers*. We then use these results to study positive modal logic over finite MV-chains algebraically, providing an algebraic completeness result. Along the way, we also introduce ordered $\mathbf{D}$-frames and argue why, from the point of view of canonicity and definability, these provide 'appropriate' semantics for our many-valued positive modal logic.

The chapter is structured as follows. In Section 5.1, we introduce and prove some basic facts about the varieties PMV_n (Subsection 5.1.2), before we establish our natural dualities for them (Subsection 5.1.3). In Section 5.2, we further study these dualities, in particular we study the relationship to Priestley duality (Subsection 5.2.2). In Section 5.3, we move on to study modal PMV_n-algebras as algebraic counterparts of $\mathbf{PŁ}_n$-valued modal logics

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[1] German original and English translation in [Wit21, Proposition 4.0641, p.74/75]

(Subsection 5.3.3). We also provide some richer semantics for this logic, over a category of ordered frames with local constraints (Subsection 5.3.2) and argue why they are adequate from the point of view of canonicity (Subsection 5.3.4). Lastly, in the concluding Section 5.4, we sketch how to further generalize some of our results to 'strongly lattice-semi-primal algebras' (Subsection 5.4.1) and positive coalgebraic logic over lattice-primal varieties (Subsection 5.4.2).

The quasi-variety of positive MV-algebras was introduced and axiomatized in [AJKV22]. Positive modal logic has been introduced in [Dun95], the idea of its semantics on *ordered frames* is due to [CJ97, CJ99]. The first part of this chapter may be seen as extended version of the author's [Poi23] and the second part may be seen as extended version of the author's [Poi24]. To the best of the author's knowledge, the latter is the first instance of a *many-valued* positive modal logic in the literature.

In the first part of this chapter, we draw heavily upon the theory of natural dualities. The standard reference is the book [CD98], in addition we also recommend [Dav15] for a good overview. In the second part of this chapter, we assume that the reader is (somewhat) familiar with positive modal logic, in particular with the papers [Dun95, CJ97, CJ99].

5.1 Natural dualities for varieties generated by finite PMV-chains

In this section, we provide a simple natural duality for the varieties generated by the negation- and implication- free reduct of a finite MV-chain. We proceed to study these varieties through the dual equivalence thus obtained. In particular, we explore the relationship between this natural duality and Priestley duality in terms of distributive skeletons and Priestley powers.

The section is structured as follows. In Subsection 5.1.1, we give an overview of the most important concepts from the theory of natural dualities for our purposes. In Subsection 5.1.2, we introduce the varieties PMV_n of positive MV_n-algebras and prove some basic facts about them. Lastly, in Subsection 5.1.3, we develop our natural dualities for the varieties PMV_n.

5.1.1 Introduction to natural duality theory

The theory of natural dualities provides a common framework to develop dual equivalences between quasi-varieties of algebras and categories of structured Stone spaces. In particular, the theory encompasses and generalizes Stone duality for Boolean algebras and Priestley duality for distributive lattices. In this subsection, we give a selective overview of this theory. For more information, we refer the reader to the book [CD98], which we often cite throughout Section 5.1.

Let $\mathbf{M}$ be a finite algebra (with underlying set M) and let $\mathcal{Q} = \mathbb{ISP}(\mathbf{M})$ be the quasi-variety it generates. An *alter ego* of $\mathbf{M}$ is a discrete topological structure (also with underlying set M) of the form

$$\underset{\sim}{\mathbf{M}} = (M, \mathcal{G}, \mathcal{H}, \mathcal{R}, \mathcal{T}_{\text{dis}}),$$

where $\mathcal{G}$ is a collection of (total) homomorphisms $\mathbf{M}^n \to \mathbf{M}$ (possibly nullary, which corresponds to *constants*), $\mathcal{H}$ is a collection of *partial homomorphisms*, that is, homomorphisms from a subalgebra of $\mathbf{M}^n$ to $\mathbf{M}$ and $\mathcal{R}$ is a collection of *algebraic relations*, that is, subalgebras $\mathbf{R} \subseteq \mathbf{M}^n$. Lastly, $\mathcal{T}_{\text{dis}}$ is the discrete topology on M.

The *topological quasi-variety* $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ generated by $\underset{\sim}{\mathbf{M}}$ consists of structured Stone spaces

$$\mathbf{X} = (X, \mathcal{G}^{\mathbf{X}}, \mathcal{H}^{\mathbf{X}}, \mathcal{R}^{\mathbf{X}}, \mathcal{T}),$$

of the same type as $\underset{\sim}{\mathbf{M}}$ which are isomorphic to a closed substructure of a non-empty product of $\underset{\sim}{\mathbf{M}}$. The category $\mathcal{X}$ with structure-preserving continuous maps as morphisms is often described using the Preservation Theorem [CD98, Theorem 1.4.3] and the Separation Theorem [CD98, Theorem 1.4.3].

By the Preduality Theorem [CD98, Theorem 1.5.2], there exists a dual adjunction between $\mathcal{Q}$ and $\mathcal{X}$ given by the contravariant hom-functors $\mathsf{D}\colon \mathcal{Q} \to \mathcal{X}$ and $\mathsf{E}\colon \mathcal{X} \to \mathcal{Q}$ defined by

$$\mathsf{D}(\mathbf{A}) = \mathcal{Q}(\mathbf{A}, \mathbf{M}) \text{ and } \mathsf{E}(\mathbf{X}) = \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$$

for all $\mathbf{A} \in \mathcal{Q}$ and $\mathbf{X} \in \mathcal{X}$. The natural transformations $e\colon 1_{\mathcal{Q}} \to \mathsf{ED}$ and $\varepsilon\colon 1_{\mathcal{X}} \to \mathsf{DE}$ corresponding to this adjunction are given by evaluations

$$\begin{aligned} e_{\mathbf{A}}(a)(u) &= u(a) && \text{for all } \mathbf{A} \in \mathcal{Q}, u \in \mathsf{D}(\mathbf{A}) \text{ and } a \in A, \\ \varepsilon_{\mathbf{X}}(x)(\alpha) &= \alpha(x) && \text{for all } \mathbf{X} \in \mathcal{X}, \alpha \in \mathsf{E}(\mathbf{X}) \text{ and } x \in X. \end{aligned}$$

If e is a natural isomorphism, we say that $\underset{\sim}{\mathbf{M}}$ *yields a duality for* $\mathcal{Q}$ (this is also known as *dual representation*). If ε is a natural isomorphism as well, we say that $\underset{\sim}{\mathbf{M}}$ yields a *full* duality for $\mathcal{Q}$ (meaning that D and E establish a *dual equivalence*). In fact, in this thesis we exclusively deal with *strong dualities* [CD98, Chapter 3], which are full dualities with the additional property that $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}$.

In particular, for lattice-based algebras, strong dualities can always be obtained via the following *NU Strong Duality Corollary* [CD98, Corollary 3.3.9].

Corollary 5.1.1 (NU Strong Duality Corollary [CD98]). *Let* $\mathbf{M}$ *have a majority term, and let all subalgebras of* $\mathbf{M}$ *be subdirectly irreducible. Then*

$$\widetilde{\mathbf{M}} = \big(M, K, P_1, \mathbb{S}(\mathbf{M}\times\mathbf{M}), \mathcal{T}_{\text{dis}}\big),$$

yields a strong duality on $\mathcal{A}$*, where* K *is the union of trivial (*i.e.*, one-element) subalgebras of* $\mathbf{M}$*, the set* P_1 *consists of all unary partial homomorphisms* $\mathbf{M}\to\mathbf{M}$ *and* $\mathbb{S}(\mathbf{M}\times\mathbf{M})$ *consists of all binary algebraic operations.*

While this corollary narrows down the structure needed to obtain a strong duality, this $\widetilde{\mathbf{M}}$ is usually still more complicated than it necessarily has to be. This is where (strong) *entailment* comes into play. We say that another alter ego $\widetilde{\mathbf{M}'} = (M, \mathcal{G}', \mathcal{H}', \mathcal{R}', \mathcal{T}_{\text{dis}})$ *strongly entails* $\widetilde{\mathbf{M}}$ if whenever $\widetilde{\mathbf{M}}$ yields a strong duality on $\mathcal{A}$, the same is true for $\widetilde{\mathbf{M}'}$. Similarly, we say that members of $\mathcal{G}'\cup\mathcal{H}'\cup\mathcal{R}'$ strongly entail members of $\mathcal{G}\cup\mathcal{H}\cup\mathcal{R}$. In the following, we give a list of admissible constructs for strong entailment relevant for this section (see [CD98, Chapter 9] for a complete list of admissible constructs for entailment).

1. Any set of relations strongly entails the full product $\mathbf{M}^2$, the diagonal $\Delta_{\mathbf{M}} = \{(m,m) \mid m\in\mathbf{M}\}$ of $\mathbf{M}$ and the identity $\mathsf{id}_{\mathbf{M}}$ on $\mathbf{M}$.
2. Any binary relation $\mathbf{R}$ strongly entails its converse $\mathbf{R}^{-1} = \{(b,a) \mid (a,b)\in\mathbf{R}\}$ and $\pi_1(\mathbf{R}\cap\Delta_{\mathbf{M}})$.
3. Relations $\mathbf{S},\mathbf{R}\subseteq\mathbf{M}^n$ strongly entail their intersection $\mathbf{S}\cap\mathbf{R}$.
4. Arbitrary relations $\mathbf{S}$ and $\mathbf{R}$ entail their product $\mathbf{S}\times\mathbf{R}$.
5. $\widetilde{\mathbf{M}'}$ strongly entails $\widetilde{\mathbf{M}}$ if it is obtained from $\widetilde{\mathbf{M}}$ by deleting a partial operation $h\in\mathcal{H}$ which has an extension in $\mathcal{G}$ and adding its domain to $\mathcal{R}$ as unary relation.

We say that $\widetilde{\mathbf{M}}$ yields an *optimal* strong duality if $\mathcal{G}\cup\mathcal{H}\cup\mathcal{R}$ is not strongly entailed by any of its proper subsets.

We illustrate the concepts introduced in this subsection by explaining how to obtain the dualities for MV_n from Section 2.1 as natural dualities (also recall Remark 2.1.3). These natural dualities for MV_n have also been explored in [Nie01]. The following example is a specific instance of the proof of the *Semi-Primal Strong Duality Theorem* [CD98, Theorem 3.3.14].

Example 5.1.2 (Strong duality for MV_n). Let $n\geq 1$. The discrete structure

$$\widetilde{\text{Ł}}_n = \big(\{0, \tfrac{1}{n}, \dots \tfrac{n-1}{n}, 1\}, \mathbb{S}(\text{Ł}_n), \mathcal{T}_{\text{dis}}\big),$$

where members of $\mathbb{S}(\text{Ł}_n)$ are understood as unary relations, yields a strong duality on MV_n.

Proof. By Corollary 5.1.1, the structure

$$\big(\{0, \tfrac{1}{n}, \dots \tfrac{n-1}{n}, 1\}, K, P_1, \mathbb{S}(\mathbf{Ł}_n \times \mathbf{Ł}_n), \mathcal{T}_{\text{dis}}\big)$$

yields a strong duality on MV_n (where K is the union of one-element subalgebras and P_1 is the collection of all unary partial homomorphisms). Since $\mathbf{Ł}_n$ is based on a bounded lattice, it has no one-element subalgebras, therefore $K = \varnothing$. Furthermore, the only homomorphism $\mathbf{Ł}_k \to \mathbf{Ł}_n$ defined on a subalgebra $\mathbf{Ł}_k \subseteq \mathbf{Ł}_n$ is the natural embedding of $\mathbf{Ł}_k$. Using the strong entailment constructs (1) and (5) above, it can be replaced by its domain $\mathbf{Ł}_k \in \mathbb{S}(\mathbf{Ł}_n)$. Every subalgebra $\mathbf{R} \in \mathbb{S}(\mathbf{Ł}_n \times \mathbf{Ł}_n)$ is simply a product of subalgebras of $\mathbf{Ł}_n$. Therefore, by (4) above, they are strongly entailed by $\mathbb{S}(\mathbf{Ł}_n)$ as well. ☐

It follows from [CD98, Theorem 9.2.6] that modifying the structure from the above example to only include the meet-irreducible members of the lattice $\mathbb{S}(\mathbf{Ł}_n)$ yields an *optimal* strong duality (also see [CD98, Theorem 8.3.2]).

In the subsections that follow, we aim to come up with a similarly simple natural duality for varieties generated by *positive* MV-chains.

5.1.2 Positive MV-chains

Following the recent paper [AJKV22], we use the term *positive* MV*-algebra* or PMV*-algebra* to refer to a negation-free (and implication-free) subreduct of an MV-algebra. In particular, we focus on finite positive MV-chains defined as follows.

Definition 5.1.3 (Finite positive MV-chain)**.** Let $n \geq 1$ be a natural number. The $(n+1)$*-element positive* MV*-chain* is given by

$$\mathbf{PŁ}_n = \langle \{0, \tfrac{1}{n}, \dots \tfrac{n-1}{n}, 1\}, \wedge, \vee, \odot, \oplus, 0, 1 \rangle,$$

understood as a reduct of $\mathbf{Ł}_n$. We write PMV_n for the variety $\mathbb{HSP}(\mathbf{PŁ}_n)$ generated by $\mathbf{PŁ}_n$, and we refer to members of PMV_n as *positive* MV_n*-algebras* or PMV_n*-algebras*.

Our first result about $\mathbf{PŁ}_n$ is that its subalgebras are the same as the subalgebras of $\mathbf{Ł}_n$ and, therefore (recall the discussion after Definition 1.2.3), the subalgebra-lattice $\mathbb{S}(\mathbf{PŁ}_n)$ is isomorphic to the bounded lattice of diviors of n.

Proposition 5.1.4. *The subalgebras of* $\mathbf{PŁ}_n$ *are exactly given by the subuniverses*

$$\mathbf{PŁ}_k \cong \{0, \tfrac{\ell}{n}, \dots, \tfrac{(k-1)\ell}{n}, 1\},$$

where $n = k \cdot \ell$.

Proof. Let $\mathbf{L} \subseteq \mathbf{PŁ}_n$ be an arbitrary subalgebra and let $\frac{\ell}{n}$ be the unique minimal element of $\mathbf{L}$ which is not zero. If $\ell = n$, then $\mathbf{L} = \mathbf{PŁ}_1$ holds, so assume $\ell < n$. Note that this implies $\frac{\ell}{n} \leq \frac{1}{2}$, since otherwise $\frac{\ell}{n} \odot \frac{\ell}{n}$ would be an element of $\mathbf{L}$ greater than zero but strictly smaller than $\frac{\ell}{n}$, contradicting our choice of ℓ. Furthermore, ℓ needs to be a divisor of n, since otherwise we can find natural numbers $x \geq 1$ and $0 < r < \ell$ with $n = x\ell + r$. But then $\frac{x\ell}{n} \odot \frac{\ell}{n} = \frac{r}{n}$ is a member of $\mathbf{L}$ above zero but strictly below $\frac{\ell}{n}$, again contradicting our choice of ℓ. Thus we showed that ℓ divides n and therefore, by closure of $\mathbf{L}$ under $\oplus$, we showed that $\mathbf{PŁ}_k$ as in the proposition is contained in $\mathbf{L}$.

Suppose towards contradiction that there is some $\frac{s}{n} \in \mathbf{L} \backslash \mathbf{PŁ}_k$. Then $\ell < s$ holds by the above assumption and we can find natural numbers $k > x > 1$ and $0 < r < \ell$ such that $s = x\ell + r$. This is equivalent to

$$r + n = s - x\ell + n = s + (k - x)\ell.$$

Therefore, we conclude that $\frac{r}{n} = \frac{s}{n} \odot \frac{(k-x)\ell}{n}$ is in $\mathbf{L}$, once more contradicting minimality in our choice of ℓ. □

The unary operations τ_d are still term-definable in $\mathbf{PŁ}_n$, since they can be defined in $\mathbf{Ł}_n$ from expressions of the form $x \odot x$ and $x \oplus x$ exclusively. This fact, shown in [Ost88], will be of high importance in many proofs later on.

Lemma 5.1.5 ([Ost88, pp. 344-345])**.** *For every $d \in \mathbf{PŁ}_n$, the unary operation $\tau_d \colon Ł_n \to Ł_n$ given by*

$$\tau_d(x) = \begin{cases} 1 & \textit{if } d \leq x, \\ 0 & \textit{otherwise} \end{cases}$$

is term-definable in $\mathbf{PŁ}_n$.

Our first goal is to show that the variety PMV_n coincides with the quasivariety $\mathbb{ISP}(\mathbf{PŁ}_n)$ generated by $\mathbf{PŁ}_n$. For this, we essentially only have to show the following.

Lemma 5.1.6. *Every subalgebra $\mathbf{PŁ}_k \subseteq \mathbf{PŁ}_n$ (including $\mathbf{PŁ}_n$ itself) is simple.*

Proof. Let θ be a congruence relation on $\mathbf{PŁ}_k$ and let $c, d \in \mathbf{PŁ}_k$ be distinct elements with $(c, d) \in \theta$. We show that θ is the trivial congruence identifying all members of $\mathbf{PŁ}_k$. Without loss of generality, we assume $c < d$. Since τ_d

from Lemma 5.1.5 is term-definable in $\mathbf{PŁ}_n$, we have $(0,1) = (\tau_d(c), \tau_d(d)) \in \theta$ and $(1,0) \in \theta$ by symmetry. Now, for arbitrary $x, y \in \mathbf{PŁ}_k$, we have

$$(x,y) = \big((1,0) \wedge (x,x)\big) \vee \big((0,1) \wedge (y,y)\big) \in \theta,$$

which implies $\theta = \mathbf{PŁ}_k^2$. □

Since PMV_n is congruence distributive (because $\mathbf{PŁ}_n$ is lattice-based and thus has a majority term), a standard application of Jónsson's Lemma [Jó67] yields the following (see, *e.g.*, [CD98, Theorem 1.3.6]).

Corollary 5.1.7. $\mathsf{PMV}_n = \mathbb{ISP}(\mathbf{PŁ}_n)$.

This allows us to study the variety PMV_n via the theory of natural dualities in what follows.

5.1.3 The natural dualities

This subsection is dedicated to finding a simple alter-ego $\widetilde{\mathbf{PŁ}}_n$ of $\mathbf{PŁ}_n$ which yields a 'useful' [CD98, Chapter 6] strong duality on PMV_n. Since $\mathbf{PŁ}_n$ has a bounded lattice reduct, it has a majority term and no trivial subalgebras. Furthermore, by Lemma 5.1.6 we know that every subalgebra of $\mathbf{PŁ}_n$ is subdirectly irreducible. Therefore, we may use Corollary 5.1.1 (*i.e.*, the NU Strong Duality Corollary [CD98, Corollary 3.3.9]) as our starting point. This states that

$$\big(\{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, P_1, \mathbb{S}(\mathbf{PŁ}_n \times \mathbf{PŁ}_n), \mathcal{T}_{\text{dis}}\big), \tag{5.1}$$

yields a strong duality for PMV_n, where P_1 is the set of all unary partial homomorphisms $\mathbf{PŁ}_n \to \mathbf{PŁ}_n$. In the following we show that, as for $\mathbf{Ł}_n$, the only partial homomorphisms of this kind are the identities of subalgebras.

Lemma 5.1.8. *Let $\mathbf{PŁ}_k \subseteq \mathbf{PŁ}_n$ be a subalgebra. Then the only homomorphism $\mathbf{PŁ}_k \to \mathbf{PŁ}_n$ is the identity on $\mathbf{PŁ}_k$ (followed by inclusion).*

Proof. Let $h\colon \mathbf{PŁ}_k \to \mathbf{PŁ}_n$ be a homomorphism. Suppose there are some $s \in \mathbf{PŁ}_k$ and $d \in \mathbf{PŁ}_n$ such that $h(s) = d$ and $s \neq d$. Recall that τ_d and τ_s from Lemma 5.1.5 are term-definable and thus preserved by h. If $s < d$ then $1 = \tau_d(h(s)) = h(\tau_d(s)) = h(0) = 0$ yields a contradiction. If $d < s$ then $1 = h(\tau_s(s)) = \tau_s(h(s)) = \tau_s(d) = 0$ also yields a contradiction. Thus no such elements s and d can exist and we showed that $h(s) = s$ holds for all $s \in \mathbf{PŁ}_k$. □

Therefore, as in Example 5.1.2, the collection P_1 of unary partial homomorphisms is strongly entailed by the collection of unary algebraic relations $\mathbb{S}(\mathbf{P\L}_n)$. Now we take a closer look at the binary algebraic relations in $\mathbb{S}(\mathbf{P\L}_n \times \mathbf{P\L}_n)$. Contrary to $\mathbf{\L}_n$, the algebra $\mathbf{P\L}_n \times \mathbf{P\L}_n$ has subalgebras which are not direct products of subalgebras of $\mathbf{P\L}_n$. For example, since all operations of $\mathbf{P\L}_n$ are order-preserving, the relation $\leq$ itself and its converse $\geq$ are clearly subalgebras of $\mathbf{P\L}_n \times \mathbf{P\L}_n$. In the following, we show that every other subalgebra of $\mathbf{P\L}_n$ which is not a direct product of subalgebras is contained in one of these.

