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Optimal strategies of critical mineral depletion and recycling☆

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ABSTRACT

Critical minerals play a crucial role in making eco-friendly and sustainable technology a reality. Yet, these essential minerals face challenges such as disruptions in their supply chain, dwindling availability, limited recycling methods, and uncertain demand, which may increase in the short term. In this study, we explore how we can better extract and recycle these important minerals through international cooperation and precommitment among different countries.

Our findings highlight a few key points: (1) While recycling technology can help reduce reliance on virgin minerals, it cannot fully replace the need for them, as recycling still depends on limited resources. (2) When deciding how to distribute these minerals in the market, international cooperation should consider both virgin and recyclable sources, prioritizing the more globally desirable option. (3) If the recyclable sources are being used up, it becomes challenging for a supranational decision-maker to decide how to best use the remaining virgin minerals. (4) With precommitment from both virgin resource suppliers and recyclers, both recycled and virgin resources can be used together until the virgin ones are completely used up.

In simpler terms, securing these crucial minerals for sustainable technology involves complex decisions about recycling, resource allocation, and cooperation among nations.

1. Introduction

The term "critical minerals" usually refers to the raw materials that are essential for modern technologies, particularly in the context of the transition to clean and sustainable technology (Pommeret et al., 2022). Examples of critical minerals include rare earth elements, which are used in the manufacturing of magnets for wind turbines, electric vehicles, smartphones and computers; lithium, cobalt, and graphite, which are essential for the production of batteries; and platinum group metals, which are used in catalytic converters in vehicles in order to reduce CO_2 emissions. The European Union (EU) has defined more than 30 critical raw materials, including lithium, cobalt, and rare earth elements. The United States has a similar definition as well.

In recent years, driven by the growth of renewable energy and electric vehicles, the demand for critical minerals has increased enormously, and at least in the near future, the trend of increasing demand will continue. An International Energy Agency (IEA, 2022) report estimates that by 2040, in order to meet the needs of clean energy technology, demand for critical minerals such as lithium, cobalt, nickel, and rare earth elements could be six times that of 2020.[1] The International Energy Agency's Sustainable Development scenario estimates that nickel demand for use in batteries for electric vehicles and backup energy storage for variable renewable electricity will grow from 196,000 tons in 2020 to 3,804,000 tons by 2040.

Given the unique properties of critical minerals, substitutions are difficult to find and reserves in the Earth's crust are limited, which makes their supply chains vulnerable to disruptions. Additionally, most

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[1] The same IEA report states that "*In a scenario that meets the Paris Agreement goals (as in the IEA Sustainable Development Scenario), their share of total demand will rise significantly over the next two decades, to over 40% for copper and rare earth elements, 60%–70% for nickel and cobalt, and almost 90% for lithium*". Additional details can be found in https://www.iea.org/reports/the-role-of-critical-minerals-in-clean-energy-transitions.

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critical minerals are concentrated in a few countries, and extraction is in the hands of just a few companies. For example, China is the world's largest producer and exporter of rare earth elements and has the largest reserves. China also has a dominant position in the production of lithium and cobalt ((IEA, 2022) special report). With large deposits of lithium, cobalt, and rare earth elements, Australia and Canada are also becoming significant producers of critical minerals. China and Russia supplied about two thirds of the total global supply of titanium in 2020 (USGS, 2020). Large international companies from various countries are also competing in Africa and South America for the large deposits of critical minerals found there. However, the development and technological progress of some countries is heavily reliant on imports from critical-mineral-rich countries and the large companies that operate in this field. The US imports about 100% of 17 important minerals and over 50% of 46 other minerals. The EU faces a similar situation: "the EU is a global manufacturing leader for products like automotive traction motors and wind turbines, it does not produce any rare earth elements itself. Of its total rare earth magnet demand, 98% is met by Chinese imports"; "Seventy-eight percent of the EU's lithium supply comes from Chile, which provides 44% of global supply".[2] Although the lithium supply from Imerys, France, which will begin in 2028, promises to reduce the EU's dependency on lithium imports, this does not significantly change the whole situation.

Reducing supply chain vulnerability and ensuring the secure supply of critical minerals has become an important issue for many countries. Not surprisingly, in the coming years we expect to see more intense competition for critical minerals among countries. Some countries and international organizations have already taken actions to reduce their dependency on imports and to promote market supply. Among these policies, recycling critical minerals from end-of-life products can help ensure a more sustainable supply. For example, both the USA and the EU have established targets for the recycling of critical minerals. Funds and policies are being devised to support the development of a circular economy for critical minerals. Nevertheless, recycling technologies for critical minerals are still in the early stages of development and are too expensive for large-scale commercial use (see Binnemans et al., 2013; Quist-Jensen et al., 2016; Zakotnik et al., 2016; Filippas et al., 2021), among others).[3]

Arguably, in facing the great challenge of reducing air pollution and supply–demand competition for critical minerals, international cooperation and commitments from different countries are crucial to managing their exploitation, distribution, and recycling. Countries should work together to develop strategies for securing the supply of critical minerals to the global market. The United States' Energy Resource Governance Initiative (ERGI) and Global Battery Alliance are examples of these kinds of cooperative efforts.

In the context of cooperation, what are the optimal strategies for the exploitation of virgin resources and the recycling of critical minerals? Would be it optimal to exhaust the global reserve first and then start recycling? And in which sense is this optimal? Furthermore, if countries cannot cooperate, what is the optimal commitment countries can make to support global demand and guarantee the transition to clean energy technology?

To answer these questions, we first propose an optimal solution for international cooperation, considering the minimum market demand for critical minerals and with recycling technology only being available in the future. Then, we propose an international competition model to investigate the optimal precommitment between a critical mineral monopoly exporting country and an importing country that will engage in developing recycling technology.

Our setting differs from the existing literature in three different ways. Firstly, critical mineral recycling is different from the classical economics literature on recycling such as Gaskins (1974), Swan (1980), Martin (1982), and Weinstein and Zeckhauser (1974), among others, which study the situation where recycling technology is already in use when the competition starts. Furthermore, this branch of the recycling literature considers competition between the virgin supplier and the recycling sector in the same economy, whereas (as noted above) the concerns regarding supply chains for critical minerals largely stem from the competition between the exporting and importing countries.

Secondly, our framework aligns with the research of Weikard and Seyhan (2009), Seyhan et al. (2012), and Kleemann et al. (2015), who extensively explore the intricacies of phosphorus recycling and distribution. Phosphorus stands as a pivotal fertilizer for agricultural land, its supply reliant upon non-renewable and constrained raw materials - phosphate rocks. With the exception of Kleemann et al. (2015), the mentioned papers use a Hotelling-type model with a finite amount of primary resource, which is also a crucial aspect we address by incorporating the initial depletable reserve as one of our constraining factors. Hoogmartens et al. (2018) have conducted a comprehensive numerical exploration of optimal extraction policies for non-renewable resources. However, their study focuses exclusively on non-competitive market scenarios, incorporating the presence of recycling technology and substitutes. These attributes diverge from the distinct landscape presented by critical minerals. In the context of critical minerals, Weigl and Young (2023) have recently made a valuable contribution by utilizing a system dynamics model—the Lithium-Ion Battery Resources Assessment—to forecast lithium-ion battery recycling trends in the United States. Yet, even in this insightful work, there is no accounting for the dynamics of market competition and constraints posed by virgin resources.

Thirdly, our analysis is also different from the literature on backstop (substitute) technology à la Dasgupta and Stiglitz (1980), Stiglitz and Dasgupta (1982), and Dasgupta et al. (1983) (among other works). In their setting, the backstop technology can support the market alone, with pricing being independent from the competing non-renewable resource. Critical mineral recycling essentially depends on the supply of virgin resource, and the price from recycling either follows the market price of the monopoly supply of the virgin resource, if the recycling sector is small or competitive, or forms a duopoly competition if the recycling sector is relatively large.

The remainder of the paper is organized as follows. Section 2 presents the model of the monopoly supply of critical virgin minerals and potential recycling by the importing country. Section 3 presents the cooperation situation where there is a supranational institute that can design the optimal supply of critical minerals. Our analysis shows that in the context of increasing exploitation costs, the updating optimal supply indicates that the more socially desirable resource — that is, the cheaper one — should be used first if neither of these two resources are being exhausted. When the recyclable resource is being exhausted, the exploitation is alternating between the two resources. More precisely, when recycling is more globally desirable than exploitation, in which case the exploitation of original natural reserves should stop, and market demand should be satisfied purely by recycling until the recyclable reserve is exhausted. Then, the not-yet-exhausted natural reserve should reopen to support the market. In this case, the supranational policy maker could face a situation of no optimal supply to the market, except switching between the virgin and the recyclable resource. Simultaneously exploiting the virgin resource and recycling is possible only when supply from the virgin resource is more socially desirable than recycling and the virgin resource is nearly exhausted. In this scenario, the virgin resource is not sufficient to satisfy market demand and recycling must be triggered.

[2] See the European Commission's website at https://single-market-economy.ec.europa.eu/sectors/raw-materials/areas-specific-interest/rare-earth-elements-permanent-magnets-and-motors-en.

[3] Binnemans et al. (2013) provide a critical review of rare earths recycling. They conclude that "...up to 2011 less than 1% of the REEs were actually recycled". Additionally, Yanamandra et al. (2022) estimate that in the US and the EU only one percent of lithium-ion batteries get recycled.

Section 4 introduces a duopoly Nash competition between the exporting and importing countries. The precommitted Nash equilibrium for the duopoly game is presented in Section 5. Our model highlights the exact time when the virgin reserve is exhausted, when the minimum input level is reached, and when there is no more resource to be employed after repeated recycling. Depending on the combination of parameters, it is possible that minimum supply is the optimal choice. Otherwise, the market supply first monotonically decreases until it reaches the minimum requirement level. From then on, supply is at the minimum level required (constant, decreasing, or increasing, depending on the market). Under precommitment, the co-existence of the two resources supplying the market is possible. Finally, Section 6 offers some concluding remarks.

2. The model

Let $S(t)$ and $x(t)(\geq 0)$ be the reserve and depletion, respectively, of a non-renewable critical mineral at time t, with known initial reserve $S_0(>0)$ and

$$\dot{S}(t) = -x(t); \tag{1}$$

$$S(t) = S_0 - X(t) = S_0 - \int_0^t x(\tau)d\tau (\geq 0),$$

where $X(t) = \int_0^t x(\tau)d\tau$ is the accumulated depletion until t. Let $y(t)$ be the recycling of the critical mineral at t that satisfies

$$0 \leq Y(t) = \int_0^t y(\tau)d\tau \leq \eta X(t), \tag{2}$$

where $\eta > 0$ is a positive constant. Following the argument in Weinstein and Zeckhauser (1974), Andre and Cerdra (2006), Hoogmartens et al. (2018), and Ba and Mahenc (2019), not all of the primary resource can be recycled. Nonetheless, as stated by Weinstein and Zeckhauser (1974), p. 76), at the aggregate level the accumulated repeatedly recycled resource could be more than the original depletion, that is, $\int_T^\infty y(\tau)d\tau \geq X(T)$ is possible, where $T \geq 0$ is the moment when recycling starts. Furthermore, in the case of repeated recycling, the maximum available recycled mineral could be

$$X(T)(\bar{\rho} + \bar{\rho}^2 + \cdots + \bar{\rho}^n + \cdots) = \frac{\bar{\rho}}{1-\bar{\rho}} X(T) > X(T), \quad \text{if} \quad 0 < \bar{\rho} < 1,$$

where $\bar{\rho}$ is the share of the recyclable mineral, so that the rest $(1-\bar{\rho})X(t)$ is permanently lost. Obviously, even with a repeated recycling process, the aggregated recyclable resource is limited. Hence, we impose the constraint in (2) for some constant $\eta \leq \bar{\rho}/(1-\bar{\rho})$.

For simplicity, we use a linear-quadratic benefit function:

$$B(x, y) = B_0(x + y) - \frac{(x+y)^2}{2}, \tag{3}$$

where $B_0 > 0$ is a positive constant. The minimum demand

$$x + y \geq x_{min} > 0 \tag{4}$$

indicates the minimum consumption requirement to capture the essentiality of the minerals.

The justification for a minimum demand encompasses several facets: First and foremost, numerous industries — ranging from electronics and renewable energy to telecommunications — rely significantly upon critical minerals for the production of contemporary technologies. As these industries expand, their growth inevitably aligns with the demand for critical minerals, thereby establishing a foundational consumption level that serves as the bedrock of minimum demand. Secondly, the constant emergence of technological innovations consistently propels the need for novel applications of critical minerals. A case in point is the surge in demand for lithium and cobalt, driven by their indispensable roles in energy storage solutions, owing to advancements in battery technology. Thirdly, escalating environmental consciousness and the implementation of stringent regulations have paved the way for cleaner technologies that pivot on critical minerals. The persistence of this demand remains steadfast, constituting the indispensable threshold required to meet prevailing environmental standards. Lastly, but certainly not least, the susceptibilities inherent in supply chains — affected by geopolitical factors or unforeseen events such as the Covid-19 pandemic — can lead to erratic fluctuations in availability. In light of these potential disruptions, the imperative of maintaining a minimum demand becomes paramount, ensuring an uninterrupted supply for industries of utmost significance.

The extraction cost at time t depends on both current extraction $x(t)$ and past accumulated extraction. Following Hotelling (1931, p. 152), " *the accumulated production affects both cost and demand. The cost of extraction increases as the mine goes deeper and durable substances by their accumulated influence the market*". Again for simplicity, we take a linear extraction cost function, following Hotelling (1931, p. 153) and Seyhan et al. (2012), p. 105). Specifically, a linear function $L(X) = \xi X$ transforms the accumulated extraction X as part of the current extraction cost. More precisely,

$$C(X, x, t) = \begin{cases} C_0 e^{-gt} x + \xi X(t) & \text{if} \quad x > 0, \\ 0 \quad \text{if} \quad x = 0, \end{cases} \tag{5}$$

where $C_0 > 0$ is a scaling parameter measuring the starting cost of the extraction, $g \geq 0$ measures cost-saving technological progress, and $\xi \geq 0$ indicates the marginal stock effect. Obviously, as long as there is extraction the accumulated past extraction matters, whereas if there is no extraction there is no cost.

Previous literature, including works by Heal (1976), Hanson (1980), Livernois and Uhler (1987), and Chakravorty and Roumasset (1990), among others, argue that extraction costs tend to increase as reserves within a single deposit are depleted. In simpler terms, extraction costs rise with aggregate extraction. Livernois and Martin (2001), Page 832) utilize a more complex extraction cost function compared to ours. However, their model explicitly accounted for the rise in total and marginal extraction costs due to depletion.

Moreover, as noted by Livernois (2009), Page 23), Hotelling acknowledged the failure of his basic model to capture the tendency for extraction costs to rise as a resource being depleted. This increase can occur both within individual deposits, as firms dig deeper or pressure declines in petroleum reservoirs, and across the industry, as firms prioritize the use of lowest-cost deposits. This phenomenon, often termed the degradation effect or stock effect of extraction, underscores the dynamic nature of extraction costs.

In a pioneering contribution, Slade (1982) developed a modified version of the Hotelling model incorporating both cost-increasing degradation effects and cost-reducing technological advancements. Furthermore, the impact of cost-increasing degradation effects on market price decisions cannot be overlooked.

Furthermore, the exploitation of certain Rare Earth Element (REE) deposits results in radioactive waste, necessitating costly management, with expenses escalating alongside accumulated exploitation.

