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**Shi-Yuan Zhou**

Born on 08 October 1995 in Düsseldorf, Germany

# NON-PARAMETRIC INFERENCE FOR STOCHASTIC PARTICLE SYSTEMS

## Dissertation defence committee

Dr. Mark Podolskij, *dissertation supervisor*
Professor, University of Luxembourg

Dr. Yannick Baraud
Professor, University of Luxembourg

Dr. Valentine Genon-Catalot
Professor, Université Paris Descartes

Dr. Marc Hoffmann
Professor, Université Paris-Dauphine

Dr. Christophe Ley, *chairman*
Professor, University of Luxembourg

Dr. Giovanni Peccati
Professor, University of Luxembourg

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# Contents

# Chapter 1

# Introduction

This thesis concerns nonparametric estimation methods for interacting particle systems and their mean-field equations. In this introductory chapter, we first present the stochastic differential equations we study in this framework and cite some probabilistic results, including existence and uniqueness of solutions, as well as a phenomenon called propagation of chaos, which links the particle systems to their associated mean-field limits. Afterwards, we present some of the literature on statistical estimation of these models that has been developed over the recent years. We finish the introduction by outlining the content of this thesis and presenting the main results.

## 1.1 Interacting Particle Systems and McKean-Vlasov Equations

For $N \in \mathbb{N}$, a $N$-**interacting particle system** $X^{1,N}, \dots, X^{N,N}$ is given by solutions of the following system of $N$ $d$-dimensional stochastic differential equations (SDEs)

$$\begin{cases} \mathrm{d}X_t^{i,N} = b(t, X_t^{i,N}, \mu_t^N)\mathrm{d}t + \sigma(t, X_t^{i,N}, \mu_t^N)\mathrm{d}B_t^i, \quad t \geq 0, \\ (X_0^i)_{1 \leq i \leq N} \sim \mu_0^{\otimes N}, \end{cases} \tag{1.1}$$

where $(B^i)_{1 \leq i \leq N}$ are independent $d$-dimensional Brownian motions, the initial random variables $(X_0^i)_i$ are independent of $(B^i)_i$, and $\mu_0^{\otimes N}$ denotes the $N$-fold product of a probability measure $\mu_0$. Unlike classical SDEs, the drift $b$ and diffusion $\sigma$ depend on the empirical measure of the trajectories, defined for each $N \geq 1$ as

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{i,N}}.$$

In other words, the dynamics of the $i$-th particle are governed by all other particles, causing the interaction. Assuming all particles have the same initial distribution, they also share the same trajectory, thus making them **exchangeable**. These systems have first been studied in [McK66] in the context of plasma physics, and have later prompted probabilistic investigations and extensions such as in [Gär88; Szn91; Mél96; Ben+98; Mal03; CGM08; Kol10], to name only a few references.

Let us present a well-posedness result given the standard assumption of Lipschitz continuity. Let $T > 0$ and $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ denote the space of probability measures on $\mathbb{R}^d$ with finite second moments, equipped with the Wasserstein-2 distance.

**Assumption 1.** The initial distribution is given by $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$. The drift $b\colon [0,T]\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$ and diffusion $\sigma\colon [0,T]\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}^{d\times d}$ are Lipschitz continuous in the sense that there exists a $L>0$ such that for all $x,y\in\mathbb{R}^d$ and $\mu,\nu\in\mathscr{P}_2(\mathbb{R}^d)$,

$$|b(x,\mu)-b(y,\nu)|+|\sigma(x,\mu)-\sigma(y,\nu)|\le L\left(|x-y|+W_2(\mu,\nu)\right).$$

By Itô's existence and uniqueness result for SDEs with Lipschitz coefficients, the solution to this SDE system is well-defined.