Lemma 5.1.9. *Every subalgebra* $\mathbf{R} \subseteq \mathbf{P\L}_n \times \mathbf{P\L}_n$ *which is not a direct product of subalgebras of* $\mathbf{P\L}_n$ *is a subalgebra of* $\leq$ *or of* $\geq$.

Proof. Suppose that $\mathbf{R}$ is neither a subset of $\leq$ nor of $\geq$. We show that this implies that $\mathbf{R}$ is a direct product of subalgebras of $\mathbf{P\L}_n$. Since $\mathbf{R}$ is not a subset of $\leq$, there is $(d_1, c_1) \in \mathbf{R}$ with $d_1 > c_1$. Similarly, there is $(c_2, d_2) \in \mathbf{R}$ with $c_2 < d_2$. This implies that $(1, 0) = \tau_{d_1}(d_1, c_1)$ and $(0, 1) = \tau_{d_2}(c_2, d_2)$ are both in $\mathbf{R}$. As in the proof of Lemma 5.1.6, with this we can show that $\mathbf{R}$ is the full direct product of its two projections $\mathsf{pr}_1(\mathbf{R})$ and $\mathsf{pr}_2(\mathbf{R})$. $\square$

Since every binary relation strongly entails its converse and all products of subalgebras of $\mathbf{P\L}_n$ are strongly entailed by $\mathbb{S}(\mathbf{P\L}_n)$, it follows that the structure

$$\big(\{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \mathbb{S}(\mathbf{P\L}_n) \cup \mathbb{S}(\leq), \mathcal{T}_{\mathrm{dis}}\big) \tag{5.2}$$

yields a strong duality for PMV_n, since it strongly entails the structure from Equation (5.1).

While the structure given in Equation (5.2) is already much simpler than that in Equation (5.1), it is still far from optimal. Therefore, we keep on studying $\mathbb{S}(\leq)$ in order to further simplify this alter ego.

A somewhat special role is played by the subalgebra of the order $\lhd \in \mathbb{S}(\leq)$ given by

$$\lhd = \{(x, y) \mid x = 0 \text{ or } y = 1\}.$$

It is easy to see that this is a subalgebra, since $0 \wedge x = 0 \odot x = 0$ and $1 \vee x = 1 \oplus x = 1$ for all $x \in \mathbf{P\L}_n$. Unfortunately, except for the case $n = 2$, this is not the only non-diagonal proper subalgebra of the order relation. However, it is minimal among those subalgebras in the following sense.

Lemma 5.1.10. *Let* $\mathbf{R} \subseteq \mathbf{P\L}_n \times \mathbf{P\L}_n$ *be a subalgebra of the order* $\leq$, *which is not the diagonal of a subalgebra of* $\mathbf{P\L}_n$, *and* $\mathbf{S} = \mathsf{pr}_1(\mathbf{R}) \times \mathsf{pr}_2(\mathbf{R})$. *Then* $\lhd|_{\mathbf{S}} \subseteq \mathbf{R} \subseteq \leq|_{\mathbf{S}}$.

Proof. Since $\mathbf{R}$ is not a diagonal, there is a pair $(x, y) \in \mathbf{R}$ with $x \neq y$, implying $x < y$. Therefore, $\tau_y(x, y) = (0, 1) \in \mathbf{R}$ as well. Now, for any $(x', y') \in \mathbf{R}$ we find that

$$(0, y') = (x', y') \wedge (0, 1) \text{ and } (x', 1) = (x', y') \vee (0, 1)$$

are also members of $\mathbf{R}$, finishing the proof. □

Since diagonals of subalgebras are already strongly entailed by $\mathbb{S}(\mathbf{PŁ}_n)$, the above tells us that we only need to consider subalgebras in-between (restrictions of) $\lhd$ and $\leq$. In order to describe these subalgebras, the following 'closure' downwards in the first and upwards in the second component will be crucial.

Definition 5.1.11. Let $\mathbf{S} = \mathbf{PŁ}_k \times \mathbf{PŁ}_{k'}$ be a product of subalgebras of $\mathbf{PŁ}_n$. Let $(x, y) \in \mathbf{S}$ with $x \leq y$. We denote by $C_{(x,y),\mathbf{S}}$ the following subset of $\mathbf{S}$ and $\leq$.

$$C_{(x,y),\mathbf{S}} = \{(x', y') \in \mathbf{S} \mid x' \leq x \text{ and } y \leq y'\}.$$

If $\mathbf{S} = \mathbf{PŁ}_n \times \mathbf{PŁ}_n$, we simply use $C_{(x,y)}$ instead of $C_{(x,y),\mathbf{S}}$.

For example, Figure 5.1 depicts the subsets $C_{(\frac{2}{6},\frac{3}{6})}$ and $C_{(\frac{2}{6},\frac{3}{6}),\mathbf{S}}$ for $\mathbf{S} = \mathbf{PŁ}_3 \times \mathbf{PŁ}_2$ as subsets of $\mathbf{PŁ}_6 \times \mathbf{PŁ}_6$.

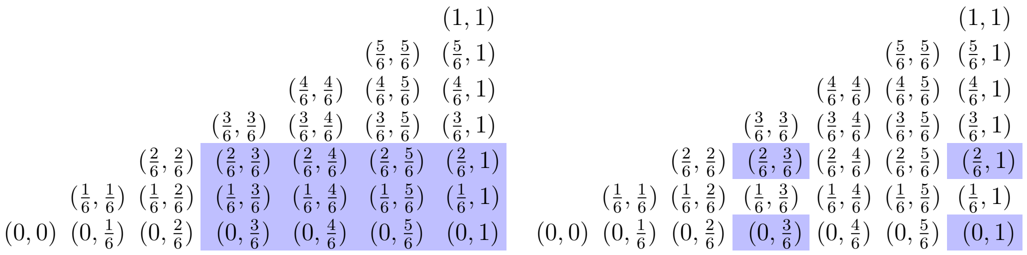


Figure 5.1: The sets $C_{(\frac{2}{6},\frac{3}{6})}$ and $C_{(\frac{2}{6},\frac{3}{6}),\mathbf{PŁ}_3\times\mathbf{PŁ}_2}$ in the case $n = 6$.

In the next lemma, we show that non-diagonal subalgebras of the order are closed under these subsets in the following sense.

Lemma 5.1.12. *Let* $\mathbf{R} \subseteq \mathbf{PŁ}_n \times \mathbf{PŁ}_n$ *be a subalgebra of the order* $\leq$ *which is not the diagonal of a subalgebra of* $\mathbf{PŁ}_n$*, and* $\mathbf{S} = \mathsf{pr}_1(\mathbf{R}) \times \mathsf{pr}_2(\mathbf{R})$. *If* $(x, y) \in \mathbf{R}$*, then* $C_{(x,y),\mathbf{S}} \subseteq \mathbf{R}$ *as well.*

Proof. By Lemma 5.1.10, we know that $\lhd|_{\mathbf{S}} \subseteq \mathbf{R}$ holds. Now let $(x, y) \in \mathbf{R}$, and say $(x', y') \in \mathbf{S}$ satisfies $x' \leq x$ and $y \leq y'$. Then $(x', y) = (x, y) \wedge (x', 1)$ is in $\mathbf{R}$ and, thus, $(x', y') = (x', y) \vee (0, y')$ is also in $\mathbf{R}$. □

Therefore, clearly every $\mathbf{R}$ as in the above lemma is a union of sets of the form $C_{(x,y),\mathbf{S}}$. However, not all unions of sets of this form necessarily yield subalgebras. In the following, we identify exactly those unions which *do* give rise to subalgebras of $\mathbf{PŁ}_n \times \mathbf{PŁ}_n$.

Proposition 5.1.13. *Let* $\mathbf{S} = \mathbf{PŁ}_k \times \mathbf{PŁ}_{k'}$ *be a product of subalgebras of* $\mathbf{PŁ}_n$.

(1) Let $\mathbf{R} \subseteq \mathbf{PŁ}_n \times \mathbf{PŁ}_n$ *be a subalgebra of* $\leq$*, which is not the diagonal of a subalgebra of* $\mathbf{PŁ}_n$*, with* $\mathsf{pr}_1(\mathbf{R}) \times \mathsf{pr}_2(\mathbf{R}) = \mathbf{S}$*. Then* $\mathbf{R}$ *can be expressed as*

$$\mathbf{R} = \bigcup_{i=0}^{k} C_{(\frac{i}{k},y_i),\mathbf{S}}$$

where y_i *is the minimal element of* $\mathbf{PŁ}_{k'}$ *with* $(\frac{i}{k}, y_i) \in \mathbf{R}$ *(in particular,* $y_0 = 0$ *and* $y_k = 1$*).*

(2) Let $y_0, \dots, y_k$ *be an increasing sequence of elements of* $\mathbf{PŁ}_{k'}$ *with* $y_0 = 0, y_k = 1$ *and* $\frac{i}{k} \leq y_i$ *for all* $i = 1, \dots, k-1$*. Then*

$$\mathbf{R} = \bigcup_{i=0}^{k} C_{(\frac{i}{k},y_i),\mathbf{S}}$$

is a subalgebra of $\mathbf{S}$ *if and only if the conditions*

$$(\tfrac{i}{k}, y_i) \odot (\tfrac{j}{k}, y_j) \in \mathbf{R} \text{ and } (\tfrac{i}{k}, y_i) \oplus (\tfrac{j}{k}, y_j) \in \mathbf{R}$$

hold for all $i, j \in \{1, \dots, k-1\}$.

Proof. (1): By Lemma 5.1.12, we have $\bigcup_{i=0}^{k} C_{(\frac{i}{k},y_i),\mathbf{S}} \subseteq \mathbf{R}$. Conversely, if $(\frac{i}{k}, y)$ is in $\mathbf{R}$, then $y_i \leq y$ by minimality of y_i and therefore $(\frac{i}{k}, y) \in C_{(\frac{i}{k},y_i),\mathbf{S}}$.

(2): Clearly these conditions are necessary for $\mathbf{R}$ to be a subalgebra. We show that they are also sufficient. So, supposing these condition hold, we want to show that $\mathbf{R}$ is indeed a subalgebra. First note that $\lhd|_{\mathbf{S}} = C_{(0,0),\mathbf{S}} \cup C_{(1,1),\mathbf{S}} \subseteq \mathbf{R}$, in particular this implies that both constants $(0,0)$ and $(1,1)$ are contained in $\mathbf{R}$. Now let (x,y) and (x',y') be two elements of $\mathbf{R}$, say $(x,y) \in C_{(\frac{i}{k},y_i),\mathbf{S}}$ and $(x',y') \in C_{(\frac{j}{k},y_j),\mathbf{S}}$. Furthermore, without loss of generality we assume $i \leq j$.

We first establish the closure under the lattice operations. To show closure under meets, we note that

$$(x,y) \wedge (x',y') = \begin{cases} (x,y) & \text{if } x \leq x', y \leq y', \\ (x',y') & \text{if } x' \leq x, y' \leq y, \\ (x',y) & \text{if } x' \leq x, y \leq y', \\ (x,y') & \text{if } x \leq x', y' \leq y. \end{cases}$$

In the first two cases the meet is obviously still in $\mathbf{R}$. In the third case the two inequalities $x' \leq x \leq \frac{i}{k}$ and $y_i \leq y$ imply $(x', y) \in C_{(\frac{i}{k}, y_i), \mathbf{S}}$. In the fourth and final case the two inequalities $x \leq x' \leq \frac{j}{k}$ and $y_j \leq y'$ imply $(x, y') \in C_{(\frac{j}{k}, y_j), \mathbf{S}}$. Closure under joins is established analogously since

$$(x, y) \vee (x', y') = \begin{cases} (x, y) & \text{if } x' \leq x, y' \leq y, \\ (x', y') & \text{if } x \leq x', y \leq y', \\ (x', y) & \text{if } x \leq x', y' \leq y, \\ (x, y') & \text{if } x' \leq x, y \leq y'. \end{cases}$$

Note that in the third case we get $(x', y) \in C_{(\frac{j}{k}, y_j), \mathbf{S}}$ and in the fourth case we get $(x, y') \in C_{(\frac{i}{k}, y_i), \mathbf{S}}$.

Now let $* \in \{\odot, \oplus\}$, and note that $(x, y) \in C_{(\frac{i}{k}, y_i), \mathbf{S}}$ and $(x', y') \in C_{(\frac{j}{k}, y_j), \mathbf{S}}$ together with monotonicity of $*$ imply

$$x * x' \leq \tfrac{i}{k} * \tfrac{j}{k} \text{ and } y_i * y_j \leq y * y'.$$

However, by assumption we have $(\frac{i}{k} * \frac{j}{k}, y_i * y_j) \in \mathbf{R}$, say it is contained in $C_{(\frac{h}{k}, y_h), \mathbf{S}}$. Thus

$$x * x' \leq \tfrac{i}{k} * \tfrac{j}{k} \leq \tfrac{h}{k} \text{ and } y_h \leq y_i * y_j \leq y * y'$$

immediately implies that $(x, y) * (x', y')$ is also contained in $C_{(\frac{h}{k}, y_h), \mathbf{S}}$, finishing the proof. □

For example, in Figure 5.2, on the left hand side the union

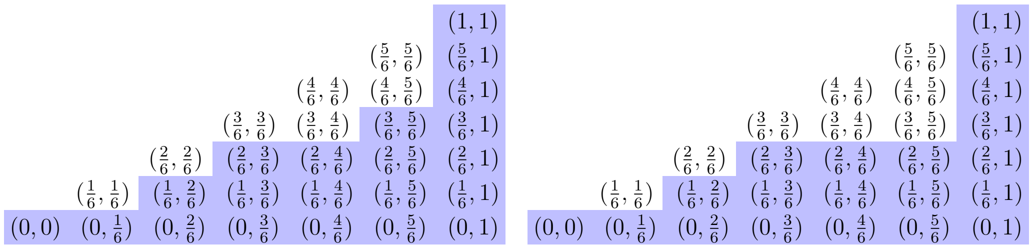
$$C_{(0,0)} \cup C_{(\frac{1}{6}, \frac{2}{6})} \cup C_{(\frac{2}{6}, \frac{3}{6})} \cup C_{(\frac{3}{6}, \frac{5}{6})} \cup C_{(\frac{4}{6}, 1)} \cup C_{(\frac{5}{6}, 1)} \cup C_{(1,1)}$$

inside $\mathbf{P Ł}_6 \times \mathbf{P Ł}_6$ is depicted. By Proposition 5.1.13, we can easily confirm that this is a subalgebra by checking that the ‘corner elements’ $(\frac{1}{6}, \frac{2}{6}), (\frac{2}{6}, \frac{3}{6})$ and $(\frac{3}{6}, \frac{5}{6})$ are closed under the operations $\odot$ and $\oplus$. On the right hand side of Figure 5.2, the union

$$C_{(0,0)} \cup C_{(\frac{1}{6}, \frac{2}{6})} \cup C_{(\frac{2}{6}, \frac{3}{6})} \cup C_{(\frac{3}{6}, 1)} \cup C_{(\frac{4}{6}, 1)} \cup C_{(\frac{5}{6}, 1)} \cup C_{(1,1)}$$

is depicted. This is not a subalgebra because $(\frac{1}{6}, \frac{2}{6}) \oplus (\frac{2}{6}, \frac{3}{6}) = (\frac{3}{6}, \frac{5}{6})$ is not contained in this union.

Now that we have a good grasp on the subalgebras of $\mathbf{P Ł}_n \times \mathbf{P Ł}_n$ thanks to Proposition 5.1.13, we aim to show that, ultimately, only subdirect products $\mathbf{R} \subseteq \mathbf{P Ł}_n \times \mathbf{P Ł}_n$ (meaning $\mathsf{pr}_1(\mathbf{R}) = \mathsf{pr}_2(\mathbf{R}) = \mathbf{P Ł}_n$) will be relevant for the natural duality. By Lemma 5.1.10, this is equivalent to saying that only the following relations will be relevant to the natural duality.



						$(1,1)$
					$(\frac{5}{6},\frac{5}{6})$	$(\frac{5}{6},1)$
				$(\frac{4}{6},\frac{4}{6})$	$(\frac{4}{6},\frac{5}{6})$	$(\frac{4}{6},1)$
			$(\frac{3}{6},\frac{3}{6})$	$(\frac{3}{6},\frac{4}{6})$	$(\frac{3}{6},\frac{5}{6})$	$(\frac{3}{6},1)$
		$(\frac{2}{6},\frac{2}{6})$	$(\frac{2}{6},\frac{3}{6})$	$(\frac{2}{6},\frac{4}{6})$	$(\frac{2}{6},\frac{5}{6})$	$(\frac{2}{6},1)$
	$(\frac{1}{6},\frac{1}{6})$	$(\frac{1}{6},\frac{2}{6})$	$(\frac{1}{6},\frac{3}{6})$	$(\frac{1}{6},\frac{4}{6})$	$(\frac{1}{6},\frac{5}{6})$	$(\frac{1}{6},1)$
$(0,0)$	$(0,\frac{1}{6})$	$(0,\frac{2}{6})$	$(0,\frac{3}{6})$	$(0,\frac{4}{6})$	$(0,\frac{5}{6})$	$(0,1)$

						$(1,1)$
					$(\frac{5}{6},\frac{5}{6})$	$(\frac{5}{6},1)$
				$(\frac{4}{6},\frac{4}{6})$	$(\frac{4}{6},\frac{5}{6})$	$(\frac{4}{6},1)$
			$(\frac{3}{6},\frac{3}{6})$	$(\frac{3}{6},\frac{4}{6})$	$(\frac{3}{6},\frac{5}{6})$	$(\frac{3}{6},1)$
		$(\frac{2}{6},\frac{2}{6})$	$(\frac{2}{6},\frac{3}{6})$	$(\frac{2}{6},\frac{4}{6})$	$(\frac{2}{6},\frac{5}{6})$	$(\frac{2}{6},1)$
	$(\frac{1}{6},\frac{1}{6})$	$(\frac{1}{6},\frac{2}{6})$	$(\frac{1}{6},\frac{3}{6})$	$(\frac{1}{6},\frac{4}{6})$	$(\frac{1}{6},\frac{5}{6})$	$(\frac{1}{6},1)$
$(0,0)$	$(0,\frac{1}{6})$	$(0,\frac{2}{6})$	$(0,\frac{3}{6})$	$(0,\frac{4}{6})$	$(0,\frac{5}{6})$	$(0,1)$

Figure 5.2: Only the subset on the left is a subalgebra of $\mathbf{PŁ}_6 \times \mathbf{PŁ}_6$.

Definition 5.1.14 (Set of relations $\mathcal{S}_n$)**.** We define $\mathcal{S}_n \subseteq \mathbb{S}(\mathbf{PŁ}_n \times \mathbf{PŁ}_n)$ to be the collection of all subalgebras $\mathbf{R} \subseteq \mathbf{PŁ}_n \times \mathbf{PŁ}_n$ which satisfy $\lhd \subseteq \mathbf{R} \subseteq \leq$.

It is clear by definition that $\mathcal{S}_n$ is a bounded sublattice of $\mathbb{S}(\mathbf{PŁ}_n \times \mathbf{PŁ}_n)$ with lower bound $\lhd$ and upper bound $\leq$.