Similarly, the recycling cost involves[4]

$$R(y, t) = R_0 e^{-\rho t} y, \tag{6}$$

where R_0 is a scaling parameter and $\rho \geq 0$ measures technological progress in recycling.

In the rest of this paper, we assume that initially the extraction cost of the virgin resource is much cheaper than recycling. In other words,

[4] It is evident that our recycling cost function represents the simplest form, featuring a constant marginal cost and overlooking the recyclable material's availability. Weikard and Seyhan (2009), as well as Seyhan et al. (2012), employ more realistic recycling cost functions where the costs depend on both y and x. The earlier work by Weikard and Seyhan (2009) provides additional rationale and support. Nevertheless, for ease of calculation in a game scenario, we opt for a considerably simpler cost function compared to the earlier literature.

there will be some time until $T_i(>0)$ during which exploitation of the virgin resource is the cheaper supply. Here, $T_i(>0)$ is the date when recycling starts, and it is known for all players without uncertainty; $i = b, n$ indicate the benchmark cooperation equilibrium and Nash precommitment equilibrium, respectively.

As in Dasgupta et al. (1983) and Olsen (1988), in this paper we do not consider R&D as a decision variable, given there are already many studies focusing on this topic pioneered by Kamien and Schwartz (1978). Nevertheless, we adopt the notation *time-cost of R&D* to indicate the delay in the development of recycling technology. Furthermore, we assume no uncertainty. Of course, the impact of the uncertainty of the arrival date is not unimportant. Nevertheless, in the current study we focus more on the reactions of different players under different competition in terms of the arrival of the recycling technology, rather than the impact of uncertainty, which deserves a separate study investigating its effect in different market structures.

3. The optimal choices for international cooperation

Suppose there is a supranational policymaker, which we will refer to as an international organization or social planner. This international organization can design (I) the optimal extraction of an exhaustible and recyclable critical mineral to supply the market and satisfy consumer demand, and (II) the invention and innovation dates of recycling technology such that the resource can be used as long as possible.

The present study centers on critical minerals for which recycling technology is either unavailable, such as rare earth elements (REEs) and certain platinum group metals, or available but economically unfeasible for widespread market adoption, as seen with lithium, cobalt, and others (Yanamandra et al., 2022; Jena et al., 2024). Consequently, immediate adoption of recycling measures remains impractical, even for a social planner. However, it is important to note that in this study, we do not address the timing of recycling technology adoption as a strategic decision. Instead, we focus on a predetermined invention date, $T_b > 0$. The optimal control problem of the social planner is choosing the extraction, $x_b(t)$, and the recycling, $y_b(t)$, to achieve maximum joint welfare:

$$\max_{x_b(t),\, y_b(t)} W_B(T_b) = \int_0^\infty e^{-rt}\left[B(x_b + y_b) - C(X_b, x_b) - R(y_b)\right] dt, \tag{7}$$

where the interest rate r is the discount rate and is subject to

$$\begin{cases} \dot{X}_b(t) = x_b(t), \quad x_b(t) \geq 0, \quad \forall t \geq 0, \\ \dot{Y}_b = y_b(t), \quad y_b(t) = 0 \ \text{for} \ 0 \leq t \leq T_b \ \text{and} \ y_b(t) \geq 0 \ \text{for} \ t \geq T_b, \\ x_b + y_b \geq x_{min} e^{mt}, \quad \forall t \geq 0, \\ 0 \leq Y_b(t) \leq \eta X_b(t). \end{cases} \tag{8}$$

Obviously, if T_b is the precise moment when the virgin resource is exhausted, the problem is rather less interesting since it becomes a two-stage optimal control scenario where, in the first stage, the optimal supply comes from the virgin reserve only and in the second period optimal supply depends solely on recycling. Thus, in the following we consider mainly the case where recycling technology is available before the exhaustion of the natural reserve. Therefore, it is interesting to know (1) which source of critical minerals should be used to supply the market — the virgin or recyclable resource, or both — and at what rate; (2) whether the virgin reserve will be exhausted; and most interestingly, (3) what is the optimal strategy, if any, when one of the reserves is being exhausted?

From the social planner's perspective, due to the given benefit functions there is no difference between the extraction of the virgin resource and recycling used mineral, except for the cost differences between extraction and recycling. Thus, the order of employing each of these sources depends on the costs: the extraction cost, the time-cost of R&D for the recycling technology, and the recycling cost.

The rest of this section is organized into two parts. Section 3.1 presents a naive solution where the international organization, as supranational policymaker, only supplies the market according to the minimum market demand. Although the minimum level may not be globally optimal, the minimum supply provides a structure with which to analyze the optimal choice. In Section 3.2, we provide conditions under which the minimum supply (at least during some period) is globally optimal, without taking into account cost-saving technology. Furthermore, conditions under which different kinds of resources should be used are investigated.

3.1. The naive solution - minimum supply only

A naive situation is that the supranational policymaker chooses the minimum level of mineral supply: $x + y = x_{\min}$, for all times $t \geq 0$. If the minimum supply is chosen and there is no technological progress, $g = \rho = 0$, it can be designed that for $t \leq T_b$, $x = x_{min}$; after T_b, $y = x_{min}$ and at T_b,

$$X(T_b) = \int_0^{T_b} x(t)dt = x_{min} T_b = S(0).$$

Thus, the virgin resource is exhausted at $T_b = \frac{S(0)}{x_{min}}$ and recycling starts. The first period of recycling ends at

$$T_{b1} = \frac{\bar{\rho} S(0)}{x_{min}} + T_b.$$

Obviously, after the recycled mineral is scrapped, re-recycling is possible. To our knowledge, Weinstein and Zeckhauser (1974) are the first to take into account repeated recycling. Although the aluminum recycling literature is generally implicit about repeated recycling, the delay of recycling is clearly modeled. Swan (1980) assumes that "*...products made of aluminum all have the same lifetime before becoming 'scrap' and therefore available for recycling*" and "*the unit of time is chosen so as to equal the 'average' period of delay between production of virgin and its subsequent conversion into secondary aluminum*". Gaudet and Van Long (2003), among others, adopt the same modeling strategy as Swan (1980). We combine Swan's assumption with the idea of Weinstein and Zeckhauser (1974). The simplest possible model of repeated recycling assumes that scrapped critical mineral (with the same average age) is piled up and waiting to be recycled. Due to permanently lost material and potentially increasing market demand, the amount of scrapped resources decreases over each generation.

Nevertheless, due to the limitation of the initial reserve and there being no substitute critical mineral, even repeated recycling cannot last forever. To see this, recall that the recycling starts at time $T_b = \frac{S(0)}{x_{min}}$ and a first round of recycling generates $\bar{\rho} S(0)$ of the critical material and ends at $T_{b1} = \frac{\bar{\rho} S(0)}{x_{min}} + T_b$. Continuing the same process, clearly the kth round of recycling generates $\bar{\rho}^k S(0)$ of the critical material and ends at $T_{bk} = \frac{\bar{\rho}^k S(0)}{x_{min}} + T_{b(k-1)}$. In the last round of recycling l, we have $x_{min} > \bar{\rho}^l S(0)$, where l is given by the integer part of $\log_{\bar{\rho}} \frac{x_{min}}{S(0)}$, i.e.,

$$l = \lfloor \log_{\bar{\rho}} \frac{x_{min}}{S(0)} \rfloor. \tag{9}$$

The above analysis leads to the conclusion that the recycling technology postpones the exhaustion of the non-renewable resource. Though the postponement could be for a long time period, $\sum_{j=0,}^{k} (T_{b(j+1)} - T_{bj})$ with $T_{b0} = T_b$ and $T_{b(k+1)} = T_{bl}$, in the long run there is no critical mineral to supply to the market, calling for a backstop substitute.

Grosse (2010) estimated the exhaustion time for a given reserve by taking into account the growth rate of demand. However, he did not investigate a repeated recycling scenario. The above analysis confirms Grosse's (2010) finding that recycling technology can postpone the exhaustion of the resource but it cannot be permanently postponed. Of course, given that repeated recycling may significantly increase the usable resource (as indicated in Section 2), the postponement could be much longer than the pessimistic estimation by Grosse (2010).

3.2. Globally optimal supply

In this section, we investigate the optimal international supply and look for a subgame perfect optimal choice. To do so, let $W(X,Y)$ be the value functions for the international organization. Recall that the dynamic equation and constraints are given in (8).

The international organization's optimal control problem faces three additional difficulties compared to standard optimal control problems: (a) one state variable depends on the other, and the boundaries of the choice domain may provide optimal choices; (b) the exploitation cost is not continuous at $x = 0$, so one has to distinguish whether the optimal choice is $x > 0$ or $x = 0$; and (c) there is minimum market demand, which is an extra inequality-control constraint. In order to make the process mathematically clear, in the following we present the Hamilton–Jacobi-Bellman (HJB) equations in detail, as well as their domains.

As mentioned above, we focus on the situation where there is no cost-saving technology available, i.e., $g = \rho = 0$.[5] The optimal control problem becomes autonomous and defined over an infinite time horizon, and then a stationary solution becomes possible.

We begin with definitions and notations. The HJB equation for stationary value function $W(X,Y)$ is

$$rW = Z\left(X, x_b^*, y_b^*\right) + x_b^* W_X + y_b^* W_Y, \tag{10}$$

where

$$\begin{aligned} Z(X,x,y) &= B(x,y) - C(X,x) - R(y) \\ &\equiv B_0(x+y) - \frac{(x+y)^2}{2} - C(X,x) - R_0 y. \end{aligned}$$

In the above HJB equation, $\left(x_b^*, y_b^*\right)$ is an optimal choice and satisfies

$$\left(x_b^*, y_b^*\right) = \underset{(x,y)\in\omega(X,Y)}{\arg\max}\, H\left(X,Y,x,y,W_X,W_Y\right) \tag{11}$$

with the Hamiltonian given by

$$\begin{aligned} H\left(X,Y,x,y,W_X,W_Y\right) &= Z(X,x,y) + \langle W_X, W_Y\rangle \cdot \langle x,y\rangle \\ &= Z(X,x,y) + x_b^* W_X + y_b^* W_Y. \end{aligned}$$

Eq. (10) is solved in the choice domain

$$\overline{\Omega} = \left\{(X,Y) \in \mathbb{R}^2 : 0 \le X \le S_0, \qquad 0 \le Y \le \eta X\right\}, \tag{12}$$

where η is the positive constant in assumption (2). To make the analysis more straightforward, we do not consider repeated recycling in our analysis of optimal choice but impose assumption (2) in the sequel. Intuitively, since there is no clear-cut division between one period of recycling and another, it is more reasonable to simply assume that the available amount for recycling at any time is proportional to the extracted amount at the moment. We use Ω to denote the interior of $\bar{\Omega}$ and let Γ_1 and Γ_2 denote the boundaries

$$\Gamma_1 = \left\{\left(S_0, Y\right) : 0 \le Y \le \eta S_0\right\}, \qquad \Gamma_2 = \left\{(X, \eta X) : 0 \le X \le S_0\right\}. \tag{13}$$

In other words, Γ_1 denotes the situation where the initial natural reserve is exhausted while recycling is still possible. Boundary Γ_2, rather, indicates the situation where there is no recyclable resource available, although the initial reserve may not yet have been exhausted.[6]

As mentioned above, the cost function $C(X,x)$ is discontinuous at $x = 0$. Therefore, the maximization problem (11) should be solved for both $x = 0$ and $x > 0$, finding the larger of the two.

For $x = 0$, there are two possibilities: Either the original reserve is exhausted or the optimal choice is to maintain some reserves by denying supply to the market, given that the extraction cost increases with the extraction and becomes more expensive than recycling. The Hamiltonian in this case is

$$H\left(X,Y,0,y,W_X,W_Y\right) = B_0 y - \frac{y^2}{2} - R_0 y + y W_Y \qquad \text{for } y \ge x_{\min}. \tag{14}$$

Its maximizer is

$$\bar{y}_b^* = \max\left\{x_{\min}, B_0 - R_0 + W_Y\right\}, \tag{15}$$

and the corresponding value of the Hamiltonian is

$$\bar{H}^*\left(t, W_Y\right) = \begin{cases} \frac{1}{2}\left[B_0 - R_0 + W_Y\right]^2 & \text{if } B_0 - R_0 + W_Y > x_{\min}, \\ \left[B_0 - R_0 + W_Y\right] x_{\min} - \frac{x_{\min}^2}{2} & \text{otherwise.} \end{cases} \tag{16}$$

For $x > 0$, it is obvious that the natural reserves of the critical mineral are not yet exhausted. Thus, both the exploitation of the original resource and recycling are possible. To obtain the optimal choice, we let

$$\tilde{C}(X,t,x) = C_0 x + \xi X \qquad \text{for } x \ge 0$$

and

$$\tilde{H}\left(X,Y,x,y,W_X,W_Y\right) = B(x,y) - \tilde{C}(X,x,) - R(y).$$

We also let $\tilde{H}^*\left(X,W_X,W_Y\right)$ denote the maximum value of $\tilde{H}$ in $\omega(X,Y)$. Note that at $x = 0$,

$$\tilde{H}\left(X,Y,0,y,W_X,W_Y\right) = H\left(X,Y,0,y,W_X,W_Y\right) - \xi X < H\left(X,Y,0,y,W_X,W_Y\right).$$

Therefore, the maximum value of $\tilde{H}$ on $\{x = 0\}$ is less than that of H. Thus,

$$\max_{(x,y)\in\omega(X,Y)} H\left(X,Y,x,y,W_X,W_Y\right) = \max\left\{\tilde{H}^*\left(X,W_X,W_Y\right), \bar{H}^*\left(W_Y\right)\right\}.$$

3.2.1. Optimal supply in the interior

Appendix A.1 demonstrates the following interior optimal choices where neither the virgin reserve nor the recyclable resource are exhausted. In order to provide a clearer intuition, consider an optimal choice (x^*, y^*) at an interior point $(X,Y) \in \Omega$. Such a choice may fall into one of three distinct types: (a) Type $(x^*, 0)$, where $x^* > x_{min}$, indicates that it is optimal to prioritize mining and postpone recycling. Despite the existence of recycling technology, its implementation is delayed. (b) Type $(0, y^*)$, with $y^* > x_{min}$, represents the opposite case to (a). Here, it is optimal to halt mining and rely entirely on recycling, signifying the high cost associated with mining. (c) The third type pertains to an interior supply (x^*, y^*) with $x^* > 0, y^* > 0$ and $x^* + y^* \ge x_{min}$.

Proposition 1. *Let T_b be the moment when the recycling technology becomes available. Suppose W is differentiable in Ω, and let $(X,Y) \in \Omega$.*

(2.1) Suppose that

$$W_Y(X,Y) - R_0 < W_X(X,Y) - C_0 \tag{17}$$

and

$$x_b^* = \max\left\{x_{\min}, B_0 - C_0 + W_X(X,Y)\right\} \tag{18}$$

satisfies

$$\tilde{H}\left(X,Y,x_b^*,0,W_X(X,Y),W_Y(X,Y)\right) \ge \bar{H}^*\left(W_Y(X,Y)\right). \tag{19}$$

Then $\left(x_b^*, 0\right)$ is an optimal control at (X,Y). If (19) does not hold, then $\left(0, \bar{y}_b^*\right)$, where $\bar{y}_b^*$ is given by (15), is an optimal control at (X,Y).

[5] A separate study will focus on the impact of cost-saving technology on the optimal choices, as well as the Markovian strategies.

