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ON SOME ASYMPTOTIC RESULTS ON FUNCTIONALS OF WEAKLY STATIONARY RANDOM FIELDS

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Abstract

Functionals of random fields have always been a central topic in probability theory, since its inception as a subject of study. The latter include, among others, partial sums of random variables and geometric quantities associated to random functions on manifolds. In this thesis, we investigate the asymptotic probabilistic behaviour of integral functionals of weakly stationary random fields on expanding Euclidean domains, with a special focus on additive (or nonlinear) functionals of stationary Gaussian fields.

In Chapter 1 we first introduce the main mathematical objects and tools encountered in this work, concluding with an overview of the state of the art and our new contributions related to the main research questions of this thesis. The two main questions are the following: first, as the integration domain expands, does a central limit theorem hold? Second, given two expanding integration domains, what is the asymptotic covariance between their integral functionals?

Chapter 2 contains the paper "Spectral central limit theorem for additive functionals of isotropic and stationary Gaussian fields", written in collaboration with Ivan Nourdin. In this chapter, we prove that a large class of additive functionals of stationary, isotropic Gaussian fields satisfies a central limit theorem if an easily verifiable condition on the spectral measure holds. This result brings to light a new class of "strongly correlated" Gaussian fields whose additive functionals satisfy a central limit theorem. This fact contradicts the intuition forged in the last four decades, starting from the seminal works by Breuer, Dobrushin, Major, Rosenblatt and Taqqu.

Chapter 3 contains the paper "Fluctuations of polyspectra in spherical and Euclidean random wave models", written in collaboration with Francesco Grotto and Anna Paola Todino. Our main result provides the variance rate of any additive functional of Euclidean (Berry's random wave model) and spherical random waves, a problem that was left as a conjecture ten years ago. To do this, we exploit a relation between random waves and Pearson's random walks.

Chapter 4 contains the paper "Asymptotic covariances for functionals of weakly stationary random fields". Here we compute the asymptotic covariances of integral functionals of weakly stationary random fields on expanding domains under assumptions that encompass the ones in the literature, deriving an explicit formula that involves the directional derivative of the cross covariogram of two domains.

Chapter 5 contains the preprint "Limit theorems for p-domain functionals of stationary Gaussian fields", written in collaboration with Nikolai Leonenko, Ivan Nourdin and Francesca Pistolato. In this chapter we consider more general families of additive functionals, which we call p-domain functionals, including as a special case spatio-temporal functionals and 1-domain functionals considered in the previous chapters. In this setting, we are able (under suitable assumptions) to reduce the study of p-domain functionals to that of some 1-domain functionals, explaining some recent findings in the literature in a new light.

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CHAPTER 1

Introduction

In this chapter, we highlight what are the novel ideas of this thesis. To do this, we proceed as follows. In Section 1.1 we define the main mathematical objects of the thesis. In Section 1.2 we provide a concise introduction to the needed mathematical tools. In Section 1.3 we give an overview of the state of the art related to our main research questions. Finally, in Section 1.4 we present our contributions in a concise form, comparing them with the existing results of the previous section.

1.1 Mathematical objects

In this section, we introduce the main mathematical objects of the thesis: functionals of random fields and the classes of random fields whose functionals will be studied. In particular, we deal with functionals of random fields in Subsection 1.1.1, invariant random fields in Subsection 1.1.2 and Gaussian fields in Subsection 1.1.3.

1.1.1 Functionals of random fields

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a set E and a measurable space $(S, \mathcal{S})$. We start with the definition of S-valued random field on E.

Definition 1.1.1. A S-valued **random field** A on E is a function

$$A : \Omega \times E \to S, \qquad (\omega, x) \mapsto A_x(\omega), \tag{1.1}$$

such that $A_x := A_x(\cdot)$ is a S-valued random variable for every $x \in E$.

Equivalently, a S-valued random field on E is a collection of S-valued random variables $A = (A_x)_{x \in E}$, or a S^E-valued random variable, where S^E is the set of all the functions $E \mapsto S$ equipped with the product sigma-algebra. Since the three definitions are equivalent, we will use them interchangeably.

Terminology. In this manuscript, a random variable is a measurable function $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel sigma-algebra. Moreover, unless otherwise stated, random *objects* (variables, vectors, fields) are all real-valued. Alternatively, if they take values in a measurable space $(S, \mathcal{S})$, we call them S-valued random *objects*. For example, a random field on E is a $\mathbb{R}^E$-valued random variable.

Random fields can model virtually every uncertain quantity A_x that depends on a variable x in a given window of values E. In applications, they appear whenever an exact prediction is impossible. For instance E may represent a time interval, a space domain, a set of space-time configurations, a set of indices associated with individuals in a given population, or every possible inputs of a neural network with random weights.

The second definition of this thesis is that of functional of a random field, which we state here in full generality.

Definition 1.1.2. Let A be a random field on E, seen as a $\mathbb{R}^E$-valued random variable $A : \Omega \to \mathbb{R}^E$, with realizations in $\mathrm{Im}(A) := \{A(\omega) : \omega \in \Omega\} \subseteq \mathbb{R}^E$. A **functional** of A is a random variable $Y = f \circ A$, where $f : \mathcal{U} \to \mathbb{R}$, with $\mathrm{Im}(A) \subseteq \mathcal{U}$.

Examples of functionals of random fields abound in the literature, and arise in every kind of application. We focus here on three examples, which can well highlight their omnipresence.

Example 1.1.1 (Sums of random variables)**.** As a first example, just consider a sequence $A_1, A_2, \dots$ of random variables on the same probability space. $A = (A_i)_{i \in \mathbb{N}}$ is obviously a random field on $\mathbb{N}$. An example of functional of A is the sum of the first $n \in \mathbb{N}$ elements of A, namely

$$Y_n := \sum_{i=1}^{n} A_i, \quad n \in \mathbb{N}. \tag{1.2}$$

In this case $Y_n = f_n \circ A$, where $f_n(a_1, a_2, \dots) = a_1 + \dots + a_n$, is a functional of A for every $n \in \mathbb{N}$, see Definition 1.1.2. The study of Y_n as $n \to \infty$ has always been a central topic in probability, since its inception as a subject of study; just think of the law of large numbers, or the classical central limit theorem. For less classical examples, see e.g. [IL71; MPU19; Pel90].

Example 1.1.2 (Geometric functionals in $\mathbb{R}^d$)**.** Let A be a random field on $\mathbb{R}^d$. In this case, numerous functionals have been studied in the literature. Under suitable regularity assumptions on A, we have that $\mathrm{Im}(A) \subseteq L^1_{loc}(\mathbb{R}^d)$, so that one can consider for example a continuous version of (1.2)

$$Y_t := \int_{tD} A_x \, dx, \quad t > 0, \tag{1.3}$$

where D is a compact domain in $\mathbb{R}^d$. Note that integral functionals can also be used to investigate the large domain geometry of random fields in growing domains, studying Y_t as $t \to \infty$. Indeed, if for example $A_x = \mathbf{1}_{[a,\infty)}(B_x)$, where B is a random field on $\mathbb{R}^d$ and $a \in \mathbb{R}$, then Y_t is the so called **excursion volume** of B at level a in a growing window tD, namely

$$Y_t = \mathrm{Vol}\left(\{x \in \mathbb{R}^d : B_x \geq a\} \cap tD\right).$$

The latter has been extensively studied in the literature, see e.g. [DER20; KS18; LO14; MN24; Not21]. Moreover, many other classes of geometric functionals have been investigated, some of which do not admit a representation (1.3). Some examples are topological quantities related to the excursion (random) set $\{x \in \mathbb{R}^d : B_x \geq a\}$, such as its number of connected components, see e.g. [BMM22; NS16; NS20], or other functionals related to lower dimensional random sets, like the number of zeros, the number of critical points, or the lower dimensional volume of level sets, see e.g. [DEL21; GS23; NPV23; NPR19; PV20; Vid21].

Example 1.1.3 (Functionals of random waves)**.** Let $(\mathcal{M}, g)$ be a compact Riemannian manifold without boundary. Fix an orthonormal basis $\{\psi_{\lambda_j}\}$ of $L^2(\mathcal{M})$, composed by eigenfunctions of the Laplace-Beltrami operator Δ_g, so that

$$\Delta_g \psi_{\lambda_j} = \lambda_j^2 \psi_{\lambda_j} \quad \langle \psi_j, \psi_k \rangle = \delta_{jk}.$$

The monochromatic random wave $A^\lambda = (A_x^\lambda)_{x\in\mathcal{M}}$ is the random field on $(\mathcal{M}, g)$ defined as the random linear combination

$$A_x^\lambda := \sum_{\lambda_j \in [\lambda, \lambda+1]} \gamma_j \, \psi_{\lambda_j},$$

where $\gamma_1, \gamma_2, \ldots$ are i.i.d. standard Gaussian random variables. Geometric functionals of random waves have been extensively studied in the literature on different manifolds, see e.g. [Cam19; CMR23; Die+23; GP23; MW14; Tod19; Zel09]. A relevant fact proved by Canzani and Hanin in [CH15; CH18] (see also [Gas23]) is that, under suitable assumptions on $\mathcal{M}$, monochromatic random waves are locally approximated by a random field on $\mathbb{R}^{\dim(\mathcal{M})}$, the so-called Berry random wave model, which will be formally defined at the end of the section. As a consequence, one may try to reduce the small scale study of Riemannian geometric functionals of monochromatic random waves, as $\lambda \to \infty$, to that of Euclidean geometric functionals of Berry's random wave model. This has been done for example in [CH20; Die+23].

Throughout the thesis we will be interested in the following heuristic question: given a random field A on E and a family of functionals $Y = (Y_t)_{t>0}$ of A, what is the probabilistic behavior of Y_t as $t \to \infty$?

To be more precise, different features will be analyzed, such as the mean, the variance, or the distribution of Y_t as $t \to \infty$. In Example 1.1.1, for instance, if $A_1, A_2, ..$ are i.i.d. (independent and identically distributed) with mean $m \in \mathbb{R}$ and variance $\sigma^2 \in (0, \infty)$, then the mean and variance of Y_n are

$$\mathbb{E}[Y_n] = m\, n, \qquad \mathrm{Var}(Y_n) = n\, \sigma^2,$$

and the **classical central limit theorem** ensures that, as $n \to \infty$,

$$\frac{Y_n - \mathbb{E}[Y_n]}{\sqrt{\mathrm{Var}(Y_n)}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{A_i - m}{\sigma} \xrightarrow{d} N(0,1),$$

where $N(0,1)$ is a standard Gaussian and $\xrightarrow{d}$ denotes the convergence in distribution. At this point of the discussion, the definition of central limit theorem follows naturally.

Definition 1.1.3. Let A be a random field on E. Let $Y = (Y_t)_{t>0}$ be a family of functionals of A. Assume that for t sufficiently large $\mathrm{Var}(Y_t) \in (0,\infty)$. Then, Y satisfies a **central limit theorem (CLT)** if

$$\frac{Y_t - \mathbb{E}[Y_t]}{\sqrt{\mathrm{Var}(Y_t)}} \xrightarrow{d} N(0,1), \qquad \text{as } t \to \infty.$$

In order to prove central limit theorems (or other asymptotic results) for a family of functionals of A, we need assumptions on its distribution. The rest of this subsection is devoted to introducing the two main classes of random fields whose functionals are considered in this thesis: invariant random fields and Gaussian fields.

1.1.2 Invariant random fields

A very popular class of assumptions for random fields are **invariance properties**, such as stationarity (invariance by translations) and isotropy (invariance by rotations). These properties can all be defined through the action of a certain group $(\mathbb{G},+)$ on the index set E of the random field.

In the classical CLT, for example, since $A_1, A_2, \ldots$ are i.i.d., A is invariant with respect to the action of any group acting on $\mathbb{N}$. Let us give the precise definition.

Definition 1.1.4. Let A be a random field on E. Let $(\mathbb{G},+)$ be a group acting on E (from the left). A is said **G-invariant** if its finite dimensional distributions are invariant under the action of $\mathbb{G}$ (from the left), that is, $\forall\, n \in \mathbb{N}$, $\forall\, x_1, \ldots, x_n \in E$,

$$(A_{x_1}, \ldots, A_{x_n}) \stackrel{d}{=} (A_{g\cdot x_1}, \ldots, A_{g\cdot x_n}) \qquad \forall g \in \mathbb{G},$$

where $\cdot$ denotes the group action and $\stackrel{d}{=}$ the equality in distribution.

Depending on the group $\mathbb{G}$ acting on E, one can have many different kinds of invariance, see e.g. [Mal13; MP11]. In particular, the invariance property that stands out above all the others in this thesis is that of stationarity.

Definition 1.1.5. Let A be a random field on E. Suppose that E has a group structure $(E,+)$, and consider the standard (left) action on itself, namely $g \cdot x = g + x$ for every $g, x \in E$. A is said **stationary** if A is E-invariant, that is, $\forall\, n \in \mathbb{N}$, $\forall\, x_1, \ldots, x_n \in E$,

$$(A_{x_1}, \ldots, A_{x_n}) \stackrel{d}{=} (A_{g+x_1}, \ldots, A_{g+x_n}) \qquad \forall g \in E. \tag{1.4}$$

The easier cases to visualize stationarity are $E \in \{\mathbb{R}^d, \mathbb{Z}^d\}$, equipped with the sum operation. Indeed, in the latter situations A stationary means that the distribution of

A is invariant by translations[1]. We shall also consider the following weaker version of stationarity.

Definition 1.1.6. Let A be a random field on E. Suppose that E has a group structure $(E, +)$, and consider the standard (left) action on itself, namely $g \cdot x = g + x$ for every $g, x \in E$. A is said **weakly stationary** if for some function $K : E \to \mathbb{R}$ we have

$$\mathbb{E}[A_x] = \mathbb{E}[A_y], \qquad \mathrm{Cov}(A_x, A_y) = K(x - y), \qquad \forall x, y \in E.$$

If this is the case, K is said the **covariance function** of A.

Another important invariance property for random fields on $\mathbb{R}^d$ (or S^{d-1}) is **isotropy**, i.e. invariance by rotations.

Definition 1.1.7. Let A be a a random field on $E = \mathbb{R}^d$ (or $E = S^{d-1}$). Consider the special orthogonal group $\mathbb{G} = SO(d)$ of rotations, equipped with the standard matrix multiplication, and acting on E with the left multiplication $g \cdot x = g\,x$, for g matrix in $SO(d)$ and $x \in E$. A is said **isotropic** if A is $SO(d)$-invariant, that is, $\forall\ n \in \mathbb{N}$, $\forall\ x_1, \dots, x_n \in E$,

$$(A_{x_1}, \dots, A_{x_n}) \overset{d}{=} (A_{g\,x_1}, \dots, A_{g\,x_n}) \qquad \forall g \in SO(d).$$

For some examples of anisotropic random fields, see e.g. [EF20; Xia09]. In the following, especially to prove central limit theorems for functionals of $A = (A_x)_{x \in \mathbb{R}^d}$, we assume that A is stationarity and/or isotropic. When these assumptions are removed, another type of invariance is usually required, for instance self-similarity (i.e. scale invariance, see e.g. [HLX23; Jar+23]).

1.1.3 Gaussian fields

We start with the definition of Gaussian field. From now on, to specify that a random field is Gaussian, we shall denote it by the letter B (instead of A).

Definition 1.1.8. A random field B on E is said to be a **Gaussian field** on E if its finite dimensional distributions are Gaussian, that is, $\forall\ n \in \mathbb{N}$, $\forall\ x_1, \dots, x_n \in E$, $(B_{x_1}, \dots, B_{x_n})$ is a Gaussian (random) vector.

Gaussian fields are popular among random fields for two main reasons. First, they often turn out to be scaling limits of families of random fields, see e.g. [NPR19, p.100] or [EF20, (2.1)], hence their interpretation as good models in many applications. Second, their Gaussian finite dimensional distributions are known explicitly and satisfy very convenient analytical properties, such as being completely determined by the **mean function** $m : E \to \mathbb{R}$,

$$m(x) := \mathbb{E}[B_x],$$

and the **covariance kernel** $\mathcal{C} : E \times E \to \mathbb{R}$,

$$\mathcal{C}(x, y) := \mathrm{Cov}(B_x, B_y).$$

[1] Following this idea, one may also say that a random field A on $E \in \{\mathbb{R}_+, \mathbb{N}\}$ is stationary if (1.4) holds for every $x_1, \dots, x_n, g \in E$, even if $(E, +)$ is not a group.

Combining this with Kolmogorov existence theorem, we have the following.

Fact 1. Let E be a set. Then, the following statements are equivalent:

1. $\mathcal{C} : E \times E \to \mathbb{R}$ is a **non-negative definite kernel** on E, that is, $\forall\, n \in \mathbb{N}$, $\forall\, x_1, \dots, x_n \in E$, $\forall\, a_1, \dots, a_n \in \mathbb{R}$,
$$\sum_{i,j=1}^{n} a_i a_j\, \mathcal{C}(x_i, x_j) \geq 0.$$
2. There exist a Gaussian field B on E with covariance kernel $\mathcal{C}$.

Since the distribution of a Gaussian field B is completely characterized by its mean function m and its covariance kernel $\mathcal{C}$, also the invariance properties considered in the previous section can be expressed in terms of m and $\mathcal{C}$. In particular, if B is a Gaussian field on E, and $(E, +)$ has a group structure, then B is stationary if and only if it is weakly stationary, with constant mean function $m : E \to \mathbb{R}$, and covariance function $C : E \to \mathbb{R}$,

$$C(x - y) := \mathcal{C}(x, y), \qquad \forall\, x, y \in E.$$

In the same vein, if B is a Gaussian field on $\mathbb{R}^d$ (or S^{d-1}), then B is isotropic if and only if $\forall\, x, y \in E, \forall\, g \in SO(d)$

$$m(x) = m(g \cdot x), \qquad \mathcal{C}(x, y) = \mathcal{C}(g \cdot x, g \cdot y). \tag{1.5}$$

Note that if B is on S^{d-1}, then (1.5) implies that $\mathcal{C}(x, y)$ only depends on the (spherical) distance between x and y, which is generally not true (considering the Euclidean distance) for an isotropic field B on $\mathbb{R}^d$. However, if B is a stationary Gaussian field on $\mathbb{R}^d$, with constant mean function m and covariance function C, then B is isotropic if and only if the covariances depend only on the distances, that is, $\forall\, x, y \in \mathbb{R}^d$,
$$\mathcal{C}(x, y) = C(x - y) =: \rho(|x - y|),$$
where $|\cdot|$ denotes the standard Euclidean norm. In this case, the function $\rho : \mathbb{R}_+ \to \mathbb{R}$ is called the **isotropic covariance function** of B.

A Gaussian field that will often recur in the thesis is Berry's random wave model, which is defined as follows.

Definition 1.1.9. The (d-dimensional) Berry random wave model is the unique (in distribution) stationary, centered (i.e. with $m \equiv 0$), isotropic Gaussian field on $\mathbb{R}^d$ with isotropic covariance function

$$\rho(r) = b_d(r) := \Gamma\left(\frac{d}{2}\right)\left(\frac{2}{r}\right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(r)$$

where Γ denotes the Gamma function, and J_ν is the Bessel function of the first kind of order ν, defined as

$$J_\nu(r) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k + \nu + 1)} \left(\frac{r}{2}\right)^{2k+\nu}.$$

Berry's random wave model can be also characterized as the limit in finite dimensional distributions of a superposition of J independent plane waves with fixed wave number 1, as $J \to \infty$; or the unique isotropic centered Gaussian field B with unit variance satisfying $\Delta B + B = 0$, see e.g. [NPR19].

As we anticipated in Example 1.1.3, this Gaussian field, introduced by Berry in [Ber77], arises as the local scaling limit of random waves on a class of manifolds, hence its importance in applications. A crucial fact is that its covariance function, $x \to b_d(|x|)$, is the Fourier transform of the uniform distribution on S^{d-1}, a property that will be exploited several times in the sequel.

1.2 Mathematical tools

This section is devoted to the introduction of the main mathematical tools which will be applied in the thesis. In particular: in Subsection 1.2.1 we focus on the spectral representations for the covariance functions of invariant random fields; in Subsection 1.2.2 we focus on chaotic decompositions and Fourth Moment theorems for functionals of Gaussian fields; in Subsection 1.2.3 we define cross covariograms and discuss their properties and importance.

1.2.1 Spectral representations

In this subsection, we introduce one of the main tool of this thesis: spectral representations. We start with the definition of non-negative definite function.

Definition 1.2.1. Fix a group $(E, +)$. A symmetric function $C : E \to \mathbb{R}$ is non-negative definite on E if $\forall\ n \in \mathbb{N}$, $\forall\ x_1, \dots, x_n \in E$, $\forall\ a_1, \dots, a_n \in \mathbb{R}$,

$$\sum_{i,j=1}^{n} a_i a_j\, C(x_i - x_j) \geq 0.$$

Equivalently, a non-negative definite function C on a group E is the covariance function of a stationary centered Gaussian field B on E (see Fact 1), namely

$$C(z) = \mathbb{E}[B_0 B_z] = \mathbb{E}[B_x B_{x+z}] \qquad \forall x \in E. \tag{1.6}$$

Using this fact, we easily obtain these useful properties.

Proposition 1.2.1. *Fix a group $(E, +)$. Let $C, C_1, \dots, C_n$ be non-negative definite functions on E and $a_1, \dots, a_n \geq 0$. Then:*

1. *$C(0) \geq 0$ and $|C(z)| \leq C(0)$.*

2. *$a_1 C_1 + \cdots + a_n C_n$ and $\prod_{i=1}^{n} C_i$ are non-negative definite functions on E.*

3. *If E is a metric space, then C is continuous if and only if C is continuous in 0.*

Proof. 1. follows by (1.6) and Cauchy-Schwarz inequality. 2. follows by the fact that $a_1 C_1 + \cdots + a_n C_n$ and $\prod_{i=1}^{n} C_i$ are the covariance functions of linear combinations and products (respectively) of n independent stationary Gaussian fields with covariance functions C_i. Finally, 3. follows by (1.6), observing that

$$|C(z) - C(z+h)| = |\mathbb{E}[B_0(B_z - B_{z+h})]| \leq \sqrt{C(0)}\sqrt{2(C(0) - C(h)}.$$

□

The most important theorem of the subsection is Bochner's theorem.

Theorem 1.2.2 (Bochner)**.** *Let $C : \mathbb{R}^d \to \mathbb{R}$ be a non-negative definite function on $\mathbb{R}^d$, with C continuous and $C(0) = 1$. Then C is the Fourier transform of a unique Borel probability measure G on $\mathbb{R}^d$, namely*

$$C(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} G(dx). \tag{1.7}$$

Note that Bochner's theorem has been generalized by Yaglom in [Yag60] for non-negative definite functions on every group E satisfying very general assumptions, see also [Mal13, Theorem 2.18]. We only consider the Euclidean case $E = \mathbb{R}^d$ for simplicity, since we will not focus on other cases.

In probabilistic terms, Bochner's theorem ensures that the covariance function $C : \mathbb{R}^d \to \mathbb{R}$ of a stationary Gaussian field, if continuous, is the characteristic function of a random vector Λ, where $\Lambda \sim G$. Moreover, considering the random norm $|\Lambda| \sim \mu$ and the random direction $\widehat{\Lambda} := \mathbf{1}_{\Lambda\neq 0}\Lambda/|\Lambda| \sim \sigma$, with μ, σ probability measures on $\mathbb{R}_+$, S^{d-1} respectively [2], then we have

$$C(z) = \mathbb{E}\left[e^{i|\Lambda|\langle z,\widehat{\Lambda}\rangle}\right] = \int_0^\infty \mu(dr)\mathbb{E}\left[e^{ir\langle z,\widehat{\Lambda}\rangle}\middle||\Lambda| = r\right]. \tag{1.8}$$

In particular, if $|\Lambda|$, $\widehat{\Lambda}$ are independent, we can write

$$C(z) = \int_0^\infty \mu(dr) \int_{S^{d-1}} \sigma(d\theta)\, e^{ir\langle z,\theta\rangle}.$$

Definition 1.2.2. Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a stationary Gaussian field on $\mathbb{R}^d$ with continuous covariance function $C : \mathbb{R}^d \to \mathbb{R}$, $C(0) = 1$. The **spectral measure** of B is the unique probability measure G on $\mathbb{R}^d$ associated to C through (1.7). The **isotropic spectral measure** of B is the unique probability measure μ on $\mathbb{R}_+$ associated to C through (1.8). The **spherical spectral measure** of B is the unique probability measure σ on S^{d-1} associated to C through (1.8).

If a given Gaussian field $B = (B_x)_{x\in\mathbb{R}^d}$ is stationary, then recall that it is isotropic if and only if its covariance function is radial, namely

$$C(z) = \rho(|z|)$$

where the function $\rho : \mathbb{R}_+ \to \mathbb{R}$ is said the **isotropic covariance function** of B. Moreover, C is radial if and only if $C(z) = C(Pz)$, or equivalently $P\Lambda \sim \Lambda$, for every

[2] If $d = 1$, then $S^0 = \{-1, 1\}$ and $\sigma = \frac{1}{2}(\delta_1 + \delta_{-1})$.

$P \in SO(d)$. Furthermore, since the latter is equivalent to say that the spherical spectral measure σ is the uniform probability measure on S^{d-1}, we obtain the following result, the so-called Schoenberg theorem.

Theorem 1.2.3 (Schoenberg). *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a stationary Gaussian field with continuous covariance function $C : \mathbb{R}^d \to \mathbb{R}$, $C(0) = 1$, spectral measure G, isotropic spectral measure μ, and spherical spectral measure σ. Let ν be the uniform probability measure on S^{d-1}. Then B is isotropic if and only if $\sigma = \nu$. Moreover, by (1.8), if B is isotropic with isotropic covariance function $\rho : \mathbb{R}_+ \to \mathbb{R}$, we have*

$$\rho(r) = \int_0^\infty b_d(rs)\mu(ds), \tag{1.9}$$

where b_d is the characteristic function of $\nu = \sigma$. In particular, we have

$$b_d(s) = \int_{S^{d-1}} e^{i\langle z,\theta\rangle}\nu(d\theta) = \Gamma\left(\frac{d}{2}\right)\left(\frac{s}{2}\right)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(s), \qquad s = |z|,$$

where $J_{\frac{d}{2}-1}$ is the Bessel function of the first kind and order $\frac{d}{2} - 1$.

(1.9) could be seen as the isotropic version of (1.7), obtained fixing the spherical spectral measure σ equal to the uniform distribution on the sphere. The two spectral representations by Bochner, (1.7), and Schoenberg, (1.9), play a crucial role in the sequel, for two main reasons:

- (1.7) is our gateway to Fourier analysis. Indeed, it allows to rewrite important quantities related to functionals of B in terms of Fourier transforms and integrals with respect to the spectral measure G. The variance of $\int_{tD} B_x dx$, for example, which has the form

$$\mathrm{Var}\left(\int_D B_x dx\right) = \int_{\mathbb{R}^d}\left|\int_D e^{i\langle x,\lambda\rangle}dx\right|^2 G(d\lambda),$$

  may (and will) be estimated under suitable conditions on G and D.

- (1.9) is used in Chapter 2, to estimate the decay of ρ under suitable conditions on μ, see in particular Lemma 2.3.2. We conjecture that one may deduce similar results for C when B is not isotropic, i.e. when σ is not the uniform distribution ν, assuming that the spherical spectral measure σ is "regular" enough. This may be done using estimates for the decay of the Fourier transform of measures supported on surfaces, see e.g. [Ste93, p.348, Theorem 1].

Remark 1.2.1. We conjecture that some of the arguments that we are going to use in the sequel, based on (1.7) and (1.9), can be extended from non-negative definite functions $C : \mathbb{R}^d \to \mathbb{R}$ to **harmonizable** non-negative definite kernels $\mathcal{C} : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ of the form

$$\mathcal{C}(x, y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle\lambda,x\rangle - i\langle\lambda',y\rangle}\mathcal{G}(d\lambda, d\lambda'), \tag{1.10}$$

where $\mathcal{G}$ is a non-negative definite bimeasure of bounded variation, see e.g. [Swi02]. Such a class of kernels is much more general than the class of translation-invariant kernels $\mathcal{C}(x,y) = C(x-y)$, which corresponds to the just discussed case of non-negative definite functions. In the latter case, $\mathcal{G}$ is supported on the diagonal $\lambda = \lambda'$, and coincides with the spectral measure G of C. In other words, we conjecture that representations of the form (1.10) would allow to extend some of the results presented in this thesis from stationary Gaussian fields to non-stationary Gaussian fields with harmonizable covariance kernel $\mathcal{C}$. Nevertheless, these possible extensions are not immediate and would require novel ideas, which are left for future research.

1.2.2 Chaotic decompositions and Fourth Moment theorems

In this subsection, we introduce some concepts and results which will allow in the sequel to decompose the functional of a Gaussian field as the sum of easier functionals, the so-called chaotic components, and reduce the study of the functional to the one of its chaotic components.

Isonormal Gaussian Processes

Let us begin by introducing isonormal Gaussian processes, a special class of Gaussian fields on Hilbert spaces. In the following we denote by $\mathcal{H}$ a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and induced norm $\| \cdot \|_{\mathcal{H}}$.

Definition 1.2.3. An **isonormal Gaussian process** on $\mathcal{H}$ is a Gaussian field $I = (I(h))_{h\in\mathcal{H}}$ satisfying two properties:

- I is centered, that is $\mathbb{E}[I(h)] = 0 \; \forall h \in \mathcal{H}$.
- $I : \mathcal{H} \to L^2(\Omega)$ is an isometry, that is, I has covariance kernel

$$\mathcal{C}(h,g) = \langle I(h), I(g) \rangle_{L^2(\Omega)} = \mathbb{E}[I(h)I(g)] = \langle h, g \rangle_{\mathcal{H}}, \qquad \forall h, g \in \mathcal{H}.$$

Remark 1.2.2. Given a real separable Hilbert space $\mathcal{H}$, we can always construct an isonormal Gaussian process on $\mathcal{H}$, see e.g. [NP12, Proposition 2.1.1]. However, this construction is generally abstract. The effectiveness of isonormal Gaussian processes as mathematical tools comes from the fact that one can identify a Gaussian field B with a subset of an isonormal Gaussian process $I = (I(h))_{h\in\mathcal{H}}$, where the Hilbert space $\mathcal{H}$ is suitably chosen depending on B and on the problem we are dealing with, see e.g. the construction provided in Proposition 1.2.4 and [NP12, Examples 2.1.3-2.1.7] for other instructive examples. Then, once B is identified as a subset of I, we can use powerful tools on I to study functionals of B.

In this thesis, we only deal with isonormal Gaussian processes as global environments containing stationary Gaussian fields on $\mathbb{R}^d$. In particular, a construction based on the spectral measure of a starting Gaussian field on $\mathbb{R}^d$ is the following.

Proposition 1.2.4. *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a centered, stationary Gaussian field, with continuous covariance function $C : \mathbb{R}^d \to \mathbb{R}$ and spectral measure G. Let us consider the real separable Hilbert space*

$$L^2(G) := \left\{ h : \mathbb{R}^d \to \mathbb{C} \, : \, \int_{\mathbb{R}^d} |h(\lambda)|^2 G(d\lambda) < \infty \, , \, \overline{h(\lambda)} = h(-\lambda) \right\}, \tag{1.11}$$

where $|\cdot|$ denotes the complex norm and $\bar{\cdot}$ the complex conjugate. Consider an isonormal Gaussian process $I = (I(h))_{h\in L^2(G)}$ (recall Remark 1.2.2). For $x \in \mathbb{R}^d$, let $e_x \in L^2(G)$ be defined as $e_x(\lambda) := e^{i\langle x,\lambda\rangle}$. Then B can be identified with a subset of I as follows

$$(B_x)_{x\in\mathbb{R}^d} \stackrel{d}{=} (I(e_x))_{x\in\mathbb{R}^d} \subseteq (I(h))_{h\in L^2(G)}.$$

Proof. The fact that the inner product $\langle h, g\rangle_{L^2(G)}$ is real follows by the symmetry of G and the fact that h, g are even. The fact that B has the same distribution of $(I(e_x))_{x\in\mathbb{R}^d}$ follows by Bochner's theorem and the definition of I

$$\mathbb{E}[B_xB_y] = C(x-y) = \int_{\mathbb{R}^d} e_x(\lambda)\overline{e_y(\lambda)}G(d\lambda) = \langle e_x, e_y\rangle_{L^2(G)} = \mathbb{E}[I(e_x)I(e_y)].$$

□

Proposition 1.2.4 allows to identify a stationary Gaussian field B on $\mathbb{R}^d$ with a subset of an isonormal Gaussian process on $L^2(G)$. This construction will be important in the sequel, in particular for proving non-central limit theorems for functionals of B.

Note that, by definition of isonormal Gaussian process, $I : \mathcal{H} \to L^2(\Omega)$ is **linear** and **closed**, since

$$\mathbb{E}\left[(I(h) - I(g))^2\right] = \|h - g\|^2_{\mathcal{H}} \qquad \forall\, h, g \in \mathcal{H}.$$

This property is crucial, and may be considered as the first step to represent functionals of Gaussian fields in terms of operators acting on Hilbert spaces. If we consider linear combinations of a stationary Gaussian field B with continuous covariance function, for example, then by Proposition 1.2.4 and the linearity of I we have

$$a_1B_{x_1} + \cdots + a_nB_{x_n} \stackrel{d}{=} I(a_1e_{x_1} + \cdots + a_ne_{x_n}), \qquad x_i \in \mathbb{R}^d,\, a_i \in \mathbb{R}.$$

Moreover, since I is closed, if $\int_D |B_x|dx$ is well defined and in $L^2(\Omega)$, then

$$\int_D B_x dx \stackrel{d}{=} \int_D I(e_x)\, dx = I\left(\int_D e_x\, dx\right).$$

Such functionals of B, as well as all those which can be represented as evaluations of I in an element of $\mathcal{H}$, are said to be **linear functionals** of B. Note in particular that every linear functional is Gaussian.

As we are going to see, every square integrable functional of B can actually be written in terms of operators acting on Hilbert spaces.

Hermite polynomials, Wiener chaoses and Wiener-Itô integrals

In Definition 1.2.3, $L^2(\Omega)$ is the space of square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, the underlying probability space on which I is defined as a Gaussian field. From now on, we will implicitly assume that

$$\mathcal{F} = \sigma(I) = \sigma(I(h), h \in \mathcal{H}),$$

or equivalently that $L^2(\Omega) = L^2(\Omega, \sigma(I), \mathbb{P})$ is the space of square integrable functionals of I. Under this assumption, every functional of I in $L^2(\Omega)$ can be decomposed as the sum of orthogonal elements in "easier" Hilbert subspaces, called Wiener chaoses. The qth **Wiener chaos** is defined as the closed linear subspace of $L^2(\Omega)$

$$\overline{H_q} := \overline{\text{span}\{H_q(I(h))\,,\, h \in \mathcal{H},\, \|h\|_{\mathcal{H}} = 1\}}^{L^2(\Omega)},$$

where H_q is the qth **Hermite polynomial**, defined by the recurrence relation

$$H_0(x) = 1, \quad H_1(x) = x, \qquad H_q(x) = xH_{q-1}(x) - (q-1)H_{q-2}(x)\,. \tag{1.12}$$

Note that $\overline{H_0} = \mathbb{R}$ and $\overline{H_1} = I = (I(h))_{h\in\mathcal{H}}$.

As stated in the following proposition, the family of Wiener chaoses $(\overline{H_q})_{q\in\mathbb{N}}$ plays in $L^2(\Omega)$ the same role that the orthogonal basis of Hermite polynomials $(H_q)_{q\in\mathbb{N}}$ plays in $L^2(\mathbb{R}, \gamma(dx))$, where γ is the standard Gaussian measure

$$\gamma(dx) := \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx\,.$$

Terminology. In the sequel, **integrating by parts** with respect to a standard Gaussian random variable $N \sim N(0,1) \sim \gamma$ will mean

$$\mathbb{E}[f(N)N] = \int_{\mathbb{R}} f(x)\,x\,\gamma(dx) = \int_{\mathbb{R}} f'(x)\gamma(dx) = \mathbb{E}[f'(N)],$$

for f absolutely continuous with $f' \in L^1(\mathbb{R}, \gamma(dx))$ (see e.g. [NP12, Lemma 1.1.1]).

Proposition 1.2.5. *Let $(H_q)_{q\in\mathbb{N}} \subseteq L^2(\mathbb{R}, \gamma(dx))$ be the family of Hermite polynomials and $(\overline{H_q})_{q\in\mathbb{N}} \subseteq L^2(\Omega)$ be the family of Wiener chaoses. Then:*

- *For $N_1, N_2 \sim N(0,1)$ jointly Gaussian, we have the **isometry property***

$$\mathbb{E}\left[H_q(N_1)H_p(N_2)\right] = \mathbf{1}_{\{q=p\}}\, q!(\mathbb{E}[N_1N_2])^q. \tag{1.13}$$

- *$(H_q)_{q\in\mathbb{N}}$ is an orthogonal basis in $L^2(\mathbb{R}, \gamma(dx))$ and $L^2(\Omega)$ coincides with the direct sum of $(\overline{H_q})_{q\in\mathbb{N}}$, namely*

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \overline{H_q}. \tag{1.14}$$

Proof. First of all, note that by definition of Hermite polynomials one can easily prove by induction that $H_q'(x) = qH_{q-1}(x)$ for $q \geq 1$. Now let us prove that $\mathbb{E}[H_q(N)] = 0$, $N \sim N(0,1)$ for $q \geq 1$. For $q = 1$ the claim is trivially true. For $q \geq 2$ we have by definition of Hermite polynomial

$$\mathbb{E}[H_q(N)] = \mathbb{E}[NH_{q-1}(N)] - (q-1)\mathbb{E}[H_{q-2}(N)] = 0, \tag{1.15}$$

where the last equality follows by integration by parts and $H_q'(x) = qH_{q-1}(x)$. Now we can prove the isometry property (1.13). Let us assume without loss of generality that $q \leq p$ and proceed by induction on q. If $q = 0$, then (1.13) is (1.15). If $q \geq 1$, by definition of H_q and inductive hypothesis we have

$$\mathbb{E}[H_q(N_1)H_p(N_2)] = \mathbb{E}[N_1 H_{q-1}(N_1)H_p(N_2)].$$

Then the proof of (1.13) is concluded observing that (N_1, N_2) has the same distribution of $\left(N_1, \rho N_1 + \sqrt{1-\rho^2} N_1'\right)$, where N_1' is an independent copy of N_1 and $\rho = \mathbb{E}[N_1 N_2]$. Indeed, integrating by parts with respect to N_1 and recalling that $H_q'(x) = qH_{q-1}(x)$, we obtain (here we use the convention $H_{-1} = 0$ if $q = 1$)

$$\begin{aligned}\mathbb{E}[H_q(N_1)H_p(N_2)] &= \mathbb{E}\left[N_1 H_{q-1}(N_1)H_p\left(\rho N_1 + \sqrt{1-\rho^2}N_1'\right)\right] \\ &= (q-1)\mathbb{E}[H_{q-2}(N_1)H_p(N_2)] + p\rho\mathbb{E}[H_{q-1}(N_1)H_{p-1}(N_2)] \\ &= p\rho\mathbb{E}[H_{q-1}(N_1)H_{p-1}(N_2)].\end{aligned}$$

To prove the second part of the statement, we need to show that $(H_q)_{q\geq 1}$ is dense in $L^2(\mathbb{R}, \gamma(dx))$ and $\operatorname{span}\{H_q(I(h)),\, q \in \mathbb{N},\, h \in \mathcal{H},\, \|h\|_{\mathcal{H}} = 1\}$ is dense in $L^2(\Omega)$. These facts follow by standard considerations, mainly based on the density of monomials in $L^2(\mathbb{R}, \gamma(dx))$, the linearity of $I : \mathcal{H} \to L^2(\Omega)$ and the fact that H_q is a polynomial of degree q for every $q \in \mathbb{N}$, see e.g. [NP12, Proposition 1.1.5, Theorem 2.2.4]. □

The previous ensures that, just as for a function $\varphi \in L^2(\mathbb{R}, \gamma(dx))$ we can write

$$\varphi = \sum_{q=0}^{\infty} a_q H_q, \qquad a_q := \frac{1}{q!}\int_{\mathbb{R}} \varphi(x) H_q(x)\gamma(dx), \tag{1.16}$$

similarly for every functional $Y \in L^2(\Omega)$ we have the orthogonal decomposition

$$Y = \sum_{q=0}^{\infty} Y[q], \qquad Y[q] := \operatorname{proj}_{|\overline{H_q}}(Y), \tag{1.17}$$

where the projection $Y[q] \in \overline{H_q(B)}$ of Y is said the qth **chaotic component** of Y and (1.17) is said the **Wiener-Itô chaotic decomposition** of Y.

Let us now focus on the single chaotic components. Note that $Y[0] = \mathbb{E}[Y]$ since $\overline{H_0} = \mathbb{R}$. Moreover, $Y[1] = I_1(h_1)$ for some $h_1 \in \mathcal{H}$ since $\overline{H_1} = I$. Thus, $I : \mathcal{H} \to \overline{H_1}$ is an isomorphism between $\mathcal{H}$ and the first Wiener chaos. Thanks to this fact, one can study the first chaotic component as an element of $\mathcal{H}$. In the same vein, we would like to study the qth chaotic component ($q \geq 2$) as an element of an Hilbert space. To do this, we introduce multiple Wiener-Itô integrals.

Let $\mathcal{H}^{\otimes q}$ be the Hilbert space obtained as the q-fold tensor product of $\mathcal{H}$, generated by the set $\{h_1 \otimes \cdots \otimes h_q \,:\, h_i \in \mathcal{H},\, \|h_i\|_{\mathcal{H}} = 1\}$, endowed with the standard tensor inner product $\langle \cdot, \cdot \rangle_q$ and norm $\|\cdot\|_q$, satisfying

$$\langle h_1 \otimes \cdots \otimes h_q, g_1 \otimes \cdots \otimes g_q \rangle_q := \prod_{i=1}^{q} \langle h_i, g_i \rangle.$$

Let $H^{\odot q}$ be the symmetrized q-fold tensor product of $\mathcal{H}$, defined as the Hilbert subspace of $\mathcal{H}^{\otimes q}$ generated by the set

$$H^{\odot q} = \overline{\text{span}\{\text{Sym}(h_1 \otimes \cdots \otimes h_q) \,:\, h_i \in \mathcal{H},\, \|h_i\|_{\mathcal{H}} = 1\}}^{\mathcal{H}^{\otimes q}}.$$

Here Sym is the symmetrization operator defined as

$$\text{Sym}(h_1 \otimes \cdots \otimes h_q) = \frac{1}{q!} \sum_{\sigma \in S_q} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(q)},$$

where S_q is the group of all the permutations of $\{1, \ldots, q\}$. In order to define Wiener-Itô integrals, we use the following classical polarization formula, whose proof can be found e.g. in [Flo97].

Lemma 1.2.6 (Polarization formula)**.** *For $q \geq 1$ and $h_0, h_1, \ldots, h_q \in \mathcal{H}$, we have*

$$\text{Sym}(h_1 \otimes \cdots \otimes h_q) = \frac{1}{2^q q!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \ldots \epsilon_q (h_0 + \epsilon_1 h_1 + \cdots + \epsilon_q h_q)^{\otimes q}.$$

where $h^{\otimes q} := h \otimes \cdots \otimes h$, or in other words

$$\text{Sym}(h_1 \otimes \cdots \otimes h_q) = \frac{1}{q!} \mathbb{E}\left[\varepsilon_1 \ldots \varepsilon_q (h_0 + \varepsilon_1 h_1 + \cdots + \varepsilon_q h_q)^{\otimes q}\right]$$

where $\epsilon_1, \ldots, \epsilon_q$ are independent Rademacher random variables[3]*. In particular*

$$H^{\odot q} = \overline{\text{span}\{h^{\otimes q} \,:\, h \in \mathcal{H},\, \|h\|_{\mathcal{H}} = 1\}}^{\mathcal{H}^{\otimes q}}. \tag{1.18}$$

Definition 1.2.4. Let the notations above prevail. Fix $q \geq 1$. The qth **Wiener-Itô integral** with respect to the isonormal Gaussian process I is the unique linear closed operator $I_q : \mathcal{H}^{\odot q} \to \overline{H_q}$ such that for every $h \in \mathcal{H}$, $\|h\| = 1$,

$$I_q(h^{\otimes q}) = H_q(I(h)).$$

In particular, $I_q/\sqrt{q!}$ is an isomorphism and $\mathcal{H}^{\odot q}, \overline{H_q}$ are isomorphic Hilbert spaces.

Proof of Definition 1.2.4. Let us prove that the previous definition is well posed and that $I_q/\sqrt{q!}$ is an isomorphism between the Hilbert spaces $H^{\odot q}$ and $\overline{H_q}$. First, we define the qth Wiener-Itô integral I_q on q-fold tensors $h^{\otimes q}$, $\|h\| = 1$, as

$$I_q(h^{\otimes q}) = H_q(I(h)),$$

and extend by linearity the definition to $\text{span}\{h^{\otimes q}, \|h\| = 1\}$. Then, recalling the isometry property of Hermite polynomials we have

$$\mathbb{E}[I_q(v_1) I_q(v_2)] = q! \langle v_1, v_2 \rangle_q, \tag{1.19}$$

$\forall\, v_1, v_2 \in \text{span}\{h^{\otimes q}, \|h\| = 1\}$. Thus, by (1.18) we can extend I_q to an operator $I_q : \mathcal{H}^{\odot q} \to \overline{H_q}$, such that the isometry property (1.19) holds for every $v_1, v_2 \in H^{\odot q}$. Moreover, by definition of Wiener chaos $\overline{H_q}$ and again (1.19), we have that every

[3]Actually the result holds more in general for $\epsilon_1, \ldots, \epsilon_q$ independent with mean 0 and variance 1, see [Flo97].

element in $\overline{H_q}$ can be written as $I_q(v)$ for some $v \in \mathcal{H}^{\odot q}$. We have proved that $I_q/\sqrt{q!}$ is a surjective isometry, i.e. an isomorphism . □

Combining the Wiener-Itô chaotic decomposition (1.17) and the just proved fact $\overline{H_q} = (I_q(h))_{h\in\mathcal{H}^{\odot q}}$, we are finally able to decompose every square integrable functional Y of an isonormal Gaussian process I, i.e. $Y \in L^2(\Omega)$, as the infinite sum of Wiener-Itô integrals evaluated in elements of $H^{\odot q}$, namely

$$Y = \mathbb{E}[Y] + \sum_{q=1}^{\infty} I_q(h_q) \tag{1.20}$$

for some unique symmetric tensors $(h_q)_{q=1}^{\infty}$, $h_q \in \mathcal{H}^{\odot q}$. This representation plays a crucial role in the study of square integrable functionals of Gaussian fields. Indeed, one can often reduce the study of Y to the one of the single chaotic components $I_q(h_q)$, which are usually much easier to deal with.

Contractions and product formula

Formula (1.20) is very general, but also abstract. In other words, even if every functional $Y \in L^2(\Omega)$ is uniquely determined by the tensors $h_q \in \mathcal{H}^{\odot q}$, the latter are not always easy to derive. One of the most powerful tools to solve this kind of issues is the **product formula**, which allows to write the product of Wiener-Itô integrals $I_q(f)I_p(g)$ as the sum of Wiener Itô integrals evaluated in the symmetrized **contractions** of f and g. As a consequence, the product formula also allows to derive explicit expression for the powers and moments of Wiener-Itô integrals. In order to introduce this important formula, we first need the definition of contraction.

Definition 1.2.5. Let $p, q \geq 1$, $1 \leq r \leq p \wedge q$. Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Let $f \in \mathcal{H}^{\odot q}$ and $g \in \mathcal{H}^{\odot p}$, with orthogonal expansions

$$f = \sum_{i_1,\dots,i_q\in\mathbb{N}} f_{i_1,\dots,i_q} e_{i_1} \otimes \cdots \otimes e_{i_q}, \qquad g = \sum_{j_1,\dots,j_p\in\mathbb{N}} g_{j_1,\dots,j_p} e_{j_1} \otimes \cdots \otimes e_{j_p},$$

where the coefficients $f_{i_1,\dots,i_q}$ and $g_{j_1,\dots,j_p}$ are invariant by permutations of the indices. Let us use the notation

$$\langle f, e_{k_1} \otimes \cdots \otimes e_{k_r} \rangle := \sum_{i_1,\dots,i_q\in\mathbb{N}} f_{i_1,\dots,i_q} e_{i_1} \otimes \cdots \otimes e_{i_{q-r}}.$$

The rth **contraction** of f and g is the tensor $f \otimes_r g$ in $\mathcal{H}^{\otimes p+q-2r}$ defined as

$$f \otimes_r g := \sum_{k_1,\dots,k_r\in\mathbb{N}} \langle f, e_{k_1} \otimes \cdots \otimes e_{k_r} \rangle \langle g, e_{k_1} \otimes \cdots \otimes e_{k_r} \rangle.$$

Remark 1.2.3 (Some useful inequalities)**.** For $r = 0$, we formally define $f \otimes_0 g = f \otimes g$. Moreover, for the rth contraction of $f \in \mathcal{H}^{\odot q}$ with itself, we have by Cauchy-Schwarz inequality (see e.g. [NP12, (B.4.3)])

$$\|f \otimes_r f\|_{2q-2r} \leq \|f \otimes_0 f\|_{2q} = \|f\|_q^2.$$

Combining the previous and the equivalence (see e.g. [NP12, Lemma 6.2.1])

$$\|f \otimes_r g\|^2_{q+p-2r} = \langle f \otimes_{q-r} f, g \otimes_{p-r} g\rangle_{2r},$$

which for $f = g$ provides the symmetry $\|f \otimes_r f\|_{2q-2r} = \|f \otimes_{q-r} \|_{2r}$, we also obtain

$$\|f \otimes_r g\|_{q+p-2r} \le \|f \otimes_{q-r} f\|_{2r} \|g \otimes_{p-r} g\|_{2r} \le \|f\|^2_q \, \|g\|^2_p.$$

Remark 1.2.4. Note that for a q-fold tensor product $h_1^{\otimes q}$ and a p-fold tensor product $h_2^{\otimes p}$, $h_1, h_2 \in \mathcal{H}$, the rth contraction $h_1^{\otimes q} \otimes_r h_2^{\otimes p}$ coincides with

$$h_1^{\otimes q} \otimes_r h_2^{\otimes p} = \langle h_1, h_2\rangle^r h_1^{\otimes q-r} \otimes h_2^{\otimes p-r}. \tag{1.21}$$

Moreover, thanks to Lemma 1.2.6, the rth contraction $\otimes_r$ may be also defined imposing (1.21) for every $h_1^{\otimes q}$, $h_2^{\otimes p}$, $h_1, h_2 \in \mathcal{H}$. Indeed, one can linearly extend the definition (1.21) to $f \in \text{span}\{h^{\otimes q}, \|h\| = 1\}$, $g \in \text{span}\{h^{\otimes p}, \|h\| = 1\}$, and then prove the relations in Remark 1.2.3 for every $f \in \text{span}\{h^{\otimes q}, \|h\| = 1\}$, $g \in \text{span}\{h^{\otimes p}, \|h\| = 1\}$, to extend the bilinear operator $f \otimes_r g$ to $f \in \mathcal{H}^{\odot q}$, $g \in \mathcal{H}^{\odot p}$ by density (see (1.18)).

We are finally ready to state the product formula.

Theorem 1.2.7 (Product formula)**.** *Let $q, p \ge 1$. Then, for $f \in \mathcal{H}^{\odot q}$ and $g \in \mathcal{H}^{\odot p}$, we have*

$$I_q(f) I_p(g) = \sum_{r=0}^{q\wedge p} r! \binom{q}{r}\binom{p}{r} I_{p+q-2r}(f \tilde{\otimes}_r g)\,,$$

where $f \tilde{\otimes}_r g$ is a concise notation for $\text{Sym}(f \otimes_r g)$. In particular, if $e_1, \dots, e_k$ are orthonormal in $\mathcal{H}$, the formula implies

$$\prod_{i=1}^k H_{p_i}(I(e_i)) = \prod_{i=1}^k I_{p_i}(e_i^{\otimes p_i}) = I_{p_1+\dots+p_k}(e_1^{\otimes p_1} \tilde{\otimes} \dots \tilde{\otimes} e_k^{\otimes p_k}).$$

Proof. By Lemma 1.2.6, Remark 1.2.4, since I_q, Sym are linear and I_q is closed, it is enough to prove the result for $f = h_1^{\otimes q}$ and $g = h_2^{\otimes p}$, $h_i \in \mathcal{H}$, $\|h_i\| = 1$. Here we prove the result only for $q = 1$, see e.g [Nua06, Proposition 1.1.3] for the inductive extension to $q \ge 1$. For $q = 1$, we have to prove

$$\begin{aligned} I(h_1) I_p(h_2^{\otimes p}) &= I_{p+1}(h_1 \tilde{\otimes} h_2^{\otimes p}) + p I_{p-1}(h_1 \tilde{\otimes}_1 h_2^{\otimes p}) \\ &= I_{p+1}(h_1 \tilde{\otimes} h_2^{\otimes p}) + p \langle h_1, h_2\rangle I_{p-1}(h_2^{\otimes p-1}). \end{aligned} \tag{1.22}$$

First of all, note that if $h_1 = h_2$ the result trivially follows by definition of Hermite polynomial and Wiener-Itô integral. Suppose now that the result holds when $\langle h_1, h_2\rangle = 0$. Then, by a Gram-Schmidt argument (writing $h_1 = h_1' + \langle h_1, h_2\rangle h_2$) and linearity of all the operators that come into play, the result holds true for general h_1. Thus, without loss of generality we can prove (1.22) for $\langle h_1, h_2\rangle = 0$. By Lemma 1.2.6 and the definitions of Wiener-Itô integral and Hermite polynomial, $I_{p+1}(h_1 \tilde{\otimes} h_2^{\otimes p})$

can be rewritten as (when $p = 1$ the third addend is 0)

$$\frac{1}{(p+1)}\left(I(h_1)I_p(h_2^{\otimes p}) + p\, I(h_2)I_p(h_1\tilde{\otimes}h_2^{\otimes p-1}) - p\,(p-1)I_{p-1}(h_1\tilde{\otimes}h_2^{\otimes p-2})\right).$$

Thus, for $p = 1$ the result is true. For $p \geq 2$, assume that (1.22) holds for $k \leq p-1$. Then the proof is conluded if the following holds

$$I(h_2)I_p(h_1\tilde{\otimes}h_2^{\otimes p-1}) - (p-1)I_{p-1}(h_1\tilde{\otimes}h_2^{\otimes p-2}) = I(h_1)I_p(h_2^{\otimes p}).$$

By using the inductive hypothesis on the LHS and the defintion of Hermite polynomial on the RHS, both sides are equal to

$$I(h_1)I(h_2)I_{p-1}(h_2^{\otimes p-1}) - (p-1)I(h_1)I_{p-2}(h_2^{\otimes p-2})$$

and the proof is concluded. □

Fourth Moment theorems and related results

A consequence of the product formula is that in a fixed chaos q we are able to derive explicit expressions for powers and moments of $I_q(f)$ in terms of f. Thus, one may use the method of moments to study the distribution of $I_q(f_t)$ as $t \to \infty$, where $(f_t)_{t>0} \subseteq H^{\odot q}$. Fortunately, thanks to Nualart and Peccati, we do not need to study every moment, but only (the second and) the fourth. Indeed, these authors proved in [NP05] a surprising result on the asymptotic behavior of Wiener-Itô integrals in a fixed Wiener chaos: the convergence in distribution to a Gaussian random variable $N(0, \sigma^2)$ is equivalent to the convergence of only the second moment to $\sigma^2 > 0$ and of the fourth moment to $3\sigma^2$. From that time to the present, the so-called Fourth Moment theorem of Nualart and Peccati has been extended, adapted and applied in many different contexts, see e.g. [Azm+16; Cam+16; DP17; DP18; HMP24; NP09; Nou+16; NP05; PT05; Zhe19].

Theorem 1.2.8 (Fourth Moment Theorem). *Let I be an isonormal Gaussian process on $\mathcal{H}$. Fix $q \geq 2$ and consider $(I_q(f_t))_{t>0} \subseteq \overline{H_q}$, with $(f_t)_{t>0} \subseteq H^{\odot q}$. Suppose that* $\mathrm{Var}(I_q(f_t)) = q!\|f_t\|_q^2 \to \sigma^2 \in (0,\infty)$. *Then, the following assertions all are equivalent:*

- $I_q(f_t) \to N(0, \sigma^2)$.
- $\mathbb{E}\left[I_q(f_t)^4\right] \to 3\sigma^2$.
- $\max_{r=1\ldots,q-1} \|f_t \otimes_r f_t\|_{2q-2r} \to 0$.

There are many proofs of the Fourth Moment Theorem. Here we refer in particular to the proof in [Nou11], which consists in a recursive use of the product formula to compute the moments of $I_q(f_t)$ and a combinatorial manipulation of the obtained expressions. In particular, one obtains (see e.g. [Nou11][Lemma 4.1])

$$c(q) \max_{r=1,\ldots,q-1} \|f_t \otimes_r f_t\|_{2q-2r}^2 \leq \mathbb{E}[I_q(f_t)^4] - 3 \leq c'(q) \max_{r=1,\ldots,q-1} \|f_t \otimes_r f_t\|_{2q-2r}^2, \tag{1.23}$$

for some constants $c'(q) > c(q) > 0$, which explains the equivalence between the last two conditions of the Fourth Moment Theorem.

The following multidimensional version of Theorem 1.2.8 was proved by Peccati and Tudor in [PT05]. In particular, it asserts that, for Wiener-Itô integrals, componentwise convergence to Gaussian random variables is equivalent to joint convergence to Gaussian random vectors.

Theorem 1.2.9 (Multidimensional Fourth Moment Theorem). *Let I be an isonormal Gaussian process on $\mathcal{H}$. Fix $n \geq 2$, $q_1, \ldots, q_n \geq 1$ and consider $(I_{q_1}(f_{1,t}), \ldots, I_{q_n}(f_{n,t}))$, with $f_{i,t} \in \mathcal{H}^{\odot q_i}$, $t > 0$. Finally, assume that the following limits exist finite for $i, j \in \{1, \ldots, n\}$*

$$\sigma_{ij} := \lim_{t\to\infty} \operatorname{Cov}\left(I_{q_i}(f_{i,t}), I_{q_j}(f_{j,t})\right) < \infty\,. \tag{1.24}$$

Let $\underline{N} = (N_1, \ldots, N_n)$ be a centered Gaussian vector with covariance matrix (1.24). Then, the following assertions are all equivalent:

- $(I_{q_1}(f_{1,t}), \ldots, I_{q_n}(f_{n,t})) \xrightarrow{d} \underline{N}$.
- *For $i = 1, \ldots, n$, we have* $\mathbb{E}\left[I_q(f_{i,t})^4\right] \to 3\sigma_{ii}$.

The previous result is not only particularly useful to prove multidimensional central limit theorems for functionals of Gaussian fields, but also, combined with the chaotic decomposition (1.20), to prove central limit theorems for general functionals of Gaussian fields, see e.g. [NP12, Theorem 6.3.1].