In the next two (somewhat technical) lemmas, we show that the set of relations $\mathcal{S}_n$ strongly entails $\mathbb{S}(\leq)$ and $\mathbb{S}(\mathbf{PŁ}_n)$. The first lemma shows that relations $\mathbf{R} \in \mathbb{S}(\leq)$ with $\mathsf{pr}_1(\mathbf{R}) \times \mathsf{pr}_2(\mathbf{R}) \neq \mathbf{PŁ}_n \times \mathbf{PŁ}_n$ are strongly entailed by $\mathbb{S}(\mathbf{PŁ}_n)$ and $\mathcal{S}_n$.

Lemma 5.1.15. *Let $\mathbf{R} \subseteq \mathbf{PŁ}_n \times \mathbf{PŁ}_n$ be a subalgebra of the order $\leq$, which is not the diagonal of a subalgebra of $\mathbf{PŁ}_n$, and let $\mathbf{S} = \mathsf{pr}_1(\mathbf{R}) \times \mathsf{pr}_2(\mathbf{R}) = \mathbf{PŁ}_k \times \mathbf{PŁ}_{k'}$ for some divisors k, k' of n. Then there exists a subalgebra $\overline{\mathbf{R}} \in \mathcal{S}_n$ with $\mathbf{R} = \overline{\mathbf{R}} \cap \mathbf{S}$.*

Proof. By Proposition 5.1.13(1), we know that $\mathbf{R}$ can be expressed as union

$$\mathbf{R} = \bigcup_{i=0}^{k} C_{(\frac{i}{k}, y_i), \mathbf{S}}$$

where y_i is the minimal element of $\mathbf{PŁ}_{k'}$ with $(\frac{i}{k}, y_i) \in \mathbf{R}$. Let $n = k \cdot \ell$. We define $\overline{\mathbf{R}}$ by

$$\overline{\mathbf{R}} = \bigcup_{j=0}^{n} C_{(\frac{j}{n}, \hat{y}_j)}$$

where we stipulate $\hat{y}_0 = 0$ and

$$\hat{y}_j = \begin{cases} y_1 & \text{if } 1 \leq j \leq \ell, \\ y_2 & \text{if } \ell + 1 \leq j \leq 2\ell, \\ \vdots & \vdots \\ y_{k-1} & \text{if } (k-2)\ell + 1 \leq j \leq (k-1)\ell, \\ y_k = 1 & \text{if } (k-1)\ell + 1 \leq j \leq n. \end{cases}$$

We show that $\overline{\mathbf{R}}$ is a subalgebra of $\mathbf{PŁ}_n \times \mathbf{PŁ}_n$ using Proposition 5.1.13(2). That is, for any $j_1, j_2 \in \{1, \dots, n-1\}$, we want to show that $(\frac{j_1}{n}, \hat{y}_{j_1}) * (\frac{j_2}{n}, \hat{y}_{j_2}) \in \overline{\mathbf{R}}$ holds for the MV-operations $* \in \{\odot, \oplus\}$.

Let $i_1, i_2 \in \{1, \dots, k\}$ be the unique elements satisfying

$$(i_1 - 1)\ell < j_1 \leq i_1\ell \text{ and } (i_2 - 1)\ell < j_2 \leq i_2\ell,$$

which by definition means $\hat{y}_{j_1} = y_{i_1}$ and $\hat{y}_{j_2} = y_{i_2}$. Since $\mathbf{R}$ is a subalgebra, we know that $(\frac{i_1}{k}, y_{i_1}) * (\frac{i_2}{k}, y_{i_2}) \in \mathbf{R}$, say it is in $C_{(\frac{h}{k}, y_h), \mathbf{S}}$. Now because $\frac{j_1}{n} \leq \frac{i_1\ell}{n} = \frac{i_1}{k}$ and similarly for j_2, i_2, we have

$$\frac{j_1}{n} * \frac{j_2}{n} \leq \frac{i_1}{k} * \frac{i_2}{k} \leq \frac{h}{k} = \frac{h\ell}{n}$$

and furthermore

$$y_h \leq y_{i_1} * y_{i_2} = \hat{y}_{j_1} * \hat{y}_{j_2}.$$

Because $\hat{y}_{h\ell} = y_h$, this shows that $(\frac{j_1}{n}, \hat{y}_{j_1}) * (\frac{j_2}{n}, \hat{y}_{j_2}) \in C_{(\frac{h\ell}{n}, \hat{y}_{h\ell})} \subseteq \overline{\mathbf{R}}$, finishing the proof. □

Our second lemma shows that the collection $\mathbb{S}(\mathbf{PŁ}_n)$ itself is already strongly entailed by $\mathcal{S}_n$.

Lemma 5.1.16. *For every* $\mathbf{PŁ}_k \in \mathbb{S}(\mathbf{PŁ}_n)$*, there exists a* $\mathbf{R} \in \mathcal{S}_n$ *such that* $\mathbf{R} \cap \Delta_{\mathbf{PŁ}_n} = \Delta_{\mathbf{PŁ}_k}$ *(where* $\Delta_{\mathbf{A}}$ *denotes the diagonal of the corresponding algebra* $\mathbf{A}$*).*

Proof. Let $n = k \cdot \ell$ and $\mathbf{PŁ}_k$ be given as in Proposition 5.1.4. We define $\mathbf{R}$ by

$$\mathbf{R} = \bigcup_{i=0}^{n} C_{(\frac{i}{n}, y_i)}$$

where we stipulate $y_0 = 0$ and

$$y_i = \begin{cases} \frac{\ell}{n} & \text{if } 1 \leq j \leq \ell, \\ \frac{2\ell}{n} & \text{if } \ell + 1 \leq j \leq 2\ell, \\ \vdots & \vdots \\ \frac{(k-1)\ell}{n} & \text{if } (k-2)\ell + 1 \leq j \leq (k-1)\ell, \\ 1 & \text{if } (k-1)\ell + 1 \leq j \leq n. \end{cases}$$

By definition, it is clear that $\mathbf{R} \cap \Delta_{\mathbf{PŁ}_n} = \Delta_{\mathbf{PŁ}_k}$, so we only have to show that $\mathbf{R}$ is a subalgebra. For this, we again use Proposition 5.1.13(2). Let $i_1, i_2 \in \{1, \dots, n-1\}$ and let j_1, j_2 be the unique elements of $\{1, \dots, k\}$ with

$$(j_1 - 1)\ell < i_1 \leq j_1\ell \text{ and } (j_2 - 1)\ell < i_2 \leq j_2\ell,$$

which means that $y_{i_1} = \frac{j_1\ell}{n}$ and $y_{i_2} = \frac{j_2\ell}{n}$. Furthermore, let $\frac{j_1\ell}{n} * \frac{j_2\ell}{n} = \frac{h\ell}{n}$ (note that such an h exists because $\mathbf{P}\mathbf{Ł}_k$ is a subalgebra). Then

$$\frac{i_1}{n} * \frac{i_2}{n} \leq \frac{j_1\ell}{n} * \frac{j_2\ell}{n} = \frac{h\ell}{n}$$

implies

$$(\tfrac{i_1}{n}, y_{i_2}) * (\tfrac{i_2}{n}, y_{i_2}) \in C_{(\frac{h\ell}{n}, \frac{h\ell}{n})} \subseteq \mathbf{R},$$

which finishes the proof. $\square$

With the above two lemmas at hand, we are ready to state and easily prove the main theorem of this subsection.

Theorem 5.1.17 (Natural duality for PMV_n)**.** *Let $n \geq 1$. The discrete relational structure*

$$\widetilde{\mathbf{P}\mathbf{Ł}}_n = \big(\{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \mathcal{S}_n, \mathcal{T}_{\text{dis}}\big)$$

yields a strong duality for PMV_n.

Proof. By the discussion after Lemma 5.1.9, we know that the structure given in Equation (5.2), that is $(\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \mathbb{S}(\mathbf{P}\mathbf{Ł}_n) \cup \mathbb{S}(\leq), \mathcal{T}_{\text{dis}})$, yields a strong duality for PMV_n. By Lemma 5.1.15, we know that every $\mathbf{R} \in \mathbb{S}(\leq)$ is an intersection of (and thus strongly entailed by) a product of subalgebras of $\mathbf{P}\mathbf{Ł}_n$ and a relation from $\mathcal{S}_n$. By Lemma 5.1.16, subalgebras of $\mathbf{P}\mathbf{Ł}_n$ are strongly entailed by $\mathcal{S}_n$ as well. $\square$

In light of Proposition 5.1.13, it is fairly straightforward to find the lattice $\mathcal{S}_n$ in a systematic way. Indeed, in [Poi23, Appendix A] we provide an easy algorithm to compute this lattice. Also note that, to obtain an *optimal duality* (see [CD98, Chapters 8 and 9]), we could simplify the above structure further by only including meet-irreducible elements of $\mathcal{S}_n$ (this follows from [CD98, Theorem 9.2.6]). However, since it won't make a significant difference in this thesis, we keep working with the alter ego from Theorem 5.1.17.

Definition 5.1.18. For all $n \geq 1$, let $\mathcal{X}_n = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\widetilde{\mathbf{P}\mathbf{Ł}}_n)$ denote the topological quasi-variety generated by the structure from Theorem 5.1.17. Furthermore, let $\mathsf{D}_n \colon \mathsf{PMV}_n \to \mathcal{X}_n$ and $\mathsf{E}_n \colon \mathcal{X}_n \to \mathsf{PMV}_n$ be the hom-functors establishing the corresponding dual equivalence.

Note that the dualities between PMV_n and $\mathcal{X}_n$ can be seen as many-valued generalizations of Priestley duality, which is recovered in the case where $n = 1$.

In the following, we collect some consequences of Theorem 5.1.17 which can be immediately derived from the general theory of natural dualities.

Corollary 5.1.19 (Consequences of the duality)**.** *The categories* PMV_n *and* $\mathcal{X}_n$ *have the following properties.*

(1) $\mathbf{PŁ}_n$ *is injective in* PMV_n *and* $\widetilde{\mathbf{PŁ}}_n$ *is injective in* $\mathcal{X}_n$*.*

(2) The injectives in PMV_n *are exactly the Boolean powers* $\mathbf{PŁ}_n[\mathbf{B}]$*, where* $\mathbf{B}$ *is a non-trivial complete Boolean algebra.*

(3) PMV_n *has the amalgamation property.*

(4) A morphism $\varphi\colon \mathbf{X}_1 \to \mathbf{X}_2$ *in* $\mathcal{X}_n$ *is an embedding (a surjection) if and only if* $\mathsf{E}_n(\varphi)$ *is a surjection (an embedding). A homomorphism* $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ *in* PMV_n *is an embedding (a surjection) if and only if* $\mathsf{D}_n(h)$ *is a surjection (an embedding).*

(5) The congruence lattice of $\mathbf{A} \in \mathsf{PMV}_n$ *is dually isomorphic to the lattice of closed substructures of* $\mathsf{D}_n(\mathbf{A})$*.*

(6) Coproducts in $\mathcal{X}_n$ *are given by direct union (*i.e.*, the duality is* logarithmic*).*

Proof. The second part of statement (1) follows from the definition of strong duality, the first part follows from [CD98, Lemma 3.2.10] and the fact that $\widetilde{\mathbf{PŁ}}_n$ is a total structure. Statement (2) follows from [CD98, Theorem 5.5.15] because all relations $\mathbf{R} \in \mathcal{S}_n$ avoid binary products. Statement (3) follows from [CD98, Lemma 5.3.4]. Statement (4) follows from (1) and [CD98, Lemmas 3.2.6 and 3.2.8]. Statement (5) follows from [CD98, Theorem 3.2.1]. Lastly, statement (6) follows from [CD98, Theorem 6.3.3]. ☐

This corollary already demonstrates how useful the dualities developed in this subsection are. In the next section, we investigate the dualities further to derive more results about the varieties PMV_n.

5.2 Further exploration of the dualities

In this section, we delve deeper into various aspects of the natural dualities established in Subsection 5.1.3. The section is structured as follows. In Subsection 5.2.1, we give a more explicit description of $\mathcal{X}_2$, the dual category of PMV_2 generated by the three-element positive MV-chain $\mathbf{PŁ}_2$. In Subsection 5.2.2, we discuss an adjunction between DL and PMV_n given by distributive skeletons (Definition 5.2.3) and Priestley powers (Definition 5.2.5). In Subsection 5.2.3, we make use of these results to describe the algebraically and existentially closed members of PMV_n, respectively (Theorem 5.2.11).

Lastly, we axiomatize the discrete category corresponding to $\mathcal{X}_n$ under the additional assumption that n is prime in Subsection 5.2.4.

5.2.1 The dual category for the three-element chain.

Among Łukasiewicz finitely-valued logics, arguably the most popular (and historically the first one considered [Łuk20]) is the three-valued logic corresponding to the variety MV_2 generated by the three-element MV-chain $\mathbf{Ł}_2$. In this section, we focus on the variety PMV_2 generated by the positive three-element MV-chain $\mathbf{PŁ}_2$. More specifically, we provide an explicit description of the category $\mathcal{X}_2$ dual to PMV_2.

Theorem 5.2.1 (Axiomatization of $\mathcal{X}_2$). *A structured Stone space $\mathbf{X} = (X, \lhd^{\mathbf{X}}, \leq^{\mathbf{X}}, \mathcal{T})$ with binary relations $\lhd^{\mathbf{X}}$ and $\leq^{\mathbf{X}}$ closed in X^2 is a member of $\mathcal{X}_2$ if and only if it satisfies the following axioms.*

(a) $x \lhd^{\mathbf{X}} y \Rightarrow x \leq^{\mathbf{X}} y$.

(b) $(X, \leq^{\mathbf{X}}, \mathcal{T})$ *is a* Priestley space, *that is,* $\leq^{\mathbf{X}}$ *is a partial order and if* $x \not\leq^{\mathbf{X}} y$*, then there exists a clopen upset* U *containing* x *but not* y*.*

(c) If $x \ntriangleleft^{\mathbf{X}} y$ *but* $x \leq^{\mathbf{X}} y$*, then there exist a clopen upset* U *and a clopen downset* D *with the following properties*

- $x \notin D$ *and* $y \notin U$*,*
- *For all* $z, z' \in X$*, if* $z \lhd^{\mathbf{X}} z'$ *then* $z \in D$ *or* $z' \in U$*.*

Proof. First we show that every member $\mathbf{X} = (X, \lhd^{\mathbf{X}}, \leq^{\mathbf{X}}, \mathcal{T})$ of $\mathcal{X}_2$ satisfies conditions (a)-(c). The formula (a) is quasi-atomic and holds in $\mathbf{PŁ}_n$, therefore, by the Preservation Theorem [CD98, Theorem 1.4.3], it also holds for all members of $\mathcal{X}_2$.

To see condition (b), stating that $(X, \leq^{\mathbf{X}}, \mathcal{T})$ is a Priestley space, assume that $x \not\leq^{\mathbf{X}} y$. By the Separation Theorem [CD98, Theorem 1.4.4], there exists a $\mathcal{X}_2$-morphism $\varphi\colon \mathbf{X} \to \widetilde{\mathbf{PŁ}}_2$ with $\varphi(x) > \varphi(y)$. If $\varphi(x) = 1$, choose $U = \varphi^{-1}(\{1\})$ and if $\varphi(x) = \frac{1}{2}$, choose $U = \varphi^{-1}(\{\frac{1}{2}\}) \cup \varphi^{-1}(\{1\})$. In both cases, U is a clopen (because $\mathbf{PŁ}_2$ carries the discrete topology and φ is continuous) upset (because φ is order-preserving) which contains x but not y.

To see (c), assume $x \ntriangleleft^{\mathbf{X}} y$ but $x \leq^{\mathbf{X}} y$. Then, again by the Separation Theorem, there exists a morphism $\varphi\colon \mathbf{X} \to \widetilde{\mathbf{PŁ}}_2$ with $\varphi(x) \ntriangleleft \varphi(y)$ but $\varphi(x) \leq \varphi(y)$. Since $\lhd = \leq \setminus \{(\frac{1}{2}, \frac{1}{2})\}$, this implies $\varphi(x) = \varphi(y) = \frac{1}{2}$. The clopen upset $U = \varphi^{-1}(\{1\})$ and the clopen downset $D = \varphi^{-1}(\{0\})$ satisfy the two subconditions of (c), the first one since $\varphi(x) = \varphi(y) = \frac{1}{2}$ and the second one since $z \lhd \mathbf{X}z'$ and $\varphi(z) = \varphi(z') = \frac{1}{2}$ would yield a contradiction $\varphi(z) \not\lhd \varphi(z')$ to φ being a morphism.

For the converse, assuming that $\mathbf{X} = (X, \lhd^{\mathbf{X}}, \leq^{\mathbf{X}}, \mathcal{T})$ satisfies (a)-(c), we want to show that it is a member of $\mathcal{X}_2$. We apply the Separation Theorem again.

Suppose $x \not\leq^{\mathbf{X}} y$. Using that $(X, \leq^{\mathbf{X}}, \mathcal{T})$ is a Priestley space, we can find a clopen upset U which contains x but not y. We define a continuous map $\varphi\colon X \to \{0, \frac{1}{2}, 1\}$ by $\varphi(z) = 1$ if $z \in U$ and $f(z) = 0$ otherwise. This clearly is order-preserving, and it also preserves $\lhd$, because $\lhd$ is a subset of $\leq$ by (a) and, in $\widetilde{\mathbf{P\text{Ł}}}_2$ the relations $\lhd$ and $\leq$ coincide on the subset $\{0, 1\}$. Clearly this morphism satisfies $\varphi(x) \not\leq \varphi(y)$.

In particular, the above covers the case where $x \neq y$ and the case where $x \not\lhd^{\mathbf{X}} y$ and $x \not\leq^{\mathbf{X}} y$ hold. Now assume $x \not\lhd^{\mathbf{X}} y$ but $x \leq^{\mathbf{X}} y$. Take a clopen upset U and a clopen downset D as given in (c). Replacing U by the clopen upset $U' := U\backslash D$, the properties of (c) are still satisfied, since $z \lhd^{\mathbf{X}} z'$ and $z \notin D$ imply $z' \in U$, and $z' \in D$ would yield the contradiction $z \in D$, so $z' \in U'$. Let the continuous map $\varphi\colon X \to \{0, \frac{1}{2}, 1\}$ be defined via

$$\varphi(z) = \begin{cases} 0 & \text{if } z \in D, \\ 1 & \text{if } z \in U', \\ \frac{1}{2} & \text{if } z \in X\backslash(D \cup U'). \end{cases}$$

This is a well-defined continuous map since D, U' and $X\backslash(D \cup C)$ form a clopen partition of X. Furthermore, since $x \notin D$ (which implies $y \notin D$) and $y \notin U$ (which implies $x \notin U$) implies $\varphi(x) = \varphi(y) = \frac{1}{2}$ (*i.e.*, $\varphi(x) \not\lhd \varphi(y)$ as desired), it remains to be shown that φ preserves $\leq$ and $\lhd$. Order-preservation follows immediately from the fact that U is an upset and D is a downset. Now suppose $z \lhd^{\mathbf{X}} z'$. Then $z \in D$, which implies $\varphi(z) = 0$ holds, or $z' \in U'$, which implies $\varphi(z') = 1$ holds. In both cases, $\varphi(z) \lhd \varphi(z')$ is assured. □

In the next subsection, we give a similar but more 'implicit' axiomatization of the categories $\mathcal{X}_n$ for $n > 2$ as well. Since (as we've already seen in the case $n = 2$) all structures $\mathbf{X} \in \mathcal{X}_n$ have underlying Priestley spaces, we then proceed to explore various functors relating our natural dualities to Priestley duality.

5.2.2 The relationship to Priestley duality

We continue to denote the functors establishing the duality from Theorem 5.1.17 by $\mathsf{D}_n\colon \mathsf{PMV}_n \to \mathcal{X}_n$ and $\mathsf{E}_n\colon \mathcal{X}_n \to \mathsf{PMV}_n$ (Definition 5.1.18).

In particular, for $n = 1$ this coincides with *Priestley duality* between the variety of *bounded distributive lattices* $\mathsf{DL} = \mathsf{PMV}_1$ and the category of Priestley spaces $\mathsf{Priest} = \mathcal{X}_1$. In this case, we simply use $\mathsf{D}\colon \mathsf{DL} \to \mathsf{Priest}$ and $\mathsf{E}\colon \mathsf{Priest} \to \mathsf{DL}$ instead of D_1 and E_1.