[6] Obviously, due to the constraints of the definition domain (12) and the discontinuity of the cost function $C(X,x)$ at $x = 0$, the usually guessed linear-quadratic value function, and thus the affine strategy, do not provide a solution to the above seemingly linear-quadratic optimal control problem.

(2.2) Suppose

$$W_Y(X,Y) - R_0 > W_X(X,Y) - C_0. \tag{20}$$

Then $(0, \bar{y}_b^*)$, with $\bar{y}_b^*$ defined in (15), is an optimal control at (X,Y).

(2.3) Suppose

$$W_Y(X,Y) - R_0 = W_X(X,Y) - C_0 \tag{21}$$

and (x_b^*, y_b^*) that satisfies

$$x_b^* + y_b^* = \max\{x_{\min}, B_0 - C_0 + W_X\}$$

and

$$\tilde{H}(X, Y, x_b^*, y_b^*, W_X(X,Y), W_Y(X,Y)) \geq \bar{H}^*(W_Y(X,Y)). \tag{22}$$

Then (x_b^*, y_b^*) is an optimal control at (X,Y). If (22) does not hold, then $(0, \bar{y}_b^*)$ is an optimal control at (X,Y).

First, note that although the finding in (2.3) of Proposition 1 seemingly suggests that the coexistence of recycling and exploitation is possible, (21) defines a one-dimensional manifold in the two-dimensional $XY-$ space. Hence, the set in which (22) holds generally has measure zero. That is, either $x_b^* = 0$ or $y_b^* = 0$ hold generically for $(X,Y) \in \Omega$. Therefore, we focus our interpretation on the first two cases.

The two inequalities (17) and (20) compare the marginal values of exploitation and recycling, which are defined as the marginal value net of the marginal cost of exploitation and recycling, respectively. When inequality (20) holds, recycling is more globally desirable and thus recycling is the only choice, as presented in (2.2). However, when (17) holds the situation is more complicated due to the cost discontinuity at $x = 0$. Although from (17) exploitation is more globally desirable, we still need to distinguish whether the optimal choice is $x = 0$ or $x > 0$. Condition (19) provides the criterion under which one of these two indeed yield a higher current value Hamiltonian. Notwithstanding, if $x = 0$ yields a higher Hamiltonian, then it is optimal to stop supplying the market from the virgin resource, even though this resource is not exhausted. But considering the minimum market demand, $x + y \geq x_{min}$, then recycling must take place to satisfy the market; while if $x_b^* > 0$ brings a higher Hamiltonian and is more globally desirable, then naturally there is no recycling until the natural reserve is no longer sufficient to support the market.

Nonetheless, given the perfect substitution in utility between the original resource and the recycled one, from the perspective of the international organization there is only one optimal way to supply the critical mineral to the market: either from the exploitation of the natural reserve or recycling, provided neither reserve is being exhausted. Thus, with the increasing cost of exploitation, it is possible that starting recycling before exhausting the initial natural reserve is more globally desirable. In the rest of this section, we provide conditions for when this unexploited resource should be extracted.

In the following two subsections, we investigate the two boundary cases, which provide more interesting results considering the scarcity of critical minerals: Either the virgin resource or the recyclable resource is being exhausted.[7]

3.2.2. Optimal supply near Γ_1

We first solve (16) on Γ_1, where the virgin resource is being exhausted while recyclable resources are still available. Appendix A.2 demonstrates the following results.

Proposition 2. *Suppose $W(X,Y)$ is differentiable in the closure $\overline{\Omega}$ of domain Ω. Also, suppose that*

$$C_0 + \frac{\xi S_0}{x_{\min}} \neq R_0, \qquad [B_0 - C_0]^2 - [B_0 - R_0]^2 \neq 2\xi S_0. \tag{23}$$

Then for any Y satisfying $0 < Y < \eta S_0$, there is $\varepsilon > 0$ such that $x_b^* = 0$ at (X,Y) with $S_0 - \varepsilon < X < S_0$.

No matter whether the central planner is lavish or frugal, this proposition shows that for $X < S_0$ but near S_0 there is no extraction if $Y > 0$. In other words, if recycling technology is not only available but also starts to supply the market, although the virgin reserve is not yet exhausted it is globally optimal to stop supply from the virgin reserve.

In other words, the above proposition indicates that at $X = S_0^-$, the only possible point at which $x_b^* > 0$ is $Y = 0$, that is, recycling has not yet started. The next proposition gives sufficient conditions for $x_b^* > 0$ near $(X,Y) = (S_0, 0)$, the proof of which is given in Appendix A.3.

Proposition 3. *Under the following conditions, $x_b^* > 0$ at $(S_0, 0)$:*

(4.1) at $X = S_0^-$, the central planner chooses $y_b^* = x_{\min}$ and

$$\frac{S_0 \xi}{x_{\min}} + C_0 \leq R_0. \tag{24}$$

(4.2) at $X = S_0^-$, the central planner allows $y_b^* > x_{\min}$ and

$$\frac{2 S_0 \xi}{x_{\min}} + C_0 \leq R_0. \tag{25}$$

This proposition is a continuation of the previous one. From a global point of view, to optimally use the original resource and prolong the duration of each recycling period, the original resource must be nearly exhausted when the recycling technology is available and being used. On the one hand, this is in line with literature on perfect backstop substitution (Dasgupta and Stiglitz, 1980; Dasgupta et al., 1983; etc.) in the sense that in order to save time-costs in the invention and innovation of recycling technology (which we do not model here), it is optimal to delay the invention of the recycling technology. But at the same time, the finding in Proposition 3 is different from those of the perfect backstop substitution literature, given that we have to take into account the minimum market demand constraint.

When the virgin reserve is nearly exhausted, the virgin supply may not be sufficient to satisfy market demand, thus recycling must be ready to be implemented. When recycling starts, by combining Propositions 1 and 2 it is clear that recycling is more globally desirable, but without exhausting the virgin reserve the duration of recycling is shorter compared to exhausting the depletable resource first.

However, there is no guarantee that the virgin resource will indeed be exhausted before recycling starts, if the recycling technology is available. The following proposition provides sufficient conditions — depending on the cost parameters and the minimum market demand combination — for the international organization to start recycling before exhaustion of the virgin resource. The proof of these findings can be found in Appendix A.4.

Proposition 4. *Under the following conditions, $x_b^* = 0$ near $(X,Y) = (S_0, 0)$:*

(5.1) at $X = S_0$ the central planner chooses $y_b^* = x_{\min}$ and

$$R_0 - C_0 < \sqrt{x_{\min}^2 + 2\xi S_0} - x_{\min}.; \tag{26}$$

(5.2) at $X = S_0$ the central planner allows $y_b^* > x_{\min}$ at $(S_0, 0)$ and

$$(B_0 - C_0)^2 + x_{\min}(R_0 - B_0) < 2\xi S_0. \tag{27}$$

We next examine the situation where recycling reaches the limit $Y = \eta X$, that is, the recyclable resource is nearly exhausted.

3.2.3. Optimal supply near Γ_2

On boundary Γ_2, the recyclable resource is exhausted although the initial depletable resource still has some reserves. In this case, one potential choice is that $y_b^* = \eta x_b^*$. With this choice, both extraction and recycling are carried out *simultaneously* and in proportion. However,

[7] At the same time, the solutions provide boundary conditions to the HJB Eq. (10).

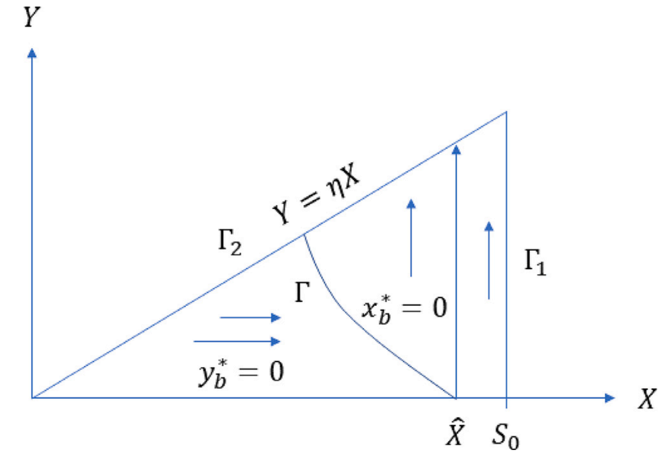

Fig. 1. Overall structure of solutions.

Proposition 6 presented in Appendix A.5 shows that this choice is not optimal.

To obtain this finding, we argue by contradiction. Given that the virgin reserve is not exhausted while recycling is, recycling must be more globally desirable than exploitation. If we assume that the optimal choice of supply from the virgin resource is when $x_b^* > 0$, it must be that $y_b^* = \eta x_b^*$. Appendix A.5 demonstrates that this kind of combination cannot be a solution to the HJB on Γ_2. Detailed calculations are given in the proof of Proposition 6 in Appendix A.5.

Intuitively, the above proportional choice pair $(x_b, y_b) = (x_b^*, \eta x_b^*)$ fails to be an optimal choice coming from the modeling assumption of there being an average lifetime of a product before it can be scrapped from the market and enter recycling (Swan, 1980). When there is no accumulated recyclable resource, that is, near the boundary Γ_2, simultaneous supply of the critical mineral from the virgin resource and from recycling is impossible. Arguably, this finding suggests that the potential optimal supply of critical minerals should involve switching between the virgin resource and the recyclable resource. Thus, in the following we construct the HJB equations that can offer this kind of periodic supply.

In Appendix A.6, we derive the HJB equations for W on Γ_2 at the points reached by recycling while extraction is paused. Let $(X, \eta X) \in \Gamma_2$ be a point reached by a vertical line (X, Y), $Y < \eta X$, along which $x_b^* = 0$.

3.2.4. The overall structure of solutions

Based on the discussion of solutions, we can describe their overall structure as follows. The triangular region Ω is divided by a cure Γ on which

$$H_1\left(X, W_X(X,Y)\right) = H_2\left(W_Y(X,Y)\right),$$

where

$$H_1\left(X, W_X\right) = \begin{cases} \left[B_0 - C_0 + W_X\right] x_{\min} - \frac{x_{\min}^2}{2} - \xi X & \text{if } B_0 - C_0 + W_X \leq x_{\min}, \\ \frac{1}{2}\left[B_0 - C_0 + W_X\right]^2 - \xi X & \text{otherwise,} \end{cases}$$

and

$$H_2\left(W_Y\right) = \begin{cases} \left[B_0 - R_0 + W_Y\right] x_{\min} - \frac{x_{\min}^2}{2} & \text{if } B_0 - R_0 + W_Y \leq x_{\min}, \\ \frac{1}{2}\left[B_0 - R_0 + W_Y\right]^2 & \text{otherwise.} \end{cases}$$

On the left side of Γ, the international organization as policymaker only extracts the natural resource and does not recycle; on the right side, the opposite is true. Furthermore, $\hat{X}$ is either less than or equal to S_0, at which point there is no extraction from the natural resource until recycling concludes. Fig. 1 illustrates these findings.

There are two cases: Either $\hat{X} = S_0$ or $\hat{X} < S_0$. Propositions 3 and 4 provide sufficient conditions for the former and latter, respectively. In the former case, recycling does not start until the virgin resource is exhausted. In the latter case, recycling starts when $X = \hat{X} < S_0$ and continues until there is nothing to recycle. After that, there are frequently switches between extraction and recycling until all resources are gone. In the last stage, globally optimal control cannot be reached, in the sense that there is no solution that satisfies the HJB equation in any non-zero measure definition domain, but the faster the switches between extraction and recycling, the greater the global benefit.

4. Duopoly competition

In this section, we consider the situation where the competing duopoly takes the quantities sold by the competitor as given, the natural resource cartel sells the virgin critical mineral, and the importing country supplies the market by recycling the used mineral.

The recycling technology is available at $T_n \geq 0$, and this is known by both the importing country and the resource cartel. Before T_n the resource cartel is a monopolist and a duopoly situation only emerges after T_n. Suppose the cartel owns the entire stock of the critical mineral and maximizes profit generated from depletion of the virgin resource.

For simplicity, the cartel's inverse demand function is

$$p_n(t) = \begin{cases} a - b x_n(t) & \text{for } 0 \leq t \leq T_n, \\ a - b[x_n(t) + y_n(t)] & \text{for } t \geq T_n, \end{cases}$$

where a, b are positive parameters with a sufficiently large such that the price is always positive and $a > bS_0$, and y_n is the recycled supply of the importing country. The instantaneous profit of the cartel is then

$$\pi_n(x_n(t)) = p_n(\cdot)x_n = \begin{cases} [a - b x_n]\, x_n & \text{for } 0 \leq t \leq T_n, \\ [a - b(x_n + y_n)]\, x_n & \text{for } t \geq T_n. \end{cases}$$

Denote $X_n(t) = \int_0^t x_n(\tau) d\tau$. The cartel's optimal control problem is the choice of monopoly supply x_n to maximize aggregate profits:

$$\begin{aligned} \max_{x_n} \Pi_n &= \int_0^{T_n} e^{-rt}\left[\pi_n(x_n(t)) - C(X_n(t), x_n(t))\right] dt \\ &\quad + \int_{T_n}^{\infty} e^{-rt}\left[\pi_n(x_n(t)) - C(X_n, x_n)\right] dt \\ &= \int_0^{T_n} e^{-rt}\left[(a - b x_n)x_n - C_0 x_n - \xi X_n(t)\right] dt \\ &\quad + \int_{T_n}^{\infty} e^{-rt}\left[(a - b(x_n + y_n))x_n - C_0 x_n - \xi X_n(t)\right] dt, \end{aligned} \tag{28}$$

subject to $x_n(t) \geq x_{min}$ for $\forall t \in [0, T_n]$, $x_n \geq 0$ $\forall t \geq T_n$, and

$\dot{X}_n = x(t)$ $\forall t$, with $X(0) = 0$.

Arguably, after recycling starts the minimum supply to the market is no longer the concern of the cartel; rather, it is the constraint of the importing country. Of course, when the recycling technology is available at T_n, the importing country has accumulated mineral reserves $X_n(T_n)$, of which the recyclable amount is $\eta X_n(T_n)$.

Consider the special case of a minimum requirement for the critical mineral. The monopoly cartel then faces the following dilemma. On the one hand, if the cartel supplies the market at the minimum level, x_{min}, it induces the importing country to make more effort to invent recycling technology if there is no direct backstop substitution.[8] Thus, the cartel should supply enough critical mineral to the market and ease the pressure on the importing country to find a substitute as soon as

[8] Platinum group metals (PGMs) seem to face this kind of fate—platinum's market position in the global automotive industry was replaced by palladium. However, there is currently a trend of returning to platinum or moving to something new, given that Russia controls about 40% of the palladium market.

possible, prolonging its monopoly position. But on the other hand, if there is too much supply to the market—the extreme example being the exhaustion of all initial natural reserves before or at the moment when the importing country's recycling technology is available—the cartel is out of the market from that moment on.