Another important generalization of Theorem 1.2.8 are quantitative bounds for the normal approximation. The first quantitative Fourth Moment Theorem was proved by Nourdin and Peccati in [NP09]. In particular, these authors considered the Fortet-Mourier, Wasserstein, Kolmogorov and total variation distances. In the following, for simplicity, we only state the result for the total variation distance between the laws of two random variables F_1 and F_2, defined (with a slight abuse of notation) as

$$d_{TV}(F_1, F_2) = \sup_{E\in\mathcal{B}(\mathbb{R})} |\mathbb{P}(F_1 \in E) - \mathbb{P}(F_2 \in E)|.$$

For the proof of the next statement, see e.g. [NP12, Theorem 5.2.6].

Theorem 1.2.10 (Quantitative Fourth Moment Theorem). *Let I be an isonormal Gaussian process on $\mathcal{H}$. Fix $q \geq 2$ and let $I_q(f) \in \overline{H_q}$ be a functional of I in the qth Wiener chaos, with $f \in H^{\odot q}$. Suppose that $\operatorname{Var}(I_q(f)) = q!\|f\|_q^2 > 0$. Then, if $N \sim N(0,1)$, we have*

$$d_{TV}\left(\frac{I_q(f)}{\sqrt{\operatorname{Var}(I_q(f))}}, N\right) \leq 2\sqrt{\frac{q-1}{3q}}\sqrt{\mathbb{E}\left[I_q(f)^4\right] - 3}\,. \tag{1.25}$$

Note that the bound (1.25) may be combined with (1.23), obtaining

$$d_{TV}\left(\frac{I_q(f)}{\sqrt{\operatorname{Var}(I_q(f))}}, N\right) \leq \sqrt{c'(q)} \max_{r=1,\ldots,q-1} \|f \otimes_r f\|_{2q-2r}\,,$$

an expression which is particularly useful in applications. Regarding the proof of Theorem 1.2.10, we remark that Nourdin and Peccati actually proved in [NP09] a more general version of the previous result, providing a bound for $d_{TV}(F, N)$ for every $F \in L^2(\Omega)$ with square integrable Malliavin derivative. This condition is equivalent to ask that

$$\sum_{q=1}^{\infty} q\, q!\, \|f_q\|_q^2 < \infty\,, \tag{1.26}$$

where the f_q are the symmetric tensors uniquely associated to F through its chaotic decomposition (1.20). To prove these general bounds under assumptions on the Malliavin derivative of F, they introduced the so-called Malliavin-Stein method, which is nowadays very popular and well-developed, and has been extended and applied in different frameworks, see e.g. [APY21; LPS16; Led12]. In a nutshell, this method consists of two main steps:

- Exploiting Stein's method to bound $d_{TV}(F, N)$ with an expression that explicitly depends only on F, i.e.

$$d_{TV}(F,N) \le \sup_{f \in \mathcal{F}_{TV}} |\mathbb{E}[f'(F)] - \mathbb{E}[F\, f(F)]|\,,$$

  where $\mathcal{F}_{TV}$ is a suitable set of absolutely continuous functions. The dependence on $N \sim N(0,1)$ is implicit, hidden in the expression of the bound. Indeed, by Stein's lemma (see e.g. [NP12, Lemma 3.1.2]), F is Gaussian if and only if $E[f'(F)] - \mathbb{E}[F\, f(F)] = 0$ for every f differentiable with $f' \in L^1(\mathbb{R}, \gamma(dx))$.

- Expressing the bound obtained in the previous step in terms of Malliavin operators, see e.g. [NP12][Proposition 5.1.1]. This yields bounds for general classes of Malliavin differentiable random variables. In particular, as an application, one obtains the quantitative Fourth Moment Theorem 1.2.10.

In this thesis, mentioning the Malliavin-Stein method, we intend to refer to results such as Theorem 1.2.10, obtained as a consequence of the two steps above.

Application to additive functionals of Gaussian fields

We conclude the section with an important application of the previous results: additive functionals of Gaussian fields.

Let us consider a stationary, continuous Gaussian field $B = (B_x)_{x\in\mathbb{R}^d}$, with $B_x \sim N(0,1)$, covariance function $C : \mathbb{R}^d \to \mathbb{R}$ and spectral measure G. The spectral measure G is well defined by Bochner's theorem, because every stationary, continuous Gaussian field has continuous covariance function. By Proposition 1.2.4, we can identify B with a subset of an isonormal Gaussian process I on $\mathcal{H} = L^2(G)$, so that

$$(B_x)_{x\in\mathbb{R}^d} \overset{d}{=} (I(e_x))_{x\in\mathbb{R}^d}, \qquad e_x(\cdot) := e^{i\langle x,\cdot\rangle}.$$

Moreover, we have:

- $\mathcal{H}^{\otimes q}$ can be identified with the space $L^2((\mathbb{R}^d)^q, G^{\otimes q})$ of complex-valued even functions that are square integrable with respect to the product measure $G^{\otimes q}$;

- $\mathcal{H}^{\odot q}$ can be identified with the space $L^2_s((\mathbb{R}^d)^q, G^{\otimes q})$ of symmetric functions in $L^2((\mathbb{R}^d)^q, G^{\otimes q})$, where $f : (\mathbb{R}^d)^q \to \mathbb{C}$ symmetric means

$$f(\lambda_1,\dots,\lambda_q) = f(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(q)}) \quad \forall\, \sigma \in S_q.$$

In particular, note that the q-fold tensor product $e_x^{\otimes q}$ can be seen as the function $e_x^{\otimes q} : (\mathbb{R}^d)^q \to \mathbb{C}$, defined as $e_x^{\otimes q}(\lambda_1, \dots, \lambda_q) = e^{i\langle x, \lambda_1 + \dots + \lambda_q \rangle}$.

Consider now a function $\varphi \in L^2(\mathbb{R}, \gamma(dx))$ and a compact domain $D \subseteq \mathbb{R}^d$, with $\mathrm{Vol}(D) > 0$. We define the family of additive functionals $Y = (Y_t)_{t>0}$ of B as

$$Y_t := \int_{tD} \varphi(B_x)dx, \qquad t > 0.$$

By (1.16) and (1.13), the chaotic decomposition of Y_t (see (1.17)) is

$$Y_t - \mathbb{E}[Y_t] = \sum_{q=1}^{\infty} a_q \int_{tD} H_q(B_x)dx \tag{1.27}$$

where we recall that a_q and H_q are (respectively) the qth coefficient of φ and the qth Hermite polynomial appearing in (1.16). Moreover, by Definition 1.2.4 (recall $I(e_x) \overset{d}{=} B_x \sim N(0,1) \implies \|e_x\| = 1$) and the continuity of I_q, we obtain

$$Y_t - \mathbb{E}[Y_t] = \sum_{q=1}^{\infty} I_q(f_{q,t}),$$

with $f_{q,t} \in H^{\odot q}$ identifiable with the symmetric function $f_{q,t} : (\mathbb{R}^d)^q \to \mathbb{C}$

$$f_{q,t} := \int_{tD} e_x^{\otimes q} dx, \qquad f_{q,t}(\lambda_1, \dots, \lambda_q) := \mathcal{F}[\mathbf{1}_{tD}](\lambda_1 + \dots + \lambda_q),$$

where $\mathcal{F}$ denotes the Fourier transform. This last identification will be particularly useful to prove non-central limit theorems for Y.

In this setting, one can often reduce the study of Y_t to that of its chaotic components $I_q(f_{q,t}) = a_q \int_{tD} H_q(B_x)dx$. Thus, recalling the aforementioned Fourth Moment theorems, it becomes important to study the contractions $f_{q,t} \otimes_r f_{q,t}$,

$$f_{q,t} \otimes_r f_{q,t} = \int_{tD} dx \int_{tD} dy\, C(x-y)^r e_x^{\otimes q-r} \otimes e_y^{\otimes q-r},$$

with squared norm

$$\|f_{q,t} \otimes_r f_{q,t}\|_{2q-2r}^2 = \int_{(tD)^4} C(x-y)^r C(u-v)^r C(x-u)^{q-r} C(y-v)^{q-r}\, dx\, dy\, du\, dv\,. \tag{1.28}$$

This expression can then be combined with the Nualart-Peccati, Peccati-Tudor and Nourdin-Peccati Fourth Moment theorems to prove central limit theorems for $I_q(f_{q,t})$. This fact will be crucial for many of the coming discussions.

1.2.3 Cross covariograms

We conclude this section with the last simple (but crucial) tool of this thesis: cross covariograms. For all the missing details and proofs, we refer to [Gal11].

Definition 1.2.6. Let $D, L \subseteq \mathbb{R}^d$ be compact sets. The cross covariogram of D, L is the function $g_{D,L} : \mathbb{R}^d \to \mathbb{R}$ defined as

$$g_{D,L}(z) := \mathrm{Vol}(D \cap (L+z)) = \mathbf{1}_D * \mathbf{1}_{-L}(z), \qquad z \in \mathbb{R}^d.$$

The covariogram of D is the function $g_D := g_{D,D}$.

Covariograms and cross covariograms are scale-invariant ($g_{tD,tL}(z) = t^d g_{D,L}(z/t)$) and uniformly continuous (with compact support $D - L$). Moreover, g_D is a Lipschitz function if and only if its perimeter is finite, and $g_{D,L}$ is Lipschitz with Lipschitz constant $\mathrm{Per}(D) \wedge \mathrm{Per}(L)$ if $\mathrm{Per}(D) \wedge \mathrm{Per}(L) < \infty$, where we recall that the definition of (generalized) perimeter is

$$\mathrm{Per}(D) := \sup \left\{ \int_D \mathrm{div}\varphi(x)\, dx : \varphi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d), \|\varphi\|_\infty \le 1 \right\}. \tag{1.29}$$

See Section 4.2.2 for additional details and properties.

The reason why we introduce them is that they play a fundamental role in our analysis. If we consider the qth chaotic component $\int_{tD} H_q(B_x)dx$ considered in (1.27), for instance, we know that its variance is

$$\mathrm{Var}\left(\int_{tD} H_q(B_x)dx \right) = q! \int_{tD} \int_{tD} C^q(x-y)dxdy = q! \int_{\mathbb{R}^d} C^q(z) g_{tD}(z)dz.$$

In other words, its variance can be written only in terms of C (which depends only on the Gaussian field B) and g_{tD} (which depends only on the domain). Analogously, one could write the covariance for two domains D, L only in terms of C and $g_{tD,tL}$:

$$\mathrm{Cov}\left(\int_{tD} H_q(B_x)dx, \int_{tL} H_q(B_y)dy \right) = q! \int_{\mathbb{R}^d} C^q(z) g_{tD,tL}(z)dz. \tag{1.30}$$

This fact, combined with the properties of cross covariograms, will allow us to derive the exact asymptotics for the (co)variances as $t \to \infty$, under simple and general assumptions on C, D, L (that encompass the classical assumptions in the literature).

A less immediate fact is that also the norms of the contractions (1.28), which have to be estimated to apply Fourth Moment theorems to $\int_{tD} H_q(B_x)dx$, can be rewritten only in terms of C and cross covariograms, namely

$$\int_{\mathbb{R}^{3d}} C(u)^r C(v)^r C(w)^{q-r} C(u+v+w)^{q-r}\, g_{D_t(-w),D_t(u)}(v) du\, dv\, dw\,, \tag{1.31}$$

where $D_t(u) := tD \cap tD + u$. This expression, combined with Bochner's theorem, yields a reformulation of (1.28) in terms of the spectral measure G of B and Fourier transforms of cross covariograms, an idea which will allow us to prove central limit theorems under general assumptions on G.

1.3 Research questions and existing results

In this section, we introduce our main research questions, giving an overview of the state of the art related to them.

From now on, our study concentrates on functionals of the form

$$\int_{tD} A_x dx, \qquad t > 0, \tag{1.32}$$

where $A = (A_x)_{x\in\mathbb{R}^d}$ is a weakly stationary random field and $D \subseteq \mathbb{R}^d$ is compact. In this section, we focus on the existing results related to our main research questions:

Question 1 (CLT for additive functionals of Gaussian fields)**.** If $A_x = \varphi(B_x)$ for some stationary Gaussian field $B = (B_x)_{x\in\mathbb{R}^d}$, $B_x \sim N(0,1)$, and $\varphi \in L^2(\mathbb{R}, \gamma(dx))$, the functionals in (1.32) are the additive functionals of B previously introduced,

$$Y_t := \int_{tD} \varphi(B_x)dx, \quad t > 0.$$

In this setting, we ask ourselves: does $Y = (Y_t)_{t>0}$ satisfy a central limit theorem? We mainly deal with Question 1 in Chapter 2 and Chapter 5.

Question 2 (Asymptotic covariances)**.** Given a family of compact sets $\mathcal{D}$ of $\mathbb{R}^d$ and $D, L \in \mathcal{D}$, can we compute the asymptotic covariances of $\left(\int_{tD} A_x dx\right)_{D\in\mathcal{D}}$? That is, can we compute the limit

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{r_t}, \frac{\int_{tL} A_y dy}{r_t}\right), \tag{1.33}$$

for some $r_t \to \infty$ suitably chosen so that the previous limit is finite and not identically 0 for every $D, L \in \mathcal{D}$? We mainly deal with Question 2 in Chapter 3 and Chapter 4.

Note that the two questions are very related to each other. Indeed, Question 2 with $D = L$ corresponds to the study of the asymptotic variance of $\int_{tD} A_x dx$, which is usually the first step to study the fluctuations of the functionals in Question 1. Moreover, multi-dimensional central limit theorems for additive functionals of Gaussian fields can often be obtained (thanks to Theorem 1.2.9) by a simple combination of CLTs as in Question 1 and exact asymptotics of the covariances in Question 2.

We split the exposition in three parts: in Subsections 1.3.1-1.3.2 we focus on the existing results related to Question 1 and Question 2 when A has short memory and long memory, respectively; in Subsection 1.3.3 we focus on Question 1 for more general families of p-domain functionals.

1.3.1 Functionals of random fields with short memory

In this section, we analyze situations where A exhibits **short memory**. Short memory for random fields is defined in many different ways in the literature, see e.g. [Mak+21] and the references therein. Heuristically, the common denominator among all these definitions is the **low global dependence** (or correlation) between the random variables composing the random field, where the "threshold" to not exceed depends on the problem under study.

In order to speak about short memory, we need to measure (or classify) the dependence or the correlation between the random variables of A. A popular way to do this are strong mixing conditions, see e.g. [IL71; Pel90]. When A is weakly stationary, one may also classify the memory of A in terms of its covariance function $K : \mathbb{R}^d \to \mathbb{R}$, see the upcoming Definition 1.3.1. Of course, whatever reasonable definition of short memory we choose, the extreme case of the classical central limit theorem, in which the random variables in A are all independent, should correspond to "zero" memory. Likewise, the opposite extreme case, where all the random variables are equal, should correspond to "infinite" memory.

The concept of short memory is convenient because it allows to give an immediate interpretation of many central limit theorems' statements. To give few examples, just consider the classical central limit theorem, where the independence of the random variables is assumed (i.e. zero memory), or the upcoming Breuer-Major theorem. In all these results, one assumes that the random field satisfies some kind of short memory condition (independence in the classical CLT, $C \in L^R$ in Theorem 1.3.1).

In our setting, where A is a weakly stationary random field, the most popular definition of short memory is the following, see e.g. [Azo09; Lav06; WC10].

Definition 1.3.1. Let $A = (A_x)_{x \in \mathbb{R}^d}$ be a weakly stationary random field, with measurable covariance function $K : \mathbb{R}^d \to \mathbb{R}$. We say that A has **short memory** if

$$\int_{\mathbb{R}^d} |K(z)| dz < \infty. \tag{1.34}$$

Terminology. Short memory is sometimes referred to as **short-range dependence** (or weak dependence). These terms are often used interchangeably, see for example [Lav06; WC10]. When we are dealing with Gaussian fields, we also use them interchangeably. Since in our analysis the weakly stationary random field A is not necessarily Gaussian, we use the word "memory" (with a slight abuse of terminology when $d \geq 2$, since memory is intuitively associated with time) instead of "dependence", to specify that (1.34) is in general not a condition on the low global dependence of the random variables in A, but rather on their low global correlation. We avoid the term "short-range correlation" because it does not seem to be a common nomenclature in the literature.

Let us start considering Question 1 when A has short memory. If $A_x = \varphi(B_x)$, with $B = (B_x)_{x \in \mathbb{R}^d}$ stationary Gaussian field, $B_x \sim N(0,1)$, and $\varphi \in L^2(\mathbb{R}, \gamma(dx))$, the functionals in (1.32) are additive functionals of B, namely

$$Y_t := \int_{tD} \varphi(B_x) dx, \qquad t > 0. \tag{1.35}$$

Since $\varphi \in L^2(\mathbb{R}, \gamma(dx))$, we have

$$\varphi = \mathbb{E}[\varphi(N)] + \sum_{q=R}^{\infty} a_q H_q, \qquad a_R \neq 0,$$

and

$$\mathrm{Var}(\varphi(N)) = \sum_{q=R}^{\infty} q! a_q^2 < \infty, \qquad N \sim N(0,1). \tag{1.36}$$

$R := \inf\{q \geq 1 \, : \, a_q \neq 0\}$ is said the **Hermite rank** of φ. To avoid trivialities, we assume that φ is not constant, so that R is a well defined finite integer.

If K, C are the covariance functions of A and B (respectively), by (1.13) we have

$$K(z) = \sum_{q=R}^{\infty} q! a_q^2 C^q(z). \tag{1.37}$$

In this setting, the most important and celebrated theorem is Breuer-Major theorem, first proved in [BM83] for discrete Gaussian fields and extended in many different directions, see e.g. [BH02; CNN20; NNP21; NPY19; NZ20].

Theorem 1.3.1 (Breuer-Major)**.** *Let B be a stationary Gaussian field with $B_x \sim N(0,1)$ and covariance function $C : \mathbb{R}^d \to \mathbb{R}$, $\varphi \in L^2(\mathbb{R}, \gamma(dx))$ with Hermite rank $R < \infty$ and consider the family $Y = (Y_t)_{t>0}$ defined in (1.35). If $C \in L^R(\mathbb{R}^d)$, then*

$$\frac{Y_t - \mathbb{E}[Y_t]}{t^{d/2}} \xrightarrow{d} N(0, \sigma^2), \quad \text{as } t \to \infty.$$

Moreover, if $\sigma^2 := \mathrm{Vol}(D) \int_{\mathbb{R}^d} K(z)dz > 0$, then $\mathrm{Var}(Y_t) \sim \sigma^2 t^d$ and Y satisfies a CLT.

$C \in L^R(\mathbb{R}^d)$ implies[4] that $K \in L^1(\mathbb{R}^d)$, i.e. that $A_x = \varphi(B_x)$ has short memory in the sense of Definition 1.3.1. Moreover, since B is Gaussian, $C \in L^R$ is also a condition on the low global dependence of the random variables in B, where the "low" depends on the Hermite rank R of φ, i.e. from the considered functional . Thus, Breuer-Major theorem provides **Gaussian fluctuations in short memory situations**.

Let us now deal with Question 2. If A has short memory, by (1.34) we have the following fact: for every $D, L \subseteq \mathbb{R}^d$ compact sets,

$$\mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2}}, \frac{\int_{tL} A_y dy}{t^{d/2}}\right) \to \mathrm{Vol}(D \cap L) \int_{\mathbb{R}^d} K(z)dz\,, \qquad \text{as } t \to \infty. \tag{1.38}$$

This simply follows by dominated convergence theorem and

$$\mathrm{Cov}\left(\int_{tD} A_x dx, \int_{tL} A_y dy\right) = \int_{tD}\int_{tL} K(x-y)dxdy = t^d \int_{\mathbb{R}^d} K(z) g_{D,L}(z/t)\, dz$$

where we recall that $g_{D,L}$ denotes the cross covariogram of D and L.

Thus, we have two possible situations:

- If $\int_{\mathbb{R}^d} K(z)dz = 0$, then $r_t = t^{d/2}$ is not a correct choice to compute the asymptotic covariances, since (1.38) is identically 0 for every D, L compact.
- If $\int_{\mathbb{R}^d} K(z)dz > 0$, then $r_t = t^{d/2}$ is a correct choice for computing the asymptotic covariances. In this case, up to a scaling factor, the asymptotic covariance structure is that of a Gaussian noise (see [AT07, p.24]), i.e. $\mathrm{Vol}(D \cap L)$.

The above analysis provides an additional possible interpretation of the short memory of A in terms of the asymptotic covariances (when $\int_{\mathbb{R}^d} K(z)dz > 0$[5]): if D, L are disjoint, then $\int_{tD} A_x dx$ and $\int_{tL} A_y dy$ (once suitably rescaled) are asymptotically uncorrelated.

[4] This follows by (1.36) and $\|C\|_\infty \leq 1$. Note that the converse implication is not true in general.

[5] See Example 4.4.1 for a situation where $\int_{\mathbb{R}^d} K(z)dz = 0$.

1.3.2 Functionals of random fields with long memory

Following the trend in the literature, if A has not short memory, i.e. $K \notin L^1(\mathbb{R}^d)$, we say that A has **long memory**. When this is the case, we usually need more specific assumptions on K in order to answer Question 1 and Question 2. The most classical family of non-integrable[6] covariance functions considered in the literature is that of radial and **regularly varying** functions with index $-\beta \in (-d, 0)$,

$$K(z) = k(|z|) := \ell(|z|)|z|^{-\beta}, \qquad \beta \in (0, d), \tag{1.39}$$

where $\ell : \mathbb{R}_+ \to \mathbb{R}$ is **slowly varying**, i.e. ℓ is definitively positive and $\ell(rs)/\ell(s) \to 1$ as $s \to \infty$, $\forall\, r > 0$. An explicit example is given by Cauchy covariance functions, of the form

$$K(z) = \frac{1}{(1 + |z|^2)^{\beta/2}}, \qquad \beta \in (0, d).$$

Let us consider Question 1 in the long-memory situations (1.39). Consider again $A_x = \varphi(B_x)$, where K is the covariance function of A. The main theorem in this setting is the following Theorem 1.3.2. In the discrete case, a first proof for functionals with Hermite rank $R = 1, 2$, was given by Taqqu in [Taq75], and then generalized to any R by Dobrushin and Major in [DM79]. The following sticks to the continuous case and a proof can be found e.g. in [LO14].

Theorem 1.3.2 (Dobrushin-Major-Taqqu)**.** *Let B be a stationary Gaussian field with $B_x \sim N(0, 1)$ and covariance function $C : \mathbb{R}^d \to \mathbb{R}$. Consider $\varphi \in L^2(\mathbb{R}, \gamma(dx))$ with Hermite rank $R < \infty$, Rth coefficient $a_R \neq 0$, see (1.16). Consider the family $Y = (Y_t)_{t>0}$ defined in (1.35). Suppose that C satisfies the following conditions:*

- *C is radial and regularly varying with index $-\beta \in (-d/R, 0)$, i.e.*

$$C(z) = \ell(z)|z|^{-\beta}, \qquad \beta \in (0, d/R). \tag{1.40}$$

- *The spectral measure G of C is absolutely continuous, i.e. $G(d\lambda) = g(\lambda)d\lambda$, and the spectral density g satisfies the following condition for some constant $c > 0$*

$$g(\lambda) \sim c\,\ell\left(|\lambda|^{-1}\right)|\lambda|^{\beta-d}, \qquad \textit{as } |\lambda| \to 0. \tag{1.41}$$

Then, as $t \to \infty$, we have

$$\widetilde{Y}(t) \xrightarrow{d} \frac{\mathrm{sgn}(\mathrm{a_R}) I_{\nu,R}\left(f_{R,D}\right)}{\sqrt{\mathrm{Var}\left(I_{\nu,R}\left(f_{R,D}\right)\right)}},$$

where ν is the measure on $\mathbb{R}^d$ defined as $\nu(dx) := |x|^{\beta-d}dx$, $f_{R,D}$ is defined as

$$f_{R,D}(\lambda_1, \ldots, \lambda_R) := \int_D e^{i\langle x, \lambda_1 + \cdots + \lambda_R\rangle} dx,$$

[6] Note that if $\beta > d$, then $K \in L^1(\mathbb{R}^d)$ and we fall in the previous case of short memory. If $\beta = d$, then $K \notin L^1(\mathbb{R}^d)$, but one has a CLT, and asymptotic covariances as in the short memory case. The case $\beta = d$ may be considered a critical case between short and long memory, at least for the class of covariance functions (1.39).

and $I_{\nu,R}$ is the Rth Wiener-Itô integral acting on $\mathcal{H}_\nu^{\odot R}$, where $\mathcal{H}_\nu := L^2(\nu)$ is defined as in (1.11). In particular, $I_{\nu,R}(f_{R,D})$ is not Gaussian as soon as $R \geq 2$.

Remark 1.3.1 (The extra spectral condition (1.41)). Note that the extra condition on the spectral density (1.41) is not needed in the discrete version of Theorem 1.3.2 proved by Taqqu and Dobrushin-Major. To the best of our knowledge, the continuous version of the result, i.e. Theorem 1.3.2, has never been proved without assumption (1.41). For more details on the relation between assumption (1.40) and (1.41) see [LO13]. In many situations, e.g. when C is a Cauchy covariance function, the two conditions both hold (see [LO13, Example 3]).

Combining relation (1.37) and assumption (1.40), we have that (1.39) is satisfied and A has long memory. Thus, Theorem 1.3.2 provides non-Gaussian fluctuations in long-memory situations, as soon as $R \geq 2$. This phenomenon is opposite to the one in the previous subsection, where Theorem 1.3.1 gives Gaussian fluctuations in short memory situations.

Let us now consider Question 2 in the long memory situations (1.39). Since $K(z) = k(|z|)$ is radial and regularly varying with index $-\beta \in (-d, 0)$, for $D, L \subseteq \mathbb{R}^d$ compact we have

$$\mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^d k(t)^{1/2}}, \frac{\int_{tL} A_y dy}{t^d k(t)^{1/2}}\right) = \int_D \int_L \frac{k(|x-y|t)}{k(t)} dx dy \to \int_D \int_L |x-y|^{-\beta} dx dy,$$

where the convergence follows by dominated convergence theorem and Potter's bounds for regularly varying functions (see Theorem 4.2.5). In particular, this means that $r_t = t^d k(t)^{1/2}$ is a correct choice for computing the asymptotic covariances (1.33), since $\int_D \int_L |x-y|^{-\beta} dx dy \in (0, \infty)$ for every D, L compact with positive volume. When $d = 1$ and $D = [0, s]$, $L = [0, r]$, then the asymptotic covariance structure is the one of a fractional Brownian motion, see Remark 4.1.2.

This provides an additional possible interpretation of the long memory of A in terms of the asymptotic covariances: if D, L are disjoint and A satisfies (1.39), then $\int_{tD} A_x dx$ and $\int_{tL} A_y dy$ are not necessarily asymptotically uncorrelated.

1.3.3 p-domain functionals of random fields

A possible extension of Question 1 is the following: consider $B = (B_x)_{x \in \mathbb{R}^d}$ and φ as in Question 1, with $d \geq 2$, but instead of integrating on a uniformly growing domain tD as in (1.35), let us consider the p-domain functional

$$Y(t_1, \dots, t_p) := \int_{t_1 D_1 \times \cdots \times t_p D_p} \varphi(B_x) dx, \qquad t_1, \dots, t_p \to \infty$$

where $2 \leq p \leq d$, the compact sets $D_i \subseteq \mathbb{R}^{d_i}$ have positive Lebesgue measure, $d_1 + \cdots + d_p = d$, and $t_1, \dots, t_p$ can grow possibly at different rates. The two main reasons to study this extended class of functionals are random field with separable covariance functions and spatio-temporal models.

Random fields with separable covariance function

In several applications, the covariance function C of B is assumed to be separable, that is, it can be written as $C = C_1 \otimes \cdots \otimes C_p$, where C_i is a non-negative definite function on $\mathbb{R}^{d_i}$. An example which has been studied in deep, see e.g. [PR16; RST12], are the Hermite variations of the rectangular increments of a fractional Brownian sheet, corresponding to $\varphi = H_q$, $p = 2$ and C_i of the form

$$C_i(x_i) = \frac{1}{2}\left(|x_i + 1|^{2H_i} + |x_i - 1|^{2H_i} - 2|x_i|^{2H_i}\right), \qquad H_i \in (0,1).$$

In [RST12], the authors observed[7] that $Y(t_1, t_2)$ satisfies a CLT as $t_1, t_2 \to \infty$ if and only if $H_i \leq 1 - 1/2q$ for at least one i. In particular, the authors exploited the Malliavin-Stein method introduced by Nourdin and Peccati (see Subsection 5.2.1) to study quantitative limit theorems. This work was followed by several papers, see e.g. [Bre11; PR16]. Other examples of separable models in hydrology and fluid dynamics can be found e.g. in [Chr92, Chapter 5].

Spatio-temporal models

In applications, B may model a random quantity of interest depending on several (also non-Euclidean) variables $x_1, \ldots, x_p$, see e.g. [AO18; BPP22; Chr92; Gne02; LRM23; MRV21] A very important class of these models are spatio-temporal ones, corresponding to $p = 2$ and $d_2 = 1$. In this case, $Y(t_1, t_2)$ is said a spatio-temporal functional and $t_1 D_1, t_2 D_2$ are respectively the space, time observation windows. An explicit example is the excursion volume of B as the space/time windows grow at different rates. Apart from the separable case previously discussed, there are also many models where C is not separable. A very popular alternative is represented for example by Gneiting covariance functions, introduced in [Gne02]. Some recent works going in this direction, i.e. studying spatio-temporal functionals for Gaussian fields with separable/Gneiting covariance functions, are [LRM23; LRM24].

1.4 New results of the thesis

In this last section, we focus on the main contributions of this thesis, presenting in a concise way the main results of the next four chapters and comparing them with the ones considered in the previous section. Each of the next four subsections is respectively based on Chapters 2, 3, 4, 5, which are themselves reproductions of the following papers/preprints:

- Chapter 2 contains the paper [MN24], "Spectral central limit theorem for additive functionals of isotropic and stationary Gaussian fields", written in collaboration with Ivan Nourdin. *Ann. Probab.* (2024), 52(2), 737 – 763.
- Chapter 3 contains the paper [GMT24], "Fluctuations of polyspectra in spherical and Euclidean random wave models", written in collaboration with Francesco Grotto and Anna Paola Todino. *Electron. Comm. Probab.* (2024), 29, 1-12.

[7] This was done in the discrete case in [RST12], but can be translated in the continuous case.

- Chapter 4 contains the paper [Mai24], "Asymptotic covariances for functionals of weakly stationary random fields". *Stoch. Proc. Appl.* (2024), 170, 104297.

- Chapter 5 contains the preprint [Leo+24], "Limit theorems for p-domain functionals of stationary Gaussian fields", written in collaboration with Nikolai Leonenko, Ivan Nourdin and Francesca Pistolato. arXiv (2024+): 2402.16701.

For all the missing details in the next subsections, we refer to Chapters 2-5.

1.4.1 Chapter 2: Spectral central limit theorem for additive functionals of isotropic and stationary Gaussian fields

In Chapter 2 we deal with Question 1, that is, we study under which conditions the family Y of additive functionals defined in (1.35) satisfies a CLT.

The **binary intuition** forged starting from the seminal works of Breuer, Dobrushin, Major, Taqqu (see Theorem 1.3.1 and Theorem 1.3.2), is that Y satisfies, in general (i.e. except possibly in "special" cases), a **central limit theorem when B has short memory**, and a **non-central limit theorem when B has long memory and $R \geq 2$**.

With "special cases" we mean for example the critical case $\beta = d/R$ in (1.40), for which we have long memory in the sense of Definition 1.3.1, but usually Gaussian fluctuations (see, e.g., [BM83, Theorem 1'], [NN20, Section 5] and [NNT10, Theorem 1]). Another Gaussian field that may be considered "special" is the Berry random wave model, for which Theorem 1.4.1 provides Gaussian fluctuations in a long memory context (when $R = 2$). Berry's random wave model may be considered special in the sense that it represents Euclidean random waves. Indeed, Gaussian fluctuations in a long memory context were already observed for random waves on the sphere, see e.g. [MW14; Tod19].

The first novelty of Chapter 2 consists in showing that this surprising phenomenon (Gaussian fluctuations in a long memory context with $R \geq 2$) can occur in a large variety of situations. More specifically, we prove in Theorem 1.4.1 that Y satisfies a CLT under an easily verifiable condition on the negative moment of the spectral measure of B, without assumptions on the covariance function. Thanks to it, we construct a family of situations showing that **the binary intuition related to memory and limit theorems for Y is wrong in general**[8] **and not only in special cases**. To highlight the fact that the main assumption in Theorem 1.4.1 involves only the spectral measure, we decided to refer to it as a *spectral* CLT.

Note that this also brings to light a second novelty introduced in Chapter 2: we are able to prove central limit theorems for Y even when the covariance function is not available, or has not a convenient mathematical expression.

Let us state the main result of the chapter in a concise and simplified form.

Theorem 1.4.1 (Spectral CLT). *Fix $d \geq 2$, let $B = (B_x)_{x\in\mathbb{R}^d}$ be stationary, isotropic, with covariance function $C : \mathbb{R}^d \to \mathbb{R}$ and isotropic spectral measure μ. Let φ be as in* (1.16)*, with Hermite rank R and second Hermite rank R' (as defined in* (2.4)*). Suppose that $\exists k > 0$ with $a_{2k} \neq 0$ and that $(R, R') \notin \{(1, 3)\} \cup \{(2k + 1, n) : k \geq 1, n \in \mathbb{N}\}$.*

[8] For a class of examples, see Section 2.6.

Finally, when $R \in \{1,2\}$, suppose that D is "nice" (see Theorem 2.1.2). If the following spectral condition holds

$$\int_0^\infty \frac{\mu(ds)}{s^{d/R}} < \infty, \tag{1.42}$$

then Y satisfies a CLT, with the following variance rate

$$\mathrm{Var}(Y_t) \asymp \begin{cases} t^d \int_{\{|x|\le t\}} C^R(x)dx & \textit{if } R \textit{ even} \\ t^d \int_{\{|x|\le t\}} C^{R'}(x)dx & \textit{if } R=1 \textit{ and } R' \in \{2,4\} \\ t^d & \textit{if } R=1 \textit{ and } R' \ge 5 \end{cases}.$$

If we consider the 2-dimensional Berry random wave model $B=(B_x)_{x\in\mathbb{R}^2}$ and $R=2$, then the covariance function is neither in $L^2(\mathbb{R}^d)$ nor regularly varying, but instead has the following oscillating asymptotic behavior (see [Kra14])

$$C(z) = J_0(|z|) = \sqrt{\frac{2}{\pi}}\frac{\cos(|z|-\pi/4)}{\sqrt{|z|}}\left(1+O\left(|z|^{-3/2}\right)\right) \qquad |z|\to\infty. \tag{1.43}$$

This makes it hard to deal with it in computations, see e.g. (1.31). On the other hand, the isotropic spectral measure of Berry's random wave model is the easiest we can think about: the degenerate measure $\mu=\delta_1$. Thus, by applying Theorem 1.4.1 for $R\in\{1,2,4\}$ and Theorem 1.3.1 for $R\ge 5$, taking φ and D as in the statement of Theorem 1.4.1, Y satisfies a CLT with variance rates

$$\mathrm{Var}(Y_t) \asymp \begin{cases} t^3 & \text{if } R=2 \text{ or } (R=1 \text{ and } R'=2) \\ t^2\log(t) & \text{if } R=4 \text{ or } (R=1 \text{ and } R'=4) \\ t^2 & \text{if } R\ge 5 \text{ or } (R=1 \text{ and } R'\ge 5) \end{cases}. \tag{1.44}$$

In view of the binary intuition discussed above, a CLT is expected when $R\ge 4$, but surprising in the long memory case $R=2$, and with surprising variance rate in the cases with $R=1$. This does not happen only for Berry's random wave model because it is "special", but whenever $C\notin L^R(\mathbb{R}^d)$ and (1.42) holds.

To prove Theorem 1.4.1, we proceed dividing the analysis in two cases: $R\ge 4$ and $R\le 2$ (note that $R\ne 3$ by assumption). Since if $C\in L^R$ we can use Breuer-Major Theorem 1.3.1, we assume that $C\notin L^R(\mathbb{R}^d)$. Then we can use reduction theorems (see in particular Proposition 2.2.2) to reduce the study from a general φ to $\varphi = H_R$ (or $\varphi = H_{R'}$, $R'\ge 2$ if $R=1$). At this point, recalling Theorem 1.2.8, (1.28) and (1.31), we need to show that for R even the following converges to 0 for every $r=1,\dots,R-1$,

$$\frac{\int_{\mathbb{R}^{3d}} C(u)^r C(v)^r C(w)^{R-r} C(u+v+w)^{R-r}\, g_{D_t(-w),D_t(u)}(v)du\,dv\,dw}{(\int_{\mathbb{R}^d} C^R(z) g_{tD}(z)dz)^2}.$$

To do this, we study separately the two following cases:

- If $R\ge 4$ even, then we can show (by using Schoenberg's theorem) that $C(z)|z|^{d/R}$ is bounded. Thus, we can use techniques exploited e.g. in [NPR19, Lemma 8.1] to show that the previous converges to 0.

- If $R=2$, then $C(z)|z|^{d/2}$ is not necessarily bounded, and we exploit Bochner's and Schoenberg's spectral representations, together with Fourier analysis, to find a bound in terms of spectral measures and the Fourier transforms of cross covariograms, showing that (1.31) converges to 0. To do this, the starting idea is rewriting the "annoying factor" $C(u+v+w)$ using Bochner's theorem,

$$\frac{\int_{\mathbb{R}^d} G(dx) \int_{\mathbb{R}^{2d}} du\, dv\, C(u)\, C(v) e^{i\langle x,u+v\rangle} \int_{\mathbb{R}^d} dw\, C(w) e^{i\langle x,w\rangle} g_{D_t(-w),D_t(u)}(v)}{(\int_{\mathbb{R}^d} C^R(z) g_{tD}(z) dz)^2},$$

recalling that $g_{D_t(-w),D_t(u)} = \mathbf{1}_{D_t(-w)} * \mathbf{1}_{-D_t(u)}$ and finally using the convolution theorem and other Fourier analysis results to simplify the expression.

- If $R = 1$, we reduce the study from φ to $\varphi - a_R H_R$ with Hermite rank $R' \geq 2$, and use the arguments above.

As we anticipated in section 1.2.1, one may probably extend Theorem 1.4.1 to non-isotropic Gaussian fields with "regular" spherical spectral measure, or maybe to non-stationary harmonizable Gaussian fields, probably under analogous (but different) spectral conditions.

We conclude with an important remark, that also introduces the next subsection.

Remark 1.4.1. The reason why we need to exclude some cases in Theorem 1.4.1, assuming for instance that R is not odd, or that the even chaoses are not all null, is the tricky nature of the odd chaoses, due to the possible oscillating behavior of the covariance function (see e.g. the asymptotics (1.43) of the Berry field). Indeed, by (1.27) the variance of additive functionals of Gaussian fields is

$$\operatorname{Var}\left(\int_{tD} \varphi(B_x) dx\right) = \sum_{q=1}^{\infty} a_q^2 \operatorname{Var}\left(\int_D H_q(B_x) dx\right).$$

Thus, since

$$\operatorname{Var}\left(\int_D H_q(B_x) dx\right) = q! \int_{\mathbb{R}^d} C^q(z) g_{tD}(z) dz,$$

one can easily intuit why the variance rate is easier to derive when $C^q \geq 0$ and harder otherwise, at least if we want to work for general C. Indeed, even in the apparently simple case $C^q \in L^1(\mathbb{R}^d)$, reasoning as in (1.38) we have

$$\operatorname{Var}\left(\int_D H_q(B_x) dx\right) \sim q! t^d \operatorname{Vol}(D) \int_{\mathbb{R}^d} C^q(z) dz,$$

which is useless without knowing that $\int_{\mathbb{R}^d} C^q(z) dz \neq 0$. If B is the Berry random wave model, for example, then the positivity of $\int_{\mathbb{R}^d} C^q(z) dz$ for every $q \geq 5$ odd was left as a conjecture in [MW14].

1.4.2 Chapter 3: Fluctuations of Polyspectra in Spherical and Euclidean Random Wave Models

In Chapter 3, we focus on Question 2 when $d \geq 2$, $D = L$ is a ball and $A_x = H_q(B_x)$, where B is the Berry random wave model. As we just discussed in Remark 1.4.1, this problem was still open for general q odd. Moreover, we also deal with the "spherical version" of this question. More precisely, we study

$$V^E_{d,R}(q,\lambda) = \mathrm{Var}\left(\int_{B^E(R)} H_q(U_\lambda(x))dx\right) = \mathrm{Var}\left(\int_{B^E(R)} H_q(B_{\lambda x})dx\right), \quad \lambda\to\infty$$

and

$$V^S_{d,R}(q,\ell) = \mathrm{Var}\left(\int_{B^S(R)} H_q(T_\ell(x))dx\right), \quad \ell\to\infty,$$

where $B^E(R), B^S(R)$ are Euclidean/spherical balls of radius $R>0$ and $(U_\lambda(x))_{x\in\mathbb{R}^d}$, $(T_\ell(x))_{x\in S^d}$ are Euclidean/spherical random wave models, defined as follows:

- The Euclidean random wave model $U_\lambda(x)$, $\lambda>0$, $x\in\mathbb{R}^d$, is $U_\lambda(x) := B_{\lambda x}$, where B is the Berry random wave model (see Definition 1.1.9).
- The (hyper)spherical random wave model $T_\ell(x)$, $\ell\in\mathbb{N}$, $x\in S^d$, is the only (in distribution) Gaussian field on S^d with covariance function

$$\mathbb{E}T_\ell(x)T_\ell(y) = \mathcal{G}_{d,\ell}(\cos d(x,y)) \quad x,y\in S^d, \tag{1.45}$$

  where $\mathcal{G}_{d,\ell}$ is the Gegenbauer polynomial of degree ℓ (see [Sze39, p. 4.7]).

U_λ and T_ℓ can be also defined in terms of random linear combinations of Laplace eigenfunctions, similarly to how we defined monochromatic random waves in Example 1.1.3 (see Chapter 3 for more details).

The main result of the chapter is the following theorem, that provides a complete picture of the asymptotic variances of the above polyspectra for $d\ge 2$ and $q\ge 2$ (the case $q=1$ is apart, because the variances defined above are 0 infinitely many times).

Theorem 1.4.2. *Let $d,q\ge 2$, $R>0$. There exist finite* ***positive*** *constants c^d_q such that:*

- *(Euclidean) as $\lambda\to\infty$,*

$$V^E_{d,R}(q,\lambda) = c^d_q\,\mathrm{Vol}(B^E(R))(1+o_{q,d}(1))\cdot\begin{cases}\lambda^{1-d} & q=2,\\ \lambda^{-2}\log(\lambda) & q=4, d=2,\\ \lambda^{-d} & \textit{all other } d\ge 2, q\ge 3,\end{cases}$$

- *(Hyperspherical) as $\ell\to\infty$, if $R\in(0,\pi)$,*

$$V^S_{d,R}(q,\ell) = c^d_q\sigma_d(B^S(R))(1+o_{q,d}(1))\cdot\begin{cases}\ell^{1-d} & q=2,\\ \ell^{-2}\log(\ell) & q=4, d=2,\\ \ell^{-d} & \textit{all other } d\ge 2, q\ge 3,\end{cases}$$

  where σ_d is the volume measure on S^d, and when $R=\pi$ (that is in the case of polyspectra obtained integrating over the whole S^d),

$$V^S_{d,\pi}(q,\ell) = 2c^d_q q!\omega_{d-1}\omega_d(1+o_{q,d}(1))\cdot\begin{cases}\ell^{1-d} & q=2,\\ \ell^{-2}\log(\ell) & q=4, d=2,\\ 0 & q,\ell \textit{ both odd},\\ \ell^{-d} & \textit{all other } d\ge 2, q\ge 3.\end{cases}$$

Some of the previous asymptotic variances were already known in the literature, see Remark 3.1.3. Before Theorem 1.4.2, the missing cases corresponded to $q \geq 5$ odd. In these situations, we have

$$c_q^d = c_d \int_{\mathbb{R}^d} C^q(z)dz\,,$$

where C is the covariance function of the Berry random wave model and c_d is a positive constant depending only on d. Our main contribution consists in showing that $\int_{\mathbb{R}^d} C^q(z)dz$ is positive for every $q \geq 3$ odd. To prove this, we exploit a relation between the latter and Pearson's random walks. The Pearson's random walk X_q^d is a random variable defined as the sum of q independent random variables uniformly distributed on S^{d-1}. Thus, if C is the covariance function of the d-dimensional Berry random wave model, i.e. the characteristic function of X_1^d, we have that C^q is the characteristic function of X_q^d. As a consequence, Kluyver's formula (see Lemma 3.2.1) implies

$$\int_{\mathbb{R}^d} C^q(z)dz = f_q^d(0),$$

where f_q^d is the density of X_q^d, i.e. the spectral density associated to C^q, which exists for $q \geq 2$. Thus, the proof is reduced to showing that $f_q^d(0)$ is positive, which is done relying on technical results in the context of Pearson's random walks.

We conclude with the following remark, that also introduces the next subsection.

Remark 1.4.2. Recall Question 2. Theorem 1.4.2 (Euclidean statement) allows to compute the asymptotic variance of $\int_{tD} A_x\,dx$ if $A_x = H_q(B_x)$, B is the Berry random wave model and $D = L$ is a ball. For $q \geq 5$, we have $C^q \in L^1$, i.e. A has short memory, and we can obtain the exact rate for $V_{d,R}^E(q,t)$ combining (1.38) and the positivity of $\int_{\mathbb{R}^d} C^q(z)dz$ mentioned above. For $q \in \{2, 3, 4\}$, $C^q \notin L^1(\mathbb{R}^d)$ and C^q is not regularly varying. This means that A has long memory, but we can not compute the asymptotic variance as we did in Section 1.3.2. Despite this, we are able to compute the asymptotic variance of $\int_{tD} H_q(B_x)dx$ with ad-hoc techniques. Since these techniques only partially rely on the specific covariance structure of the Berry random wave model, a natural question arises: can we use analogous techniques to compute the asymptotic (co)variances of other random fields? This question is thoroughly investigated in Chapter 4.

1.4.3 Chapter 4: Asymptotic covariances for functionals of weakly stationary random fields

In Chapter 4 we answer to Question 2. More precisely, given a measurable, weakly stationary random field A on $\mathbb{R}^d$ with covariance function $K : \mathbb{R}^d \to \mathbb{R}$, our goal is giving the minimal assumptions on K and $\mathcal{D}$ (a class of compact sets of $\mathbb{R}^d$) to compute the asymptotic covariances

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} A_x\,dx}{r_t}, \frac{\int_{tL} A_y\,dy}{r_t}\right), \qquad D, L \in \mathcal{D}, \tag{1.46}$$

where $r_t \to \infty$ is chosen so that the previous limit is finite and not identically 0 for every $D, L \in \mathcal{D}$.

To do this, let us consider the **integral covariance function** $w : \mathbb{R}_+ \to \mathbb{R}$,

$$w_t := \int_{\{|z| \le t\}} K(z) dz \,,$$

and note the following facts:

- If $K \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} K(z)dz \neq 0$, then we saw in Subsection 1.3.1 that $r_t = t^{d/2}$ is a correct choice for computing the asymptotic covariances. Moreover, also $r_t = t^{d/2} w_t^{1/2}$ is a correct choice and we have $\forall\, D, L$ compact

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} A_x\, dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_y\, dy}{t^{d/2} w_t^{1/2}} \right) = \mathrm{Vol}\,(D \cap L)\;, \qquad D, L \in \mathcal{D}. \tag{1.47}$$

- If $K(z) = k(|z|)$ is radial and regularly varying with index $-\beta \in (-d, 0)$, see (1.39), then we saw in Subsection 1.3.2 that $r_t = t^d k(t)^{1/2}$ is a correct choice for computing the asymptotic covariances. Moreover, by Proposition 4.2.4 $r_t = t^{d/2} w_t^{1/2}$ is also a correct choice, and we have $\forall\, D, L$ compact

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} A_x\, dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_y\, dy}{t^{d/2} w_t^{1/2}} \right) = \frac{(d-\beta)}{\omega_{d-1}} \int_D \int_L |x-y|^{-\beta} dx dy. \tag{1.48}$$

Thus, in both the classical situations analyzed in Subsections 1.3.1-1.3.2, we have that w_t is regularly varying (with index 0 in the first case and $(d-\beta) \in (0, d)$ in the second case) and $r_t = t^{d/2} w_t^{1/2}$ is a correct choice for computing the asymptotic covariances. For this reason, we may conjecture that if w_t is regularly varying, then we can compute (1.46) with rate $r_t = t^{d/2} w_t^{1/2}$. We prove that this is the case in the two following situations:

Case 1. K is **radial** and $\mathcal{D}$ is the class of all compact sets in $\mathbb{R}^d$ with **finite perimeter** (recall the definition of perimeter given in (1.29)), i.e. for some $k : \mathbb{R}_+ \to \mathbb{R}$ we have

$$K(z) = k(|z|), \quad \mathcal{D} = \{D \subseteq \mathbb{R}^d : D \text{ compact}, \mathrm{Per}(D) < \infty\}. \tag{1.49}$$

Case 2. $\mathcal{D}$ is the class of **closed balls centered** at a fixed point $x_0 \in \mathbb{R}^d$, namely

$$\mathcal{D} = \left\{ \left\{ x \in \mathbb{R}^d : |x - x_0| \le r \right\} : r \in \mathbb{R}_+ \right\}. \tag{1.50}$$

The main result of Chapter 4 is the following.

Theorem 1.4.3. *Let $\mathcal{D}$ be a collection of compact sets in $\mathbb{R}^d$ and $A = (A_x)_{x\in\mathbb{R}^d}$ be a measurable, weakly stationary random field with covariance function $K : \mathbb{R}^d \to \mathbb{R}$. Furthermore, let K and $\mathcal{D}$ satisfy the assumption (1.49) or the assumption (1.50), and assume that w is* ***regularly varying*** *with index $\alpha \in (-1, d]$, that is*

$$w_t = \ell(t) t^\alpha \quad \forall t \in \mathbb{R}_+ \tag{1.51}$$

where $\ell : \mathbb{R}_+ \to \mathbb{R}$ is slowly varying. Then, for all $D, L \in \mathcal{D}$, we have as $t \to \infty$

$$\mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_x dx}{t^{d/2} w_t^{1/2}}\right) \to \frac{1}{\omega_{d-1}} \int_{S^{d-1}} d\theta \int_0^\infty dl \left(-\frac{d}{dl} g_{D,L}(l\theta)\right) l^\alpha. \quad (1.52)$$

Theorem 1.4.3 allows to compute the asymptotic covariances (1.46) under assumptions on K that encompass the classical ones discussed in Subsections 1.3.1-1.3.2. Note that (1.52) is a new expression for the asymptotic covariances that generalizes (1.47)-(1.48), see Remark 4.1.2 for the details.

Moreover, combining Theorem 1.4.1, Theorem 1.4.3 and the Peccati-Tudor Fourth Moment Theorem 1.2.9, we can prove new multi-dimensional central limit theorems for additive functionals of Gaussian fields, assuming only (1.51). If for instance B is Berry's random wave model, $\varphi \in L^2(\mathbb{R}, \gamma(dx))$ with $a_2 \neq 0$ in (1.16) and $A_x = \varphi(B_x)$, then w_t is regularly varying with index $\alpha = 1$ and we have (see Example 4.4.5 for the details)

$$\left(\frac{\int_{tD} A_x dx}{t^{d/2} w_t^{1/2}}\right)_{D \in \mathcal{D}_O} \overset{f.d.d.}{\to} G = (G(D))_{D \in \mathcal{D}_O}$$

where $\overset{f.d.d.}{\to}$ denotes the convergence in finite dimensional distributions, $\mathcal{D}_O$ is a suitable class of compact sets with finite perimeter and G is a centered Gaussian field on $\mathcal{D}_O$ with covariance kernel

$$\mathbb{E}\left[G(D)G(L)\right] = \frac{1}{\omega_{d-1}} \int_D \int_L |x-y|^{1-d} dx dy = \frac{1}{\omega_{d-1}} \int_{\mathbb{R}^d} g_{D,L}(z) |z|^{1-d} dz.$$

For other examples and applications of Theorem 1.4.3, see Sections 4.4-4.5.

1.4.4 Chapter 5: Limit theorems for p-domain functionals of stationary Gaussian fields

In the last Chapter 5, we deal with the p-domain generalization of Question 1 introduced in Subsection 1.3.3, that is, we study the fluctuations of

$$Y(t_1, \dots, t_p) := \int_{t_1 D_1 \times \dots \times t_p D_p} \varphi(B_x) dx, \qquad t_1, \dots, t_p \to \infty \quad (1.53)$$

where $2 \leq p \leq d$, the compact sets $D_i \subseteq \mathbb{R}^{d_i}$ have positive Lebesgue measure, $d_1 + \dots + d_p = d$, and $t_1, \dots, t_p$ can grow possibly at different rates.

We start assuming that the covariance function $C : \mathbb{R}^d \to \mathbb{R}$ is **separable**, that is

$$C(x_1, \dots, x_p) = \prod_{i=1}^p C_i(x_i), \quad x_i \in \mathbb{R}^{d_i},$$

where $C_i : \mathbb{R}^{d_i} \to \mathbb{R}$ are non-negative definite functions on $\mathbb{R}^{d_i}$, with $C_i(0) = 1$. In this setting, we thoroughly investigate under what conditions the study of the fluctuations of (1.53) can be reduced to that of its marginal functionals $Y_i(t_i)$,

$$Y_i(t_i) := \int_{t_i D_i} \varphi(B^{(i)}_{x_i}) dx_i,$$

where $B^{(i)}$ is a stationary centered Gaussian field on $\mathbb{R}^{d_i}$ with covariance function C_i. The main theorem of Chapter 5 is the following.

Theorem 1.4.4. *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Consider $\varphi \in L^2(\mathbb{R}, \gamma(dx))$ with Hermite rank $R \geq 1$. Let $C : \mathbb{R}^d \to \mathbb{R}$ be the covariance function of B, assume it is separable, and also that it satisfies, for each i:*

$$C^R \geq 0 \quad \text{and} \quad C_i \in \bigcup_{M=1}^{\infty} L^M(\mathbb{R}^{d_i}).$$

Let us consider $\widetilde{Y}$ given by (5.3) and $\widetilde{Y}_i$ given by (5.6), the rescaled versions of Y, Y_i. Then, the following two assertions are equivalent:

(a) $\widetilde{Y}_i(t_i) \xrightarrow{d} N(0,1)$ *as* $t_i \to \infty$ *for at least one* $i \in \{1,\dots,p\}$*;*

(b) $\widetilde{Y}(t_1,\dots,t_p) \xrightarrow{d} N(0,1)$ *as* $t_1,\dots,t_p \to \infty$*.*

The implication $(a) \implies (b)$ in Theorem 1.4.4 ensures that the p-domain functional (1.53) has Gaussian fluctuations if one of the 1-domain marginal functionals does. This criterion is quite useful because, contrarily to p-domain ones, 1-domain functionals have been extensively studied in the literature, see e.g. Subsections 1.3.1-1.3.2 and the references therein. It is noteworthy that a specific instance of this implication had previously been observed in the papers [RST12] and [PR16]; however, it was restricted to a very specific context – rectangular increments of a fractional Brownian sheet – and was not part of a comprehensive systematic investigation, as we undertake in Chapter 5.

The theorem is proved relying on a p-domain version of Breuer-Major Theorem 1.3.1 when $C \in L^R$ and reducing the problem to $\varphi = H_R$ if $C \notin L^R$.

Indeed, in the particular case $\varphi = H_q$, the equivalence in Theorem 1.4.4 is true without integrability assumptions on C. Moreover, exploiting Theorem 1.2.10 and some inequalities involving the fourth moment in a fixed Wiener chaos, **if $\boldsymbol{\varphi = H_q}$** we prove that (see Proposition 5.3.3)

$$d_{TV}(\widetilde{Y}(t_1,\dots,t_p), N) \leq c_q \prod_{i=1}^{p} \sqrt{\mathbb{E}[\widetilde{Y}_i(t_i)^4] - 3}\,. \tag{1.54}$$

This inequality also allows to improve some bounds for the Hermite variations of the rectangular increments of the fractional Brownian sheet obtained in [RST12], see Example 5.4.1 for the details.

Note that Theorem 1.4.4 also brings to light another large class of Gaussian fields for which the intuition "long memory and $R \geq 2$ imply Gaussian fluctuations" is wrong (recall the discussion at the beginning of Subsection 1.4.1). Indeed, if we take for instance $R = 2$, $C_1 \in L^R(\mathbb{R}^d)$ and C_i of the form (1.39) if $i \neq 1$, then combining Theorem 1.3.1 on Y_1 and Theorem 1.4.4 we have Gaussian fluctuations even if we are in a long memory setting.

Conversely, when C_i is of the form (1.39) for every i, we are able to prove a generalization of Theorem 1.3.2 (see Theorem 5.1.2) for p-domain functionals, ensuring non-Gaussian fluctuations as soon as $R \geq 2$.

It is worth noting that all our results obtained assuming that C is separable may be adapted to situations where some of the domains are fixed, see Remark 5.1.4.

When C is **not separable**, it is easy to construct examples illustrating that the normal convergence of the functionals $\widetilde{Y}_i(t_i)$ is in general not enough to determine the behavior of $\widetilde{Y}(t_1,\dots,t_p)$ (see Example 5.5.1).

For this reason, we go beyond separability, studying two non-separable classes of covariance functions (in the specific case $\varphi = H_q$):

- **Gneiting covariance functions**, of the form

$$C(x_1,x_2) = C_2(x_2)C_1\left(x_1 C_2(x_2)^{2/d_1}\right), \qquad x_1 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2},$$

  where C_1, C_2 have "nice" properties. In this case, C can be wedged between two separable covariance functions. This fact allows to prove Theorem 5.5.1, that brings to light a comparable phenomenon to the one observed in the separable case.

- **Additively separable covariance functions**, of the form

$$C(x_1,x_2) = K_1(x_1) + K_2(x_2), \qquad x_1 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2},$$

  where K_1, K_2 are positive covariance functions on $\mathbb{R}^{d_1}, \mathbb{R}^{d_2}$. In this case, we are still able to prove a kind of "reduction" theorem (see Theorem 5.5.2), but with marginal functionals completely different from the Y_i's of Theorem 1.4.4.

In particular, a remarkable fact observed in the additively separable case (see Example 5.5.2) is that $Y(t_1,t_2)$ has (or has not) Gaussian fluctuations depending on the growth rates t_1, t_2. This is very different from what we observe in the separable case, where the interplay between the growth rates of the domains plays no role.

CHAPTER **2**

Spectral central limit theorem for additive functionals of isotropic and stationary Gaussian fields

This chapter contains the paper [MN24], "Spectral central limit theorem for additive functionals of isotropic and stationary Gaussian fields", written in collaboration with Ivan Nourdin. *Ann. Probab.* (2024), 52(2), 737 – 763.

2.1 Introduction

Fix a dimension $d \geq 2$, and consider a real-valued almost surely continuous Gaussian field $(B_x)_{x\in\mathbb{R}^d}$ defined on $\mathbb{R}^d$. Assume furthermore that B is **stationary**, that is, there is a function $C : \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathrm{Cov}(B_x, B_y) = C(x-y), \quad x, y \in \mathbb{R}^d, \tag{2.1}$$

and suppose that $B_x \sim N(0,1)$ for all $x \in \mathbb{R}^d$ or, equivalently, that $\mathbb{E}[B_x] = 0$ and $C(0) = 1$.