Similarly to what we described in Section 2.2 (and slightly abusing notation), in this subsection, we show that there are functors $\mathfrak{S}\colon \mathsf{PMV}_n \to \mathsf{DL}$ taking the *distributive skeleton* and $\mathfrak{P}\colon \mathsf{DL} \to \mathsf{PMV}_n$ taking a *Priestley power* with $\mathfrak{S}$ being left-adjoint to $\mathfrak{P}$. Later on in this chapter, the main property of the distributive skeleton becomes quite useful, for example to prove an algebraic completeness theorem for positive modal logic over $\mathbf{PŁ}_n$ (see Subsection 5.3.3).

While, in theory, the Separation Theorem [CD98, Theorem 1.4.3] always gives an 'implicit' description of a topological quasi-variety generated by an alter-ego, the reader can imagine that for $n > 2$, it gets increasingly complicated to come up with more 'explicit' descriptions of the categories $\mathcal{X}_n$ like the one in Theorem 5.2.1. Therefore, in these cases we content ourselves with the following.

Proposition 5.2.2. *A structured Stone space* $\mathbf{X} = (X, (\mathbf{R}^{\mathbf{X}} \mid \mathbf{R} \in \mathcal{S}_n), \mathcal{T})$ *with closed binary relations* $\mathbf{R}^{\mathbf{X}}$ *is a member of* $\mathcal{X}_n$ *if and only if it satisfies the following:*

(a) $x\mathbf{R}_1^{\mathbf{X}}y \Rightarrow x\mathbf{R}_2^{\mathbf{X}}y$ *for all* $\mathbf{R}_1 \subseteq \mathbf{R}_2$ *in* $\mathcal{S}_n$.

(b) $(X, \leq^{\mathbf{X}}, \mathcal{T})$ *is a Priestley space.*

(c) For all $\mathbf{R} \in \mathcal{S}_n \backslash \{\leq\}$, *if* $(x, y) \notin \mathbf{R}^{\mathbf{X}}$, *then there is a structure-preserving continuous map* $\varphi\colon \mathbf{X} \to \widetilde{\mathbf{PŁ}}_n$ *with* $\big(\varphi(x), \varphi(y)\big) \notin \mathbf{R}$.

Proof. Every member $\mathbf{X}$ of $\mathcal{X}_n$ satisfies the quasi-atomic formulas from (a). Furthermore, both (b) and (c) are immediate consequences of the Separation Theorem. To see that $(X, \leq^{\mathbf{X}}, \mathcal{T})$ is a Priestley space, assume $x \not\leq^{\mathbf{X}} y$. By the Separation Theorem there is a morphism $\varphi\colon \mathbf{X} \to \widetilde{\mathbf{PŁ}}_n$ with $\varphi(x) \not\leq \varphi(y)$. Let $\varphi(x) = \frac{i}{n}$. Then $U = \varphi^{-1}(\{\frac{1}{n}\}) \cup \varphi^{-1}(\{\frac{i+1}{n}\}) \cup \cdots \cup \varphi^{-1}(\{\frac{n-1}{n}\}) \cup \varphi^{-1}(\{1\})$ is a clopen upset which contains x but not y. The converse is also a straightforward application of the Separation Theorem. □

Therefore, for every n there is a well-defined forgetful functor $\mathsf{U}\colon \mathcal{X}_n \to \mathsf{Priest}$ sending an object of $\mathcal{X}_n$ to its underlying Priestley space and a $\mathcal{X}_n$-morphism to itself. In the following, we show that the dual of U is given by the *distributive skeleton functor* $\mathfrak{S}\colon \mathsf{PMV}_n \to \mathsf{DL}$. It is defined very similarly to the *Boolean skeleton functor* $\mathsf{MV}_n \to \mathsf{BA}$ from Subsection 2.2.2. The distributive skeleton of a PMV_n algebra is defined completely analogous to the Boolean skeleton of an MV_n algebra (see, *e.g.*, [CDM00, Section 1.5].

Definition 5.2.3 (Distributive Skeleton)**.** Let $\mathbf{A} \in \mathsf{PMV}_n$ be a positive MV_n-algebra. The *distributive skeleton* of $\mathbf{A}$ is the bounded distributive lattice

$$\mathfrak{S}(\mathbf{A}) = \langle \mathfrak{S}(A), \wedge, \vee, 0, 1 \rangle$$

defined on the carrier set $\mathfrak{S}(A) = \{a \in A \mid a \oplus a = a\}$, with the operations $\wedge, \vee$ and constants $0, 1$ inherited from $\mathbf{A}$.

To turn this into a functor $\mathfrak{S}\colon \mathsf{PMV}_n \to \mathsf{DL}$, for a homomorphism $h\colon \mathbf{A} \to \mathbf{A}'$ between PMV_n-algebras, similarly to Subsection 2.2.2, let $\mathfrak{S}h\colon \mathfrak{S}(\mathbf{A}) \to \mathfrak{S}(\mathbf{A}')$ be the homomorphism defined via restriction $\mathfrak{S}h = h|_{\mathfrak{S}(\mathbf{A})}$. The functor thus arising is called the *distributive skeleton functor*.

Theorem 5.2.4 ($\mathfrak{S}$ is dual to U)**.** *The distributive skeleton functor $\mathfrak{S}$ is dual to the forgetful functor $\mathsf{U}\colon \mathcal{X}_n \to \mathsf{Priest}$, that is, $\mathsf{D}\mathfrak{S}$ is naturally isomorphic to $\mathsf{U}\mathsf{D}_n$.*

Proof. By definition, natural in the choice of $\mathbf{A} \in \mathsf{PMV}_n$, we want to find an order-preserving homeomorphism

$$\Phi_{\mathbf{A}}\colon \big(\mathsf{PMV}_n(\mathbf{A}, \mathbf{P}\mathbf{Ł}_n), \leq\big) \to \big(\mathsf{DL}(\mathfrak{S}(\mathbf{A}), \mathbf{2}), \leq\big),$$

where $\mathbf{2}$ denotes the two-element distributive lattice. We claim that

$$\Phi_{\mathbf{A}}(u) = u|_{\mathfrak{S}(\mathbf{A})}$$

has the desired properties.

To see that $\Phi_{\mathbf{A}}$ is injective, suppose that $u \neq u'$ are two distinct homomorphisms $\mathbf{A} \to \mathbf{P}\mathbf{Ł}_n$. Let $a \in \mathbf{A}$ be such that $u(a) \neq u'(a)$, without loss of generality say $u(a) < u'(a)$. Then, for $d = u'(a)$, we have $u(\tau_d(a)) = \tau_d(u(a)) = 0$ and $u'(\tau_d(a)) = \tau_d(u'(a)) = 1$. Since $\tau_d(a) \in \mathfrak{S}(\mathbf{A})$ holds, this shows that $\Phi(u) \neq \Phi(u')$.

Now we show that Φ is surjective. Let $p\colon \mathfrak{S}(\mathbf{A}) \to \mathbf{2}$ be a homomorphism. We construct a homomorphism $u_p\colon \mathbf{A} \to \mathbf{P}\mathbf{Ł}_n$ with $\Phi_{\mathbf{A}}(u_p) = p$. Given $a \in \mathbf{A}$, define

$$u_p(a) = \bigvee \{d \mid p\big(\tau_d(a)\big) = 1\}.$$

Clearly u_p preserves 0 and 1. Now let $a_1, a_2 \in \mathbf{A}$, let $u_p(a_1) = d_1$ and $u_p(a_2) = d_2$. We want to show that, for $* \in \{\wedge, \vee, \odot, \oplus\}$, $u_p(a_1 * a_2) = d_1 * d_2$. In other words, we want to show that $p(\tau_{d_1 * d_2}(a_1 * a_2)) = 1$ and $p(\tau_{d'}(a_1 * a_2)) = 0$ for all $d' > d_1 * d_2$. Since $*$ is order-preserving we know that $\mathbf{P}\mathbf{Ł}_n$ satisfies

$$\tau_{d_1}(x_1) \wedge \tau_{d_2}(x_2) \leq \tau_{d_1 * d_2}(x_1 * x_2).$$

Since this can be expressed as an equation, it also holds in $\mathbf{A}$. Therefore, we get

$$1 = p\big(\tau_{d_1}(a_1) * \tau_{d_2}(a_2)\big) \leq p\big(\tau_{d_1 * d_2}(a_1 * a_2)\big).$$

Now let $d' > d_1 * d_2$. Then, since $d_1 * d_2 \neq 1$, we can choose minimal $d_1' > d_1$ and $d_2' \geq d_2$ with $d_1' * d_2' \geq d'$. By minimality, $\mathbf{P\L}_n$ satisfies the equation corresponding to

$$\tau_{d_1}(x_1) \wedge \tau_{d_2}(x_2) \wedge \tau_{d'}(x_1 * x_2) \leq \tau_{d_1'}(x_1),$$

which is therefore also satisfied in $\mathbf{A}$. But now, if we assume that $p(\tau_{d'}(a_1 * a_2)) = 1$, then

$$1 = p(\tau_{d_1}(a_1) \wedge \tau_{d_2}(a_2) \wedge \tau_{d'}(a_1 * a_2)) \leq p(\tau_{d_1'}(a_1))$$

implies $p(\tau_{d_1'}(a_1)) = 1$, which is a contradiction to $u_p(a_1) = d_1$. Therefore, u_p is a homomorphism. The restriction of u_p to $\mathfrak{S}(\mathbf{A})$ is equal to p because $a \in \mathfrak{S}(\mathbf{A})$ is equivalent to $\tau_d(a) = a$ for all $d \in \mathbf{P\L}_n \backslash \{0\}$.

Thus we showed that $\Phi_{\mathbf{A}}$ is bijective. It is also continuous (and therefore a homeomorphism) since a subbasis of the topology on $\mathsf{D}_1\mathfrak{S}(\mathbf{A})$ is given by the sets of the form $[a : e] = \{p\colon \mathfrak{S}(\mathbf{A}) \to \mathbf{2} \mid p(a) = e\}$ where a ranges over $\mathfrak{S}(\mathbf{A})$ and e ranges over $\mathbf{2}$. The preimage $\Phi^{-1}([a : e])$ is exactly the corresponding subbase element $[a : e] = \{h\colon \mathbf{A} \to \mathbf{P\L}_n \mid h(a) = e\}$ of the topology on $\mathsf{UD}_n(\mathbf{A})$. The fact that $\Phi_{\mathbf{A}}$ is order-preserving follows directly from its definition, so it only remains to show that Φ defines a natural transformation $\mathsf{UD}_n \Rightarrow \mathsf{D}_1\mathfrak{S}$. Let $h\colon \mathbf{A} \to \mathbf{A}'$ be a homomorphism. We need to show that the square

$$\begin{tikzcd}
\mathsf{PMV}_n(\mathbf{A}', \mathbf{P\L}_n) \arrow[r, "\Phi_{\mathbf{A}'}"] \arrow[d, "\mathsf{UD}_n h"'] & \mathsf{DL}(\mathfrak{S}(\mathbf{A}'), \mathbf{2}) \arrow[d, "\mathsf{D}_1\mathfrak{S}h"] \\
\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n) \arrow[r, "\Phi_{\mathbf{A}}"'] & \mathsf{DL}(\mathfrak{S}(\mathbf{A}'), \mathbf{2})
\end{tikzcd}$$

commutes. By definition, for a homomorphism $u\colon \mathbf{A}' \to \mathbf{P\L}_n$ we have

$$\Phi_{\mathbf{A}} \circ \mathsf{UD}_n h(u) = \Phi_{\mathbf{A}}(u \circ h) = (u \circ h)|_{\mathfrak{S}(\mathbf{A})}$$

and

$$\mathsf{D}_1\mathfrak{S}h \circ \Phi_{\mathbf{A}'}(u) = \mathsf{D}_1\mathfrak{S}h(u|_{\mathfrak{S}}\mathbf{A}') = u|_{\mathfrak{S}(\mathbf{A}')} \circ h|_{\mathfrak{S}(\mathbf{A})}$$

which makes it easy to see that these two coincide, finishing the proof. □

In the following we show that, similarly to what we showed in Subsection 2.2.3, the distributive skeleton functor has a right-adjoint which takes *Priestley powers* defined as follows.

Definition 5.2.5 (Priestley power)**.** Let $\mathbf{L} \in \mathsf{DL}$ be a distributive lattice and let $\mathbf{M}$ be a finite ordered algebra. The *Priestley power*, $\mathbf{M}[\mathbf{L}]$, is given by the collection

$$\mathbf{M}[\mathbf{L}] = \mathsf{Priest}\big(\mathsf{D}(\mathbf{L}), (M, \leq, \mathcal{T}_{\text{dis}})\big)$$

of continuous order-preserving maps from the dual of $\mathbf{L}$ to the discrete Priestley space $(M, \leq, \mathcal{T}_{\text{dis}})$.

A more constructive defintion of Priestley powers, similarly to the one of Boolean powers in Definition 2.2.6, is given and shown to be equivalent to the above definition in [Len86] (where they are called *distributive extensions*). We also emphasize that our notion of Priestley power differs from the one introduced in [Jip09].

Similarly to the Boolean power (but with the constraint that all operations of $\mathbf{M}$ need to be order-preserving), we get the following.

Lemma 5.2.6. *Let* $\mathbf{M}$ *be a finite ordered algebra, all of whose operations are order-preserving. Then, for every distributive lattice* $\mathbf{L} \in \mathsf{DL}$*, the Priestley power* $\mathbf{M}[\mathbf{L}]$ *with component-wise operations is a subalgebra of* $\mathbf{M}^{\mathsf{D}(\mathbf{L})}$*.*

Proof. Let f be an n-ary operation of $\mathbf{M}$ and let $\alpha_1, \dots, \alpha_n \in \mathbf{M}[\mathbf{L}]$. We need to show that $\alpha \colon \mathsf{D}(\mathbf{L}) \to M$ defined by $\alpha(x) = f(\alpha_1(x), \dots, \alpha_n(x))$ is continuous and order-preserving. Order-preservation is easy, since if $x \leq y$ we know that $\alpha_i(x) \leq \alpha_i(y)$ for all i and since f is order-preserving we have

$$\alpha(x) = f\big(\alpha_1(x), \dots, \alpha_n(x)\big) \leq f\big(\alpha_1(y), \dots, \alpha_n(y)\big) \leq \alpha(y).$$

To see that α is continuous, we show that $\alpha^{-1}(\{m\})$ is clopen for every $m \in M$. Let $N \subseteq M^n$ be the finite set of all tuples $(m_1, \dots, m_n)$ with $f(m_1, \dots, m_n) = m$. Then we have

$$\alpha^{-1}(\{m\}) = \bigcup_{(m_1,\dots,m_n) \in N} \alpha_1^{-1}(\{m_1\}) \cap \dots \cap \alpha_n^{-1}(\{m_n\}),$$

which is clopen because N is finite and all α_i are continuous. $\square$

Therefore, it is easily seen that the following *Priestly power functor* similar to the Boolean power functor from Definition 2.2.7 is well-defined.

Definition 5.2.7. We define the *Priestley power functor* $\mathfrak{P}\colon \mathsf{DL} \to \mathsf{PMV}_n$ as follows. For a distributive lattice $\mathbf{L} \in \mathsf{DL}$, let $\mathfrak{P}(\mathbf{L}) = \mathbf{P\L}_n[\mathbf{L}]$ be the Priestley power, and for a homomorphism $h\colon \mathbf{L}_1 \to \mathbf{L}_2$ let $\mathfrak{P}h\colon \mathfrak{P}(\mathbf{L}_1) \to \mathfrak{P}(\mathbf{L}_2)$ be defined by $\alpha \mapsto \alpha \circ \mathsf{D}h$.

We now show by duality that this functor is right-adjoint to the distributive skeleton functor from Definition 5.2.3.

Theorem 5.2.8. *The Priestley power functor* $\mathfrak{P}\colon \mathsf{DL} \to \mathsf{PMV}_n$ *is right-adjoint to the distributive skeleton functor* $\mathfrak{S}\colon \mathsf{PMV}_n \to \mathsf{DL}$.

Proof. Our proof strategy consists of the following two steps. We first define a functor $\mathsf{P}\colon \mathsf{Priest} \to \mathcal{X}_n$ and show that it is left-adjoint to the forgetful functor U_n. Then we show that $\mathfrak{P}$ is the dual of P. By Theorem 5.2.4 and the uniqueness of an adjoint up to natural isomorphism, the theorem follows.

$$\begin{tikzcd}
\mathcal{X}_n \arrow[r, bend left, "\mathsf{E}_n"] \arrow[d, bend left, "\mathsf{U}"] & \mathsf{PMV}_n \arrow[l, bend left, "\mathsf{D}_n"] \arrow[d, bend left, "\mathfrak{S}"] \\
\mathsf{Priest} \arrow[u, bend left, "\mathsf{P}"] \arrow[u, phantom, "\dashv"] \arrow[r, bend left, "\mathsf{E}"] & \mathsf{DL} \arrow[l, bend left, "\mathsf{D}"] \arrow[u, bend left, "\mathfrak{P}"] \arrow[u, phantom, "\vdash"]
\end{tikzcd}$$

Let $\mathsf{P}\colon \mathsf{Priest} \to \mathcal{X}_n$ be defined as follows. For a Priestley space $(X, \leq)$, define $\mathsf{P}(X, \leq)$ to be the structured topological space $(X, \leq, (\mathbf{R}^{\mathbf{X}} = \emptyset \mid \mathbf{R} \in \mathcal{S}_n \backslash \{\leq\}))$, which is a well-defined member of $\mathcal{X}_n$ by Proposition 5.2.2. Furthermore, define $\mathsf{P}\varphi = \varphi$ on morphisms. It is easy to see that P is left-adjoint to U, since for every Priestley space $(X, \leq)$ and structure $\mathbf{Y} \in \mathcal{X}_n$, by definition of P, morphisms in $\mathcal{X}_n(\mathsf{P}(X, \leq), \mathbf{Y})$ clearly coincide with continuous order-preserving maps $X \to Y$, that is, morphisms in $\mathsf{Priest}((X, \leq), \mathsf{U}(\mathsf{Y}))$.

We now show that P is dual to $\mathfrak{P}$, more specifically, we show that there is a natural isomorphism $\mathsf{E}_n\mathsf{P} \cong \mathfrak{P}\mathsf{E}$. For this we simply note that, for a Priestley space $(X, \leq)$, we have the following natural isomorphisms

$$\begin{aligned} \mathsf{E}_n\mathsf{P}(X, \leq) = \mathcal{X}_n\big(\mathsf{P}(X, \leq), \widetilde{\mathbf{PL}}_n\big) &\cong \\ \cong \mathsf{Priest}\big((X, \leq), \mathsf{U}(\widetilde{\mathbf{P\L}}_n)\big) &\cong \\ \cong \mathsf{Priest}\big(\mathsf{DE}(X, \leq), \mathsf{U}(\widetilde{\mathbf{P\L}}_n)\big) &= \mathfrak{P}\mathsf{E}(X, \leq), \end{aligned}$$

where we used $\mathsf{P} \dashv \mathsf{U}$ established above and the definition of the Priestley power $\mathfrak{P}(\mathbf{L}) = \mathsf{Priest}(\mathsf{D}(\mathbf{L}), \mathsf{U}(\widetilde{\mathbf{P\L}}_n))$. This finishes the proof. $\square$

One simple consequence of (the proof of) this theorem is the following.

Corollary 5.2.9. *Every algebra* $\mathbf{A} \in \mathsf{PMV}_n$ *is a subalgebra of a Priestley power. More specifically, there is an embedding* $\mathbf{A} \hookrightarrow \mathfrak{P}\mathfrak{S}(\mathbf{A})$.

Proof. Let P be the dual of $\mathfrak{P}$ as in the proof of Theorem 5.2.8. It is easy to see that the counit of the adjunction $\mathsf{P} \dashv \mathsf{U}$ is the identity map id_x as a morphism $\mathsf{PU}(\mathbf{X}) \to \mathbf{X}$ on every component. Therefore, it is a component-wise epimorphism in $\mathcal{X}_n$. Dually, this implies that the unit of the adjunction $\mathfrak{S} \dashv \mathfrak{P}$ is a component-wise monomorphism, and therefore yields an embedding $\mathbf{A} \hookrightarrow \mathfrak{P}\mathfrak{S}(\mathbf{A})$ for every PMV_n-algebra $\mathbf{A} \in \mathsf{PMV}_n$ as desired. ☐

In the next subsection, we describe the algebraically and existentially closed members of PMV_n via their duals. For this, Boolean powers (rather than Priestley powers) play an essential role. However, since Boolean powers arise as special cases of Priestley powers, the results of this subsection still prove useful towards this end.