The importing country's objective is similar to that described in Section 3, but the extraction cost is replaced by the importing cost and we ignore the transportation cost. Thus, the importing country faces the following optimal control:

$$\begin{aligned}\max_{y_n(t)} W_n &= \int_0^{T_n} e^{-rt}\left[B(x_n) - \pi_n(x_n)\right]dt + \int_{T_n}^{\infty} e^{-rt}\left[B(x_n+y_n) - \pi_n(x_n) - R_n(y_n)\right]dt \\ &= \int_0^{T_n} e^{-rt}\left[B_0 x_n - \frac{x_n^2}{2} - (a - bx_n)x_n\right]dt \\ &\quad + \int_{T_n}^{\infty} e^{-rt}\left[B_0(x_n+y_n) - \frac{(x_n+y_n)^2}{2} - [a - b(x_n+y_n)]x_n - R_0 y_n\right]dt\end{aligned} \tag{29}$$

subject to $y_n(t) = 0$ for $0 \le t < T_n$, $x_n + y_n \ge x_{min}$ and $y_n \ge 0$ for $t \ge T_n$,

$$Y_n(t) \equiv \int_{T_n}^{t} y_n(\tau)d\tau \le \eta X_n(t) = \eta \int_0^t x_n(\tau)d\tau, \tag{30}$$

and

$$\dot{Y}_n = y(t) \quad t \ge T_n \quad \text{with} \quad Y_n(T_n) = 0, \tag{31}$$

where x_n is taken as given from the cartel's optimal choice. Although the first part of the integral in (29) does not have any choice variables for the importing country, it provides information regarding which situation is more beneficial.

The importing country faces a dilemma similar to that of the cartel: On the one hand, earlier availability of recycling technology will partially ease dependence on the cartel's virgin resource stock, although this means increasing the time-cost of R&D for the development of recycling technology. On the other hand, with the earlier arrival of recycling technology the recyclable resource accumulated for later use may not be sufficient. With depreciation (i.e., the permanently lost portion) the importing country still relies on supply from the cartel until the original reserve is exhausted.

5. Precommitted Nash equilibrium

Given the central theme of this paper, which centers around the design of international cooperation for the exploitation and recycling of critical minerals, Section 3 above presents an ideal scenario characterized by comprehensive cooperation. In this section, we delve into a more pragmatic situation wherein exporting cartels and importing countries base their actions on precommitments established through market competition. In this context, the term "precommitted" (also referred to as open-loop) Nash equilibrium pertains to the equilibrium in which each player dedicates their optimal choices, contingent solely upon time, while considering the optimal choice of the competing player as given.

With the two players making their precommitments, the cartel supplies the market following a monopoly's profit maximization choice, initially without constraints but at a level no lower than the minimum requirement. In order to simultaneously solve the above two optimal control problems, we define the Hamiltonian of the cartel as

$$\begin{aligned}H_{c,I}(x_n, X_n, \lambda_x, \eta_x, \nu_x, \mu_x) &= [(a - b(x_n+y_n))x_n - C_0 x_n - \xi X_n] \\ &+ \lambda_x x_n + \eta_x x_n + \nu_x(\eta X_n - Y_n) + \mu_x(x_n - x_{min})\end{aligned}$$

and the Hamiltonian of the importer as

$$\begin{aligned}H_I(y_n, Y_n, \lambda_y, \eta_y, \nu_y, \mu_y) &= \left[B_0(x_n+y_n) - \frac{(x_n+y_n)^2}{2}\right] \\ &- [a - b(x_n+y_n)]x_n - R_0 y_n + \lambda_y y_n + \eta_y y_n \\ &+ \nu_y(\eta X_n - Y_n) + \mu_y(x_n + y_n - x_{min}),\end{aligned}$$

with λ_x, λ_y being co-state variables and $\eta_x, \eta_y, \nu_x, \nu_y, \mu_x, \mu_y$ being Kuhn–Tucker multipliers associated with the inequality constraints.

The solution is obtained via backward induction. Thus, we start from the moment when $t \ge T_n$. To complete this section, we have to model the period when $t \le T_n$, namely, Period I. But in Period I, the game reduces to the cartel's optimal control problem, given that the importing country's recycling technology is not yet available. We leave the details to Appendix A.8.

Appendix A.7 demonstrates the following results.

Proposition 5. *Let $t \ge T_n$. If there is no cost-saving technology, i.e., $g = 0$ and $\rho = 0$, then there exist finite times $(T_n \le)T_{n2}, T_{n3} \le T_{n4} \le T_{n5}$ such that there is an open-loop Nash equilibrium, $(x_n^o(t)), y_n^o(t)$, given by the following:*

$$x_n^o(t) = \begin{cases} \dfrac{1}{b(b+1)}\left[a - b(B_0 - R_0) - C_0 + \lambda_x - b\lambda_y\right], T_n \le t \le T_{n3}, \\ 0, \qquad t \ge T_{n3}. \end{cases} \tag{32}$$

(6.1) If $T_{n2} < T_{n3}$,

$$y_n^o(t) = \begin{cases} B_0 - R_0 + (b-1)x_n^o(t) + \lambda_y, T_n \le t \le T_{n2}, \\ x_{min} - x_n^o(t), \quad T_{n2} \le t \le T_{n3}, \\ x_{min}, \quad T_{n3} \le t \le T_{n5}; \end{cases}$$

(6.2) if $T_{n2} > T_{n3}$

$$y_n^o(t) = \begin{cases} B_0 - R_0 + (b-1)x_n^o(t) + \lambda_y, T_n \le t \le T_{n3}, \\ B_0 - R_0 + \lambda_y(t), \quad T_{n3} \le t \le T_{n4}, \\ x_{min}, \quad T_{n4} \le t \le T_{n5}; \end{cases}$$

(6.3) at $t = T_{n5}$, the recyclable resource is exhausted.

The shadow value λ_x measures the value of the cartel's accumulated supply in the market, $X_n(t)$. Thus, it is the value of lost mineral to the importer—obviously, it is always negative for the cartel. Similarly, λ_y measures the value of the importing country's recycled minerals. Although the recycled mineral can be recycled again, due to the lost material following each round of recycling, some value is lost. Additionally, the two costs have different impacts on the cartel's optimal market supply: Its own marginal cost, C_0, gives the cartel an incentive to decrease market supply, while the importer's recycling cost augments the cartel's supply—the more costly recycling is, the larger the market share for the cartel. Nevertheless, the impact of cost on the importing country also depends on the market structure—whether $b > 1$ or $b < 1$.

In the above, there is an implicit condition $x_n^o > 0$ for $T_n \le t \le T_{n3}$, that is, at least when $t = T_n$,

$$a - bB_0 + bR_0 - C_0 + \lambda_x(T_n) - b\lambda_y(T_n) > 0. \tag{33}$$

Of course, it may occur that at $t = T_n$, $S(T_n) = 0$, that is, the natural reserve is exhausted when recycling starts. The conditions for this situation are presented in Appendix A.8 for Period I. If this is the case, the supply in the second period depends only on recycling and $x_n^o(t) = 0$ for $t \ge T_n$.

Suppose (33) holds. It is straightforward that the joint market supply of the mineral is

$$x_n^* + y_n^* = \frac{1}{b+1}\left[a + B_0 + (\lambda_x + \lambda_y) - (R_0 + C_0)\right], \tag{34}$$

which follows

$$\frac{d(x_n^* + y_n^*)}{dt} = \frac{1}{b+1}\left[\dot{\lambda}_x + \dot{\lambda}_y\right].$$

Given that both shadow values decrease at the rate of interest, the joint supply in the market also decreases over time at the same rate. Thus, the market price

$$p_n(x_n^* + y_n^*) = a - b(x_n^* + y_n^*) = \frac{1}{b+1}\left[a - bB_0 - b(\lambda_x + \lambda_y) + b(R_0 + C_0)\right] \tag{35}$$

follows

$$\frac{dp_n}{dt} = -b\frac{d(x_n^* + y_n^*)}{dt}.$$

In other words, without cost-saving technological progress, i.e., $\rho = 0$ and $g = 0$, the price follows Hotelling's rule by increasing over time at the rate of interest, and the total supply from the cartel and recycler decreases at the same rate over time until the minimum market requirement level is reached.

Keeping this monotonic changing in mind, the intuition of time T_{ni} is clear. T_{n2} is the moment when the (monotonically decreasing) joint supply, $x_n^o + y_n^o$, reaches the minimum market demand level, x_{min}. Thus, from that moment on the supply has to follow this minimum level and:

$$x_n^o(T_{n2}) + y_n^o(T_{n2}) = B_0 - R_0 + bx_n^o(T_{n2}) + \lambda_y(T_{n2}) = x_{min}. \tag{36}$$

Two possibilities appear: (a) At T_{n2}, the cartel's original reserve is not yet exhausted, and (b) the natural reserve is exhausted before T_{n2}. In the first case, the cartel continues to supply the market until the reserve is exhausted at $T_{n3} > T_{n2}$. Thus, during the time interval $[T_{n2}, T_{n3}]$, the recycler complements the cartel's insufficient supply to maintain market supply at its minimum level:

$$\int_{T_n}^{T_{n3}} x_n^o(t)dt = S(0) - X(T_n). \tag{37}$$

In the second case, the cartel must supply the market alone for $t \geq T_{n2}$. If the monotonically decreasing recycling supply is above the minimum level, that is, if

$$y_n^o(t) = B_0 - R_0 + \lambda_y(t) > x_{min}, \quad t \geq \max\{T_{n2}, T_{n3}\},$$

it can only last until T_{n4} (and thus, $y_n^o(T_{n4}) = x_{min}$). From that moment on, the recycling is fixed at the minimum market-demand level until T_{n5}:

$$\int_{T_n}^{T_{n5}} y_n^o(t)dt = \frac{\bar{\rho}S(0)}{1-\bar{\rho}}. \tag{38}$$

In other words, the duration of each recycling process shrinks with repeated recycling until, at a finite time, the recyclable resource is exhausted, such that either the supply is not sufficient to satisfy the minimum market demand or the amount of unused critical mineral is greater than the total stock of mineral to be recycled. Thus, the recycling process can no longer continue.

Of course, it is not impossible that from the beginning $t = T_n$, $x_n(T_n) + y_n(T_n) \leq x_{min}$. In this case, $x_n^o(t) + y_n^o(t) = x_{min}$ for all $t \geq T_n$. The rest of T_{ni}, $i = 2, 3, 4, 5$ can be similarly defined.

6. Conclusion

Critical minerals are essential components of many modern technologies, and especially those used for renewable energy and electric vehicles. As the demand for these minerals grows, so does the need for a reliable and secure supply. In this article, we looked into the best ways to obtain these minerals, either by working together globally or by competing in a certain way.

The main findings are as follows. (i) When both the virgin resource and recyclable reserves are abundant, it is globally optimal to rely on only one resource to satisfy market demand. Obviously, this main finding differs from the existing literature, such as, Hoogmartens et al. (2018) and Seyhan et al. (2012) etc., where both extraction and recycling could supply the market simultaneously. The fundamental reason are two fold: under our setting the (shadow) price is monotonic, which is not the case within the framework of Seyhan et al. (2012); and which resource to supply the market is an endogenous decision instead of exogenous assumption ((Hoogmartens et al., 2018), etc.). (ii) The more globally desirable resource, i.e., the cheaper one, should supply the market first. (iii) The two resources supply the market at the same time only if the cheaper one is being exhausted and is not sufficient to satisfy market demand. (iv) When the recyclable reserve is being exhausted but the virgin reserve is not, it must be that recycling is cheaper than exploitation. In this case, the international organization, as policymaker, has no optimal choice as to how to exploit the remain resource. (v) In contrast, under precommitment both resources could supply the market at the same time.

Our paper takes a first step toward a comprehensive framework that can be used to understand the complexity of supply chains for many critical minerals, in order to reduce their vulnerability to disruptions and geopolitical risks.

CRediT authorship contribution statement

Weihua Ruan: Writing – review & editing, Writing – original draft. **Benteng Zou:** Writing – review & editing, Writing – original draft.

Declaration of competing interest

none.

Data availability

No data was used for the research described in the article.

Appendix A

A.1. Proof of Proposition 1

We first find maximizers of $\tilde{H}$ in each case. In the interior Ω, a maximizer $\left(\tilde{x}_b^*, \tilde{y}_b^*\right)$ of $\tilde{H}$ satisfies

$$\nabla\tilde{H} + \mu\nabla\left[x + y - x_{\min}\right] + \nu\nabla x + \gamma\nabla y = 0,$$

where ∇ is the gradient with respect to (x, y) and μ, ν, and γ are Kuhn–Tucker multipliers. The above equation is equivalent to

$$\begin{aligned} &B_0 - (x+y) - C_0e^{-gt} + W_X + \mu + \nu = 0,\\ &B_0 - (x+y) - R_0e^{-\rho t} + W_Y + \mu + \gamma = 0,\\ &\mu \geq 0, \quad x + y \geq x_{\min}, \quad \mu\left(x + y - x_{\min}\right) = 0,\\ &\nu \geq 0, \quad x \geq 0, \quad \nu x = 0,\\ &\gamma \geq 0, \quad y \geq 0, \quad \gamma y = 0. \end{aligned} \tag{39}$$

Thus,

$$B_0 + \mu - (x+y) = C_0e^{-gt} - W_X - \nu = R_0e^{-\rho t} - W_Y - \gamma. \tag{40}$$

By subtracting the first two equations in (39), we find

$$-C_0e^{-gt} + W_X + \nu + R_0e^{-\rho t} - W_Y - \gamma = 0. \tag{41}$$

Suppose (17) holds. Then

$$\gamma = \nu - C_0e^{-gt} + W_X - \left[-R_0e^{-\rho t} + W_Y\right] > 0.$$

By the last line in (39), $\tilde{y}_b^* = 0$. Also, by (40),

$$\tilde{x}_b^* = B_0 + \mu - C_0e^{-gt} + W_X.$$

Either $\mu = 0$ or $\mu > 0$. In the former case,

$$\tilde{x}_b^* = B_0 - C_0e^{-gt} + W_X,$$

and by the third line in (39),

$$\tilde{x}_b^* = \tilde{x}_b^* + \tilde{y}_b^* \geq x_{\min}.$$

In the latter case, by the same relation we have

$$\tilde{x}_b^* = x_{\min} > B_0 - C_0 e^{-gt} + W_X.$$

This proves that $(\tilde{x}_b^*, 0)$, where

$$\tilde{x}_b^* = \max\{x_{\min}, B_0 - C_0 e^{-gt} + W_X\}$$

is a maximizer of $\tilde{H}$.

If (19) holds, then $(\tilde{x}_b^*, 0)$ is a maximizer of H. Otherwise, $(0, \tilde{y}_b^*)$ is a maximizer of H. This proves Part 1.

Suppose (20) holds. By (41),

$$\nu = \gamma + C_0 e^{-gt} - W_X - R_0 e^{-\rho t} + W_Y > 0.$$

Hence, by the fourth line in (39), $x_b^* = 0$. Furthermore, by (40),

$$y_b^* = B_0 + \mu - R_0 e^{-\rho t} + W_Y.$$

If $\mu = 0$, then

$$y_b^* = B_0 - R_0 e^{-\rho t} + W_Y > x_{\min}.$$

If $\mu > 0$, then

$$y_b^* = x_{\min} > B_0 - R_0 e^{-\rho t} + W_Y.$$

Hence, $y_b^* = \tilde{y}_b^*$. This proves Part 2.

Finally, suppose (21) holds. Then $\nu = \gamma$. It is not possible for both to be positive, because if so, by the fourth and fifth lines in (39), $\tilde{x}_b^* = \tilde{y}_b^* = 0$, contradicting the third line. Thus, $\nu = \gamma = 0$. By (40),

$$\tilde{x}_b^* + \tilde{y}_b^* = B_0 + \mu - C_0 e^{-gt} + W_X = B_0 + \mu - R_0 e^{-\rho t} + W_Y.$$

Either $\mu = 0$ or $\mu > 0$. If $\mu = 0$, then

$$\tilde{x}_b^* + \tilde{y}_b^* = B_0 - C_0 e^{-gt} + W_X = B_0 - R_0 e^{-\rho t} + W_Y \ge x_{\min}.$$

Otherwise,

$$\tilde{x}_b^* + \tilde{y}_b^* = x_{\min} > B_0 - C_0 e^{-gt} + W_X = B_0 - R_0 e^{-\rho t} + W_Y.$$

Hence, $(\tilde{x}_b^*, \tilde{y}_b^*)$ is a maximizer of $\tilde{H}$. If (22) holds, then $(\tilde{x}_b^*, \tilde{y}_b^*)$ is also a maximizer of H. Otherwise, $(0, \tilde{y}_b^*)$ is a maximizer of H.