As a second ingredient, consider a measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}\left[\varphi(N)^2\right] < \infty, \quad \text{for } N \sim N(0,1). \tag{2.2}$$

Our object of interest in this paper is

$$Y_t = \int_{tD} \varphi(B_x)dx, \quad t > 0, \tag{2.3}$$

where $D \subset \mathbb{R}^d$ is compact with $\mathrm{Vol}(D) > 0$ and $tD := \{tx | x \in D\}$. Well-posedness of (2.3) as a random variable in $L^2(\Omega)$ is ensured by Proposition 2.2.1 and the almost sure continuity of $(B_x)_{x\in\mathbb{R}^d}$. We also observe that the continuity of B, together with its stationarity, implies[1] the continuity of its covariance function C, a property that will be needed to evoke Bochner's theorem later in (2.14).

Studying the asymptotic behavior of functionals of the form (2.3) dates back from the eighties, with seminal works by Breuer and Major [BM83], Dobrushin and Major [DM79], Rosenblatt [Ros60] and Taqqu [Taq79]. Since then, limit theorems for (2.3) have been constantly investigated, and represent nowadays a central theme in the modern probability theory.

We note that many interesting geometric quantities associated with the Gaussian field $(B_x)_{x\in\mathbb{R}^d}$ can be represented as functionals of the form (2.3). For instance, the choice $\varphi = \mathbf{1}_{(-\infty,u]}$ (resp. $\varphi = \mathbf{1}_{[u,\infty)}$), $u \in \mathbb{R}$ corresponds to the volume of the lower (resp. upper) level sets of $(B_x)_{x\in\mathbb{R}^d}$.

Since (2.2) holds, we can decompose φ in Hermite polynomials (see, e.g., [NP12, Section 1.4]) as

$$\varphi = \mathbb{E}[\varphi(N)] + \sum_{q=R}^{\infty} a_q H_q, \quad \text{with } R \geq 1 \text{ such that } a_R \neq 0, \tag{2.4}$$

where H_q denotes the qth Hermite polynomial and $a_q = a_q(\varphi) = \frac{1}{q!}\mathbb{E}\left[\varphi(N)H_q(N)\right] \in \mathbb{R}$. The integer $R \geq 1$ is called the **Hermite rank** of φ. We also define the **second Hermite rank** $R' \geq 2$ of φ as the Hermite rank of $\varphi(x) - \mathbb{E}[\varphi(N)] - a_R H_R(x)$ (if $\varphi(x) = \mathbb{E}[\varphi(N)] + a_R H_R(x)$, we set $R' = \infty$).

In the present paper, we are more specifically interested in the asymptotic behavior of

$$\frac{Y_t - m_t}{\sigma_t}, \quad t \to \infty, \tag{2.5}$$

where we have $Y_t \in L^2(\Omega)$ for all t, and where we note $m_t = \mathbb{E}[Y_t]$ and $\sigma_t = \sqrt{\mathrm{Var}(Y_t)} > 0$. For simplicity, to ensure that $\sigma_t > 0$ for all t, we will assume for the rest of the paper the existence of some $k \geq 1$ such that $a_{2k} \neq 0$[2], or equivalently that φ is **not an odd function**. As an illustration of what may happen when φ is odd, see (2.22).

Throughout all the paper, for two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ we write

$$f(t) \asymp g(t) \tag{2.6}$$

to indicate that $f(t) = O(g(t))$ and $g(t) = O(f(t))$ as $t \to \infty$.

[1] Indeed, since $|B_0(B_{x+h} - B_x)| \leq B_0^2 + \frac{1}{2}B_x^2 + \frac{1}{2}B_{x+h}^2$, by applying the generalized dominated convergence theorem we have $C(x+h) - C(x) = \mathbb{E}\left[B_0(B_{x+h} - B_x)\right] \to 0$ as $h \to 0$.

[2] This comes from the fact that $\sigma_t^2 = \mathrm{Var}(Y_t) = \sum_{q=R}^{\infty} q! a_q^2 \int_{(tD)^2} C^q(x-y)dxdy$.

2.1.1 Previous results

Given its importance in our paper, we start with the celebrated Breuer-Major theorem, stated here in its continuous form.

Theorem 2.1.1 (Breuer, Major [BM83]). *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, assumed to be stationary and to have unit-variance. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$, let R be the Hermite rank of φ, consider Y_t defined by (2.3), and recall the definition (2.1) of the covariance function C.*

If $\int_{\mathbb{R}^d} |C(x)|^R dx < \infty$, then $t^{-\frac{d}{2}}(Y_t - \mathbb{E}[Y_t]) \overset{\text{law}}{\to} N(0,\sigma^2)$ where

$$\sigma^2 = \mathrm{Vol}(D) \sum_{q=R}^{\infty} q! a_q^2 \int_{\mathbb{R}^d} C(z)^q dz \geq 0. \tag{2.7}$$

In particular, if φ is not odd, then $\sigma^2 > 0$, $\sigma_t^2 \asymp t^d$ and

$$\frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1).$$

To describe the asymptotic behavior in the case where $C \notin L^R(\mathbb{R}^d)$ (that is, when we cannot apply the Breuer-Major Theorem 2.1.1), we have to be more precise on the behavior of C at infinity. In the papers studying limit theorem for (2.5) in a *general* framework (i.e. not for a *particular* model), it is often (if not always) assumed that

$$C(x) = |x|^{-\beta} L(|x|), \tag{2.8}$$

with $\beta \in (0,\infty)$ and $L : \mathbb{R}_+ \to \mathbb{R}$ **slowly varying** (that is, satisfying $L(\lambda r)/L(\lambda) \to 1$ as $\lambda \to \infty$, for every fixed $r > 0$). The following three different situations (i)-(ii)-(iii) then occur:

1. If $\beta > \frac{d}{R}$ then $C \in L^R(\mathbb{R}^d)$ and we say that we are in the **short-memory case**. We deduce from Breuer-Major Theorem 2.1.1 that $t^{-d/2}(Y_t - \mathbb{E}[Y_t]) \overset{\text{law}}{\to} N(0,\sigma^2)$.

2. If $\beta = \frac{d}{R}$ we are in a **critical case**. Although $C \notin L^R(\mathbb{R}^d)$, fluctuations of Y_t around its mean are still asymptotically Gaussian (see, e.g., [BM83, Theorem 1'], [NN20, Section 5] and [NNT10, Theorem 1]).

3. If $\beta < \frac{d}{R}$ then $C \notin L^R(\mathbb{R}^d)$ and we say that we are in the **long-memory case**. A theorem of Dobrushin and Major [DM79] asserts that $t^{-(d-\frac{R\beta}{2})}L^{-R/2}(t)(Y_t - \mathbb{E}[Y_t]) \overset{\text{law}}{\to} Z$ where, up to a multiplicative constant, Z is the Hermite distribution of order R and self-similarity index $H \in (0,1)$ (with H depending only on β). Since we do not use it in the sequel, we do not give its precise definition here. (The interested reader can e.g. consult [Tud13].) Let us only stress here that the Hermite distribution of order R belongs to the Rth Wiener chaos, and so is **not Gaussian** as soon as $R \geq 2$.

2.1.2 Motivating examples

The intuition we can naturally develop from the previous (i)-(ii)-(iii) (and that represents the common intuition forged by the papers written on the subject over the last forty years) is that Y_t defined by (2.3) displays *Gaussian* (resp. *non-Gaussian*) fluctuations when the Gaussian field B has *short* (resp. *long*) memory, and this whatever the function φ with Hermite rank $R \geq 2$. (The case $R = 1$ is apart.[3])

As anticipated, we will show in this paper that this intuition can be wrong, and not only marginally or in critical cases. We will indeed state a central limit theorem (Theorem 2.1.2 below) whose conclusion is valid provided that a certain spectral condition is satisfied. As we will see, this may lead to Gaussian fluctuations in a long memory context, in total contrast with Dobrushin and Major [DM79] (see (iii) above).

Recently, **Berry's Random Wave Model** (BRWM) has attracted a lot of attention. It is defined as follows. Choose the dimension $d = 2$ and consider the centered continuous Gaussian field B on $\mathbb{R}^2$ with covariance $\mathbb{E}[B_x B_y] = C(x-y) = J_0(|x-y|)$, with J_0 the Bessel function of the first kind of order 0 (see Section 2.2.2 for its definition and some properties); in particular, we have

$$C(x) = \sqrt{\frac{2}{\pi}}\,|x|^{-\frac{1}{2}} \cos\left(|x| - \frac{\pi}{4}\right) + O\left(|x|^{-\frac{3}{2}}\right) \quad \text{as } |x| \to \infty, \tag{2.9}$$

see e.g. [Kra14, Theorem 4]. This field, called in this way in honor of Berry who introduced it in the seminal paper [Ber77], can be seen as a universal Gaussian field emerging as the local scaling limit of a number of random fields on two-dimensional manifolds, see e.g. [Die+23] and the references therein. It is widely regarded as a popular model for the Laplacian eigenfunctions (with large eigenvalue t^2) of classically chaotic billiards, hence its importance in quantum mechanics. Indeed, integrating $\varphi(B_x)$ over tD is the same as integrating $\varphi(B_{tx})$ over the fixed domain D, after a change of variable. As we will see, our Theorem 2.1.2 will allow to prove the Gaussian fluctuations of $\int_D \varphi(B_{tx})dx$ for many functionals that were never investigated in the literature, in particular for the cases $R = 2$ and $R = 1$ in (2.10), enriching the knowledge on the geometric properties of the Berry's random wave model. Indeed, to the best of our knowledge, so far Gaussian fluctuations were proved only for (2.3) under the assumption $R \geq 4$ (see e.g. [Not21]), or for nodal set volumes (see [NPR19]), where the latter cannot be expressed in the form (2.3).

If we compare (2.9) with (2.8), it is like we have $\beta = \frac{1}{2}$ and that we had replaced the slowly varying function L in (2.8) by the bounded and oscillatory function $\cos(\cdot - \frac{\pi}{4})$. As we will see, this replacement is all but insignificant in the presence of long memory. Indeed, taking for example D compact such that $D = \overline{\mathring{D}}$ and with smooth ∂D and non-vanishing Gaussian curvature, if we apply our main result (Theorem 2.1.2 below) for $R \in \{1, 2, 4\}$ and Breuer-Major theorem for $R \geq 5$, we obtain (recalling that R, resp. R', denotes the Hermite rank, resp. the second Hermite

[3]The case where $R = 1$ is apart since, whatever the memory, we generally obtain Gaussian fluctuations. But this is for different reasons that we understand better in the functional version: in the short memory case the limit is a standard Brownian motion, while in the long memory case the limit is a fractional Brownian motion.

rank, of the **non-odd** function φ):

$$\sigma_t^2 \asymp \begin{cases} t^3 & \text{if } R = 2 \text{ or } (R = 1 \text{ and } R' = 2) \\ t^2 \log(t) & \text{if } R = 4 \text{ or } (R = 1 \text{ and } R' = 4) \\ t^2 & \text{if } R \geq 5 \text{ or } (R = 1 \text{ and } R' \geq 5) \end{cases} \tag{2.10}$$

and

$$\frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1) \quad \text{as } t \to \infty. \tag{2.11}$$

Indeed, Gaussian fluctuations (2.11) and asymptotics (2.10) follow directly from (2.9) and the Breuer-Major Theorem 2.1.1 when $R \geq 5$, because in this case we have $\int_{\mathbb{R}^d} |C(x)|^R dx < \infty$. The case $R = 4$ is comparable with the situation (ii) in the model (2.8), since $\beta = \frac{1}{2} = \frac{d}{R}$; Gaussian fluctuations (2.11) and asymptotic (2.10) displaying a logarithmic correction are then not surprising, since in agreement with what is usually observed in critical cases. In constrast, the fact that we still have Gaussian fluctuations in (2.11) with unconventional rate when $R \in \{1, 2\}$ is very surprising. Indeed, since $\beta = \frac{1}{2} < 1 \leq \frac{d}{R}$ in this case, we are in the long-memory case where non-Gaussian fluctuations are usually the rule, see (iii). This last case turns out to be the most important, since $R = 1$ and $R = 2$ are the most common values in applications.

As we will see, what we just described (Gaussian fluctuations in a long memory framework) for BRWM is not an isolated phenomenon. In fact, we are going to state and prove Theorem 2.1.2 (our main result), which not only explains in a clear way the phenomenon observed for BRWM, but also bring to light an easy-to-check condition on the spectral measure of a general Gaussian field B (see (2.20) below) to imply Gaussian fluctuations for Y_t.

We conclude this section by emphasing that Gaussian fluctuations in presence of long memory have already been observed in the literature in other (non-Euclidean) contexts, in particular for integral functionals of random Laplace eigenfunctions on the sphere $\mathbb{S}^2$, see [MW14] for more details.

2.1.3 Main result

In order to state Theorem 2.1.2, we have to introduce a certain number of further quantities and notations. We continue to let B, φ and Y_t be as described in (2.1), (2.2) and (2.3) respectively, and we recall the Hermite decomposition (2.4) of φ defining its Hermite rank R and its second Hermite rank R'.

Fix $t > 0$ and $q \geq R$, and set

$$Y_{q,t} = \int_{tD} H_q(B_x) dx. \tag{2.12}$$

Using (2.4) , we immediately get that

$$Y_t = \mathbb{E}[Y_t] + \sum_{q=R}^{\infty} a_q Y_{q,t}, \quad t \geq 0. \tag{2.13}$$

Moreover, a direct computation (making use of the isometry properties of Hermite polynomials, see, e.g., [NP12, Section 1.4]) yields that

$$\mathrm{Var}(Y_{q,t}) = q! t^d v_{q,t},$$

where

$$v_{q,t} = \int_{\mathbb{R}^d} C(z)^q g_D\left(\frac{z}{t}\right) dz = \int_{\{|z|\le \mathrm{diam}(D)t\}} C(z)^q g_D\left(\frac{z}{t}\right) dz \ge 0,$$

with $g_D(x)$ the **covariogram** of D at $x \in \mathbb{R}^d$, defined as the Lebesgue measure of $D \cap (x + D)$, and $\mathrm{diam}(D) := \sup\{|x - y| : x, y \in D\} < \infty$ since D is compact. When $C \in L^q(\mathbb{R}^d)$, we deduce by dominated convergence (using that $\|g_D\|_\infty \le \mathrm{Vol}(D)$ and $g_D(\frac{z}{t}) \to \mathrm{Vol}(D)$ as $t \to \infty$ for all fixed $z \in \mathbb{R}^d$) that

$$v_{q,t} \to \mathrm{Vol}(D) \int_{\mathbb{R}^d} C(z)^q dz \ge 0.$$

Since our field B is stationary and its covariance function C is continuous, Bochner's theorem yields the existence of a finite measure G on $\mathbb{R}^d$ such that

$$C(x) = \int_{\mathbb{R}^d} e^{i\langle \lambda, x\rangle} G(d\lambda), \tag{2.14}$$

with $\langle \cdot, \cdot \rangle$ the usual scalar product on $\mathbb{R}^d$. The measure G is called the **spectral measure** of B, and it will be our gateway towards the Fourier analysis techniques developed in the sequel.

At this stage, let us make a further assumption on B, by supposing that it is also **isotropic**. In our framework, this is equivalent to suppose that the quantity $C(x)$ only depends on the norm $|x|$, namely that there is a function $\rho : \mathbb{R}_+ \to \mathbb{R}$ such that

$$C(x) = \rho(|x|), \quad x \in \mathbb{R}^d.$$

Now, set

$$\mu(s) = G(\{|x| \le s\}), \quad s \in (0, \infty). \tag{2.15}$$

Since μ is increasing and bounded, it defines a finite measure on $\mathbb{R}_+$, called the **isotropic spectral measure** of B. Because $C(x)$ only depends of $|x|$, we can write, with $w_d = \int_{S^{d-1}} d\xi$ (S^{d-1} being the unit sphere of $\mathbb{R}^d$):

$$\begin{aligned} \rho(r) &= \frac{1}{w_d} \int_{S^{d-1}} C(r\xi) d\xi = \frac{1}{w_d} \int_{S^{d-1}} d\xi \int_{\mathbb{R}^d} e^{i\langle \lambda, r\xi\rangle} G(d\lambda) \\ &= \int_{\mathbb{R}^d} b_d(r|\lambda|) G(d\lambda) = \int_0^\infty b_d(rs) \mu(ds), \end{aligned} \tag{2.16}$$

where

$$b_d(|\lambda|) = \frac{1}{w_d} \int_{S^{d-1}} e^{i\langle \lambda, \xi\rangle} d\xi. \tag{2.17}$$

By d-dimensional polar coordinates, we readily find (see [Sch38, page 815]) that

$$b_d(r) = c_d\, r^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(r), \quad r > 0, \tag{2.18}$$

with J_ν denoting the Bessel function of the first kind of order ν (see Section 2.2.2) and $c_d > 0$ a constant depending only of d.

We are at last in a position to state our main result, which we have decided to call the **spectral central limit theorem**, because it leads to Gaussian fluctuations on the one hand (hence 'central') and the main assumption we have to check is the spectral condition (2.20) on the other hand (hence 'spectral').

Theorem 2.1.2 (Spectral CLT). *Fix $d \geq 2$, let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary, isotropic and has unit-variance. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be **not odd** and such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$, let R (resp. R') be the Hermite rank (resp. second Hermite rank) of φ, where $(R, R') \notin \{(1,3)\} \cup \{(2k+1, n) : k \geq 1, n \in \mathbb{N}\}$. Consider Y_t defined by (2.3), where $D \subset \mathbb{R}^d$ compact and $\mathrm{Vol}(D) > 0$. Set $m_t = \mathbb{E}[Y_t]$ and $\sigma_t = \sqrt{\mathrm{Var}(Y_t)} > 0$, and recall the definition (2.15) of the isotropic spectral measure μ. Set*

$$w_{q,t} = \int_{\{|z|\leq t\}} C(z)^q dz, \tag{2.19}$$

and assume that the following spectral condition holds

$$\int_0^\infty s^{-\frac{d}{R}} \mu(ds) < \infty. \tag{2.20}$$

Finally, when $R = 2$ assume that $|\mathcal{F}[\mathbf{1}_D](x)| = O\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$, and when $R = 1$ assume that $|\mathcal{F}[\mathbf{1}_D](x)| = o\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$, with $\mathcal{F}$ the Fourier transform (since we assumed D to be compact, these two assumptions are satisfied for example when $D = \overline{\mathring{D}}$ and ∂D is smooth with non-vanishing Gaussian curvature, see e.g. [BHI03]. Then we have:

$$\sigma_t^2 \asymp \begin{cases} t^d w_{R,t} & \text{if } R \text{ even} \\ t^d w_{R',t} & \text{if } R = 1 \text{ and } R' \in \{2,4\} \\ t^d & \text{if } R = 1 \text{ and } R' \geq 5 \end{cases}$$

and

$$\frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1) \quad \text{as } t \to \infty.$$

By a simple Fubini argument, we observe that the spectral condition (2.20) is equivalent to

$$\int_0^\infty s^{-1-\frac{d}{R}} \mu(s) ds < \infty. \tag{2.21}$$

When $\rho(r) = r^{-\beta} L(r)$ with $\beta \in (0, \frac{d}{R})$ and L slowly varying as $r \to \infty$, we have $\mu(s) \sim s^\beta L(s^{-1})$ as $s \to 0$, see [Leo99, Thm 1.4.3]. We deduce that (2.20)-(2.21) is not satisfied, which is of course perfectly consistent with the conclusion of the Dobrushin-Major noncentral limit theorem [DM79].

Also, let us observe that the assumption "φ is not odd" is equivalent to "$\exists\, k \geq 1$ such that $a_{2k} \neq 0$", and so is needed only in the case where $R = 1$ and $R' \geq 5$. Moreover, the cases not covered by our Theorem 2.1.2 are $R \geq 3$ odd, $(R, R') = (1,3)$ and φ odd. In these situations, very peculiar things can happen. Consider for example $\varphi(x) = x$, which gives $Y_t = \int_{tD} B_x dx$. Using (2.29), (2.14) and Fubini we have, with

$\mathcal{F}$ the Fourier transform,

$$\mathrm{Var}(Y_t) = \int_{\mathbb{R}^d} C(z) g_{tD}(z) dz = \int_{\mathbb{R}^d} \mathcal{F}[g_{tD}](\lambda) G(d\lambda).$$

Using that $g_{tD} = \mathbf{1}_{tD} * \mathbf{1}_{-tD}$ we obtain

$$\mathrm{Var}(Y_t) = \text{const} \int_{\mathbb{R}^d} \frac{t^d}{|\lambda|^d} |t\lambda|^d |\mathcal{F}[\mathbf{1}_D](t\lambda)|^2 G(d\lambda). \tag{2.22}$$

In particular, for $D = \{|z| \leq 1\} \subset \mathbb{R}^d$ and Berry's Random Wave Model $B = (B_x)_{x\in\mathbb{R}^2}$ we get (using (2.26))

$$\sigma_t^2 = \mathrm{Var}(\mathrm{Y_t}) = \text{const}\, t^2 J_1^2(t);$$

although Y_t is Gaussian the conclusion of Theorem 2.1.2 cannot hold in this case, the limit of $\frac{Y_t - m_t}{\sigma_t}$ being ill-defined due to the fact that $\mathrm{Card}\{t : \sigma_t = 0\} \cap [T, \infty) = +\infty$ for all $T > 0$.

To understand the significance of our spectral CLT, let us go back to BRWM and explain how Theorem 2.1.2 together with Breuer-Major theorem allow to prove (2.11). Since $C(x) = \rho(|x|) = J_0(|x|)$, it follows immediately from the representation (2.16) that the isotropic spectral measure associated with BRWM $B = (B_x)_{x\in\mathbb{R}^2}$ is $\mu = \delta_1$, with δ_1 the Dirac mass at 1. In particular, the spectral condition (2.20) is obviously satisfied, whatever the value of R. The convergence (2.11) thus follows from the Breuer-Major Theorem 2.1.1 (see also (2.9)) if $R \geq 5$ and from Theorem 2.1.2 in all the other cases in (2.10).

2.1.4 Possible natural extensions of Theorem 2.1.2

As a natural extension of the present work, it would be interesting to study the *joint convergence* associated with Theorem 2.1.2. More precisely, taking $D_1, \ldots, D_n$ compact domains in $\mathbb{R}^d$, is it possible to identify conditions that resemble those in Theorem 2.1.2 ensuring that the random vector

$$\left(\int_{tD_1} \varphi(B_x) dx, \ldots, \int_{tD_n} \varphi(B_x) dx \right)$$

converges, after proper normalization, to a Gaussian random vector? In other words, can we prove a *multivariate* spectral central limit theorem? Such an extension is not immediate, because the spectral condition (2.20) alone looks too general to capture the asymptotic behavior of the covariances between the components of the random vector. A partial answer to this question (covering also Berry's model) will be given in the forthcoming work [Mai24] by the first author. Note that a multivariate result was obtained in the particular case of the nodal length of Berry's model restricted to a finite collection of smooth compact domains $D_1, \ldots, D_n$ of the plane in the recent paper [PV20].

A further stronger generalization of Theorem 2.1.2 would be to prove a *functional* spectral central limit theorem. This problem has been investigated in the particular case of nodal set volumes in [NPV23] by ad hoc techniques. In our general framework, we believe however that proving such a functional extension would require novel ideas that go beyond the scope of the present paper.

2.1.5 Plan of the paper

Apart from Section 2.6 that illustrates a use of Theorem 2.1.2, the rest of the paper is fully devoted to the proof of this latter. More precisely, after a few needed preliminaries given in Section 2.2,

1. in Section 2.3 we will first consider the situation where the Hermite rank R of φ is even and bigger or equal than 4. As we will see, in this case we have that $\int_{tD} |C(x)|^R dx = O(\log t)$, meaning that we are somehow in the 'domain of attraction' of the Breuer-Major theorem 2.1.1.;

2. we will then deal with the remaining cases, namely $R = 2$ in Section 2.4 and ($R = 1$, φ non-odd and $R' \neq 3$) in Section 2.5. We regard this part as the most important contribution of this paper, especially since among the Hermite ranks, $R = 1$ and $R = 2$ are the most common values encountered in practice. They turn out to be also the most difficult cases. To deal with them, we will have to introduce novel ideas, by making heavy use of Fourier techniques in a way that, to the best of our knowledge, has never been introduced before this work.

2.2 A few preliminaries for the proof of Theorem 2.1.2

2.2.1 Well-posedness of Y_t

The following proposition explains why the random variable Y_t defined by (2.3) is well-defined for all t.

Proposition 2.2.1. *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, assumed to be stationary and to have unit-variance. Let $D \subset \mathbb{R}^d$ be compact with $\mathrm{Vol}(D) > 0$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$. Consider the Hermite expansion of φ given by (2.4), and set*

$$\varphi_n = \mathbb{E}[\varphi(N)] + \sum_{q=R}^{n} a_q H_q, \quad n \geq R.$$

For each fixed $t > 0$, the sequence $\int_{tD} \varphi_n(B_x)dx$, $n \geq R$, is a.s. well-defined and converges in $L^2(\Omega)$. The limit is noted Y_t and we write, possibly with a slight abuse of notation:

$$Y_t = \int_{tD} \varphi(B_x)dx.$$

Proof. That $\int_{tD} \varphi_n(B_x)dx$ is a.s. well-defined is because tD is compact, φ_n is a polynomial and B is continuous. For any $n, m \geq R$ and $q \in \mathbb{N}$, we have (since $|C(z)| \leq C(0) = 1$, $a_0 = \mathbb{E}[\varphi(N)]$ and $a_q = 0$ if $q \in \{1, \dots, R-1\}$)

$$\left| \mathbf{1}_{[0,n\wedge m]}(q) \int_{(tD)^2} q! a_q^2 C(x-y)^q dxdy \right| \leq \mathrm{Vol}(tD)^2 q! a_q^2,$$

with $\sum_{q=0}^{\infty} \mathrm{Vol}(tD)^2 q! a_q^2 = \mathrm{Vol}(tD)^2 \mathbb{E}[\varphi(N)^2] < \infty$ by assumption. Then, by dominated convergence we obtain as $n, m \to \infty$

$$\mathbb{E}\left[\int_{tD} \varphi_n(B_x)dx \int_{tD} \varphi_m(B_y)dy\right] = \sum_{q=0}^{n\wedge m} q! a_q^2 \int_{(tD)^2} C(x-y)^q dxdy$$
$$\to \mathbb{E}[\varphi(N)]^2 \mathrm{Vol}(tD)^2 + \sum_{q=R}^{\infty} q! a_q^2 \int_{(tD)^2} C(x-y)^q dxdy.$$

That is, $\{\int_{tD} \varphi_n(B_x)dx\}_{n\geq R}$ is an $L^2(\Omega)$-Cauchy sequence, and the desired conclusion follows. □

2.2.2 Bessel functions and Fourier transform of the indicator of the unit ball

The Bessel function J_ν, $\nu \geq 0$, is defined by the series

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{j=0}^{\infty} (-1)^j \frac{(t^2/4)^j}{j!\Gamma(j+\nu+1)}, \quad t \in \mathbb{R},$$

and it is a solution to the ODE

$$t^2 J_\nu''(t) + tJ_\nu'(t) + (t^2 - \nu^2)J_\nu(t) = 0.$$

It satisfies the classical Schläfli's representation

$$J_\nu(t) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - t\sin\theta)d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\theta - t\sinh\theta}d\theta, \tag{2.23}$$

as well as the Mehler-Sonine formula

$$J_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 e^{its}(1-s^2)^{\nu-\frac{1}{2}}ds. \tag{2.24}$$

Schläfli's representation (2.23) immediately implies that J_ν is bounded on $\mathbb{R}$. This, combined with an inequality found e.g. in [Kra14] (see also the references therein), leads to

$$\sup_{t\in\mathbb{R}_+} \sqrt{t}|J_\nu(t)| < \infty \quad \text{for all } \nu \geq 0, \tag{2.25}$$

a simple property that we will use several times in the forthcoming proof of Theorem 2.1.2.

The Fourier transform of the indicator of the unit ball is given by

$$\mathcal{F}\left[\mathbf{1}_{\{|x|\leq 1\}}\right](y) = \int_{\{|x|\leq 1\}} e^{i\langle x,y\rangle}dx.$$

Since $\mathcal{F}\left[\mathbf{1}_{\{|x|\leq 1\}}\right]$ is rotationally symmetric, we can write, with α_d the volume of the unit ball $\{|x| \leq 1\} \subseteq \mathbb{R}^d$,

$$\begin{aligned} \mathcal{F}\left[\mathbf{1}_{\{|x|\leq 1\}}\right](y) &= \mathcal{F}\left[\mathbf{1}_{\{|x|\leq 1\}}\right]((0,\dots,0,|y|)) \\ &= \int_{-1}^{1} (1-x_d^2)^{\frac{d-1}{2}} \alpha_{d-1} e^{ix_d|y|} dx_d \\ &= \alpha_{d-1}\sqrt{\pi}\,\Gamma\left(\frac{d+1}{2}\right) 2^{\nu}\,|y|^{-\frac{d}{2}} J_{d/2}(|y|), \end{aligned} \tag{2.26}$$

the last equality being a consequence of (2.24).

2.2.3 Reduction to the Rth chaos

In the short memory case, that is when $C \in L^R(\mathbb{R}^d)$, the Breuer-Major theorem 2.1.1 yields Gaussian fluctuations for Y_t. In its modern proof given by [NPP11] (see also [NP12, Chapter 7]) the chaotic expansion of $Y_t - m_t$ is considered, namely

$$Y_t - m_t = \sum_{q=R}^{\infty} a_q Y_{q,t}, \tag{2.27}$$

and the proof goes as follows. It is first shown that $t^{-d}\sigma_t^2 \to \sigma^2$ in (2.7) by means of the isometry property of Hermite polynomials. Then, it is proved using the Fourth Moment Theorem (see [NP12, Theorem 5.2.7]) that $t^{-d/2}Y_{q,t} \overset{\text{law}}{\to} N(0,\sigma_q^2)$ for *all* $q \geq R$ from which it is deduced that $t^{-\frac{d}{2}}(Y_t - m_t) \overset{\text{law}}{\to} N(0, \sum_{q=R}^{\infty} a_q^2\sigma_q^2)$ thanks to [NP12, Theorem 5.2.7]. In particular, we observe that no term is asymptotically dominant in (2.27): they all contribute to the limit.

As the following result will show, the situation is totally opposite in the critical and long memory cases (when R is even): here, it is the term $Y_{R,t}$ alone which is responsible of the limit.

Proposition 2.2.2 (Reduction to the Rth chaos)**.** *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary and has unit-variance (note that $d \geq 2$ and isotropy are not required here). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\mathbb{E}[\varphi(N)^2] < \infty$ (with $N \sim N(0,1)$) and have Hermite decomposition (2.4) and Hermite rank R, and consider Y_t defined by (2.3), where $D \subset \mathbb{R}^d$ compact and* $\mathrm{Vol}(D) > 0$. *Set $m_t = \mathbb{E}[Y_t]$, $\sigma_t = \sqrt{\mathrm{Var}(Y_t)} > 0$, and recall the definition (2.1) of $C(x)$ and the definition (2.12) of $Y_{q,t}$.*

*If R is **even**, if $t^{-d}\mathrm{Var}(Y_{R,t}) \to \infty$ and if $\int_{\mathbb{R}^d} |C(x)|^M dx < \infty$ for some $M \geq R+1$ then, as $t \to \infty$, we have $\sigma_t^2 \asymp \mathrm{Var}(Y_{R,t})$ and*

$$Y_{R,t}/\sqrt{\mathrm{Var}(Y_{R,t})} \overset{\text{law}}{\to} N(0,1) \quad \Longrightarrow \quad \frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1). \tag{2.28}$$

Proof. We have

$$\mathrm{Var}(Y_{q,t}) = \int_{(tD)^2} \mathbb{E}[H_q(B_y)H_q(B_x)]dxdy = q! \int_{(tD)^2} C(x-y)^q dxdy.$$

Applying the change of variable $z = y - x$ and then Fubini, we obtain that

$$\mathrm{Var}(Y_{q,t}) = q!t^d v_{q,t},$$

where

$$v_{q,t} = \int_{\{|z| \le \mathrm{diam}(D)t\}} C(z)^q \, g_D\left(\frac{z}{t}\right) dz, \tag{2.29}$$

where $\mathrm{diam}(D) = \sup\{|x-y| : x, y \in D\} < \infty$ (because D is compact) and $g_D(x)$ is the covariogram of D at $x \in \mathbb{R}^d$, that is, $g_D(x)$ is the Lebesgue measure of $D \cap (x+D)$. Also, set

$$\widetilde{v}_{q,t} = \int_{\{|z| \le \mathrm{diam}(D)t\}} |C(z)|^q \, g_D\left(\frac{z}{t}\right) dz, \tag{2.30}$$

and observe that $v_{R,t} = \widetilde{v}_{R,t} > 0$ since R is even.

For any $q > R$, we have

$$\frac{\mathrm{Var}(Y_{q,t})}{\mathrm{Var}(Y_{R,t})} = \frac{q! v_{q,t}}{R! v_{R,t}}. \tag{2.31}$$

Applying Cauchy-Schwarz inequality we obtain, for $q > R$,

$$\begin{aligned} \frac{|v_{q,t}|}{v_{R,t}} &\le \frac{\int_{\{|z| \le \mathrm{diam}(D)t\}} |C(z)|^q g_D(\frac{z}{t}) dz}{v_{R,t}} \\ &= \frac{\int_{\{|z| \le \mathrm{diam}(D)t\}} |C(z)|^{q-\frac{R}{2}} |C(z)|^{\frac{R}{2}} g_D\left(\frac{z}{t}\right) dz}{v_{R,t}} \\ &\le \left(\frac{\int_{\{|z| \le \mathrm{diam}(D)t\}} |C(z)|^{2q-R} g_D\left(\frac{z}{t}\right) dz}{v_{R,t}} \right)^{1/2} = \left(\frac{\widetilde{v}_{2q-R,t}}{v_{R,t}} \right)^{1/2}. \end{aligned}$$

Applying Cauchy-Schwarz again, but this time with $2q - R > R$ instead of $q > R$, we obtain

$$\left(\frac{\widetilde{v}_{2q-R,t}}{v_{R,t}} \right)^{1/2} \le \left(\frac{\widetilde{v}_{4q-3R,t}}{v_{R,t}} \right)^{\frac{1}{4}}.$$

By iterating the process, we get, for every $n \ge 3$

$$\frac{|v_{q,t}|}{v_{R,t}} \le \left(\frac{\widetilde{v}_{2q-R,t}}{v_{R,t}} \right)^{1/2} \le \left(\frac{\widetilde{v}_{4q-3R,t}}{v_{R,t}} \right)^{\frac{1}{4}} \le \cdots \le \left(\frac{\widetilde{v}_{R+2^n(q-R),t}}{v_{R,t}} \right)^{\frac{1}{2^n}}.$$

When $q > R$, we have $R + 2^n(q-R) \ge 2^n$, so we may and will choose n large enough so that $R + 2^n(q-R) \ge M$ for all $q > R$ (recall from the statement of Proposition 2.2.2 that M is an integer supposed to be such that $\int_{\mathbb{R}^d} |C(x)|^M dx < \infty$). Using $|C(x)| \le C(0) = 1$ and $g_D(x) \le \mathrm{Vol}(D)$ for all $x \in \mathbb{R}^d$, we deduce that $\widetilde{v}_{R+2^n(q-R),t} \le \mathrm{Vol}(D) \int_{\mathbb{R}^d} |C(x)|^M dx$. Combining all these facts together, we finally get that

$$\frac{|v_{q,t}|}{v_{R,t}} \le \left(\mathrm{Vol}(D) \int_{\mathbb{R}^d} |C(x)|^M dx \right)^{\frac{1}{2^n}} v_{R,t}^{-\frac{1}{2^n}} \quad \text{for all } q > R. \tag{2.32}$$

Since $\mathrm{Var}(Y_t) = \sum_{q=R}^{\infty} a_q^2 \,\mathrm{Var}(Y_{q,t})$, we deduce

$$\left| \frac{\mathrm{Var}(Y_t)}{\mathrm{Var}(Y_{R,t})} - a_R^2 \right| \le \frac{1}{R!} \left(\mathrm{Vol}(D) \int_{\mathbb{R}^d} |C(x)|^M dx \right)^{\frac{1}{2n}} \left(\sum_{q=R+1}^{\infty} a_q^2 q! \right) v_{R,t}^{-\frac{1}{2n}}, \quad (2.33)$$

from which it comes that $\mathrm{Var}(Y_t) \sim a_R^2 \mathrm{Var}(Y_{R,t})$ as $t \to \infty$, since $v_{R,t} = \frac{1}{R!} t^{-d} \mathrm{Var}(Y_{R,t}) \to \infty$ by assumption.

To conclude the proof of Proposition 2.2.2, it remains to prove (2.28). Using the decomposition

$$\begin{aligned} &\frac{Y_t - m_t}{\sqrt{\mathrm{Var}(Y_t)}} - \mathrm{sgn}(a_R) \frac{Y_{R,t}}{\sqrt{\mathrm{Var}(Y_{R,t})}} \\ = \; &\frac{\mathrm{sgn}(a_R)(Y_t - m_t - a_R Y_{R,t})}{a_R \sqrt{\mathrm{Var}(Y_{R,t})}} + \frac{Y_t - m_t}{\sqrt{\mathrm{Var}(Y_t)}} \left\{ 1 - \frac{1}{|a_R|} \sqrt{\frac{\mathrm{Var}(Y_t)}{\mathrm{Var}(Y_{R,t})}} \right\} \end{aligned}$$

we get that

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{Y_t - m_t}{\sqrt{\mathrm{Var}(Y_t)}} - \mathrm{sgn}(a_R) \frac{Y_{R,t}}{\sqrt{\mathrm{Var}(Y_{R,t})}} \right)^2 \right] \\ \le \; & 2 \frac{\mathbb{E}[(Y_t - m_t - a_R Y_{R,t})^2]}{a_R^2 \mathrm{Var}(Y_{R,t})} + 2 \left(1 - \frac{1}{|a_R|} \sqrt{\frac{\mathrm{Var}(Y_t)}{\mathrm{Var}(Y_{R,t})}} \right)^2 . \end{aligned} \quad (2.34)$$

Since $\mathbb{E}\left[(Y_t - m_t - a_R Y_{R,t})^2 \right] = \sum_{q=R+1}^{\infty} a_q^2 \,\mathrm{Var}(Y_{q,t})$, we deduce from (2.31) and (2.32) that

$$\begin{aligned} &\frac{\mathbb{E}[(Y_t - m_t - a_R Y_{R,t})^2]}{a_R^2 \mathrm{Var}(Y_{R,t})} \\ \le \; & \frac{1}{R! a_R^2} \left(\mathrm{Vol}(D) \int_{\mathbb{R}^d} |C(x)|^M dx \right)^{\frac{1}{2n}} \left(\sum_{q=R+1}^{\infty} q! a_q^2 \right) v_{R,t}^{-\frac{1}{2n}}, \end{aligned}$$

and this tends to zero as $t \to \infty$. By plugging this into (2.34) and taking into account that (2.33) holds, we deduce that

$$\frac{Y_t}{\sqrt{\mathrm{Var}(Y_t)}} - \mathrm{sgn}(a_R) \frac{Y_{R,t}}{\sqrt{\mathrm{Var}(Y_{R,t})}} \xrightarrow{L^2(\Omega)} 0 \quad \text{as } t \to \infty,$$

from which the implication (2.28) now follows easily. □

2.2.4 Elements of Malliavin calculus and the Fourth Moment theorem

To obtain the $N(0,1)$ distribution in the limit in Theorem 2.1.2, we rely on the Fourth Moment Theorem of Nualart and Peccati (see [NP12, Theorem 5.2.7]). Before stating and reformulating it in our framework, we start with some notions on Malliavin calculus. For all the missing details we refer to [NP12] and [Nua06].

The Wiener-Itô integral

Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary with unit-variance. Define

$$\mathcal{H} = \overline{\text{span}\{B_x,\, x \in \mathbb{R}^d\}}^{L^2(\Omega)}.$$

Since $\mathcal{H}$ is a real, separable Hilbert space, there is an isometry $\Phi : \mathcal{H} \to L^2(\mathbb{R}_+)$. If we set $e_x := \Phi(B_x)$ for every $x \in \mathbb{R}^d$, we have

$$\mathbb{E}[B_x B_y] = C(x-y) = \langle e_x, e_y\rangle_{L^2(\mathbb{R}_+)}.$$

Consider now the Gaussian noise $W = \{W(h), h \in L^2(\mathbb{R}_+)\}$, i.e. a family of centered Gaussian random variables with covariance given by

$$\mathbb{E}[W(h)W(g)] = \langle h, g\rangle_{L^2(\mathbb{R}_+)}.$$

Since in this paper we are only interested in distributions, we can assume without loss of generality that $(B_x)_{x\in\mathbb{R}^d} = (W(e_x))_{x\in\mathbb{R}^d}$.

For $q \geq 1$, we also define the qth Wiener chaos as the linear subspace of $L^2(\Omega)$ generated by $\{H_q(W(h)), h \in L^2(\mathbb{R}_+)\}$. For every $h \in L^2(\mathbb{R}_+)$ with $\|h\|_{L^2(\mathbb{R}_+)} = 1$, we define the qth **Wiener-Itô integral**

$$I_q(h^{\otimes q}) = H_q(W(h)),$$

where $h^{\otimes q} : \mathbb{R}_+^q \to \mathbb{R}$ is defined by

$$h^{\otimes q}(x_1, \dots, x_q) = \prod_{r=1}^{q} h(x_r).$$

Note that the definition of I_q can be extended to every function in the space $L_s^2(\mathbb{R}_+^q)$ of symmetric functions in $L^2(\mathbb{R}_+^q)$ so that $I_q : L_s^2(\mathbb{R}_+^q) \to L^2(\Omega)$ is a linear map, because $\text{span}\{h^{\otimes q}, h \in L^2(\mathbb{R}_+)\}$ is dense in $L_s^2(\mathbb{R}_+^q)$ (see, e.g., [Flo97]).

Contractions and the Fourth Moment Theorem

For $q \in \mathbb{N}$, $r \in \{1, \dots, q-1\}$ and h_1, h_2 symmetric functions with unit norm in $L^2(\mathbb{R}_+)$, we can define the rth **contraction** of $h_1^{\otimes q}$ and $h_2^{\otimes q}$ as the (non-symmetric) element of $L^2(\mathbb{R}_+^{2q-2r})$ given by

$$h_1^{\otimes q} \otimes_r h_2^{\otimes q} = \langle h_1, h_2\rangle_{L^2(\mathbb{R}_+)}^r\, h_1^{\otimes q-r} \otimes h_2^{\otimes q-r}.$$

Here again this definition can be extended (taking the closure in $L^2(\mathbb{R}_+^q)$) to every h_1, h_2 in $L_s^2(\mathbb{R}_+^q)$. We will denote the norm in the space $L^2(\mathbb{R}_+^q)$ as $\|\cdot\|_q$. We can finally state the celebrated **Fourth Moment Theorem**, proved in [NP05] by Nualart and Peccati.

Theorem 2.2.3 (Fourth Moment Theorem). *Fix $q \geq 2$, consider $(h_t)_{t>0} \subset L_s^2(\mathbb{R}_+^q)$ and assume that $\mathbb{E}[I_q(h_t)^2] \to 1$ as $t \to \infty$. Then the following assertions are equivalent:*

- $I_q(h_t)$ *converges in distribution to a standard Gaussian* $N \sim N(0,1)$.
- $\mathbb{E}[I_q(h_t)^4] \to 3 = \mathbb{E}[N^4]$*, where* $N \sim N(0,1)$.
- $\|h_t \otimes_r h_t\|_{2q-2r} \to 0$ *as* $t \to \infty$*, for all* $r = 1, \dots, q-1$.

An important consequence of the previous result (in our framework) is the following.

Theorem 2.2.4. *Let* $B = (B_x)_{x\in\mathbb{R}^d}$ *be a real-valued continuous centered Gaussian field on* $\mathbb{R}^d$*, and assume that* B *is stationary and has unit-variance. Assume that* $D \subset \mathbb{R}^d$ *is compact with* $\mathrm{Vol}(D) > 0$*. Recall the definition (2.1) of* C*. Fix also* $q \geq 2$*, recall the definition (2.12) of* $Y_{q,t}$ *and assume* $\mathrm{Var}(Y_{q,t}) > 0$ *for all* t *large enough. If we have, for any* $r \in \{1, \dots, q-1\}$*, that*

$$\frac{t^d}{\mathrm{Var}^2(Y_{q,t})} \int_{\{|u|\leq diam(D)t\}^3} |C(x)|^r |C(y)|^r |C(z)|^{q-r} |C(x+y+z)|^{q-r} dxdydz \quad (2.35)$$

converges to 0 *as* $t \to \infty$*, then* $Y_{q,t}/\sqrt{\mathrm{Var}(Y_{q,t})} \overset{law}{\to} N(0,1)$ *as* $t \to \infty$*.*

Proof. Let us first write $Y_{q,t}$ as a q-th multiple Wiener-Itô integral with respect to B:

$$Y_{q,t} = \int_{tD} H_q(B_x)dx = \int_{tD} H_q(W(e_x))dx = \int_{tD} I_q(e_x^{\otimes q})dx = I_q(f_{t,q}),$$

where

$$f_{t,q} = \int_{tD} e_x^{\otimes q} dx.$$

Since by definition

$$f_{t,q} \otimes_r f_{t,q} = \int_{tD}\int_{tD} C^r(x-y) e_x^{\otimes q-r} e_y^{\otimes q-r} dxdy,$$

we obtain

$$\begin{aligned} &\|f_{t,q} \otimes_r f_{t,q}\|^2_{2q-2r} \\ = &\int_{(tD)^4} C(x_1-x_3)^r C(x_2-x_4)^r C(x_1-x_2)^{q-r} \\ &\qquad\qquad \times C(x_3-x_4)^{q-r} dx_1dx_2dx_3dx_4. \end{aligned}$$

Applying the change of variable $x = x_3 - x_1$, $y = x_2 - x_4$, $z = x_1 - x_2$, $a = x_4$ (whose Jacobian is equal to 1) and using the symmetry of C, we get that

$$\begin{aligned} &\|f_{t,q} \otimes_r f_{t,q}\|^2_{2q-2r} \leq \mathrm{Vol}(D) \\ &t^d \int_{\{|u|\leq \mathrm{diam}(D)t\}^3} |C(x)|^r |C(y)|^r |C(z)|^{q-r} |C(x+y+z)|^{q-r} dxdydz. \end{aligned}$$

The Fourth Moment Theorem asserts that $Y_{q,t}/\sqrt{\mathrm{Var}(Y_{q,t})}$ converges in distribution to $N(0,1)$ if (and only if) $\|f_{t,q} \otimes_r f_{t,q}\|_{2q-2r} / \mathrm{Var}(Y_{q,t}) \to 0$ for all $r = 1, \dots, q-1$. The desired conclusion thus follows from (2.35) and the previous bound for $\|f_{t,q} \otimes_r f_{t,q}\|^2_{2q-2r}$. □

2.3 Proof of Theorem 2.1.2 when $R \geq 4$ even

This section is devoted to the proof of Theorem 2.1.2 when $R \geq 4$ is even, which is equivalent to say that $\frac{d}{R} \leq \frac{d-1}{2}$ and R even. It represents the 'easy' part of Theorem 2.1.2.

To ease the exposition, we write in the following proposition the statement obtained when, in Theorem 2.1.2, we suppose that $\frac{d}{R} \leq \frac{d-1}{2}$ and R even.

Proposition 2.3.1. *Fix $d \geq 2$, let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary, isotropic and has unit-variance. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$, let R be the Hermite rank of φ and consider Y_t defined by (2.3), where D compact and $\mathrm{Vol}(D) > 0$. Set $m_t = \mathbb{E}[Y_t]$ and $\sigma_t = \sqrt{\mathrm{Var}(Y_t)} > 0$, and recall the definition (2.15) of the isotropic spectral measure μ and the definition (2.19) of $w_{R,t}$:*

$$w_{R,t} = \int_{\{|z|\leq t\}} C^R(z)dz.$$

If (2.20) holds, if R even and $\frac{d}{R} \leq \frac{d-1}{2}$ (i.e. $R \geq 4$ and R even), then $\sigma_t^2 \asymp t^d w_{R,t}$ and

$$\frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1) \quad \text{as } t \to \infty.$$

The goal of this Section 2.3 is to prove Proposition 2.3.1. As we will see, it will be a direct consequence of Lemma 2.3.2 and Proposition 2.3.3 below.

We start with Lemma 2.3.2.

Lemma 2.3.2. *Let ρ and μ be associated as in (2.16) and consider an exponent $\beta \in (0, \frac{d-1}{2}]$. If $\int_0^\infty s^{-\beta}\mu(ds) < \infty$, then $\sup_{r\in\mathbb{R}_+} r^\beta|\rho(r)| < \infty$.*

Proof. Since $\rho(|x|) = \mathbb{E}[B_0 B_x]$ for all $x \in \mathbb{R}^d$, we deduce from Cauchy-Schwarz that $|\rho|$ is bounded by 1, and thus $\sup_{r\in[0,T]} r^\beta|\rho(r)| < \infty$ for all fixed $T > 0$.

On the other hand, using the representation (2.16), we can write

$$r^\beta \rho(r) = \int_0^\infty (rs)^\beta b_d(rs)\frac{\mu(ds)}{s^\beta}.$$

But $J_{\frac{d}{2}-1}$ is bounded and satisfies $J_{\frac{d}{2}-1}(u) = O(u^{-1/2})$ as $u \to \infty$ (see (2.25)). So $\sup_{u\in\mathbb{R}_+} u^\beta|b_d(u)| < \infty$, and the desired conclusion follows. □

Now, let us state and prove Proposition 2.3.3, which may be of independent interest.

Proposition 2.3.3. *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary and has unit-variance (note that we did not assume isotropy and $d \geq 2$). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$, let R be the Hermite rank of φ and consider Y_t defined by (2.3), where D compact and $\mathrm{Vol}(D) > 0$. Set $m_t = \mathbb{E}[Y_t]$ and $\sigma_t = \sqrt{\mathrm{Var}(Y_t)} > 0$, and recall the definition (2.1) of the covariance C and the definition (2.19) of $w_{R,t}$.*

If R is even, if $\lim_{t\to\infty} w_{R,t} = \infty$ and if

$$\sup_{x\in\mathbb{R}^d} |x|^{d/R}|C(x)| < \infty, \tag{2.36}$$

then $\sigma_t^2 \asymp t^d w_{R,t}$ and

$$\frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1) \quad \textit{as } t \to \infty. \tag{2.37}$$

Proof. The proof is divided into several steps. Recall the definition (2.12) of $Y_{q,t}$.

Step 1. We claim that if R even, then

$$v_{R,t} \asymp w_{R,t}, \tag{2.38}$$

where $v_{R,t}$ is defined by (2.29). To prove it, below we let $c > 0$ denote a constant independent of t whose value can change from one instance to another. Using (2.29), we have on the one hand

$$\begin{aligned} v_{R,t} &= \frac{1}{R!}t^{-d}\mathrm{Var}(Y_{R,t}) = \int_{\{|z|\leq \mathrm{diam}(D)t\}} C(z)^R g_D\left(\frac{z}{t}\right) dz \\ &\leq \mathrm{Vol}(D) w_{R,\mathrm{diam}(D)t} \leq c\, w_{R,t} \end{aligned}$$

where the last equality follows from the so-called doubling conditions at the origin for non-negative positive definite functions (see [GT19]).

On the other hand, g_D is uniformly continuous (in particular continuous in 0) according to [Gal11]. We deduce that $g_D\left(\frac{z}{t}\right) \geq \mathrm{Vol}(D) - \frac{1}{2}\mathrm{Vol}(D) = \frac{1}{2}\mathrm{Vol}(D)$ for all $z \in \{|z| \leq \delta_D t\}$ for some $\delta_D > 0$ depending only on D and for every $t > 0$. As a result,

$$\begin{aligned} v_{R,t} = \frac{1}{R!}t^{-d}\mathrm{Var}(Y_{R,t}) &\geq \int_{\{|z|\leq \delta_D t\}} C(z)^R g_D\left(\frac{z}{t}\right) dz \\ &\geq \frac{1}{2}\mathrm{Vol}(D) w_{R,\delta_D t} \geq c w_{R,t}, \end{aligned}$$

where the last inequality follows again from the doubling conditions proved in [GT19]. The announced claim (2.38) follows.

Step 2. Since $w_{R,t} \to \infty$, it follows from Step 1 that $t^{-d}\mathrm{Var}(Y_{R,t}) \to \infty$. Moreover, since (2.36) holds, we have that $\int_{\mathbb{R}^d} |C(x)|^{R+1} dx < \infty$. Applying Proposition 2.2.2, we obtain that $\sigma_t^2 \asymp t^d w_{R,t}$ and also that (2.37) will follow if we prove that $Y_{R,t}/\sqrt{\mathrm{Var}(Y_{R,t})} \to N(0,1)$.

Step 3. In this last step, we prove that $Y_{R,t}/\sqrt{\mathrm{Var}(Y_{R,t})} \to N(0,1)$, which will complete the proof of Proposition 2.3.3, see the conclusion of Step 2. To do this, we use Theorem 2.2.4, following the same approach as in [NPR19, Lemma 8.1]. Fix a contraction index $r \in \{1, \ldots, R-1\}$. Using the inequality $u^r v^{R-r} \leq u^R + v^R$ for

$u, v \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\int_{\{|u|\le \mathrm{diam}(D)t\}^3} |C(x)|^r|C(y)|^r|C(z)|^{R-r}|C(x+y+z)|^{R-r}dxdydz \\
\le\ & 2\int_{\{|u|\le \mathrm{diam}(D)t\}^3} |C(x)|^r|C(y)|^R|C(x+y+z)|^{R-r}dxdydz \\
\le\ & c\widetilde{w}_{r,t}w_{R,t}\widetilde{w}_{R-r,t}
\end{aligned}$$

where the last inequality follows from the change of variable $a = x + y + z$ and doubling conditions in [GT19], and

$$\widetilde{w}_{q,t} = \int_{\{|z|\le t\}} |C(z)|^q dz.$$

We deduce that (2.35) is bounded by

$$\frac{c}{t^d}\frac{\widetilde{w}_{r,t}w_{R,t}\widetilde{w}_{R-r,t}}{v_{R,t}^2} = O\left(t^{-d}\frac{\widetilde{w}_{r,t}\widetilde{w}_{R-r,t}}{w_{R,t}}\right),$$

where the big O comes from (2.38). We deduce from (2.36) that $\widetilde{w}_{q,t} = O\left(t^{d-d\frac{q}{R}}\right)$ for $q < R$. Using this last fact, we get

$$O\left(t^{-d}\frac{\widetilde{w}_{r,t}\widetilde{w}_{R-r,t}}{w_{R,t}}\right) = O\left(t^{-d}\frac{t^{d-d\frac{r}{R}}t^{d-d\frac{R-r}{R}}}{w_{R,t}}\right) = O\left(\frac{1}{w_{R,t}}\right)$$

and then (2.35) converges to 0 as $t \to \infty$. Therefore, the convergence $Y_{R,t}/\sqrt{\mathrm{Var}(Y_{R,t})} \to N(0,1)$ follows from Theorem 2.2.4, and completes the proof of Proposition 2.3.3. □

We can now proceed with the proof of Proposition 2.3.1, that is, the proof of Theorem 2.1.2 when $R \ge 4$ and R even (or equivalently, $\frac{d}{R} \le \frac{d-1}{2}$ and R even).

Proof of Proposition 2.3.1. If $w_{R,t}$ is convergent, then the result follows applying the Breuer-Major theorem 2.1.1. So we can assume that $w_{R,t} \to \infty$. Now, by comparing the statements of Proposition 2.3.1 and Proposition 2.3.3, we see that we are left to check that, if (2.20) holds with $\frac{d}{R} \le \frac{d-1}{2}$, then (2.36) holds. Since this is a mere application of Lemma 2.3.2 with $\beta = \frac{d}{R}$, the proof of Proposition 2.3.1 is complete. □

2.4 Proof of Theorem 2.1.2 when $R = 2$

This section is devoted to the proof of Theorem 2.1.2 when $R = 2$. It requires the introduction of novel ideas with respect to the existing literature, mainly Fourier arguments.

To ease the exposition, we write in the following proposition the statement obtained when, in Theorem 2.1.2, we additionally suppose that $R = 2$ and $|\mathcal{F}[\mathbf{1}_D](x)| = O\left(\frac{1}{|x|^{d/2}}\right)$.

Proposition 2.4.1. *Fix $d \ge 2$, let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary, isotropic and has unit-variance.*

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$, and assume that φ has Hermite rank $R = 2$. Consider Y_t defined by (2.3), where D compact and $\mathrm{Vol}(D) > 0$. Set $m_t = \mathbb{E}[Y_t]$ and $\sigma_t = \sqrt{\mathrm{Var}(Y_t)} > 0$, and recall the definition (2.15) of the isotropic spectral measure μ and the definition (2.19) of $w_{q,t}$.

If $|\mathcal{F}[\mathbf{1}_D](x)| = O\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$[4], with $\mathcal{F}$ the Fourier transform, and if the spectral condition holds

$$\int_0^\infty s^{-\frac{d}{2}} \mu(ds) < \infty \tag{2.39}$$

then $\sigma_t^2 \asymp t^d w_{2,t}$ and

$$\frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1) \quad \text{as } t \to \infty.$$

Before proving Proposition 2.4.1 (which is the goal of this section), let us recall the definition (2.12) of $Y_{2,t}$ and let us state and prove some preliminary results. We start with Lemma 2.4.2, reformulating in a spectral form the norm of the contractions introduced in section 2.2.4. Note that the following result has an analogous version for a general rth contraction in the case $Y_{q,t}$, but we skip this unnecessary extension for the sake of brevity.

Lemma 2.4.2. *Let $B = (B_x)_{x \in \mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary and has unit-variance (note that we did not assume isotropy and $d \geq 2$). Assume that $D \subset \mathbb{R}^d$ is compact with $\mathrm{Vol}(D) > 0$, recall the notions introduced in section 2.2.4 and write*

$$Y_{2,t} = \int_{tD} H_2(B_x) dx = I_2(f_t),$$

where

$$f_t = \int_{tD} e_x^{\otimes 2} dx.$$

Recall the definition (2.1) of C and define

$$C_t(u) := C(u) \mathbf{1}_{\{|u| \leq \mathrm{diam}(D)t\}}(u), \quad t > 0, \quad u \in \mathbb{R}^d, \tag{2.40}$$

and for $D \subset \mathbb{R}^d$ compact

$$D_t(u) := tD \cap (tD + u), \quad u \in \mathbb{R}^d. \tag{2.41}$$

Finally, recall the definition (2.14) of G. Then

$$\|f_t \otimes_1 f_t\|_2^2 = \int_{\mathbb{R}^d} G(dx) \int_{\mathbb{R}^d} dy \mathcal{F}[C_t](x-y) \left| \int_{\mathbb{R}^d} C_t(w) e^{i\langle x, w \rangle}, \mathcal{F}\left[\mathbf{1}_{D_t(-w)}\right](y) dw \right|^2 \tag{2.42}$$

where $\mathcal{F}$ is the Fourier transform, and

$$\int_{\mathbb{R}^d} C_t(w) e^{i\langle x, w \rangle} \mathcal{F}\left[\mathbf{1}_{D_t(-w)}\right](y) dw = \int_{\mathbb{R}^d} \mathcal{F}[C_t](x-z) \mathcal{F}[\mathbf{1}_{tD}](y-z) \mathcal{F}[\mathbf{1}_{tD}](z) dz. \tag{2.43}$$

[4]Since we assumed D to be compact, this happens for example when $D = \overline{\mathring{D}}$ and ∂D is smooth with non-vanishing Gaussian curvature, see e.g. [BHI03].