**Theorem 1.1** ([Lac], Lemma 3.2)**.** *Under Assumption 1, the system of SDEs (1.1) admits a unique strong solution for any $N\in\mathbb{N}$.*

Associated to these particles is a so-called **mean-field equation**, which is a $d$-dimensional SDE given by

$$\begin{cases} \mathrm{d}X_t = b(t,X_t,\mu_t)\mathrm{d}t+\sigma(t,X_t,\mu_t)\mathrm{d}B_t, \\ \mathrm{Law}(X_t)=\mu_t, \end{cases} \tag{1.2}$$

where $B$ is a $d$-dimensional Brownian motion. These equations are known in the literature as **McKean-Vlasov SDEs**, distribution dependent SDEs (DDSDEs), or nonlinear SDEs in the sense of McKean. The nonlinearity is given by the fact that, due to the distribution dependence in the coefficients, solutions to these equations no longer satisfy the Markov property, and the flow of measures $(\mu_t)_{t\ge 0}$ solves a nonlinear Fokker-Planck equation

$$\partial_t\mu_t=\frac{1}{2}\nabla\nabla^\top\left(\sigma\sigma^\top(\cdot,\cdot,\mu_t)\,\mu_t\right)-\nabla\left(b(\cdot,\cdot,\mu_t)\mu\right),$$

where $\nabla$ denotes the divergence operator with respect to the space variable $x$. These equations have been studied from multiple angles, either from a probabilistic perspective as the mean-field limit of the aforementioned particle systems, or as DDSDEs [HRW21; GHM22], or through their associated PDE [CMV03; BGG13; BR18; BR20].

To specify the relationship between interacting particle systems and their mean-field limit, let us introduce the phenomenon of propagation of chaos. Heuristically, as the name suggests, this means that particles that start out chaotically, i.e. $(X_0^i)_{1\le i\le N}\sim\mu_0^{\otimes N}$, will propagate this chaos in time more as the number of particles grows. There are a number of ways to define propagation of chaos, we refer to the exhaustive reviews [CD22a; CD22b] for further literature.

**Definition 1.2** (Propagation of Chaos)**.** Let $X_t^{i,N}$ be the solution of (1.1), and $X_t^i$ the solution of

$$\begin{cases} \mathrm{d}X_t^i = b(t,X_t^i,\mu_t)\mathrm{d}t+\sigma(t,X_t^i,\mu_t)\mathrm{d}B_t^i, \\ \mathrm{Law}(X_t^i)=\mu_t, \end{cases}$$

where the initial values $(X_0^i)_i$ and the Brownian motions $(B^i)_i$ are the same as in (1.1). We say that (pointwise) propagation of chaos holds if

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\left|X_t^{i,N}-X_t^i\right|^2\right]=0.$$

Propagation of chaos in particular implies the weak convergence $\mu_t^N\to\mu_t$, pointwise in $t$. More precisely, the random variable $\mu_t^N$ converges in law to the deterministic measure $\mu_t$. By coupling the trajectories of the particle system with i.i.d. copies of the mean-field equation, one can show that the classical case of Lipschitz coefficients indeed exhibits propagation of chaos. Existence and uniqueness of solutions to the mean-field equation can be proved using a fixed point argument.

**Theorem 1.3** ([Lac], Theorem 3.3). *Under Assumption 1, the mean-field equation (1.2) admits a unique strong solution. Furthermore, propagation of chaos in the sense of Definition 1.2 holds for the associated interacting particle system given by (1.1).*

A special case of interacting particle systems, and one we will be focusing on in this thesis, is given by time-homogeneous drifts of convolution type, meaning that there exist differentiable potentials $V, W : \mathbb{R}^d \to \mathbb{R}$ such that

$$b(x,\mu) = V'(x) + W' * \mu(x),$$

where $f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\mathrm{d}y$ denotes the convolution. In this case, $V$ and $W$ are called confinement (or friction) potential and interaction potential, respectively. The associated Fokker-Planck equation is known in the literature as the granular media equation. These types of equations have been extensively studied, some works already mentioned include [Ben+98; CMV03; Mal03; CGM08; BGG13].

## 1.2 Results on Parametric Inference

The field of statistical inference of interacting particle systems and McKean-Vlasov equations has only recently garnered attention. Originating as models in plasma physics and first studied in [McK66], physical observations of the microscopic systems were not available, resulting in an initial lack of interest in the development of statistical programmes. When these models found more widespread applications in the recent decades, see for example [GH11; Bal+12; CFT12; FS13; Cha+17; GSS20; Dje+22], the need for statistical inference became more apparent. Since then, there have been many contributions made in a variety of directions. We highlight some important works pertinent to this thesis in the present and following section.