This completes the proof.

A.2. Proof of Proposition 2

The proof is complete in two steps: we first rewrite the HJB equation for this boundary case; then we provide a detailed proof of the proposition.

Step 1. The HJB equation.

The HJB equation takes the form

$$rW(S_0, Y) = \begin{cases} [B_0 - R_0 + W_Y]\, x_{\min} - \frac{x_{\min}^2}{2} & \text{if } B_0 - R_0 + W_Y \le x_{\min}, \\ \frac{1}{2}[B_0 - R_0 + W_Y]^2 & \text{otherwise.} \end{cases} \tag{42}$$

There are two cases: Either $B_0 - R_0 \le x_{\min}$ or $B_0 - R_0 > x_{\min}$. In the first case, if we choose

$$W(S_0, \eta S_0) \le \frac{x_{\min}}{r}\left[B_0 - R_0 - \frac{x_{\min}}{2}\right] \tag{43}$$

and solve the linear differential equation of $W(Y)$ in terms of Y,

$$rW = [B_0 - R_0 + W_Y]\, x_{\min} - \frac{x_{\min}^2}{2}, \tag{44}$$

then

$$W(S_0, Y) = e^{\frac{r(Y - \eta S_0)}{x_{\min}}}\left\{W(S_0, \eta S_0) - \frac{x_{\min}}{r}\left[B_0 - R_0 - \frac{x_{\min}}{2}\right]\right\} + \frac{x_{\min}}{r}\left[B_0 - R_0 - \frac{x_{\min}}{2}\right], \tag{45}$$

with initial condition $W(S_0, \eta S_0)$ undetermined but satisfying constraint (43). By (43), this solution satisfies

$$W(S_0, Y) \le \frac{x_{\min}}{r}\left[B_0 - R_0 - \frac{x_{\min}}{2}\right] \quad \text{for } Y < \eta S_0.$$

Hence,

$$B_0 - R_0 + W_Y(S_0, Y) = \frac{r}{x_{\min}} W(S_0, Y) + \frac{x_{\min}}{2} \le B_0 - R_0 \le x_{\min}$$

for all Y such that $0 < Y < \eta S_0$. Therefore, W satisfies Eq. (42) for all $Y < \eta S_0$.

In the case where $B_0 - R_0 > x_{\min}$, by (44) and (45)

$$\begin{aligned} B_0 - R_0 + W_Y(S_0, Y) &= \frac{r}{x_{\min}} W(S_0, Y) + \frac{x_{\min}}{2} \\ &= e^{\frac{r(Y - \eta S_0)}{x_{\min}}}\left[B_0 - R_0 + W_Y(S_0, \eta S_0)\right] + \left(1 - e^{\frac{r(Y - \eta S_0)}{x_{\min}}}\right)[B_0 - R_0] \\ &> x_{\min} \end{aligned}$$

if S_0 is sufficiently large and Y is sufficiently small, that is, at the early stage of recycling. Let $\hat{Y}$ be such that

$$B_0 - R_0 + W_Y(S_0, \hat{Y}) = x_{\min}.$$

In the case where $\hat{Y} > 0$, then for $Y < \hat{Y}$ (42) becomes

$$rW = \frac{1}{2}[B_0 - R_0 + W_Y]^2. \tag{46}$$

With such a value function,

$$y_b^* = B_0 - R_0 + W_Y > x_{\min} \quad \text{for } Y < \hat{Y}.$$

So when the virgin resource is exhausted, the central planner may be lavish at first but then becomes frugal.

Alternatively, we can choose $W(S_0, \eta S_0)$ to be sufficiently negative such that

$$B_0 - R_0 + W_Y(S_0, Y) \le x_{\min} \quad \text{for } 0 \le Y \le \eta S_0.$$

In this way, (44) is valid for the entire Γ_1. This solution represents a frugal central planner who provides only the minimum supply when the virgin resource is exhausted.

Step 2. The proof of Proposition 2.

We first show that if there is such a Y at which no ϵ exists, then there is a sequence (Y_k) such that $\lim_{k\to\infty} Y_k = Y$ and $x_b^* = 0$ at (X, Y_k) for $S_0 - \delta < X \le S_0$. Assume that the opposite holds, so that there are $a, b > 0$ such that $a < b$ and $x_b^* > 0$ at (X, Y) for $S_0 - \delta < X \le S_0$ and $a \le Y \le b$ for some $\delta > 0$. Then W satisfies

$$rW = \begin{cases} [B_0 - C_0 + W_X]\, x_{\min} - \frac{x_{\min}^2}{2} - \xi X & \text{if } B_0 - C_0 + W_X \le x_{\min}, \\ \frac{1}{2}[B_0 - C_0 + W_X]^2 - \xi X & \text{otherwise} \end{cases} \tag{47}$$

for $(X, Y) \in [S_0 - \delta, S_0] \times [a, b]$. The equation is supplemented with the boundary value $W(S_0, Y)$ solved from (42) at $X = S_0$. Reducing ϵ and the interval $[a, b]$ if necessary, we can assume that either

$$rW(X, Y) = [B_0 - C_0 + W_X(X, Y)]\, x_{\min} - \frac{x_{\min}^2}{2} - \xi X \tag{48}$$

or

$$rW = \frac{1}{2}[B_0 - C_0 + W_X]^2 - \xi X. \tag{49}$$

holds in the domain $[S_0 - \delta, S_0] \times [a, b]$. In addition, we can assume that (42) is either (44) or (46) for the entire interval $[a, b]$.

We first consider the case where (44) and (48) hold. In this case, from these two equations we derive

$$\begin{aligned} B_0 - C_0 + W_X(S_0, Y) &= \frac{r}{x_{\min}} W(S_0, Y) + \frac{x_{\min}}{2} + \frac{\xi S_0}{x_{\min}}, \\ B_0 - R_0 + W_Y(S_0, Y) &= \frac{r}{x_{\min}} W(S_0, Y) + \frac{x_{\min}}{2} \end{aligned} \tag{50}$$

for $Y \in [a,b]$. Hence

$$B_0 - C_0 + W_X\left(S_0, Y\right) - \frac{\xi S_0}{x_{\min}} = B_0 - R_0 + W_Y\left(S_0, Y\right) \qquad \text{for } Y \in [a,b]. \tag{51}$$

We next differentiate the two sides of (44) to derive

$$rW_X\left(S_0, Y\right) = x_{\min} W_{XY}\left(S_0, Y\right), \qquad rW_Y\left(S_0, Y\right) = x_{\min} W_{YY}\left(S_0, Y\right). \tag{52}$$

The solutions are

$$W_X\left(S_0, Y\right) = e^{\frac{r}{x_{\min}}(Y-a)} W_X\left(S_0, a\right), \qquad W_Y\left(S_0, Y\right) = e^{\frac{r}{x_{\min}}(Y-a)} W_Y\left(S_0, a\right) \tag{53}$$

for $Y \in [a,b]$. Hence,

$$\begin{aligned} B_0 - C_0 + W_X\left(S_0, Y\right) &= e^{\frac{r(Y-a)}{x_{\min}}}\left[B_0 - C_0 + W_X\left(S_0, a\right)\right] \\ &\quad + \left(1 - e^{\frac{r(Y-a)}{x_{\min}}}\right)\left[B_0 - C_0\right], \\ B_0 - R_0 + W_Y\left(S_0, Y\right) &= e^{\frac{r(Y-a)}{x_{\min}}}\left[B_0 - R_0 + W_Y\left(S_0, a\right)\right] \\ &\quad + \left(1 - e^{\frac{r(Y-a)}{x_{\min}}}\right)\left[B_0 - R_0\right]. \end{aligned} \tag{54}$$

From the first equation, we obtain

$$\begin{aligned} B_0 - C_0 + W_X\left(S_0, Y\right) - \frac{\xi S_0}{x_{\min}} &= e^{\frac{r(Y-a)}{x_{\min}}}\left[B_0 - C_0 + W_X\left(S_0, a\right) - \frac{\xi S_0}{x_{\min}}\right] \\ &\quad + \left(1 - e^{\frac{r(Y-a)}{x_{\min}}}\right)\left[B_0 - C_0 - \frac{\xi S_0}{x_{\min}}\right]. \end{aligned}$$

Hence, by (51),

$$B_0 - C_0 - \frac{\xi S_0}{x_{\min}} = B_0 - R_0.$$

This contradicts (23).

Note that (46) and (48) cannot both hold because the former leads to

$$rW\left(S_0, Y\right) = \left[B_0 - C_0 + W_X\left(S_0, Y\right)\right] x_{\min} - \frac{x_{\min}^2}{2} - \xi S_0 \le \frac{x_{\min}^2}{2} - \xi S_0,$$

and the latter leads to

$$rW\left(S_0, Y\right) = \frac{1}{2}\left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right]^2 \ge \frac{x_{\min}^2}{2}.$$

We next consider the case where (44) and (49) hold. In this case,

$$\begin{aligned} \frac{1}{2}\left[B_0 - C_0 + W_X\left(S_0, Y\right)\right]^2 - \xi S_0 &= rW\left(S_0, Y\right) \\ &= \left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right] x_{\min} - \frac{x_{\min}^2}{2} \end{aligned}$$

for $Y \in [a,b]$. This cannot be true because by (45) $W\left(S_0, Y\right)$ is a linear function of $e^{\frac{rY}{x_{\min}}}$, but by (54) the right-hand is a linear combination of $e^{\frac{2rY}{x_{\min}}}$ and $e^{\frac{rT}{x_{\min}}}$.

We now consider the case where (46) and (49) both hold. In this case,

$$\left[B_0 - C_0 + W_X\left(S_0, Y\right)\right]^2 - 2\xi S_0 = \left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right]^2 \tag{55}$$

for $Y \in [a,b]$. Differentiating both sides with respect to Y, we obtain

$$\left[B_0 - C_0 + W_X\left(S_0, Y\right)\right] W_{XY}\left(S_0, Y\right) = \left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right] W_{YY}\left(S_0, Y\right). \tag{56}$$

In addition, differentiating both sides of (46), we obtain

$$\begin{aligned} rW_X\left(S_0, Y\right) &= \left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right] W_{XY}\left(S_0, Y\right), \\ rW_Y\left(S_0, Y\right) &= \left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right] W_{YY}\left(S_0, Y\right) \end{aligned} \tag{57}$$

for $Y \in [a,b]$. Hence, (56) leads to

$$W_X\left(S_0, Y\right)\left[B_0 - C_0 + W_X\left(S_0, Y\right)\right] = W_Y\left(S_0, Y\right)\left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right].$$

Hence, by (55),

$$\begin{aligned} &\left[B_0 - C_0\right]\left[B_0 - C_0 + W_X\left(S_0, Y\right)\right] - \left[B_0 - R_0\right]\left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right] \\ &= \left[B_0 - C_0 + W_X\left(S_0, Y\right)\right]^2 - \left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right]^2 = 2\xi S_0. \end{aligned}$$

On the other hand, solving equations in (57) we obtain

$$W_X\left(S_0, Y\right) = \mu(Y) W_X\left(S_0, a\right), \qquad W_Y\left(S_0, Y\right) = \mu(Y) W_Y\left(S_0, a\right),$$

where

$$\mu(Y) = e^{\int_a^Y \frac{r\,ds}{B_0 - R_0 + W_Y\left(S_0, s\right)}}.$$

Hence,

$$\begin{aligned} &\left[B_0 - C_0\right]\left[B_0 - C_0 + W_X\left(S_0, Y\right)\right] - \left[B_0 - R_0\right]\left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right] \\ &= \left[B_0 - C_0\right]\left[B_0 - C_0 + \mu W_X\left(S_0, a\right)\right] - \left[B_0 - R_0\right]\left[B_0 - R_0 + \mu W_Y\left(S_0, a\right)\right]. \end{aligned}$$

This leads to

$$2\mu(Y)\xi S_0 + (1 - \mu(Y))\left\{\left[B_0 - C_0\right]^2 - \left[B_0 - R_0\right]^2\right\} = 2\xi S_0.$$

This is only possible if

$$\left[B_0 - C_0\right]^2 - \left[B_0 - R_0\right]^2 = 2\xi S_0,$$

contradicting (23).

This proves the existence of a sequence $\left(Y_k\right)$ of the asserted property.

We next show that Y does not exist. If such a Y exists. Then, W satisfies (47) at (X, Y) for $X \in \left[S_0 - \epsilon, S_0\right]$. We first assume (44) and (48) both hold. Thus,

$$B_0 - C_0 + W_X(X, Y) - \frac{\xi X}{x_{\min}} \ge B_0 - R_0 + W_Y(X, Y) \tag{58}$$

for $X \in \left[S_0 - \epsilon, S_0\right]$. However, since $Y_k \to Y$ and the reversed inequality holds with Y replaced by Y_k, the two sides are equal. By differentiating (48), we obtain

$$rW_X(X, Y) = x_{\min} W_{XX}(X, Y) - \xi, \qquad rW_Y(X, Y) = x_{\min} W_{YX}(X, Y). \tag{59}$$

Hence,

$$\begin{aligned} W_X(X, Y) &= e^{\frac{r\left(X - S_0\right)}{x_{\min}}}\left[W_X\left(S_0, Y\right) + \frac{\xi}{r}\right] - \frac{\xi}{r}, \\ W_Y(X, Y) &= e^{\frac{r\left(X - S_0\right)}{x_{\min}}} W_Y\left(S_0, Y\right). \end{aligned} \tag{60}$$

This leads to

$$\begin{aligned} B_0 - C_0 + W_X(X, Y) &= e^{\frac{r\left(X - S_0\right)}{x_{\min}}}\left[B_0 - C_0 + W_X\left(S_0, Y\right)\right] \\ &\quad + \left(1 - e^{\frac{r\left(X - S_0\right)}{x_{\min}}}\right)\left[B_0 - C_0 - \frac{\xi}{r}\right], \\ B_0 - R_0 + W_Y(X, Y) &= e^{\frac{r\left(X - S_0\right)}{x_{\min}}}\left[B_0 - R_0 + W_Y\left(S_0, Y\right)\right] \\ &\quad + \left(1 - e^{\frac{r\left(X - S_0\right)}{x_{\min}}}\right)\left[B_0 - R_0\right]. \end{aligned}$$

Substituting these into (58), the two sides are different types of Functions and thus cannot be equal on the interval.

If (44) and (49) hold, then

$$\left[B_0 - C_0 + W_X(X,Y)\right] x_{\min} - \frac{x_{\min}^2}{2} - \xi X = \frac{1}{2}\left[B_0 - R_0 + W_Y(X,Y)\right]^2$$

for $X \in \left[S_0 - \varepsilon, S_0\right]$. Moreover, W_X and W_Y satisfy (60). We again find that the two side are different types of functions and, therefore, cannot be equal on an interval.