5.2.3 Algebraically and existentially closed algebras

A standard application of natural dualities is the classification of algebraically closed and existentially closed algebras via their duals (see, *e.g.*, [CD98, Sections 5.3 and 5.4]). In this subsection, we give full classifications of the algebraically closed and existentially closed members of PMV_n via Boolean powers. Note that, for a *complemented* bounded distributive lattice $\mathbf{B}$, the Priestley power $\mathbf{P\L}_n[\mathbf{B}]$ from Definition 5.2.5 coincides with the usual Boolean power $\mathbf{P\L}_n[\mathbf{B}]$. Since the structure $\widetilde{\mathbf{P\L}}_n$ is total, we can use the *AC-EC Theorem* [CD98, Theorem 5.3.5] to characterize algebraically and existentially closed members of PMV_n.

Before we state this theorem, we recall that $\mathbf{X} \in \mathcal{X}_n$ has the *dual finite homomorphism property* (FHP)* if, for all finite $\mathbf{Y}, \mathbf{Z} \in \mathcal{X}_n$ and surjective morphisms $\varphi\colon \mathbf{X} \to \mathbf{Z}$, $\psi\colon \mathbf{Y} \to \mathbf{Z}$, there exists a morphism $\lambda\colon \mathbf{X} \to \mathbf{Y}$ such that $\varphi = \psi \circ \lambda$.

$$\begin{tikzcd}
\mathbf{X} \arrow[r, two heads, "\varphi"] \arrow[dr, dashed, "\exists\lambda"'] & \mathbf{Z} \\
& \mathbf{Y} \arrow[u, two heads, "\psi"']
\end{tikzcd} \tag*{(FHP)*}$$

The *dual finite embedding property* (FEP)* is similar, the only difference being that λ is also required to be surjective.

Theorem 5.2.10 (AC-EC-Theorem [CD98, Theorem 5.3.5])**.** *For every algebra* $\mathbf{A} \in \mathsf{PMV}_n$*, the following hold.*

(1) $\mathbf{A}$ *is algebraically closed if and only if* $\mathsf{D}_n(\mathbf{A})$ *has the dual finite homomorphism property* (FHP)*.

(2) **A** *is existentially closed if and only if* $\mathsf{D}_n(\mathbf{A})$ *has the finite embedding property* (FEP)*.

We now show that algebraically and existentially closed members of PMV_n stem from Boolean algebras in the following sense.

Theorem 5.2.11 (Algebraically & existentially closed PMV_n-algebras)**.** *For every algebra* $\mathbf{A} \in \mathsf{PMV}_n$ *the following hold.*

(1) **A** *is algebraically closed if and only if* **A** *is isomorphic to a Boolean power* $\mathbf{PŁ}_n[\mathbf{B}]$*, where* $\mathbf{B} \in \mathsf{BA}$ *is an arbitrary Boolean algebra.*

(2) **A** *is existentially closed if and only if* **A** *is isomorphic to a Boolean power* $\mathbf{PŁ}_n[\mathbf{B}]$*, where* $\mathbf{B} \in \mathsf{BA}$ *is an atomless Boolean algebra.*

Proof. By (the proof of) Theorem 5.2.8, we know that the duals of Boolean powers $\mathbf{PŁ}_n[\mathbf{B}]$ in $\mathcal{X}_n$ are exactly the structures isomorphic to some $\mathbf{X} \in \mathcal{X}_n$ where $\leq^{\mathbf{X}}$ is the discrete order and $\mathbf{R}^{\mathbf{X}}$ is empty for all other $\mathbf{R} \in \mathcal{S}_n$. We first show by contrapositive that if **X** has the finite homomorphism property, then it needs to be of this form.

Let $\mathbf{X} \in \mathcal{X}_n$ not be of the form described above. If $\leq$ is not discrete, there are distinct $x, y \in \mathbf{X}$ with $x < y$. Let U be an upset of X containing y but not x. Define $\mathbf{Z} \in \mathcal{X}_n$ to consist of two points $\{a, b\}$ with order $a < b$ (and all other relations empty), and let **Y** consist of two points $\{a', b'\}$ with the discrete order (and all other relations empty). Let $\varphi\colon \mathbf{X} \to \mathbf{Z}$ be the morphism sending U to b and $X \backslash U$ to a. Let $\psi\colon \mathbf{Y} \to \mathbf{Z}$ be the morphism sending a' to a and b' to b. Now if there was a morphism λ witnessing (FHP)*, it would have to satisfy $\lambda(x) = a'$ and $\lambda(y) = b'$. However, this is impossible since this would mean $x \leq y$ and $\lambda(x) \not\leq \lambda(y)$, contradicting that λ needs to be order-preserving.

Now assume that **X** has the discrete order-relation and there is some other relation $\mathbf{R}^{\mathbf{X}}$ which is non-empty. Choose **R** minimal in $\mathcal{S}_n$ such that there is some $x \in \mathbf{X}$ with $x\mathbf{R}^{\mathbf{X}}x$. Define $\mathbf{Z} \in \mathcal{X}_n$ to consist of one point $\{a\}$ with $a\mathbf{R}^{\mathbf{X}}a$ and let **Y** consist of one point $\{a'\}$ with $a' \leq^{\mathbf{Y}} a'$ and all other $\mathbf{R}^{\mathbf{Y}}$ empty. Let $\varphi\colon \mathbf{X} \to \mathbf{Z}$ and $\psi\colon \mathbf{Y} \to \mathbf{Z}$ be the unique morphisms. The unique map $\lambda\colon X \to Z$ is not a morphism because otherwise $x\mathbf{R}^{\mathbf{X}}x$ would imply $a'\mathbf{R}a'$. Therefore **X** does not satisfy (FHP)*.

Thus we showed that if **A** is algebraically closed, then it is isomorphic to some Boolean power $\mathbf{PŁ}_n[\mathbf{B}]$. For the converse of (1), one has to show that every $\mathbf{X} \in \mathcal{X}_n$ with discrete order and all other relations empty has the finite homomorphism property. For (2), one has to show such an **X** has the finite embedding property if and only if $\leq$ has no isolated points. However, this is easy, since both of these can be proven completely analogous to [CD98, Theorem 5.4.1]. $\square$

In particular, for $n = 1$ we recover the known description of algebraically closed and existentially closed distributive lattices as complemented distributive lattices and atomless complemented distributive lattices, respectively [Sch79].

5.2.4 The discrete dual category

At the beginning of Subsection 5.2.2, we noted that it seems difficult to come up with 'explicit' descriptions of $\mathcal{X}_n$ with increasing n. In this subsection, we show that, at least for n prime, this difficulty is due to the *topological aspects* of the duality. More specifically, we give a fairly simple explicit characterization of the *discrete* dual category $\mathcal{X}_n^{\mathrm{dis}}$ corresponding to $\mathcal{X}_n$ in this case.

Discrete natural dualities (or rather, natural dualities where the *algebras* carry the additional topological structure) have been extensively studied, for example, in [Dav06, DHP12]. For our purposes, an important result is [Dav06, Lemma 3.4]. Roughly speaking, it states that a collection of quasi-atomic formulas fully characterises the discrete dual category of $\mathcal{X}$ whenever it fully characterises the category of finite members of $\mathcal{X}$. In our case, this collection of quasi-atomic formula is the following.

Definition 5.2.12 (The category $\mathcal{X}_n^{\mathrm{dis}}$)**.** For n prime, we define a category $\mathcal{X}_n^{\mathrm{dis}}$ as follows. The objects are (exclusively binary) relational structures $(X, (\mathbf{R}^X \mid \mathbf{R} \in \mathcal{S}_n))$, satisfying the following conditions.

(a) $(X, \leq^X)$ is a poset.

(b) $x\mathbf{R}_1^X y \Rightarrow x\mathbf{R}_2^X y$ for all $\mathbf{R}_1 \subseteq \mathbf{R}_2$ in $\mathcal{S}_n$.

(c) $x {\leq^X} x' \mathbf{R}^X y' {\leq^X} y \Rightarrow x\mathbf{R}^X y$ for all $\mathbf{R} \in \mathcal{S}_n$.

(d) $x\mathbf{R}^X x \Rightarrow x \lhd^X x$ for all $\mathbf{R} \in \mathcal{S}_n \backslash \{\leq\}$.

The main result of this subsection is that $\mathcal{X}_n^{\mathrm{dis}}$ is indeed the discrete category corresponding to $\mathcal{X}_n$.

Theorem 5.2.13. *Let n be a prime number. Then $\mathcal{X}_n^{\mathrm{dis}}$ is the discrete counterpart of $\mathcal{X}_n$.*

Proof. Using [Dav06, Lemma 3.4], we only have to show that the finite members of $\mathcal{X}_n^{\mathrm{dis}}$ coincide with the finite members of $\mathcal{X}_n$. All finite members of $\mathcal{X}_n$ are in $\mathcal{X}_n^{\mathrm{dis}}$ because (a)-(d) are quasi-atomic formulas satisfied by $\widetilde{\mathbf{PŁ}}_n$ (in particular, (d) is satisfied because $\mathbf{PŁ}_n$ only has $\mathbf{PŁ}_1$ as subalgebra).

Conversely, by Proposition 5.2.2, we only need to show that for $(X, (\mathbf{R}^X \mid \mathbf{R} \in \mathcal{S}_n))$ finite, whenever $(x_1, x_2) \notin \mathbf{R}^X$ there is a structure-preserving map $\varphi\colon X \to \widetilde{\mathbf{P\L}}_n$ with $(\varphi(x_1), \varphi(x_2)) \notin \mathbf{R}$. To see this, we choose some

$$(a, b) \in \bigcap\{\mathbf{R}' \mid \mathbf{R} \subsetneq \mathbf{R}'\}\backslash\mathbf{R}.$$

Note that such an element exists, since (as described in the paragraph after Theorem 5.1.17 we can assume that) $\mathbf{R}$ is meet-irreducible in $\mathbb{S}(\leq)$, and $\bigcap\{\mathbf{R}' \in \mathcal{S}_n \mid \mathbf{R} \subsetneq \mathbf{R}'\}\backslash\mathbf{R} = \varnothing$ would imply that $\mathbf{R}$ is a meet. Now define $\varphi\colon \mathbf{X} \to \mathbf{P\L}_n$ via

$$\varphi(z) = \begin{cases} 1 & \text{if } x_1 < z \not\leq x_2, \\ b & \text{if } x_1 < z \leq x_2, \\ a & \text{if } z = x_1, \\ 0 & \text{otherwise.} \end{cases}$$

We show that φ is structure-preserving, starting with order-preservation. Let $z_1 {\leq^X} z_2$. If $\varphi(z_1) = a$, then $z_1 = x_1$ and $x_1 \leq z_2$ implies $\varphi(z_2) \in \{1, b, a\}$. If $\varphi(z_1) = b$, then $x_1 < z_1 \leq x_2$ implies $\varphi(z_2) \in \{1, b\}$ and if $\varphi(z_1) = 1$ then $x_1 < z_1$ and $z_1 \not\leq x_2$ imply that $x_1 < z_2$ and $z_2 \not\leq x_2$ as well, so $\varphi(z_2) = 1$. In any case, $\varphi(z_1) \leq \varphi(z_2)$ holds. We now show that φ also preserves the other remaining relations.

Let $\mathbf{S} \in \mathcal{S}_n\backslash\{\leq\}$ and suppose $z_1\mathbf{S}^X z_2$. Note that if $x_1 \leq z_1 \leq z_2 \leq x_2$, then $z_1 \neq z_2$, since otherwise due to (d) we have $z_1 \lhd^X z_1$ and due to (b) and (c) we have

$$x_1 \leq^X z_1 \lhd^X z_2 \leq^X x_2 \Rightarrow x_1 \lhd^X x_2 \Rightarrow x_1\mathbf{R}^X x_2,$$

a contradiction to our initial assumption. Thus, if $\mathbf{R} \subsetneq \mathbf{S}$, then $(a, b) \in \mathbf{S}$ ensures $(\varphi(z_1), \varphi(z_2)) \in \mathbf{S}$.

If, on the other hand, $\mathbf{S} \subseteq \mathbf{R}$. Then $(\varphi(z_1), \varphi(z_2)) = (a, b)$ would mean

$$z_1 = x_1 \text{ and } x_1 < z_2 \leq x_2.$$

But this implies

$$z_1 = x_1\mathbf{S}^X z_2 \leq x_2,$$

which in turn implies $x_1\mathbf{S}^X x_2$ by (c) and thus $x_1\mathbf{R}^X x_2$ by (b), again contradicting our initial assumption. This finishes the proof. □

In order to prove something similar for arbitrary $n \in \mathbb{N}$, it seems like (d) in Definition 5.2.12 needs to be replaced by a more sophisticated collection of axioms, taking reflexivity with respect to other relations into account.

In the next section, we study *modal extensions* of PMV_n algebras, generalizing two-valued positive modal logic over DL to many-valued positive modal logic over PMV_n.

5.3 Many-valued positive modal logic

In this section, we introduce and study the positive (*i.e.*, the negation-free and implication-free) fragment of many-valued modal logic over MV_n as introduced in Chapter 3. We introduce richer relational semantics based on partially ordered sets with local constraints on admissible valuations (Definitions 5.3.4 and 5.3.6) and study this logic algebraically via the varieties of modal PMV_n-algebras (Definition 5.3.8). Utilizing the results from the previous sections, we prove an algebraic completeness theorem (Theorem 5.3.14). Furthermore, we illustrate how the richer relational semantics presented in this thesis are 'better-behaved' with respect to the positive many-valued modal logic than Kripke (or other intermediate) semantics are (Subsection 5.3.4). This section may be seen as extended version of the author's [Poi24].

Our terminology and notation in this section sometimes digress from the ones used in previous chapters. For example, we refer to Kripke frames as 'Set-*frames*', and to their ordered versions [CJ97] as 'Pos-*frames*'. We also use 'Set_n' instead of $\mathsf{Set}_{\mathbf{Ł}_n}$ for simplicity, and instead of $\mathbf{Ł}_n$-frames we now say 'Set_n-*frames*', in order to distinguish it from their ordered counterparts, which we accordingly call 'Pos_n-*frames*'.

The section is structured as follows. In Subsection 5.3.1, we give an introduction to (classical) positive modal logic. In Subsection 5.3.2, we introduce its $\mathbf{PŁ}_n$-valued analogue, together with its richer semantics over the category Pos_n. In Subsection 5.3.3, we study this logic algebraically. Lastly, in Subsection 5.3.4 we give an example of a consequence pair which is only canonical with respect to the semantics over Pos_n-frames.

5.3.1 Introduction to positive modal logic

Positive modal logic (with truth-constants), introduced by Dunn [Dun95], is the $\{\wedge, \vee, 0, 1, \Box, \Diamond\}$-fragment of classical modal logic. Algebraically speaking, removing the negation from the language amounts to replacing the underlying variety BA of Boolean algebras by the variety DL of bounded distributive lattices. Since $\Box$ and $\Diamond$ are then no longer inter-definable, their interplay is instead described by Dunn's *positivity-axioms*, that is, the axioms (P1)-(P2) in the following.

Definition 5.3.1 (Modal distributive lattice). A *modal (bounded) distributive lattice* is an algebra $\langle \mathbf{D}, \Box, \Diamond \rangle$, where $\mathbf{D} \in \mathsf{DL}$ is a bounded distributive lattice and $\Box, \Diamond \colon D \to D$ are unary operations satisfying the equations

(B1) $\Box 1 = 1$, (D1) $\Diamond 0 = 0$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$, (D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,

(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$, (P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

We denote the variety of modal distributive lattices by mDL.

The exact relationship between the variety mDL and positive modal logic was described by Jansana in [Jan02].

While Dunn [Dun95] only considered the usual relational semantics on Kripke-frames (from now on, we use the term Set-*frame* instead of *Kripke frame* to avoid confusion), Celani and Jansana [CJ97] proposed some 'better-behaved' semantics based on the category of quasi-orders, which we now recall. For a slight increase in simplicity, in this thesis we work with the category Pos of *posets* instead of quasi-orders.

Given a poset $(X, \leq)$, we use $\leq_{\mathrm{EM}}$ to denote the *Egli-Milner lifting* of $\leq$ to the powerset $\mathcal{P}(X)$, defined for $A, B \subseteq X$ by

$$A \leq_{\mathrm{EM}} B \Leftrightarrow \begin{cases} \forall a \in A \colon \exists b \in B \colon a \leq b \text{ and} \\ \forall b \in B \colon \exists a \in A \colon a \leq b. \end{cases}$$

With this convention, we may paraphrase [CJ97, Definition 4.1] as follows.

Definition 5.3.2 (Pos-frame & -model). A Pos-*frame* is a structure $(X, \leq, R)$, where $(X, \leq)$ is a poset and $R \subseteq X^2$ is a binary relation which satisfies that $R[x] := \{x' \in X \mid xRx'\}$ is always convex and for all $x, y \in X$ it holds that

$$x \leq y \Rightarrow R[x] \leq_{\mathrm{EM}} R[y].$$

Fixing a countable set Prop of propositional variables, a Pos-*model* is a structure $(X, \leq, R, \mathsf{Val})$ consisting of a Pos-frame together with a Pos-*valuation* $\mathsf{Val} \colon X \times \mathsf{Prop} \to \{0, 1\}$, which satisfies

$$x \leq y \Rightarrow \mathsf{Val}(x, p) \leq \mathsf{Val}(y, p)$$

for all $x, y \in X$ and $p \in \mathsf{Prop}$.

The condition on Pos-frames ensures that the extension of Val to all formulas φ built from Prop and connectives in $\{\wedge, \vee, 0, 1, \Box, \Diamond\}$ also has the property that $\mathsf{Val}(-, \varphi)$ is always order-preserving.

As pointed out in [CJ97], one considerable advantage these Pos-based semantics have over Set-based semantics comes to light in the study of canonicity. For example, the two consequence pairs

$$\Box p \vdash p \text{ and } p \vdash \Diamond p$$

both define the class of reflexive Set-frames but are not mutually interderivable in positive modal logic. In the richer semantics, however, both these formulas are canonical and correspond to the classes of Pos-frames $(X, \leq, R)$ where

$$R_{\Box} := R \circ \leq \text{ and } R_{\Diamond} := R \circ \leq^{-1}$$

are reflexive, respectively [CJ97, Sections 5,7]. The advantages of Pos-based semantics over Set-based ones also become apparent in the study of positive modal logic via duality theory [CJ99] and coalgebras [Pal04, BKV13, DK17]. In particular, the latter fully embraces this view and proposes positive coalgebraic logic as the *logic of ordered transition systems.*

5.3.2 Positive modal logic over finite MV-chains

We work in the language $\mathcal{L}^{\Box\Diamond}_{\mathsf{PMV}} = \{\odot, \oplus, \wedge, \vee, 0, 1, \Box, \Diamond\}$, that is, the signature of PMV together with two unary operation symbols $\Box$ and $\Diamond$. We define the set $\mathsf{Form}^{\Box\Diamond}_{\mathsf{PMV}}$ of modal PMV-formulas inductively from a countable collection Prop of propositional variables and the connectives in $\mathcal{L}^{\Box\Diamond}_{\mathsf{PMV}}$.

Similarly to what is discussed in Subsection 3.2.1 and the previous subsection, we consider 'expanded' relational semantics on frames with additional structure. For these frames we choose the following base-category, which arises as a blend of Pos and Set_n.

Definition 5.3.3 (The category Pos_n)**.** We define the category Pos_n as follows. The objects of Pos_n are structures of the form $(X, \leq, v)$, such that $(X, \leq)$ is a poset and $(X, v) \in \mathsf{Set}_n$, that is, $v\colon X \to \mathbb{S}(\mathbf{P{\L}}_n)$.

A morphism $f\colon (X_1, \leq_1, v_1) \to (X_2, \leq_2, v_2)$ in Pos_n is both an order-preserving map $f\colon (X_1, \leq_1) \to (X_2, \leq_2)$ and a Set_n-morphism, meaning that $v_2(f(x)) \subseteq v_1(x)$ holds for all $x \in X_1$.