Proof. Proceeding exactly as in the proof of Theorem 2.2.4, we have

$$\|f_t \otimes_1 f_t\|_2^2 = \int_{(tD)^4} C(x_1 - x_3)C(x_2 - x_4)C(x_1 - x_2) \times C(x_3 - x_4)dx_1dx_2dx_3dx_4.$$

Applying the change of variable $u = x_1 - x_3$, $v = x_3 - x_4$, $w = x_4 - x_2$, $z = x_2$ we have: $x_2 = z$, $x_4 = w + x_2 = w + z$, $x_3 = v + x_4 = v + w + z$ and $x_1 = u + x_3 = u + v + w + z$. Then

$$\begin{aligned}
&\|f_t \otimes_1 f_t\|_2^2 \\
= &\int_{\mathbb{R}^{3d}} C(u)C(v)C(w)C(u+v+w) \\
&\times \left(\int_{\mathbb{R}^d} \mathbf{1}_{tD}(z)\mathbf{1}_{tD}(w+z)\mathbf{1}_{tD}(v+w+z)\mathbf{1}_{tD}(u+v+w+z)dz\right) dudvdw \\
= &\int_{\mathbb{R}^{3d}} C(u)C(v)C(w)C(u+v+w) \\
&\times \left(\int_{\mathbb{R}^d} \mathbf{1}_{tD}(z)\mathbf{1}_{tD}(-w+z)\mathbf{1}_{tD}(-v-w+z)\mathbf{1}_{tD}(-u-v-w+z)dz\right) dudvdw \\
= &\int_{\mathbb{R}^{3d}} C(u)C(v)C(w)C(u+v+w) \\
&\times \left(\int_{\mathbb{R}^d} \mathbf{1}_{tD\cap(tD+w)\cap(tD+w+v)\cap(tD+u+v+w)}(z)dz\right) dudvdw \\
= &\int_{\mathbb{R}^{3d}} C_t(u)C_t(v)C_t(w)C(u+v+w) \\
&\times \mathrm{Vol}((tD-w)\cap tD \cap (tD+v)\cap(tD+u+v))dudvdw,
\end{aligned}$$

where in the last equality we used the translation invariance of the Lebesgue measure (subtracting w) and the definition (2.40), justified by the fact that $tD \cap (tD + a)$ is empty when $|a| > \mathrm{diam}(D)t$. Now recall the definition (2.41). We have

$$\begin{aligned}
&\|f_t \otimes_1 f_t\|_2^2 \\
= &\int_{\mathbb{R}^{3d}} C_t(u)C_t(v)C_t(w)C(u+v+w)\mathrm{Vol}\left(D_t(-w)\cap(D_t(u)+v)\right) dudvdw \\
= &\int_{\mathbb{R}^{3d}} C_t(u)C_t(v)C_t(w)C(u+v+w)\left(\mathbf{1}_{D_t(-w)} * \mathbf{1}_{-D_t(u)}\right)(v)dudvdw,
\end{aligned}$$

where in the last expression we used that

$$\begin{aligned}
&\mathrm{Vol}\left(D_t(-w)\cap(D_t(u)+v)\right) \\
= &\int_{\mathbb{R}^d} \mathbf{1}_{D_t(-w)}(z)\mathbf{1}_{D_t(u)}(z-v)dz = \left(\mathbf{1}_{D_t(-w)} * \mathbf{1}_{-D_t(u)}\right)(v).
\end{aligned}$$

Now, using the spectral representation (2.14)

$$C(u+v+w) = \int_{\mathbb{R}^d} e^{i\langle x, u+v+w\rangle} G(dx),$$

we have that

$$\begin{aligned}
&\|f_t \otimes_1 f_t\|_2^2 \\
&= \int_{\mathbb{R}^d} G(dx) \int_{\mathbb{R}^{2d}} dudw C_t(u) C_t(w) e^{i\langle x, u+w\rangle} \int_{\mathbb{R}^d} e^{i\langle x, v\rangle} C_t(v) \\
&\times \left(\mathbf{1}_{D_t(-w)} * \mathbf{1}_{-D_t(u)}\right)(v) dv.
\end{aligned} \tag{2.44}$$

Fix x, u and w, and let us focus in (2.44) on the integral with respect to v. Using the properties of the Fourier transform $\mathcal{F}$ with respect to convolution and products, we can write

$$\begin{aligned}
& \int_{\mathbb{R}^d} e^{i\langle x,v\rangle} C_t(v) (\mathbf{1}_{D_t(-w)} * \mathbf{1}_{-D_t(u)})(v) dv = \mathcal{F}[C_t \cdot (\mathbf{1}_{D_t(-w)} * \mathbf{1}_{-D_t(u)})](x) \\
= & \left(\mathcal{F}[C_t] * \mathcal{F}[\mathbf{1}_{D_t(-w)} * \mathbf{1}_{-D_t(u)}]\right)(x) = \left(\mathcal{F}[C_t] * \left(\mathcal{F}[\mathbf{1}_{D_t(-w)}]\mathcal{F}[\mathbf{1}_{-D_t(u)}]\right)\right)(x) \\
= & \int_{\mathbb{R}^d} \mathcal{F}[C_t](x-y) \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) \mathcal{F}[\mathbf{1}_{-D_t(u)}](y) dy.
\end{aligned}$$

Putting everything in (2.44), we get

$$\begin{aligned}
\|f_t \otimes_1 f_t\|_2^2 = & \int_{\mathbb{R}^d} G(dx) \int_{\mathbb{R}^{2d}} dudw C_t(u) C_t(w) e^{i\langle x, u+w\rangle} \\
& \times \int_{\mathbb{R}^d} dy \mathcal{F}[C_t](x-y) \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) \mathcal{F}[\mathbf{1}_{-D_t(u)}](y).
\end{aligned} \tag{2.45}$$

Exchanging integrals in (2.45) yields

$$\begin{aligned}
&\|f_t \otimes_1 f_t\|_2^2 \\
&= \int_{\mathbb{R}^d} G(dx) \int_{\mathbb{R}^d} dy \mathcal{F}[C_t](x-y) \\
&\times \left(\int_{\mathbb{R}^d} dw C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y)\right) \left(\int_{\mathbb{R}^d} du C_t(u) e^{i\langle x,u\rangle} \mathcal{F}[\mathbf{1}_{-D_t(u)}](y)\right) \\
&= \int_{\mathbb{R}^d} G(dx) \int_{\mathbb{R}^d} dy \mathcal{F}[C_t](x-y) \left|\int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw\right|^2,
\end{aligned}$$

which is exactly (2.42). Now, let us focus on the term $\int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw$ in (2.42). Using again basic Fourier analysis and Fubini theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw = \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{tD} \mathbf{1}_{tD-w}](y) dw \\
&= \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \left(\int_{\mathbb{R}^d} \mathcal{F}[\mathbf{1}_{tD}](y-z) \mathcal{F}[\mathbf{1}_{tD-w}](z) dz\right) dw \\
&= \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \left(\int_{\mathbb{R}^d} \mathcal{F}[\mathbf{1}_{tD}](y-z) \mathcal{F}[\mathbf{1}_{tD}](z) e^{i\langle -w,z\rangle} dz\right) dw \\
&= \int_{\mathbb{R}^d} \mathcal{F}[\mathbf{1}_{tD}](y-z) \mathcal{F}[\mathbf{1}_{tD}](z) \left(\int_{\mathbb{R}^d} C_t(w) e^{i\langle x-z,w\rangle} dw\right) dz \\
&= \int_{\mathbb{R}^d} \mathcal{F}[C_t](x-z) \mathcal{F}[\mathbf{1}_{tD}](y-z) \mathcal{F}[\mathbf{1}_{tD}](z) dz,
\end{aligned}$$

which is exactly (2.43). □

Lemma 2.4.3. *Fix $d \geq 2$, let $B = (B_x)_{x \in \mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is isotropic, stationary and has unit-variance. Assume that $D \subset \mathbb{R}^d$ is compact with $\mathrm{Vol}(D) > 0$. Recall the definition (2.1) of C, the definition (2.14) of G, the definition (2.19) of $w_{2,t}$, the definition (2.40) of C_t and the definition (2.41) of $D_t(u)$. If the spectral condition (2.39) holds, then*

$$\int_{\mathbb{R}^d} G(dx) \int_{|x-y| \geq |x|/2} dy |\mathcal{F}[C_t](x-y)| \left| \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw \right|^2$$
$$\leq \text{const}\, w_{2,t}^{3/2}\, t^{2d}.$$

Proof. First, call

$$I := \int_{\mathbb{R}^d} G(dx) \int_{|x-y| \geq |x|/2} dy |\mathcal{F}[C_t](x-y)| \left| \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw \right|^2.$$

Using polar coordinates, we can write

$$\mathcal{F}[C_t](x) = \int_{\{|u| \leq \mathrm{diam}(D)t\}} C(u) e^{i\langle x,u\rangle} du = t^d \int_0^{\mathrm{diam}(D)} dr \rho(rt) r^{d-1} \int_{S^{d-1}} e^{irt\langle x,\theta\rangle} d\theta.$$

Thanks to (2.17)-(2.18) and (2.25) we have for $|x| > 1$

$$\left| \int_{S^{d-1}} e^{i\langle x,\theta\rangle} d\theta \right| = \text{const}\, |x|^{1-\frac{d}{2}} \left| J_{\frac{d}{2}-1}(|x|) \right| \leq \text{const}\, |x|^{1-\frac{d}{2}} |x|^{-\frac{1}{2}} \leq \frac{\text{const}}{\sqrt{|x|}}.$$

We deduce

$$|\mathcal{F}[C_t](x)| \leq \frac{\text{const}}{\sqrt{|x|}} t^d \int_0^{\mathrm{diam}(D)} |\rho(rt)| \frac{r^{d-1}}{\sqrt{rt}} dr = \frac{\text{const}}{\sqrt{|x|}} \int_0^{\mathrm{diam}(D)t} |\rho(r)| r^{\frac{d-1}{2}+\frac{d}{2}-1} dr$$
$$\leq \frac{\text{const}}{\sqrt{|x|}} t^{\frac{d}{2}-1} \int_0^{\mathrm{diam}(D)t} |\rho(r)| r^{\frac{d-1}{2}} dr \leq \frac{\text{const}}{\sqrt{|x|}} t^{\frac{d-1}{2}} \sqrt{\int_0^{\mathrm{diam}(D)t} \rho^2(r) r^{d-1} dr}$$
$$= \frac{\text{const}}{\sqrt{|x|}} t^{\frac{d-1}{2}} w_{R,\mathrm{diam}(D)t}^{1/2} \leq \frac{\text{const}}{\sqrt{|x|}} t^{\frac{d-1}{2}} w_{R,t}^{1/2}$$

where the last inequality follows from the doubling conditions in [GT19]. With this estimate, we have that I satisfies:

$$
\begin{aligned}
I &\le \int_{\mathbb{R}^d} G(dx) \int_{|y-x|\ge|x|/2} dy\, |\mathcal{F}[C_t](x-y)| \\
&\quad \times \left| \int_{\mathbb{R}^d} dw C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) \right|^2 \\
&\le \text{const} \cdot t^{\frac{d-1}{2}} w_{R,t}^{1/2} \int_{\mathbb{R}^d} \frac{G(dx)}{\sqrt{|x|}} \int_{\mathbb{R}^d} dy \left| \int_{\mathbb{R}^d} dw C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) \right|^2 \\
&= \text{const} \cdot t^{\frac{d-1}{2}} w_{R,t}^{1/2} \int_{\mathbb{R}^{2d}} dudw C_t(u) C_t(w) \\
&\quad \times \underbrace{\left(\int_{\mathbb{R}^d} \frac{G(dx)}{\sqrt{|x|}} e^{i\langle x,u+w\rangle} \right)}_{:=\bar{C}(u+w)} \int_{\mathbb{R}^d} dy \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) \mathcal{F}[\mathbf{1}_{-D_t(u)}](y) \\
&\le \text{const} \cdot t^{\frac{d-1}{2}} w_{R,t}^{1/2} \int_{\mathbb{R}^{2d}} dudw |C_t(u) C_t(w) \bar{C}(u+w)| \\
&\quad \times \sqrt{\mathrm{Vol}(tD \cap (tD+w))} \sqrt{\mathrm{Vol}(tD \cap (tD+u))} \\
&\le \text{const} \cdot t^{\frac{3}{2}d-\frac{1}{2}} w_{R,t}^{1/2} \int_{\mathbb{R}^{2d}} dudw |C_t(u) C_t(w) \bar{C}(u+w)|,
\end{aligned}
$$

where in the third inequality we used Cauchy-Schwarz inequality and Plancherel theorem.

Recall now that $\int_{\mathbb{R}^d} |x|^{-\frac{d}{2}} G(dx) = \int_0^\infty s^{-\frac{d}{2}} \mu(ds) < \infty$ by assumption, see (2.39). By definition, the spectral measure of $\bar{C}$ is $\frac{G(d\lambda)}{\sqrt{|\lambda|}}$, see (2.14), and the associated isotropic spectral measure is $\bar{\mu}(ds) = \frac{\mu(ds)}{\sqrt{s}}$. Since $\int_0^\infty s^{-\frac{d}{2}} \mu(ds) = \int_0^\infty s^{-\frac{d-1}{2}} \bar{\mu}(ds) < \infty$, we deduce from Lemma 2.3.2 that $\sup_{r\in\mathbb{R}_+} r^{\frac{d-1}{2}} |\bar{\rho}(r)| = \sup_{u\in\mathbb{R}^d} |u|^{\frac{d-1}{2}} |\bar{C}(u)| < \infty$, that is, $\bar{C}(u) \le \text{const}\, |u|^{-\frac{d-1}{2}}$. Using the inequality $|ab| \le a^2 + b^2$, this yields

$$
\begin{aligned}
I &\le \text{const} \cdot t^{\frac{3}{2}d-\frac{1}{2}} w_{R,t}^{1/2} \int_{\{|u|,|w|\le \mathrm{diam}(D)t\}} dudw C^2(u) |\bar{C}(u+w)| \\
&\le \text{const} \cdot t^{\frac{3}{2}d-\frac{1}{2}} w_{R,t}^{1/2} w_{R,\mathrm{diam}(D)t} \int_{\{|z|\le 2diam(D)t\}} |\bar{C}(z)| dz \\
&\le \text{const} \cdot t^{\frac{3}{2}d-\frac{1}{2}} w_{R,t}^{3/2} \int_0^{2diam(D)t} r^{\frac{d-1}{2}} dr = \text{const} \cdot t^{2d} w_{R,t}^{3/2}.
\end{aligned}
$$

where in the last inequality we used again doubling conditions in [GT19]. □

Lemma 2.4.4. *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary and has unit-variance (note that we did not assume isotropy and $d \ge 2$). Assume that $D \subset \mathbb{R}^d$ is compact with $\mathrm{Vol}(D) > 0$. Recall the definition (2.1) of C, the definition (2.14) of G, the definition (2.19) of $w_{2,t}$, the definition (2.40) of C_t and the definition (2.41) of $D_t(u)$. If $|\mathcal{F}[\mathbf{1}_D](x)| = O\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$[5], with $\mathcal{F}$ the Fourier transform, and if the analogous of the spectral condition (2.39) holds (note that here the isotropic spectral measure μ is not defined, because B is*

[5] Since we assumed D to be compact, this happens for example when $D = \overline{\mathring{D}}$ and ∂D is smooth with non-vanishing Gaussian curvature, see e.g. [BHI03].

not necessarily isotropic, but G is defined)

$$\int_{\mathbb{R}^d} |x|^{-\frac{d}{2}} G(dx) < \infty,$$

then

$$\int_{\mathbb{R}^d} G(dx) \int_{|y|\geq|x|/2} dy |\mathcal{F}[C_t](x-y)| \left| \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw \right|^2 \leq \text{const}\, w_{2,t}^{3/2}\, t^{2d}.$$

Proof. First, by (2.43) in Lemma 2.4.2, we can write

$$\begin{aligned} II := & \int_{\mathbb{R}^d} G(dx) \int_{|y|\geq|x|/2} dy |\mathcal{F}[C_t](x-y)| \left| \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw \right|^2 \\ = & \int_{\mathbb{R}^d} G(dx) \\ & \int_{|y|\geq|x|/2} dy |\mathcal{F}[C_t](x-y)| \left| \int_{\mathbb{R}^d} \mathcal{F}[C_t](x-z) \mathcal{F}[\mathbf{1}_{tD}](y-z) \mathcal{F}[\mathbf{1}_{tD}](z) dz \right|^2 \end{aligned}$$

Using Cauchy-Schwarz inequality two times (first with respect to y, then with respect to z) and then Plancherel theorem, II satisfies, with $\|\cdot\|_2$ the L^2-norm:

$$\begin{aligned} II \leq & \|\mathcal{F}[C_t]\|_2 \int_{\mathbb{R}^d} G(dx) \\ & \times \left(\int_{|y|\geq|x|/2} dy \left| \int_{\mathbb{R}^d} dz \mathcal{F}[C_t](x-z) \mathcal{F}[\mathbf{1}_{tD}](y-z) \mathcal{F}[\mathbf{1}_{tD}](z) \right|^4 \right)^{1/2} \\ \leq & \|\mathcal{F}[C_t]\|_2 \int_{\mathbb{R}^d} G(dx) \\ & \times \left(\int_{|y|\geq|x|/2} dy\, \|\mathcal{F}[C_t]\|_2^4 \left| \int_{\mathbb{R}^d} dz\, |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y-z)\, |\mathcal{F}[\mathbf{1}_{tD}]|^2 (z) \right|^2 \right)^{1/2} \\ = & \|\mathcal{F}[C_t]\|_2^3 \int_{\mathbb{R}^d} G(dx) \left(\int_{|y|\geq|x|/2} dy \left| |\mathcal{F}[\mathbf{1}_{tD}]|^2 * |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y) \right|^2 \right)^{1/2} \\ \leq & \|\mathcal{F}[C_t]\|_2^3 \int_{\mathbb{R}^d} G(dx) \left(\sup_{|y|\geq|x|/2} |\mathcal{F}[\mathbf{1}_{tD}]|^2 * |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y) \right)^{1/2} \\ & \times \left(\int_{\mathbb{R}^d} dy \left| |\mathcal{F}[\mathbf{1}_{tD}]|^2 * |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y) \right| \right)^{1/2} \\ \leq & \|\mathcal{F}[C_t]\|_2^3 \int_{\mathbb{R}^d} G(dx) \left(\sup_{|y|\geq|x|/2} |\mathcal{F}[\mathbf{1}_{tD}]|^2 * |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y) \right)^{1/2} \\ & \times \left(\int_{\mathbb{R}^d} |\mathcal{F}[\mathbf{1}_{tD}](y)|^2 \, dy \right) \\ = & \text{const} \cdot t^d \, \|\mathcal{F}[C_t]\|_2^3 \int_{\mathbb{R}^d} G(dx) \left(\sup_{|y|\geq|x|/2} |\mathcal{F}[\mathbf{1}_{tD}]|^2 * |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y) \right)^{1/2}, \end{aligned}$$

where the last inequality comes from Young convolution inequalities and the last equality from Plancherel theorem.

Since by assumption $|\mathcal{F}[\mathbf{1}_D](x)| = O\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$, in particular $\sup_{x\in\mathbb{R}^d} |x|^{d/2}|\mathcal{F}[\mathbf{1}_D](x)| < \infty$. We deduce, for all $t > 0$ and all $y \in \mathbb{R}^d$, that

$$|\mathcal{F}[\mathbf{1}_{tD}](y)|^2 = t^{2d}|\mathcal{F}[\mathbf{1}_D](ty)|^2 \leq \text{const}\, t^d|y|^{-d}.$$

This implies

$$\begin{aligned} &\sup_{|y|\geq|x|/2} |\mathcal{F}[\mathbf{1}_{tD}]|^2 * |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y) = \sup_{|y|\geq|x|/2} \left|\int_{\mathbb{R}^d} |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y-z)\, |\mathcal{F}[\mathbf{1}_{tD}]|^2 (z)dz\right| \\ \leq\ & \text{const}\cdot \frac{t^d}{|x|^d} \sup_{|y|\geq|x|/2} \left|\int_{|z|\geq|x|/4} |\mathcal{F}[\mathbf{1}_{tD}]|^2 (y-z)dz + \int_{|y-z|\geq|x|/4} |\mathcal{F}[\mathbf{1}_{tD}]|^2 (z)dz\right| \\ \leq\ & \text{const}\cdot \frac{t^d}{|x|^d}\int_{\mathbb{R}^d} |\mathcal{F}[\mathbf{1}_{tD}]|^2 (z)dz = \text{const}\frac{t^{2d}}{|x|^d}. \end{aligned}$$

We deduce that

$$\begin{aligned} II &\leq \text{const}\cdot t^{2d}\, \|\mathcal{F}[C_t]\|_2^3 \int_{\mathbb{R}^d} \frac{G(dx)}{|x|^{d/2}} \\ &= \text{const}\cdot t^{2d} w^{3/2}_{R,\text{diam}(D)t}\int_{\mathbb{R}^d}\frac{G(dx)}{|x|^{d/2}} \leq \text{const}\cdot t^{2d}w^{3/2}_{R,t}\int_{\mathbb{R}^d}\frac{G(dx)}{|x|^{d/2}}, \end{aligned}$$

where the last equality comes from Plancherel theorem and the last inequality from doubling conditions in [GT19].

Since $\int_{\mathbb{R}^d} |x|^{-\frac{d}{2}}G(dx) < \infty$ by assumption, our bound for II is

$$II \leq \text{const}\cdot t^{2d}w^{3/2}_{R,t}.$$

and the proof is concluded. □

Now we can proceed with the proof of Proposition 2.4.1.

Proof of Proposition 2.4.1. First, we assume without loss of generality that $w_{2,t} \to \infty$, since otherwise the statement follows from theorem 2.1.1. Moreover, throughout all the proof we freely use that $v_{2,t} \asymp w_{2,t}$, see (2.38).

Starting from now, the proof is divided into three steps.

Step 1: Reduction of the proof. We claim that it is enough to check that $Y_{2,t}/\sqrt{\text{Var}(Y_{2,t})} \to N(0,1)$ in order to prove Proposition 2.4.1. Indeed, since (2.39) holds and given that $\frac{d}{2} > \frac{d-1}{2}$, we deduce from Lemma 2.3.2 that $\sup_{r\in\mathbb{R}_+}\{r^{\frac{d-1}{2}}\rho(r)\} < \infty$. Proposition 2.2.2 implies the statement on σ_t^2 and justifies that we are left to prove that $Y_{2,t}/\sqrt{\text{Var}(Y_{2,t})} \to N(0,1)$.

As done in the proof of Proposition 2.3.1, in order to prove the convergence $Y_{2,t}/\sqrt{\text{Var}(Y_{2,t})} \to N(0,1)$ we make use of the Fourth Moment Theorem ([NP12, Theorem 5.2.7]); this requires checking that the only involved contraction goes to zero. To do this, we will use the novel ideas from Fourier analysis introduced in

Lemma 2.4.2, Lemma 2.4.3 and Lemma 2.4.4.

As in the proof of Theorem 2.2.4, we can first rewrite $Y_{2,t}$ as a double Wiener-Itô integral with respect to B:

$$Y_{2,t} = I_2(f_t), \quad \text{where } f_t = \int_{tD} e_x^{\otimes 2} dx,$$

with e_x such that $B_x = I_1(e_x)$. We know from the Fourth Moment Theorem stated in section 2.2.4 that $Y_{2,t}/\sqrt{\mathrm{Var}(Y_{2,t})} \to N(0,1)$ if and only if $\|f_t \otimes_1 f_t\|_2 / \mathrm{Var}(Y_{2,t}) \to 0$.

Step 2: A two-term error bound for the norm of the contraction. Here we apply Lemma 2.4.2. Noting that for $x, y \in \mathbb{R}^d$ either $|x-y| \geq |x|/2$ or $|y| \geq |x|/2$, we deduce from (2.42) and (2.43) the following two terms error bound:

$$\begin{aligned}
&\|f_t \otimes_1 f_t\|_2^2 \\
\leq& \int_{\mathbb{R}^d} G(dx) \int_{|x-y|\geq|x|/2} dy \, |\mathcal{F}[C_t](x-y)| \left| \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw \right|^2 \\
&+ \int_{\mathbb{R}^d} G(dx) \int_{|y|\geq|x|/2} dy \, |\mathcal{F}[C_t](x-y)| \left| \int_{\mathbb{R}^d} C_t(w) e^{i\langle x,w\rangle} \mathcal{F}[\mathbf{1}_{D_t(-w)}](y) dw \right|^2 \\
=& I + II.
\end{aligned}$$

Step 3: The norm of the contraction divided by the variance goes to 0. To conclude the proof of Proposition 2.4.1, we will now check (see the conclusion of Step 1) that $\|f_t \otimes_1 f_t\|_2 / \mathrm{Var}(Y_{2,t}) \to 0$. From (2.38), we have

$$\mathrm{Var}(Y_{2,t}) \asymp t^d w_{R,t}. \tag{2.46}$$

We deduce from Step 2, Lemma 2.4.3 and Lemma 2.4.4, that

$$\frac{\|f_t \otimes_1 f_t\|_2^2}{\mathrm{Var}(Y_{2,t})^2} \leq \text{const} \cdot \frac{w_{R,t}^{3/2}}{w_{R,t}^2} = \text{const} \cdot \frac{1}{w_{R,t}^{1/2}},$$

and the right-hand size goes to 0, since we assumed at the beginning of the proof that $w_{2,t} \to \infty$. □

2.5 Proof of theorem 2.1.2 when $R = 1$

This section is devoted to the proof of Theorem 2.1.2 in the remaining cases, namely φ non-odd, $R = 1$ and $R' \neq 3$.

To ease the exposition, we write in the following proposition the statement obtained when, in Theorem 2.1.2, we additionally suppose that we are in the cases just mentioned above.

Proposition 2.5.1. *Fix $d \geq 2$, let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued continuous centered Gaussian field on $\mathbb{R}^d$, and assume that B is stationary, isotropic and has unit-variance. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be not odd and such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$. Assume*

φ has Hermite rank $R = 1$ and let $R' \geq 2$, $R' \neq 3$, be its second Hermite rank. Consider Y_t defined by (2.3), where D is compact and $\mathrm{Vol}(D) > 0$. Set $m_t = \mathbb{E}[Y_t]$ and $\sigma_t = \sqrt{\mathrm{Var}(Y_t)} > 0$, and recall the definition (2.15) of the isotropic spectral measure μ, the definition (2.12) of $Y_{q,t}$ and the definition (2.19) of $w_{q,t}$. Assume that the spectral condition holds

$$\int_0^\infty s^{-d}\mu(ds) < \infty. \tag{2.47}$$

and that $|\mathcal{F}[\mathbf{1}_D](x)| = o\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$ [6]*, with $\mathcal{F}$ the Fourier transform. Then*

$$\sigma_t^2 \asymp \begin{cases} t^d w_{R',t} & \text{if } R' \in \{2,4\} \\ t^d & \text{if } R' \geq 5 \end{cases}$$

and

$$\frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1) \quad \text{as } t \to \infty.$$

Proof. Set $\widehat{\varphi}(x) = \varphi(x) - a_0 - a_1 x$. By the very definition of R', the function $\widehat{\varphi}$ has Hermite rank R'. Let us define

$$\widehat{Y}_t = \int_{tD} \widehat{\varphi}(B_x)dx = \sum_{q=R'}^{\infty} a_q Y_{q,t} = Y_t - \mathbb{E}[Y_t] - a_1 Y_{1,t} \tag{2.48}$$

and its variance

$$\widehat{\sigma}_t^2 = \mathrm{Var}(\widehat{Y}_t) = \sum_{q=R'}^{\infty} a_q^2 \mathrm{Var}(Y_{q,t}) = \sigma_t^2 - a_1^2 \mathrm{Var}(Y_{1,t}). \tag{2.49}$$

The proof is divided into two steps.

Step 1: CLT for $\widehat{Y}_t$. We claim that $\widehat{\sigma}_t^2 \asymp t^d w_{R',t}$ and

$$\frac{\widehat{Y}_t - m_t}{\widehat{\sigma}_t} \overset{\text{law}}{\to} N(0,1) \quad \text{as } t \to \infty. \tag{2.50}$$

First of all, observe that (2.47) implies (2.20) for $R' \geq 2$. Moreover, note that we can have three different situations:

- If $R' \geq 5$, then by (2.47) and Lemma 2.3.2 we have $\sup_{r\in\mathbb{R}_+} r^{\frac{d-1}{2}}|\rho(r)| = \sup_{u\in\mathbb{R}^d} |u|^{\frac{d-1}{2}}|C(u)| < \infty$, $C \in L^{R'}(\mathbb{R}^d)$, and the claim follows immediately by Theorem 2.1.1 and the fact that φ is not odd.
- If $R' = 4$, then the claim follows by Proposition 2.3.1.
- If $R' = 2$, since $|\mathcal{F}[\mathbf{1}_D](x)| = o\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$ implies $|\mathcal{F}[\mathbf{1}_D](x)| = O\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$, then the claim follows by Proposition 2.4.1.

[6] Since we assumed D to be compact, this happens for example when $D = \overline{\mathring{D}}$ and ∂D is smooth with non-vanishing Gaussian curvature, see e.g. [BHI03].

Step 2: CLT for Y_t. We claim that $\sigma_t \sim \widehat{\sigma}_t$ as $t \to \infty$ and

$$\mathbb{E}\left[\left(\frac{Y_t - m_t}{\sigma_t} - \frac{\widehat{Y}_t}{\widehat{\sigma}_t}\right)^2\right] \to 0. \tag{2.51}$$

The proof of Proposition 2.5.1 thus follows as soon as these two claims are shown to be true. From (2.22) we have

$$t^{-d}\mathrm{Var}(Y_{1,t}) = \mathrm{const} \int_{\mathbb{R}^d} \frac{G(d\lambda)}{|\lambda|^d} |t\lambda|^d |\mathcal{F}[\mathbf{1}_D](t\lambda)|^2.$$

Then, combining (2.47), dominated convergence theorem and the fact that $|\mathcal{F}[\mathbf{1}_D](x)| = o\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$, we have that $\mathrm{Var}(Y_{1,t}) = o(t^d)$ as $t \to \infty$. Since in Step 1 we proved that $\widehat{\sigma}_t^2 \asymp t^d$ or $\widehat{\sigma}_t^2 \asymp t^d w_{R',t}$, we have by (2.49) that $\widehat{\sigma}_t \sim \sigma_t$. It remains to prove (2.51). By orthogonality of chaotic projections we have

$$\mathbb{E}\left[\left(\frac{Y_t - m_t}{\sigma_t} - \frac{\widehat{Y}_t}{\widehat{\sigma}_t}\right)^2\right] = a_1^2 \frac{\mathrm{Var}(Y_{1,t})}{\mathrm{Var}(Y_t)} + \left(\frac{\widehat{\sigma}_t}{\sigma_t} - 1\right)^2.$$

Since $\mathrm{Var}(Y_{1,t}) = o(t^d)$, the first addend converges to 0. On the other hand, the second term vanishes because $\widehat{\sigma}_t \sim \sigma_t$ as $t \to \infty$.

□

2.6 An example of application of Theorem 2.1.2

In this section, we illustrate a possible use of Theorem 2.1.2. In order to introduce our class of fields of interest, recall the definition of the Bessel function J_ν given in Section 2.2.2 and define, for every $\nu \geq 0$, the normalized Bessel function function $\rho_\nu : [0, \infty) \to \mathbb{R}$ as

$$\rho_\nu(r) = c_\nu \frac{J_\nu(r)}{r^\nu},$$

where c_ν is chosen so that $\rho_\nu(0) = 1$.

Note that $\rho_{\frac{d}{2}-1}$ is equal to the function b_d defined in (2.18), and in particular $\rho_0 = b_2 = J_0$ when $d = 2$.

Throughout this section, we define the **Bessel Gaussian field of order ν and dimension** d as the real-valued continuous centered Gaussian field $B^\nu = (B_x^\nu)_{x \in \mathbb{R}^d}$ with covariance function

$$\mathbb{E}[B_x^\nu B_y^\nu] = \rho_\nu(|x-y|).$$

In particular, **d-dimensional Berry's Random Wave Model** is defined as the Bessel Gaussian field of order $\frac{d}{2} - 1$ and dimension d.

The existence of the Bessel Gaussian field of order ν and dimension $d \geq 2$ is neither obvious, nor always true. The following result provides a complete picture, and also shows that d-dimensional Berry's Random Wave Model appears to be the critical case for the existence of the Bessel Gaussian field.

Proposition 2.6.1. *Fix $d \geq 2$ and $\nu \geq 0$. There exists a Bessel Gaussian field $B^\nu = (B^\nu_x)_{x\in\mathbb{R}^d}$ of order ν and dimension d if and only if $\nu \geq \frac{d}{2} - 1$. In this case, the isotropic spectral measure associated to B^ν (see (2.16)) is*

$$\mu_\nu(ds) = \begin{cases} c_{d,\nu} s^{d-1}(1-s^2)^{\nu-\frac{d}{2}} \mathbf{1}_{(0,1)} ds & \text{if } \nu > \frac{d}{2}-1 \\ \delta_1(ds) & \text{if } \nu = \frac{d}{2}-1 \end{cases}, \tag{2.52}$$

where $c_{d,\nu} > 0$ is chosen so that μ_ν is a probability measure.

Proof. Observe that ρ_ν has representation

$$\rho_\nu(r) = c_\nu \frac{J_\nu(r)}{r^\nu} = \text{const} \times \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\nu+1)(r^2/4)^j}{j!\Gamma(j+\nu+1)}.$$

Combining this representation with [GMO18, Proposition 2.2], one has that ρ_ν is not positive definite when $\nu < \frac{d}{2} - 1$, showing that the Bessel Gaussian field does not exist in this case.

The statement for the critical case $\nu = \frac{d}{2} - 1$ immediately follows from (2.16) and (2.18), after observing that $\rho_{\frac{d}{2}-1} = b_d$.

For $\nu > \frac{d}{2} - 1$, one can actually check that μ_ν given in the statement is the distribution of the square root of the Beta random variable $\beta(\frac{d}{2}, \nu - \frac{d}{2} + 1)$. Then, using Fubini and $b_d = \rho_{\frac{d}{2}-1}$ we have

$$\begin{aligned} \int_0^1 b_d(rt)\mu_\nu(dr) &= \text{const} \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(d/2)}{\Gamma(k+d/2)k!} ((rt)^2/4)^k \right) \mu_\nu(dr) \\ &= \text{const} \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(d/2)}{\Gamma(k+d/2)k!} \left(\frac{t}{2}\right)^{2k} \int_0^1 r^{2k} \mu_\nu(dr) \\ &= \text{const} \sum_{k=0}^{\infty} \frac{\Gamma(\nu+1)(-1)^k}{\Gamma(\nu+k+1)k!} \left(\frac{t}{2}\right)^{2k} = \rho_\nu(t), \end{aligned}$$

where the second-last equality comes from the fact that the k-th moment of a random variable $\beta(\frac{d}{2}, \nu - \frac{d}{2} + 1)$ is

$$\frac{\Gamma(d/2+k)\Gamma(\nu+1)}{\Gamma(d/2)\Gamma(\nu+k+1)}.$$

This means (see (2.16)) that ρ_ν defines a Bessel Gaussian field when $\nu > \frac{d}{2} - 1$ and that its spectral measure is μ_ν as defined in the statement.

The fact that any Bessel Gaussian field is continuous can be proved using standard results, see e.g. [AT07, Theorem 1.4.1]. □

Now let us apply our Theorem 2.1.2 to the Bessel Gaussian field with parameter $\nu \geq \frac{d}{2} - 1$ and dimension $d \geq 2$.

Corollary 2.6.2 (Spectral CLT applied to B^ν). *Fix $d \geq 2$ and $\nu \geq \frac{d}{2} - 1$. Let $B^\nu = (B^\nu_x)_{x\in\mathbb{R}^d}$ be a Bessel Gaussian field on $\mathbb{R}^d$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be not odd and such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$, let R be the Hermite rank of φ and consider Y_t defined by (2.3), where $B = B^\nu$ and D is compact, $Vol(D) > 0$. Set $m_t = \mathbb{E}[Y_t]$ and $\sigma_t = \sqrt{\mathrm{Var}(Y_t)} > 0$. If $R = 2$, assume that $|\mathcal{F}[\mathbf{1}_D](x)| = O\left(\frac{1}{|x|^{d/2}}\right)$ as $|x| \to \infty$. Then,* *if $R = 2$ or $R \geq 4$, we have that $\sigma_t^2 \asymp t^d w_{R,t}$ and*

$$\frac{Y_t - m_t}{\sigma_t} \overset{\text{law}}{\to} N(0,1) \quad \textit{as } t \to \infty.$$

Proof. Using standard asymptotics for Bessel functions (see e.g. [Kra14]), one has

$$\rho_\nu(r) = c_\nu \frac{\cos(r - a_\nu)}{r^{\nu+1/2}} + O\left(\frac{1}{r^{\nu+3/2}}\right) \quad \text{as } r \to \infty$$

for some constants c_ν and a_ν. Then, it follows that

$$\int_0^{2t} |\rho_\nu(r)|^q r^{d-1} dr = O\left(\int_1^{2t} \frac{dr}{r^{q\nu + \frac{q}{2} + 1 - d}}\right) \quad \text{as } t \to \infty.$$

Since $\nu \geq \frac{d}{2} - 1$, when $R \geq 5$ we have that $|\rho|^q(r) r^{d-1}$ is integrable, and the result follows by Breuer Major Theorem 2.1.1. For $R = 2$ and $R = 4$, the result follows by a direct application of Theorem 2.1.2, since the negative moment of order d/R of μ_ν exists for every $d \geq 2$ and $R \geq 2$. □

Remark 2.6.1. For simplicity, in the previous statement we did not consider all the cases which can be solved applying our Theorem 2.1.2. For example, the spectral condition (2.20) for $R = 1$ is not verified by μ_ν if $\nu > \frac{d}{2} - 1$, but holds in the d-dimensional Berry case $\nu = \frac{d}{2} - 1$. Moreover, relying on a slight variation of the proof of Proposition 2.5.1, it would have not been very difficult to also cover the non-critical cases where $\nu \in [\frac{d}{2} - 1, \frac{d}{2} - \frac{1}{2})$. We skip the details for the sake of brevity.

CHAPTER 3

Fluctuations of polyspectra in spherical and Euclidean random wave models

This chapter contains the paper [GMT24], "Fluctuations of polyspectra in spherical and Euclidean random wave models", written in collaboration with Francesco Grotto and Anna Paola Todino. *Electron. Comm. Probab.* (2024), 29, 1-12.

3.1 Introduction and Main Results

A *Random Wave* model (RW) can be defined as a random field on a Riemannian manifold given by a random linear combination of eigenfunctions of the Laplace-Beltrami operator at a fixed frequency, or within a small frequency bandwidth. The study of such stochastic objects dates back to the works of Berry and Zelditch [Ber77; Zel09] and it is now a well-established area of research, in which properties of nodal sets of RWs and the comparison with their deterministic counterpart have garnered significant focus (*cf.* [Wig23] for a recent survey). In homogeneous spaces such as $\mathbb{R}^d$, the hypersphere S^d or the hyperbolic space $\mathbb{H}^d$ (*cf.* respectively [GP23; MN24; MP11] for specific discussions), RWs also appear naturally in the spectral decomposition of isotropic Gaussian random fields, that is Gaussian fields whose law (thus, its covariance function) is invariant under isometries of the underlying manifold, as detailed in [CL12].

In this note, we focus on Gaussian RWs on Euclidean space $\mathbb{R}^d$ and on the (hyper)sphere $S^d \subset \mathbb{R}^{d+1}$. Our main result, Theorem 3.1.1, establishes asymptotics for (variances of) polynomial transforms of those RWs. In particular, we complete the high-frequency asymptotic description of the so-called *polyspectra*, that is integrals of Hermite polynomials of the RWs on fixed space domains: the case of Hermite polynomials of large odd order was left as a conjecture on hyperspheres in [MR15; MW14] and was not discussed in works (such as [MN24; NPR19]) in the Euclidean case (see Remark 3.1.3 for a precise comparison with previous results). Polyspectra constitute the elementary objects in the Wiener chaos decomposition of functionals of Gaussian RWs, therefore their asymptotics play a crucial role in the Wiener chaos approach to limit theorems.

In order to prove Theorem 3.1.1 we will exploit a peculiar relation with uniform random walks on Euclidean space, and the well-established theory of those random processes. To some extent, this resembles a connection with short random walks considered in the case of random spherical harmonics in [MP08; MP10].

Notation

We regard the d-dimensional sphere $S^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ as an embedded manifold of codimension 1; we denote by σ_d the induced volume measure and by $\omega_d = \sigma_d(S^d) = 2\pi^{(d+1)/2}/\Gamma\left(\frac{d+1}{2}\right)$ the total volume. The geodesic distance on S^d is given by $d(x,y) = \cos^{-1}(x \cdot y)$, $x \cdot y$ denoting the canonical scalar product in $\mathbb{R}^{d+1}$.

We denote by $B^E_d(x,R) \subset \mathbb{R}^d$ the ball of radius R in $\mathbb{R}^d$ centered at the point x. When the context allows it, we lighten notation omitting dependencies, writing for instance B_R to denote a ball of radius R whose center can be chosen arbitrarily. The same applies to geodesic balls $B^S_d(x,R) \subset S^d$. The symbol χ_A denotes the indicator function of a set A.

The symbol C denotes a *positive* constant, possibly differing in any of its occurrences, and depending only on eventual subscripts, as in $C_{a,b}$. Landau's O and o symbols have their usual meaning, with constants involved in upper and lower bounds depending again only on eventual subscripts, as in $O_{a,b}(1)$.

3.1.1 Random Wave Models

Berry's model, that is the RW model on Euclidean space $\mathbb{R}^d$, is the centered Gaussian random field $U_\lambda(x)$, $\lambda > 0$, $x \in \mathbb{R}^d$, with covariance function

$$\mathbb{E}\left[U_\lambda(x)U_\lambda(y)\right] = j_d(\lambda|x-y|), \quad x, y \in \mathbb{R}^d, \tag{3.1}$$

where j_d is (a multiple of) the Fourier transform of the volume measure σ_{d-1} of S^{d-1}, regarded as a generalized function on the ambient space $\mathbb{R}^d$,

$$j_d(|x|) = \frac{1}{\omega_{d-1}} \int_{S^{d-1}} e^{ix\cdot\theta} d\sigma_{d-1}(\theta) = \frac{\nu! 2^\nu}{(|x-y|)^\nu} J_\nu(|x-y|), \quad \nu = \nu_d := \frac{d}{2} - 1, \tag{3.2}$$

J_ν being the Bessel function of first kind and order ν. In particular, $j_2(r) = J_0(r)$ and $j_3(r) = \text{sinc}(r) = \frac{\sin(r)}{r}$. Notation $\nu = d/2 - 1$ will remain valid throughout the paper.

The Gaussian field U_λ is named after M. Berry, who introduced it in order to describe local behavior of high frequency eigenstates in quantum billiards [Ber77;

Ber02]. It arises as a central limit of random linear combinations of planar waves on $\mathbb{R}^d$ of the form

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\cos\left(\lambda x\cdot y_i+\varphi_i\right),\quad x\in\mathbb{R}^d,\quad N\to\infty,$$

with the y_i's and φ_i's being i.i.d. uniform random variables respectively in S^{d-1} and $[0,2\pi]$ (*cf.* [Ber02; NPR19]). Samples of U_λ are themselves smooth λ^2-eigenfunctions of the Laplace operator.

Random (Hyper)Spherical Harmonics, that is the RW model on the sphere S^d, consist in the centered Gaussian random field $T_\ell(x)$, $\ell\in\mathbb{N}$, $x\in S^d$, with covariance

$$\mathbb{E}\left[T_\ell(x)T_\ell(y)\right]=\mathcal{G}_{d,\ell}(\cos d(x,y))=\binom{\ell+\nu}{\ell}^{-1}P_\ell^{\nu,\nu}(\cos d(x,y))\quad x,y\in S^d, \tag{3.3}$$

where $\mathcal{G}_{d,\ell}$ is the normalized Gegenbauer polynomial of degree ℓ (see [Sze39, p. 4.7]), the right-hand side providing an alternative representation in terms of Jacobi polynomials. This random field can be regarded as the analog of Berry's model on $\mathbb{R}^d$. Indeed, the role of Euclidean planar waves is played on the sphere S^d by hyperspherical harmonics, that is Laplace-Beltrami eigenfunctions

$$\{Y_{\ell,m,d}\}_{m=1,\dots,\eta_{\ell,d}}\subset C^\infty(S^d),\ \ell\in\mathbb{N},\quad \Delta_{S^d}Y_{\ell,m,d}+\ell(\ell+d-1)Y_{\ell,m,d}=0,$$

forming a complete orthonormal basis of $L^2(S^d,\sigma_d)$, spanning at each fixed $\ell\in\mathbb{N}$ a distinct eigenspace[1] of dimension $\eta_{\ell,d}=\frac{2\ell+d-1}{\ell}\binom{\ell+d-2}{\ell-1}$. The Gaussian field T_ℓ has the law of a random superposition of waves at a fixed wavenumber:

$$T_\ell(x)\sim\sum_{m=1}^{\eta_{\ell,d}}a_{\ell,m,d}Y_{\ell,m,d}(x),\quad x\in S^d,$$

with $\{a_{\ell,m,d}\}_{m=1}^{\eta_{\ell,d}}$ being i.i.d. Gaussian variables $N\left(0,\omega_d/\eta_{\ell,d}\right)$. The index ℓ parametrizes the spectrum of Laplace-Beltrami operator on S^d through $\ell(\ell+d-1)$, thus in the limit $\ell\to\infty$ it plays the same asymptotic role of $\lambda\to\infty$ in the Euclidean setting, where λ^2 is the eigenvalue of the wave $x\mapsto e^{i\lambda x\cdot\theta}$.

3.1.2 Polyspectra

Relevant geometric functionals of U_λ, such as the $(d-1)$-dimensional volume of the nodal set $\{U_\lambda=0\}$, depend in general also on derivatives (*i.e.* the gradient) of the random field. However, already in the case of integral functionals of the form

$$\mathcal{F}_\lambda=\int_D F(U_\lambda(x))dx,\quad F:\mathbb{R}\to\mathbb{R},\ \lambda>0,\ D\subset\mathbb{R}^d, \tag{3.4}$$

(and the analog for spherical RWs on measurable subsets $D\subseteq S^d$) the description of the asymptotic behavior as $\lambda\to\infty$ is not trivial, essentially because covariance

[1] In fact, the arbitrary choice of an orthogonal basis of the eigenspace relative to $\ell(\ell+d-1)$ is irrelevant in our scope.

functions do not satisfy integrability conditions allowing direct applications of basic results (*e.g.* [BM83]), and they oscillate between positive and negative values.

Remark 3.1.1. The covariance function (3.1) depends on $\lambda|x-y|$, therefore the high-frequency limit for functionals of the form (3.4) is equivalent to a (fixed-frequency) large domain limit. Indeed, the random fields $B_\lambda(\cdot) \sim B_1(\lambda\cdot)$ have the same law, and thus integral functionals

$$\int_D F(U_\lambda(x))dx \sim \frac{1}{\lambda^d}\int_{\lambda D} F(U_1(x))dx,$$

are also equidistributed. There is no equivalent infinite-domain limit on the compact S^d.

Since the RWs we consider are Gaussian random fields, functionals of the form (3.4) can be studied considering their Wiener chaos expansion. Hermite polynomials, $H_n(t) = (-1)^n\varphi^{(n)}(t)/\varphi(t)$, $n \in \mathbb{N}$, form an orthogonal basis of $L^2(\mathbb{R}, \varphi(t)dt)$, φ being the p.d.f. of the standard Gaussian variable. If $F \in L^2(\mathbb{R}; \varphi(x)dx)$, the Wiener chaos decomposition of $\mathcal{F}_\lambda$ is given by (*cf.* [NPR19])

$$\mathcal{F}_\lambda = \sum_{q=0}^{\infty} \frac{1}{q!}\left(\int_{\mathbb{R}} F(t)H_q(t)\varphi(t)dt\right)\left(\int_D H_q(U_\lambda(x))dx\right).$$

The stochastic terms $\int_D H_q(U_\lambda(x))dx$ in the decomposition are called *polyspectra*: Fourth Moment Theorems –by now a standard tool in Gaussian analysis, *cf.* [NP12]– allow to deduce Central Limit Theorems for single polyspectra and functionals of the above form as $\lambda \to \infty$, provided that asymptotics of variance and fourth cumulants are available.

3.1.3 Variance Asymptotics

For $d, q \geq 2$, $R > 0$, we will write:

$$V^E_{d,R}(q,\lambda) = \mathrm{Var}\left(\int_{B^E(x_0,R)} H_q(U_\lambda(x))dx\right), \quad \lambda > 0,$$
$$V^S_{d,R}(q,\ell) = \mathrm{Var}\left(\int_{B^S(x_0,R)} H_q(T_\ell(x))d\sigma_d(x)\right), \quad \ell \in \mathbb{N}.$$

Remark 3.1.2. The case $q = 0$ needs no discussion. As for $q = 1$, it turns out that both $V^E_{d,R}(1,\lambda)$ and $V^S_{d,R}(1,\ell)$ have an oscillatory behavior as $\lambda, \ell \to \infty$ and they can vanish (see respectively [MN24; Tod19] in the two geometrical settings). Anyways, when $q = 1$ the polyspectrum is a Gaussian variable, and the study of its variance is simpler, so we will omit that case in our discussion.

Our main result is the following:

Theorem 3.1.1. *Let $d, q \geq 2$, $R > 0$. There exist finite **positive** constants c^d_q such that:*

- *(Euclidean) as $\lambda \to \infty$,*

$$V^E_{d,R}(q,\lambda) = c_q^d q! \omega_d \omega_{d-1} R^d (1+o_{q,d}(1)) \cdot \begin{cases} \lambda^{1-d} & q=2 \\ \lambda^{-2}\log(\lambda) & q=4, d=2 \\ \lambda^{-d} & \textit{all other } d\geq 2, q\geq 3 \end{cases},$$

- *(Hyperspherical) as $\ell \to \infty$, if $R \in (0,\pi)$,*

$$V^S_{d,R}(q,\ell) = c_q^d q! \omega_{d-1} \sigma_d(B^S_R)(1+o_{q,d}(1)) \cdot \begin{cases} \ell^{1-d} & q=2, \\ \ell^{-2}\log(\ell) & q=4, d=2, \\ \ell^{-d} & \textit{all other } d\geq 2, q\geq 3, \end{cases}$$

and when $R=\pi$ (that is in the case of polyspectra obtained integrating over the whole S^d),

$$V^S_{d,\pi}(q,\ell) = 2c_q^d q! \omega_{d-1} \omega_d (1+o_{q,d}(1)) \cdot \begin{cases} \ell^{1-d} & q=2, \\ \ell^{-2}\log(\ell) & q=4, d=2, \\ 0 & q,\ell \textit{ both odd,} \\ \ell^{-d} & \textit{all other } d\geq 2, q\geq 3. \end{cases}$$

The proof of Theorem 3.1.1 is the content of the forthcoming Section 3.3.

Remark 3.1.3. Let us clarify the relation of Theorem 3.1.1 with the previous literature.

- (Euclidean case) [NPR19] proved the result in dimension $d=2$ for even $q\geq 2$ and upper bounds for odd $q\geq 3$, for integration domains $D\subset \mathbb{R}^2$ with boundary of class C^1; [Not21] generalizes the discussion to arbitrary $d\geq 2$ and even $q\geq 2$; for $d\geq 2$ and even $q\geq 2$ [Mai24] considers compact domains $D\subset\mathbb{R}^d$ with finite perimeter, treating the larger class of weakly stationary random fields and establishing asymptotics for covariances of functionals;

- (case of S^2) [MW11; MW14] proved the above asymptotics on the whole S^2 (case $R=\pi$) leaving as an open problem the positivity of c_q^2 for odd $q\geq 5$; [Tod19] discussed the case $R<\pi$ for $q=2$ and provided upper bounds for larger $q\geq 3$;

- (Hyperspherical case) concerning polyspectra over the whole S^d ($R=\pi$), in larger dimension the above asymptotic was established for all even $q\geq 2$ in [MR15] and for $q=3$ [Ros19], leaving positivity of c_q^d for odd $q\geq 5$ as a conjecture (see [MR15, p. 2386]).

The remainder of this note is organized as follows. Section 3.2 details the relation between variance of polyspectra and the density of uniform random walks, and introduces results on the latter that we can exploit to control the former. We prove Theorem 3.1.1 in Section 3.3.

3.2 Pearson's Random Walk

Let X_n^d, $n = 1, 2, \dots$ denote the uniform random walk on $\mathbb{R}^d$, that is

$$X_n^d = U_1^d + \cdots + U_n^d,$$

with $(U_k^d)_{k\in\mathbb{N}}$ a sequence of i.i.d. uniform random variables on $S^{d-1} \subset \mathbb{R}^d$. The stochastic process X_n^d is known as *Pearson's Random Walk*, because of a very early mention in [Pea05].

Much is known on Pearson's random walk: in particular, its distribution and moments for a small number of steps n are of interest in Number Theory because of their relation with Mahler measures. We can thus rely on the series of works [BSV16; BSW13; Bor+11; Bor+12], to which we refer for a comprehensive background on the matter and the related literature.

The basic result on the distribution of X_n for small n is a representation formula for the p.d.f. dating back to the seminal work [Klu06].

Lemma 3.2.1 (Kluyver's Formula). *For $n \geq 2$ the law μ_n^d of X_n^d is absolutely continuous with respect to Lebesgue's measure on $\mathbb{R}^d$, and its density is a radial function,* i.e. *$\mu_q^d(dx) = f_q^d(\|x\|)dx$ with $f_q^d(\|\cdot\|) \in L^1(\mathbb{R}^d)$.*

The (Euclidean) distance from the origin $\left\|X_n^d\right\|$ is absolutely continuous with respect to Lebesgue's measure on $(0, \infty)$, and its density is given by

$$\rho_n^d(r) = \frac{1}{(\nu!)^2 4^\nu} \int_0^\infty (tr)^{2\nu+1} j_d(tr) j_d(t)^n dt, \quad r > 0, \quad \nu = \frac{d}{2} - 1.$$

To be precise, the integral in Kluyver's formula has to be regarded as an improper integral that either converges to a non-negative number (for large enough d, n even absolutely) or diverges to $+\infty$ (see Remark 3.2.2 below). We refer to [BSV16, Theorem 2.1] for a proof of Lemma 3.2.1 and a thorough general discussion. Let us note that f_q^d and ρ_q^d are related by

$$\omega_{d-1} \int_0^s r^{d-1} f_q^d(r) dr = \int_{B_s} \mu_q^d(dx) = \mathbb{P}\left(X_q^d \in B_s\right) = \mathbb{P}\left(\|X_q^d\| \leq s\right) = \int_0^s \rho_q^d(r) dr. \tag{3.5}$$

3.2.1 Random Walks and Polyspectra

Before we outline further properties of Pearson's random walk to be used in the last Section, let us explain heuristically how one is led to consider this apparently unrelated object in studying RW's polyspectra. The following computation concerns the Euclidean case only, and has no direct analog in the case of hyperspherical RWs. Nevertheless, in the high-frequency limiting regime the Euclidean and hyperspherical RW models turn out to exhibit the same behavior. It holds

$$V_{d,R}^E(q, \lambda) = q! \int_{B_R} \int_{B_R} j_d(\lambda|x - y|)^q dx dy,$$

since $j_d(\lambda|x-y|)$ is the covariance function of U_λ, and if X, Y are two centered jointly Gaussian variables $\mathbb{E}\left[H_q(X)H_q(Y)\right] = q!\mathbb{E}\left[XY\right]^q$. By (3.2), the function

$$j_d(|x|) = \mathcal{F}[\mu_1^d](x) := \int_{\mathbb{R}^d} e^{ix\cdot y}\mu_1^d(dy) \tag{3.6}$$

can be represented as the Fourier transform of μ_1^d. Hence we can write

$$\begin{aligned} V_{d,R}^E(q,\lambda) &= q!\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\chi_{B_{\lambda R}}(x)\chi_{B_{\lambda R}}(y)}{\lambda^{2d}}\mathcal{F}[\mu_1^d](x-y)^q dxdy \\ &= q!\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\chi_{B_{\lambda R}}(x)\chi_{x+B_{\lambda R}}(z)}{\lambda^{2d}}\mathcal{F}[\underbrace{\mu_1^d * \cdots * \mu_1^d}_{q \text{ times}}](z)dxdz, \end{aligned}$$

where we recognize the q-fold convolution of μ_1^d with itself as the law μ_q^d of X_q^d. As $\lambda \to \infty$, indicator functions in the last displayed equation converge pointwise to 1 on $\mathbb{R}^d$, therefore we expect —at least intuitively— that the limiting integral

$$\int_{\mathbb{R}^d} \mathcal{F}[\mu_q^d](z)dz = f_q^d(0)$$

plays a relevant role in determining the asymptotic behavior of $V_{d,R}^E(q,\lambda)$. By (3.5) and Kluyver's formula we can also write

$$\omega_{d-1}f_q^d(0) = \lim_{r\to 0}\frac{\rho_q^d(r)}{r^{d-1}} = \frac{1}{(\nu!)^2 4^\nu}\int_0^\infty j_d(t)^q t^{d-1}dt = \rho_{q-1}^d(1),$$

therefore the crucial objects appear to be the integrals

$$I_q^d := \int_0^\infty j_d(t)^q t^{d-1}dt = \frac{\rho_{q-1}^d(1)}{(\nu!)^2 4^\nu}, \tag{3.7}$$

and these numbers can *a priori* take any value in $[0,\infty]$.

In fact, I_q^d *coincides with the coefficient* c_q^d *appearing in* Theorem 3.1.1 in the cases $d \ge 2, q \ge 3$ except $d = 2, q = 4$. Before we prove it in the last Section, we need to discuss finiteness and positivity of the I_q^d's, that is the task for which we need to resort to previous results on Pearson's random walk.

3.2.2 Density of short random walks

We report an explicit expression for the densities ρ_2^d and a recursion formula allowing to treat the density at higher n. Representations for ρ_n^d, $n = 3,4$, in terms of hypergeometric functions are available in [BSV16, Equations (74), (79)]. We recall that $\nu = d/2 - 1$ in all its occurences.