Finally, if (46) and (49) hold, then

$$\frac{1}{2}\left[B_0 - C_0 + W_X(X,Y)\right]^2 - \xi X = \frac{1}{2}\left[B_0 - R_0 + W_Y(X,Y)\right]^2 \tag{61}$$

for $X \in \left[S_0 - \varepsilon, S_0\right]$. By differentiation with respect to X, it follows that

$$\left[B_0 - C_0 + W_X(X,Y)\right] W_{XX}(X,Y) - \xi = \left[B_0 - R_0 + W_Y(X,Y)\right] W_{YX}(X,Y). \tag{62}$$

Differentiating (49), we find

$$rW_X = \left[B_0 - C_0 + W_X\right] W_{XX} - \xi, \qquad rW_Y = \left[B_0 - C_0 + W_X\right] W_{YX}. \tag{63}$$

The solutions are

$$\begin{aligned} W_X(X,Y) &= v(X,Y)\left[W_X\left(S_0,Y\right) + \frac{\xi}{r}\right] - \frac{\xi}{r}, \\ W_Y(X,Y) &= v(X,Y)\, W_Y\left(S_0,Y\right), \end{aligned} \tag{64}$$

where

$$v(X,Y) = e^{\int_{S_0}^{X} \frac{r\,ds}{B_0 - C_0 + W_X(s,Y)}}.$$

From (62) and (63), we derive

$$W_X(X,Y)\left[B_0 - C_0 + W_X(X,Y)\right] = W_Y(X,Y)\left[B_0 - R_0 + W_Y(X,Y)\right].$$

Then, by (61),

$$\left(B_0 - C_0\right)\left[B_0 - C_0 + W_X\right] - \left(B_0 - R_0\right)\left[B_0 - R_0 + W_Y\right] = 2\xi X.$$

Finally, by (64),

$$\left(B_0 - C_0\right)^2 - \left(B_0 - R_0\right)^2 - \frac{\xi}{r} = 2\xi X \qquad \text{for } X \in \left[S_0 - \varepsilon, S_0\right].$$

Since the left-hand side is a constant and the right-hand side is a linear function, the two sides are not equal on the interval.

Therefore, there is no positive Y such that $x_b^* > 0$ at (X,Y) near Γ_1. The proof is complete.

A.3. Proof of Proposition 3

Let W satisfy (42) on Γ_1. We find values of W at $(X,0)$ for $X < S_0$ and near S_0 by solving (47) with $Y = 0$ and $X < S_0$, that is, $X = X_0^-$. There are three Possibilities: Either

$$rW\left(S_0,0\right) \le \frac{x_{\min}^2}{2} - \xi S_0, \tag{65}$$

or

$$\frac{x_{\min}^2}{2} - \xi S_0 < rW\left(S_0,0\right) \le \frac{x_{\min}^2}{2}, \tag{66}$$

or

$$rW\left(S_0,0\right) > \frac{x_{\min}^2}{2}. \tag{67}$$

Note that if the central planner chooses $y_b^* = x_{\min}$ at $X = S_0$, either (65) or (66) holds. (67) holds only if the central planner allows $y_b^* > x_{\min}$ at $X = S_0$.

Suppose (65) holds. Then Eq. (47) takes the form

$$rW(X,0) = \left[B_0 - C_0 + W_X(X,0)\right] x_{\min} - \frac{x_{\min}^2}{2} - \xi X \qquad \text{for } X < S_0. \tag{68}$$

Since (65) implies that (42) takes the form (44), it is necessary that

$$B_0 - C_0 + W_X(X,0) - \frac{\xi X}{x_{\min}} \ge B_0 - R_0 + W_Y(X,0) \tag{69}$$

for X near S_0. Let $F(X)$ and $G(X)$ denote the left- and right-hand sides of the above inequality, respectively. Differentiating both sides of (48) with respect to X and Y, we obtain

$$rW_X(X,0) = x_{\min} W_{XX}(X,0) - \xi, \qquad rW_Y(X,0) = x_{\min} W_{YX}(X,0).$$

In addition, W_X and W_Y satisfies the boundary conditions

$$\begin{aligned} B_0 - C_0 + W_X\left(S_0,0\right) &= \tfrac{r}{x_{\min}} W\left(S_0,0\right) + \tfrac{x_{\min}}{2} + \tfrac{\xi S_0}{x_{\min}}, \\ B_0 - R_0 + W_Y\left(S_0,0\right) &= \tfrac{r}{x_{\min}} W\left(S_0,0\right) + \tfrac{x_{\min}}{2}. \end{aligned}$$

Solving the boundary value problems, we obtain

$$\begin{aligned} W_X(X,0) &= e^{\frac{r}{x_{\min}}(X - S_0)}\left[W_X\left(S_0,0\right) + \frac{\xi}{r}\right] - \frac{\xi}{r}, \\ W_Y(X,0) &= e^{\frac{r}{x_{\min}}(X - S_0)} W_Y\left(S_0,0\right). \end{aligned}$$

Therefore,

$$\begin{aligned} F(X) &= e^{\frac{r}{x_{\min}}(X - S_0)}\left[B_0 - C_0 + W_X\left(S_0,0\right)\right] - \frac{\xi X}{x_{\min}} + \left(1 - e^{\frac{r}{x_{\min}}(X - S_0)}\right)\left[B_0 - C_0 - \frac{\xi}{r}\right], \\ G(X) &= e^{\frac{r}{x_{\min}}(X - S_0)}\left[B_0 - R_0 + W_Y\left(S_0,0\right)\right] + \left(1 - e^{\frac{r}{x_{\min}}(X - S_0)}\right)\left[B_0 - R_0\right]. \end{aligned}$$

By (50),

$$F\left(S_0\right) = G\left(S_0\right).$$

Hence, (69) holds if

$$F_X(X) < G_X(X).$$

By differentiation,

$$\begin{aligned} F_X(X) &= \frac{r}{x_{\min}} e^{\frac{r}{x_{\min}}(X - S_0)}\left\{\left[B_0 - C_0 + W_X\left(S_0,0\right)\right] - \left[B_0 - C_0 - \frac{\xi}{r}\right]\right\} - \frac{\xi}{x_{\min}}, \\ G_X(X) &= \frac{r}{x_{\min}} e^{\frac{r}{x_{\min}}(X - S_0)}\left\{\left[B_0 - R_0 + W_Y\left(S_0,0\right)\right] - \left[B_0 - R_0\right]\right\}. \end{aligned}$$

In particular, by (50),

$$\begin{aligned} F_X\left(S_0\right) &= \frac{r}{x_{\min}}\left\{\left[B_0 - C_0 + W_X\left(S_0\right)\right] - \left[B_0 - C_0\right]\right\} \\ &= \frac{r}{x_{\min}}\left\{\frac{r}{x_{\min}} W\left(S_0,0\right) + \frac{x_{\min}}{2} + \frac{\xi S_0}{x_{\min}} - B_0 + C_0\right\}, \\ G_X\left(S_0\right) &= \frac{r}{x_{\min}}\left\{\frac{r}{x_{\min}} W\left(S_0,0\right) + \frac{x_{\min}}{2} - B_0 + R_0\right\}. \end{aligned}$$

In view of (24), $F_X\left(S_0\right) > G_X\left(S_0\right)$. Thus, (69) holds in a neighborhood of S_0.

Suppose (66) holds. Eq. (47) takes the form (49), and (42) takes the form of (44). For this solution to be valid, it is necessary that

$$\frac{1}{2}\left[B_0 - C_0 + W_X(X,0)\right]^2 - \xi X \ge \left[B_0 - R_0 + W_Y(X,0)\right] x_{\min} - \frac{x_{\min}^2}{2}. \tag{70}$$

We let $L(X)$ and $M(X)$ denote the left- and right-hand sides, respectively. By differentiating both sides of (49) with respect to X and Y, we obtain

$$\begin{aligned} rW_X(X,0) &= \left[B_0 - C_0 + W_X(X,0)\right] W_{XX} - \xi, \\ rW_Y(X,0) &= \left[B_0 - C_0 + W_X(X,0)\right] W_{YX}. \end{aligned} \tag{71}$$

Let

$$A(X) = B_0 - C_0 + W_X(X,0), \qquad \mu(X) = e^{\int_{S_0}^{X} \frac{r\,ds}{A(s)}}. \tag{72}$$

We find the solutions to (71) in the form

$$\begin{aligned} W_X(X,0) &= \mu(X)\left[W_X\left(S_0,0\right) + \frac{\xi}{r}\right] - \frac{\xi}{r}, \\ W_Y(X,0) &= \mu(X)\, W_Y\left(S_0,0\right). \end{aligned} \tag{73}$$

This leads to

$$\begin{aligned} B_0 - C_0 + W_X(X,0) &= \mu \left[B_0 - C_0 + W_X(S_0,0)\right] + (1-\mu)\left[B_0 - C_0 - \tfrac{\xi}{r}\right], \\ B_0 - R_0 + W_Y(X,0) &= \mu \left[B_0 - R_0 + W_Y(S_0,0)\right] + (1-\mu)\left[B_0 - R_0\right]. \end{aligned} \tag{74}$$

Hence, (70) is equivalent to

$$\begin{aligned} &\tfrac{1}{2}\left\{\mu(X)\left[B_0 - C_0 + W_X(S_0,0)\right] + (1-\mu(X))\left[B_0 - C_0 - \tfrac{\xi}{r}\right]\right\}^2 - \xi X \\ &\geq \left\{\mu(X)\left[B_0 - R_0 + W_Y(S_0,0)\right] + (1-\mu(X))\left[B_0 - R_0\right]\right\} x_{\min} - \tfrac{x_{\min}^2}{2}. \end{aligned} \tag{75}$$

Note that by (44) and (49),

$$\frac{1}{2}\left[B_0 - C_0 + W_X(S_0,0)\right]^2 - \xi S_0 = rW(S_0,0) = \left[B_0 - R_0 + W_Y(S_0,0)\right] x_{\min} - \frac{x_{\min}^2}{2}. \tag{76}$$

Hence,

$$L(S_0) = M(S_0).$$

It suffices to prove

$$L_X(S_0) < M_X(S_0).$$

By differentiation, we find

$$\begin{aligned} L_X(S_0) &= \frac{r}{A(S_0)}\left[B_0 - C_0 + W_X(S_0,0)\right]\left\{\left[B_0 - C_0 + W_X(S_0,0)\right]\right. \\ &\qquad \left. - \left[B_0 - C_0 - \frac{\xi}{r}\right]\right\} - \xi \\ M_X(S_0) &= \frac{r x_{\min}}{A(S_0,0)}\left\{\left[B_0 - R_0 + W_Y(S_0,0)\right] - \left[B_0 - R_0\right]\right\}. \end{aligned}$$

In view of (72),

$$A(S_0) = \left[B_0 - C_0 + W_X(S_0,0)\right].$$

Hence, by (76)

$$\begin{aligned} L_X(S_0) &= rA(S_0) - r\left[B_0 - C_0\right], \\ M_X(S_0) &= r\left[B_0 - R_0 + W_Y(S_0,0)\right] = r\left(\frac{A(S_0)^2 + x_{\min}^2}{2x_{\min}} - \frac{\xi S_0}{x_{\min}}\right) - r\left[B_0 - R_0\right]. \end{aligned}$$

Using (24) and the inequality

$$\frac{A(S_0)^2 + x_{\min}^2}{2x_{\min}} \geq A(S_0),$$

we obtain

$$M_X(S_0) > L_X(S_0).$$

This proves that $L(X) > M(X)$ for X in a neighborhood of S_0.

Finally, suppose (67) holds. Then (42) and (47) take the forms (46) and (49), respectively. For the solution to be valid, it is necessary that

$$\frac{1}{2}\left[B_0 - C_0 + W_X(X,0)\right]^2 - \xi X = \frac{1}{2}\left[B_0 - R_0 + W_Y(X,0)\right]^2 \tag{77}$$

for $X < S_0$ and near S_0. We let $U(X)$ and $V(X)$ denote the left- and right-hand sides, respectively. By differentiating the two sides of (49), we obtain Eq. (71). Hence, (74) holds. Therefore,

$$\begin{aligned} U(X) &= \tfrac{1}{2}\left\{\mu\left[B_0 - C_0 + W_X(S_0,0)\right] + (1-\mu)\left[B_0 - C_0 - \tfrac{\xi}{r}\right]\right\}^2 - \xi X \\ V(X) &= \tfrac{1}{2}\left\{\mu\left[B_0 - R_0 + W_Y(S_0,0)\right] + (1-\mu)\left[B_0 - R_0\right]\right\}^2. \end{aligned} \tag{78}$$

By (46) and (49), we have

$$\frac{1}{2}\left[B_0 - C_0 + W_X(S_0,0)\right]^2 - \xi S_0 = rW(S_0,0) = \frac{1}{2}\left[B_0 - R_0 + W_Y(S_0,0)\right]^2. \tag{79}$$

Thus, $U(S_0) = V(S_0)$. We differentiate U and V with respect to X to obtain

$$\begin{aligned} U_X(S_0) &= \frac{r}{A(S_0)}\left[B_0 - C_0 + W_X(S_0,0)\right]\left\{\left[B_0 - C_0 + W_X(S_0,0)\right] - \left[B_0 - C_0 - \tfrac{\xi}{r}\right]\right\} - \xi, \\ V_X(S_0) &= \frac{r}{A(S_0)}\left[B_0 - R_0 + W_Y(S_0,0)\right]\left\{\left[B_0 - R_0 + W_Y(S_0,0)\right] - \left[B_0 - R_0\right]\right\}. \end{aligned}$$

Using (72), we find

$$\begin{aligned} U_X(S_0) &= r\left\{A(S_0) - \left[B_0 - C_0 - \frac{\xi}{r}\right]\right\} - \xi = r\left\{A(S_0) - \left[B_0 - C_0\right]\right\}, \\ V_X(S_0) &= \frac{r}{A(S_0)}\left\{A(S_0)^2 - 2\xi S_0 - \left[B_0 - R_0 + W_Y(S_0,0)\right]\left[B_0 - R_0\right]\right\}. \end{aligned}$$

By (25) and (79),

$$A(S_0) > x_{\min}, \qquad \frac{B_0 - R_0 + W_Y(S_0,0)}{A(S_0)}\left[B_0 - R_0\right] < B_0 - R_0.$$

Hence,

$$V_X(S_0) \geq r\left\{A(S_0) - \frac{2\xi S_0}{x_{\min}} - \left[B_0 - R_0\right]\right\} \geq r\left\{A(S_0) - \left[B_0 - C_0\right]\right\} = U_X(S_0).$$

This proves (77).

This completes the proof.

A.4. Proof of Proposition 4

We show that any solution W that satisfies (47) at $(X,0)$ for X in a neighborhood of S_0 is invalid.

There are three cases, either (65), (66), or (67) holds at such points. The first two correspond to Part 1 and the last one, Part 2. Suppose (65) holds. Then (47) takes the form (68). For this solution to be valid, it is necessary that (69) holds. As in the proof of Proposition 3, the left- and right-hand sides of (69) satisfy

$$G_X(S_0) - F_X(S_0) = \frac{r}{x_{\min}}\left\{-\frac{\xi S_0}{x_{\min}} + R_0 - C_0\right\}.$$

Since by (26)

$$R_0 - C_0 < \frac{2\xi S_0}{x_{\min}} \frac{1}{x_{\min} + \sqrt{x_{\min}^2 + 2\xi S_0}} < \frac{\xi S_0}{x_{\min}},$$

it follows that

$$F_X(S_0) > G_X(S_0).$$

Hence,

$$F(X) < G(X)$$

for X near S_0. This contradicts (69). Therefore, such a solution does not exist.