Note that we can really consider the reduct (X, v) of an object of Pos_n as a member of Set_n, since the subuniverses of $\mathbf{P{\L}}_n$ coincide with the ones of $\mathbf{{\L}}_n$, as shown in Proposition 5.1.4. Similar to Section 5.1, we also think of $\mathbf{P{\L}}_n$ itself as a member of Pos_n which we denote by

$$\widetilde{\mathbf{P{\L}}}_n = \big(\{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\}, \leq, \langle\cdot\rangle\big),$$

where $\leq$ is the the usual chain-order and $\langle \ell \rangle$ takes the subalgebra of $\mathbf{P{\L}}_n$ generated by $\{\ell\}$.

Members of Pos_n endowed with a compatible accessibility relation constitute our semantics for positive modal logic over finite MV-chains.

Definition 5.3.4 (Pos_n-frame)**.** A Pos_n*-frame* is a structure of the form

$$\mathfrak{F} = (X, \leq, v, R),$$

where $(X, \leq, v) \in \mathsf{Pos}_n$ and $R \subseteq X^2$ is a binary relation such that for every $x \in X$ the set $R[x]$ is convex and the following two *compatibility conditions* are satisfied:

- For all $x, y \in X$ it holds that
$$x \leq y \Rightarrow R[x] \leq_{\mathrm{EM}} R[y].$$
- Whenever $x, y \in X$ satisfy $y \in R[x]$, there exist $y', y'' \in R[x]$ with
$$y' \leq y \leq y'' \text{ and } v(y'), v(y'') \subseteq v(x).$$

Here, in the first condition the subscript EM denotes the Egli-Milner lifting of $\leq$ to $\mathcal{P}(X)$ as defined in the paragraph before Definition 5.3.2.

As shown in the following, Pos_n-frames generalize Set_n-frames and Pos-frames (thus, in particular, they generalize Set-frames as well).

Example 5.3.5 (Special Pos_n-frames)**.** The following are special examples of Pos_n-frames.

(1) Every Pos-frame $(X, \leq, R)$ as in Definition 5.3.2 can be identified with the Pos_n-frame $(X, \leq, v^{\mathbf{PŁ}_n}, R)$, where $v^{\mathbf{PŁ}_n}$ are the 'trivial constraints' $v^{\mathbf{PŁ}_n}(x) = \mathbf{PŁ}_n$ for all $x \in X$.

(2) Every Set_n-frame (X, v, R) as in Definition 3.2.1 can be identified with the Pos_n-frame $(X, \leq^{\mathrm{dis}}, v, R)$, where $\leq^{\mathrm{dis}}$ is the discrete order on X.

(3) In particular, combining (1) and (2), every Set-frame (X, R) can be identified with the Pos_n-frame $(X, \leq^{\mathrm{dis}}, v^{\mathbf{PŁ}_n}, R)$. ■

Not surprisingly, the corresponding Pos_n-models are also defined as a blend of Pos-models and Set_n-models.

Definition 5.3.6 (Pos_n-model)**.** A Pos_n*-model* is a structure of the form

$$\mathfrak{M} = (X, \leq, v, R, \mathsf{Val}),$$

where $(X, \leq, v, R)$ is a Pos_n-frame and the Pos_n*-valuation* $\mathsf{Val}\colon X \times \mathsf{Prop} \to \mathbf{PŁ}_n$ satisfies the following two conditions:

- If $x \leq y$, then $\mathsf{Val}(x, p) \leq \mathsf{Val}(y, p)$ for all $p \in \mathsf{Prop}$.

- $\mathsf{Val}(x,p) \in v(x)$ for all $x \in X$ and $p \in \mathsf{Prop}$.

For example, every Pos-model, Set_n-model or Set-model can be seen as a Pos_n-model analogous to Example 5.3.5.

Given a Pos_n-model $(X, \leq, v, R, \mathsf{Val})$, as before we inductively extend its Pos_n-valuation to a map

$$\mathsf{Val}\colon X \times \mathsf{Form}^{\Box\Diamond}_{\mathsf{PMV}} \to \mathbf{P\text{Ł}}_n,$$

which is defined in the obvious way for the PMV-connectives and for formulas of shape $\Box\varphi$ or $\Diamond\varphi$ via

$$\mathsf{Val}(x, \Box\varphi) = \bigwedge\{\mathsf{Val}(x', \varphi) \mid xRx'\} \text{ and } \mathsf{Val}(x, \Diamond\varphi) = \bigvee\{\mathsf{Val}(x', \varphi) \mid xRx'\}.$$

The two conditions on Pos_n-models of Definition 5.3.6 are equivalent to the fact that $\mathsf{Val}(-,p)\colon (X, \leq, v) \to \widetilde{\mathbf{P\text{Ł}}_n}$ always is a morphism in Pos_n (where we identify $\mathbf{P\text{Ł}}_n$ with a member of Pos_n as described after Definition 5.3.3). In the following, we show that the compatibility conditions required of Pos_n-frames (Definition 5.3.4) assure that this property extends to $\mathsf{Val}(-,\varphi)$ for every modal formula $\varphi \in \mathsf{Form}^{\Box\Diamond}_{\mathsf{PMV}}$.

Proposition 5.3.7. *Let $(X, \leq, v, R, \mathsf{Val})$ be a Pos_n-model. Then for every modal PMV-formula $\varphi \in \mathsf{Form}^{\Box\Diamond}_{\mathsf{PMV}}$ the following two conditions hold.*

- *If $x \leq y$ then $\mathsf{Val}(x,\varphi) \leq \mathsf{Val}(y,\varphi)$ for all $x, y \in X$.*
- *$\mathsf{Val}(x,\varphi) \in v(x)$ for all $x \in X$.*

Proof. We proceed by induction on the formula φ. The case $\varphi = p \in \mathsf{Prop}$ is covered by Definition 5.3.6. If $\varphi = \psi_1 * \psi_2$ with $* \in \{\odot, \oplus, \wedge, \vee\}$ being a PMV-connective, the first condition holds because all of these connectives are order-preserving and the second one holds because $v(x)$ always is a subalgebra of $\mathbf{P\text{Ł}}_n$.

This leaves us with the case $\varphi = \Box\psi$ (the case $\varphi = \Diamond\psi$ being analogous). To prove condition one, suppose that $\mathsf{Val}(y,\varphi) < 1$. Then there exists some $y' \in R[y]$ with $\mathsf{Val}(y,\varphi) = \mathsf{Val}(y',\psi)$. Since $R[x] \leq_{\mathrm{EM}} R[y]$, there exists some $x' \in R[x]$ with $x' \leq y'$ and therefore, by the inductive hypothesis, $\mathsf{Val}(x',\psi) \leq \mathsf{Val}(y',\psi)$. This yields

$$\mathsf{Val}(x, \Box\psi) = \bigwedge\{\mathsf{Val}(\tilde{x}, \psi) \mid xR\tilde{x}\} \leq \mathsf{Val}(x', \psi) \leq \mathsf{Val}(y', \psi) = \mathsf{Val}(y, \Box\psi)$$

as desired.

To prove condition two, suppose towards contradiction that $\mathsf{Val}(x, \Box\psi) \notin v(x)$. Since $1 \in v(x)$, there needs to exist some $y \in R[x]$ with $\mathsf{Val}(x,\varphi) =$

$\mathsf{Val}(y, \psi)$. By Definition 5.3.4, there is some $y' \in R[x]$ with $y' \leq y$ and $v(y') \subseteq v(x)$. By inductive hypothesis we have $\mathsf{Val}(y', \psi) \neq \mathsf{Val}(y, \psi)$ (since otherwise $\mathsf{Val}(y', \psi) \notin v(y') \subseteq v(x)$). This yields

$$\mathsf{Val}(y', \psi) < \mathsf{Val}(y, \psi) = \mathsf{Val}(x, \Box\psi) = \bigwedge\{\mathsf{Val}(\tilde{y}, \psi) \mid xR\tilde{y}\},$$

a contradiction to xRy'. $\Box$

We say that a modal PMV-formula $\varphi \in \mathsf{Form}_{\mathsf{PMV}}^{\Box\Diamond}$ is *true* at $x \in X$ in a Pos_n-model $(X, \leq, v, R, \mathsf{Val})$ if $\mathsf{Val}(x, \varphi) = 1$. As usual, we say that φ is *valid* in a Pos_n-frame if it is satisfied at every state in every model based on that frame. We denote by $\Lambda_{\mathsf{PMV}_n}^{\Box\Diamond}$ the set of all modal PMV-formulas which are valid in every Pos_n-frame. More generally, similar to [Dun95], we consider *consequence pairs* $\psi \vdash \varphi$ and say that such a consequence pair is *valid in a* Pos_n*-frame* if $\mathsf{Val}(x, \psi) = 1$ implies $\mathsf{Val}(x, \varphi) = 1$ in every Pos_n-model based on that frame. We say the consequence pair is *valid* if it is valid in all Pos_n-frames. In this case, we write $\psi \models_n \varphi$. The valid formulas $\varphi \in \Lambda_{\mathsf{PMV}_n}^{\Box\Diamond}$ are thus precisely the ones for which $1 \models_n \varphi$ holds.

5.3.3 Algebraic framework

In this subsection, we study the logic introduced in the previous section by algebraic means. The corresponding variety of modal PMV_n-algebras arises as a combination of the axioms of modal distributive lattices (see Definition 5.3.1) and modal MV_n-algebras (see Definition 3.3.1) as follows.

Definition 5.3.8 (Modal PMV_n-algebra)**.** A *modal* PMV_n*-algebra*, or simply mPMV_n-algebra, is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$, where $\mathbf{A} \in \mathsf{PMV}_n$ is a positive MV_n-algebra and $\Box, \Diamond \colon A \to A$ are unary operations satisfying the equations

(B1) $\Box 1 = 1$, (D1) $\Diamond 0 = 0$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$, (D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,

(B3) $\tau_d(\Box x) = \Box \tau_d(x)$, (D3) $\tau_d(\Diamond x) = \Diamond \tau_d(x)$,

(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$, (P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

We denote the variety of modal PMV_n-algebras by mPMV_n.

Similar to Example 3.3.10, the following alternative axiomatization of mPMV_n might be considered more ‘pleasing’ in its usage of the signature of PMV. Nevertheless, Definition 5.3.8 is usually more convenient for our purposes.

Fact. An algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ with $\mathbf{A} \in \mathsf{PMV}_n$ is a modal PMV_n-algebra if and only if it satisfies the following equations

(B1) $\Box 1 = 1$, (D1) $\Diamond 0 = 0$,

(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$, (D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,

(B$\oplus$) $\Box(x \oplus x) = \Box x \oplus \Box x$, (D$\oplus$) $\Diamond(x \oplus x) = \Diamond x \oplus \Diamond x$,

(B$\odot$) $\Box(x \odot x) = \Box x \odot \Box x$, (D$\odot$) $\Diamond(x \odot x) = \Diamond x \odot \Diamond x$,

(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$, (P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

The proof of this fact is completely analogous to the proof in Example 3.3.10.

Next, similarly to Definition 3.3.3, we explain how to obtain Pos_n-frames from mPMV_n-algebras.

Definition 5.3.9 (Canonical frame of a mPMV_n-algebra)**.** Let $\langle \mathbf{A}, \Box, \Diamond \rangle$ be a modal PMV_n-algebra. The *canonical* Pos_n*-frame of* $\langle \mathbf{A}, \Box, \Diamond \rangle$ is the Pos_n-frame

$$\langle \mathbf{A}, \Box, \Diamond \rangle^+ := \big(\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n), \leq^{\mathrm{pw}}, \mathsf{im}, R^{\mathrm{m}}\big),$$

where $\leq^{\mathrm{pw}}$ is the point-wise order, im takes a homomorphism u to its image $\mathsf{im}(u) = u(\mathbf{A}) \subseteq \mathbf{P\L}_n$ and the binary relation R^{m} is defined via

$$uR^{\mathrm{m}}u' \Leftrightarrow \forall a \in \mathbf{A} : u(\Box a) \leq u'(a) \leq u(\Diamond a).$$

This definition is justified, since in the following we show that canonical frames of mPMV_n-algebras really are Pos_n-frames.

Proposition 5.3.10. *Let* $\langle \mathbf{A}, \Box, \Diamond \rangle$ *be a modal* PMV_n*-algebra. Then its canonical frame* $\langle \mathbf{A}, \Box, \Diamond \rangle^+$ *is a* Pos_n*-frame.*

Proof. It is clear by definition that $(\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n), \leq^{\mathrm{pw}}, \mathsf{im})$ is an object of the category Pos_n. We need to show that R^{m} satisfies the two compatibility conditions from Definition 5.3.4.

To verify condition one, first suppose that $u, w \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n)$ are homomorphisms for which $u \leq^{\mathrm{pw}} w$ and $wR^{\mathrm{m}}w'$ hold. Then we need to find u' which satisfies $uR^{\mathrm{m}}u' \leq w'$. We define a filter $F \subseteq \mathfrak{S}(\mathbf{A})$ on the distributive skeleton of $\mathbf{A}$ (recall Definition 5.2.3) by

$$F = \{b \in \mathfrak{S}(\mathbf{A}) \mid u(\Box b) = 1\}$$

and an ideal $J \subseteq \mathfrak{S}(\mathbf{A})$ generated by the set

$$\{b \in \mathfrak{S}(\mathbf{A}) \mid w'(b) = 0 \text{ or } u(\Diamond b) = 0\}.$$

If $b \in J$, there are $b', b'' \in \mathfrak{S}(\mathbf{A})$ with $b \leq b' \vee b''$, $w'(b') = 0$ and $u(\Diamond b'') = 0$. We then compute

$$\begin{aligned} u(\Box b) &\leq u\big(\Box(b' \vee b'')\big) \\ &\leq u(\Box b') \vee u(\Diamond b'') && \text{(P1)} \\ &\leq w(\Box b') \vee u(\Diamond b'') && (u \leq^{\mathrm{pw}} w) \\ &\leq w'(b') \vee u(\Diamond b'') = 0. && (wR^{\mathrm{m}}w') \end{aligned}$$

Therefore, $b \notin J$ and we showed that F and J are disjoint. There is a prime filter U' extending F with $U' \cap J = \varnothing$, and by Theorem 5.2.4 there is a homomorphism $u' \colon \mathbf{A} \to \mathbf{PŁ}_n$ which extends (the characteristic function of) U'.

We show that u' has the desired properties. To see that $u'(a) \leq w'(a)$, in case $w'(a) \neq 1$ we can use $\ell = w'(a) + \frac{1}{n}$ and find $\tau_\ell(a) \in J$, which yields

$$\tau_\ell\big(u'(a)\big) = u'\big(\tau_\ell(a)\big) = 0,$$

that is, $u'(a) < \ell$ as desired. The argument for $u'(a) \leq u(\Diamond a)$ is similar, using (D3), and for $u(\Box a) \leq u'(a)$ using the construction of F and (B3).

For the second half of condition one, assuming that $u \leq^{\mathrm{pw}} w$ and $uR^{\mathrm{m}}u'$ hold, we want to find w' with $wR^{\mathrm{m}}w'$ and $u' \leq^{\mathrm{pw}} w'$. This time, let $F \subseteq \mathfrak{S}(\mathbf{A})$ be the filter generated by

$$\{b \in \mathfrak{S}(\mathbf{A}) \mid u'(b) = 1 \text{ or } w(\Box b) = 1\},$$

and let $J \subseteq \mathfrak{S}(\mathbf{A})$ be the ideal

$$J = \{b \in \mathfrak{S}(\mathbf{A}) \mid w(\Diamond b) = 0\}.$$

Given $b \in F$, we have $b \geq b' \wedge b''$ for some $b', b'' \in \mathfrak{S}(\mathbf{A})$ which satisfy $u'(b') = w(\Box b'') = 1$. We compute

$$\begin{aligned} w(\Diamond b) &\geq w\big(\Diamond(b' \wedge b'')\big) \\ &\geq w(\Diamond b') \wedge w(\Box b'') && \text{(P2)} \\ &\geq u(\Diamond b') \wedge w(\Box b'') && (u \leq^{\mathrm{pw}} w) \\ &\geq u'(b') \wedge w(\Box b'') = 1. && (uR^{\mathrm{m}}u') \end{aligned}$$

Therefore, $b \notin J$ and we showed that J and F are disjoint. The filter F can be extended to a prime filter $W' \subseteq \mathfrak{S}(\mathbf{A})$ which is disjoint from J, and by

Theorem 5.2.4, there is a homomorphism $w'\colon \mathbf{A} \to \mathbf{PŁ}_n$ which restricts to (the characteristic function of) W' on $\mathfrak{S}(\mathbf{A})$.

To see that w' has the desired properties,

$$u'(a) \leq w'(a)$$

follows from $\tau_\ell(a) \in F$ with $\ell = u'(a)$ and similarly we get $w(\Box a) \leq w'(a)$. To see that $w'(a) \leq w(\Diamond a)$ holds, suppose $w(\Diamond a) \neq 1$ and set $\ell = w(\Diamond a) + \frac{1}{n}$. Then $\tau_\ell(a) \in J$ implies $0 = w'(\tau_\ell(a)) = \tau_\ell(w'(a))$, which means $w'(a) \leq w(\Diamond a)$ by our choice of ℓ.

We now verify condition two of Definiton 5.3.4, using an argument similar to [HT13, Lemma 7.4]. Let $u \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_n)$ with $\mathsf{im}(u) = \mathbf{PŁ}_k$ and let $uR^{\mathrm{m}}w$. Suppose towards contradiction that there is no $w' \leq^{\mathrm{pw}} w$ with $uR^{\mathrm{m}}w'$ and $\mathsf{im}(w') \leq \mathbf{PŁ}_k$. Choose $\tilde{w} \in R^{\mathrm{m}}[u]$ and $a \in \mathbf{A}$ such that $\tilde{w} \leq^{\mathrm{pw}} w$ and $\tilde{w}(a) = \frac{1}{m} \in \mathbf{PŁ}_n \backslash \mathbf{PŁ}_k$ is the minimal value obtained by a member of $R^{\mathrm{m}}[u]$ below w. By the duality for PMV_n established in Section 5.1, we can think of $(\mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_n), \leq^{\mathrm{pw}})$ as Priestley space in which the subsets $\mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_k)$ and $R^{\mathrm{m}}[u]$ are closed. The algebra $\mathbf{A}$ can be identified with the collection of structure-preserving continuous maps $\mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_n) \to \widetilde{\mathbf{PŁ}}_n$ via the isomorphism $a \mapsto \mathsf{ev}_a$. Now consider the closed sets

$$F = R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_k) \text{ and } G = R^{\mathrm{m}}[u] \cap \tilde{w}{\downarrow}.$$

We have $F{\uparrow} \cap G{\downarrow} = \varnothing$, since otherwise there would be $w' \in R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_k)$ with $w' \leq^{\mathrm{pw}} \tilde{w} \leq^{\mathrm{pw}} w$, contradicting our initial assumption. Therefore, there exists a clopen down-set Ω which contains G and is disjoint from F. Now we define the element

$$a' = a|_{\Omega} \cup 1|_{\mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_n) \backslash \Omega},$$

which is a well-defined member of $\mathbf{A}$ since Ω is downwards closed. Now by Lemma 5.3.11 below we find

$$u(\Box a') = \bigwedge \{v(a') \mid v \in R^{\mathrm{m}}[u]\} = \bigwedge \{v(a) \mid v \in R^{\mathrm{m}}[u] \cap \Omega\} = \tilde{w}(a),$$

which yields a contradiction to $\mathsf{im}(u) = \mathbf{PŁ}_k$, since $\tilde{w}(a) \notin \mathbf{PŁ}_k$.