Lemma 3.2.2. *[BSV16, Lemma 4.1] For* $d \ge 2$ *and* $0 < r < 2$ *it holds*

$$\rho_2^d(r) = \frac{2}{\pi\binom{2\nu}{\nu}} r^{2\nu}(4-r^2)^{\nu-1/2}\chi_{(0,2)}(r).$$

Lemma 3.2.3. *[BSV16, Theorem 2.9] Let* $d, n \ge 2$*,* $r > 0$ *and define* $\psi_n^d(r) =$

$\rho_n^d(r)/r^{d-1}$. *It holds, for* $n \geq 3$*,*

$$\psi_n^d(r) = \frac{(\nu!)^2 4^\nu}{\pi(2\nu)!} \int_{-1}^{1} \psi_{n-1}^d \left(\sqrt{1+2sr+r^2}\right) (1-s^2)^{\nu-1/2} ds, \quad r \in (0,n). \tag{3.8}$$

Corollary 3.2.4. *For any* $d \geq 2$ *and* $n \geq 3$*,* ψ_n^d *is strictly positive (possibly infinite) on* $(0,n)$*.*

Proof. By induction on n, the case $n = 2$ follows by definition of ψ_2^d and Lemma 3.2.2. Assume now that $\psi_{n-1}^d(r) > 0$ for $r \in (0, n-1)$ (by definition it vanishes for $r > n-1$); by (3.8) if the open set

$$\{s \in (-1,1) : \psi_{n-1}^d \left(\sqrt{1+2sr+r^2}\right) > 0\} = \{s \in (-1,1) : 1+2sr+r^2 < (n-1)^2\}$$

is not empty, then $\psi_n^d(r) > 0$. This is in fact the case, because $p(s) = 1+2sr+r^2$ is increasing and continuous and $p(-1) = 1-2r+r^2 = (r-1)^2 < (n-1)^2$ for $r < n$. □

Moving back to the specific integrals I_q^d we are interested in, we combine the latter Corollary and the following approximation result on Bessel functions to show that they are both positive and finite.

Lemma 3.2.5. *[Kra14, Theorem 4, Equation 6] Let* $d \geq 2$*. There exist constants* $\varphi_d > 0$ *such that*

$$0 < \sup_{r \geq 0} r^{3/2} \left| J_\nu(r) - \sqrt{\frac{2}{\pi r}} \cos(r - \varphi_d) \right| < \infty. \tag{3.9}$$

In particular by (3.2)*,*

$$j_d(r) = C_d r^{-(d-1)/2} \cos(r - \varphi_d) + O_d(r^{-(d+1)/2}), \quad r \to \infty. \tag{3.10}$$

Lemma 3.2.6. *For any* $d \geq 2, q \geq 3$ *except the single case* $d = 2, q = 4$*,* $I_q^d \in (0,\infty)$ *is positive and finite.*

Proof. By Corollary 3.2.4, for all $d \geq 2, q \geq 3$ it holds

$$I_n^d = \frac{\rho_{n-1}^d(1)}{(\nu!)^2 4^\nu} = \frac{\psi_{n-1}^d(1)}{(\nu!)^2 4^\nu} > 0.$$

Concerning finiteness, by (3.10) and (3.7) we have

$$I_q^d \leq C_{d,q} \int_1^\infty r^{(d-1)(1-q/2)} dr;$$

in particular, the integral in (3.7) is absolutely convergent if $q > \frac{d}{d-1}$. We are left to discuss the cases $d = 2,3, q = 3$, since we are excluding $d = 2, q = 4$ (see Remark 3.2.2 below). A classical result on Bessel functions (*cf.* [Wat95, Section 13.46]) implies $I_3^2 = \frac{2}{\pi\sqrt{3}}$, and $I_3^3 = \int_0^\infty t^{-1} \sin(t)^3 dt = \pi/4$ is easily computed reducing it to Dirichlet's integral. □

Remark 3.2.1. The following exact expressions

$$I_3^d = \frac{2}{\pi\sqrt{3}} \cdot \frac{12^\nu(\nu!)^4}{(2\nu)!}, \; d \geq 2, \qquad I_5^2 = \frac{\sqrt{5}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{40\pi^4},$$

can be deduced respectively from [BSV16, Proposition 4.3] and [Bor+12, Theorem 5.1] (concerning the asymptotics of $\rho_n^d(r)$ as $r \to 0$). To the best of our knowledge, no representation is available in the general case.

Remark 3.2.2. In the case $d = 2, q = 4$ excluded from Lemma 3.2.6, one can prove by means of Lemma 3.2.5 that $I_4^2 = \int_0^\infty J_0(t)^4 t dt = +\infty$, coherently with the fact that the density $\rho_2^3(1) = I_2^4 = +\infty$ is not finite. This is actually the only case of a density ρ_n^d, $d, n \geq 2$, taking value $+\infty$ at some point, as for larger n the regularizing effect of the multiple convolution defining the law of Pearson's random walk prevails (*cf.* again [BSV16]).

3.3 Fluctuation Asymptotics and Integration Domains

3.3.1 Variance of Euclidean Polyspectra

We first rewrite the variances we are interested in as integrals of a single variable.

Lemma 3.3.1. *Let $R > 0$, $B_R = B^E(x_0, R) \subseteq \mathbb{R}^d$ a ball of radius R centered at a (fixed, arbitrary) point $x_0 \in \mathbb{R}^d$, and $f \in L^1_{loc}(\mathbb{R})$; it holds*

$$\int_{B_R}\int_{B_R} f(|x-y|)dxdy = \int_0^{2R} f(r) W_{d,R}(r) r^{d-1} dr, \tag{3.11}$$

where, denoting by m_d Lebesgue's measure on $\mathbb{R}^d$,

$$W_{d,R}(r) = \omega_{d-1} m_d(B(x,R) \cap B(y,R)), \quad |x-y| = r, \tag{3.12}$$

whose value does not depend on the choice of points $x, y \in \mathbb{R}^d$.

The proof is a consequence of the fact that translations on $\mathbb{R}^d$ are isometries of the whole space: it holds

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_{B_R}(x)\chi_{B_R}(y) f(|x-y|)dxdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_{B_R}(x)\chi_{B_R+z}(x) f(|z|)dxdz,$$

from which (3.11) follows moving to polar coordinates for z. It is easy to observe that $W_{d,R} : [0,\infty) \to [0,\infty)$ is supported by $[0, 2R]$, it is differentiable on $(0,\infty) \setminus \{2R\}$ and $W_{d,R} = \omega_{d-1} m_d(B_R)$, $W_{d,R}(2R) = 0$.

Proof of Theorem 3.1.1, Euclidean part. In sight of Lemma 3.3.1 it holds,

$$V_{d,R}^E(q,\lambda) = q! \int_{B_R}\int_{B_R} j_d(\lambda|x-y|)^q dxdy = q! \int_0^{2R} j_d(\lambda r)^q W_{d,R}(r) r^{d-1} dr. \tag{3.13}$$

For $q = 2$, (3.10) implies that, as $\lambda \to \infty$,

$$V^E_{d,R}(2,\lambda) = C_d \int_0^{2R} \frac{\cos(\lambda r - \varphi_d)^2}{\lambda^{d-1}} W_{d,R}(r) dr + O_{d,R}\left(\frac{1}{\lambda^d}\right) = C_{d,R} \frac{1 + o_{d,R}(1)}{\lambda^{d-1}},$$

the second step following by expanding $2\cos(\theta)^2 = 1+\cos(2\theta)$ and applying Riemann-Lebesgue lemma to oscillatory terms.

We also treat separately the case $d = 2, q = 4$, since it involves a logarithmic discrepancy with the general case $q \geq 3$. (3.10) implies that, as $\lambda \to \infty$,

$$V^E_{2,R}(4,\lambda) = C_d \int_0^{2R} \frac{\cos(\lambda r - \varphi_d)^4}{\lambda^2 r} W_{d,R}(r) dr + O_R\left(\frac{1}{\lambda^3}\right) = C_R \frac{\log(\lambda) + o_R(1)}{\lambda^2},$$

where, as in the case $q = 2$, the last equality follows by $8\cos(\theta)^4 = 3 + 4\cos(2\theta) + \cos(4\theta)$ and Riemann-Lebesgue lemma.

Finally, we consider the general case $q \geq 3$, excluding $d = 2, q = 4$. By (3.13),

$$\begin{aligned} V^E_{d,R}(q,\lambda) &= q! \int_0^{2R} j_d(\lambda r)^q r^{d-1} \left(\int_r^{2R} -W'_{d,R}(s) ds \right) dr \\ &= -q! \int_0^{2R} W'_{d,R}(s) \left(\int_0^s j_d(\lambda r)^q r^{d-1} dr \right) ds \\ &= \frac{q!}{\lambda^d} \int_0^{2R} -W'_{d,R}(s) \left(\int_0^{s\lambda} j_d(r)^q r^{d-1} dr \right) ds, \end{aligned}$$

integrating by parts in the second step. Since $|W'_{d,R}(r)|$ is bounded and supported by $[0, 2R]$, the thesis follows by dominated convergence and Lemma 3.2.6 (implying convergence of the inner integral on the right-hand side as $\lambda \to \infty$). □

3.3.2 Variance of Hyperspherical Polyspectra

The forthcoming computations are in fact close analogs of the ones in the previous paragraph, only in a perhaps less familiar geometrical setting. The change of variables of Lemma 3.3.1 takes on S^d the following form, with flat translations of $\mathbb{R}^d$ being replaced by isometries of S^d.

Lemma 3.3.2. *Let $R \in [0, \pi]$, $B_R = B^S(x_0, R) \subseteq S^d$ a ball of radius R centered at a (fixed, arbitrary) point $x_0 \in S^d$, and $f \in L^1(\mathbb{R})$; it holds*

$$\int_{B_R} \int_{B_R} f(d(x,y)) d\sigma_d(x) d\sigma_d(y) = \int_0^{\pi} f(r) \tilde{W}_{d,R}(r) \sin(r)^{d-1} dr, \tag{3.14}$$

where

$$\tilde{W}_{d,R}(r) = \omega_{d-1} \sigma_d(B(x,R) \cap B(y,R)), \quad d(x,y) = r, \tag{3.15}$$

whose value does not depend on the choice of points $x, y \in S^d$.

The proof is conceptually analogous to the one of Lemma 3.3.1. First, let us recall that the isometry group of S^d can be identified with that of linear orthogonal transformations of the ambient R^{d+1}, and it acts transitively on S^d. The volume σ_d and the spherical geodesic distance are invariant under isometries, that is $I_\# \sigma_d =$

$\sigma_d \circ I = \sigma_d$ and $d(a,b) = d(I(a), I(b))$ for any isometry $I : S^d \to S^d$ and points $a, b \in S^d$.

Proof. We first observe that for any isometry $I : S^d \to S^d$ it holds

$$\begin{aligned}\sigma_d(B(Ix,R) \cap B(Iy,R)) &= \sigma_d \{z : d(Ix,z), d(Iy,z) \le R\} \\ &= \sigma_d \left\{z : d(x, I^{-1}z), d(y, I^{-1}z) \le R\right\} = \sigma_d \{Iw : d(x,w), d(y,w) \le R\} \\ &= I_\#^{-1}\sigma_d \{w : d(x,w), d(y,w) \le R\} = \sigma_d(B(x,R) \cap B(y,R)),\end{aligned}$$

therefore the right-hand side only depends on $d(x,y)$, as it is a function of x, y invariant under isometries. In particular, $W_{d,R}(r)$ in (3.15) is well-defined.

For any two points $x, y \in S^d$ we can choose isometries $I_x, I_y : S^d \to S^d$ such that $I_x x_0 = x$ and $I_y x_0 = y$. Letting $z = I_y I_x^{-1} x_0$, by Fubini-Tonelli Theorem it holds

$$\begin{aligned}&\int_{S^d}\int_{S^d} \chi\{x \in B_R\}\chi\{y \in B_R\} f(d(x,y)) d\sigma_d(x) d\sigma_d(y) \\ &\qquad = \int_{S^d} d\sigma_d(z) f(d(x_0,z)) \int_{S^d} d\sigma_d(x) \chi\{x \in B_R\}\chi\{I_x z \in B_R\},\end{aligned}$$

where the inner integral can be rewritten as follows: since $I_x^{-1}x_0 = I_x^{-2}x$,

$$\begin{aligned}\int_{S^d} \chi\{x \in B_R\}\chi\{I_x z \in B_R\} d\sigma_d(x) &= \sigma_d\{x : d(x,x_0), d(I_x z, x_0) \le R\} \\ &= I_\#^2 \sigma_d \left\{x : d(x_0, I_x^{-1}x_0), d(z, I_x^{-1}x_0) \le R\right\} \\ &= \sigma_d \left\{I_x^{-2}x : d(x_0, I_x^{-1}x_0), d(z, I_x^{-1}x_0) \le R\right\} \\ &= \sigma_d\{w : d(x_0,w), d(z,w) \le R\},\end{aligned}$$

thus in particular it only depends on $d(x_0, z)$, thanks to the argument at the beginning of the proof. Since the volume element of S^d reduces to

$$\int_{S^d} h(d(x_0,x)) d\sigma_d(x) = \omega_{d-1} \int_0^\pi h(r) \sin(r)^{d-1} dr, \quad h \in L^1([0,\pi]),$$

for radial functions, combining the above equations concludes the proof. □

The function $\tilde{W}_{d,R} : [0,\pi] \to [0,\infty)$ is differentiable, and attains its maximum value in $\tilde{W}_{d,R}(0) = \omega_{d-1}\sigma_d(B(x_0,R))$ and minimum in $\tilde{W}_{d,R}(\pi)$ (which, unlike the Euclidean $W_{d,R}(2R)$, does not vanish in general). We can now proceed to complete the proof of Theorem 3.1.1 by reducing ourselves to the computations performed in the Euclidean case thanks to the following:

Lemma 3.3.3 (Hilb's Asymptotic Formula). *[Sze39, Theorem 8.21.12] Let $d \ge 2$, $\ell \in \mathbb{N}$. Given $a \in (0,\pi)$, for all $\theta \in (0,a)$ it holds:*

$$(\sin\theta)^\nu \mathcal{G}_{d,\ell}(\cos\theta) = \frac{2^\nu}{\binom{\ell+\nu}{\ell}} \left(\frac{\Gamma(\ell + d/2)}{L^\nu \ell!} \left(\frac{\theta}{\sin\theta}\right)^{1/2} J_\nu(L\theta) + \delta(\theta) \right), \tag{3.16}$$

where $L = \ell + \frac{d-1}{2}$ and $\delta(\theta) \le C_{d,a}(\chi_{(0,1/\ell)}(\theta)\theta^{\nu+2}\ell^\nu + \chi_{(1/\ell,a)}(\theta)\sqrt{\theta}\ell^{-3/2})$.

Corollary 3.3.4. *Let $d \geq 2$, $\ell \in \mathbb{N}$. For all $a \in (0, \pi)$ it holds, as $\ell \to \infty$,*

$$\int_0^a \mathcal{G}_{d,\ell}(\cos r)^q (\sin r)^{d-1}\, dr = (1+o(1)) \begin{cases} C_d \ell^{1-d} & q=2 \\ C\log(\ell)/\ell^2 & q=4, d=2 \\ I_q^d/\ell^d & \text{all other } d \geq 2, q \geq 3 \end{cases} \tag{3.17}$$

with I_q^d being as in Lemma 3.2.6 and $C, C_d > 0$ positive constants.

Remark 3.3.1. The limit case $a = \pi$ can be reduced to the one with $a = \pi/2$ splitting the integration domain in half and exploiting the following symmetry property: for all $d \geq 2$,

$$\mathcal{G}_{d,\ell}(t) = (-1)^\ell \mathcal{G}_{d,\ell}(-t), \quad t \in \mathbb{R}, \ell \in \mathbb{N}.$$

In particular, the integral in Corollary 3.3.4 identically vanishes if $a = \pi$ and both q, ℓ are odd. If this is not the case, then the thesis of Corollary 3.3.4 holds with an additional factor 2 on the right-hand side.

Corollary 3.3.4 actually collects computations already appeared in [MR15; MW11; Wig10]. Because of this and the similarity with computations in the previous paragraph, we only present a sketch of the proof.

Proof. We first observe that as $\ell \to \infty$,

$$\frac{\Gamma(\ell + d/2)}{L^\nu \ell!} = 1 + o(1), \quad \binom{\ell+\nu}{\ell} = \frac{\ell^\nu}{\nu!}(1 + o(1)). \tag{3.18}$$

When $q = 2$, Lemma 3.3.3 and (3.18) imply

$$\int_0^a \mathcal{G}_{d,\ell}(\cos r)^2 (\sin r)^{d-1} dr = \frac{4^\nu (\nu!)^2}{\ell^{2\nu}}(1+o(1)) \int_0^a J_\nu(Lr)^2 r dr + O\left(\frac{1}{\ell^{d-2}} \int_0^a |J_\nu(Lr)| \delta(r) \sin(r) dr\right).$$

The thesis now follows by applying Lemma 3.2.5 in the same fashion of the Euclidean case (the remainder term on the right-hand side is proved to be $O(\ell^{-d})$). The analysis of the case $d = 2, q = 4$ is similar, we refer again to [Wig10] for full details.

As for the remaining cases, by Lemma 3.3.3, (3.18) and the definition of j_d in (3.2),

$$\int_0^a \mathcal{G}_{d,\ell}(\cos r)^q (\sin r)^{d-1} dr = (1+o(1)) \int_0^a \left(\frac{r}{\sin r}\right)^{q\nu + q/2 - (d-1)} j_d(Lr)^q r^{d-1} dr + O\left(\frac{1}{\ell^{q\nu}} \int_0^a (\sin r)^{-q\nu} |J_\nu(Lr)|^{q-1} \delta(r) (\sin r)^{d-1} dr\right).$$

The remainder term can be shown to be $O(\ell^{-(d+2)\wedge(q(d/2-1/2)-1)}) = o(\ell^{-d})$ by means of Lemma 3.2.5 (changing variables and splitting the integration domain similarly to the previous case). The thesis now follows if

$$\int_0^a \left(\frac{r}{\sin r}\right)^{q\nu + q/2 - (d-1)} j_d(Lr)^q r^{d-1} dr = (1+o(1)) \int_0^\infty j_d(r)^q r^{d-1} dr, \quad \ell \to \infty,$$

which can be proved with the same computation exploiting integration by parts we used at the end of the proof of Theorem 3.1.1, Euclidean case, that is replacing $W_{d,R}$ of that computation with $\chi_{[0,a]}(r)(r/\sin r)^{q\nu+q/2-(d-1)}$. □

Proof of Theorem 3.1.1, Hyperspherical part. Let $R \in (0, \pi)$ (the case $R = \pi$ is dealt with similarly, by Remark 3.3.1). By Lemma 3.3.2,

$$V^S_{d,R}(q,\ell) = \mathrm{Var}\left(\int_{B_R} H_q(T_\ell(x))d\sigma_d(x)\right)$$
$$= q! \int_{B_R}\int_{B_R} \mathcal{G}_{d,\ell}(x,y)^q \, d\sigma_d(x)\, d\sigma_d(y) = q! \int_0^\pi \mathcal{G}_{d,\ell}(\cos r)^q \sin(r)^{d-1}\tilde{W}_{d,R}(r)dr.$$

Integrating by parts as in Section 3.3.1 we rewrite:

$$\int_0^\pi \mathcal{G}_{d,\ell}(\cos r)^q(\sin r)^{d-1}\tilde{W}_{d,R}(r)dr = \tilde{W}_{d,R}(\pi)\int_0^\pi \mathcal{G}_{d,\ell}(\cos r)^q(\sin r)^{d-1}dr$$
$$-\int_0^\pi\left(\int_0^r \mathcal{G}_{d,\ell}(\cos s)^q(\sin s)^{d-1}ds\right)\tilde{W}'_{d,R}(r)dr.$$

The thesis now follows from Corollary 3.3.4. We observe in particular that when we are not in the cases $q = 2$ nor in the one $d = 2, q = 4$, by Corollary 3.3.4 and dominated convergence we have

$$\ell^d \int_0^\pi \mathcal{G}_{d,\ell}(\cos r)^q(\sin r)^{d-1}\tilde{W}_{d,R}(r)dr$$
$$\xrightarrow{\ell\to\infty} I^d_q\tilde{W}_{d,R}(\pi) - I^d_q\int_0^\pi \tilde{W}_{d,R}(r)'dr = I^d_q\tilde{W}_{d,R}(0). \quad \square$$

CHAPTER 4

Asymptotic covariances for functionals of weakly stationary random fields

This chapter contains the paper [Mai24], "Asymptotic covariances for functionals of weakly stationary random fields". *Stoch. Proc. Appl.* (2024), 170, 104297.

4.1 Introduction

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a dimension $d \geq 1$ and let $A = (A_x)_{x \in \mathbb{R}^d}$ be a **measurable** random field on $(\Omega, \mathcal{F}, \mathbb{P})$, that is,

$$A\colon\ (\Omega, \mathcal{F}) \times \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))\,, \qquad (\omega, x) \longmapsto A_x(\omega),$$

is a measurable function, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sigma-algebra on $\mathbb{R}^d$. In addition, assume that A is **weakly stationary**, i.e.

$$\mathbb{E}\left[A_x\right] = \mathbb{E}[A_y], \qquad \operatorname{Cov}(A_x, A_y) = K(x-y), \qquad \forall x, y \in \mathbb{R}^d,$$

with **covariance function** $K : \mathbb{R}^d \to \mathbb{R}$. Note that under the above assumptions K is measurable and bounded, hence locally integrable. Therefore, we can define the **integral covariance function** $w_\cdot : \mathbb{R}_+ \to \mathbb{R}$ as

$$w_t := \int_{\{|z| \leq t\}} K(z) dz, \qquad t > 0, \tag{4.1}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

In this paper, we will focus on **functionals** of A of the form

$$\left(\int_{tD} A_x dx\right)_{D\in\mathcal{D}}, \qquad \text{as } t\to\infty, \tag{4.2}$$

where $\mathcal{D}$ is a suitable class of compact sets in $\mathbb{R}^d$. The well-posedness of (4.2) under the above assumptions is ensured by Proposition 4.2.1.

The asymptotic behavior of (4.2) has been extensively investigated in probability, and plays an important role in many applications. In Statistics, for instance, functionals of the form $\int_{tD} A_x dx$ are often involved in parameter estimation, and tD plays the role of an increasing observation window, see e.g. [DOV22; MY22].

Moreover, note that (4.2) may represent non-linear functionals of stationary Gaussian fields, that will be discussed in more detail in Section 4.4. The latter corresponds to the choice $A_x = \varphi(B_x)$, where $B = (B_x)_{x\in\mathbb{R}^d}$ is a stationary Gaussian field and $\varphi : \mathbb{R}\to\mathbb{R}$ is a function which varies depending on the framework. Just to mention two examples (among many), φ could be a polynomial in view of statistical applications, see e.g. [RST12; TV09], or an indicator function if one is interested in the geometry (and in particular the excursion volumes) of B, see e.g. [LRM23; LO14; MN24].

The goal of this paper will be to give the minimal assumptions on K and $\mathcal{D}$ in order to compute exactly the **asymptotic covariances** of (4.2), i.e. the limit

$$\lim_{t\to\infty} \operatorname{Cov}\left(\frac{\int_{tD} A_x dx}{r_t}, \frac{\int_{tL} A_y dy}{r_t}\right), \qquad D, L\in\mathcal{D}, \tag{4.3}$$

where $r_t\to\infty$ is chosen so that the limit (4.3) exists finite (and not identically equal to 0 for all $D, L\in\mathcal{D}$) for every $D, L\in\mathcal{D}$. The problem (4.3) often arises when studying the fluctuations of (4.2), in particular in the case of non-linear functionals of stationary Gaussian fields (see Section 4.4). Indeed, the first step to prove limit theorems for (4.2), i.e.

$$\frac{\int_{tD}(A_x - \mathbb{E}[A_x])dx}{r_t} \xrightarrow{\text{d}} Z(D), \qquad \text{as } t\to\infty,$$

where $Z(D)$ is a limiting random variable, is usually to study the asymptotic variance of $\int_{tD} A_x dx$, which corresponds to (4.3) when $D = L$. Moreover, to extend such results to a multi-dimensional setting, i.e.

$$\left(\frac{\int_{tD_1}(A_x - \mathbb{E}[A_x])dx}{r_t}, \dots, \frac{\int_{tD_n}(A_x - \mathbb{E}[A_x])dx}{r_t}\right) \xrightarrow{\text{d}} (Z(D_1), \dots, Z(D_n)), \qquad \text{as } t\to\infty,$$

it is often necessary to compute (4.3) for $D = D_i$ and $L = D_j$, for all $i, j = 1, \dots, n$.

A number of different methods have been used in the literature to compute (4.3), such as:

- Spectral representations and Fejer-type kernels or approximate identity for convolutions, see e.g. [ALS15].

- Spectral representations and Abelian-Tauberian theorems, see e.g. [LO13].
- When K is radial, the method of geometric probabilities, see e.g. [LRM23].
- A direct approach. By Fubini's theorem one has [1]

$$\mathrm{Cov}\left(\int_{tD} A_x dx, \int_{tL} A_y dy\right) = \int_{tD}\int_{tL} K(x-y)dxdy. \tag{4.4}$$

In this paper, we will focus on this latter direct approach, combining it with regularity conditions for cross covariograms developed in [Gal11] and standard properties of regularly varying functions. This method will allow to compute (4.3) under assumptions that encompass the classical ones found in the literature.

4.1.1 Classical assumptions, existing literature, motivating examples

To highlight the main novelty introduced by this paper, let us look at the classical assumptions considered in the literature.

The **first classical assumption**, probably the most popular in the literature (see e.g. [BH02; BM83; CNN20; KS18; LO14; NPP11]), is

$$K \in L^1(\mathbb{R}^d), \tag{4.5}$$

see also Section 4.4.1. In this case, it is well known that for every $D, L \subseteq \mathbb{R}^d$ compact, choosing $r_t = t^{d/2}$, we have

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2}}, \frac{\int_{tL} A_y dy}{t^{d/2}}\right) = \mathrm{Vol}(D\cap L)\int_{\mathbb{R}^d} K(z)dz, \tag{4.6}$$

where the latter follows by (4.4), a change of variable $z = x - y$ and dominated convergence. Then, we have two possibilities:

- If $\int_{\mathbb{R}^d} K(z)dz = 0$, then $r_t = t^{d/2}$ is not a correct choice for computing the asymptotic covariances, since the limit is identically 0 for every D, L compact (see also Example 4.4.1).
- If $\int_{\mathbb{R}^d} K(z)dz \neq 0$, then $r_t = t^{d/2}$ is a correct choice for computing the asymptotic covariances (4.3), as well as $r_t = t^{d/2}w_t^{1/2}$ (recall the definition (4.1) of w and note that $K \in L^1(\mathbb{R}^d)$), and we have

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2}w_t^{1/2}}, \frac{\int_{tL} A_y dy}{t^{d/2}w_t^{1/2}}\right) = \mathrm{Vol}(D\cap L). \tag{4.7}$$

The **second classical assumption**, which covers some of the cases where $K \notin L^1(\mathbb{R}^d)$, is K radial and regularly varying with index $-\beta \in (-d, 0)$ (see e.g. [ALO15;

[1] Here the equality is ensured by Proposition 4.2.1.

DM79; LO14; Ros60; Taq79]), namely

$$K(z) = k(|z|) = \frac{\ell(|z|)}{|z|^\beta}, \qquad \beta \in (0, d), \tag{4.8}$$

for some $k : \mathbb{R}_+ \to \mathbb{R}$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$ and $\ell : \mathbb{R}_+ \to \mathbb{R}$ is a slowly varying function, (i.e. ℓ is definitively positive and $\ell(rs)/\ell(r) \to 1$ for $s > 0$, as $r \to \infty$, see the seminal book [BGT89] or Section 4.2.1 for more details). In this case, it is a standard fact that for every $D, L \subseteq \mathbb{R}^d$ compact, choosing $r_t = t^d k(t)^{1/2} = t^{d-\frac{\beta}{2}}\ell(t)^{1/2}$, we have

$$\lim_{t\to\infty} \text{Cov}\left(\frac{\int_{tD} A_x dx}{t^d k(t)^{1/2}}, \frac{\int_{tL} A_y dy}{t^d k(t)^{1/2}}\right) = \lim_{t\to\infty} \int_D \int_L \frac{k(t|x-y|)}{k(t)} dx dy = \int_D \int_L |x-y|^{-\beta} dx dy, \tag{4.9}$$

where the latter follows by (4.4), Theorem 4.2.5 and dominated convergence (see also Remark 4.2.1). This means that $r_t = t^d k(t)^{1/2}$ is a correct choice to compute the asymptotic covariances (4.9). Moreover, by Theorem 4.2.4 we have [2]

$$w_t = \omega_{d-1} \int_0^t k(r) r^{d-1} dr \sim \frac{\omega_{d-1}}{(d-\beta)} t^d k(t) = \frac{\omega_{d-1}}{(d-\beta)} t^{d-\beta}\ell(t), \qquad \text{as } t \to \infty,$$

which implies that $r_t = t^{d/2} w_t^{1/2}$ is also a correct choice, and

$$\lim_{t\to\infty} \text{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_y dy}{t^{d/2} w_t^{1/2}}\right) = \frac{(d-\beta)}{\omega_{d-1}} \int_D \int_L |x-y|^{-\beta} dx dy. \tag{4.10}$$

Summarizing, and excluding the case $K \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} K(z)dz = 0$, we have:

1. Under both assumptions, $r_t = t^{d/2} w_t^{1/2}$ is a correct choice for computing the asymptotic covariances.

2. Under both assumptions w_t is a regularly varying function, with index: $\alpha = 0$ if $K \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} K(z)dz \neq 0$; $\alpha = d - \beta \in (0, d)$ if K is radial and regularly varying with index $-\beta \in (-d, 0)$.

The intuition we can develop from (i)-(ii) is: if w_t is regularly varying, then $r_t = t^{d/2} w_t^{1/2}$ is a correct choice for computing (4.3). This fact will be proved in our main result, Theorem 4.1.1, together with an explicit expression for the asymptotic covariances (see (4.16)), which generalizes (4.7) and (4.10) for D, L in a suitable class $\mathcal{D}$ of compact domains (see (4.13) and (4.14)).

It is important to note that, excluding the case $K \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} K(z)dz = 0$, our assumption "$w_t$ regularly varying" (see (4.15)) in Theorem 4.1.1 is **strictly more general than the two forementioned classical assumptions**. This is the main novelty provided by this paper.

In fact, in many cases (see e.g. [GMT24, Lemma 2.6], [LO13, Example 5], or [MN24, Section 6]) the covariance function K is neither integrable nor radial

[2]Here $\omega_0 = 2$ and ω_{d-1} is the surface area of the sphere S^{d-1} in $\mathbb{R}^d$ when $d \geq 2$.

and regularly varying, but the integrated covariance function w is regularly varying. Consider, for instance, the situations where K is not absolutely integrable, but $\lim_{t\to\infty} w_t = \int_{\mathbb{R}^d} K(z)dz \in (0,\infty)$; or $w_t \sim t^{\alpha}$, $\alpha > 0$, but K is not radial and regularly varying. Some explicit examples will be given in Section 4.4.3.

The case $K \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} K(z)dz = 0$ was not considered in the discussion above, since even if it falls under the first classical assumption, one is not able to compute the asymptotic covariances with (4.6). However, if w is regularly varying, Theorem 4.1.1 allows to derive an expression for the asymptotic covariances, with the rate $r_t = t^{d/2} w_t^{1/2}$. An explicit example (see Example 4.4.1) will be given in Section 4.4.1.

As mentioned above, the main novelty provided by Theorem 4.1.1 is the computation of (4.3) in many cases which do not fall under the two classical assumptions.

However, as we will better explain in Section 4.4, Theorem 4.1.1 can also be combined with the Peccati-Tudor theorem [PT05] to prove **new multi-dimensional central limit theorems for non-linear functionals of Gaussian fields**.

An example of this fact, that stands out for its importance in quantum mechanics, is the **Berry's random wave model**, a Gaussian field whose functionals have been extensively studied in the literature (see Section 4.4.3 and the references therein). Recently, it was proved in [MN24] that a large class of functionals of this field (and of many other fields satisfying a specific spectral condition) in the form (4.30) have Gaussian fluctuations. Nevertheless, the authors were not able to extend their result to a multi-dimensional central limit theorem (see also [MN24, Section 1.4]). This paper also aims to partially fill this gap. Indeed, as we will better explain in Section 4.4.3, by means of Theorem 4.1.1 we will prove new multi-dimensional central limit theorems for the Berry's random wave model (see in particular Example 4.4.3 and Example 4.4.5), using the fact that w is regularly varying (i.e. (4.15) holds) even if K does not always satisfy the forementioned classical assumptions.

4.1.2 Main result

In order to state our main result, we need to introduce some notations, quantities and assumptions.

We denote by "Vol" the Lebesgue measure and by "Per" the perimeter in generalized sense in $\mathbb{R}^d$, defined for $D \subseteq \mathbb{R}^d$ measurable as (see [Gal11])

$$\mathrm{Per}(D) = \sup\left\{ \int_D \mathrm{div}\varphi(x)\, dx : \varphi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d), \|\varphi\|_\infty \le 1 \right\},$$

where $C_c^1(\mathbb{R}^d, \mathbb{R}^d)$ is the set of continuously differentiable functions with compact support. Note that, denoting by $\mathcal{H}^{d-1}(\partial D)$ the $(d-1)$-Hausdorff measure of the topological boundary ∂D, we have $\mathrm{Per}(D) \le \mathcal{H}^{d-1}(\partial D)$, which becomes an equality under additional assumptions on D (see [Gal11] for more details).

Furthermore, for $f : \mathbb{R}^d \to \mathbb{R}$ and $r > 0$,

$$\int_{S^{d-1}} f(r\theta) d\theta$$

is the integral of $\theta \mapsto f(r\theta)$ (when defined) with respect to the uniform measure on S^{d-1} and $\omega_{d-1} = \int_{S^{d-1}} d\theta$ is the surface measure of S^{d-1}. (For $d = 1$ we use the formalism

$$\int_{S^0} f(r\theta)d\theta = f(r) + f(-r), \qquad \omega_0 = 2.) \tag{4.11}$$

For $D, L \in \mathcal{D}$, we consider the **cross covariogram** $g_{D,L} : \mathbb{R}^d \to \mathbb{R}$,

$$g_{D,L}(z) = \text{Vol}(D \cap (L + z)), \tag{4.12}$$

which, by Proposition 4.2.2, is such that all its directional derivatives

$$\frac{d}{dl} g_{D,L}(l\theta), \quad \theta \in S^{d-1},$$

are well-defined in $L^1(\mathbb{R}_+)$ and bounded by $\text{Per}(D) \wedge \text{Per}(L)$.

Finally, we will always assume to be in one of the following cases:

Case 1. K is **radial** and $\mathcal{D}$ is the class of all compact sets in $\mathbb{R}^d$ with **finite perimeter**, i.e. for some $k : \mathbb{R}_+ \to \mathbb{R}$ we have

$$K(z) = k(|z|), \quad \mathcal{D} = \{D \subseteq \mathbb{R}^d : D \text{ compact}, \text{Per}(D) < \infty\}. \tag{4.13}$$

If $d = 1$, K is always radial.

Case 2. $\mathcal{D} = \mathcal{D}_{x_0}$ is the class of the **closed balls centered** at a fixed point $x_0 \in \mathbb{R}^d$, namely

$$\mathcal{D} = \left\{ \left\{ x \in \mathbb{R}^d : |x - x_0| \le r \right\} : r \in \mathbb{R}_+ \right\}. \tag{4.14}$$

We are now ready to state our main result.

Theorem 4.1.1. *Let $\mathcal{D}$ be a collection of compact sets in $\mathbb{R}^d$ and $A = (A_x)_{x \in \mathbb{R}^d}$ be a measurable, weakly stationary random field with covariance function $K : \mathbb{R}^d \to \mathbb{R}$. Furthermore, let K and $\mathcal{D}$ satisfy the assumption (4.13) or the assumption (4.14), and assume that w in (4.1) is **regularly varying** with index $\alpha \in (-1, d]$, that is*

$$w_t = \ell(t)t^\alpha \quad \forall t \in \mathbb{R}_+ \tag{4.15}$$

where $\ell : \mathbb{R}_+ \to \mathbb{R}$ is slowly varying (see Definition 4.2.1 and Lemma 4.2.3).

Then, for all $D, L \in \mathcal{D}$, we have as $t \to \infty$

$$\text{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_x dx}{t^{d/2} w_t^{1/2}} \right) \to \frac{1}{\omega_{d-1}} \int_{S^{d-1}} d\theta \int_0^\infty dl \left(-\frac{d}{dl} g_{D,L}(l\theta) \right) l^\alpha. \tag{4.16}$$

Remark 4.1.1. (Restriction of the parameter α) Note that in Theorem 4.1.1 we consider $\alpha \in (-1, d]$ instead of $\alpha \in \mathbb{R}$. This happens for different reasons:

- $\alpha > -1$ is assumed for technical reasons. For example, if $D = L = \{|z| \le 1\} \subseteq$

$\mathbb{R}^d$, we have for all $\theta \in S^{d-1}$ and some constant $c_d > 0$

$$g_{D,D}(l\theta) = c_d \int_{\frac{l}{2}}^{1} \left(1 - r^2\right)^{\frac{d-1}{2}} dr \ \mathbf{1}_{[0,2]}(l),$$

$$-\frac{d}{dl} g_{D,D}(l\theta) = \frac{c_d}{2} \left(1 - \left(\frac{l}{2}\right)^2\right)^{\frac{d-1}{2}} \mathbf{1}_{[0,2]}(l),$$

implying that the RHS of (4.16) is not a finite integral if $\alpha \le -1$. Anyway, if (4.15) holds with $\alpha \le -1$, another scaling could yield a different limit.

- Since $|K(z)| \le K(0) < \infty$ for every $z \in \mathbb{R}^d$, we have

$$|w_t| \le \int_{\{|z| \le t\}} |K(z)| dz \le \mathrm{const}\, K(0) t^d,$$

implying that w_t can not be regularly varying with index $\alpha > d$.

Remark 4.1.2. (Comparison to known covariance structures) Before going on, it is worth discussing analogies and differences with some known covariance structures, highlighting when (4.16) can be seen as their generalization. First of all, note that when $d = 1$, $D = [0, s]$ and $L = [0, r]$ with $r > s > 0$, then for $l > 0$ we have

$$\begin{aligned} g_{D,L}(l) &= (s-l)\mathbf{1}_{[0,s]}(l) \\ g_{D,L}(-l) &= s\mathbf{1}_{[0,r-s]}(l) + (r-l)\mathbf{1}_{[r-s,r]}(l) \\ -\frac{d}{dl}(g_{D,L}(l) + g_{D,L}(-l)) &= \mathbf{1}_{[0,s]}(l) + \mathbf{1}_{[r-s,r]}(l) \end{aligned}$$

and the RHS of (4.16) becomes

$$\frac{1}{2}\left(\int_0^s l^\alpha dl + \int_{r-s}^r l^\alpha dl\right) = \frac{1}{2(\alpha+1)}\left(s^{\alpha+1} + r^{\alpha+1} - |r-s|^{\alpha+1}\right). \tag{4.17}$$

In (4.17) we observe the covariance structure of a fractional Brownian motion $(B_s^H)_{s>0}$ with Hurst index $H = \frac{\alpha+1}{2}$. Note that this is also the covariance structure of other (non-Gaussian) stochastic processes, like the Hermite process (see e.g. [Tud13, Section 3]).

For $\alpha = 0$, the RHS of (4.16) has the special form

$$\mathrm{Vol}(D \cap L), \tag{4.18}$$

which is the covariance structure of a Gaussian noise (a set-indexed generalization of the Brownian motion, see e.g. [AT07, Section 1.4.3]). Note that if D is fixed and $u, v > 0$, (4.16) becomes

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tDu^{1/d}} A_x dx}{t^{d/2}}, \frac{\int_{tDv^{1/d}} A_y dy}{t^{d/2}}\right) = \mathrm{Vol}(D)\ \min\{u, v\},$$

which is, up to a scaling factor, the covariance structure of a Brownian motion (see also (4.37)). This rescaling is used in the literature to prove multidimensional central limit theorems (to a Brownian motion) for functionals of Gaussian fields, see e.g.

[IL89, Theorem 2.3.1] and the discussion in Section 4.4.1.

For $\alpha > 0$, the RHS of (4.16) coincides with

$$\frac{1}{\omega_{d-1}}\int_{S^{d-1}} d\theta \int_0^\infty dl \left(-\frac{d}{dl}g_{D,L}(l\theta)\right) l^\alpha = \frac{\alpha}{\omega_{d-1}}\int_{S^{d-1}} d\theta \int_0^\infty dl g_{D,L}(l\theta) l^{\alpha-1} \quad (4.19)$$

$$= \frac{\alpha}{\omega_{d-1}}\int_{\mathbb{R}^d} g_{D,L}(z)|z|^{\alpha-d}dz, \quad (4.20)$$

where the first equality follows integrating by parts and the second passing from polar to standard coordinates. Note that

$$\int_{\mathbb{R}^d} g_{D,L}(z)|z|^{\alpha-d}dz = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \mathbf{1}_D(x)\mathbf{1}_L(x-z)dx\right)|z|^{\alpha-d}dz = \int_D\int_L |x-y|^{\alpha-d}dxdy.$$

Thus, (4.19) is exactly the RHS of (4.10) with $\beta = d-\alpha$, which is perfectly consistent with the discussion in Section 4.1.1.

Finally, choosing $d \geq 2$, $D = [0, \underline{s}]$ and $L = [0, \underline{r}]$, with $\underline{r}, \underline{s} \in \mathbb{R}^d$, we have:

- If $\alpha = 0$, then (4.18) is the covariance structure of a d-dimensional Brownian sheet.
- If $\alpha \neq 0$, then the RHS of (4.16) is not the covariance structure of a d-dimensional fractional Brownian sheet (see e.g. [Wan07] for the definition).

4.1.3 Plan of the paper

The rest of the paper is organized as follows. In Section 4.2 we ensure the well-posedness of (4.2), prove (4.4) and give some preliminaries about cross covariograms and regularly varying functions. In Section 4.3 we prove Theorem 4.1.1. In Section 4.4 we apply Theorem 4.1.1 to prove multi-dimensional central limit theorems for non-linear functionals of stationary Gaussian fields, giving several examples. Finally, in Section 4.5 we show that (4.3) for A with continuous covariance function K can be reduced to the same problem for a radial, continuous covariance function K_{iso}, providing a class of non-Gaussian, non-stationary, weakly stationary random fields where this reduction principle can be applied.

4.2 Preliminaries for the proof of Theorem 4.1.1

The goal of this section is to ensure the well-posedness of (4.2), prove (4.4) and give preliminary results on cross covariograms and regularly varying functions.

4.2.1 Well-posedness of (4.2) and proof of (4.4)

The following proposition explains why (4.2) is well posed and (4.4) holds.

Proposition 4.2.1. *Fix $d \geq 1$ and let $A = (A_x)_{x\in\mathbb{R}^d}$ be a measurable (in the sense that $(\omega, x) \mapsto A_x(\omega)$ is measurable), weakly stationary random field, with covariance function $K : \mathbb{R}^d \to \mathbb{R}$. Then:*

1. *We have*
$$\mathbb{P}\left(\int_D |A_x|dx < \infty, \quad \forall\, D \subseteq \mathbb{R}^d \text{ compact}\right) = 1$$
which implies that (4.2) *is well defined.*

2. *For every* $D, L \subseteq \mathbb{R}^d$ *compact we have*
$$\operatorname{Cov}\left(\int_D A_x dx, \int_L A_y dy\right) = \int_D \int_L K(x-y) dx dy,$$
which implies (4.4).

Proof. For simplicity, since A is weakly stationary, let us use the notation $m := \mathbb{E}[A_x]$, $\forall\, x \in \mathbb{R}^d$. Note that the function $(\omega, x, y) \mapsto |A_x(\omega) - m||A_y(\omega) - m|$ is measurable because we are assuming that $(\omega, x) \mapsto A_x(\omega)$ is measurable. Moreover, we have

$$|A_x - m||A_y - m| \leq \frac{1}{2}\left(|A_x - m|^2 + |A_y - m|^2\right),$$

which implies

$$\mathbb{E}\left[\int_D \int_L |A_x - m|\, |A_y - m| dx dy\right] \leq \frac{1}{2}\left(\operatorname{Var}(A_x) + \operatorname{Var}(A_y)\right) \operatorname{Vol}(D)\operatorname{Vol}(L) \tag{4.21}$$
$$= K(0)\operatorname{Vol}(D)\operatorname{Vol}(L) < \infty. \tag{4.22}$$

In particular, for $D \subseteq \mathbb{R}^d$ compact we have

$$\mathbb{E}\left[\int_D |A_x - m| dx\right] \leq \mathbb{E}\left[\left(\int_D |A_x - m| dx\right)^2\right]^{1/2}$$
$$= \mathbb{E}\left[\int_D \int_D |A_x - m|\, |A_y - m| dx dy\right]^{1/2} < \infty,$$

implying that for almost every $\omega \in \Omega$ the function $x \mapsto A_x(\omega)$ is integrable on the compact D. To prove that for almost every $\omega \in \Omega$ the function $x \mapsto A_x(\omega)$ is integrable on every compact set of $\mathbb{R}^d$ (i.e. the almost sure local integrability stated in (i)), note that as $n \to \infty$

$$\Omega_n := \left\{\omega \in \Omega : \int_{\{|x| \leq n\}} |A_x(\omega)| dx < \infty\right\}$$
$$\searrow \Omega_\infty := \left\{\omega \in \Omega : \int_D |A_x(\omega)| dx < \infty, \forall D \subseteq \mathbb{R}^d \text{ compact}\right\}.$$

Thus, since $\mathbb{P}(\Omega_n) = 1$ for every $n \in \mathbb{N}$, we have $\mathbb{P}(\Omega_\infty) = 1$ and (i) is proved.

Since (4.2) is now well defined by (i) and (4.21) holds, by Fubini-Tonelli we obtain

$$\operatorname{Cov}\left(\int_D A_x dx, \int_L A_y dy\right) = \mathbb{E}\left[\int_D \int_L (A_x - m)(A_y - m) dx dy\right]$$
$$= \int_D \int_L \operatorname{Cov}(A_x, A_y) dx dy = \int_D \int_L K(x-y) dx dy,$$

for every D, L compact domains, which concludes the proof of (ii). □

4.2.2 Cross covariograms

First of all, let us recall the definition (4.12) of the cross covariogram $g_{D,L}$ of two compact sets D, L in $\mathbb{R}^d$. In particular, when $D = L$, denote by g_D the **covariogram** of D

$$g_D(z) := g_{D,D}(z) = \mathrm{Vol}(D \cap (D+z)), \qquad z \in \mathbb{R}^d.$$

Moreover, for a bounded set $E \subseteq \mathbb{R}^d$, recall the definition of diameter

$$\mathrm{diam}(E) := \sup_{x,y \in E} |x-y| < \infty. \tag{4.23}$$

Let us now list some properties related to cross covariograms, which are easily derived from the results in [Gal11] and will be needed in the sequel.

Proposition 4.2.2. *Consider D, L compact sets in $\mathbb{R}^d$. Then we have:*

1. $g_{D,L}(z) = g_{L,D}(-z)$ *and in particular g_D is symmetric.*

2. *For every $z, h \in \mathbb{R}^d$*

$$|g_{D,L}(z+h) - g_{D,L}(z)| \leq 2\min\{g_D(0) - g_D(h), g_L(0) - g_L(h)\}.$$

3. $g_{D,L}(z) = 0$ *if* $|z| > \mathrm{diam}(D \cup L)$. *In particular, since* $\mathrm{diam}(D \cup L) < \infty$, $g_{D,L}$ *has compact support.*

4. $\mathrm{Per}(D) < \infty$ *if and only if g_D is Lipschitz continuous (with Lipschitz constant* $\mathrm{Per}(D)/2$*).*

5. *If* $\mathrm{Per}(D) < \infty$ *or* $\mathrm{Per}(L) < \infty$, *then $g_{D,L}$ is Lipschitz with Lipschitz constant* $\mathrm{Per}(D) \wedge \mathrm{Per}(L)$. *Moreover, all the directional derivatives of $g_{D,L}$ exist almost everywhere and are bounded by* $\mathrm{Per}(D) \wedge \mathrm{Per}(L)$. *In addition, for every $\theta \in S^{d-1}$ the function $r \to g_{D,L}(r\theta)$ is absolutely continuous and we have*

$$g_{D,L}(r\theta) = \int_r^{\mathrm{diam}(D\cup L)} \left(-\frac{d}{dl} g_{D,L}(l\theta)\right) dl, \qquad \forall r > 0. \tag{4.24}$$

Proof. (i) follows from $\mathrm{Vol}(D \cap (L+z)) = \mathrm{Vol}((D-z) \cap L)$ and (ii) from

$$|\mathrm{Vol}(D \cap (L+z+h)) - \mathrm{Vol}(D \cap (L+z))| \leq \int_{\mathbb{R}^d} |\mathbf{1}_{L+z}(x) - \mathbf{1}_{L+z+h}(x)| dx$$
$$= \int_{\mathbb{R}^d} |\mathbf{1}_{L+z}(x) - \mathbf{1}_{L+z+h}(x)|^2 dx$$
$$= \int_{\mathbb{R}^d} \mathbf{1}_{L+z}(x) + \mathbf{1}_{L+z+h}(x) - 2\mathbf{1}_{L+z}(x)\mathbf{1}_{L+z+h}(x)\, dx = 2(g_L(0) - g_L(h))$$

and using (i). (iii) simply follows by definition, since $g_{D,L}(z) \neq 0$ is possible only if $z \in D - L \subseteq (D \cup L) - (D \cup L)$, and (iv) is proved in [Gal11, Theorem 14]. Regarding (v), the fact that $g_{D,L}$ is Lipschitz with Lipschitz constant $\mathrm{Per}(D) \wedge \mathrm{Per}(L)$ easily follows from (iv) and (ii). As a consequence, for every fixed $\theta \in S^{d-1}$ the function

$$l \mapsto g_{D,L}(l\theta)$$

is Lipschitz, its derivative exists almost everywhere and is bounded by $\mathrm{Per}(D)\wedge\mathrm{Per}(L)$. Moreover, the fact that every Lipschitz function is absolutely continuous, together with (iii), implies (4.24).

□

4.2.3 Regularly varying functions

Definition 4.2.1. A measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ is said **regularly varying** if h is positive on $[a, \infty)$ for some $a > 0$ and if we have, for all $l > 0$

$$\frac{h(tl)}{h(t)} \to g(l) \qquad \text{as } t \to \infty.$$

In particular, if $g \equiv 1$ then h is said **slowly varying**.

Definition 4.2.1 is one of the equivalent definitions used for regularly varying functions, as explained in Lemma 4.2.3 (see [BGT89, Section 1.4]).

Lemma 4.2.3. *Consider a measurable function $h : \mathbb{R}_+ \to \mathbb{R}$. Then the following statements are equivalent:*

1. *h is regularly varying.*
2. *h is regularly varying with limit $g(l) = l^\alpha$, for some $\alpha \in \mathbb{R}$.*
3. *$h(l) = \ell(l)l^\alpha$ for some $\alpha \in \mathbb{R}$ and $\ell : \mathbb{R}_+ \to \mathbb{R}$ slowly varying.*

*If (one of) the three statements hold, we say that h is **regularly varying with index** α.*

The following proposition says that the condition (4.15) in Theorem 4.1.1 is more general than the classical assumption "$K(z) = k(|z|)$ with k regularly varying with index $\beta \in (-d, 0)$", discussed in Section 4.1.1.

Proposition 4.2.4. *[BGT89, Proposition 1.5.11] If $k : \mathbb{R}_+ \to \mathbb{R}$ is regularly varying with index $-\beta \in (-d, 0)$ and locally bounded on $\mathbb{R}_+$, then*

$$\frac{t^d k(t)}{\int_0^t k(r) r^{d-1} dr} \to (d - \beta) \qquad \textit{as } t \to \infty$$

and $w_t := \omega_{d-1} \int_0^t k(r) r^{d-1} dr$ is regularly varying with index $\alpha = d - \beta$.

A fundamental tool for the proof of Theorem 4.1.1 will be Potter's theorem (see e.g. [BGT89, Theorem 1.5.6]).

Theorem 4.2.5. *(Potter's Theorem) If $h : \mathbb{R}_+ \to \mathbb{R}$ is regularly varying with index $\alpha \in \mathbb{R}$, then for every $A > 1$, $\delta > 0$, $\exists\, X = X(h, A, \delta) > 0$ such that*

$$|h(lt)/h(t)| \le A \max\{l^{\alpha+\delta}, l^{\alpha-\delta}\} \qquad \forall l > \frac{X}{t}, \quad \forall t > X.$$

Remark 4.2.1. **(Justifying equation (4.9))** Note that if $k : \mathbb{R}_+ \to \mathbb{R}$ is measurable, bounded and regularly varying with index $-\beta \in (-d, 0)$, by Theorem 4.2.5 we have for $D, L \subseteq \mathbb{R}^d$ compact sets

$$\mathbf{1}_{\{|x-y|\geq X/t\}}\mathbf{1}_D(x)\mathbf{1}_L(y)\frac{k(t|x-y|)}{k(t)} \leq c\,\mathbf{1}_D(x)\mathbf{1}_L(y)\max\left\{|x-y|^{-\beta-\delta}, |x-y|^{-\beta+\delta}\right\}$$

where $X, c > 0$ are suitable constants. Moreover, since $\beta \in (0, d)$, choosing $\delta > 0$ small enough, the function on the RHS is integrable in $\mathbb{R}^{2d}$. Therefore, by dominated convergence theorem, we have as $t \to \infty$

$$\int_D \int_L \mathbf{1}_{\{|x-y|\geq X/t\}} \frac{k(t|x-y|)}{k(t)} dxdy \to \int_D \int_L |x-y|^{-\beta} dxdy.$$

This last fact, together with

$$\int_D \int_L \mathbf{1}_{\{|x-y|\leq X/t\}} \frac{k(t|x-y|)}{k(t)} dxdy \leq c\,\frac{1}{k(t)t^d} = \frac{1}{\ell(t)t^{d-\beta}} \to 0, \qquad \text{as } t \to \infty,$$

concludes the proof of (4.9).

Corollary 4.2.6. *Consider $K : \mathbb{R}^d \to \mathbb{R}$ measurable, bounded function and $w : \mathbb{R}_+ \to \mathbb{R}$ as defined in (4.1). Assume that w is regularly varying with index $\alpha > -1$. Then for every $U \in (0, \infty)$*

$$\int_0^U |w_{lt}/w_t| dl \to \int_0^U l^\alpha dl \qquad \text{as } t \to \infty.$$

Proof. First of all, observe that by Potter's Theorem 4.2.5, choosing $A = 2$ and $\delta = \frac{\alpha+1}{2} > 0$, $\exists\, X = X(w, A, \delta) = X(K, \alpha) > 0$ such that

$$|w_{lt}/w_t| \leq 2\max\left\{l^{\frac{3\alpha+1}{2}}, l^{\frac{\alpha-1}{2}}\right\} \qquad \forall l > \frac{X}{t}, \quad \forall t > X, \tag{4.25}$$

with $l \mapsto \max\{l^{\frac{3\alpha+1}{2}}, l^{\frac{\alpha-1}{2}}\}$ integrable on $[0, U]$ because $\alpha > -1$. Moreover, since w is of the form (4.1), for $0 < l < X/t$ we have

$$|w_{lt}/w_t| \leq \frac{\max_{x\in\mathbb{R}^d}|K(x)|}{|w_t|}\omega_{d-1}X^d \quad \forall\, 0 < l < X/t.$$

Putting all together, we obtain

$$\int_0^U |w_{lt}/w_t| dl = \int_0^{X/t} |w_{lt}/w_t| dl + \int_{X/t}^U |w_{lt}/w_t| dl.$$

Note that $w_t = \ell(t)t^\alpha$ with $\alpha > -1$, so $\ell(t)t^{1+\alpha} \to \infty$ and

$$\int_0^{X/t} |w_{lt}/w_t| dl \leq \frac{\max_{x\in\mathbb{R}^d}|K(x)|}{t|w_t|}\omega_{d-1}X^{d+1} = \frac{\max_{x\in\mathbb{R}^d}|K(x)|}{|\ell(t)t^{1+\alpha}|}\omega_{d-1}X^{d+1} \to 0.$$

Moreover, by (4.25) and dominated convergence theorem we have

$$\int_{X/t}^U |w_{lt}/w_t| dl \to \int_0^U l^\alpha dl,$$

which concludes the proof. □

4.3 Proof of Theorem 4.1.1

Proof. (of Theorem 4.1.1) By Proposition 4.2.1 and the change of variable $x = x$, $y = x - z$, we have

$$\begin{aligned}
&\mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_y dy}{t^{d/2} w_t^{1/2}}\right) && (4.26)\\
&= \int_{tD}\int_{tL} K(x-y)\frac{dxdy}{t^d w_t}\\
&= \int_{\mathbb{R}^d} K(z) g_{tD,tL}(z)\frac{dz}{t^d w_t}\\
&= \int_{\{|z|\le \mathrm{diam}(D\cup L)t\}} K(z) g_{D,L}\left(\frac{z}{t}\right)\frac{dz}{w_t}
\end{aligned}$$

where $g_{D,L}$ is defined in (4.12), $\mathrm{diam}(D\cup L)$ is defined in (4.23) and the last equality follows from (iii) of Proposition 4.2.2 and

$$g_{tD,tL}(z) = \mathrm{Vol}(tD\cap(tL+z)) = t^d \mathrm{Vol}\left(D\cap\left(L+\frac{z}{t}\right)\right) = t^d g_{D,L}\left(\frac{z}{t}\right).$$

Passing to polar coordinates (if $d = 1$, recall the notation (4.11)), we have

$$\begin{aligned}
&\int_{\{|z|\le \mathrm{diam}(D\cup L)t\}} K(z) g_{D,L}\left(\frac{z}{t}\right) dz\\
&= \int_{S^{d-1}} d\theta \int_0^{\mathrm{diam}(D\cup L)t} dr\, r^{d-1} K(r\theta) g_{D,L}\left(\frac{r}{t}\theta\right),
\end{aligned}$$

and since $D, L \in \mathcal{D}$ have finite perimeter in both ***Case 1.*** (4.13) and ***Case 2.*** (4.14), by (v) in Proposition 4.2.2 the covariance (4.26) becomes

$$\begin{aligned}
&\mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_y dy}{t^{d/2} w_t^{1/2}}\right)\\
&= w_t^{-1}\int_{S^{d-1}} d\theta \int_0^{\mathrm{diam}(D\cup L)} dl \left(-\frac{d}{dl}(g_{D,L}(l\theta))\right)\left(\int_0^{tl} r^{d-1} K(r\theta) dr\right).
\end{aligned}$$

Now we distinguish the two cases in the statement of Theorem 4.1.1.