Suppose (66) holds. We follow the proof of Proposition 3 to obtain

$$L_X(S_0) = r\left[A - B_0 + C_0\right], \qquad M_X(S_0) = r\left[\frac{A^2 + x_{\min}^2}{2x_{\min}} - \frac{\xi S_0}{x_{\min}} - B_0 + R_0\right],$$

where

$$A = B_0 - C_0 + W_X(S_0,0).$$

Hence

$$M_X(S_0) - L_X(S_0) = r\left\{\frac{A^2 + x_{\min}^2}{2x_{\min}} - \frac{\xi S_0}{x_{\min}} - A + R_0 - C_0\right\}. \tag{80}$$

The right-hand side is a quadratic function in A with the minimum at $A = x_{\min}$. Note that (66) implies that

$$A > x_{\min}, \qquad \frac{A^2 + x_{\min}^2}{2x_{\min}} - \frac{\xi S_0}{x_{\min}} \leq x_{\min}.$$

The second inequality leads to

$$A \leq \sqrt{x_{\min}^2 + 2\xi S_0}.$$

By (26) and (80),

$$M_X(S_0) - L_X(S_0) \le r\left\{x_{\min} - \sqrt{x_{\min}^2 + 2\xi S_0} + R_0 - C_0\right\} < 0.$$

Therefore,

$$L(X) < M(X)$$

for X near S_0.

Suppose (67) holds. We follow the proof of Proposition 3 to obtain

$$\begin{aligned} U_X(S_0) &= r\{A - B_0 + C_0\}, \\ V_X(S_0) &= \frac{r}{A}\left\{A^2 - 2\xi S_0 - \left[B_0 - R_0 + W_Y(S_0, 0)\right]\left[B_0 - R_0\right]\right\}. \end{aligned}$$

Hence,

$$U_X(S_0) > V_X(S_0)$$

if

$$\left[B_0 - C_0\right] A < 2\xi S_0 + \left[B_0 - R_0 + W_Y(S_0, 0)\right]\left[B_0 - R_0\right]. \tag{81}$$

It is reasonable to assume that $W_X \le 0$. Hence,

$$A = B_0 - C_0 + W_X(S_0, 0) \le B_0 - C_0.$$

Also, by assumption,

$$B_0 - R_0 + W_Y(S_0, 0) = y_b^* > x_{\min}.$$

Therefore, (81) holds if (27) holds.

This completes the proof.

A.5. Proposition 6 and its proof

Proposition 6. *Let $X_0 \in (0, S_0)$. If $x_b^* = 0$ along the vertical line (X_0, Y) approaching $(X_0, \eta X_0) \in \Gamma_2$, then $y_b^* = 0$ at $(X_0, \eta X_0)$.*

Proof.

Suppose the opposite is true. That is, suppose $y_b^* > 0$ at $(X_0, \eta X_0)$. Then it is necessary that $y_b^* = \eta x_b^*$ at $(X_0, \eta X_0)$. As

$$\begin{aligned} rW(X_0, \eta X_0) &= B_0\left(x_b^* + y_b^*\right) - \frac{\left(x_b^* + y_b^*\right)^2}{2} - C_0 x_b^* - \xi X_0 - R_0 y_b^* \\ &\quad + x_b^* W_X(X_0, \eta X_0) + y_b^* W_Y(X_0, \eta X_0) \\ &= x_b^*\left\{\left[B_0 - C_0 + W_X(X_0, \eta X_0)\right] + \eta\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right]\right\} \\ &\quad - \frac{(1+\eta)^2\left(x_b^*\right)^2}{2} - \xi X_0, \end{aligned}$$

Either

$$\begin{aligned} rW(X_0, \eta X_0) = \frac{x_{\min}}{1+\eta}&\left\{\left[B_0 - C_0 + W_X(X_0, \eta X_0)\right]\right. \\ &\left. + \eta\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right]\right\} - \frac{x_{\min}^2}{2} - \xi X_0 \end{aligned} \tag{82}$$

if

$$\left[B_0 - C_0 + W_X(X_0, \eta X_0)\right] + \eta\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right] \le x_{\min} \tag{83}$$

or

$$\begin{aligned} rW(X_0, \eta X_0) = \frac{1}{2(1+\eta)^2}&\left\{\left[B_0 - C_0 + W_X(X_0, \eta X_0)\right]\right. \\ &\left. + \eta\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right]\right\}^2 - \xi X_0 \end{aligned} \tag{84}$$

if

$$\left[B_0 - C_0 + W_X(X_0, \eta X_0)\right] + \eta\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right] > x_{\min}. \tag{85}$$

There are three cases: Either

$$rW(X_0, \eta X_0) \le \frac{x_{\min}^2}{2} - \xi X_0, \tag{86}$$

or

$$\frac{x_{\min}^2}{2} - \xi X_0 < rW(X_0, \eta X_0) \le \frac{x_{\min}^2}{2}, \tag{87}$$

or

$$rW(X_0, \eta X_0) > \frac{x_{\min}^2}{2}. \tag{88}$$

We examine each case below.

Suppose (86) holds. At points $(X, Y) \in \Omega$ near $(X_0, \eta X_0)$, W satisfies either (42) or (47). If $W(X_0, Y)$ satisfies (42) for Y near ηX_0, because of (86), (42) becomes (44) near $(X_0, \eta X_0)$. For the solution to be valid, it is necessary that

$$B_0 - C_0 + W_X(X_0, Y) - \frac{\xi X_0}{x_{\min}} \le B_0 - R_0 + W_Y(X_0, Y). \tag{89}$$

Note that $W(X_0, \eta X_0)$ satisfies both (44) and (82). The two equations lead to

$$\begin{aligned} B_0 - C_0 + W_X(X_0, Y) - \frac{\xi X_0}{x_{\min}} &= \frac{r}{x_{\min}} W(X_0, \eta X_0) + \frac{x_{\min}}{2} + \frac{\eta \xi X_0}{x_{\min}}, \\ B_0 - R_0 + W_Y(X_0, \eta X_0) &= \frac{r}{x_{\min}} W(X_0, \eta X_0) + \frac{x_{\min}}{2}. \end{aligned}$$

Hence, (89) is false for Y near ηX_0. This means there is no solution with $x_b^* = 0$ near Γ_2 if (86) holds.

Suppose (87) holds. Then (42) still takes the form of (44). The solution is valid if

$$\frac{1}{2}\left[B_0 - C_0 + W_X(X_0, Y)\right]^2 - \xi X_0 \le \left[B_0 - R_0 + W_Y(X_0, Y)\right] x_{\min} - \frac{x_{\min}^2}{2}. \tag{90}$$

We let $F(Y)$ and $G(Y)$ denote the left- and right-hand sides of the above inequality, respectively. Note that $W(X_0, \eta X_0)$ satisfies both (44) and either (82) if (83) holds, and (84) if (85) holds. In the former case,

$$\begin{aligned} B_0 - R_0 + W_Y(X_0, \eta X_0) &= \frac{r}{x_{\min}} W(X_0, \eta X_0) + \frac{x_{\min}}{2}, \\ B_0 - C_0 + W_X(X_0, \eta X_0) &= \frac{r}{x_{\min}} W(X_0, \eta X_0) + \frac{x_{\min}}{2} + \frac{1+\eta}{x_{\min}} \xi X_0. \end{aligned}$$

It follows that

$$F(\eta X_0) = \frac{1}{2x_{\min}^2}\left\{G(\eta X_0) + (1+\eta)\xi X_0 + \frac{x_{\min}^2}{2}\right\}^2 - \xi X_0.$$

Inequality (90) is equivalent to

$$\frac{K^2}{2x_{\min}^2} - K + \eta \xi X_0 + \frac{x_{\min}^2}{2} \le 0,$$

where

$$K = G(Y) + (1+\eta)\xi X_0 + \frac{x_{\min}^2}{2}.$$

However, since

$$\frac{K^2}{2x_{\min}^2} - K + \eta \xi X_0 + \frac{x_{\min}^2}{2} = \frac{1}{2}\left(\frac{K}{x_{\min}} - x_{\min}\right)^2 + \eta \xi X_0 > 0,$$

the inequality cannot hold. So no valid solution exists in this case.

If (84) and (85) both hold, we let

$$\begin{aligned} U &= B_0 - C_0 + W_X(X_0, \eta X_0) = \sqrt{2F(\eta X_0) + 2\xi X_0}, \\ V &= \sqrt{2G(\eta X_0) + 2\xi X_0}. \end{aligned}$$

Then (84) and (85) lead to

$$(1+\eta)V = U + \eta\left[\frac{G(\eta X_0)}{x_{\min}} + \frac{x_{\min}}{2}\right] = U + \eta\left[\frac{V^2}{2x_{\min}} - \frac{\xi X_0}{x_{\min}} + \frac{x_{\min}}{2}\right].$$

Inequality $F \le G$ is equivalent to $U \le V$. Thus,

$$(1+\eta)V - \eta\left[\frac{V^2}{2x_{\min}} - \frac{\xi X_0}{x_{\min}} + \frac{x_{\min}}{2}\right] \le V.$$

This is equivalent to

$$-\frac{V^2}{2x_{\min}} + V - \frac{x_{\min}}{2} + \frac{\xi X_0}{x_{\min}} \le 0.$$

This inequality is satisfied for

$$V \le x_{\min} - \sqrt{2\xi X_0} \text{ or } V \ge x_{\min} + \sqrt{2\xi X_0}. \quad (91)$$

On the other hand, by (66),

$$B_0 - R_0 + W_Y(x, \eta X_0) \le x_{\min} < B_0 - C_0 + W_X(x, \eta X_0).$$

It follows that

$$x_{\min} \le U \le V \le \sqrt{x_{\min}^2 + 2\xi X}. \quad (92)$$

Since

$$\sqrt{x_{\min}^2 + 2\xi X} < x_{\min} + \sqrt{2\xi X},$$

(92) contradicts (91). Thus, there is no valid solution in this case.

Suppose (88) holds. Then (42) takes the form

$$rW = \frac{1}{2}\left[B_0 - R_0 + W_Y\right]^2 \qquad \text{for } Y < \eta X \quad (93)$$

and the solution is valid if

$$\frac{1}{2}\left[B_0 - C_0 + W_X(X_0, Y)\right]^2 - \xi X_0 \le \frac{1}{2}\left[B_0 - R_0 + W_Y(X_0, Y)\right]^2. \quad (94)$$

At $Y = \eta X_0$, both (84) and (93) hold. Hence,

$$\begin{aligned} &\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right]^2 = 2rW(X_0, \eta X_0) \\ = &\ \frac{1}{(1+\eta)^2}\left\{\left[B_0 - C_0 + W_X(X_0, \eta X_0)\right] + \eta\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right]\right\}^2 \\ &- 2\xi X_0. \end{aligned}$$

This implies

$$\begin{aligned} B_0 - C_0 + W_X(X_0, \eta X_0) &= (1+\eta)\sqrt{\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right]^2 + 2\xi X_0} \\ &\quad - \eta\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right] \\ &> \sqrt{\left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right]^2 + 2\xi X_0}. \end{aligned}$$

As a result,

$$\left[B_0 - C_0 + W_X(X_0, \eta X_0)\right]^2 - 2\xi X_0 > \left[B_0 - R_0 + W_Y(X_0, \eta X_0)\right]^2.$$

This contradicts (94) for Y near ηX_0. Hence, there is no valid solution in this case.

This completes the proof.

A.6. Optimal controls and HJB equations on Γ_2

Below, we derive the HJB equations for W on Γ_2 at the points reached by recycling while extraction is paused. Let $(X, \eta X) \in \Gamma_2$ be a point reached by a vertical line (X, Y), $Y < \eta X$, along which $x_b^* = 0$. By Proposition 6, $y_b^* = 0$ at $(X, \eta X)$. Hence, the central planner must choose a value $x \ge x_{\min}$ at $(X, \eta X)$ for a brief moment Δt. This will change the state to $(X + x\Delta t, \eta X)$. Since $y_b^* > 0$ in the interior near $(X, \eta X)$, the central planner must choose $y \ge x_{\min}$ to return to Γ_2. The time $\Delta\tilde{t}$ to return to Γ_2 satisfies

$$y\Delta\tilde{t} = \eta x \Delta t. \quad (95)$$

This leads to

$$\Delta\tilde{t} = \frac{\eta x}{y}\Delta t, \qquad \eta X + y\Delta\tilde{t} = \eta(X + x\Delta t). \quad (96)$$

Let $\overline{W}(X) = W(X, \eta X)$. After the zig-zag steps that take an amount of time

$$\Delta t + \Delta\tilde{t} = \left(1 + \frac{\eta x}{y}\right)\Delta t, \quad (97)$$

the state is $(X + x\Delta t, \eta(X + x + \Delta)) \in \Gamma_2$. Therefore, by dynamic programming,

$$\begin{aligned} \overline{W}(X) = \max_{x, y \ge x_{\min}} &\left\{\left[B_0 x - \frac{x^2}{2} - C_0 x - \xi X\right]\Delta t + \left[B_0 y - \frac{y^2}{2} - R_0 y\right]\Delta\tilde{t}\right. \\ &\left. + e^{-r(\Delta t + \Delta\tilde{t})} Z(X + x\Delta t)\right\} \\ = \max_{x, y \ge x_{\min}} &\left\{\left[B_0(1+\eta)x - \frac{1}{2}(x^2 + xy) - \left[C_0 + \eta R_0\right]x - \xi X\right]\Delta t\right. \\ &\left. + e^{-r\left(1 + \frac{\eta x}{y}\right)\Delta t} Z(X + x\Delta t)\right\}. \end{aligned}$$

We use the Taylor expansion

$$e^{-r\left(1+\frac{\eta x}{y}\right)\Delta t}\overline{W}(X + x\Delta t) = \overline{W}(X) + \left\{-r\left(1 + \frac{\eta x}{y}\right)\overline{W}(X) + \overline{W}_X(X)x\right\}\Delta t + o(\Delta t).$$

Subtracting $\overline{W}(X)$ from both sides, dividing the result by Δt, and letting $\Delta t \to 0$, we obtain

$$\begin{aligned} r\overline{W}(X) + \xi X = \max_{x, y \ge x_{\min}} &\left\{\left[B_0(1+\eta)x - \frac{1}{2}(x^2 + xy) - \left[C_0 + \eta R_0\right]x\right.\right. \\ &\left.\left. - \frac{x}{y}r\eta\overline{W}(X) + \overline{W}_X(X)x\right]\right\}. \end{aligned}$$

Then, four possible outcomes appear:

(I) If

$$B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \sqrt{2r\eta\overline{W}} > x_{\min}, \qquad \sqrt{2r\eta\overline{W}} > x_{\min},$$

the maximizer (x_b^*, y_b^*) is

$$x_b^* = B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \sqrt{2r\eta\overline{W}}, \qquad y_b^* = \sqrt{2r\eta\overline{W}}.$$

In this case,

$$r\overline{W} + \xi X = \frac{1}{2}\left[B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \sqrt{2r\eta\overline{W}}\right]^2.$$

(II) If

$$B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \frac{r\eta\overline{W}}{x_{\min}} - \frac{x_{\min}}{2} > x_{\min}, \qquad \sqrt{2r\eta\overline{W}} \le x_{\min},$$

then

$$x_b^* = B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \frac{r\eta\overline{W}}{x_{\min}} - \frac{x_{\min}}{2}, \qquad y_b^* = x_{\min}$$

and

$$r\overline{W} + \xi X = \frac{1}{2}\left[B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \frac{r\eta\overline{W}}{x_{\min}} - \frac{x_{\min}}{2}\right]^2.$$

(III) If

$$B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \sqrt{2r\eta\overline{W}} \le x_{\min}, \qquad \sqrt{2r\eta\overline{W}}, > x_{\min},$$

then

$$x_b^* = x_{\min}, \qquad y_b^* = \sqrt{2r\eta\overline{W}}$$

and

$$r\overline{W} + \xi X = \left[B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \sqrt{2r\eta\overline{W}}\right]x_{\min} - \frac{x_{\min}^2}{2}.$$

(IV) If

$$B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X - \sqrt{2r\eta\overline{W}} \le x_{\min}, \qquad \sqrt{2r\eta\overline{W}} \le x_{\min},$$

then $displaystyle x_b^* = y_b^* = x_{\min}$ and

$$r\overline{W} + \xi X = \left[B_0(1+\eta) - (C_0 + \eta R_0) + \overline{W}_X\right]x_{\min} - x_{\min}^2 - r\eta\overline{W}.$$

This completes the derivation of the HJB equations for $\overline{W}(X) = W(X, \eta X)$.