The proof of the second half of condition two is similar. Take $u \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_n)$ with $\mathsf{im}(u) = \mathbf{PŁ}_k$ and $uR^{\mathrm{m}}w$ and suppose towards contradiction that there is no $w \leq^{\mathrm{pw}} w''$ with $uR^{\mathrm{m}}w''$ and $\mathsf{im}(w'') \subseteq \mathbf{PŁ}_k$. Choose $\tilde{w} \in R^{\mathrm{m}}[u] \cap w{\uparrow}$ and $a \in \mathbf{A}$ such that $\tilde{w}(a) = \frac{m-1}{m}$ is the maximal value obtained by a member of $R[u]$ above w. Separate the closed sets

$$F = R^{\mathrm{m}}[u] \cap \tilde{w}{\uparrow} \text{ and } G = R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_k),$$

by a clopen upset $\Omega \supseteq F$ and take

$$a' = a|_{\Omega} \cup 0|_{\mathsf{PMV}_n(\mathbf{A},\mathbf{P}\mathbf{Ł}_n)\setminus\Omega},$$

which is a well-defined member of $\mathbf{A}$ since Ω is upwards closed. Then, using Lemma 5.3.11, we compute

$$u(\Diamond a') = \bigvee\{v(a') \mid v \in R^{\mathrm{m}}[u]\} = \bigvee\{v(a) \mid v \in R^{\mathrm{m}}[u] \cap \Omega\} = \tilde{w}(a),$$

which yields $u(\Diamond a') = \frac{m-1}{m} \notin \mathbf{P}\mathbf{Ł}_k$, a contradiction to our assumption $\mathsf{im}(u) = \mathbf{P}\mathbf{Ł}_k$. $\Box$

In order to obtain an algebraic completeness result, the following truth lemma will be the most important ingredient.

Lemma 5.3.11 (Truth Lemma). *Let $\langle \mathbf{A}, \Box, \Diamond\rangle$ be a mPMV_n-algebra. Then, in the canonical frame $\langle \mathbf{A}, \Box, \Diamond\rangle^+$, it holds that*

$$u(\Box a) = \bigwedge\{u'(a) \mid uR^{\mathrm{m}}u'\} \text{ and } u(\Diamond a) = \bigvee\{u'(a) \mid uR^{\mathrm{m}}u'\}$$

for all $u \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{P}\mathbf{Ł}_n)$ and $a \in \mathbf{A}$.

Proof. We start with the first equation. It is clear by definition that $u(\Box a) \leq \bigwedge\{u'(a) \mid uR^{\mathrm{m}}u'\}$ holds. Suppose towards contradiction that this inequality is strict, say

$$u(\Box a) < \bigwedge\{u'(a) \mid uR^{\mathrm{m}}u'\} =: d$$

holds for some $a \in \mathbf{A}$. We define a filter F and an ideal J on the distributive skeleton $\mathfrak{S}(\mathbf{A})$ as follows. Set

$$F = \{b \in \mathfrak{S}(\mathbf{A}) \mid u(\Box b) = 1\},$$

which is a filter due to axioms (B1) and (B2) and let $J \subseteq \mathfrak{S}(\mathbf{A})$ be the ideal generated by the set

$$\{\tau_d(a) \vee b' \in \mathfrak{S}(\mathbf{A}) \mid u(\Diamond b') = 0\}.$$

If $b \in J$, then $b \leq \tau_d(a) \vee b'_1 \vee \cdots \vee b'_n$ for some $b'_1, \dots b'_n$ with $u(\Diamond b'_i) = 0$ for all $i = 1, \dots, n$. Then we use the fact that u is a homomorphism and the properties of modal PMV_n-algebras to find

$$\begin{aligned}
u(\Box b) &\leq u\big(\Box(\tau_d(a) \vee (b'_1 \vee \cdots \vee b'_n))\big) && \text{(B2)}\\
&\leq u\big(\Box\tau_d(a) \vee \Diamond(b'_1 \vee \cdots \vee b'_n)\big) && \text{(P1)}\\
&= u\big(\Box\tau_d(a)\big) \vee u(\Diamond b'_1 \vee \cdots \vee \Diamond b'_n) && \text{(D2)}\\
&= \tau_d\big(u(\Box a)\big) \vee u(\Diamond b'_1) \vee \cdots \vee u(\Diamond b'_n) = 0. && \text{(B3)}
\end{aligned}$$

Therefore, $u(\Box b) = 0$, which implies $b \notin F$. So we showed that F and J are disjoint. By the prime ideal theorem for distributive lattices, we can find a prime filter $W \subseteq \mathfrak{S}(\mathbf{A})$ with $F \subseteq W$ and $W \cap J = \varnothing$. By Theorem 5.2.4, there is a homomorphism $w\colon \mathbf{A} \to \mathbf{P}\mathbf{Ł}_n$ which restricts to (the characteristic function of) W on $\mathfrak{S}(\mathbf{A})$.

We now show that $uR^{\mathrm{m}}w$ holds. Indeed, take $a' \in \mathbf{A}$ and set $u(\Box a') = \ell$. Then

$$\tau_\ell\big(w(a')\big) = w\big(\tau_\ell(a')\big) = 1$$

because $u(\Box \tau_\ell(a')) = \tau_\ell(u(\Box a')) = 1$ implies $\tau_\ell(a') \in F$. Therefore, we have

$$u(\Box a') \leq w(a') \text{ for all } a' \in \mathbf{A}.$$

Similarly, if $u(\Diamond a') \neq 1$, set $\ell = u(\Diamond a') + \frac{1}{n}$ and find

$$\tau_\ell\big(w(a')\big) = w\big(\tau_\ell(a')\big) = 0$$

because $\tau_\ell(a') \leq \tau_d(a) \vee \tau_\ell(a') \in J$ implies $\tau_\ell(a') \in J$. Therefore, we have $w(a') < u(\Diamond a') + \frac{1}{n}$, which implies

$$w(a') \leq u(\Diamond a') \text{ for all } a' \in \mathbf{A}$$

as desired.

So we showed that $uR^{\mathrm{m}}w$ holds. However, note that $\tau_d(a) \in J$ implies that

$$\tau_d\big(w(a)\big) = w\big(\tau_d(a)\big) = 0,$$

which by our choice of d means $w(a) < \bigwedge\{v(a) \mid uR^{\mathrm{m}}v\}$, a contradiction. This finishes the proof of the first equation.

The proof of the second equation is similar. Suppose towards contradiction that

$$\bigvee\{u'(a) \mid uR^{\mathrm{m}}u'\} < u(\Diamond a)$$

holds for some $a \in \mathbf{A}$. Set $d := \bigvee\{u'(a) \mid uR^{\mathrm{m}}u'\} + \frac{1}{n}$. Define a filter F and an ideal J of $\mathfrak{S}(\mathbf{A})$ as follows. Let

$$J = \{b \in \mathfrak{S}(\mathbf{A}) \mid u(\Diamond b) = 0\},$$

which is an ideal due to axioms (D1)-(D2) and let F be the filter generated by the set

$$\{\tau_d(a) \wedge b' \mid u(\Box b') = 1\}.$$

If $b \in F$, then $\tau_d(a) \wedge b'_1 \wedge \cdots \wedge b'_n \leq b$ for some $b'_1, \ldots b'_n$ with $u(\Box b'_i) = 1$ for all $i = 1, \ldots, n$. We compute

$$\begin{aligned} u(\Diamond b) &\geq u\big(\Diamond(\tau_d(a) \wedge (b'_1 \wedge \cdots \wedge b'_n))\big) && \text{(D2)} \\ &\geq u\big(\Diamond\tau_d(a) \wedge \Box(b'_1 \wedge \cdots \wedge b'_n)\big) && \text{(P2)} \\ &= u\big(\Diamond\tau_d(a)\big) \wedge u(\Box b'_1 \wedge \cdots \wedge \Box b'_n) && \text{(B2)} \\ &= \tau_d\big(u(\Diamond a)\big) \wedge u(\Box b'_1) \wedge \cdots \wedge u(\Box b'_n) = 1. && \text{(D3)} \end{aligned}$$

Therefore, $u(\Diamond b) = 1$, which implies $b \notin J$ and we showed that F and J are disjoint. As before, we can find a homomorphism $w\colon \mathbf{A} \to \mathbf{PŁ}_n$ with $w(F) = \{1\}$ and $w(J) = \{0\}$.

We show that $uR^{\mathrm{m}}w$ holds. Given $a' \in \mathbf{A}$, let $\ell = u(\Box a')$. Then $\tau_\ell(a') \geq \tau_d(a) \wedge \tau_\ell(a') \in F$ implies $w(\tau_\ell(a')) = 1$ and thus we showed

$$u(\Box a') \leq w(a') \text{ for all } a \in \mathbf{A}.$$

Conversely, if $u(\Diamond a') \neq 1$, take $\ell = u(\Diamond a') + \frac{1}{n}$. Then $u(\Diamond\tau_\ell(a')) = 0$ yields $\tau_\ell(a') \in J$ and thus $w(\tau_\ell(a')) = 0$. This means $w(a') < u(\Diamond a) + \frac{1}{n}$. Thus we showed

$$w(a') \leq u(\Diamond a') \text{ for all } a' \in \mathbf{A},$$

and altogether that $uR^{\mathrm{m}}w$ holds.

However, since $\tau_d(a) \in F$ holds, we have $\tau_d(w(a)) = w(\tau_d(a)) = 1$, which by our choice means $\bigvee\{u'(a) \mid uR^{\mathrm{m}}u'\} < w(a)$, a contradiction to $uR^{\mathrm{m}}w$. □

In the converse direction to Definition 5.3.9, we now explain how to obtain mPMV_n-algebras from Pos_n-frames (similarly to Definition 3.3.20).

Definition 5.3.12 (Complex algebra of a Pos_n-frame)**.** Let $\mathfrak{F} = (X, \leq, v, R)$ be a Pos_n-frame. The *complex* mPMV_n*-algebra* of $\mathfrak{F}$ is the mPMV_n-algebra

$$\mathfrak{F}_+ := \langle \widetilde{\mathbf{PŁ}}_n^{\mathfrak{F}}, \Box_R, \Diamond_R \rangle,$$

where $\widetilde{\mathbf{PŁ}}_n^{\mathfrak{F}}$ consists of all Pos_n-morphisms $(X, \leq, v) \to \widetilde{\mathbf{PŁ}}_n$ with point-wise PMV_n-operations and

$$(\Box_R\alpha)(x) = \bigwedge_{xRx'} \alpha(x') \text{ and } (\Diamond_R\alpha)(x) = \bigvee_{xRx'} \alpha(x').$$

It is easy to check that this really is a mPMV_n-algebra.

Every mPMV_n-algebra can be embedded in the complex algebra of its canonical frame and vice versa.

Theorem 5.3.13 (Representations). *Let* $\mathbf{A}$ *be a* mPMV_n*-algebra and let* $\mathfrak{F}$ *be a* Pos_n*-frame.*

(1) There is an embedding $\mathbf{A} \hookrightarrow (\mathbf{A}^+)_+$ *via evaluations* $a \mapsto \mathsf{ev}_a$*.*

(2) There is an embedding $\mathfrak{F} \hookrightarrow (\mathfrak{F}_+)^+$ *via evaluations* $x \mapsto \mathsf{ev}_x$*.*

Proof. The 'non-modal parts' of (1) and (2) are both consequences of the dualities established in Section 5.1.

For the 'modal part' of (1), we need to show that $\Box_{R^{\mathrm{m}}}\mathsf{ev}_a = \mathsf{ev}_{\Box a}$ and $\Diamond_{R^{\mathrm{m}}}\mathsf{ev}_a = \mathsf{ev}_{\Diamond a}$. This follows from a direct computation

$$\Box_{R^{\mathrm{m}}}\mathsf{ev}_a(u) = \bigwedge_{uR^{\mathrm{m}}v} \mathsf{ev}_a(v) = \bigwedge_{uR^{\mathrm{m}}v} v(a) = u(\Box a) = \mathsf{ev}_{\Box a},$$

where we used Lemma 5.3.11. The other equation is shown completely analogous.

For the 'modal part' of (2), we need to show

$$xRx' \Leftrightarrow \mathsf{ev}_x R^{\mathrm{m}} \mathsf{ev}_{x'}.$$

The direction '$\Rightarrow$' is immediate by definition since

$$\begin{aligned} \mathsf{ev}_x R^{\mathrm{m}} \mathsf{ev}_{x'} &\Leftrightarrow \mathsf{ev}_x(\Box_R \alpha) \leq \mathsf{ev}_{x'}(\alpha) \leq \mathsf{ev}_x(\Diamond_R \alpha) \\ &\Leftrightarrow \bigwedge_{xRy} \alpha(y) \leq \alpha(x') \leq \bigvee_{xRy} \alpha(y), \end{aligned}$$

and the latter clearly holds for all α if xRx'.

For the direction '$\Leftarrow$', suppose $x' \notin R[x]$. Then $R[x] \cap x'{\uparrow} = \varnothing$ or $R[x] \cap x'{\downarrow} = \varnothing$ needs to hold (since otherwise xRx' holds by convexity of $R[x]$). In the former case, define $\alpha\colon \mathfrak{F} \to \widetilde{\mathbf{P}\mathbf{Ł}}_n$ by $\alpha(y) = 0$ if $y \in R[x]{\downarrow}$ and $\alpha(y) = 1$ otherwise. Then

$$\mathsf{ev}_{x'}(\alpha) = \alpha(x') = 1 \text{ but } \mathsf{ev}_x(\Diamond_R \alpha) = \bigvee_{xRy} \alpha(y) = 0$$

shows $\mathsf{ev}_{x'} \notin R^{\mathrm{m}}[\mathsf{ev}_x]$. In the other case a similar argument works with $\alpha(y) = 1$ if $y \in R[x]{\uparrow}$ and $\alpha(y) = 0$ otherwise. $\Box$

We are now ready to state the main result of this subsection. In the presence of the prior results of this subsection (in particular Lemma 5.3.11), the proof is fairly routine.

Theorem 5.3.14 (Algebraic Completeness). *Let* $\psi, \varphi \in \mathsf{Form}^{\Box\Diamond}_{\mathsf{PMV}}$*.*

(1) $\psi \models_n \varphi$ *if and only if* $\mathsf{mPMV}_n \models \psi \leq \varphi$.

(2) In particular, $\varphi \in \Lambda^{\Box\Diamond}_{\mathsf{PMV}_n}$ *if and only if* $\mathsf{mPMV}_n \models \varphi \approx 1$.

Proof. Let $\langle \mathbf{F}, \Box, \Diamond \rangle = \mathsf{Free}_{\mathsf{mPMV}_n}(\mathsf{Prop})$ be the free mPMV_n-algebra generated by the countable set of propositional variables Prop. The *canonical* Pos_n*-model* is based on its canonical frame $\langle \mathbf{F}, \Box, \Diamond \rangle^+$ together with the *canonical* Pos_n*-valuation* defined by

$$\mathsf{Val}^{\mathrm{c}}(u, p) = u\big([p]\big),$$

where $[p]$ denotes the equivalence class of p. By Proposition 5.3.10, we know that $\langle \mathbf{F}, \Box, \Diamond \rangle^+$ really is a Pos_n-frame. Furthermore, we can easily check that the canonical Pos_n-model really is a Pos_n-model (Definition 5.3.6) since

- If $u \leq^{\mathrm{pw}} v$, then $\mathsf{Val}^{\mathrm{c}}(u, p) = u([p]) \leq v([p]) = \mathsf{Val}^{\mathrm{c}}(v, p)$.
- $\mathsf{Val}^{\mathrm{c}}(u, p) = u([p]) \in \mathsf{im}(u)$.

Using the Truth Lemma (Lemma 5.3.11), it is easy to verify that the property

$$\mathsf{Val}^{\mathrm{c}}(u, \varphi) = u\big([\varphi]\big)$$

extends to all modal PMV-formulas $\varphi \in \mathsf{Form}^{\Box\Diamond}_{\mathsf{PMV}}$.

Now suppose that $\mathsf{mPMV}_n \not\models \psi \leq \varphi$. Then in particular $[\psi] \leq [\varphi]$ does not hold in $\mathbf{F}$. Thus, there exists a homomorphism $u \colon \mathbf{F} \to \mathbf{P Ł}_n$ with $u([\psi]) = 1$ and $u([\varphi]) = 0$. This means that $\mathsf{Val}^{\mathrm{c}}(u, [\psi]) = 1$ and $\mathsf{Val}^{\mathrm{c}}(u, [\varphi]) = 0$, witnessing that $\psi \not\models_n \varphi$ in the canonical Pos_n-model. $\Box$

For the minimal $\mathbf{P Ł}_n$-valued modal logic, the additional structure of Pos_n-frames (over that of Set-frames) is irrelevant (which is shown similarly to Proposition 3.2.3). However, in the next section we study a consequence pair which is only canonical with respect to the semantics over Pos_n.

5.3.4 A case study in canonicity

In this subsection, we give an example of a consequence pair which is canonical with respect to the semantics over Pos_n-frames but isn't with respect to either of the semantics over Pos-frames or Set_n-frames.

From now on, for simplicity we assume that n is a prime number, which implies $\mathbb{S}(\mathbf{P Ł}_n) = \{\mathbf{P Ł}_n, \mathbf{P Ł}_1\}$. We consider the consequence pairs

$$\Box(x \oplus x) \vdash \Box x \text{ and } \Diamond(x \oplus x) \vdash \Diamond x.$$

As shown in [HT13, Example 8.28], over MV_n (*i.e.*, with negation) the formula $\Box(x \oplus x) \to \Box x$ is canonical and defines the class of Set_n-frames in which '*all successors are crisp*'. In particular, this means that the formula $\Diamond(x \oplus x) \to \Diamond x$ which defines the same class of Set_n-frames is derivable from the corresponding axiomatic extension of modal logic over MV_n. In this section, we show that in modal logic over PMV_n this is not the case any more. In the following, we identify the classes of Pos_n-frames which are defined by the above consequence pairs. As it turns out, the former consequence pair defines Pos_n-frames in which '*every successor has a crisp successor of the same element below*' while the latter one defines the ones in which '*every successor has a crisp successor of the same element above*'. Note that a Pos_n-frame $(X, \leq, v, R)$ can be identified with a first-order structure where v is described by unary relations for all $\mathbf{P Ł}_k \in \mathbb{S}(\mathbf{P Ł}_n)$.

Proposition 5.3.15. *Let $\mathfrak{F} = (X, \leq, v, R)$ be a Pos_n-frame.*

(1) The consequence pair $\Box(p \oplus p) \vdash \Box p$ is valid in $\mathfrak{F}$ if and only if $\mathfrak{F}$ satisfies

$$\forall x \forall y \colon \big(xRy \to \exists y' : (xRy' \wedge y' \leq y \wedge v(y') = \mathbf{P Ł}_1)\big).$$

(2) The consequence pair $\Diamond(p \oplus p) \vdash \Diamond p$ is valid in $\mathfrak{F}$ if and only if $\mathfrak{F}$ satisfies

$$\forall x \forall y \colon \big(xRy \to \exists y'' : (xRy'' \wedge y \leq y'' \wedge v(y'') = \mathbf{P Ł}_1)\big).$$

Proof. Starting with the proof of (1), assume that $\mathfrak{F}$ satisfies the first-order condition in the statement and let $(X, \leq, v, R, \mathsf{Val})$ be an arbitrary Pos_n-model based on $\mathfrak{F}$ which satisfies $\mathsf{Val}(x, \Box(p \oplus p)) = 1$ and $R[x] \neq \varnothing$ (otherwise, the consequence pair is trivially satisfied at x). If $y \in R[x]$, there is some $y' \in R[x] \cap y{\downarrow}$ with $v(y') = \mathbf{P Ł}_1$. Then it must hold that $\mathsf{Val}(y', p) = 1$ (since otherwise $\mathsf{Val}(y', p) = 0$ implies $\mathsf{Val}(x, \Box(p \oplus p)) = 0$). Since $y' \leq y$, this now implies $\mathsf{Val}(y, p) = 1$ as well. Since y was an arbitrary successor of x, we get that $\mathsf{Val}(x, \Box p) = 1$ as desired.

For the converse, assume that $\mathfrak{F}$ does not satisfy the first-order formula of the statement. In other words, there are xRy such that no xRy' with $y' \leq y$ satisfies $v(y') = \mathbf{P Ł}_1$. Then choose some $\ell \in \mathbf{P Ł}_n$ with $\frac{1}{2} \leq \ell < 1$ and consider any Pos_n-valuation which satisfies

$$\mathsf{Val}(w, p) = \begin{cases} \ell & \text{if } xRw \text{ and } w \leq y \\ 1 & \text{if } xRw \text{ and } w \not\leq y. \end{cases}$$

The Pos_n-model thus arising witnesses $\mathsf{Val}(x, \Box(p \oplus p)) = 1$ but $\mathsf{Val}(x, \Box p) = \ell$, which shows that the consequence pair is not valid in $\mathfrak{F}$.