Case 1. If $K(z) = k(|z|)$ for some $k : \mathbb{R}_+ \to \mathbb{R}$, then $K(r\theta) = k(r)$ and we have (recall the definition (4.1) of w)

$$\begin{aligned}
&\mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_y dy}{t^{d/2} w_t^{1/2}}\right)\\
&= \frac{1}{\omega_{d-1}}\int_{S^{d-1}} d\theta \int_0^{\mathrm{diam}(D\cup L)} dl \left(-\frac{d}{dl}(g_{D,L}(l\theta))\right)\frac{w_{lt}}{w_t}
\end{aligned}$$

Case 2. If D, L are balls centered in the same point $x_0 \in \mathbb{R}^d$, then $g_{D,L}(r\theta) = g_{D,L}(r\theta')$ for every $\theta, \theta' \in S^{d-1}$ and by a Fubini-Tonelli argument (and (v) in Proposition 4.2.2) we have, for all $\theta' \in S^{d-1}$

$$\begin{aligned} &\operatorname{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} A_y dy}{t^{d/2} w_t^{1/2}}\right) \\ &= \int_0^{\operatorname{diam}(D\cup L)} \left(-\frac{d}{dl}(g_{D,L}(l\theta'))\right) \frac{w_{lt}}{w_t} dl \\ &= \frac{1}{\omega_{d-1}} \int_{S^{d-1}} d\theta \int_0^{\operatorname{diam}(D\cup L)} dl \left(-\frac{d}{dl}(g_{D,L}(l\theta))\right) \frac{w_{lt}}{w_t} \end{aligned} \tag{4.27}$$

which is equal to what we obtained in Case 1.

Since the expression for the covariance is the same in both cases, to prove (4.16) (and conclude the proof of Theorem 4.1.1) we only need to show

$$\begin{aligned} &\int_{S^{d-1}} d\theta \int_0^{\operatorname{diam}(D\cup L)} dl \left(-\frac{d}{dl}(g_{D,L}(l\theta))\right) \frac{w_{lt}}{w_t} \\ &\to \int_{S^{d-1}} d\theta \int_0^{\operatorname{diam}(D\cup L)} dl \left(-\frac{d}{dl}(g_{D,L}(l\theta))\right) l^\alpha \quad \text{as } t \to \infty, \end{aligned} \tag{4.28}$$

where α is the index of regular variation of w_t. First of all, note that $w_{lt}/w_t \to l^\alpha$ for every $l > 0$, because w is regularly varying with index α, implying the point-wise convergences (for almost every $l > 0, \theta \in S^{d-1}$)

$$F_t(l,\theta) := \left(-\frac{d}{dl}(g_{D,L}(l\theta))\right) \frac{w_{lt}}{w_t} \to F(l,\theta) := \left(-\frac{d}{dl}(g_{D,L}(l\theta))\right) l^\alpha, \qquad \text{as } t \to \infty$$

and

$$M_t(l) := \operatorname{Per}(D) \wedge \operatorname{Per}(L)\, |w_{lt}/w_t| \to M(l) := \operatorname{Per}(D) \wedge \operatorname{Per}(L)\, l^\alpha, \qquad \text{as } t \to \infty.$$

Moreover, by Proposition 4.2.2 the inequality

$$|F_t(l,\theta)| = \left|\left(-\frac{d}{dl}(g_{D,L}(l\theta))\right) \frac{w_{tl}}{w_t}\right| \le M_t(l) := \operatorname{Per}(D) \wedge \operatorname{Per}(L)\, |w_{lt}/w_t|$$

holds for almost every $l > 0, \theta \in S^{d-1}$, and by Corollary 4.2.6 we have

$$\int_{S^{d-1}} d\theta \int_0^{\operatorname{diam}(D\cup L)} dl\, M_t(l) \to \int_{S^{d-1}} d\theta \int_0^{\operatorname{diam}(D\cup L)} dl M(l).$$

Putting all together, by the generalized dominated convergence theorem we obtain

$$\int_{S^{d-1}} d\theta \int_0^{\operatorname{diam}(D\cup L)} dl\, F_t(l,\theta) dl \to \int_{S^{d-1}} d\theta \int_0^{\operatorname{diam}(D\cup L)} dl\, F(l,\theta), \qquad \text{as } t \to \infty,$$

which is exactly (4.28). $\square$

4.4 Non-linear functionals of stationary Gaussian fields

In this section, we show how Theorem 4.1.1 can be applied in the setting of non-linear functionals of stationary Gaussian fields to obtain multi-dimensional limit theorems. Through several examples, we will also compare the results obtained to the ones found in the existing literature.

Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a continuous, centered, stationary Gaussian field with $B_x \sim N(0,1)\ \forall\, x \in \mathbb{R}^d$ and (continuous) covariance function

$$C : \mathbb{R}^d \to \mathbb{R}, \qquad \mathrm{Cov}(B_x, B_y) = C(x-y), \quad |C(z)| \le C(0) = 1. \tag{4.29}$$

Denote again by $\mathcal{D}$ a collection of compact sets in $\mathbb{R}^d$ and consider the non-linear functional[3] of B

$$\int_{tD} \varphi(B_x)dx, \qquad D \in \mathcal{D}, \quad t > 0, \tag{4.30}$$

where $tD := \{tx : x \in D\}$, $\varphi : \mathbb{R} \to \mathbb{R}$ is not constant (to avoid trivialities), $\varphi \in L^2(\mathbb{R}, \gamma(dx))$, $\int_{\mathbb{R}} \varphi(x)\gamma(dx) = 0$ and γ is the standard Gaussian measure on $\mathbb{R}$. Then, we consider the L^2-decomposition of φ (see e.g. [NP12, Section 1.4])

$$\varphi = \sum_{q=1}^{\infty} a_q H_q,$$

where H_q is the q-th Hermite polynomial and

$$\|\varphi\|^2_{L^2(\mathbb{R},\gamma(dx))} = \sum_{q=1}^{\infty} q! a_q^2 < \infty. \tag{4.31}$$

Therefore, $A_x := \varphi(B_x)$ can be expressed as

$$A_x := \varphi(B_x) = \sum_{q=R}^{\infty} a_q H_q(B_x), \qquad a_R \neq 0,$$

where $R = \inf\{q \ge 1 : a_q \neq 0\} < \infty$ denotes the **Hermite rank** of φ and the equality holds in $L^2(\Omega)$ sense. Moreover, we have the isometry property (see e.g. [NP12, Section 1.4])

$$\mathbb{E}[H_q(B_x)H_r(B_y)] = q! C^q(x-y)\delta_{qr}$$

where δ_{qr} is the Kronecker delta. As a consequence, the covariance function $K(x-y) = \mathrm{Cov}(A_x, A_y)$ of $A = (A_x)_{x\in\mathbb{R}^d}$ is

$$K(z) = \sum_{q=R}^{\infty} q! a_q^2 C^q(z). \tag{4.32}$$

Note that A is obviously weakly stationary and K is continuous, because uniform limit of continuous functions (thanks to (4.31) and (4.29)). Moreover, w in (4.1) in

[3] (4.30) is always well-posed in the L^2-sense, see e.g. [MN24, Proposition 3].

this case is

$$w_t = \int_{\{|z|\le t\}} K(z)dz = \sum_{q=R}^{\infty} a_q^2 w_{q,t}, \qquad t > 0, \tag{4.33}$$

where we additionally introduced the notation

$$w_{q,t} = q! \int_{\{|z|\le t\}} C^q(z)dz.$$

Now we can finally understand what new results we obtain in this setting applying Theorem 4.1.1, dividing the study in different cases.

4.4.1 The Breuer-Major case.

If $C \in L^R(\mathbb{R}^d)$, we are in the Breuer-Major case. Since $C \in L^R(\mathbb{R}^d)$ implies $K \in L^1(\mathbb{R}^d)$ (see (4.29) and (4.32)), reasoning as in (4.6) we get

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD}\varphi(B_x)dx}{t^{d/2}}, \frac{\int_{tL}\varphi(B_y)dy}{t^{d/2}}\right) = \mathrm{Vol}(D\cap L)\int_{\mathbb{R}^d} K(z)dz. \tag{4.34}$$

Note that if D is fixed and $u, v \in [0,1]$, we have

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tDu^{1/d}}\varphi(B_x)dx}{t^{d/2}}, \frac{\int_{tDv^{1/d}}\varphi(B_y)dy}{t^{d/2}}\right) = \mathrm{Vol}(D)\ \min\{u,v\}\int_{\mathbb{R}^d} K(z)dz,$$

which is, up to a scaling factor, the covariance function of a Brownian motion (see also (4.37)). Moreover, we have the following fundamental result, proved for the first time in the discrete setting by Breuer and Major in their seminal paper [BM83], and extended to different settings by several authors, see e.g. [BH02], [CNN20], [IL89], [LO14], [NP09], [NPP11], [NZ20].

Theorem 4.4.1. *(Breuer-Major) If $C \in L^R(\mathbb{R}^d)$, then*

$$t^{-d/2}\int_{tD}\varphi(B_x)dx \overset{\text{law}}{\to} N(0, \sigma^2 \mathrm{Vol}(D)) \tag{4.35}$$

where

$$\sigma^2 = \int_{\mathbb{R}^d} K(z)dz = \sum_{q=R}^{\infty} q! a_q^2 \int_{\mathbb{R}^d} C(z)^q dz \ge 0.$$

Note that also multi-dimensional (see e.g. [NZ20, Theorem 1.2]) and stronger type (see e.g. [CNN20, Theorem 1.1]) central limit theorems of the form

$$\left(t^{-d/2}\int_{tD}\varphi(B_x)dx\right)_{D\in\mathcal{D}'} \overset{\text{f.d.d.}}{\to} \sigma G = (\sigma G(D))_{D\in\mathcal{D}'} \tag{4.36}$$

have been proved in the literature, where G is a Gaussian noise, with

$$\mathrm{Cov}(G(D), G(L)) = \mathrm{Vol}(D\cap L),$$

and $\mathcal{D}'$ is a suitable class of compact sets. In particular, considering $\mathcal{D}' = \{Du^{1/d}, u \in$

$[0,1]\}$ with fixed $D = \{x \in \mathbb{R}^d : |x| \leq 1\}$, we have (see [IL89, Theorem 2.3.1])

$$\left(t^{-d/2} \int_{tDu^{1/d}} \varphi(B_x)dx\right)_{u\in[0,1]} \overset{\text{f.d.d.}}{\to} \sigma \mathrm{Vol}(D) W = (\sigma \mathrm{Vol}(D) W_u)_{u\in[0,1]}, \tag{4.37}$$

where W is a standard Brownian motion.

If $\sigma^2 > 0$, then (4.15) holds with $\alpha = 0$ and Theorem 4.1.1 implies

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} \varphi(B_x)dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} \varphi(B_y)dy}{t^{d/2} w_t^{1/2}}\right) = \mathrm{Vol}(D \cap L), \tag{4.38}$$

which is (4.34). Moreover, Theorem 4.4.1 (and its multi-dimensional generalizations) ensure Gaussian fluctuations, which are not guaranteed by Theorem 4.1.1. Nevertheless, our Theorem 4.1.1 allows to obtain the asymptotic covariances (4.38) in many other situations (see e.g. Example 4.4.3 and 4.4.4), whenever (4.15) holds with $\alpha = 0$, i.e. w_t is slowly varying.

Conversely, if $\sigma^2 = 0$, (4.34) and (4.35) only imply that $t^{d/2}$ is not the correct normalization if we hope in a non-degenerate limit in distribution (see also the discussion in Section 4.1.1). However, w_t could be regularly varying, allowing us to apply Theorem 4.1.1. This is exactly what happens in the following example.

Example 4.4.1. (Breuer-Major case, $\sigma^2 = 0$.) Fix $d = 1$, consider a fractional Brownian motion $(W_x^H)_{x\in\mathbb{R}}$ with Hurst index $H \in (0, 1/2)$ and recall its covariance structure

$$\mathrm{Cov}(W_x^H, W_y^H) = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right).$$

Furthermore, consider the associated continuous, stationary Gaussian process $B = (B_x)_{x\in\mathbb{R}^d}$, defined by

$$B_x := W_{x+1}^H - W_x^H.$$

Note that we consider the 1-increment process B instead of W^H because B is stationary (hence weakly stationary, and we can apply Theorem 4.1.1), while W^H is not even weakly stationary (it only has stationary increments). A standard computation yields that the covariance function $C : \mathbb{R} \to \mathbb{R}$ of B is given by

$$C(z) = \frac{1}{2}\left(|1+z|^{2H} + |1-z|^{2H} - 2|z|^{2H}\right)$$

and $C \in L^1(\mathbb{R}^d)$, because as $|z| \to \infty$ we have (by Taylor expansion)

$$C(z) \sim 2H(2H-1)|z|^{2H-2}, \qquad 2H - 2 \in (-2, -1).$$

If we choose $\varphi(x) = x$, then $\int_{tD} \varphi(B_x)dx = \int_{tD} B_x dx$ is Gaussian for every $t > 0$. The important fact to note in this example is that

$$\sigma^2 = \int_{\mathbb{R}^d} K(z)dz = \int_{\mathbb{R}^d} C(z)dz = 0,$$

implying that (4.34) and the Breuer-Major theorem only say that $t^{d/2} = t^{1/2}$ is not the correct normalization. Indeed, as $t \to \infty$ we have

$$\begin{aligned} w_t &= w_{1,t} = 2\int_0^t C(z)dz = \int_0^t |1+z|^{2H} + |1-z|^{2H} - 2|z|^{2H} dz \\ &= \frac{1}{2H+1}\left((t+1)^{2H+1} + (t-1)^{2H+1} - 2t^{2H+1}\right) \sim 2Ht^{2H-1}, \end{aligned}$$

which means that w_t is regularly varying with index $\alpha = 2H - 1 \in (-1, 0)$. Therefore, applying Theorem 4.1.1 we obtain that the correct normalization is $t^{1/2}w_t^{1/2} \sim \sqrt{2H}t^H$ and $\forall D, L \in \mathcal{D}$

$$\mathrm{Cov}\left(\frac{\int_{tD} B_x dx}{t^{1/2}w_t^{1/2}}, \frac{\int_{tL} B_y dy}{t^{1/2}w_t^{1/2}}\right) \to \frac{1}{2}\int_0^\infty -\frac{d}{dl}(g_{D,L}(l) + g_{D,L}(-l))l^{2H-1}dl$$

which, if $D = [0, r]$ and $L = [0, s]$, is the covariance function of a fractional Brownian motion with Hurst index $H \in (0, 1/2)$ (see Remark 4.1.2).

4.4.2 The long memory case.

Another frequent assumption is $C(z) = \rho(|z|)$ radial with ρ regularly varying with index $-\beta \in (-d/R, 0)$ (see e.g. [LO13] for some examples), that is:

$$\rho(r) = \frac{\ell(r)}{r^\beta}, \qquad r > 0,$$

with $\ell : \mathbb{R}_+ \to \mathbb{R}$ slowly varying (see Section 4.2.3). In this case, we say that we are in the long-memory case. As $|z| \to \infty$, we have

$$K(z) = k(|z|) = \sum_{q=R}^{\infty} q! a_q^2 \rho^q(|z|) \sim R! a_R^2 \rho^R(|z|) = R! a_R^2 \frac{\ell^R(|z|)}{|z|^{R\beta}},$$

implying that $k : \mathbb{R}_+ \to \mathbb{R}$ is regularly varying with index $-R\beta \in (-d, 0)$. Therefore, reasoning as in (4.9) we obtain

$$\begin{aligned} \lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} \varphi(B_x)dx}{t^d k(t)^{1/2}}, \frac{\int_{tL} \varphi(B_y)dy}{t^d k(t)^{1/2}}\right) &= \int_{\mathbb{R}^d} g_{D,L}(z)|z|^{-R\beta}dz \\ &= \int_D\int_L |x-y|^{-R\beta}dxdy, \end{aligned} \tag{4.39}$$

and a correct normalization turns out to be

$$t^d k(t)^{1/2} \sim \sqrt{R!\, a_R^2}\, t^{d-\frac{R\beta}{2}} \ell^{R/2}(t), \qquad \text{as } t \to \infty,$$

i.e. the one usually observed in the long memory context (see e.g. [LO14]). Note that $C \notin L^R(\mathbb{R}^d)$, so we are not in the Breuer-Major case, and

$$w_t = \sum_{q=R}^{\infty} a_q^2 w_{q,t} \sim a_R^2 w_{R,t} \qquad \text{as } t \to \infty, \tag{4.40}$$

which by Proposition 4.2.4 is regularly varying with index $\alpha = d - R\beta \in (0, d)$. For this reason, we can apply Theorem 4.1.1, obtaining

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD}\varphi(B_x)dx}{t^{d/2}w_t^{1/2}}, \frac{\int_{tL}\varphi(B_y)dy}{t^{d/2}w_t^{1/2}}\right) = \frac{\alpha}{\omega_{d-1}}\int_D\int_L |x-y|^{\alpha-d}dxdy,$$

which is exactly (4.39) (indeed, $w_t \sim \frac{\omega_{d-1}}{\alpha}t^d k(t)$ by Proposition 4.2.4).

The first limit theorems in the long memory context were proved in the seminal works [DM79], [Ros60], [Taq79] and then extended in many directions. What happens in this case is that (4.30) (suitably normalized by $t^d k(t)$) has asymptotically an Hermite distribution of order R, which is not Gaussian if $R \geq 2$. Moreover, the result can be extended to multi-dimensional (and stronger type) central limit theorems, see e.g. [LO14].

Summarizing, if B has a radial covariance function $C(z) = \rho(|z|)$ with ρ regularly varying with index $-\beta \in (-d/R, 0)$, then we can compute the asymptotic covariances of B using (4.39) or Theorem 4.1.1 (since w_t is regularly varying with index $\alpha = d - R\beta$). Moreover, the forementioned limit theorems apply, yielding an Hermite limiting distribution. However, Theorem 4.1.1 allows to obtain the asymptotic covariances in many other situations in which (4.39) does not hold, whenever w_t is regularly varying with index $\alpha > 0$ and k is not regularly varying (see Example 4.4.5).

4.4.3 The Berry's case.

The Berry's random wave model is a Gaussian field which arises as the local scaling limit of a variety of random fields on two-dimensional manifolds, see [Die+23] for the details. It was first introduced by Berry in [Ber77], and is used in quantum mechanics to model the Laplace eigenfunctions of classically chaotic billiards with large eigenvalue (see also [NPR19] and the references therein). It is defined as a smooth, stationary, centered Gaussian field B on the plane ($d = 2$) with radial covariance function

$$C(z) = \rho(|z|) = J_0(|z|),$$

where J_0 is the Bessel function of the first kind of order 0 (see e.g. [Kra14]), with series expansion

$$J_0(r) = \sum_{j=0}^{\infty}\frac{(-1)^j}{(j!)^2}\left(\frac{r}{2}\right)^{2j}$$

and asymptotic behavior

$$J_0(r) = \sqrt{\frac{2}{\pi}}\, r^{-\frac{1}{2}}\cos\left(r - \frac{\pi}{4}\right) + O\left(r^{-\frac{3}{2}}\right) \quad \text{as } r \to \infty. \tag{4.41}$$

Note that ρ^q is not regularly varying. This means that we can not use Proposition 4.2.4 to prove that $w_{q,t}$ is regularly varying, as we did in Section 4.4.2. Nevertheless, this latter fact is given by the following lemma (for $q \neq 1$) and is a consequence of the results in [GMT24].

Remark 4.4.1. For $q = 1$, $w_{1,t}$ is not regularly varying. To prove this fact, consider

J_1, the Bessel function of the first kind of order 1 (see e.g. [Kra14]), with series expansion

$$J_1(r) = \left(\frac{r}{2}\right)\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+1)!}\left(\frac{r}{2}\right)^{2j}$$

and asymptotic behavior

$$J_1(r) = \sqrt{\frac{2}{\pi}}\, r^{-\frac{1}{2}} \cos\left(r - \frac{3}{4}\pi\right) + O\left(r^{-\frac{3}{2}}\right) \quad \text{as } r \to \infty.$$

Note that using the series expansions of J_0 and J_1 we have

$$\frac{d}{dr}\left(J_1(r)r\right) = J_0(r)r.$$

Therefore, since

$$w_{1,t} = 2\pi\int_0^t J_0(r)r\,dr = J_1(t)t = 2\pi\sqrt{\frac{2}{\pi}}\, t^{\frac{1}{2}} \cos\left(t - \frac{3}{4}\pi\right) + O\left(t^{-\frac{1}{2}}\right) \quad \text{as } t \to \infty,$$

$w_{1,t}$ is not a regularly varying function, and Theorem 4.1.1 can not be applied to compute the asymptotic covariances of $\int_{tD} H_1(B_x)dx = \int_{tD} B_x dx$ and $\int_{tL} H_1(B_x)dx = \int_{tL} B_x dx$ (recall that $H_1(x) = x$ is the first Hermite polynomial).

Nevertheless, with a different argument, something can be said about the variance of $\int_{tD} B_x dx = \int_{tD} H_1(B_x)dx$. Indeed, we have the formula (see e.g. [Sch38])

$$C(z) = J_0(|z|) = \frac{1}{\omega_{d-1}}\int_{S^{d-1}} e^{i\langle x,\theta\rangle}d\theta.$$

and after applying Fubini-Tonelli theorem we obtain

$$\begin{aligned}\text{Var}\left(\int_{tD} B_x dx\right) &= \int_{tD}\int_{tD} J_0(|x-y|)dxdy = \\ \frac{1}{\omega_{d-1}}\int_{S^{d-1}} |\mathcal{F}[\mathbf{1}_{tD}]|^2(\theta)d\theta &= \frac{t^{2d}}{\omega_{d-1}}\int_{S^{d-1}} |\mathcal{F}[\mathbf{1}_{D}]|^2(t\theta)d\theta,\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. Therefore, if $|\mathcal{F}[\mathbf{1}_D](x)| = o(|x|^{-d/2})$, we have

$$\text{Var}\left(\int_{tD} B_x dx\right) = o(t^d), \quad \text{as } t \to \infty.$$

This fact will be needed in Example 4.4.6.

Lemma 4.4.2. *Let $(B_x)_{x\in\mathbb{R}^2}$ be the Berry's random wave model, and recall the notation introduced in the first part of Section 4.4. Then for $q \geq 2$, as $t \to \infty$, we have*

$$w_{q,t} \sim \begin{cases} c\,t & \text{if } q = 2 \\ c\log(t) & \text{if } q = 4 \\ c & \text{if } q = 3 \text{ or } q \geq 5 \end{cases},$$

where $c \in (0, \infty)$ depends on q, but does not depend on t.

Proof. In the sequel, let c be a positive constant which may change value. For $q = 2$

even, by (4.41) and the trigonometric identity $2\cos^2(x) = 1 + \cos(2x)$ we have

$$\begin{aligned} w_{2,t} &= 2\pi \int_0^t J_0^2(r) r\, dr = c \left(\int_1^t \cos^2(r - \pi/4) dr + O\left(\int_0^t \frac{dr}{r} \right) \right) \\ &= c\left(t - 1 + \int_1^t \cos(2r - \pi/2) dr + O\left(\int_0^t \frac{dr}{r} \right) \right) \\ &= c\, (t + O(\log(t))) \sim ct. \end{aligned}$$

For $q = 4$, by (4.41) and the trigonometric identity $8\cos^4(x) = 2(1 + \cos(2x))^2 = 2 + 4\cos(2x) + 2\cos^2(2x) = 2 + 4\cos(2x) + 1 + \cos(4x)$, we have

$$\begin{aligned} w_{4,t} &= 2\pi \int_0^t J_0^4(r) r\, dr = c \left(\int_1^t \frac{\cos^4(r - \pi/4)}{r} dr + O\left(\int_0^t \frac{dr}{r^2} \right) \right) \\ &= c\left(3\log(t) + 4\int_1^t \cos(2r - \pi/2) dr + \int_1^t \cos(4r - \pi) dr + O\left(\int_0^t \frac{dr}{r^2} \right) \right) \\ &= c\, (\log(t) + O(1)) \sim c\log(t). \end{aligned}$$

If $q \geq 6$ is even, $w_{q,t} > 0$, $w_{q,t}$ increasing in t, and $w_{q,t} = O(1)$ (see Remark 4.4.2). Therefore, we have $w_{q,t} \sim c > 0$ for some positive constant $c > 0$.

If $q \geq 3$ is odd, the fact that $w_{q,t}$ converges to a positive constant is more difficult to see, and is proved in [GMT24] by means of Pearson's random walks. □

Remark 4.4.2. Note that since $|C(z)| = |J_0(|z|)| \leq 1$ and (4.41) holds, we have

$$\left| J_0^5(r) \right| \leq c \left(1 \wedge \frac{1}{r^{5/2}} \right), \quad r > 0,$$

where c is a positive constant. Moreover, since $|C(z)| = |J_0(|z|)| \leq 1$ we have

$$\int_{\mathbb{R}^d} |C^q(z)|\, dz \leq \int_{\mathbb{R}^d} \left| C^5(z) \right| dz = 2\pi \int_0^\infty \left| J_0^5(r) \right| r\, dr \leq c \int_1^\infty \frac{dr}{r^{3/2}} < \infty, \qquad q \geq 5.$$

Therefore, we obtain the uniform bound

$$\sum_{q=5}^{\infty} a_q^2 w_{q,t} \leq \|\varphi\|^2_{L^2(\mathbb{R}_+, \gamma(dx))} \int_{\mathbb{R}^d} \left| C^5(x) \right| dx < \infty. \tag{4.42}$$

Combining Lemma 4.4.2 with Theorem 4.1.1, we are now able to compute the asymptotic covariances

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} \varphi(B_x) dx}{t^{d/2} w_t^{1/2}}, \frac{\int_{tL} \varphi(B_y) dy}{t^{d/2} w_t^{1/2}} \right),$$

where $B = (B_x)_{x\in\mathbb{R}^2}$ is the Berry's random wave model, for a large class of functions φ and compact domains D, L. If we further apply [MN24, Theorem 2] and the Peccati-Tudor multi-dimensional fourth moment theorem (see [PT05]), we are able to prove multi-dimensional central limit theorems for functionals of the Berry's random wave model. All these facts are explained in the upcoming Examples 4.4.2-4.4.6. Before we move on to the latter, to ease the exposition, let us state a simplified version of the Peccati-Tudor theorem in our context.

Theorem 4.4.3. *[PT05, Proposition 1] Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a continuous, centered, stationary Gaussian field with covariance function C. Let $\mathcal{D}'$ be a class of compact sets in $\mathbb{R}^d$. Assume that the following three conditions hold:*

1. $\operatorname{Var}\left(\int_{tD} H_q(B_x)dx\right) > 0$*, for every $D \in \mathcal{D}'$ and t large enough.*
2. *For every $D \in \mathcal{D}'$*
$$\frac{\int_{tD} H_q(B_x)dx}{\sqrt{\operatorname{Var}\left(\int_{tD} H_q(B_x)dx\right)}} \overset{law}{\to} N,$$
where $N \sim N(0,1)$ has a standard Gaussian distribution.
3. *For every $D, L \in \mathcal{D}'$, we have*
$$\operatorname{Cov}\left(\frac{\int_{tD} H_q(B_x)dx}{\sqrt{\operatorname{Var}\left(\int_{tD} H_q(B_x)dx\right)}}, \frac{\int_{tL} H_q(B_y)dy}{\sqrt{\operatorname{Var}\left(\int_{tL} H_q(B_y)dy\right)}}\right) \to b(D,L),$$
where $b(D,L)$ is a constant depending only on D and L.

Then, as $t \to \infty$ we have the multi-dimensional central limit theorem

$$\left(\frac{\int_{tD} H_q(B_x)dx}{\sqrt{\operatorname{Var}\left(\int_{tD} H_q(B_x)dx\right)}}\right)_{D\in\mathcal{D}'} \overset{f.d.d.}{\to} G = (G(D))_{D\in\mathcal{D}'}$$

where G is a centered, set-indexed Gaussian field with covariances

$$\operatorname{Cov}(G(D), G(L)) = b(D,L).$$

Example 4.4.2. (Berry with $R \geq 5$.) If $R \geq 5$, then $C \in L^R(\mathbb{R}^2)$ and we are in the Breuer-Major case of Section 4.4.1. In particular, by Lemma 4.4.2 we have that $\sigma^2 > 0$ and w_t is slowly varying (see (4.42)). Therefore, the asymptotic covariances are given by (4.34) or (applying Theorem 4.1.1) by (4.38), and Gaussian fluctuations follow by multi-dimensional Breuer-Major theorems of the form (4.36)-(4.37).

Example 4.4.3. (Berry with $R = 4$.) When $R = 4$, note that $K(z) = k(|z|)$ is radial, non-integrable and k is not regularly varying, implying that we are not in the classical cases discussed in Sections 4.4.1 and 4.4.2. Despite this, w_t is slowly varying (see Lemma 4.4.2, (4.33) and (4.42)) and in particular

$$w_t \sim a_4^2 w_{4,t} \sim c\log(t) \qquad \text{as } t \to \infty,$$

where c is a positive constant. Then, Theorem 4.1.1 can be applied to obtain the asymptotic covariances

$$\lim_{t\to\infty} \operatorname{Cov}\left(\frac{\int_{tD}\varphi(B_x)dx}{t^{d/2}w_t^{1/2}}, \frac{\int_{tL}\varphi(B_y)dy}{t^{d/2}w_t^{1/2}}\right) = \operatorname{Vol}(D\cap L), \tag{4.43}$$

for every $D, L \in \mathcal{D}$, where $\mathcal{D}$ is the class of compact sets with finite perimeter introduced in (4.13). Moreover, by using reduction techniques (see e.g. the proof of

[MN24, Proposition 4]) in (4.44) and the spectral CLT [MN24, Theorem 2] in (4.45), we have for $D \subseteq \mathbb{R}^2$ compact with $\mathrm{Vol}(D) > 0$:

$$\mathbb{E}\left[\left(\frac{\int_{tD} \varphi(B_x)dx}{\sqrt{\mathrm{Var}\left(\int_{tD} \varphi(B_x)dx\right)}} - sgn(a_4)\frac{\int_{tD} H_4(B_x)dx}{\sqrt{\mathrm{Var}\left(\int_{tD} H_4(B_x)dx\right)}}\right)^2\right] \to 0 \tag{4.44}$$

and

$$\frac{\int_{tD} H_4(B_x)dx}{\sqrt{\mathrm{Var}\left(\int_{tD} H_4(B_x)dx\right)}} \overset{law}{\to} N(0,1). \tag{4.45}$$

Combining (4.43), (4.44), (4.45) and Theorem 4.4.3 with $\mathcal{D}' = \mathcal{D}$, we obtain

$$\left(\frac{\int_{tD} \varphi(B_x)dx}{t^{d/2}w_t^{1/2}}\right)_{D\in\mathcal{D}} \overset{f.d.d.}{\to} (G(D))_{D\in\mathcal{D}}, \tag{4.46}$$

where $G = (G(D))_{D\in\mathcal{D}}$ is a Gaussian noise with covariances given by (4.43). Note that we have proved (4.46) on $\mathcal{D}$, excluding the compact sets with infinite perimeter, since in the latter cases Theorem 4.1.1 can not be applied as in (4.43) to verify condition (iii) of Theorem 4.4.3. Note also that (4.46) could have been obtained (for a less general class of domains $\mathcal{A}$ defined in [PV20]) combining the results proved in [PV20] for the nodal lengths of the Berry's random wave model and the reduction principle in [Vid21].

Example 4.4.4. (Berry with $R = 3$.) If $a_4 \neq 0$, then $w_t \sim a_4^2 w_{4,t}$ and we can reason exactly as in Example 4.4.3. So we can assume that $R = 3$ and $a_4 = 0$. If this is the case, by Lemma 4.4.2, (4.33) and (4.42) we have that w_t converges to a positive constant, so that it is slowly varying. Then, by Theorem 4.1.1 we have for every D, L in the class $\mathcal{D}$ of compact sets with finite perimeter (see (4.13))

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD} \varphi(B_x)dx}{t^{d/2}w_t^{1/2}}, \frac{\int_{tL} \varphi(B_y)dy}{t^{d/2}w_t^{1/2}}\right) = \mathrm{Vol}(D \cap L).$$

Note that in this case $C \notin L^3(\mathbb{R}^d)$, so we are not in the Breuer-Major case, where the covariances can be obtained for D, L compact domains as in (4.34). In fact, in this case $\int_{\mathbb{R}^d} C^3(z)dz$ is only conditionally convergent, but not absolutely convergent. Nevertheless, excluding compact domains with infinite perimeter, we can compute the asymptotic covariances using Theorem 4.1.1.

To the best of our knowledge, limit theorems in the case $R = 3$ and $a_4 = 0$ have never been proved.

Example 4.4.5. (Berry with $R = 2$.) When $R = 2$, we have (by Lemma 4.4.2, (4.33) and (4.42)), c being a positive constant,

$$w_t \sim a_2^2 w_{2,t} \sim c\,t \qquad \text{as } t \to \infty.$$

Again, we are not in the classical cases discussed in Section 4.4.1 or 4.4.2, but w_t is regularly varying with index $\alpha = 1$. Then, by Theorem 4.1.1 we have for every D, L

in the class $\mathcal{D}$ of compact sets with finite perimeter (see (4.13))

$$\lim_{t\to\infty} \mathrm{Cov}\left(\frac{\int_{tD}\varphi(B_x)dx}{t^{d/2}w_t^{1/2}}, \frac{\int_{tL}\varphi(B_y)dy}{t^{d/2}w_t^{1/2}}\right) = \frac{1}{2\pi}\int_{S^1} d\theta \int_0^\infty dl \left(-\frac{d}{dl}g_{D,L}(l\theta)\right) l \quad (4.47)$$
$$= \frac{1}{2\pi}\int_D\int_L \frac{dxdy}{|x-y|}. \quad (4.48)$$

For the last equality, see Remark 4.1.2. Note that the integral on the RHS of (4.47) is finite, since $g_{D,L}$ has bounded derivative and compact support if D, L are compact sets with finite perimeter (see Proposition 4.2.2).

Let us define the following class of compact sets:

$$\mathcal{D}_O := \left\{D \subseteq \mathbb{R}^d \text{ compact } : \mathrm{Vol}(D) > 0, |\mathcal{F}[\mathbf{1}_D](x)| = O(|x|^{-d/2}) \text{ as } |x| \to \infty\right\},$$

where $\mathcal{F}$ denotes the Fourier transform. For example, $D \in \mathcal{D}_O$ when D is compact, $D = \bar{\mathring{D}}$ and ∂D is smooth with non-vanishing Gaussian curvature, see [BHI03]. By using reduction techniques (see e.g. the proof of [MN24, Proposition 4]) in (4.49) and the spectral CLT [MN24, Theorem 2], we obtain for every $D \in \mathcal{D}_O$:

$$\mathbb{E}\left[\left(\frac{\int_{tD}\varphi(B_x)dx}{\sqrt{\mathrm{Var}\left(\int_{tD}\varphi(B_x)dx\right)}} - sgn(a_2)\frac{\int_{tD}H_2(B_x)dx}{\sqrt{\mathrm{Var}\left(\int_{tD}H_2(B_x)dx\right)}}\right)^2\right] \to 0 \quad (4.49)$$

and

$$\frac{\int_{tD}H_2(B_x)dx}{\sqrt{\mathrm{Var}\left(\int_{tD}H_2(B_x)dx\right)}} \overset{law}{\to} N(0,1). \quad (4.50)$$

Moreover, combining (4.47), (4.49), (4.50) and Theorem 4.4.3 with $\mathcal{D}' = \mathcal{D}\cap\mathcal{D}_O$, we obtain the multi-dimensional central limit theorem

$$\left(\frac{\int_{tD}\varphi(B_x)dx}{t^{d/2}w_t^{1/2}}\right)_{D\in\mathcal{D}\cap\mathcal{D}_O} \overset{f.d.d.}{\to} (G'(D))_{D\in\mathcal{D}\cap\mathcal{D}_O}, \quad (4.51)$$

where the set-indexed Gaussian field $G' = (G'(D))_{D\in\mathcal{D}\cap\mathcal{D}_O}$ has covariances given by (4.47). Note that the restriction on $\mathcal{D}$ in (4.51) is needed to exclude the compact sets with infinite perimeter, since in the latter cases Theorem 4.1.1 can not be applied as in (4.47) to verify condition (iii) of Theorem 4.4.3. Moreover, the restriction on $\mathcal{D}_O$ in (4.51) is needed to conclude (4.50) by means of Theorem [MN24, Theorem 2], verifying the condition (ii) in Theorem 4.4.3.

Example 4.4.6. (Berry with $R = 1$.) If φ is **linear**, then $\varphi(x) = a_1x = a_1H_1(x)$, where $H_1(x) = x$ is the first Hermite polynomial and $a_1 \neq 0$ by definition of Hermite rank R. In this case, $w_t = a_1^2 w_{1,t}$ is not regularly varying, see Remark 4.4.1. Therefore, we can not apply Theorem 4.1.1 to compute the asymptotic covariances.

Let us now assume that $\varphi(x)$ is **not linear**, splitting the functional as follows

$$\int_{tD}\varphi(B_x)dx = a_1\int_{tD}H_1(B_x)dx + \int_{tD}\hat{\varphi}(B_x)dx,$$

where $\hat{\varphi} := \varphi - a_1H_1 \neq 0$ because φ is not linear (i.e. $\exists q \geq 2$ with $a_q \neq 0$). In addition, recall the notation $\mathcal{D}$ for the class of compact sets with finite perimeter, and consider

$$\mathcal{D}_o := \left\{ D \subseteq \mathbb{R}^d \text{ compact } : \mathrm{Vol}(D) > 0, |\mathcal{F}[\mathbf{1}_D](x)| = o(|x|^{-d/2}) \text{ as } |x| \to \infty \right\}.$$

For example, $D \in \mathcal{D}_o$ when D is compact, $D = \bar{\mathring{D}}$ and ∂D is smooth with non-vanishing Gaussian curvature, see [BHI03].

Then, we can study the asymptotic covariances and multi-dimensional fluctuations of $\int_{tD} \varphi(B_x)dx$, as we did in the previous examples, following two steps:

1. We observe that as $t \to \infty$ (note that the equality holds by definition of $\hat{\varphi}$)

$$\mathbb{E}\left[\left(\frac{\int_{tD} \varphi(B_x)dx}{t^{d/2}\left(\sum_{q=2}^{\infty} a_q^2 w_{q,t}\right)^{1/2}} - \frac{\int_{tD} \hat{\varphi}(B_x)dx}{t^{d/2}\left(\sum_{q=2}^{\infty} a_q^2 w_{q,t}\right)^{1/2}} \right)^2 \right] \tag{4.52}$$

$$= \frac{\mathrm{Var}\left(\int_{tD} a_1 H_1(B_x)dx\right)}{t^d \sum_{q=2}^{\infty} a_q^2 w_{q,t}} \to 0, \qquad \forall D \in \mathcal{D}_o. \tag{4.53}$$

In fact, by Remark 4.4.1 we have (note that $t^d = O\left(t^d \sum_{q=2}^{\infty} a_q^2 w_{q,t}\right)$ by Lemma 4.4.2, since $\exists\, q \geq 2$ with $a_q \neq 0$)

$$\mathrm{Var}\left(a_1 \int_{tD} H_1(B_x)dx\right) = o\left(t^d \sum_{q=2}^{\infty} a_q^2 w_{q,t}\right), \qquad \forall D \in \mathcal{D}_o.$$

2. Since the asymptotic L^2-equivalence (4.52) holds, the problem of studying the asymptotic covariances and the multi-dimensional Gaussian fluctuations of $\int_{tD} \varphi(B_x)dx$ is reduced to the same problem for $\int_{tD} \hat{\varphi}(B_x)dx$. Hence, since $\hat{\varphi}$ has Hermite rank greater or equal than 2, one can solve the problem using the results in the previous examples. For instance, if the Hermite rank of $\hat{\varphi}$ is 2, then by Example 4.4.5 we have

$$\left(\frac{\int_{tD} \hat{\varphi}(B_x)dx}{t^{d/2}\left(\sum_{q=2}^{\infty} a_q^2 w_{q,t}\right)^{1/2}} \right)_{D \in \mathcal{D} \cap \mathcal{D}_O} \overset{f.d.d.}{\to} (G'(D))_{D \in \mathcal{D} \cap \mathcal{D}_O},$$

and combining the latter with (4.52) we obtain (note that $\mathcal{D}_o \subseteq \mathcal{D}_O$, so we add an additional restriction on the class of domains in order to use (4.52)).

$$\left(\frac{\int_{tD} \varphi(B_x)dx}{t^{d/2}\left(\sum_{q=2}^{\infty} a_q^2 w_{q,t}\right)^{1/2}} \right)_{D \in \mathcal{D} \cap \mathcal{D}_o} \overset{f.d.d.}{\to} (G'(D))_{D \in \mathcal{D} \cap \mathcal{D}_o},$$

where the set-indexed Gaussian field $G' = (G'(D))_{D \in \mathcal{D} \cap \mathcal{D}_o}$ has covariances given by (4.47).

Remark 4.4.3. Note that all the arguments above could be generalized to prove analogous multi-dimensional central limit theorems for the d-dimensional Berry's random wave model (and partially extended to a larger class of fields, see [MN24, Section 6]). We only focused on the case $d = 2$ for the sake of brevity.

4.5 Non-radial covariance functions.

When $d \geq 2$, the covariance function K of the field A could be non-radial, but Theorem 4.1.1 can still be applied taking $\mathcal{D}$ as the set of balls centered at a fixed point, see (4.14).

The goal of this section is showing that the problem (4.3) for non-radial, continuous K can be reduced (for D, L balls centered at a fixed point) to the same problem for a radial covariance function K_{iso}, which will be defined in the sequel. This reduction principle will be proved in Proposition 4.5.1.

In order to state Proposition 4.5.1, we need to introduce some additional quantities.

Let $A = (A_x)_{x\in\mathbb{R}^d}$ be a measurable, weakly stationary random field with **continuous** covariance function $K : \mathbb{R}^d \to \mathbb{R}$, $K(0) = 1$. By Bochner's theorem, there exists a unique, symmetric, Borel probability measure G on $\mathbb{R}^d$ such that

$$K(z) = \int_{\mathbb{R}^d} e^{i\langle x,\lambda\rangle} G(d\lambda).$$

G is called the **spectral measure** of A (or associated to K). In other words, K is the characteristic function of Λ, where $\Lambda \sim G$ is a random variable with values in $\mathbb{R}^d$.

Note that we can always write

$$K(z) = \mathbb{E}\left[e^{i\langle z,\Lambda\rangle}\right] = \mathbb{E}\left[e^{i|\Lambda|\langle z,\hat{\Lambda}\rangle}\right],$$

where $|\Lambda| \sim \mu$ and $\hat{\Lambda} := \mathbf{1}_{\Lambda\neq 0}\, \Lambda/|\Lambda| \sim \sigma$ are respectively the random norm and the random direction of Λ. We will refer respectively to the probability measures μ on $\mathbb{R}_+$ and σ on S^{d-1} as the **isotropic spectral measure** and **spherical spectral measure** of A (or associated to K or G).

Remark 4.5.1. Let us denote by ν the uniform probability measure on S^{d-1}. Note that K is radial if and only if $\sigma = \nu$. Indeed, K is radial, i.e. $K(z) = K(Pz)$ for every P orthogonal matrix, if and only if $P\Lambda \sim \Lambda$ for every P orthogonal matrix, that is $\hat{\Lambda}$ is uniformly distributed on S^{d-1} (i.e. $\sigma = \nu$). In particular, given a probability measure μ on $\mathbb{R}_+$, there is a unique continuous, radial covariance function with isotropic spectral measure μ (and spherical spectral measure ν).

Thanks to Remark 4.5.1, we can now give the crucial definition of this section.

Definition 4.5.1. Let $A = (A_x)_{x\in\mathbb{R}^d}$ be a weakly stationary random field with continuous covariance function $K : \mathbb{R}^d \to \mathbb{R}$, isotropic spectral measure μ and spherical spectral measure σ, that is

$$K(z) = \int_0^\infty \mu(dr) \int_{S^{d-1}} \sigma(d\theta)\, e^{i\langle r\theta,z\rangle}.$$

The **isotropic covariance function** of A is the only continuous, radial covariance function $K_{\text{iso}} : \mathbb{R}^d \to \mathbb{R}$ with isotropic spectral measure μ, namely (see Remark 4.5.1)

$$K_{\text{iso}}(z) := \int_0^\infty \mu(dr) \int_{S^{d-1}} \nu(d\theta)\, e^{i\langle r\theta,z\rangle},$$

where $\nu(d\theta) = d\theta/\omega_{d-1}$ is the uniform probability measure on S^{d-1}.

The main result of this section is the following.

Proposition 4.5.1. *Let $A = (A_x)_{x\in\mathbb{R}^d}$ be a measurable, weakly stationary random field with continuous covariance function $K : \mathbb{R}^d \to \mathbb{R}$ and isotropic covariance function K_{iso}. Then we have (recall the definition (4.1) of w)*

$$w_t^{iso} := \int_{\{|z|\le t\}} K_{iso}(z)dz = \int_{\{|z|\le t\}} K(z)dz = w_t, \qquad t > 0, \tag{4.54}$$

and for D, L balls centered at the same point

$$\int_D \int_L K(x-y)dxdy = \int_D \int_L K_{iso}(x-y)dxdy. \tag{4.55}$$

In particular, if $w_t = w_t^{iso}$ is regularly varying with index $\alpha \in (-1, d]$ (see (4.15)), by Theorem 4.1.1 we have, for D, L balls centered at the same point (see (4.14)), as $t \to \infty$

$$\begin{aligned}
\mathrm{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2}w_t^{1/2}}, \frac{\int_{tL} A_x dx}{t^{d/2}w_t^{1/2}}\right) &= \frac{\int_{tD}\int_{tL} K(x-y)dxdy}{t^d w_t} \\
&= \frac{\int_{tD}\int_{tL} K_{iso}(x-y)dxdy}{t^d w_t^{iso}} \qquad (4.56)\\
&\to \frac{1}{\omega_{d-1}} \int_{S^{d-1}} d\theta \int_0^\infty dl \left(-\frac{d}{dl} g_{D,L}(l\theta)\right) l^\alpha. \qquad (4.57)
\end{aligned}$$

Proof. By Fubini, we have

$$\begin{aligned}
w_t = \int_{\{|z|\le t\}} K(z)dz &= \int_{S^{d-1}} \sigma(d\theta) \int_0^\infty \mu(dr) \int_{\{|z|\le t\}} e^{i\langle z, r\theta\rangle} \\
&= \int_{S^{d-1}} \sigma(d\theta) \int_0^\infty \mu(dr) \mathcal{F}[\mathbf{1}_{\{|z|\le t\}}](r\theta),
\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. Since $\mathcal{F}[\mathbf{1}_{\{|z|\le t\}}](r\theta)$ is a radial function (i.e. does not depend on θ) and σ, ν are probability measures on S^{d-1}, we have

$$\begin{aligned}
\int_{S^{d-1}} \sigma(d\theta) \int_0^\infty \mu(dr)\mathcal{F}[\mathbf{1}_{\{|z|\le t\}}](r\theta) &= \int_{S^{d-1}} \nu(d\theta) \int_0^\infty \mu(dr)\mathcal{F}[\mathbf{1}_{\{|z|\le t\}}](r\theta) \\
&= \int_{\{|z|\le t\}} K_{\text{iso}}(z)dz = w_t^{\text{iso}},
\end{aligned}$$

where the last equality follows again by Fubini. Therefore, (4.54) is proved. As a consequence of (4.54) and (4.27), we also obtain (4.55). Moreover, (4.56) follows by (4.54), (4.55) and a direct application of Theorem 4.1.1. □

Proposition 4.5.1 allows to reduce the problem (4.3) for K non-radial (and D, L balls centered at the same point) to the easiest problem (4.3) for K_{iso} radial.

Note that we can also make the argument in the opposite direction: if K_{iso} is a radial covariance function and we can check (4.15) for K_{iso}, then by Proposition 4.5.1 we can apply Theorem 4.1.1 not only for random fields with covariance function K_{iso}, but for every random field with isotropic covariance function K_{iso}.

To be more concrete about the possible applications of Proposition 4.5.1, we conclude with the following example.

Example 4.5.1. Fix $d \geq 2$ and consider the **non-Gaussian** random field $A = (A_x)_{x\in\mathbb{R}^d}$ given by the rescaled sum of N independent random waves with random phases φ_i, random directions θ_i and random wavenumbers k_i. Namely, we have

$$A_x := \sqrt{\frac{2}{N}} \sum_{i=1}^{N} \cos\left(k_i\langle x, \theta_i\rangle + \varphi_i\right), \tag{4.58}$$

where $\varphi_1, \theta_1, k_1, \dots, \varphi_N, \theta_N, k_N$ are all independent and for all $i = 1, \dots, N$ we have: φ_i is uniformly distributed on $[0, 2\pi]$; $\theta_i \sim \sigma$ symmetric probability measure on S^{d-1}; $k_i \sim \mu$ probability measure on $(0, \infty)$.

Note that when σ is the uniform distribution on S^{d-1} and $\mu = \delta_k$ for some constant $k > 0$, random fields of the form (4.58) are called random superposition of independent waves, and have been extensively studied as good local models for wavefunctions (i.e. eigenfunctions of the Laplacian), see e.g. [Ber77; NPR19; PV20; Vid21].

Moreover, note that A is **not stationary**, since A_x and A_y have not the same distribution in general. For instance, if $N = 1$, in general $A_0 = \cos(\varphi_1)$ has not the same distribution of $A_x = \cos(k_1\langle x, \theta_1\rangle + \varphi_1)$ for every $x \neq 0$.

However, A is **weakly stationary**, implying that Theorem 4.1.1 can be applied to compute the asymptotic covariances of functionals of A. Indeed, we have

$$\begin{aligned}
\mathbb{E}[A_x] &= \sqrt{2N}\mathbb{E}\left[\cos\left(k_1\langle x, \theta_1\rangle + \varphi_1\right)\right] \\
&= \sqrt{2N}\left(\mathbb{E}\left[\cos\left(k_1\langle x, \theta_1\rangle\right)\right]\underbrace{\mathbb{E}\left[\cos(\varphi_1)\right]}_{=0} - \mathbb{E}\left[\sin\left(k_1\langle x, \theta_1\rangle\right)\right]\underbrace{\mathbb{E}\left[\sin(\varphi_1)\right]}_{=0}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}(A_x, A_y) = \mathbb{E}\left[A_x A_y\right] &= 2\mathbb{E}\left[\cos\left(k_1\langle x, \theta_1\rangle + \varphi_1\right)\cos\left(k_1\langle y, \theta_1\rangle + \varphi_1\right)\right] \\
&= 2\mathbb{E}\left[\cos\left(k_1\langle x, \theta_1\rangle\right)\cos\left(k_1\langle y, \theta_1\rangle\right)\right]\underbrace{\mathbb{E}\left[\cos^2(\varphi_1)\right]}_{=1/2} \\
&+ 2\mathbb{E}\left[\sin\left(k_1\langle x, \theta_1\rangle\right)\sin\left(k_1\langle y, \theta_1\rangle\right)\right]\underbrace{\mathbb{E}\left[\sin^2(\varphi_1)\right]}_{=1/2} \\
&- 2\mathbb{E}\left[\cos\left(k_1\langle x, \theta_1\rangle\right)\sin\left(k_1\langle y, \theta_1\rangle\right)\right]\underbrace{\mathbb{E}\left[\cos(\varphi_1)\sin(\varphi_1)\right]}_{=0} \\
&- 2\mathbb{E}\left[\sin\left(k_1\langle x, \theta_1\rangle\right)\cos\left(k_1\langle y, \theta_1\rangle\right)\right]\underbrace{\mathbb{E}\left[\sin(\varphi_1)\cos(\varphi_1)\right]}_{=0} \\
&= \mathbb{E}\left[\cos\left(k_1\langle x - y, \theta_1\rangle\right)\right].
\end{aligned}$$

Therefore, denoting by K the covariance function of A, by the symmetry of θ_1 we have

$$K(z) = \mathbb{E}\left[\cos\left(k_1\langle z, \theta_1\rangle\right)\right] = \mathbb{E}\left[e^{ik_1\langle z, \theta_1\rangle}\right] = \int_{\mathbb{R}_+\times S^{d-1}} e^{i\langle z, r\theta\rangle}(\mu\times\sigma)(dr, d\theta),$$

implying that the isotropic (resp. spherical) spectral measure of A is μ (resp. σ). By Proposition 4.5.1 the integral covariance function w defined in (4.1) does not depend on the spherical spectral measure. In other words, whatever is the distribution of the random directions $\theta_1, \ldots, \theta_N$ fixed in the definition (4.58) of A, one can apply Theorem 4.1.1 with the isotropic covariance function of A, i.e.

$$K_{\text{iso}}(z) = \int_{\mathbb{R}_+ \times S^{d-1}} e^{i\langle z, r\theta\rangle} (\mu \times \nu)(dr, d\theta),$$

where ν is the uniform probability measure on S^{d-1}. More precisely, whatever is the covariance function K, whatever is the spherical spectral measure σ, if K_{iso} satisfies

$$w_t^{\text{iso}} = \int_{\{|z| \le t\}} K_{\text{iso}}(z) dz = \ell(t) t^\alpha, \qquad \alpha \in (-1, d],$$

with $\ell : \mathbb{R}_+ \to \mathbb{R}$ slowly varying, we have for D, L balls centered at the same point (see (4.14)), as $t \to \infty$

$$\text{Cov}\left(\frac{\int_{tD} A_x dx}{t^{d/2} (w_t^{\text{iso}})^{1/2}}, \frac{\int_{tL} A_x dx}{t^{d/2} (w_t^{\text{iso}})^{1/2}} \right) \to \frac{1}{\omega_{d-1}} \int_{S^{d-1}} d\theta \int_0^\infty dl \left(-\frac{d}{dl} g_{D,L}(l\theta) \right) l^\alpha.$$

CHAPTER 5

Limit theorems for p-domain functionals of stationary Gaussian fields

This chapter contains the preprint [Leo+24], "Limit theorems for p-domain functionals of stationary Gaussian fields", written in collaboration with Nikolai Leonenko, Ivan Nourdin and Francesca Pistolato. arXiv (2024+): 2402.16701.

5.1 Introduction

Gaussian fields are widely used to model random quantities arising in different applications, see e.g. [Chr92; DR07; Gne02; LRM23]. These random quantities may depend for instance on space $x_1 \in \mathbb{R}^{d_1}$ and time $x_2 \in \mathbb{R}_+$ (e.g. [Gne02]) or more generally on several variables $x_1, x_2, x_3, \dots$ belonging to (possibly non-Euclidean) spaces of different dimensions (e.g. [BPP22; MRV21; MRV24]).

Throughout this paper, we consider a continuous, stationary, real-valued, centered Gaussian field $B = (B_x)_{x\in\mathbb{R}^d}$ with unit variance, where $d \geq 2$. We denote by $C : \mathbb{R}^d \to \mathbb{R}$ the **covariance function** of B, defined as

$$\mathrm{Cov}(B_x, B_y) = C(x - y), \qquad x, y \in \mathbb{R}^d.$$

We also consider a non constant measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\mathbb{E}[\varphi^2(N)] < \infty$ for $N \sim N(0, 1)$. As is well-known, φ admits an Hermite decomposition (see, e.g. [NP12, Section 1.4]) of the form

$$\varphi = \mathbb{E}[\varphi(N)] + \sum_{q=R}^{\infty} a_q H_q, \quad \text{with } R \geq 1 \text{ such that } a_R \neq 0, \tag{5.1}$$

where H_q is the qth Hermite polynomial and $a_q = a_q(\varphi) = \frac{1}{q!}\mathbb{E}\left[\varphi(N)H_q(N)\right]$. The integer $R \geq 1$ is called the **Hermite rank** of φ.

Finally, we consider a family of compact subsets $D_i \subseteq \mathbb{R}^{d_i}$, $1 \leq i \leq p$, satisfying $\mathrm{Vol}(D_i) > 0$ for each i. The number $d = d_1 + \cdots + d_p$ is called the total dimension.

The main object of interest of our paper is the additive functional

$$Y(t_1, \ldots, t_p) := \int_{t_1 D_1 \times \cdots \times t_p D_p} \varphi(B_x) dx \quad \text{for } t_1, \ldots, t_p > 0. \tag{5.2}$$

We remark that the integral in (5.2) is well defined thanks to the continuity assumption made on B as well as the square integrability of φ with respect to the standard Gaussian measure (see e.g. [MN24, Proposition 3]).

Under sufficient conditions ensuring that $\mathrm{Var}(Y(t_1, \ldots, t_p)) > 0$ for every $t_1, \ldots, t_p > 0$ large enough, we study the limit in distribution as $t_1, \ldots, t_p \to \infty$ of the normalized version of the functional $Y(t_1, \ldots, t_p)$ defined as

$$\widetilde{Y}(t_1, \ldots, t_p) := \frac{Y(t_1, \ldots, t_p) - \mathbb{E}[Y(t_1, \ldots, t_p)]}{\sqrt{\mathrm{Var}(Y(t_1, \ldots, t_p))}}. \tag{5.3}$$

While the case $p = 1$ has been extensively studied since the eighties (see e.g. the seminal works [BM83], [DM79], [Ros60], [Taq79]), not much literature can be found when the integration domain of (5.2) does not grow uniformly with respect to all its directions (that is, when $p \geq 2$ and when $t_1, \ldots, t_p$ go to infinity at possibly different rates). In fact, the extended class of functionals (5.2) has aroused interest only in the past decade (see e.g. [RST12], [PR16], [Bre11], [LRM23], [AO18], [PS17]), because of the more and more important role that **spatio-temporal** functionals of random fields or random fields with **separable covariance function** (e.g. the fractional Brownian sheet) play now in applications.

In our work, the number p of growing domains and their dimensions d_i can be arbitrary. This is why we refer to (5.2) as a p-**domain** (rather than a *spatio-temporal*) **functional**. To give at least one explicit motivation for studying the asymptotic behavior of this type of functionals, let us consider the case where $\varphi = \mathbf{1}_{[a,\infty)}$. It corresponds to the excursion volume of B at level $a \in \mathbb{R}$ in the observation window $t_1 D_1 \times \cdots \times t_p D_p$, namely

$$Y(t_1, \ldots, t_p) = \mathrm{Vol}\left(\{(x_1, \ldots, x_p) \in t_1 D_1 \times \ldots \times t_p D_p \,:\, B_{(x_1,\ldots,x_p)} \geq a\}\right),$$

which is a (random) geometrical object that has been extensively studied in the literature, see e.g [AT07]. Our approach offers more flexibility than the usual one (corresponding to $p = 1$), as it gives the possibility to study the excursion sets of B when the parameters are in domains $t_1 D_1, \ldots, t_p D_p$ that can grow at different rates.

In a broader setting, the aim of the present work is to enhance our understanding of the asymptotic behaviour of p-domain functionals associated with stationary

Gaussian fields, when the covariance function has a specific form. In particular, we will show how, in some cases, this asymptotic behaviour can be simply obtained from that of 1-domain functionals, explaining in a new light and in a more systematic way some results from the recent literature (e.g. those contained in [RST12]).

5.1.1 Separable case

In this section, we investigate the asymptotic behavior of (5.3) when we assume that the covariance function of B is separable.

A covariance function $C : \mathbb{R}^d \to \mathbb{R}$ is said **separable** when it can be written as $C = C_1 \otimes \ldots \otimes C_p$, that is, when

$$C(x_1, \ldots, x_p) = \prod_{i=1}^{p} C_i(x_i), \quad x_i \in \mathbb{R}^{d_i}, \quad 1 \le i \le p, \tag{5.4}$$

for some functions $C_i : \mathbb{R}^{d_i} \to \mathbb{R}$ satisfying $C_i(0) = 1$ for each i. It is easy to check that C is non-negative definite if and only if C_i is non-negative definite for every i. Also, by stationarity of B on $\mathbb{R}^d$, we have that C_i is the covariance function of the field $B^{(i)} := (B_{(x_1,\ldots,x_i,\ldots,x_p)})_{x_i \in \mathbb{R}^{d_i}}$ for any fixed values of x_j, $j \neq i$. Since we are only interested in distributions, we can define the **marginal functionals**

$$Y_i(t_i) := \int_{t_i D_i} \varphi(B^{(i)}_{x_i}) dx_i, \quad i = 1, \ldots p, \tag{5.5}$$

and their normalized versions

$$\widetilde{Y}_i(t_i) := \frac{Y_i(t_i) - \mathbb{E}[Y_i(t_i)]}{\sqrt{\mathrm{Var}(Y_i(t_i))}}, \quad i = 1, \ldots p, \tag{5.6}$$

with the convention that, for any given i, the values of x_j, $j \neq i$, in (5.5) are arbitrary but fixed.