We begin the exposition by outlining some works on parametric inference in interacting particle systems. Let us first focus on the estimation in the asymptotic regime of $N \to \infty$. Consider a system of $N$ one-dimensional SDEs given by

$$\begin{cases} \mathrm{d}X_t^{i,N} = b(\theta, t, X_t^{i,N}, \mu_t^N)\mathrm{d}t + \sigma(t, X_t^{i,N})\mathrm{d}B_t^i, \quad t \in [0,T] \\ (X_0^i)_{1\le i\le N} \sim \mu_0^{\otimes N}. \end{cases} \tag{1.3}$$

Here the parameter of interest is given by $\theta \in \Theta \subset \mathbb{R}^p$. Using Girsanov's theorem, the log-likelihood of this system can be written explicitly, serving as a natural contrast function for the estimation of $\theta$, and motivating other contrast functions. Let

$$X^{(N)} = (X^{1,N}, \dots, X^{N,N})$$

be the canonical process on $\mathscr{C}^N = C([0,T], \mathbb{R}^{dN})$, equipped with its Borel $\sigma$-algebra $\mathscr{B}^N$. Denote by $\mathbb{P}_\theta^N$ and $\mathbb{P}_0^N$ be the probability measures on $(\mathscr{C}^N, \mathscr{B}^N)$ such that

$$\begin{aligned} &X^{(N)} \text{ solves (1.3) with respect to } \mathbb{P}_\theta^N, \\ &X^{(N)} \text{ solves } \mathrm{d}X_t^{i,N} = \sigma(t, X^{i,N})\mathrm{d}B_t^i \text{ with respect to } \mathbb{P}_0^N. \end{aligned} \tag{1.4}$$

Then the Radon-Nikodym derivative of these two measures is given by

$$L_N(\theta) := \log\left(\frac{\mathrm{d}\mathbb{P}_\theta^N}{\mathrm{d}\mathbb{P}_0^N}\right)(X^{(N)}) = \sum_{i=1}^N \left( \int_0^T \frac{b(\theta, t, X_t^{i,N}, \mu_t^N)}{\sigma(t, X_t^{i,N})}\mathrm{d}X_t^{i,N} - \frac{1}{2}\int_0^T \left(\frac{b(\theta, t, X_t^{i,N}, \mu_t^N)}{\sigma(t, X_t^{i,N})}\right)^2 \mathrm{d}t \right). \tag{1.5}$$

The maximum likelihood estimator (MLE) is defined as the maximiser of the above likelihood.

$$\hat{\theta}_N^{\mathrm{MLE}} = \operatorname*{argmax}_{\theta\in\Theta} L_N(\theta).$$

Furthermore, if $\theta \mapsto L_N(\theta)$ is sufficiently regular, the Fisher information matrix of the experiment $(\mathscr{C}^N, \mathscr{B}^N, (\mathbb{P}_\theta^N)_{\theta\in\Theta})$ is defined as $I(\theta) = \mathbb{E}_{\mathbb{P}_\theta^N}[-\mathscr{H}_\theta L_N(\theta)]$, where $\mathscr{H}_\theta L_N$ is the Hessian matrix of $L_N$ with the derivatives taken with respect to $\theta$.

Our primary interest is the study of the MLE in the high-dimensional limit, that is as $N \to \infty$. Note that this is different from the classical literature of estimation in SDEs, where the asymptotic usually considered is $T \to \infty$. We begin this literature review by presenting the first work covering high-dimensional parametric statistics for interacting particle systems, [Kas90], where the MLE was studied in particle systems with a linear dependence on the parameters. In [DH23], the MLE has been shown to exhibit the LAN property. The joint estimation of parameters appearing both in the drift and diffusion functions has been investigated in [Amo+23]. Different settings are discussed afterwards: [Sha+23; PZ22b; PZ22a] study the asymptotic behaviour of estimators in particle systems in large time. The works [Wen+16; LQ22; GL21a; GL21b; GL23a; GL23b] present estimation procedures based on observations of the mean-field equation of particle systems.

### 1.2.1 Maximum Likelihood Estimation in the Case of Linear Dependence

[Kas90] was the first paper of its kind to analyse parametric estimation in interacting particle systems by deriving and studying the maximum likelihood estimator. The author showed that for an interacting particle system with a linear dependence on the parameter of interest $\theta$, the MLE can be computed explicitly, given knowledge of the drift and diffusion functions. Furthermore, the MLE is shown to be a consistent and asymptotically normal estimator as the number of particles goes to infinity.