To prove statement (2) of the proposition, first suppose that $\mathfrak{F}$ satisfies the first-order formula in the statement, and suppose $(X, \leq, v, R, \mathsf{Val})$ is a Pos_n-model based on $\mathfrak{F}$ with $\mathsf{Val}(x, \Diamond(p \oplus p)) = 1$. Then there exists some successor $y \in R[x]$ with $\mathsf{Val}(y, p) \geq \frac{1}{2}$. By our assumption, there exists another successor $y'' \in R[x]$ with $y \leq y''$ and $v(y'') = \mathbf{P\L}_1$. Since $\mathsf{Val}(y, p) \neq 0$ and y'' is crisp, we necessarily have $\mathsf{Val}(y'', p) = 1$, which implies $\mathsf{Val}(x, \Diamond p) = 1$ as desired.

For the converse, assume that the $\mathfrak{F}$ does not satisfy the first-order condition in the statement. Then there exist some xRy such that all $y' \in R[x]$ with $y \leq y'$ satisfy $v(y') = \mathbf{P\L}_n$. Now pick some $\ell \in \mathbf{P\L}_n$ with $\frac{1}{2} \leq \ell < 1$ and consider any valuation which satisfies

$$\mathsf{Val}(y', p) = \begin{cases} \ell & \text{if } xRy' \text{ and } y \leq y' \\ 0 & \text{if } xRy' \text{ and } y \not\leq y'. \end{cases}$$

The Pos_n-model thus arising witnesses $\mathsf{Val}(x, \Diamond(p \oplus p)) = 1$ but $\mathsf{Val}(x, \Diamond p) = \ell$, which shows that the consequence pair is not valid in $\mathfrak{F}$. $\Box$

The assumption that n is prime really is needed for the above. For example, the Pos_6-frame depicted in Figure 5.3, consisting of x with $v(x) = \mathbf{P\L}_6$ and $R[x] = \{x_i \mid i \in \mathbb{N}\}$ with (x_i) being a decreasing sequence with $v(x_i)$ alternating between $\mathbf{P\L}_3$ (for odd i) and $\mathbf{P\L}_2$ (for even i) has no crisp states and still satisfies $\Box(p \oplus p) \vdash \Box p$. The latter holds because whenever $\mathsf{Val}(x_i, p) \neq 1$ for some i, then $\mathsf{Val}(x_{i+4}, p) = 0$. However, if n is not prime, then Proposition 5.3.15 can simply be adapted with the condition that '*every successor state y has a successor state y' of the same element below such that $v(y)$ and $v(y')$ correspond correspond to coprime divisors of n*'.

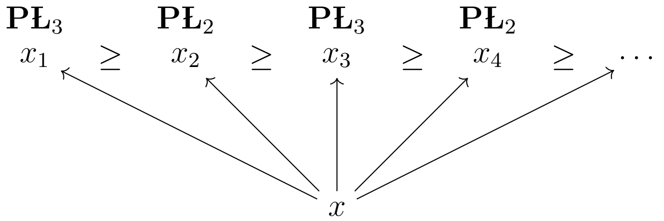


Figure 5.3: A Pos_6-frame satisfying $\Box(p \oplus p) \vdash \Box p$ without crisp states.

To conclude this subsection, we show that the consequence pair $\Box(p \oplus p) \vdash \Box p$ is canonical with respect to Pos_n-frames. Together with the results from the previous subsection, this is a consequence of the following proposition.

Proposition 5.3.16. *Let* $\langle \mathbf{A}, \Box, \Diamond \rangle$ *be a modal* PMV_n*-algebra which satisfies the equation* $\Box(x \oplus x) \leq \Box x$*. Then the canonical frame* $\langle \mathbf{A}, \Box, \Diamond \rangle^+$ *satisfies the first-order condition from Proposition 5.3.15.*

Proof. The idea is similar to the proof of part two of Proposition 5.3.10. Suppose towards contradiction that there are $u, u' \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_n)$ with $uR^{\mathrm{m}}u'$ and $R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_1) \cap u'{\downarrow} = \varnothing$. Consider the closed sets

$$F = R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{PŁ}_1) \text{ and } G = R^{\mathrm{m}}[u] \cap u'{\downarrow}.$$

As in the proof of Proposition 5.3.10, we can separate these sets by a clopen downset $\Omega \supseteq G$ with $\Omega \cap F = \varnothing$. Thus, from $a \in \mathbf{A}$ with $u'(a) = \frac{1}{n}$ (which exists because $\mathsf{im}(u') = \mathbf{PŁ}_n$), we can construct a witness for $u(\Box a') = u'(a) = \frac{1}{n}$, which yields $u(\Box(a' \oplus a')) > u(\Box a')$, contradicting the initial assumption that $\Box(a' \oplus a') \leq \Box a'$ holds in $\mathbf{A}$. $\Box$

Together with Proposition 5.3.15 and the algebraic treatment from Subsection 5.3.3, this immediately implies that the consequence pair $\Diamond(p \oplus p) \vdash \Diamond p$ is not derivable in the corresponding axiomatic extension. While the semantics over Pos_n reflect this situation adequately, the semantics restricted to Set, Set_n or Pos all fail to do so.

5.4 Conclusion of Chapter 5

In the first part of this chapter, we developed a logarithmic optimal natural duality for the variety PMV_n of positive MV_n-algebras, generated by $\mathbf{PŁ}_n$, the negation-free reduct of the finite MV-chain $\mathbf{Ł}_n$. We explored the relationship between this duality and Priestley duality, showing that there is an adjunction between DL and PMV_n given by the distributive skeleton functor $\mathfrak{S}\colon \mathsf{PMV}_n \to \mathsf{DL}$ and the Priestley power functor $\mathfrak{P}\colon \mathsf{DL} \to \mathsf{PMV}_n$. Specializing this relationship to Boolean powers, we gave a full characterization of algebraically and existentially closed members of PMV_n.

In the second part of this chapter, we introduced the positive fragment of $\mathbf{Ł}_n$-valued modal logic and studied it via its algebraic counterpart, modal PMV_n-algebras. We introduced relational semantics on Pos_n-frames for this logic and showed that canonical frames of modal PMV_n-algebras are Pos_n-frames. In an example, we illustrated how these richer semantics are 'well-behaved' in the context of canonicity. Note that for our semantics on Pos_n-frames, we only used a 'fragment' of the structure of the dual categories obtained in Section 5.1. It would be interesting to obtain a full 'Jónsson-Tarski style' duality between the variety mPMV_n and a category of adequately defined binary relations on $\mathcal{X}_n$-objects (generalizing the duality for mDL established in [CJ99]). On the logical side, we also call for a more thorough study of canonicity and definability over Pos_n-frames, and the study of positive fragments of modal logics over *infinite* algebras of truth-degrees.

In the remainder of this subsection, we go into slightly more detail on two further open topics for future research related to this chapter. In particular, we focus on possible generalizations of the results of this chapter from $\mathbf{PŁ}_n$ to algebras which we call 'strongly lattice-semi-primal' (Subsection 5.4.1) and to coalgebraic logic over lattice-primal varieties (Subsection 5.4.2).

5.4.1 Towards a notion of lattice-semi-primality

Note that many of the results of Section 5.1 do not only hold for $\mathbf{PŁ}_n$, but for any algebra $\mathbf{L}$ which satisfies the following property. Our terminology is motivated by the characterization of semi-primality of Theorem 1.1.12 and the definition of lattice-primal algebras (recall Definition 1.1.13).

Definition 5.4.1 (Strongly lattice-semi-primal algebra)**.** A *strongly lattice-semi-primal algebra* is an algebra $\mathbf{L}$ which is based on a bounded lattice, such that every primitive operation of $\mathbf{L}$ is order-preserving and such that, for every $\ell \in \mathbf{L}$, the unary operation τ_ℓ is term-definable in $\mathbf{L}$.

The justification for this name is that strong lattice-semi-primality is slightly stronger than the following generalization of Definition 1.1.5. To the best of the author's knowledge, the only other prior notion of order-semi-primality in the literature is found in [Sch94]. However, the notion therein seems too restrictive, as it does not account for potential subalgebras of $\leq$, as prominently featured in this chapter. Therefore, we propose the following notion of (weak) lattice-semi-primality.

Definition 5.4.2 (Weakly lattice-semi-primal algebra)**.** A *weakly lattice-semi-primal algebra* is an algebra $\mathbf{L}$ which is based on a bounded lattice, such that every primitive operation of $\mathbf{L}$ is order-preserving and such that every operation $f\colon L^n \to L$ which preserves subalgebras of $\mathbf{L}$ and $\leq$ is term-definable in $\mathbf{L}$.

This weak form of lattice-semi-primality also arises as the following generalization of the 'Baker-Pixley' characterization of semi-primality from Theorem 1.1.11(2).

Proposition 5.4.3. *Let $\mathbf{L}$ be a finite algebra based on a bounded lattice, all of whose primitive operations are order-preserving. Then the following are equivalent.*

(i) $\mathbf{L}$ *is weakly lattice-semi-primal.*

(ii) Every subalgebra of $\mathbf{L}^2$ *is either a product of subalgebras of* L*, a subalgebra of* $\leq$ *or a subalgebra of* $\geq$*.*

With this characterization, it is also easy to see that strong lattice-semi-primality implies weak lattice-semi-primality, by a starightforward adaptation of the proof of Lemma 5.1.9. In fact, the exact relationship between the two notions is the following.

Fact. A weakly lattice-semi-primal algebra $\mathbf{L}$ is strongly lattice-semi-primal if and only if every non-diagonal subalgebra of $\mathbf{L}^2$ contains $(0, 1)$ or $(1, 0)$.

As mentioned above, natural dualities similar to the ones from Section 5.1 can be established for all strongly lattice-semi-primal algebras. Furthermore, it is easy to see that every such algebra becomes lattice-primal after adding truth-constants for every element. Thus, they become subject to the study sketched in the following subsection.

5.4.2 Towards many-valued positive coalgebraic logic

In this subsection, we give an outline of how to carry out research similar to the one conducted in Section 4.2, moving from primal varieties to lattice-primal varieties. In particular, this covers the case of $\mathbf{PŁ}_n$ expanded with a constant $\hat{\ell}$ for every $\ell \in \mathbf{PŁ}_n$.

First, we note that for every lattice-primal algebra $\mathbf{L}$, the duality from [Qua79b] (Theorem 1.1.14) between $\mathbb{HSP}(\mathbf{L})$ and DL can be obtained by combining the natural duality from the NU-Strong Duality Corollary (Corollary 5.1.1) between $\mathbb{HSP}(\mathbf{L})$ and Priest with the *Priestley skeleton* and *Priestley power* functors $\mathfrak{S}$ and $\mathfrak{P}$ which are defined analogously to Definitions 5.2.3 and 5.2.5, except that we again use T_1 to define $\mathfrak{S}$ as in the semi-primal case.

Now, coalgebraic logics over the dual adjunction between DL and Pos, as for example studied in [BKV13, DK17], can be lifted to ones on $\mathbb{HSP}(\mathbf{L})$ completely analogous to Definition 4.2.3. The proof of inheritance of one-step completeness and expressivity (Theorem 4.2.4) can be easily adapted to this case.

In particular, (two-valued) positive modal logic for the convex powerset functor $\mathcal{P}^{\mathrm{cv}} \colon \mathsf{Pos} \to \mathsf{Pos}$ corresponds to the functor $\mathsf{L}_{\mathrm{m}} \colon \mathsf{DL} \to \mathsf{DL}$ which carries the presentation of modal distributive lattices (Definition 5.3.1). The corresponding lifting $\mathsf{L}'_{\mathrm{m}} \colon \mathbb{HSP}(\mathbf{L}) \to \mathbb{HSP}(\mathbf{L})$ again has a presentation by the same equations together with $\tau_\ell(\Box x) = \Box\tau_\ell(x)$ and $\eta_\ell(\Diamond x) = \Diamond\eta_\ell(x)$.

The case where $\mathbf{L}$ is (strongly) lattice-semi-primal seems to be more involved, since the dual category $\mathcal{X}$ of $\mathbb{HSP}(\mathbf{L})$ is now more complicated and it is not obvious whether a similar representation of objects of $\mathcal{X}$ from ones of Priest (similar to the coend representation in the semi-primal case) can be obtained here as well. We leave this question open for future research.

Another potential approach to positive modal logic over lattice-semi-primal varieties would be to mimic the approach of [DK17], where, *e.g.*, positive modal logic is obtained as the *positivication* of classical modal logic. Similarly, one might for example study positivications of coalgebraic logics over $\mathbf{Ł}_n$ to obtain ones over $\mathbf{PŁ}_n$. Similarly to [DK17], this would likely involve the use of *enriched category theory*.

Conclusion

> *I think I now have explained sufficiently the mathematical aspect of the situation and can turn to the philosophical implications. Of course, in consequence of the undeveloped state of philosophy in our days, you must not expect these inferences to be drawn with mathematical rigour.*
>
> – KURT GÖDEL
> (1951)[1]

In this thesis, we studied the category-theoretical relationship between semi-primal varieties and the variety of Boolean algebras (Chapter 2) and utilized various aspects of their close correlations in order to systematically study modal logic over semi-primal algebras (Chapter 3) and, more generally, coalgebraic logics over semi-primal varieties (Chapter 4). We showed how these many-valued coalgebraic logics can arise as liftings of classical coalgebraic logics, in which case important properties regarding completeness, expressivity and definability are inherited. Lastly, specializing to the case of finite MV-chains and based on techniques from natural duality theory, we studied the positive (*i.e.*, negation- and implication-free) fragment of the many-valued modal logic by algebraic means (Chapter 5).

The individual chapter conclusions (Sections 1.3, 2.4, 3.4, 4.4 and 5.4) already contain many explicit open questions which indicate various possible directions for future research related to the respective chapters. The nature of the following concluding remarks may be considered more 'philosophical' or 'speculative'. Indeed, in the first part of this conclusion we discuss the results of the thesis in the context of the philosophical debate around 'Suszko's thesis' on the relationship between two-valued and many-valued logic, in the second part we discuss the potential generalizations to the infinite case and in the last part we move on to give some ideas for possible applications of this research in theoretical computer science.

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[1] [Gö51, p.311]

Two-valued versus many-valued

A recurring theme throughout this thesis is the relationship between classical (two-valued) logic and many-valued logic. This relationship has also been discussed in a 'philosophical' debate around what is usually referred to as *Suszko's thesis* (see, *e.g.*, [WS08]). Based on Suszko's (rather polemic) paper [Sus77], Suszko's thesis essentially says that there are no logical values beyond 'true' and 'false', and that many-valued systems still are bivalent in some sense. This claim is usually backed up by the fact that any many-valued structural consequence relation has an equivalent two-valued one, based on the (two-valued) distinction between designated and non-designated elements of truth-degrees (this is often called *Suszko's reduction*). However, the actual 'effectiveness' of this reduction has been challenged (see, *e.g.*, [Mal94, Tsu98, WS08]). Without getting into the philosophical debates around the 'nature' or even 'existence' of many-valuedness, we remark that the results of this thesis indicate that an effective reduction of a many-valued (modal) logic to classical (modal) logic can be obtained at least if the algebra of truth-degrees is semi-primal. Since these algebras of truth-degrees are precisely the ones in which every potential collection of designated elements can be defined internally as a crisp predicate, the converse might also hold. To make these thoughts more 'mathematically precise', one would first have to clarify what is precisely meant by an *effective reduction* of a many-valued logic to two-valued logic. To this end, it could be interesting to re-investigate what is described in Chapter 2 with the interpretation of adjunctions between varieties of algebras as *translations* between their respective logics as described in [Mor18]. A translation of finitely-valued modal logic to classical modal logic is also used in the recent paper [BCN23], where it is shown that finitely-valued and two-valued modal definability (of crisp Kripke frames) coincide in many cases (also recall Corollary 3.3.25 and its coalgebraic generalization Corollary 4.3.34). This translation (albeit defined 'semantically') highly resembles the construction of the Boolean power (extended to modalities), and is used to show that every class of frames definable by a finitely-valued modal logic is also classically definable [BCN23, Theorem 5]. It is also shown that the converse holds whenever the algebra of truth-degrees 'interprets a Boolean algebra' [BCN23, Theorem 10]. In the case of semi-primal algebras, this is likely to correspond to the Boolean skeleton. It might more generally correspond to the existence of an adjoint of the Boolean power functor. For now, we end this section with the (philosophical) conjecture that a finitely-valued modal logic can be effectively reduced to classical modal logic if and only if its algebra of truth-degrees is semi-primal. Likely more challenging seems the question of the relationship (and possible effective reductions) of infinitely-valued (modal) logics to classical (modal) logic. We discuss some potential approaches to generalizations of results of this thesis in the infinite case in the next section.

From finite to infinite

In this thesis, taking the first steps towards many-valued coalgebraic logic via the 'abstract algebra-coalgebra' approach, we focused on finite 'well-behaved' algebras of truth-degrees. In the author's opinion, the most important (long-term) generalization following up this research would be to entail *fuzzy modal logics* based on the real interval $[0,1]$, like infinite Łukasiewicz, Gödel and product logic [Há98] as well. In particular, analysing the case of *infinitely-valued Łukasiewicz logic* based on the standard MV-algebra $[0,1]_{\mathbf{Ł}}$ seems like a promising endeavour. This algebra still has similarities with the finite MV-chains and may be considered 'well-behaved' in some regards. Thus, it might also be connected to the search for 'appropriate' analogues of (semi-)primality for *infinite algebras*. While a direct adaptation of the 'term-definability' definition of (semi-)primality in the infinite case necessarily requires an infinite signature (this is, for instance, investigated in [vN14] in the case of primality), in the author's opinion, it would be interesting to find some analogues of variations of primality which work for infinite algebras with finite signatures. In particular, motivated by the examples $\mathbf{Ł}_n$, is there a generalization of semi-primality which applies to the standard MV-chain $[0,1]_{\mathbf{Ł}}$? For example, while it is not possible to define the unary operations τ_ℓ anymore in this algebra, it is still possible to find *separating terms*, in the sense that for all distinct $r_1, r_2 \in [0,1]$, there exists a unary MV-term $T_{r_1,r_2}(x)$ with $r_1 \mapsto 1$ and $r_2 \mapsto 0$. This 'term-separation' property (something similar is also occurring in [MM18]), which is equivalent to the characterization of semi-primality of Theorem 1.1.12 in the finite case, might also be what is required in the infinite case. One obstacle here may be the lack of a full topological duality for the variety MV, although dualities for subcategories of MV could already suffice. In this context, it might also be worthwhile to investigate the *bounded Boolean power* construction, which is usually used for infinite algebras [Bur75]. To summarize, for future research we propose to extract the precise algebraic ingredients necessary to extend the research of this thesis to infinite algebras, in particular to encompass modal logic over the standard MV-chain as algebra of truth-degrees from [HT13] to the level of coalgebraic logic via the abstract algebra-coalgebra approach. This could also open up some further potential applications in theoretical computer science, as discussed in the following section.

Applications in computer science

Although the author's motivation for this research was mostly 'purely mathematical', it does have some promising applications. For example, the coalgebraic generalization broadens the range of applications of many-valued modal logic in modelling knowledge in the presence of multiple experts [Fit92], fuzzy preferences [VEG20] or coalitional power [KT17]. In general, the applications of non-classical logic in computer science are far reaching, for example including artificial intelligence and natural language processing, cyber-physical systems, formal verification and hardware design. Finitely-valued logic is of interest in the representation of approximate or vague knowledge and decision-making, as well as in the design and verification of logic circuits with multiple states (see, *e.g.*, [Rin77, BB03]). Coalgebras are, for example, convenient to deal with transition systems and observational behaviour, infinite data-structures and coinduction (see, *e.g.*, [Rut00, Jac16]). Modal logic (in a broad sense) offers expressive, yet decidable logical systems. In computer science, they are most notably used for program verification via various fragments of the modal μ-calculus like propositional dynamic logic or linear temporal logic (see, *e.g.*, [BS07]). Not surprisingly, coalgebraic investigations of these topics have also emerged in recent years (see, *e.g.*, [SV10, Cîr11, HK15, ESV19]). For future work building on this thesis, the author plans to furhter extend its results to these coalgebraic fixpoint logics, giving rise to coalgebraic treatments of many-valued fixpoint logics, for example many-valued propositional dynamic logic [Teh14, Sed21] or linear temporal logic. There are also interesting connections of this topic to automata theory to be explored, for example in the case of linear temporal logic and Büchi automata [Var96]. In the future this research program could, for example, lead to novel tools in model checking and verification in settings where it is important to take errors or uncertainty into account.

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