Let us denote by $\xrightarrow{d}$ the convergence in distribution. The main result of this section is the following.

Theorem 5.1.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $\mathbb{E}[\varphi^2(N)] < \infty$ for $N \sim N(0,1)$, with Hermite rank $R \ge 1$. Let $B = (B_x)_{x \in \mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $C : \mathbb{R}^d \to \mathbb{R}$ be the covariance function of B, assume it is separable in the sense of (5.4), and also that it satisfies, for each i:*

$$C^R \ge 0 \quad \text{and} \quad C_i \in \bigcup_{M=1}^{\infty} L^M(\mathbb{R}^{d_i}).$$

Let us consider $\widetilde{Y}$ given by (5.3) and $\widetilde{Y}_i$ given in (5.6). Then, the following two assertions are equivalent:

(a) $\widetilde{Y}_i(t_i) \xrightarrow{d} N(0,1)$ *as* $t_i \to \infty$ *for at least one* $i \in \{1, \ldots, p\}$*;*

(b) $\widetilde{Y}(t_1, \ldots, t_p) \xrightarrow{d} N(0,1)$ *as* $t_1, \ldots, t_p \to \infty$.

Remark 5.1.1. The integrability assumptions on C_i for Theorem 5.1.1 to hold may be removed when $\varphi = H_q$; moreover, in this case we also have **quantitative results** for the convergence in distribution (see Subsection 5.3.1).

Remark 5.1.2. Given a *discrete* stationary centered Gaussian field $B = (B_k)_{k\in\mathbb{Z}^d}$ with unit variance, the definition of separable covariance function $C : \mathbb{Z}^d \to \mathbb{R}$ is similar, i.e. $C(z) = \prod_{i=1}^p C_i(z_i)$ with $C_i : \mathbb{Z}^{d_i} \to \mathbb{R}$ for $i = 1, \dots, p$, whereas the functionals (5.2) and (5.5) are defined as follows:

$$Y(n_1, \dots, n_p) = \sum_{k\in(\prod_{i=1}^p [0,n_i]^{d_i})\cap\mathbb{Z}^d} \varphi(B_k), \qquad Y_i(n_i) = \sum_{k_i\in[0,n_i]^{d_i}\cap\mathbb{Z}^{d_i}} \varphi(B_{k_1,\dots,k_p}),$$

where $n_i \in \mathbb{N}$ for every $i = 1, \dots, p$. In this setting, analogous results to Theorem 5.1.1 could be obtained, see e.g. Remark 5.3.1 where we highlight that Proposition 5.3.3 directly translates with straightforward modifications for a discrete Gaussian field.

Remark 5.1.3. The previous result provides a large class of fields B with long-range dependence, i.e. with covariance function $C \notin L^R$, such that the functional $Y_t := Y(t, \dots, t)$ exhibits Gaussian fluctuations around its mean. See e.g. Example 5.4.3. We refer to [MN24] and the references therein for a deeper analysis and further examples of this phenomenon.

Since central limit theorems for 1-domain functionals have been extensively explored in the literature, it is not difficult to imagine how useful and powerful the implication $(a) \Rightarrow (b)$ in Theorem 5.1.1 can be. It is noteworthy that a specific instance of this implication had previously been observed in the papers [RST12] and [PR16]; however, it was restricted to a very specific context – rectangular increments of a fractional Brownian sheet – and was not part of a comprehensive systematic investigation, as we undertake in this work.

Since Theorem 5.1.1 establishes that (5.2) displays Gaussian fluctuations (if and only) if at least one of its marginal functionals does, a natural question arises: what happens when none of the $Y_i(t_i)$'s exhibits Gaussian fluctuations? We investigate this problem in the classical framework of regularly varying covariance functions.

Given two functions f and g, we write $f(x) \sim g(x)$ to indicate that $\lim_{\|x\|\to 0} f(x)g(x)^{-1} = 1$. Recalling that R denotes the Hermite rank of φ (see (5.1)), we consider the following conditions:

- C_i is regularly varying with parameter $-\beta_i \in (-d_i/R, 0)$, that is

$$C_i(z_i) = \rho_i(\|z_i\|) = L_i(\|z_i\|)\|z_i\|^{-\beta_i} \qquad \beta_i \in (0, d_i/R), \tag{5.7}$$

  where $L_i : (0, \infty) \to \mathbb{R}$ is slowly varying, i.e. $L_i(rs)/L_i(s) \to 1$ as $s \to \infty$, $\forall r > 0$.

- C_i admits an absolutely continuous spectral measure $G_i(d\lambda_i) = g_i(\lambda_i)d\lambda_i$ on $\mathbb{R}^{d_i}$, and for some constant $c_i > 0$ we have

$$g_i(\lambda_i) \sim c_i L_i(1/\|\lambda_i\|)\|\lambda_i\|^{\beta_i - d_i} \qquad \text{as } \|\lambda_i\| \to 0. \tag{5.8}$$

If (5.7)-(5.8) hold and $R \geq 2$, then it is known that $\widetilde{Y}_i(t_i)$ converges in distribution to a non-Gaussian random variable (see Theorem 5.2.4). In the following Theorem 5.1.2, we give conditions so that Y has non-Gaussian fluctuations either.

Theorem 5.1.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $\mathbb{E}[\varphi^2(N)] < \infty$ for $N \sim N(0,1)$, with Hermite rank $R \geq 1$ (in particular, we have $a_R \neq 0$, see (5.1)). Let $B = (B_x)_{x \in \mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $C : \mathbb{R}^d \to \mathbb{R}$ be the covariance function of B, assume it is separable in the sense of (5.4), and satisfies*

$$C^R \geq 0 \qquad \text{and} \qquad (5.7)\text{-}(5.8) \text{ hold} \quad \forall i \in \{1, \dots, p\}.$$

Let us consider $\widetilde{Y}$ given by (5.3). Then, as $t_1, \dots, t_p \to \infty$, we have

$$\widetilde{Y}(t_1, \dots, t_p) \xrightarrow{d} \frac{\operatorname{sgn}(a_R) I_{\nu_1 \times \ldots \times \nu_p, R}\left(f_{R, t_1 D_1 \times \ldots \times t_p D_p}\right)}{\sqrt{\operatorname{Var}\left(I_{\nu_1 \times \ldots \times \nu_p, R}\left(f_{R, t_1 D_1 \times \ldots \times t_p D_p}\right)\right)}}, \tag{5.9}$$

where $f_{R, t_1 D_1 \times \ldots \times t_p D_p} : (\mathbb{R}^d)^R \to \mathbb{C}$ is defined by

$$f_{R, t_1 D_1 \times \ldots \times t_p D_p}(\lambda_1, \dots, \lambda_R) := \int_{t_1 D_1 \times \ldots \times t_p D_p} e^{i \langle x, \lambda_1 + \cdots + \lambda_R \rangle} dx, \tag{5.10}$$

the product measure $\nu_1 \times \ldots \times \nu_p$ on $\mathbb{R}^d$ is defined by $\nu_i(dx_i) = \|x_i\|^{\beta_i - d_i} dx_i$ and $I_{\nu_1 \times \ldots \times \nu_p, R}$ denotes the Rth Wiener-Itô integral with respect to a σ-finite measure (see Section 5.2). In particular, the limit (5.9) is not Gaussian as soon as $R \geq 2$.

Theorem 5.1.2 represents a generalization from 1-domain to p-domain of the celebrated Dobrushin-Major-Taqqu (see Theorem 5.2.4). We remark that depending on the choice of the domains, the function f can be computed explicitly (see e.g. Remark 5.2.1). In the particular case of the rectangular increments of a fractional Brownian sheet, we note that Theorem 5.1.2 for $p = 2$ is already proved in [RST12] and [PR16].

Remark 5.1.4. Analogous results (both central and non-central) hold if we fix some of the domains, i.e. considering $t_1 D_1 \times \cdots \times t_{p-1} D_{p-1} \times D_p$ and $Y(t_1, \dots, t_{p-1}, 1)$. In this setting, it is possible to prove the following.

- Under the same assumptions of Theorem 5.1.1, the following two assertions are equivalent:
  - (a) $\widetilde{Y}_i(t_i) \xrightarrow{d} N(0,1)$ as $t_i \to \infty$ for at least one $i \in \{1, \dots, p-1\}$;
  - (b) $\widetilde{Y}(t_1, \dots, t_{p-1}, 1) \xrightarrow{d} N(0,1)$ as $t_1, \dots, t_{p-1} \to \infty$.
- Analogously to Theorem 5.1.2, if (5.7)-(5.8) hold for $i = 1, \dots, p-1$ and $C^R \geq 0$, if G_p is the spectral measure of C_p, then the convergence of Theorem 5.1.2 still holds replacing ν_p with G_p, namely

$$\widetilde{Y}(t_1, \dots, t_{p-1}, 1) \xrightarrow{d} \frac{\operatorname{sgn}(a_R) I_{\nu_1 \times \ldots \times \nu_{p-1} \times G_p, R}\left(f_{R, D_1 \times \ldots \times D_p}\right)}{\sqrt{\operatorname{Var}\left(I_{\nu_1 \times \ldots \times \nu_{p-1} \times G_p, R}\left(f_{R, D_1 \times \ldots \times D_p}\right)\right)}}, \; t_1, \dots, t_{p-1} \to \infty \,.$$

The proofs of these facts follow from similar arguments to the ones of Theorem 5.1.1 and Theorem 5.1.2 (see Section 5.3).

5.1.2 Non separable case

It is easy to construct examples illustrating that the normal convergence of functionals $\widetilde{Y}_i(t_i)$ is in general not enough to determine the behavior of $\widetilde{Y}(t_1, \ldots, t_p)$ when the separability of the covariance function (5.4) is dropped. See Example 5.5.1 for such a situation.

This is why we examine in Section 5.5 what can happen when we go beyond the separable case, by investigating two different classes: Gneiting covariance functions (Section 5.5.1) and additively separable covariance functions (Section 5.5.2).

Although not separable, the covariance functions belonging to the Gneiting class are wedged between two separable functions. It is therefore not surprising that a comparable phenomenon can still be proven in this context, see Theorem 5.5.1.

In contrast, the situation in the additively separable case is much more complicated. We still manage to prove a kind of "reduction" theorem, see Theorem 5.5.2, with however a big difference: the marginal functionals to be considered in the additively separable case are really different from the Y_i's of Theorem 5.1.1.

We refer to Section 5 for details.

5.1.3 Plan of the paper

The paper is organized as follows. Section 5.2 contain some needed preliminaries. In Section 5.3 we prove Theorem 5.1.1 and Theorem 5.1.2. In Section 5.4 we provide several examples where our results apply, and we compare them with the existing literature. In Section 5.5 we go beyond the separability assumption by investigating two other frameworks. Finally, in the Appendix we prove some auxiliary results.

5.2 Preliminaries

In this section we briefly present selected results on Malliavin-Stein method and classical results for 1-domain functionals.

5.2.1 Elements of Malliavin-Stein method

Theorem 5.1.1 is proved using the Fourth Moment Theorem by Nualart and Peccati (see [NP05]) and its quantitative version by Nourdin and Peccati (see [NP12, Theorem 5.2.6]). For all the missing details on Malliavin calculus we refer to [Nua06] or [NP12].

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a continuous, stationary, centered Gaussian field $B = (B_x)_{x\in\mathbb{R}^d}$ with unit variance and covariance function $C : \mathbb{R}^d \to \mathbb{R}$. We assume that $\mathcal{F}$ is the σ-field generated by B. By continuity and stationarity of B, the covariance function C is continuous. As a result, Bochner's theorem yields the existence of a unique real, symmetric[1], finite measure G on $\mathbb{R}^d$ endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, called the spectral measure of B, satisfying

$$C(x) = \int_{\mathbb{R}^d} e^{i\langle x,\lambda\rangle} G(d\lambda), \quad x \in \mathbb{R}^d. \tag{5.11}$$

[1]In the sense that $G(A) = G(-A)$ for every $A \in \mathcal{B}(\mathbb{R}^d)$.

We define the real separable Hilbert space

$$\mathcal{H} := L^2(G) = \left\{ h : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |h(\lambda)|^2 G(d\lambda) < \infty,\ \overline{h(\lambda)} = h(-\lambda) \right\}, \quad (5.12)$$

where $|\cdot|$ denotes the complex norm, endowed with the inner product[2]

$$\langle h, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} h(\lambda)\overline{g(\lambda)} G(d\lambda) = \int_{\mathbb{R}^d} h(\lambda) g(-\lambda) G(d\lambda). \quad (5.13)$$

Thanks to [NP12, Proposition 2.1.1] we may consider an isonormal Gaussian process X on $\mathcal{H}$, with covariance kernel

$$\mathbb{E}\left[X_G(h) X_G(g)\right] = \langle h, g \rangle_{\mathcal{H}}. \quad (5.14)$$

Moreover, by (5.11), the field $(X_G(e_x))_{x \in \mathbb{R}^d}$, where $e_x := \mathrm{e}^{i\langle x, \cdot \rangle}$, is stationary and Gaussian, with covariance function C and spectral measure G. Hence, the two fields share the same distribution, that is

$$(X_G(e_x))_{x \in \mathbb{R}^d} \stackrel{d}{=} (B_x)_{x \in \mathbb{R}^d}. \quad (5.15)$$

Since we study limit theorems in distribution, we assume from now on that $B_x = X_G(e_x)$ for any $x \in \mathbb{R}^d$. For $q \geq 1$, we also define the qth Wiener chaos as the linear subspace of $L^2(\Omega)$ generated by $\{H_q(X_G(h)) : h \in \mathcal{H}\}$, and for every $h \in \mathcal{H}$ such that $\|h\|_{\mathcal{H}} = 1$, we define the qth Wiener-Itô integral as

$$I_q(h^{\otimes q}) = H_q(X_G(h)), \quad (5.16)$$

where $h^{\otimes q} : (\mathbb{R}^d)^q \to \mathbb{R}$ is such that

$$h^{\otimes q}(x_1, \ldots, x_q) = \prod_{l=1}^{q} h(x_l). \quad (5.17)$$

By density the definition of I_q can be linearly extended to every function in the space $\mathcal{H}^{\odot q} = L^2_s((\mathbb{R}^d)^q, G^{\otimes q})$ of the *symmetric* functions in $\mathcal{H}^{\otimes q} = L^2((\mathbb{R}^d)^q, G^{\otimes q})$. This allows one to write

$$\int_D H_q(B_x)\, dx = I_q(f), \quad (5.18)$$

where $f : (\mathbb{R}^d)^q \to \mathbb{R}$ is given by

$$f(\lambda_1, \ldots, \lambda_q) = \int_D \mathrm{e}^{i\langle x, \lambda_1 + \cdots + \lambda_q \rangle} dx. \quad (5.19)$$

In the following, we write $I_{G,q}$ when we need to explicitly refer to the qth Wiener-Itô integral acting on $\mathcal{H}^{\odot q}$ with respect to the measure G.

For $q \in \mathbb{N}$, $r = 1, \ldots, q-1$ and h, g symmetric functions with unit norm in $\mathcal{H}$, we can define the rth contraction of $h^{\otimes q}$ and $g^{\otimes q}$ as the (generally non-symmetric)

[2] Note that $\langle h, g \rangle_{\mathcal{H}}$ is real for every $h, g \in \mathcal{H}$, because G is symmetric and h, g are even.

element of $\mathcal{H}^{\otimes 2q-2r}$ given by

$$h^{\otimes q} \otimes_r g^{\otimes q} = \langle h, g\rangle_{\mathcal{H}}^r \, h^{\otimes q-r} \otimes g^{\otimes q-r}. \tag{5.20}$$

We then extend the definition of contraction to every pair of elements in $\mathcal{H}^{\odot q}$. We will denote the norm in this space by $\|\cdot\|_q$.

We are finally ready to state the celebrated Fourth Moment Theorem.

Theorem 5.2.1 (Fourth Moment Theorem, [NP05])**.** *Fix $q \geq 2$, consider $(h_t)_{t>0} \subset \mathcal{H}^{\odot q}$ and assume that $\mathbb{E}[I_q(h_t)^2] \to 1$ as $t \to \infty$. Then, the following three assertions are equivalent:*

- *$I_q(h_t)$ converges in distribution to a standard Gaussian random variable $N \sim N(0,1)$;*
- *$\mathbb{E}[I_q(h_t)^4] \to 3 = \mathbb{E}[N^4]$, where $N \sim N(0,1)$;*
- *$\|h_t \otimes_r h_t\|_{2q-2r} \to 0$ as $t \to \infty$, for all $r = 1, \dots, q-1$.*

We will also need a quantitative version of the Fourth Moment Theorem, which can be stated as follows (see [NP12, Theorem 5.2.6 and (5.2.6)]):

Theorem 5.2.2. *Fix $q \geq 2$ and consider $h \in \mathcal{H}^{\odot q}$. Then*

$$\mathbb{E}[I_q(h)^4] - 3\mathbb{E}[I_q(h)^2]^2 = \frac{3}{q}\sum_{r=1}^{q-1} r r!^2 \binom{q}{r}^4 (2q-2r)! \|h \widetilde{\otimes}_r h\|_{2q-2r}^2 \tag{5.21}$$

$$= \sum_{r=1}^{q-1} q!^2 \binom{q}{r}^2 \left\{ \|h \otimes_r h\|_{2q-2r}^2 + \binom{2q-2r}{q-r} \|h \widetilde{\otimes}_r h\|_{2q-2r}^2 \right\} \tag{5.22}$$

and, with $N \sim N(0,1)$,

$$d_{TV}(I_q(h), N) \leq \frac{2}{\sqrt{3}} \sqrt{\mathbb{E}[I_q(h)^4] - 3\mathbb{E}[I_q(h)^2]^2}. \tag{5.23}$$

5.2.2 Classical results for 1-domain functionals

In this section, we state the two most popular results in the framework of limit theorems for 1-*domain* functionals. The first result, the celebrated Breuer-Major theorem, was first proved in [BM83] in a discrete version, and then extended to several settings. Here we state the continuous version of the result. Its proof can be found e.g. in [NZ20].

Theorem 5.2.3 (Breuer-Major)**.** *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued, continuous, centered Gaussian field on $\mathbb{R}^d$, assumed to be stationary and to have unit-variance. Let C denotes its covariance function. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $\mathbb{E}[\varphi^2(N)] < \infty$, $N \sim N(0,1)$, with Hermite rank R. Let us consider*

$$Y(t) := \int_{tD} \varphi(B_x)dx, \tag{5.24}$$

where $D \subset \mathbb{R}^d$ is a compact set, and $t \geq 0$. If $C \in L^R(\mathbb{R}^d)$, then, as $t \to \infty$,

$$\frac{Y(t) - \mathbb{E}[Y(t)]}{t^{d/2}} \xrightarrow{d} N(0, \sigma^2), \tag{5.25}$$

where $\sigma^2 = \mathrm{Vol}(D) \sum_{q=R}^{\infty} q! a_q^2 \int_{\mathbb{R}^d} C(x)^q dx \geq 0$. In particular, if $\sigma^2 > 0$, then $\mathrm{Var}(Y(t)) \sim \sigma^2 t^d$ and a central limit theorem holds for $(Y(t) - \mathbb{E}[Y(t)])/\sqrt{\mathrm{Var}(Y(t))}$.

The idea behind Breuer-Major theorem is that if the fields is not "too correlated" at infinity (precisely, if $\int_{\mathbb{R}^d} |C(x)|^R dx < \infty$), then the fluctuations of the functional Y are Gaussian. Conversely, if this is not the case, then one can have non-Gaussian fluctuations (this does not mean that we necessarily have non-Gaussian fluctuations, see e.g. [MN24]). The following Theorem 5.1.2 provides a non-central limit theorem for functionals of Gaussian fields having a regularly varying covariance function, see (5.7), satisfying (5.8). In the discrete case, a first proof for functionals with Hermite rank $R = 1, 2$, was given by Taqqu in [Taq75], and then generalized to any R by Dobrushin and Major in [DM79]. The following sticks to the continuous case and a proof can be found e.g. in [LO14].

Theorem 5.2.4 (Dobrushin-Major-Taqqu)**.** *Let $B = (B_x)_{x \in \mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $\mathbb{E}[\varphi^2(N)] < \infty$, $N \sim N(0,1)$, with Hermite rank R and Rth coefficient $a_R \neq 0$, see (5.1). Let us consider*

$$Y(t) := \int_{tD} \varphi(B_x) dx,$$

where D is a compact set with $\mathrm{Vol}(D) > 0$ and $t \geq 0$. Let $C : \mathbb{R}^d \to \mathbb{R}$ be the covariance function of B and suppose that (5.7)-(5.8) hold for C with parameter $-\beta \in (-d/R, 0)$. Then, as $t \to \infty$, we have

$$\widetilde{Y}(t) \xrightarrow{d} \frac{\mathrm{sgn}(\mathrm{a_R}) I_{\nu,R}(f_{R,D})}{\sqrt{\mathrm{Var}(I_{\nu,R}(f_{R,D}))}}, \tag{5.26}$$

where ν is the measure on $\mathbb{R}^d$ defined as $\nu(dx) := |x|^{\beta - d} dx$, and $f_{R,D}$ is defined as

$$f_{R,D}(\lambda_1, \dots, \lambda_R) := \int_D e^{i\langle x, \lambda_1 + \cdots + \lambda_R \rangle} dx. \tag{5.27}$$

In particular, the limit $I_{\nu,R}(f_{R,D})$ is not Gaussian as soon as $R \geq 2$.

Remark 5.2.1. The previous theorem shows that the limit $I_{\nu,R}(f_{R,D})$ depends on R, d, β and the domain D. In fact, the integrand is $f_{R,D}(\lambda_1, \dots, \lambda_R) = \mathcal{F}[\mathbf{1}_D](\lambda_1 + \cdots + \lambda_R)$, where $\mathcal{F}[\mathbf{1}_D]$ is the Fourier transform of the indicator function of the compact set D. The most common choices for D are the rectangles $D = \times_{i=1}^d [0, u_i]$, $u_i \in \mathbb{R}_+$, with Fourier transform

$$\mathcal{F}[\mathbf{1}_D](\lambda) = \prod_{j=1}^d \int_0^{u_j} e^{i\lambda_j x_j} dx_j = \prod_{j=1}^d \frac{e^{i\lambda_j u_j} - 1}{i\lambda_j},$$

and domains $D = \{x \in \mathbb{R}^d : \|x\| \le u\}$, $u \in \mathbb{R}_+$, with Fourier transform

$$\mathcal{F}[\mathbf{1}_D](\lambda) = c_d\, J_{d/2}(u\|\lambda\|) \left(\frac{u}{\|\lambda\|}\right)^{d/2},$$

where $J_{d/2}$ is a Bessel function of the first kind and order $d/2$, and c_d is a positive constant, see e.g. [MN24, Section 2.2].

5.3 Proof of Theorems 5.1.1 and 5.1.2

We split the proofs of Theorems 5.1.1 and 5.1.2 in four subsections. In Subsection 5.3.1, we prove Theorem 5.1.1 when $\varphi = H_q$, under weaker assumptions, providing also a quantitative result. In Subsection 5.3.2, we extend Theorem 5.2.3 to p-domain functionals. Finally, in Subsection 5.3.3 and 5.3.4 we prove Theorem 5.1.1 and Theorem 5.1.2, respectively.

5.3.1 Proof of Theorem 5.1.1 when $\varphi = H_q$

When $\varphi = H_q$, the functional (5.2) is given by

$$Y(t_1,\dots,t_p) = Y(t_1,\dots,t_p)[q] := \int_{t_1 D_1 \times \ldots \times t_p D_p} H_q(B_x)dx. \tag{5.28}$$

Therefore, see Section 5.2, we may express it as follows

$$\int_{t_1 D_1 \times \ldots \times t_p D_p} H_q(B_x)dx = I_q(f(t_1,\dots,t_p)), \tag{5.29}$$

where $I_q : \mathcal{H}^{\odot q} \to L^2(\Omega)$ stands for the qth Wiener-Ito integral (5.16), and $f(t_1,\dots,t_p) \in \mathcal{H}^{\odot q}$ is given by

$$f(t_1,\dots,t_p) := \int_{t_1 D_1 \times \ldots \times t_p D_p} e_x^{\otimes q} dx. \tag{5.30}$$

Let us recall the definition of the marginal functionals (5.5), and, accordingly, set

$$Y_i(t_i)[q] = I_q\left(\int_{t_i D_i} (e^{(i)}_{x_i})^{\otimes q} dx_i\right) = I_q(f_i(t_i)), \quad i = 1,\dots,p. \tag{5.31}$$

Proposition 5.3.1. *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $C : \mathbb{R}^d \to \mathbb{R}$ be the covariance function of B and assume it is separable in the sense of* (5.4)*. Then*

$$\mathrm{Var}(Y(t_1,\dots,t_p)[q]) = (q!)^{1-p} \prod_{i=1}^{p} \mathrm{Var}(Y_i(t_i)[q]). \tag{5.32}$$

Moreover, the following holds: if there exists $i \in \{1,\dots,p\}$ such that

$$\widetilde{Y}_i(t_i)[q] \xrightarrow{d} N(0,1) \qquad \textit{as } t_i \to \infty, \tag{5.33}$$

then

$$\widetilde{Y}(t_1,\dots,t_p)[q] \xrightarrow{d} N(0,1) \qquad \text{as } t_1,\dots,t_p \to \infty. \tag{5.34}$$

Proof. Note that $\mathrm{Var}(Y_i(t_i)[q]) = q!\|f_i(t_i)\|_q^2$. As a result, (5.32) follows from the separability of C:

$$\mathrm{Var}(Y(t_1,\dots,t_p)[q]) = q!\int_{(t_1D_1\times\ldots\times t_pD_p)^2} C(x_1-y_1,\dots,x_p-y_p)^q dx\,dy \tag{5.35}$$

$$= \prod_{i=1}^p \int_{(t_iD_i)^2} C_i(x_i-y_i)^q dx_i dy_i = (q!)^{1-p}\prod_{i=1}^p \mathrm{Var}(Y_i(t_i)[q]). \tag{5.36}$$

If $q=1$ everything is Gaussian and there is nothing more to show. Thus, let us assume that $q\geq 2$ and let us compute the norm of the rth contraction of $f(t_1,\dots,t_p)$ with itself, for $r=1,\dots,q-1$:

$$\|f(t_1,\dots,t_p)\otimes_r f(t_1,\dots,t_p)\|_{2q-2r}^2 = \tag{5.37}$$

$$= \int_{(t_1D_1\times\cdots\times t_pD_p)^4} C(x-z)^r C(y-u)^r C(x-y)^{q-r} C(z-u)^{q-r} dxdydudz \tag{5.38}$$

$$= \prod_{i=1}^p \int_{(t_iD_i)^4} C_i(x_i-z_i)^r C_i(y_i-u_i)^r C_i(x_i-y_i)^{q-r} C_i(z_i-u_i)^{q-r} dx_idy_idu_idz_i, \tag{5.39}$$

where x_i denotes the projection of x onto the ith block of $\mathbb{R}^d = \prod_{i=1}^p \mathbb{R}^{d_i}$. Let us define $\tilde{f} := \frac{f}{\|f\|}$. By linearity, it immediately follows that $\widetilde{Y}(t_1,\dots,t_p)[q] = I_q(\tilde{f}(t_1,\dots,t_p))$ and $\widetilde{Y}_i(t_i)[q] = I_q(\tilde{f}_i(t_i))$. Then, combining (5.37) and (5.32), we obtain

$$\|\tilde{f}(t_1,\dots,t_p)\otimes_r \tilde{f}(t_1,\dots,t_p)\|_{2q-2r}^2 = \prod_{i=1}^p \|\tilde{f}_i(t_i)\otimes_r \tilde{f}_i(t_i)\|_{2q-2r}^2. \tag{5.40}$$

Since for every $f\in\mathcal{H}^{\otimes q}$ and $r=1,\dots,q-1$, we have

$$\|f\otimes_r f\|_{2q-2r}^2 \leq \|f\|_q^4, \tag{5.41}$$

we obtain

$$\|\tilde{f}(t_1,\dots,t_p)\otimes_r \tilde{f}(t_1,\dots,t_p)\|_{2q-2r}^2 \leq \min\{\|\tilde{f}_i(t_i)\otimes_r \tilde{f}_i(t_i)\|_{2q-2r}^2 : i=1,\dots,p\}. \tag{5.42}$$

It remains to apply Theorem 5.2.1 to conclude that, if $\widetilde{Y}_i(t_i)\to N(0,1)$ as $t_i\to\infty$, then $\|\tilde{f}_i(t_i)\otimes_r\tilde{f}_i(t_i)\|_{2q-2r}\to 0$ for all $r\in\{1,\dots,q-1\}$, implying that $\|\tilde{f}(t_1,\dots,t_p)\otimes_r \tilde{f}(t_1,\dots,t_p)\|_{2q-2r}\to 0$ for all $r\in\{1,\dots,q-1\}$, implying finally that $\widetilde{Y}(t_1,\dots,t_p)\xrightarrow{d} N(0,1)$. □

The following result provides the converse implication of Proposition 5.3.1, under additional assumptions.

Proposition 5.3.2. *Let $B=(B_x)_{x\in\mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $C:\mathbb{R}^d\to\mathbb{R}$ be the covariance function of B and* *assume it is separable in the sense of* (5.4)*. Finally, assume either that* $C^q \geq 0$ *or* $q = 3$*. Then, the following holds: if*

$$\widetilde{Y}(t_1,\dots,t_p)[q] \xrightarrow{d} N(0,1) \qquad \text{as } t_1,\dots,t_p \to \infty, \tag{5.43}$$

then there exists at least one $i \in \{1,\dots,p\}$ *such that*

$$\widetilde{Y}_i(t_i)[q] \xrightarrow{d} N(0,1) \qquad \text{as } t_i \to \infty. \tag{5.44}$$

Proof. When $q = 1$, we have that $\widetilde{Y}$ and the $\widetilde{Y}_i$'s are Gaussian, meaning that the statement is correct but empty.

So, let us assume from now on that $q \geq 2$ and that $\widetilde{Y}(t_1,\dots,t_p)[q] \xrightarrow{d} N(0,1)$. By the Fourth Moment Theorem 5.2.1 one has that $\|\widetilde{f}(t_1,\dots,t_p) \otimes_r \widetilde{f}(t_1,\dots,t_p)\|_{2q-2r} \to 0$ as $t_1,\dots,t_p \to \infty$ for any $r \in \{1,\dots,q-1\}$.

When $q = 2$, there is only one contraction to consider. Looking at (5.40), we deduce that at least one of the factors in (5.40) must go to 0 which, by Theorem 5.2.1, implies that at least one among the $\widetilde{Y}_i(t_i)$'s must have a Gaussian limit.

Consider now the case where $q = 3$. By Fubini, we have for every $i = 1,\dots,p$

$$\|\widetilde{f}_i(t_i) \otimes_1 \widetilde{f}_i(t_i)\|_4^2 = \|\widetilde{f}_i(t_i) \otimes_2 \widetilde{f}_i(t_i)\|_2^2, \tag{5.45}$$

and (5.40) allows again to conclude.

Finally, let us suppose that $q \geq 4$ and $C^q \geq 0$, that is either q even, or q odd and $C \geq 0$. Since the second contraction satisfies

$$\|\widetilde{f}(t_1,\dots,t_p) \otimes_2 \widetilde{f}(t_1,\dots,t_p)\|_{2q-4} \longrightarrow 0, \quad \text{as } t_1,\dots,t_p \to \infty, \tag{5.46}$$

we deduce from (5.40) that there exists $i \in \{1,\dots,p\}$ such that

$$\|\widetilde{f}_i(t_i) \otimes_2 \widetilde{f}_i(t_i)\|_{2q-4}^2 \to 0 \text{ as } t_i \to \infty. \tag{5.47}$$

Let us show that this is sufficient to conclude about the asymptotic normality of $\widetilde{Y}_i(t_i)[q]$ when $C^q \geq 0$. For this, let us consider a positive sequence $(a_{t_i})_{t_i>0}$ with $a_{t_i} \to \infty$ as $t_i \to \infty$ (the exact expression of a_{t_i} will be made precise later) and define the subset A_{t_i} of $\mathbb{R}^{4d_i}$ by

$$A_{t_i} = \Big\{(x_i,y_i,u_i,z_i) \in \mathbb{R}^{4d_i} : |C_i(x_i - z_i)C_i(y_i - u_i)| \leq a_{t_i}|C_i(x_i - y_i)C_i(z_i - u_i)|\Big\}.$$

We can then write, for every $3 \leq r \leq q-1$:

$$\|f_i(t_i) \otimes_r f_i(t_i)\|^2_{2q-2r} = \tag{5.48}$$

$$= \int_{(t_iD_i)^4} C_i(x_i - z_i)^r C_i(y_i - u_i)^r C_i(x_i - y_i)^{q-r} C_i(z_i - u_i)^{q-r} dx_i dy_i du_i dz_i \tag{5.49}$$

$$\leq \int_{(t_iD_i)^4 \cap A_{t_i}} |C_i(x_i - z_i)^r C_i(y_i - u_i)^r C_i(x_i - y_i)^{q-r} C_i(z_i - u_i)^{q-r}| dx_i dy_i du_i dz_i \tag{5.50}$$

$$+ \int_{(t_iD_i)^4 \cap (\mathbb{R}^{4d_i} \setminus A_{t_i})} |C_i(x_i - z_i)^r C_i(y_i - u_i)^r C_i(x_i - y_i)^{q-r} C_i(z_i - u_i)^{q-r}| dx_i dy_i du_i dz_i \tag{5.51}$$

$$\leq a_{t_i}^{r-2} \int_{(t_iD_i)^4} C_i(x_i - z_i)^2 C_i(y_i - u_i)^2 |C_i(x_i - y_i)|^{q-2} |C_i(z_i - u_i)|^{q-2} dx_i dy_i du_i dz_i \tag{5.52}$$

$$+ a_{t_i}^{-(q-r)} \int_{(t_iD_i)^4} |C_i(x_i - z_i)|^q |C_i(y_i - u_i)|^q dx_i dy_i du_i dz_i \tag{5.53}$$

$$= a_{t_i}^{r-2} \int_{(t_iD_i)^4} C_i(x_i - z_i)^2 C_i(y_i - u_i)^2 C_i(x_i - y_i)^{q-2} C_i(z_i - u_i)^{q-2} dx_i dy_i du_i dz_i \tag{5.54}$$

$$+ a_{t_i}^{-(q-r)} \int_{(t_iD_i)^4} C_i(x_i - z_i)^q C_i(y_i - u_i)^q dx_i dy_i du_i dz_i \tag{5.55}$$

$$= a_{t_i}^{r-2} \|f_i(t_i) \otimes_2 f_i(t_i)\|^2_{2q-4} + a_{t_i}^{-(q-r)} \|f_i(t_i)\|^4_q. \tag{5.56}$$

Summarizing, we have, for every $r \in \{3, \ldots, q-1\}$,

$$\|\widetilde{f}_i(t_i) \otimes_r \widetilde{f}_i(t_i)\|^2_{2q-2r} \leq a_{t_i}^{r-2} \|\widetilde{f}_i(t_i) \otimes_2 \widetilde{f}_i(t_i)\|^2_{2q-4} + a_{t_i}^{-(q-r)}. \tag{5.57}$$

Now, let us choose $a_{t_i} = (\|\widetilde{f}_i(t_i) \otimes_2 \widetilde{f}_i(t_i)\|_{2q-4})^{\frac{2}{2-q}}$ and observe that $a_{t_i} \to \infty$ as $t_i \to \infty$. Plugging into (5.57) allows to obtain, for every $r \in \{3, \ldots, q-1\}$,

$$\|\widetilde{f}_i(t_i) \otimes_r \widetilde{f}_i(t_i)\|^2_{2q-2r} \leq 2(\|\widetilde{f}_i(t_i) \otimes_2 \widetilde{f}_i(t_i)\|_{2q-4})^{\frac{2(q-r)}{q-2}} \to 0 \quad \text{as } t_i \to \infty,$$

But this convergence of the rth contraction to zero also holds for $r = 1$, since the norms of the 1-contraction and the $(q-1)$-contraction are equal. Finally, the desired conclusion follows from the Fourth Moment Theorem 5.2.1. □

Proposition 5.3.3. *Let the same notations and assumptions of Proposition 5.3.1 prevail. In particular $N \sim N(0,1)$. Then, the following estimate holds:*

$$d_{TV}(\widetilde{Y}(t_1, \ldots, t_p)[q], N) \leq c_q \prod_{i=1}^{p} \sqrt{\mathbb{E}[\widetilde{Y}_i[q]^4] - 3}, \tag{5.58}$$

where $c_q = \sqrt{\frac{4}{q} \sum_{r=1}^{q-1} r r!^2 \binom{q}{r}^4 (2q-2r)!}$.

Proof of Proposition 5.3.3. Recall the result of Theorem 5.2.2, namely

$$d_{TV}(\widetilde{Y}(t_1, \ldots, t_p)[q], N) \leq \frac{2}{\sqrt{3}} \sqrt{\mathbb{E}[(\widetilde{Y}(t_1, \ldots, t_p)[q])^4] - 3}, \tag{5.59}$$

and

$$\mathbb{E}[(\widetilde{Y}(t_1,\ldots,t_p)[q])^4]-3 \tag{5.60}$$
$$=\frac{3}{q}\sum_{r=1}^{q-1} r r!^2\binom{q}{r}^4 (2q-2r)!\|\widetilde{f}(t_1\ldots,t_p)\widetilde{\otimes}_r\widetilde{f}(t_1\ldots,t_p)\|^2_{2q-2r} \tag{5.61}$$
$$=\sum_{r=1}^{q-1} q!^2\binom{q}{r}^2 \{\|\widetilde{f}(t_1\ldots,t_p)\otimes_r\widetilde{f}(t_1\ldots,t_p)\|^2_{2q-2r} \tag{5.62}$$
$$+\binom{2q-2r}{q-r}\|\widetilde{f}(t_1\ldots,t_p)\widetilde{\otimes}_r\widetilde{f}(t_1\ldots,t_p)\|^2_{2q-2r}\}. \tag{5.63}$$

We deduce

$$d_{TV}(\widetilde{Y}(t_1,\ldots,t_p)[q],N)\leq\frac{2}{\sqrt{3}}\sqrt{\mathbb{E}[(\widetilde{Y}(t_1,\ldots,t_p)[q])^4]-3} \tag{5.64}$$
$$\leq c_q\max_{r=1,\ldots,q-1}\|\widetilde{f}(t_1\ldots,t_p)\otimes_r\widetilde{f}(t_1\ldots,t_p)\|^2_{2q-2r}. \tag{5.65}$$

But, thanks to (5.40) and recalling (5.41), we have

$$\max_{r=1,\ldots,q-1}\|\widetilde{f}(t_1\ldots,t_p)\otimes_r\widetilde{f}(t_1\ldots,t_p)\|^2_{2q-2r}=\max_{r=1,\ldots,q-1}\prod_{i=1}^{p}\|\widetilde{f}_i(t_i)\otimes_r\widetilde{f}_i(t_i)\|^2_{2q-2r} \tag{5.66}$$
$$\leq\prod_{i=1}^{p}\max_{r=1,\ldots,q-1}\|\widetilde{f}_i(t_i)\otimes_r\widetilde{f}_i(t_i)\|^2_{2q-2r}. \tag{5.67}$$

Also, as a consequence of the second equality in (5.60) (with $\widetilde{Y}_i$ instead of $\widetilde{Y}$), we have

$$\|\widetilde{f}_i(t_i)\otimes_r\widetilde{f}_i(t_i)\|^2_{2q-2r}\leq\mathbb{E}[\widetilde{Y}_i[q]^4]-3,$$

and the desired conclusions now easily follows. □

Remark 5.3.1. In Example 5.4.1 below we will use the previous result to improve the bound for the central convergence of the rescaled qth Hermite variation of the rectangular increments of the fractional Brownian sheet obtained in [RST12].

5.3.2 A Breuer-Major theorem for p-domain functionals

In this subsection, we provide an extension of the celebrated Breuer-Major theorem from the classical setting of 1-domain functionals (see Theorem 5.2.3) to the setting of p-domain functionals.

Theorem 5.3.4. *Let $B=(B_x)_{x\in\mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a measurable function satisfying $\mathbb{E}[\varphi^2(N)]<\infty$, $N\sim N(0,1)$, with Hermite rank $R\geq 1$. Let us consider $\widetilde{Y}$ as in (5.3). Let $C:\mathbb{R}^d\to\mathbb{R}$ be the covariance function of B and assume it is separable in the sense* *of* (5.4). *If* $C_i \in L^R(\mathbb{R}^{d_i})$ *for any* $i \in \{1,\dots,p\}$, *then, as* $t_1,\dots,t_p \to \infty$,

$$\frac{Y(t_1,\dots,t_p) - \mathbb{E}[Y(t_1,\dots,t_p)]}{t_1^{d_1/2}\dots t_p^{d_p/2}} \xrightarrow{d} N(0,\sigma^2), \tag{5.68}$$

where

$$\sigma^2 = \sum_{q\geq R} a_q^2 q! \prod_{i=1}^{p} \mathrm{Vol}(D_i) \int_{\mathbb{R}^{d_i}} C_i^q(z_i)dz_i. \tag{5.69}$$

Moreover, if $\sigma^2 > 0$ *then*

$$\mathrm{Var}(Y(t_1,\dots,t_p)) \sim \sigma^2\, t_1^{d_1}\dots t_p^{d_p} \tag{5.70}$$

and we have a central limit theorem for (5.3), *that is*

$$\widetilde{Y}(t_1,\dots,t_p) \xrightarrow{d} N(0,1) \quad \text{as } t_1,\dots,t_p \to \infty.$$

Remark 5.3.2. The previous theorem may be seen as a Breuer-Major theorem for p-domains. Its proof follows from similar arguments to the ones in [NP12, Theorem 7.2.4]. However, we observe that the separable assumption allows us to recover the rate of the variance as a function of the (possibly distinct) growth rates of the domains.

Proof of Theorem 5.3.4. We proceed as in the proof of [NP12, Theorem 7.2.4], using [NP12, Theorem 6.3.1]. First of all, we have

$$\frac{Y(t_1,\dots,t_p) - \mathbb{E}[Y(t_1,\dots,t_p)]}{t_1^{d_1/2}\dots t_p^{d_p/2}} = \frac{\sum_{q=R}^{\infty} a_q \int_{t_1D_1\times\dots\times t_pD_p} H_q(B_x)dx}{t_1^{d_1/2}\dots t_p^{d_p/2}} \tag{5.71}$$

$$= \sum_{q=R}^{\infty} I_q(f_q(t_1,\dots,t_p)), \tag{5.72}$$

where

$$f_q(t_1,\dots,t_p) := \frac{a_q \int_{t_1D_1\times\dots\times t_pD_p} e_x^{\otimes q}dx}{t_1^{d_1/2}\dots t_p^{d_p/2}}.$$

To conclude the proof we only need to check the conditions (a)-(d) of [NP12, Theorem 6.3.1].

Condition (a): We need to check that

$$\sigma^2[q] := \lim_{t_1,\dots,t_p\to\infty} q!\|f_q(t_1,\dots,t_p)\|^2$$

exists in $[0,\infty)$ for each $q \geq R$. Since $C_i \in L^R(\mathbb{R}^{d_i})$ (implying $C_i \in L^q(\mathbb{R}^{d_i})$) for every $i \in \{1,\dots,p\}$, the change of variable $z_i = x_i - y_i$ yields

$$\int_{(t_iD_i)^2} C_i^q(x_i - y_i)dx_idy_i = t_i^{d_i} \int_{\mathbb{R}^{d_i}} C_i^q(z_i)\,\mathrm{Vol}(D_i \cap (D_i + z_i/t_i))dz_i \tag{5.73}$$

$$\sim t_i^{d_i}\,\mathrm{Vol}(D_i) \int_{\mathbb{R}^{d_i}} C_i^q(z_i)dz_i, \qquad \text{as } t_i \to \infty. \tag{5.74}$$

Combining the previous equivalent with the separability of C, we obtain

$$q!\|f_q(t_1,\dots,t_p)\|^2 = q!a_q^2 \frac{\prod_{i=1}^p \int_{(t_iD_i)^2} C_i^q(x_i-y_i)dx_idy_i}{t_1^{d_1/2}\dots t_p^{d_p/2}} \to \sigma^2[q], \tag{5.75}$$

as $t_1,\dots,t_p \to \infty$, where

$$\sigma^2[q] := q!a_q^2 \prod_{i=1}^p \mathrm{Vol}(D_i) \int_{\mathbb{R}^{d_i}} C_i^q(z_i)dz_i.$$

Condition (b): We need to check that

$$\sum_{q=R}^{\infty} \sigma^2[q] < \infty.$$

Since $\|C_i\|_\infty \le 1$ for every $i=1,\dots,p$, we have

$$\sum_{q=R}^{\infty} \sigma^2[q] \le \left(\sum_{q=R}^{\infty} q!a_q^2\right) \prod_{i=1}^p \left(\mathrm{Vol}(D_i)\int_{\mathbb{R}^{d_i}} |C_i(z_i)|^R dz\right).$$

Thanks to $\mathrm{Var}(\varphi(N)) = \sum_{q=R}^\infty q!a_q^2 < \infty$, the claim follows.

Condition (c): We need to check that for every $q \ge R$ and every $r = 1,\dots,q-1$

$$\|f_q(t_1,\dots,t_p)\otimes_r f_q(t_1,\dots,t_p)\|^2 \to 0 \qquad \text{as } t_1,\dots,t_p \to \infty.$$

Since C is separable, we deduce from (5.37) that

$$\|f_q(t_1,\dots,t_p)\otimes_r f_q(t_1,\dots,t_p)\|^2 = \prod_{i=1}^p \|f_{i,q}(t_i)\otimes_r f_{i,q}(t_i)\|^2,$$

where $f_{i,q}(t_i)$ is given by

$$f_{i,q}(t_i) := \int_{t_iD_i} (e_{x_i}^{(i)})^{\otimes q} dx_i.$$

Therefore, it is enough to prove that

$$\|f_{i,q}(t_i)\otimes_r f_{i,q}(t_i)\|^2 \to 0 \tag{5.76}$$

for at least one $i \in \{1,\dots,p\}$ as $t_i \to \infty$. But (5.76) is actually true for any $i \in \{1,\dots,p\}$, see indeed point (c) in the proof of [NP12, Theorem 7.2.4] (which uses that $C_i \in L^R(\mathbb{R}^{d_i})$).

Condition (d): We need to check that

$$\lim_{N\to\infty} \sup_{t_1,\dots,t_p\ge 1} \sum_{q=N+1}^{\infty} q!\|f_q(t_1,\dots,t_p)\|^2 = 0.$$

By (5.73) and since $\|C_i\|_\infty \leq 1$ for every $i = 1, \dots, p$, we have

$$\begin{aligned}
&\sup_{t_1,\dots,t_p \geq 1} \sum_{q=N+1}^{\infty} q! \|f_q(t_1, \dots, t_p)\|^2 \\
&= \sup_{t_1,\dots,t_p \geq 1} \sum_{q=N+1}^{\infty} q! a_q^2 \frac{\prod_{i=1}^p \int_{(t_i D_i)^2} C_i^q(x_i - y_i) dx_i dy_i}{t_1^{d_1/2} \dots t_p^{d_p/2}} \\
&\leq \left(\sum_{q=N+1}^{\infty} q! a_q^2 \right) \prod_{i=1}^{p} \left(\mathrm{Vol}(D_i) \int_{\mathbb{R}^{d_i}} |C_i(z_i)|^R dz_i \right) \to 0 \qquad \text{as } N \to \infty,
\end{aligned}$$

where $\left(\sum_{q=N+1}^{\infty} q! a_q^2\right) \to 0$ being the tail of a convergent series. □

5.3.3 Reduction to Rth chaos and proof of Theorem 5.1.1

The chaotic decomposition of (5.2) is

$$Y(t_1, \dots, t_p) = \mathbb{E}[Y(t_1, \dots, t_p)] + \sum_{q \geq R} a_q Y(t_1, \dots, t_p)[q], \tag{5.77}$$

see (5.28)-(5.29). In particular, this decomposition gives

$$\mathrm{Var}(Y(t_1, \dots, t_p)) = \sum_{q=R}^{\infty} a_q^2 \, \mathrm{Var}(Y(t_1, \dots, t_p)[q]). \tag{5.78}$$

In order to prove Theorem 5.1.1, we reduce the study of $\widetilde{Y}(t_1, \dots, t_p)$ to that of $\widetilde{Y}(t_1, \dots, t_p)[R]$, the normalization of $Y(t_1, \dots, t_p)[R]$, thanks to the following extension of [MN24, Proposition 4].

Proposition 5.3.5. *Let $B = (B_x)_{x \in \mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $\mathbb{E}[\varphi^2(N)] < \infty$, $N \sim N(0,1)$, with Hermite rank R. Let us consider $\widetilde{Y}$ as in (5.3), and $\widetilde{Y}[R]$ as above. Let $C : \mathbb{R}^d \to \mathbb{R}$ be the covariance function of B, and assume that it is separable in the sense of (5.4) and that it satisfies the following two hypotheses:*

1. *$C^R \geq 0$, that is, $C_i^R \geq 0$ for every $i = 0, \dots, p$;*
2. *for some $j \in \{1, \dots, p\}$, we have $C_j \in \bigcup_{M=R+1}^{\infty} L^M(\mathbb{R}^{d_j}) \setminus L^R(\mathbb{R}^{d_j})$.*

Then, with $\mathrm{sgn}(a_R)$ denoting the sign of the Rth Hermite coefficient in (5.1),

$$\mathbb{E}\left[\left(\mathrm{sgn}(a_R) \widetilde{Y}(t_1, \dots, t_p)[R] - \widetilde{Y}(t_1, \dots, t_p) \right)^2 \right] \to 0 \qquad \text{as } t_1, \dots, t_p \to \infty. \tag{5.79}$$

Proof. We divide the proof in three steps.

Step 1: upper and lower bounds for the variance. By assumption, their exist j and $M \geq R+1$ such that $C_j \notin L^R(\mathbb{R}^{d_j})$, but $C_j \in L^M(\mathbb{R}^{d_j})$. Since $C_j^R \geq 0$ by assumption, by doubling conditions for non-negative definite functions (see [GT19]) and properties of covariograms (see [Gal11]), reasoning exactly as in the first step of the proof of [MN24, Proposition 9], there exist two positive constants $c_1 > c_2 > 0$ such that

$$c_2 \int_{\{\|x_j\|\le t_j\}} C_j^R(z_j)dz_j \le t_j^{-d_j}\,\mathrm{Var}(Y_j(t_j)[R]) \le c_1 \int_{\{\|x_j\|\le t_j\}} C_j^R(z_j)dz_j.$$

In particular, this implies

$$t_j^{-d_j}\,\mathrm{Var}(Y_j(t_j)[R]) \to \infty \qquad \text{as } t_j \to \infty. \tag{5.80}$$

Step 2: we prove that $\mathrm{Var}(Y(t_1,\dots,t_p)) \sim a_R^2 \mathrm{Var}(Y(t_1,\dots,t_p)[R])$ *as* $t_1,\dots,t_p \to \infty$. Since $|C| \le 1$ and by assumption $C_i^R \ge 0$ for $i \in \{1,\dots,p\}$, we have that, for any $q > R$ and any $j \in \{1,\dots,p\}$,

$$\frac{\mathrm{Var}(Y(t_1,\dots,t_p)[q])}{\mathrm{Var}(Y(t_1,\dots,t_p)[R])} = \frac{q!}{R!}\prod_{i=1}^p \frac{\int_{(t_iD_i)^2} C_i^q(x_i-y_i)dx_idy_i}{\int_{(t_iD_i)^2} C_i^R(x_i-y_i)dx_idy_i} \tag{5.81}$$

$$\le \frac{q!}{R!}\frac{\int_{(t_jD_j)^2} C_j^q(x_j-y_j)dx_jdy_j}{\int_{(t_jD_j)^2} C_j^R(x_j-y_j)dx_jdy_j} = \frac{\mathrm{Var}(Y(t_j)[q])}{\mathrm{Var}(Y(t_j)[R])}. \tag{5.82}$$

Now, by applying Cauchy-Schwarz n times, we obtain

$$\begin{aligned}
\frac{\int_{(t_jD_j)^2} C_j^q(x_j-y_j)dx_jdy_j}{\int_{(t_jD_j)^2} C_j^R(x_j-y_j)dx_jdy_j} &\le \left(\frac{\int_{(t_jD_j)^2} |C_j(x_j-y_j)|^{2q-R}dx_jdy_j}{\int_{(t_jD_j)^2} C_j^R(x_j-y_j)dx_jdy_j}\right)^{1/2} \\
&\le \left(\frac{\int_{(t_jD_j)^2} |C_j(x_j-y_j)|^{4q-3R}dx_jdy_j}{\int_{(t_jD_j)^2} C_j^R(x_j-y_j)dx_jdy_j}\right)^{1/4} \\
&\le \dots \\
&\le \left(\frac{\int_{(t_jD_j)^2} |C_j(x_j-y_j)|^{R+2^n(q-R)}dx_jdy_j}{\int_{(t_jD_j)^2} C_j^R(x_j-y_j)dx_jdy_j}\right)^{1/2^n} \\
&\le \left(\mathrm{Vol}(D_j)t_j^{d_j}\frac{\int_{\{\|x_j\|\le \mathrm{diam}(D_j)t_j\}} |C_j(z_j)|^{R+2^n}dz_j}{\int_{(t_jD_j)^2} C_j^R(x_j-y_j)dx_jdy_j}\right)^{1/2^n} \\
&= \left(\mathrm{Vol}(D_j)t_j^{d_j}R!\frac{\int_{\{\|x_j\|\le \mathrm{diam}(D_j)t_j\}} |C_j(z_j)|^{R+2^n}dz_j}{\mathrm{Var}(Y_j(t_j)[R])}\right)^{1/2^n}
\end{aligned}$$

where the last inequality follows by a change of variable $x_j - y_j = z_j$, and the fact that $q - R \ge 1$. Since $C_j \in L^{R+2^n}(\mathbb{R}^{d_j})$ for n sufficiently large, we deduce from (5.80) for every $q > R$

$$\frac{\mathrm{Var}(Y(t_1,\dots,t_p)[q])}{\mathrm{Var}(Y(t_1,\dots,t_p)[R])} \le cq!\left(t_j^{d_j}/\,\mathrm{Var}(Y_j(t_j)[R])\right)^{1/2^n},$$

where $c > 0$ depends on R and n but not on q. Combining this with (5.78), we get

$$\left| \frac{\mathrm{Var}(Y(t_1,\dots,t_p))}{\mathrm{Var}(Y(t_1,\dots,t_p)[R])} - a_R^2 \right| \le c \left(\sum_{q=R+1}^{\infty} a_q^2 q! \right) \left(t_j^{d_j} / \mathrm{Var}(Y_j(t_j)[R]) \right)^{1/2^n}. \quad (5.83)$$

Using (5.80), this implies $\mathrm{Var}(Y(t_1,\dots,t_p)) \sim a_R^2 \mathrm{Var}(Y(t_1,\dots,t_p)[R])$ as $t_1,\dots,t_p \to \infty$.

Step 3: $\mathbb{E}\left[\left(\mathrm{sgn}(a_R)\widetilde{Y}(t_1,\dots,t_p)[R] - \widetilde{Y}(t_1,\dots,t_p)\right)^2\right] \to 0$. To prove this last step, considering the decomposition

$$\widetilde{Y}(t_1,\dots,t_p) - \mathrm{sgn}(a_R)\widetilde{Y}(t_1,\dots,t_p)[R] \quad (5.84)$$

$$= \frac{\mathrm{sgn}(a_R)(Y(t_1,\dots,t_p) - \mathbb{E}[Y(t_1,\dots,t_p)] - a_R Y(t_1,\dots,t_p)[R])}{a_R\sqrt{\mathrm{Var}(Y(t_1,\dots,t_p)[R])}} \quad (5.85)$$

$$+ \frac{Y(t_1,\dots,t_p) - \mathbb{E}[Y(t_1,\dots,t_p)]}{\sqrt{\mathrm{Var}(Y(t_1,\dots,t_p))}} \left\{ 1 - \frac{1}{|a_R|}\sqrt{\frac{\mathrm{Var}(Y(t_1,\dots,t_p))}{\mathrm{Var}(Y(t_1,\dots,t_p)[R])}} \right\}, \quad (5.86)$$

we get that

$$\mathbb{E}\left[\left(\widetilde{Y}(t_1,\dots,t_p) - \mathrm{sgn}(a_R)\widetilde{Y}(t_1,\dots,t_p)[R]\right)^2\right]$$

$$\le 2\frac{\mathbb{E}[(Y(t_1,\dots,t_p) - \mathbb{E}[Y(t_1,\dots,t_p)] - a_R Y(t_1,\dots,t_p)[R])^2]}{a_R^2 \mathrm{Var}(Y(t_1,\dots,t_p)[R])} \quad (5.87)$$

$$+ 2\left(1 - \frac{1}{|a_R|}\sqrt{\frac{\mathrm{Var}(Y(t_1,\dots,t_p))}{\mathrm{Var}(Y(t_1,\dots,t_p)[R])}}\right)^2. \quad (5.88)$$

By the previous step, the second addend converges to 0. Regarding the first addend, since by (5.78) we have

$$\mathbb{E}\Big[(Y(t_1,\dots,t_p) - \mathbb{E}[Y(t_1,\dots,t_p)] - a_R Y(t_1,\dots,t_p)[R])^2\Big] = \sum_{q=R+1}^{\infty} a_q^2 \,\mathrm{Var}(Y(t_1,\dots,t_p)[q]),$$

we deduce from the previous step that

$$\frac{\mathbb{E}[(Y(t_1,\dots,t_p) - \mathbb{E}[Y(t_1,\dots,t_p)] - a_R Y(t_1,\dots,t_p)[R])^2]}{a_R^2 \mathrm{Var}(Y(t_1,\dots,t_p)[R])} \to 0. \quad (5.89)$$

This concludes the proof of Step 3 and Proposition 5.3.5.

□

We are now in a position to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. Let us recall that $B = (B_x)_{x\in\mathbb{R}^d}$ is a stationary and continuous Gaussian random field, and that its covariance function is

$$C(x) = C_1(x_1)\cdot\dots\cdot C_p(x_p), \quad x_i \in \mathbb{R}^{d_i},\ i = 1,\dots,p, \quad (5.90)$$

where $C_i : \mathbb{R}^{d_i} \to \mathbb{R}$ is a non-negative definite function such that $C_i(0) = 1$ for every $i = 1, \dots, p$.

The functionals $\widetilde{Y}(t_1, \dots, t_p)$ and $\widetilde{Y}_i(t_i)$ are well-defined when C^R is positive. Indeed, by the decomposition (5.78) and the formula (5.36), we get that their variances are strictly positive.