Consider continuous-time observations $(X_t^{(N)})_{t\in[0,T]}$ on a fixed time horizon $t \in [0, T]$ of the $N$-particle system

$$\begin{cases} \mathrm{d}X_t^{i,N} = \sum_{k=1}^p \theta_k b_{ik}(X_t^{(N)})\mathrm{d}t + \sigma_i(X_t^{(N)})\mathrm{d}B_t^i, \\ (X_0^{i,N})_{1\le i\le N} \sim \mu_0^{\otimes N}. \end{cases}$$

The drift and diffusion functions $b_{ik}, \sigma_i \colon \mathbb{R}^N \to \mathbb{R}$ are assumed to be known and be such that the interacting particle system admits a unique strong solution. We present the case of exchangeable particles, however the original result also holds under more general conditions. Denote the true parameter as $\theta_0$ and the observed information as $I^N(\theta_0) = -\mathscr{H}_\theta L_N(\theta_0)$. Since the drift is linear in $\theta$, the MLE can be computed as the solution of

$$I^N(\theta_0)(\hat{\theta}_N^{\mathrm{MLE}} - \theta_0) = M_T^N,$$

where $(M_t^N)_{t\in[0,T]}$ is a local martingale. By Rebodello's central limit theorem, the quadratic variation of $M^N$ satisfies $\langle M^N \rangle_T \to I(\theta_0)$ almost surely as $N \to \infty$, which is the main argument needed to show consistency and asymptotic normality of the MLE in this context.

**Theorem 1.4** ([Kas90], Theorem 1). *$\hat{\theta}_N^{\mathrm{MLE}}$ is a consistent estimator of $\theta_0$ and*

$$\sqrt{N}(\hat{\theta}_N^{\mathrm{MLE}} - \theta_0) \xrightarrow{\mathrm{d}} \mathcal{N}(0, I^{-1}(\theta_0)).$$

This result can also be used to test for interaction given a particle system.

### 1.2.2 The LAN Property

[DH23] studied the MLE for a multivariate interacting particle system with a parameter $\theta \in \Theta \subset \mathbb{R}^p$ in the drift as in (1.3) and proved that the MLE is locally asymptotically normal as $N \to \infty$. The LAN property implies Hájek's theorem, which is used to show minimax optimality of the MLE under some conditions.

Consider the experiments $\mathscr{E}^N$ generated by the canonical process $X^{(N)}$ and the measures $\mathbb{P}_\theta^N$ defined as in (1.4):

$$\mathscr{E}^N = \left(\mathscr{C}^N, \mathscr{B}^N, (\mathbb{P}_\theta^N)_{\theta\in\Theta}\right).$$

Under some conditions, in particular on the regularity of the drift function $b$ with respect to $\theta$, the experiment $\mathscr{E}^N$ is regular in the sense of Ibragimov and Has'minskii, implying that the Fisher information matrix $I_{\mathscr{E}^N}(\theta)$ is well-defined. Furthermore, the limit $N^{-1}I_{\mathscr{E}^N}(\theta) \to I_{\mathscr{G}}(\theta)$ exists for any $\theta$ in the interior of $\Theta$, see [DH22, Proposition 11]. Under an additional identifiability assumption on the experiment, the information matrix is non-degenerate and the authors show the LAN property, following the strategy of Ibragimov and Has'minskii [IH81].

**Theorem 1.5** ([DH23], Theorem 14). *For every $\theta$ in the interior of $\Theta$, the sequence of experiments $(\mathscr{E}^N)_{N\in\mathbb{N}}$ is locally asymptotically normal at $\theta$ with information rate $NI_{\mathscr{G}}(\theta)$. That is to say, for any $u \in \mathbb{R}^p$ such that $\theta + (NI_{\mathscr{G}}(\theta))^{-1/2}u \in \Theta$, it is*

$$\log\left(\frac{\mathrm{d}\mathbb{P}^N_{\theta+(NI_{\mathscr{G}}(\theta))^{-1/2}u}}{\mathrm{d}\mathbb{P}^N_\theta}\right) = u^\top \xi_\theta^N - \frac{1}{2}|u|^2 + r_N(\theta,u),$$

*where $\xi_\theta^N \to \mathscr{N}(0_p, \mathbf{1}_{p\times p})$ in distribution and $r_N(\theta,u)\to 0$ in probability, and both convergences are with respect to $\mathbb{P}_\theta^N$.*

Due to Hájek's theorem [DH23, Corollary 18], the LAN property implies some strong results for the MLE regarding its minimax optimality.