A.7. Proof Proposition 5 - Open-loop

The objective of the cartel is

$$\begin{aligned} \max_{x_n} \Pi_n &= \int_0^\infty e^{-rt}\left[\pi_n(x_n) - C(X_n, x, t)\right]dt \\ &= \int_0^{T_n} e^{-rt}\left[(a - bx_n)x_n - C_0 x - \xi X_n\right]dt \\ &\quad + \int_{T_n}^\infty e^{-rt}\left[(a - b(x_n + y_n))x_n - C_0 x - \xi X_n\right]dt \end{aligned}$$

subject to

$$\dot{X}_n = x_n(t), \quad t \geq 0, \quad X_n(0) = 0,$$

$$x_n(t) \geq x_{min} \quad \forall t \in [0, T_n] \qquad \text{and} \qquad x_n(t) \geq 0 \quad \forall t \geq T_n.$$

The objective of the importing country is

$$\max_{x_n} W_n = \int_0^\infty e^{-rt}\left[B_n(x_n, y_n) - \pi_n(x_n) - R(y_n, t)\right] dt$$
$$= \int_0^{T_n} e^{-rt}\left[B_0 x_n - \frac{x_n^2}{2} - (a - bx_n)x_n\right] dt$$
$$+ \int_{T_n}^\infty e^{-rt}\left[B_0(x_n + y_n) - \frac{(x_n + y_n)^2}{2} - (a - b(x_n + y_n))x_n - R_0 y\right] dt$$

subject to

$$\dot{Y}_n = y_n(t), \quad t \geq T_n, \quad Y(T_n) = 0,$$

and $y_n \geq 0$, $Y_n(t) \leq \eta X(t)$ and $x_n + y_n \geq x_{\min}$.

Though intuitively, the last inequality constraints, $Y_n(t) \leq \eta X(t)$ and $x_n + y_n \geq x_{\min}$, are not the concern of the cartel but, rather, the recycler, the cartel also needs to take into account the market demand as well as the supply to the market, as mentioned above in regards to the dilemma both players face.

Repeat the Hamiltonian of the cartel as

$$H_{c,I}(x_n, X_n, \lambda_x, \eta_x, \nu_x, \mu_x) = [(a - b(x_n + y_n))x_n - C_0 x_n - \xi X_n]$$
$$+\lambda_x x_n + \eta_x x_n + \nu_x(\eta X_n - Y_n) + \mu_x(x_n - x_{min})$$

and the Hamiltonian of the importer as

$$H_I(y_n, Y_n, \lambda_y, \eta_y, \nu_y, \mu_y) = \left[B_0(x_n + y_n) - \frac{(x_n + y_n)^2}{2}\right]$$
$$-[a - b(x_n + y_n)]x_n - R_0 y_n + \lambda_y y_n + \eta_y y_n + \nu_y(\eta X_n - Y_n) + \mu_y(x_n + y_n - x_{min}).$$

The first order conditions yield

$$\begin{cases} \dfrac{\partial H_{c,I}}{\partial x_n} = \left[a - b(2x_n + y_n) - C_0\right] + \lambda_x + \eta_x + \mu_x = 0 \\ \dot{\lambda}_x = r\lambda_x - \xi + \eta \nu_x, \quad \lim_{t\to\infty} e^{-rt}\lambda_x X_n(t) = 0, \\ \mu_x \geq 0, \quad x_n \geq x_{min}, \quad \mu_x(x_n - x_{min}) = 0, \quad \forall t \in [0, T_n] \\ \eta_x \geq 0, \quad x_n \geq 0, \quad x_n\eta_x = 0, \quad \forall t \geq T_n, \\ \nu_x \geq 0, \quad \eta X \geq Y, \quad \nu_x(\eta X_n - Y_n) = 0, \\ \dot{X}_n = x_n, \quad X_n(0) = 0 \end{cases} \tag{98}$$

and

$$\begin{cases} \dfrac{\partial H_I}{\partial y_n} = \left[B_0 - (x_n + y_n) - R_0 + bx_n\right] + \lambda_y + \eta_y + \nu_y + \mu_y = 0 \\ \dot{\lambda}_y = r\lambda_y - \nu_y, \quad \lim_{t\to\infty} e^{-rt}\lambda_y Y_n(t) = 0, \\ \mu_y \geq 0, \quad x_n + y_n \geq x_{min}, \quad \mu_y(x_n + y_n - x_{min}) = 0, \quad \forall t \geq T_n, \\ \nu_y \geq 0, \quad \eta X \geq Y, \quad \nu_y(\eta X_n - Y_n) = 0, \quad \forall t \geq T_n, \\ \eta_y \geq 0, \quad y_n \geq 0, \quad \eta_y y_n = 0, \quad \forall t \geq T_n, \\ \dot{Y}_n = y_n, \quad Y_n(T_n) = 0. \end{cases} \tag{99}$$

Note that the shadow value λ_x is non-positive given the state variable $X_n(t)$ measures the accumulated supply to the market. Thus, from the cartel's point of view, $X_n(t)$ is the loss of windfall resource and its shadow value cannot be positive. Similarly, λ_y is the shadow value of state variable Y_n, measuring the recycled used material; thus, its value for the recycler is also non-positive.

We complete the analysis in a few steps. First, period II, $t \geq T_n$: Step 1, normal case where both $x_n > 0$, $y_n > 0$ and $x_n + y_n \geq x_{min}$; Step 2, $x_n > 0$, $y_n > 0$ and $x_n + y_n = x_{min}$; and then period I, $0 \leq t \leq T_n$.

Except for $x_n > 0$, $y_n > 0$, and $x_n + y_n \geq x_{min}$, we also suppose that $Y_n < \eta X_n$. Then it follows that $\eta_x = \eta_y = 0$, $\mu_x = \mu_y = 0$, $\nu_x = \nu_y = 0$. The FOCs (98) and (99) can be reduced to the following simplified forms:

$$\begin{cases} a - b(2x_n + y_n) - C_0 = -\lambda_x, \\ B_0 - (x_n + y_n) - R_0 + bx_n = -\lambda_y, \\ \dot{\lambda}_x = r\lambda_x - \xi, \\ \dot{\lambda}_y = r\lambda_y, \end{cases} \tag{100}$$

with state equation

$$\begin{cases} \dot{X}_n = x_n, \quad \forall t \geq 0, \quad X_n(0) = 0, \\ \dot{Y}_n = y_n, \quad \forall t \geq T_n, \quad Y_n(T_n) = 0. \end{cases}$$

First, note in (100) that the left-hand side of the first equation is the cartel's marginal gain net of marginal costs and the right-hand side is the shadow value of supplying to the market. Thus, $\lambda_x \leq 0$. Similarly, the second equation's right-hand side is the importing country's shadow value of the recycling mineral. The left-hand side is the marginal benefit net of marginal costs for the importing country, in which the term bx_n comes from the decrease in prices by providing y_n of recycled mineral to the market and thus no payment to the cartel. Thus, $\lambda_y \leq 0$.

Obviously, the solution from (100), x_n, y_n, λ_x, and λ_y are independent of the state variables X_n and Y_n, at least in the second period. Thus, the solution provided by (100) is not only open-loop but is also a Markovian strategy (although it may not be subgame perfect if the equilibrium depends on the initial condition at T_n).

It is easy to see that the two co-state variables are

$$\begin{cases} \lambda_x(t) = \left(\lambda_x(T_n) - \dfrac{xi}{r}\right)e^{r(t-T_n)} + \dfrac{\xi}{r}, \\ \lambda_y(t) = \lambda_y(T_n)e^{r(t-T_n)} \end{cases} \qquad \forall t \geq T_n,$$

where $\lambda_x(T_n)$ and $\lambda_y(T_n)$ are given by

$$\begin{cases} \lambda_x(T_n) = -a + b(2x_n(T_n) + y_n(T_n)) + C_0(\leq 0), \\ \lambda_y(T_n) = -B_0 + (x_n(T_n) + y_n(T_n)) - bx_n(T_n) + R_0(\leq 0), \end{cases}$$

with $x_n(T_n)$ and $y_n(T_n)$ undetermined.

Thus, considering that λ_x and λ_y are known functions, for $t \geq T_n$,

$$\begin{cases} x_n(t) = \dfrac{1}{b(b+1)}\left[a - bB_0 + bR_0 e^{-\rho t} - C_0 + \lambda_x(t) - b\lambda_y(t)\right], \\ \quad = \dfrac{1}{b(b+1)}\left[A + [\lambda_x(T_n) - \dfrac{\xi}{r} - b\lambda_y(T_n)]e^{r(t-T_n)}\right](\geq 0), \\ y_n(t) = B_0 - R_0 + (b-1)x_n + \lambda_y(t), \end{cases}$$

where constant $A = a - bB_0 + bR_0 - C_0$.

Thus,

$$x_n^o(t) + y_n^o(t) = \frac{1}{(b+1)}\left[A_1 + \lambda_x(t) + \lambda_y(t)\right] \geq x_{min}, \tag{101}$$

where $A_1 = a + B_0 - R_0 - C_0$, and it is easy to see that $A_1 = A + (1 + b)(B_0 - R_0) > 0$.

Given that both $\lambda_x(t)$ and λ_y converge to $-\infty$ for $t \to +\infty$ and that x_n and y_n are monotonically decreasing over time, there must exist a finite time $T_{n2} > T_n$ such that $\forall t \geq T_{n2}$,

$$x_n^o(t) + y_n^o(t) = x_{\min},$$

and T_{n2} is given by

$$A_1 + \frac{\xi}{r} - (b+1)x_{min} = \left[\frac{\xi}{r} - \lambda_x(T_n) - \lambda_y(T_n)\right]e^{r(T_{n2}-T_n)}. \tag{102}$$

A necessary condition for the above to be true is that

$$A_1 - (b+1)x_{min} > -(\lambda_x(T_n) + \lambda_y(T_n))(> 0). \tag{103}$$

Furthermore, there must exist a finite time $T_{n3} \geq T_n$ such that at T_{n3} the virgin resource is exhausted and after that the cartel is out of the supply market:

$$x_n^o(t) = 0 \qquad t \geq T_{n3},$$

and thus T_{n3} is given by

$$\int_{T_n}^{T_{n3}} x_n^o(t)dt = S(0) - X_n(T_n). \tag{104}$$

Obviously, if $T_{n3} \geq T_{n2}$, it follows that

$$x_n^o(t) + y_n^o(t) = x_{min}, \qquad T_{n2} \leq t \leq T_{n3}.$$

After the virgin resource is exhausted, the recycling supply must satisfy

$$y_n^o(t) = B_0 - R_0 + \lambda_y(t) \geq x_{min}$$

until $T_{n4}(\geq T_n)$. At T_{n4},

$$\lambda_y(T_n)e^{r(T_{n4}-T_n)} = x_{min} + R_0 - B_0$$

and

$$y_n^o(t) = x_{min}, \qquad t \geq T_{n4}.$$

Finally, at $T_{n5} \geq T_{n4}$ recycling has exhausted all of the mineral :

$$\int_{T_{n4}}^{T_{n5}} y_n^o(t)dt = x_{min}[T_{n5} - T_{n4}] = \eta S(0).$$

The conclusion of the above analysis is presented in Proposition . That completes the proof.

A.8. Period I

In this period, define the cartel's Hamiltonian as

$$H_{c,I}(x_n, X_n; \lambda_x, \mu_x) = (a - bx_n)x_n - C_0x_n - \xi X_n + \lambda_x x_n + \mu_x(x_n - x_{min}).$$

The first-order condition for optimality can be simplified as

$$\begin{cases} \left[a - 2bx_n - C_0\right] + \lambda_x + \mu_x = 0, \\ \dot{\lambda}_x = r\lambda_x - \xi, \\ \mu_x \geq 0, \quad x_n \geq x_{min}, \quad \mu_x(x_n - x_{min}) = 0, \quad \forall t \in [0, T_n], \\ \dot{X}_n = x_n, \quad X_n(0) = 0, \end{cases} \tag{105}$$

with the transversality condition

$$\lambda_x(T_n^-) = \lambda_x(T_n^+). \tag{106}$$

Considering $x_n \geq x_{min}$, $\mu_x = 0$. Applying the transversality condition (106), the first equation in (105) yields

$$x_n^*(t) = \max\left\{x_{min}, \frac{1}{2b}\left(a - C_0 + \lambda_x(t)\right) = \frac{1}{2b}\left[a - C_0 + \left(\lambda_x(T_n) - \frac{\xi}{r}\right)e^{-r(T_n-t)} + \frac{\xi}{r}\right]\right\}. \tag{107}$$

It is easy to see that $x_n(t)$ is monotonically decreasing over time. Thus, if at $t = 0$, $x_n(0) \leq x_{min}$, we must have $x_n(t) = x_{min}$ for all $t \in [0, T_n]$. Inequality condition $x_n(0) \leq x_{min}$ holds if and only if

$$\lambda_x(T_n) \leq \left(-a - \frac{\xi}{r} + 2bx_{min} + C_0\right)e^{rT_n} + \frac{\xi}{r} \equiv \underline{\lambda_x(T_n)}(< 0).$$

In this case, for any $t \in [0, T_n]$,

$$x_n(t) = x_{min} \quad \text{and} \qquad X(T_n) = T_n x_{min}(\leq S(0)).$$

If $\lambda_x(T_n) > \underline{\lambda_x(T_n)}$, there are two possibilities: (a) For all $t \leq T_n$, $x_n(t) > x_{min}$; or (b) there exists $T_{n1} \leq T_n$ such that $x_n(T_{n1}) = x_{min}$, and thus optimal exploiting is

$$x_n(t) = \begin{cases} \frac{1}{2b}\left[a + \left(\lambda_x(T_n) - \frac{\xi}{r}\right)e^{-r(T_n-t)} + \frac{\xi}{r} - C_0\right](> x_{min}), & 0 \leq t < T_{n1}, \\ x_{min}, \quad T_{n1} \leq t \leq T_n. \end{cases}$$

Either way,

$$X(T_n) = \int_0^{T_n} x(t)dt \leq S(0).$$

Combining Period I and Period II together, the transversality condition

$$H_{c,I}((T_n^-)) = H_{c,II}(T_n^+). \tag{108}$$

will uniqueness determines the left unknown $\lambda_x(T_n)$.

Finally, by combining Period I and Period II, i.e., combining the transversality condition with Eqs. (36), (37), (38), and (106), all unknowns T_{n2}, T_{n3}, T_{n5} and $\lambda_x(T_n)$, $\lambda_x(T_n)$ can be determined.

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