Let us now distinguish two cases:

(i) The first one is when $C_i \in L^R(\mathbb{R}^{d_i})$ for every i. In this case, each functional $Y_i(t_i)$ satisfies Theorem 5.2.3, hence $\widetilde{Y}_i(t_i)$ has a Gaussian limit. Moreover, Theorem 5.3.4 applies, too, and we can conclude the same for $\widetilde{Y}(t_1, \dots, t_p)$.

(ii) Secondly, let us suppose that for at least one i one has $C_i \in L^M(\mathbb{R}^{d_i}) \setminus L^R(\mathbb{R}^{d_i})$ with $M > R$. Then, by Proposition 5.3.5, it is equivalent to study the limit in distribution of $\widetilde{Y}(t_1, \dots, t_p)[R]$ only. Being in a fixed chaos and because $C_j^R \geq 0$, both Proposition 5.3.1 and Proposition 5.3.2 apply. Hence, if there exist j (possibly equal or distinct from i) such that $\widetilde{Y}_j(t_j)[R]$ converges in distribution to a standard Gaussian, we can deduce the same for the functional $\widetilde{Y}(t_1, \dots, t_p)[R]$, and vice-versa. To conclude, it remains to show that $\widetilde{Y}_j(t_j)$ converges in distribution to a standard Gaussian random variable if and only if $\widetilde{Y}_j(t_j)[R]$ does. When $C_j \in L^R(\mathbb{R}^{d_j})$, it is a consequence of the usual Breuer-Major theorem (Theorem 5.2.3), since in this case both $\widetilde{Y}_j(t_j)$ and $\widetilde{Y}_j(t_j)[R]$ are converging to $N(0, 1)$. When, on the contrary, we have $C_j \in L^M(\mathbb{R}^{d_j}) \setminus L^R(\mathbb{R}^{d_j})$, it is a consequence of the reduction theorem proved in [MN24, Proposition 4]. □

5.3.4 Proof of Theorem 5.1.2

We state two results on Wiener-Itô integrals, which are needed for our proof of Theorem 5.1.2. We note that they should not be considered as new, but rather as a generalization of well-known results contained in, e.g., [Maj81]. However, for completeness we provide their detailed proofs in the Appendix.

Lemma 5.3.6 (Change of variable formula). *Let ν, ν' be real, σ-finite measures on $\mathbb{R}^d$ endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and let us define the real separable[3] Hilbert spaces $\mathcal{H}_\nu := L^2(\nu), \mathcal{H}_{\nu'} := L^2(\nu')$ as in (5.12). Let us suppose that*

$$\nu(dx) = |a(x)|^2 \nu'(dx) \tag{5.91}$$

where a is a complex valued, even function (i.e. $a(-x) = \overline{a(x)}$). Then, for every $h \in \mathcal{H}_\nu{}^{\odot q}$ we have

$$I_{\nu,q}(h) \overset{\text{law}}{=} I_{\nu',q}(a^{\otimes q} h),$$

where $a^{\otimes q}$ is as in (5.17).

Lemma 5.3.7. *Let us denote by I_q the Wiener-Itô integral acting on $\mathcal{H}^{\odot q}$, with respect to the Lebesgue measure dx on $\mathbb{R}^d$. Then, for every $h \in \mathcal{H}^{\odot q} = L^2_s((\mathbb{R}^d)^q, dx)$ and $s_1, \dots, s_d > 0$, we have*

$$I_q(h) \overset{\text{law}}{=} s_1^{-q/2} \dots s_d^{-q/2} I_q\left(\tilde{h}(s_1, \dots, s_d)\right),$$

[3] Note that if ν is real, σ-finite and symmetric, then $\mathcal{H}_\nu$ is a real separable Hilbert space, and we can consider an isonormal Gaussian process on it, as well as every notion and result introduced in Section 5.2.1 for $L^2(G)$.

where $\tilde{h}(s_1,\ldots,s_d) \in \mathcal{H}^{\odot q}$ *is such that*

$$\tilde{h}(s_1,\ldots,s_d)(x_1,\ldots,x_q) := h(x_{11}/s_1,\ldots,x_{1d}/s_d,\ldots,x_{q1}/s_1,\ldots,x_{qd}/s_d). \quad (5.92)$$

Proof of Theorem 5.1.2. We divide the proof in four steps. In the sequel, c will denote a positive constant which may vary depending on the instance.

Step 1: Reduction to the Rth chaos. Since $C^R \geq 0$ and $C_i \in \bigcup_{M=R+1}^{\infty} L^M(\mathbb{R}^{d_i}) \setminus L^R(\mathbb{R}^{d_i})$ for every i, by Proposition 5.3.5 we are left to study the limit in distribution of $\widetilde{Y}(t_1,\ldots,t_p)[R]$.

Step 2: Variance analysis. It is a standard fact (see e.g. [LO13], [LO14], [Leo99] or [Mai24]) that if (5.7) holds for i, then

$$\mathrm{Var}(Y_i(t_i)[R]) \sim c\, L_i(t_i)^R t_i^{2d_i - R\beta_i} \quad \text{as } t_i \to \infty.$$

We conclude that

$$\mathrm{Var}(Y(t_1,\ldots,t_p)[R]) = \frac{1}{(R!)^{p-1}} \prod_{i=1}^{p} \mathrm{Var}(Y_i(t_i)[R]) \sim c \prod_{i=1}^{p} L_i(t_i)^R t_i^{2d_i - R\beta_i}.$$

Step 3: A suitable expression (in law) for $Y(t_1,\ldots,t_p)[R]$. Recall the definition (5.10). By assumption (5.8), we have $G_i(d\lambda_i) = g_i(\lambda_i)d\lambda_i$ for every i. Let us define $g : \mathbb{R}^d \to \mathbb{R}_+$ as

$$g(x_1,\ldots,x_p) = \prod_{i=1}^{p} g_i(x_i).$$

We get from Lemma 5.3.6 that

$$Y(t_1,\ldots,t_p)[R] = I_{G,R}(f_{R,t_1D_1\times\cdots\times t_pD_p}) \stackrel{\text{law}}{=} I_R((\sqrt{g})^{\otimes R} f_{R,t_1D_1\times\cdots\times t_pD_p}),$$

where I_R is the Rth Wiener-Itô integral with respect to the Lebesgue measure. By applying Lemma 5.3.7, we also obtain

$$I_R((\sqrt{g})^{\otimes R} f_{R,t_1D_1\times\cdots\times t_pD_p}) = \left(\prod_{i=1}^{p} t_i^{d_i - \frac{Rd_i}{2}}\right) I_R\left(\left(\sqrt{g(t_1,\ldots,t_p)}\right)^{\otimes R} f_{R,D_1\times\cdots\times D_p}\right) \quad (5.93)$$

where $g(t_1,\ldots,t_p) : \mathbb{R}^d \to \mathbb{R}_+$ is given by

$$g(t_1,\ldots,t_p)(x_1,\ldots,x_p) = \prod_{i=1}^{p} g_i(x_i/t_i).$$

Then, we have obtained the following expression in distribution for $\widetilde{Y}(t_1,\ldots,t_p)[R]$:

$$\widetilde{Y}(t_1,\ldots,t_p)[R] \stackrel{\text{law}}{=} \frac{I_R\left(\left(\sqrt{g(t_1,\ldots,t_p)}\right)^{\otimes R} f_{R,D_1\times\cdots\times D_p}\right)}{\sqrt{\mathrm{Var}\left(I_R\left(\left(\sqrt{g(t_1,\ldots,t_p)}\right)^{\otimes R} f_{R,D_1\times\cdots\times D_p}\right)\right)}}. \quad (5.94)$$

Step 4: Proving the L^2 convergence. The last step of the proof consists in showing the following L^2 convergence:

$$\frac{I_R\left(\left(\sqrt{g(t_1,\dots,t_p)}\right)^{\otimes R} f_{R,D_1\times\cdots\times D_p}\right)}{\sqrt{\mathrm{Var}\left(I_R\left(\left(\sqrt{g(t_1,\dots,t_p)}\right)^{\otimes R} f_{R,D_1\times\cdots\times D_p}\right)\right)}} \overset{L^2(\Omega)}{\to} \frac{I_R((\sqrt{g'})^{\otimes R} f_{R,D_1\times\cdots\times D_p})}{\sqrt{\mathrm{Var}\left(I_R((\sqrt{g'})^{\otimes R} f_{R,D_1\times\cdots\times D_p})\right)}}$$
$$\overset{\text{law}}{=} \frac{I_{\nu_1\times\cdots\times\nu_p,R}\left(f_{R,D_1\times\cdots\times D_p}\right)}{\sqrt{\mathrm{Var}\left(I_{\nu_1\times\cdots\times\nu_p,R}\left(f_{R,D_1\times\cdots\times D_p}\right)\right)}}, \tag{5.95}$$

where the equality in distribution follows from Lemma 5.3.6 and where $g'(x) := \prod_{i=1}^p \|x_i\|^{\beta_i-d_i}$. By the previous steps, it is enough to prove that for some positive constant c we have

$$c\prod_{i=1}^p t_i^{\frac{R(\beta_i-d_i)}{2}} L_i(t_i)^{-R/2} I_R\left(\left(\sqrt{g(t_1,\dots,t_p)}\right)^{\otimes R} f_{R,D_1\times\cdots\times D_p}\right) \tag{5.96}$$
$$\overset{L^2(\Omega)}{\to} I_R\left((\sqrt{g'})^{\otimes R} f_{R,D_1\times\cdots\times D_p}\right). \tag{5.97}$$

It is equivalent to show that

$$\left(\prod_{i=1}^p Q_{t_i}(\lambda_{1i},\dots,\lambda_{qi})\right) f_{R,D_1\times\cdots\times D_p}(\lambda_1,\dots,\lambda_R) \prod_{j=1}^R \prod_{i=1}^p \|\lambda_{ji}\|^{\frac{\beta_i-d_i}{2}} \tag{5.98}$$
$$\overset{L^2((\mathbb{R}^d)^R)}{\longrightarrow} f_{R,D_1\times\cdots\times D_p}(\lambda_1,\dots,\lambda_R) \prod_{j=1}^R \prod_{i=1}^p \|\lambda_{ji}\|^{\frac{\beta_i-d_i}{2}}, \tag{5.99}$$

where the $Q_{t_i} : (\mathbb{R}^{d_i})^R \to \mathbb{C}$ are defined as

$$Q_{t_i}(\lambda_{1i},\dots,\lambda_{Ri}) : = \sqrt{\prod_{j=1}^R \|\lambda_{ji}/t_i\|^{d_i-\beta_i} c_i^{-1} L_i^{-1}(t_i) g_i(\lambda_{ji}/t_i)},$$

and c_i are some positive constant. In particular, we will prove (5.98) with the constant c_i given by assumption (5.8). To prove (5.98), we proceed by induction on p. If $p=1$, then (5.98) reduces to

$$\int_{(\mathbb{R}^{d_1})^R} |Q_{t_1}(\lambda_{11},\dots,\lambda_{R1})-1|^2 \, |f_{R,D_1}(\lambda_{11},\dots,\lambda_{R1})|^2 \frac{d\lambda_{11}\dots d\lambda_{R1}}{\prod_{j=1}^R \|\lambda_{j1}\|^{d_1-\beta_1}} \to 0,$$

and this convergence is shown in the proof of [LO14, Theorem 5]. Now, let us assume that (5.98) holds for $p-1$, and let us prove the result for p. For this, note that $xy-1=(x-1)(y-1)+(x-1)+(y-1)$, implying

$$\left(\prod_{i=1}^{p} Q_{t_i}(\lambda_{1i},\dots,\lambda_{Ri})\right)-1=\left(\left(\prod_{i=1}^{p-1} Q_{t_i}(\lambda_{1i},\dots,\lambda_{Ri})\right)-1\right)\left(Q_{t_p}(\lambda_{1p},\dots,\lambda_{Rp})-1\right) \tag{5.100}$$

$$+\left(\left(\prod_{i=1}^{p-1} Q_{t_i}(\lambda_{1i},\dots,\lambda_{Ri})\right)-1\right) \tag{5.101}$$

$$+\left(Q_{t_p}(\lambda_{1p},\dots,\lambda_{Rp})-1\right). \tag{5.102}$$

Then, by applying the triangular inequality in $L^2((\mathbb{R}^d)^R)$, we get that (5.98) holds by inductive hypothesis. □

5.4 Examples

In this Section we collect some examples.

Example 5.4.1 (Hermite variations of a fractional Brownian sheet (fBs))**.** Let us consider a fBs

$$W^{\alpha,\beta}=(W^{\alpha,\beta})_{(x_1,x_2)\in\mathbb{R}^2_+} \tag{5.103}$$

with parameter $(\alpha,\beta)\in\mathbb{R}^2_+$ and its rectangular increments:

$$R_{x_1,x_2}:=W^{\alpha,\beta}_{x_1+1,x_2+1}-W^{\alpha,\beta}_{x_1,x_2+1}-W^{\alpha,\beta}_{x_1+1,x_2}+W^{\alpha,\beta}_{x_1,x_2}. \tag{5.104}$$

The field R defined as above is a stationary centered Gaussian field, with covariance function given by

$$\operatorname{Cov}(R_{x_1,x_2},R_{y_1,y_2})=r_\alpha(x_1-y_1)r_\beta(x_2-y_2) \tag{5.105}$$

where

$$r_H(u)=\frac{1}{2}\left(|u+1|^{2H}+|u-1|^{2H}-2|u|^{2H}\right),\qquad u\in\mathbb{R},\ H\in(0,1). \tag{5.106}$$

Notice that the covariance function of R is separable in the sense of (5.4). Moreover, r_α (resp. r_β) is the covariance function of the process defined as the (one-dimensional) increments of a fractional Brownian motion (fBm) with Hurst index α (resp. β). We briefly recall some results about this process, see e.g. [NP12]. A fractional Brownian motion $W^H=(W^H_t)_{t\in\mathbb{R}}$ of Hurst index H, is a centered Gaussian process such that

$$\mathbb{E}[W^H_{x_1}W^H_{y_1}]=\frac{1}{2}\left(|x_1|^{2H}+|y_1|^{2H}-|x_1-y_1|^{2H}\right),\quad x_1,y_1\in\mathbb{R}. \tag{5.107}$$

Its increment process

$$X_u=W^H_{u+1}-W^H_k,\quad u\in\mathbb{R}_+ \tag{5.108}$$

is known as fractional Gaussian noise. It is a centered, stationary Gaussian process with covariance function as in (5.106). Note that (5.106) is regularly varying with parameter $2H-2$ (see [NP12, (7.4.3)], or [Mai24, Example 1]), i.e. it behaves asymptotically as

$$r_H(u) = H(2H-1)|u|^{2H-2} + o(|u|^{2H-2}), \quad \text{as } |u| \to \infty. \tag{5.109}$$

Breuer-Major theorem (see Theorem 5.2.3) and [NP12, Theorem 7.4] allow us to deduce that the sequence

$$Y_N = \sum_{k=0}^{N-1} H_q(X_k), \quad N \geq 1, \tag{5.110}$$

under a proper renormalization, converges to a Gaussian distribution if $H \leq 1-1/2q$. Moreover, in [NP09] and [BN08], the authors quantify the convergence in total variation distance.

Theorem 5.4.1 (Theorem 1.1, 1.2 in [BN08]). *If $H \in (0, 1-1/2q)$ and $V_N = \frac{Y_N}{\sqrt{\mathrm{Var}(Y_N)}}$, then*

$$d_{TV}(V_N, N(0,1)) \leq c_{H,q} \begin{cases} N^{-1/2} & \text{if } H \in (0, \frac{1}{2}) \\ N^{H-1} & \text{if } H \in \left[\frac{1}{2}, \frac{2q-3}{2q-2}\right] \\ N^{2Hq-2q+1/2} & \text{if } H \in \left(\frac{2q-3}{2q-2}, 1-\frac{1}{2q}\right) \\ (\log N)^{-1/2} & \text{if } H = 1-\frac{1}{2q} \end{cases} =: c_{H,q} \cdot g(q,H,N). \tag{5.111}$$

In the previous, $c_{H,q}$ a positive constant, that may vary, only dependent on q and H.

If $H > 1-1/2q$, we have convergence towards a non Gaussian distribution (see [DM79]).

In [RST12], the authors show that a proper renormalization of

$$V_{N,M} = \sum_{x_1=0}^{N-1} \sum_{x_2=0}^{M-1} H_q(R_{x_1,x_2}), \quad N, M \in \mathbb{N} \tag{5.112}$$

converges to a Gaussian distribution as $N, M \to \infty$ when $\alpha \leq 1-1/2q$ or $\beta \leq 1-1/2q$[4]. Moreover, they bound the total variation distance between $V_{N,M}$ and its Gaussian limit.

Theorem 5.4.2 (Theorem 3.1 in [RST12]). *Let us denote by $c_{\alpha,\beta}$ a generic positive constant which depends on α, β and q, but which is independent of N and M. We have*

(1) If both $0 < \alpha, \beta < 1-1/2q$, then $\widetilde{V}_{N,M}$ converges in law to $N(0,1)$ with normalization $\varphi(\alpha,\beta,N,M) = \sqrt{\frac{q!}{s_\alpha s_\beta}} N^{\alpha q-1/2} M^{\beta q-1/2}$. In addition

$$d_{TV}(\widetilde{V}_{N,M}, N(0,1)) \leq c_{\alpha,\beta} \sqrt{N^{-1} + N^{2\alpha-2} + N^{2\alpha q-2q+1} + M^{-1} + M^{2\beta-2} + M^{2\beta q-2q+1}}. \tag{5.113}$$

[4]More precisely, they showed it for the Hermite variations of the field $N^\alpha M^\beta R^{\alpha,\beta}_{i/N,j/M}$. However, by self-similarity of the process R (see [RST12, Definition 2.3]), the two share the same law.

(2) *If* $0 < \alpha < 1 - 1/2q$ *and* $\beta = 1 - 1/2q$, *then* $\widetilde{V}_{N,M}$ *converges in law to* $N(0,1)$ *with normalization* $\varphi(\alpha, \beta, N, M) = \sqrt{\frac{q!}{s_\alpha \iota_\beta}} N^{\alpha q - 1/2} M^{q-1} (\log M)^{-1/2}$. *In addition*

$$d_{TV}(\widetilde{V}_{N,M}, N(0,1)) \leq c_{\alpha,\beta} \sqrt{N^{-1} + N^{2\alpha - 2} + N^{2\alpha q - 2q + 1} + (\log M)^{-1}}. \tag{5.114}$$

If $0 < \beta < 1 - 1/2q$ *and* $\alpha = 1 - 1/2q$, *then we get an analogous estimate as the previous one.*

(3) *If both* $\alpha = \beta = 1 - 1/2q$, *then* $\widetilde{V}_{N,M}$ *converges in law to* $N(0,1)$ *with normalization* $\varphi(\alpha, \beta, N, M) = \sqrt{\frac{q!}{\iota_\alpha \iota_\beta}} N^{q-1} (\log N)^{-1/2} M^{q-1} (\log M)^{-1/2}$. *In addition*

$$d_{TV}(\widetilde{V}_{N,M}, N(0,1)) \leq c_{\alpha,\beta} \sqrt{(\log N)^{-1} + (\log M)^{-1}}. \tag{5.115}$$

(4) *If* $\alpha < 1 - 1/2q$ *and* $\beta > 1 - 1/2q$, *then* $\widetilde{V}_{N,M}$ *converges in law to* $N(0,1)$ *with normalization* $\varphi(\alpha, \beta, N, M) = \sqrt{\frac{q!}{s_\alpha \kappa_\beta}} N^{\alpha q - 1/2} M^{q-1}$. *In addition*[5]

$$d_{TV}(\widetilde{V}_{N,M}, N(0,1)) \leq c_{\alpha,\beta} \sqrt{N^{-1} + N^{2\alpha - 2} + N^{2\alpha q - 2q + 1} + M^{-(2\beta q - 2q + 1)}}. \tag{5.116}$$

(5) *If* $\alpha = 1 - 1/2q$ *and* $\beta > 1 - 1/2q$, *then* $\widetilde{V}_{N,M}$ *converges in law to* $N(0,1)$ *with normalization* $\varphi(\alpha, \beta, N, M) = \sqrt{\frac{q!}{\iota_\alpha \kappa_\beta}} N^{q-1} (\log N)^{-1/2} M^{q-1}$. *In addition*

$$d_{TV}(\widetilde{V}_{N,M}, N(0,1)) \leq c_{\alpha,\beta} \sqrt{(\log N)^{-1} + M^{-(2\beta q - 2q + 1)}} \tag{5.117}$$

Our Theorem 5.1.1 translates in the discrete setting, too, and returns the same qualitative phenomenon described above. Moreover, by using the bounds obtained in the proof of [NP09, Theorem 4.1], Proposition 5.3.3 allows us to improve the previous rates.

Corollary 5.4.3. *Recall the definition of* g *in* (5.111).

(1) *If both* $0 < \alpha, \beta \leq 1 - 1/2q$, *then*

$$d_{TV}(\widetilde{V}_{N,M}, N(0,1)) \leq c_{\alpha,\beta,q}\, g(q, \alpha, N) \cdot g(q, \beta, M). \tag{5.118}$$

(2) *If* $\alpha \leq 1 - 1/2q$ *and* $\beta > 1 - 1/2q$, *then*

$$d_{TV}(\widetilde{V}_{N,M}, N(0,1)) \leq c_{\alpha,\beta,q}\, g(q, \alpha, N), \tag{5.119}$$

where $c_{\alpha,\beta,q}$ *are constants depending on* α, β, q *that may differ from the ones in [RST12, Theorem 3.1], and the normalization terms are as before.*

Example 5.4.2 (Tensor product of regularly varying covariance functions)**.** Let us define a centered Gaussian field $B = (B_x)_{x \in \mathbb{R}^d}$ with separable covariance function as

[5] The exponent in red is the correction of a typo in [RST12, Theorem 3.1].

in (5.4), with $p \geq 2$ and $C_i : \mathbb{R}^{d_i} \to \mathbb{R}$ defined as

$$C_i(x_i) := \frac{1}{(1 + \|x_i\|^2)^{\beta_i/2}}, \quad \beta_i > 0, \quad i = 1, \dots, p. \tag{5.120}$$

Let us consider the quadratic variation of this field, that is, the functional $Y(t_1, \dots, t_p)[2]$ as in (5.29) as well as its rescaled version $\widetilde{Y}$. The parameters β_i determine the behavior of the functional. Indeed, combining Theorem 5.2.3 and Proposition 5.3.1, it is enough to have $\beta_i > d_i/2$ for at least one i for $\widetilde{Y}(t_1, \dots, t_p)[2]$ to have a Gaussian limit. Moreover, by the application of Proposition 5.3.3, we may also deduce an upper bound for the rate of convergence in total variation. For a fixed i satisfying $\beta_i > d_i/2$, an upper bound for the rate of convergence in total variation distance of $\widetilde{Y}_i(t_i)[2]$ (recall (5.6)) towards a Gaussian distribution is given by the rate of convergence to 0 of the norm of the contraction:

$$\|\widetilde{f}_i(t_i) \otimes_1 \widetilde{f}_i(t_i)\|_2^2 \lesssim \frac{\left(\int_{\|x_i\| \leq t_i} C_i(x_i) dx_i\right)^2}{t_i^{d_i}} \asymp \begin{cases} t_i^{-d_i} & \text{if } \beta_i > d_i \\ \log(t_i)^2 t_i^{-d_i} & \text{if } \beta_i = d_i \\ t_i^{d_i - 2\beta_i} & \text{if } \beta_i \in (d_i/2, d_i) \end{cases} =: g(\beta_i, t_i)^2, \tag{5.121}$$

where we denote by $\lesssim$ the inequality up to a positive constant. From the above equivalence and (5.58), we obtain that, if $\beta_i > d_i/2$,

$$d_{TV}(\widetilde{Y}_i(t_i)[2], N(0,1)) \lesssim g(\beta_i, t_i). \tag{5.122}$$

Denoting $J = \{i \in \{1, \dots, p\} : \beta_i > d_i/2\}$, we can deduce that

$$d_{TV}(\widetilde{Y}(t_1, \dots, t_p)[2], N(0,1)) \lesssim \prod_{i \in J} g(\beta_i, t_i). \tag{5.123}$$

Example 5.4.3 (Gaussian fluctuations in a long-range dependence setting)**.** Let us define a centered Gaussian field $B = (B_x)_{\mathbb{R}^2}$ with separable covariance function

$$C(x_1, x_2) = \frac{1}{(1 + |x_1|^2)^{1/4}} \cdot \frac{1}{(1 + |x_2|^2)^{3/2}}. \tag{5.124}$$

Let us consider

$$Y(t_1, t_2) = \int_{t_1 D_1 \times t_2 D_2} H_2(B_x) dx, \qquad t_1, t_2 > 0, \tag{5.125}$$

where H_2 is the second Hermite polynomial and $D_i \subset R$ are compact sets. Then, $R = 2$. We have that $C \notin L^2(\mathbb{R}^2)$, hence we are in the long-range dependence case. However, the marginal functional

$$\widetilde{Y}_2(t_2) = \frac{\int_{t_2 D_2} H_2(B_x^{(2)}) dx}{\sqrt{\mathrm{Var}\left(\int_{t_2 D_2} H_2(B_x^{(2)}) dx\right)}}, \tag{5.126}$$

see also (5.5), where $(B_x^{(2)})_{\mathbb{R}}$ is centered Gaussian field with covariance function $C_2(x_2) = \frac{1}{(1+|x_2|^2)^{3/2}}$, exhibits Gaussian fluctuations as $t_2 \to \infty$ by Theorem 5.2.3.

Hence, by Theorem 5.1.1 also $\widetilde{Y}(t_1, t_2)$ does.

5.5 Going beyond the separability assumption

While the separability assumption holds true in numerous applications (see, e.g., [Chr92, Chapter 5] for examples in hydrology and fluid dynamics, or think of the many frameworks where the fractional Brownian sheet arises, see, e.g., [ØZ01; SW17]), there also many instances where it does not.

In this section, we aim to examine what can happen when we go beyond the separable case. First, let us give a counterexample that shows that we cannot always expect a result as simple as Theorem 5.1.1 in the non separable context.

Example 5.5.1 (Separability matters!). Let us consider $p = 2$, $d_1 = d_2 = 1$, and thus $d = 2$. Fix $R \geq 2$ and consider $\alpha \in (1/R, 2/R)$. Let $B = (B_x)_{\mathbb{R}^d}$ be a centered stationary Gaussian field with unit-variance and covariance function $C : \mathbb{R}^d \to \mathbb{R}$ as

$$C(x) = \frac{L(\|x\|)}{\|x\|^\alpha}, \quad x \in \mathbb{R}^d \setminus \{0\}, \tag{5.127}$$

with L a slowly varying function, such that C is continuous in 0. Assuming that C satisfies (5.8), since $\alpha < \frac{2}{R}$, Theorem 5.2.4 applies to the functional

$$Y(t,t)[R] = \int_{t(D_1 \times D_2)} H_R(B_x) dx \tag{5.128}$$

and we obtain that $\widetilde{Y}(t,t)[R]$ is asymptotically not Gaussian. However, both $\widetilde{Y}_1(t)[R]$ and $\widetilde{Y}_2(t)[R]$ constructed as in (5.6) have Gaussian fluctuations as $t \to \infty$. Indeed, they both have covariance functions as follows:

$$C_i(x_i) = \frac{L(|x_i|)}{|x_i|^\alpha}, \quad i = 1, 2; \tag{5.129}$$

and since $\alpha > 1/R$, they both satisfy Theorem 5.2.3. This shows how the convergence of both functionals $Y_i(t_i)$ is in general not enough to determine the behavior of $Y(t_1, \ldots, t_p)$, when the separability of the covariance function (5.4) is not satisfied.

In what follows, we drop the separability assumption by examining two different classes: Gneiting covariance functions (Section 5.5.1) and additively separable covariance functions (Section 5.5.2). We will explore whether, akin to the separable case, it is possible to simplify the asymptotic study of $Y(t_1, \ldots, t_p)$ given by (5.2) by reducing it to that of simpler functionals $Y_i(t_i)$.

5.5.1 Gneiting covariance functions

The class of Gneiting covariance functions was first introduced by Gneiting in [Gne02]. These functions are very popular in many applications, including geostatistics, environmental science, climatology and meteorology. A **Gneiting covariance function**

$C : \mathbb{R}^d \to \mathbb{R}$ has the form

$$C(x_1, x_2) = \psi\left(\|x_2\|^2\right)^{-d_1/2} \varphi\left(\frac{\|x_1\|^2}{\psi\left(\|x_2\|^2\right)}\right), \qquad x_1 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2}, \tag{5.130}$$

where $\varphi : [0,\infty) \to (0,\infty)$ is a **completely monotone function**, that is

$$(-1)^n \varphi^{(n)}(t) \geq 0 \qquad \forall n \in \mathbb{N},\ t \geq 0,$$

and $\psi : [0,\infty) \to \mathbb{R}_+$ has completely monotone derivative. The fact that (5.130) defines a non-negative definite function on $\mathbb{R}^{d_1+d_2}$ if $d_2 = 1$ was proved in [Gne02], and can be extended to $d_2 \geq 1$ using similar arguments. Assuming that B is a centered Gaussian random field with unit-variance and covariance function as in (5.130), we may infer that $C(0,0) = \psi(0)^{-d_1/2}\varphi(0) = 1$, and suppose, without loss of generality, that $\varphi(0) = \psi(0) = 1$. As a consequence, the Gaussian fields $(B^{(1)}_{x_1})_{x_1\in\mathbb{R}^{d_1}} := (B_{x_1,0})_{x_1\in\mathbb{R}^{d_1}}$ and $(B^{(2)}_{x_2})_{x_2\in\mathbb{R}^{d_2}} := (B_{0,x_2})_{x_2\in\mathbb{R}^{d_2}}$ have the following covariance functions

$$C_1(x_1) := C(x_1, 0) = \varphi\left(\|x_1\|^2\right); \tag{5.131}$$

$$C_2(x_2) := C(0, x_2) = \psi\left(\|x_2\|^2\right)^{-d_1/2}. \tag{5.132}$$

For this reason, we rewrite (5.130) as

$$C(x_1, x_2) = C_2(x_2) C_1\left(x_1 C_2(x_2)^{2/d_1}\right), \qquad x_1 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2}. \tag{5.133}$$

From the expression (5.133), since $C_1(x_1)$, $C_2(x_2)$ are positive and non-increasing in the norms $\|x_1\|$, $\|x_2\|$, we deduce that

$$C_2(x_2)C_1(x_1) \leq C(x_1, x_2) \leq C_2(x_2) C_1\left(x_1\, \underline{C_2}(D_2)^{2/d_1}\right), \tag{5.134}$$

where D_2 denotes the domain of the variable x_2, and

$$\underline{C_2}(D_2) = \inf_{x_2 \in (D_2 - D_2)} C_2(x_2) = \psi\left(\operatorname{diam}(D_2)^2\right)^{-d_1/2}. \tag{5.135}$$

Therefore, $C(x_1, x_2)$ is wedged between two separable covariance functions, $C_1(x_1)C_2(x_2)$ and $C_1(x_1\, \underline{C_2}(D_2)^{2/d_1})C_2(x_2)$. This suggests that the Gneiting case may be studied combining the bounds (5.134) and Theorem 5.1.1. We will partially formalize this intuition in Theorem 5.5.1, and we conjecture that the latter could be extended to more general functionals, growing domains and classes of non-separable covariance functions satisfying properties analogous to (5.134). These extensions are left for future research.

Analogously to the version of Theorem 5.1.1 explained in Remark 5.1.4, Theorem 5.5.1 allows to reduce the study of

$$Y(t_1, t_2)[q] = \int_{t_1 D_1 \times t_2 D_2} H_q(B_x)\,dx, \qquad \text{as } t_i \to \infty,$$

for only one index $i \in \{1, 2\}$ (i.e. only the i-th domain is growing), to that of the respective marginal functional

$$Y_i(t_i)[q] = \int_{t_i D_i} H_q(B^{(i)}_{x_i}) dx_i, \qquad \text{as } t_i \to \infty,$$

where $D_i \subseteq \mathbb{R}^{d_i}$ are compact sets with $\mathrm{Vol}(D_i) > 0$ and $(B^{(i)}_{x_i})_{x_i \in \mathbb{R}^{d_i}}$ are Gaussian fields with covariance functions C_i, as defined in (5.131).

Theorem 5.5.1. *Let $B = (B_x)_{x \in \mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $C : \mathbb{R}^d \to \mathbb{R}$ be the covariance function of B, and assume that C is a Gneiting covariance function, see (5.130)-(5.133). Let $Y(t_1, t_2)[q]$, $Y_i(t_i)[q]$ be defined as above, and $\widetilde{Y}$, $\widetilde{Y}_i$ their normalized versions (see e.g. (5.3)). Fix $j \in \{1, 2\}$. Then, the following holds:*

$$\widetilde{Y}_j(t_j)[q] \xrightarrow{d} N(0,1), \qquad \text{as } t_j \to \infty$$

implies

$$\widetilde{Y}(t_1, t_2)[q] \xrightarrow{d} N(0,1), \qquad \text{as } t_j \to \infty$$

where t_k is fixed, for $k \neq j$.

Proof of Theorem 5.5.1. The case $q = 1$ is trivial because everything is Gaussian, so let us focus on $q \geq 2$. We will show the convergence by means of the Fourth Moment Theorem 5.2.1. First, recall that

$$\mathrm{Var}\left(Y(t_1, t_2)[q]\right) = q! \int_{(t_1 D_1 \times t_2 D_2)^2} C^q(x - y) dx dy.$$

Thus, by (5.134) and (5.32), we have

$$\mathrm{Var}\left(Y_1(t_1)[q]\right) \mathrm{Var}\left(Y_2(t_2)[q]\right) \leq q! \, \mathrm{Var}\left(Y(t_1, t_2)[q]\right). \tag{5.136}$$

For the sake of brevity, let use denote $h(x_2) := C_2(x_2)^{2/d_1}$. We have

$$\begin{aligned} \|f(t_1, t_2) \otimes_r f(t_1, t_2)\|^2 = \int_{(t_1 D_1 \times t_2 D_2)^4} & C_2(x_2 - y_2)^r C_2(z_2 - u_2)^r C_2(x_2 - z_2)^{q-r} C_2(y_2 - u_2)^{q-r} \\ & \times C_1((x_1 - y_1) h(x_2 - y_2))^r C_1((z_1 - u_1) h(z_2 - u_2))^r \\ & \times C_1((x_1 - z_1) h(x_2 - z_2))^{q-r} C_1((y_1 - u_1) h(y_2 - u_2))^{q-r} dx\, dy\, dz\, du. \end{aligned} \tag{5.137}$$

If we fix t_1 and let $t_2 \to \infty$, we get from $0 \leq C_1 \leq 1$ that

$$\|f(t_1, t_2) \otimes_r f(t_1, t_2)\|^2 \leq \|f_2(t_2) \otimes_r f_2(t_2)\|^2 \, \mathrm{Vol}(t_1 D_1)^4. \tag{5.138}$$

If we fix t_2 and let $t_1 \to \infty$, setting $A := \underline{C_2}(t_2 D_2)^{2/d_1} < \infty$, by (5.134) we obtain

$$\|f(t_1,t_2)\otimes_r f(t_1,t_2)\|^2 \leq \int_{(t_1D_1\times t_2D_2)^4} C_2(x_2-y_2)^r C_2(z_2-u_2)^r C_2(x_2-z_2)^{q-r} C_2(y_2-u_2)^{q-r} \tag{5.139}$$

$$\times C_1((x_1-y_1)A)^r C_1((z_1-u_1)A)^r\ C_1((x_1-z_1)A)^{q-r} C_1((y_1-u_1)A)^{q-r} \tag{5.140}$$

$$dx\ dy\ dz\ du \tag{5.141}$$

$$= A^{-4d_1}\int_{(At_1D_1\times t_2D_2)^4} C_2(x_2-y_2)^r C_2(z_2-u_2)^r C_2(x_2-z_2)^{q-r} C_2(y_2-u_2)^{q-r} \tag{5.142}$$

$$\times C_1(x_1-y_1)^r C_1(z_1-u_1)^r C_1(x_1-z_1)^{q-r} C_1(y_1-u_1)^{q-r} dx\ dy\ dz\ du \tag{5.143}$$

$$\leq A^{-4d_1}\ \mathrm{Vol}(t_2D_2)^4 \|f_1(t_1)\otimes_r f_1(t_1)\|^2, \tag{5.144}$$

where in the last inequality we used that $A \leq 1$ and the positivity of the integrand. Applying the Fourth-Moment Theorem 5.2.1 leads to the desired conclusion. □

5.5.2 Additively separable covariance functions

In this subsection we consider **additively separable covariance functions** of the form

$$C(x_1,x_2) = K_1(x_1) + K_2(x_2), \qquad x_1 \in \mathbb{R}^{d_1},\ x_2 \in \mathbb{R}^{d_2}, \tag{5.145}$$

where K_1 and K_2 are the covariance functions of two stationary, continuous, centered Gaussian fields $(B^{(1)})_{x_1\in\mathbb{R}^{d_1}}$ and $(B^{(2)})_{x_2\in\mathbb{R}^{d_2}}$, with $K_1(0), K_2(0) > 0$. From the point of view of applications, the study of Gaussian fields with additively separable covariance function (5.145) is motivated by the fact that they can model the sum of two independent Gaussian fields. Unlike the Gneiting class considered in the previous subsection, which is comparable to the separable case (see (5.134)), here reduction theorems have to be developed in a different way. In this case, we define the marginal functionals as

$$A_i(t_i)[q] = \int_{t_iD_i} H_q\left(\frac{B^{(i)}_{x_i}}{K_i(0)}\right) dx_i. \tag{5.146}$$

We introduce new quantities to describe the interplay between the growth rates of the volumes and the variances, that are the quotients

$$\gamma^i_{t_i}[q] := \frac{\int_{(t_iD_i)^2} K_i^q(x_i-y_i)dx_idy_i}{\mathrm{Vol}(t_iD_i)^2}. \tag{5.147}$$

Theorem 5.5.2. *Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a real-valued, continuous, centered, stationary Gaussian field with unit-variance. Let $C : \mathbb{R}^d \to \mathbb{R}$ be the covariance function of B, and assume it is additively separable in the sense of (5.145), with $K_1, K_2 \geq 0$. Let $\widetilde{Y}$ be as* *in* (5.3). *For* $i = 1, 2$, *let* $A_i(t_i)[q]$ *be as in* (5.146), $\widetilde{A}_i$ *its normalized versions (see e.g.* (5.3)*), and recall the definition* (5.147) *of* $\gamma^i_{t_i}[q]$. *Finally, assume that*

$$\frac{\gamma^1_{t_1}[q]}{\gamma^2_{t_2}[q]} \to 0 \quad \textit{as } t_1, t_2 \to \infty.$$

Then, the following holds:

$$\widetilde{A}_2(t_2)[q] \xrightarrow{d} N(0,1), \qquad \textit{as } t_2 \to \infty,$$

if and only if

$$\widetilde{Y}_1(t_1,t_2)[q] \xrightarrow{d} N(0,1), \qquad \textit{as } t_1, t_2 \to \infty.$$

Remark 5.5.1. Note that, by symmetry, the roles of K_1 and K_2 can be exchanged in the statement of Theorem 5.5.2, obtaining an equivalence between $\widetilde{A}_1(t_1)[q] \xrightarrow{d} N(0,1)$ and $\widetilde{Y}(t_1,t_2)[q] \xrightarrow{d} N(0,1)$ as $t_1, t_2 \to \infty$, if $\frac{\gamma^2_{t_2}[q]}{\gamma^1_{t_1}[q]} \to 0$.

Remark 5.5.2. To ease the computation of the quotients $\gamma^i_{t_i}[q]$, $i = 1, 2$, as noted in Equation (5.188), assuming that K_i is both non-negative and non-negative definite, one can observe that

$$\gamma^i_{t_i}[q] \asymp \frac{\int_{\{\|x\| \le t_i\}} K_i^q(x_i)dx_i}{{t_i}^{d_i}}. \tag{5.148}$$

Just like Theorem 5.1.1 and Theorem 5.5.1, Theorem 5.5.2 should be interpreted as a reduction theorem, since it allows to reduce the asymptotic problem of a 2-domain functional $Y(t_1,t_2)[q]$ to that of a 1-domain functional. The difference here is that the marginal functionals to be considered in the additively separable case are not the same considered for the separable and Gneiting classes. Moreover, unlike the latter cases, here the growth rates of the integration domains come into play by means of the quotients $\gamma^i_{t_i}[q]$ in (5.147) (see also Example 5.5.2). In order to prove Theorem 5.5.2, we state and prove the following lemma.

Lemma 5.5.3. *Under the notations and assumptions of Theorem* 5.5.2*, we have that*

$$\begin{aligned} \mathrm{Var}(Y(t_1,t_2)[q]) = \mathrm{Vol}(t_1D_1)^2\,\mathrm{Var}(A_2(t_2)[q]) + \mathrm{Var}(A_1(t_1)[q])\,\mathrm{Vol}(t_2D_2)^2 \\ + \sum_{k=1}^{q-1}\binom{q}{k}^2 \mathrm{Var}(A_1(t_1)[k])\,\mathrm{Var}(A_2(t_2)[q-k]). \end{aligned} \tag{5.149}$$

In addition, assuming $q \ge 2$, *we have that*

$$\frac{\mathrm{Var}(Y(t_1,t_2)[q])}{\mathrm{Vol}(t_1D_1)^2\,\mathrm{Var}(A_2(t_2)[q])} \longrightarrow 1 \qquad \textit{as } t_1, t_2 \to \infty. \tag{5.150}$$

Proof. Recall $q \geq 2$. Applying Newton's binomial formula, we may write

$$\operatorname{Var}(Y(t_1,t_2)[q]) = q! \int_{(t_1D_1\times t_2D_2)^2} C(x-y)^q dxdy \tag{5.151}$$

$$= q! \int_{(t_1D_1\times t_2D_2)^2} (K_1(x_1-y_1)+K_2(x_2-y_2))^q dx_1dx_2dy_1dy_2 \tag{5.152}$$

$$= \sum_{k=0}^{q} q! \binom{q}{k} \int_{(t_1D_1)^2} K_1(x_1-y_1)^k dx_1dy_1 \cdot \int_{(t_2D_2)^2} K_2(x_2-y_2)^{q-k} dx_2dy_2, \tag{5.153}$$

which is (5.149). Now assume that $\gamma^1_{t_1}[q]/\gamma^2_{t_2}[q] \to 0$. Then, as $t_1, t_2 \to \infty$

$$\frac{\operatorname{Var}(Y(t_1,t_2)[q])}{\operatorname{Vol}(t_1D_1)^2 \operatorname{Var}(A_2(t_2)[q])} \longrightarrow 1.$$

Indeed,

$$\frac{\operatorname{Var}(A_1(t_1)[q]) \operatorname{Vol}(t_2D_2)^2}{\operatorname{Vol}(t_1D_1)^2 \operatorname{Var}(A_2(t_2)[q])} = \frac{\gamma^1_{t_1}[q]}{\gamma^2_{t_2}[q]} \longrightarrow 0$$

by assumption. By Jensen inequality in t_i and recalling that $K_i \geq 0$, we have that

$$\operatorname{Var}(A_i(t_i)[k]) = k! \int_{(t_iD_i)^2} K_i^k(x_i-y_i)dx_idy_i \lesssim \operatorname{Vol}(t_iD_i)^{2(q-k)/q} \operatorname{Var}(A_i(t_i)[q])^{k/q}. \tag{5.154}$$

Therefore, the proof is concluded by observing that, for $k>0$, we have

$$\frac{\operatorname{Var}(A_1(t_1)[k]) \operatorname{Var}(A_2(t_2)[q-k])}{\operatorname{Vol}(t_1D_1)^2 \operatorname{Var}(A_2(t_2)[q])} \lesssim \frac{\operatorname{Var}(A_1(t_1)[q])^{k/q} \operatorname{Vol}(t_2D_2)^{2k/q}}{\operatorname{Vol}(t_1D_1)^{2k/q} \operatorname{Var}(A_2(t_2)[q])^{k/q}} = \frac{\gamma^1_{t_1}[q]^{k/q}}{\gamma^2_{t_2}[q]^{k/q}} \longrightarrow 0. \tag{5.155}$$

□

Proof of Theorem 5.5.2. By assumption, $\gamma^1_{t_1}[q]/\gamma^2_{t_2}[q] \to 0$. By Lemma 5.5.3, as $t_1, t_2 \to \infty$ we have

$$\operatorname{Var}(Y(t_1,t_2)[q]) \sim \operatorname{Vol}(t_1D_1)^2 \operatorname{Var}(A_2(t_2)[q]).$$

Regarding contractions, by using the Newton binomial formula, we have

$$\|f(t_1,t_2)\otimes_r f(t_1,t_2)\|^2 \tag{5.156}$$

$$= \int_{(t_1D_1\times t_2D_2)^4} C(x-y)^r C_2(z-u)^r C(x-z)^{q-r} C_2(y-u)^{q-r} dx\, dy\, dz\, du \tag{5.157}$$

$$= \int_{(t_1D_1\times t_2D_2)^4} dx\, dy\, dz\, du \tag{5.158}$$

$$(K_1(x_1-y_1)+K_2(x_2-y_2))^r (K_1(z_1-u_1)+K_2(z_2-u_2))^r \tag{5.159}$$

$$\times (K_1(x_1-z_1)+K_2(x_2-z_2))^{q-r} (K_1(y_1-u_1)+K_2(y_2-u_2))^{q-r} \tag{5.160}$$

$$= \sum_{k_1,k_2=0}^{r} \sum_{k_3,k_4=0}^{q-r} \binom{r}{k_1}\binom{r}{k_2}\binom{q-r}{k_3}\binom{q-r}{k_4} \tag{5.161}$$

$$\int_{(t_1D_1)^4} K_1(x_1-y_1)^{k_1} K_1(z_1-u_1)^{k_2} K_1(x_1-z_1)^{k_3} K_1(y_1-u_1)^{k_4}\, dx_1\, dy_1\, dz_1\, du_1 \tag{5.162}$$

$$\int_{(t_2D_2)^4} K_2(x_2-y_2)^{r-k_1} K_2(z_2-u_2)^{r-k_2} K_2(x_2-z_2)^{q-r-k_3} K_2(y_2-u_2)^{q-r-k_4}\, dx_2\, dy_2\, dz_2\, du_2 \tag{5.163}$$

$$= \sum_{k_1,k_2=0}^{r} \sum_{k_3,k_4=0}^{q-r} \mathbf{1}_{\{(k_1,k_2,k_3,k_4)\neq(0,0,0,0)\}} \binom{r}{k_1}\binom{r}{k_2}\binom{q-r}{k_3}\binom{q-r}{k_4} \tag{5.164}$$

$$\int_{(t_1D_1)^4} K_1(x_1-y_1)^{k_1} K_1(z_1-u_1)^{k_2} K_1(x_1-z_1)^{k_3} K_1(y_1-u_1)^{k_4} \tag{5.165}$$

$$\int_{(t_2D_2)^4} K_2(x_2-y_2)^{r-k_1} K_2(z_2-u_2)^{r-k_2} K_2(x_2-z_2)^{q-r-k_3} K_2(y_2-u_2)^{q-r-k_4} \tag{5.166}$$

$$+ \mathrm{Vol}(t_1D_1)^4 \|f_2(t_2)\otimes_r f_2(t_2)\|^2 . \tag{5.167}$$

Moreover, we have

$$\sum_{k_1,k_2=0}^{r}\sum_{k_3,k_4=0}^{q-r}\mathbf{1}_{\{(k_1,k_2,k_3,k_4)\neq(0,0,0,0)\}}\binom{r}{k_1}\binom{r}{k_2}\binom{q-r}{k_3}\binom{q-r}{k_4} \quad (5.168)$$

$$\int_{(t_1D_1)^4} K_1(x_1-y_1)^{k_1}K_1(z_1-u_1)^{k_2}K_1(x_1-z_1)^{k_3}K_1(y_1-u_1)^{k_4}\,dx_1\,dy_1\,dz_1\,du_1 \quad (5.169)$$

$$\int_{(t_2D_2)^4} K_2(x_2-y_2)^{r-k_1}K_2(z_2-u_2)^{r-k_2}K_2(x_2-z_2)^{q-r-k_3}K_2(y_2-u_2)^{q-r-k_4} \quad (5.170)$$

$$\lesssim \sum_{k_1,k_2=0}^{r}\sum_{k_3,k_4=0}^{q-r}\mathbf{1}_{\{(k_1,k_2,k_3,k_4)\neq(0,0,0,0)\}}\binom{r}{k_1}\binom{r}{k_2}\binom{q-r}{k_3}\binom{q-r}{k_4}\times \quad (5.171)$$

$$\times\int_{(t_1D_1)^2} K_1(x_1-y_1)^{k_1+k_3}dx_1dy_1\int_{(t_2D_2)^2} K_2(x_2-y_2)^{q-k_1-k_3}dx_2dy_2 \quad (5.172)$$

$$\times\int_{(t_1D_1)^2} K_1(z_1-u_1)^{k_2+k_4}dz_1du_1\int_{(t_2D_2)^2} K_2(z_2-u_2)^{q-k_2-k_4}dz_2du_2, \quad (5.173)$$

where the last inequality follows from the positivity of K_1, K_2 and Lemma 5.5.4 applied to every term of the sum. Then, using Jensen as in the proof of Lemma 5.5.3, we obtain that, as $t_1, t_2 \to \infty$,

$$\|\widetilde{f}(t_1,t_2)\otimes_r\widetilde{f}(t_1,t_2)\|^2_{2q-2r} = (q!)^2\frac{\|f(t_1,t_2)\otimes_r f(t_1,t_2)\|^2_{2q-2r}}{\mathrm{Var}(Y(t_1,t_2))^2} \sim \frac{\|f(t_1,t_2)\otimes_r f(t_1,t_2)\|^2}{\mathrm{Vol}(t_1D_1)^4\,\mathrm{Var}(A_2(t_2))^2} \quad (5.174)$$

$$= \frac{\|f_2(t_2)\otimes_r f_2(t_2)\|^2}{\mathrm{Var}(A_2(t_2))^2} \quad (5.175)$$

$$+\,O\left(\sum_{k_1,k_2=0}^{r}\sum_{k_3,k_4=0}^{q-r}\mathbf{1}_{\{(k_1,k_2,k_3,k_4)\neq(0,0,0,0)\}}\left(\frac{\gamma^1_{t_1}[q]}{\gamma^2_{t_2}[q]}\right)^{(k_1+k_3)/q}\left(\frac{\gamma^1_{t_1}[q]}{\gamma^2_{t_2}[q]}\right)^{(k_2+k_4)/q}\right) \quad (5.176)$$

$$\sim (q!)^2\frac{\|f_2(t_2)\otimes_r f_2(t_2)\|^2_{2q-2r}}{\mathrm{Var}(A_2(t_2))^2} = \|\widetilde{f}_2(t_2)\otimes_r\widetilde{f}_2(t_2)\|^2_{2q-2r}. \quad (5.177)$$

Therefore, the proof is again concluded by means of the Fourth Moment theorem 5.2.1. □

We conclude the section with an example.

Example 5.5.2. Let us consider a Gaussian random field $B=(B_x)_{x\in\mathbb{R}^{d_1+d_2}}$ with covariance function of the form (5.145), choosing

$$K_1(x_1)=\frac{1}{(1+\|x_1\|^2)^{\beta_1/2}}, \qquad K_2(x_2)=\frac{1}{(1+\|x_2\|^2)^{\beta_2/2}}. \quad (5.178)$$

and consider its quadratic variation, that is

$$Y(t_1, t_2)[2] = \int_{t_1 D_1 \times t_2 D_2} H_2(B_x) dx. \tag{5.179}$$

Suppose that $\beta_1 \in (0, d_1/2)$and $\beta_2 > d_2/2$. Note that K_1 satisfies both (5.7) and (5.8) (see [LO13, Example 3]). Then, by Theorem 5.2.4 the functional $\widetilde{A}_1(t_1)[2]$, see (5.146) properly normalized, is not asymptotically Gaussian; on the other hand, $\widetilde{A}_2(t_2)[2]$ is asymptotically Gaussian, thanks to Theorem 5.2.3. Note that, up to constants, recalling Remark 5.5.2,

$$\frac{\gamma^1_{t_1}[2]}{\gamma^2_{t_2}[2]} \sim \frac{t_1^{-2\beta_1}}{t_2^{-d_2}}. \tag{5.180}$$

Therefore, we may observe two different behaviors depending on the choice of the rate for t_1 and t_2: applying Theorem 5.5.2, whenever $t_1^{-2\beta_1} = o(t_2^{-d_2})$, we have that $\widetilde{Y}(t_1, t_2)$ is not asymptotically Gaussian; conversely, when $t_2^{-d_2} = o(t_1^{-2\beta_1})$ we have a Gaussian limiting behavior.

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Appendix

Proof of Lemma 5.3.6. The following proof is merely a reformulation of [Maj81, Theorem 4.5] with our notation. Since $h \in \mathcal{H}_\nu^{\odot q}$, we have the equality

$$h = \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q} \operatorname{Sym}(e_{i_1} \otimes \cdots \otimes e_{i_q}),$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_\nu$ and Sym is the symmetrization operator. If we define $e'_i(x) := e_i(x)\, a(x) \in L^2_E(\nu')$, then $\{e'_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_{\nu'}$ and we obtain

$$a^{\otimes q} h = \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q} \operatorname{Sym}(e'_{i_1} \otimes \cdots \otimes e'_{i_q}).$$

Applying the operators $I_{\nu,q}$ and $I_{\nu',q}$ we obtain

$$I_{\nu,q}(h) = \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q}\, I_{\nu,q} \left(\operatorname{Sym}(e_{i_1} \otimes \cdots \otimes e_{i_q})\right)$$

and

$$I_{\nu',q}(a^{\otimes q}\, h) = \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q}\, I_{\nu',q} \left(\operatorname{Sym}(e'_{i_1} \otimes \cdots \otimes e'_{i_q})\right).$$

Therefore, we conclude by means of product formula (see e.g. [NP12]), observing that

$$I_{\nu,q}(h) = \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q} X_\nu(e_{i_1})\dots X_\nu(e_{i_q}) \overset{d}{=} \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q} N_{i_1}\dots N_{i_q}$$

and

$$I_{\nu',q}(a^{\otimes q}h) = \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q} X_\nu(e'_{i_1})\dots X_\nu(e'_{i_q}) \overset{d}{=} \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q} N_{i_1}\dots N_{i_q},$$

where $\{N_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. standard Gaussian random variables. □

Proof of Lemma 5.3.7. Acting as in the proof of Lemma 5.3.6, we have

$$I_q(h(\cdot)) = \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q}\, X(e_{i_1}(\cdot))\dots X(e_{i_q}(\cdot))),$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_\lambda$. Note that using the multi-index notation $\underline{s}^\alpha = \prod_i s_i^\alpha$ for $s \in \mathbb{N}^d$ and $\alpha \in \mathbb{R}$, also the family $\{\underline{s}^{-1/2} e_i(\cdot/s_1,\dots,\cdot/s_d)\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_\lambda$. Therefore, the two Gaussian families $(X(e_i(\cdot)))_{i=1}^{\infty}$ and $(X(\underline{s}^{-1/2} e_i(\cdot/s_1,\dots,\cdot/s_d)))_{i=1}^{\infty}$ share the same law, and, by extension, the following random variables are equally distributed:

$$I_q(h(\cdot)) \overset{\text{law}}{=} \sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q}\, X(\underline{s}^{-1/2} e_{i_1}(\cdot/s_1,\dots,\cdot/s_d))\dots\dots X(\underline{s}^{-1/2} e_{i_q}(\cdot/s_1,\dots,\cdot/s_d)). \tag{5.181}$$

Moreover, by product formula (see e.g. [NP12]), we have that

$$\sum_{i_1,\dots,i_q=1}^{\infty} c_{i_1,\dots,i_q}\, X(\underline{s}^{-1/2} e_{i_1}(\cdot/s_1,\dots,\cdot/s_d))\dots X(\underline{s}^{-1/2} e_{i_q}(\cdot/s_1,\dots,\cdot/s_d)) \overset{d}{=} I_q(\underline{s}^{-\frac{q}{2}} \tilde{h}(s_1,\dots,s_d)). \tag{5.182}$$

By linearity, $I_q(\underline{s}^{-\frac{q}{2}} \tilde{h}(s_1,\dots,s_d)) = \underline{s}^{-\frac{q}{2}} I_q(\tilde{h}(s_1,\dots,s_d))$, which concludes the proof. □

Lemma 5.5.4. *Let $K : \mathbb{R}^d \to \mathbb{R}$ be a non-negative definite function such that $K \geq 0$. Suppose that $D \subseteq \mathbb{R}^d$ compact with $\mathrm{Vol}(D) > 0$. Then, for every $k_1, k_2, k_3, k_4 \in \mathbb{N}$, there exists a positive constant c, depending on K, D and the exponents k_i, $i = 1,2,3,4$, satisfying for all $t > 0$ the following inequality:*

$$\int_{(tD)^4} K(x-y)^{k_1} K(z-u)^{k_2} K(x-z)^{k_3} K(y-u)^{k_4} dxdydzdu$$
$$\leq c \int_{(tD)^2} K(x-y)^{k_1+k_3} dxdy \int_{(tD)^2} K(z-u)^{k_2+k_4} dzdu. \tag{5.183}$$

Proof. First, by using the inequality $x^a y^b \leq x^{a+b} + y^{a+b}$, the LHS of the inequality

(5.183) can be bounded by

$$2\int_{(tD)^4} K(x-y)^{k_1+k_3}K(z-u)^{k_2+k_4}dxdydzdu+2\int_{(tD)^4} K(x-y)^{k_1+k_3}K(y-u)^{k_2+k_4}dxdydzdu. \tag{5.184}$$

Then, by the change of variable $a = x - y$ and $b = y - u$ and compactness of the domain D, we may also bound the second term in the previous equation, up to a positive constant, by

$$\begin{aligned}\mathrm{Vol}(tD)\int_{(tD-tD)^2} &K(a)^{k_1+k_3}K(b)^{k_2+k_4}\,\mathrm{Vol}(tD\cap(tD-a)\cap(tD-b))dadb\\ &\leq \mathrm{Vol}(tD)^2\int_{tD-tD} K(a)^{k_1+k_3}da\int_{tD-tD} K(b)^{k_2+k_4}db. \end{aligned}\tag{5.185}$$

To conclude, we need to show that

$$\mathrm{Vol}(tD)\int_{tD-tD} K(a)^{k_1+k_3}da \lesssim \int_{(tD)^2} K(x-y)^{k_1+k_3}dxdy. \tag{5.186}$$

By positivity of the integrand and the doubling conditions for non-negative definite functions that are also non-negative proved in [GT19], we deduce that

$$\int_{tD-tD} K(a)^{k_1+k_3}da \leq \int_{\{\|x\|\leq t\cdot \mathrm{diam}(D)\}} K(a)^{k_1+k_3}da \lesssim \int_{\{\|x\|\leq t\}} K(a)^{k_1+k_3}da. \tag{5.187}$$

Reasoning as in [MN24, Proof of Proposition 9, Step 1], we have that

$$t^d\int_{\{\|x\|\leq t\}} K(a)^{k_1+k_3}da \asymp \int_{(tD)^2} K(x-y)^{k_1+k_3}dxdy, \tag{5.188}$$

which is enough to conclude. □

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