**Theorem 1.6** ([DH23], Theorem 19). *For any polynomial loss function, the MLE is exactly locally asymptotically minimax optimal at any $\theta$ in the interior of $\Theta$. Furthermore, on any open subset $\Theta_0 \subset \Theta$, the MLE is asymptotically minimax optimal.*

### 1.2.3 Joint Estimation of Drift and Diffusion using Discrete Observations

The work [Amo+23] studies the joint estimation of parameters in the drift and diffusion function given discrete observations of the interacting particle system. Unlike the previously presented works, the diffusion is also allowed to depend on the empirical measure and contribute to the interaction in this framework. The discretised observations make the likelihood-based approach unfeasible, asking instead for a pseudo-likelihood based approach. The resulting estimator is shown to be consistent and asymptotically normal as the number of particles goes to infinity and the discretisation step goes to zero.

Consider $\mathbb{R}$-valued particles on $t \in [0,T]$ given by

$$\begin{cases} \mathrm{d}X_t^{i,N} = b(\theta_1, X_t^{i,N}, \mu_t^N)\mathrm{d}t + \sigma(\theta_2, X_t^{i,N}, \mu_t^N)\mathrm{d}B_t^i, \\ (X_0^{i,N})_{1\le i\le N} \sim \mu_0^{\otimes N}. \end{cases}$$

The vector of parameters is given by $\theta = (\theta_1, \theta_2) \in \Theta \subset \mathbb{R}^{p_1+p_2}$. The observations $(X_{t_j^n}^{(N)})_{1\le j\le n}$ are assumed to be given by equidistant time steps $t_j^n = j\Delta_n$ with discretisation step $\Delta_n = \frac{T}{n}$, and the asymptotic studied is $N \to \infty$ and $\Delta_n \to 0$. The estimator is defined as the minimiser

$\hat{\theta}_n^N = \text{argmin}_{\theta\in\Theta} S_n^N(\theta)$ of a contrast function that is motivated by the Gaussian quasi-likelihood function:

$$S_n^N(\theta) = \sum_{i=1}^{N}\sum_{j=1}^{n} \frac{\left(X^{i,N}_{t_j^n} - X^{i,N}_{t_{j-1}^n} - \Delta_n b\left(\theta_1, X^{i,N}_{t_{j-1}^n}, \mu^N_{t_{j-1}^n}\right)\right)^2}{\Delta_n \sigma\left(\theta_2, X^{i,N}_{t_{j-1}^n}, \mu^N_{t_{j-1}^n}\right)^2} + \log\left(\sigma\left(\theta_2, X^{i,N}_{t_{j-1}^n}, \mu^N_{t_{j-1}^n}\right)^2\right).$$

Under some assumptions on the regularity of $b$ and $\sigma$, the estimator is shown to be consistent in probability as $n, N \to \infty$, see [Amo+23, Theorem 3.1]. Furthermore, assuming the diffusion coefficient depends linearly on the measure argument via some integration kernel, the authors also prove asymptotic normality of their estimator, provided the limiting covariance matrix is non-degenerate.

**Theorem 1.7** ([Amo+23], Theorem 3.2)**.** *Let $N$ and $\Delta_n$ be such that $N\Delta_n \to 0$ as $n, N \to \infty$. Then*

$$\begin{pmatrix} \sqrt{N}\left(\hat{\theta}^N_{n,1} - \theta_{0,1}\right) \\ \sqrt{\frac{N}{\Delta_n}}\left(\hat{\theta}^N_{n,2} - \theta_{0,2}\right) \end{pmatrix} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 2\Sigma(\theta_0)^{-1}),$$

*where $\Sigma(\theta_0) = \text{diag}(\Sigma_1(\theta_0), \Sigma_2(\theta_0))$ is an invertible block diagonal matrix with entries $\Sigma_i(\theta_0) \in \mathbb{R}^{p_i\times p_i}$.*

The main tool used to show asymptotic normality is based on studying the derivatives of the contrast functional $\nabla_\theta S_n^N$ using a central limit theorem for martingale difference triangular arrays. Define the block diagonal matrix $M_n^N = \text{diag}(N^{-1/2}\mathbb{1}_{p_1\times p_1}, \ (\Delta_n/N)^{1/2}\mathbb{1}_{p_2\times p_2})$. Using the Taylor expansion, one can show that

$$\mathrm{I}_n^N (M_n^N)^{-1}(\hat{\theta}_n^N - \theta_0) = -M_n^N \nabla_\theta S_n^N(\theta_0),$$

where $\mathrm{I}_n^N$ is a $\mathbb{R}^{p\times p}$-valued random variable that converges to $\Sigma(\theta_0)$ in probability as $N, n \to \infty$. On the other hand, using the aforementioned CLT, it is

$$M_n^N \nabla_\theta S_n^N(\theta_0) \xrightarrow{\mathrm{d}} \mathcal{N}(0, 2\Sigma(\theta_0)),$$

which yields the claimed asymptotic normality of the MLE.

### 1.2.4 Large Time and Small Variance Statistics

In [GL21a; GL21b], the authors study the following one-dimensional McKean-Vlasov equation

$$\begin{cases} \mathrm{d}X_t = [V(\alpha, X_t) - \varphi(\beta, \cdot) * \mu_t(X_t)]\,\mathrm{d}t + \varepsilon \mathrm{d}B_t, \\ X_0 = x_0 \in \mathbb{R}, \quad \text{Law}(X_t) = \mu_t. \end{cases} \tag{1.6}$$

The parameter of interest is given by $\theta = (\alpha, \beta) \in \Theta_\alpha \times \Theta_\beta \subset \mathbb{R}^2$. In their first work, [GL21b], the authors showed that in the small noise limit $\varepsilon \to 0$, given continuous observations on a fixed time interval $(X_t)_{t\in[0,T]}$, only the parameter $\alpha$ can be estimated, whereas $\beta$ cannot. Consequently, the authors next studied the joint estimation of the parameters $\theta$ in the asymptotic regime of $T \to \infty$, $\varepsilon \to 0$, given a path $(X_t)_{t\geq 0}$ as an observation. We present some of the results from this second work, [GL21a].

As a contrast function, the authors consider an approximation of the log-likelihood, using a self-stabilisation property of the SDE. Under some conditions, it is

$$\lim_{\varepsilon\to 0} b(\theta, t, \varepsilon, x) = \varphi(\beta, x - x_t(\alpha)),$$

where $b(\theta,t,\varepsilon,x)=\varphi(\beta,\cdot)*\mu_t(x)$ and $x_t(\alpha)$ solves an ODE of the form

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}x_t(\alpha)=V(\alpha,x_t(\alpha)),\\ x_0(\alpha)=x_0.\end{cases}$$

Therefore, substituting the true drift with $H(\theta,t,x)=V(\alpha,x)-\varphi(\beta,x-x_t(\alpha))$ gives an approximate likelihood function which can be minimised.

$$(\hat{\alpha}_{\varepsilon,T},\hat{\beta}_{\varepsilon,T})=\underset{(\alpha,\beta)\in\Theta}{\operatorname{argmax}}\,\frac{1}{\varepsilon}\int_0^T H(\theta,t,X_t)\,\mathrm{d}X_t-\frac{1}{2\varepsilon^2}\int_0^T H(\theta,t,X_t)^2\mathrm{d}t.$$

The asymptotics of these estimators depend on the behaviour of the root of $V$, which is assumed to be unique. For every $\alpha\in\Theta_\alpha$, define $x^*(\alpha)$ as the (unique) solution of $V(\alpha,x^*(\alpha))=0$. The rates of convergence vary depending on whether said root $x^*(\alpha)$ is constant with respect to $\alpha$. Either case results in asymptotic normality, but with different convergence rates and covariance matrices.

**Theorem 1.8** ([GL21a], Theorem 3.3)**.** *Let $\varepsilon\to 0$ and $T\to\infty$ in such a way that $\varepsilon\sqrt{T}\to 0$. Then the following convergences hold with respect to $\mathbb{P}_{\theta_0}$.*

1. *If $x^*(\alpha)$ is not constant with respect to $\alpha$, then for every $\theta_0\in\Theta$ there exists a diagonal matrix $I^{(1)}(\theta_0)$ such that*

$$\begin{pmatrix}\varepsilon^{-1}\sqrt{T}\left(\hat{\alpha}_{\varepsilon,T}-\alpha_0\right)\\ \sqrt{T}\left(\hat{\beta}_{\varepsilon,T}-\beta_0\right)\end{pmatrix}\xrightarrow{\mathrm{d}}\mathcal{N}\left(0,I^{(1)}(\theta_0)^{-1}\right).$$

2. *If $x^*(\alpha)=x^*$ for all $\alpha\in\Theta_\alpha$, then for every $\theta_0\in\Theta$ there exists a diagonal matrix $I^{(2)}(\theta_0)$ such that*

$$\begin{pmatrix}\varepsilon^{-1}\left(\hat{\alpha}_{\varepsilon,T}-\alpha_0\right)\\ \sqrt{T}\left(\hat{\beta}_{\varepsilon,T}-\beta_0\right)\end{pmatrix}\xrightarrow{\mathrm{d}}\mathcal{N}\left(0,I^{(2)}(\theta_0)^{-1}\right).$$

From this theorem, we indeed see that the joint asymptotics $\varepsilon^{-1}$, $T\to\infty$ are needed to obtain asymptotic statements for the joint estimation. The estimators are also shown to be asymptotically efficient [GL21a, Section 3.6], meaning that no information was lost when considering this approximation of the likelihood function.

### 1.2.5 Further Works

The presented literature on parametric estimation in interacting particle systems is certainly not exhaustive, and so we mention a few other works related to this field.

Maximum likelihood estimation in the mean-field setting is studied in [Wen+16]. Given an observation of the McKean-Vlasov equation, it is shown that the MLE is consistent in probability as $T\to\infty$. Furthermore, the experiments associated to the likelihood of the mean-field equation are shown to be locally asymptotically mixed normal. This study has later been generalised to a path-dependent setting by [LQ22], where almost sure consistency of the MLE was proved.

[Che21] study the MLE for so-called linear elasticity interacting particle systems where the interaction potential is quadratic, meaning the drift is given by a linear transformation of the particles. Given continuous observations of the system, the MLE is shown to be consistent and optimal in the joint limit of $N,T\to\infty$.

A method of moments estimator is studied in [PZ22a] based on observing the path of a single particle in an interacting particle system. The drift and diffusion are assumed to be polynomials and to depend linearly on the unknown parameters. The estimator is given as the solution of a certain inverse problem that is constructed using approximations of the moments of the invariant measure and the quadratic variation of the observed path. Consistency in $L^1(\mathbb{R})$ as $N, T \to \infty$ is shown.

In [PZ22b], an eigenfunction martingale estimator is presented that relies on discrete observations of a single particle in an interacting particle system. Their method assumes the existence of an ergodic measure, which is granted under some convexity assumptions on the confinement and interaction potentials. The estimator is designed to approximate the eigenfunctions of a linear operator, where the latter is obtained by linearising the generator of the mean-field equation using the aforementioned invariant measure. This procedure is shown to result in an estimator that is consistent in probability and asymptotically normal in the joint limit of $N \to \infty$ and discretisation step $\Delta_n \to 0$. Convergence rates are given with high probability.

[Sha+23] study online estimation methods for both interacting particle systems as well as their mean-field limits. The maximiser of an approximate log-likelihood, which is updated using a stochastic gradient descent scheme, is shown to be consistent as $N, T \to \infty$, and as $T \to \infty$.

Two more recent studies treating parameter estimation in McKean-Vlasov models in large time have been undertaken by the same authors in [GL23a; GL23b]. When considering observations solving the McKean-Vlasov equation, the true likelihood is a theoretical quantity due to the distribution dependence in the drift. Therefore, the authors use two-step approaches to construct pseudo-likelihood functions: The observations are assumed to be on a interval $[0, 2T]$, whereby the path on $[0, T]$ is used to approximate the unknown distribution, and the likelihood is computed with respect to the remaining path on $[T, 2T]$ using said approximation. In addition to their constructed estimators, the authors also study the true likelihood function in their frameworks. In the first work, [GL23a], the parametric inference of the polynomial interaction functions is studied and the invariant distribution is approximated using moment estimators. In their second and more general work, [GL23b], both the confinement and interaction functions are allowed depend on the unknown parameters, and the invariant measure is approximated by a kernel estimator of bandwidth $T$. In both works, the resulting estimators are shown to be consistent and asymptotically normal.

## 1.3 Results on Nonparametric Inference

Results on nonparametric methods are noticeably fewer and more recent than their parametric counterparts. One of the first works, [Bis11], extended a linear case studied in [Kas90] to consider parameters with a dependence in time. In [DH22], kernel estimators for the empirical measure and the drift function are considered, alongside an estimator for the interaction function, if the measure-dependence of the drift is of convolution type. A semiparametric setup was considered in [BPP23], which served as motivation for the second work presented in this thesis. A projection estimator is considered in [CG23], sharing similarities the first work of this thesis, albeit in a less general setting.

### 1.3.1 Sieve Estimator and an Approximate MLE

[Bis11] generalised the setting in [Kas90] to a nonparametric one by considering

$$\begin{cases} \mathrm{d}X_t^{i,N} = \sum_{k=1}^{d} \theta_k(t) b_{ik}(X_t^{(N)})\mathrm{d}t + \sigma_i(X_t^{(N)})\mathrm{d}B_t^i, \\ X_0^{i,N} = x_i \in \mathbb{R}, \end{cases}$$

where the parameters $\theta \in L^2([0,T],\mathbb{R}^d)$ now depend on time. In that case, the log-likelihood $L_N : L^2([0,T],\mathbb{R}^d) \to \mathbb{R}$ is still explicit and coincides with the likelihood in the parametric case given by (1.5). The authors use sieves to obtain an estimator for $\theta$. Furthermore, in the parametric setup, an approximate likelihood estimator, defined by minimising a time-discretised version of the likelihood, is considered. We choose to skip the discussion of the latter and focus on the nonparametric problem. For the sieve estimator, let $V_N$, $N \in \mathbb{N}$ be increasing subspaces of $L^2([0,T],\mathbb{R}^d)$ with $\dim(V_N) = d_N$ and such that $\bigcup_{N\in\mathbb{N}} V_N \subset L^2([0,T],\mathbb{R}^d)$ is dense. Then the sieve estimator $\hat{\theta}_N^s$ is defined by

$$\hat{\theta}_N^s = \operatorname*{argmax}_{f\in V_N} L_N(f).$$

Using similar methods as for the sieve estimator in a setting without interaction, the estimator is shown to be consistent. Furthermore, its coefficients with respect to a given orthonormal basis are asymptotically normal.

**Theorem 1.9** ([Bis11], Theorems 3.1 and 3.2). *The sieve estimator satisfies the following statements.*

1. *Let $(d_N)_{N\in\mathbb{N}}$ be such that $d_N \to \infty$ and $\frac{d_N^2}{N} \to 0$ as $N \to \infty$. Then*

$$\left\| \hat{\theta}_N^s - \theta_0 \right\|_{L^2}^2 \to 0,$$

*where the convergence is in probability.*

2. *Let $\hat{\theta}_{N,k}^s$, $\theta_{0,k}$ denote the coefficients of $\hat{\theta}_N^s, \theta_0$ with respect to an orthonormal basis of $L^2(\mathbb{R}^d)$. Let $(d_N)_{N\in\mathbb{N}}$ be such that $d_N \to \infty$ and $\frac{d_N^3}{N} \to 0$ as $N \to \infty$. Then, for any $k \in \mathbb{N}$,*

$$\sqrt{N}\left(\hat{\theta}_{N,k}^s - \theta_{0,k}\right) \xrightarrow{\mathrm{d}} \mathcal{N}(0, I^{-1}(\theta_0)).$$

### 1.3.2 Kernel Estimators and Minimax Optimal Oracle Inequalities

In [DH22], the authors construct kernel estimators for the empirical measure and the drift function where the kernel bandwidth is chosen via the Goldenshluger-Lepski method. The associated oracle inequalities are derived and are shown to be minimax optimal in anisotropic Hölder classes, save for a logarithmic factor. Additionally, in the setting of interaction through convolution, the authors present a consistent estimator for the interaction function.

Consider a $d$-dimensional interacting particle system as in (1.1) with a non-interactive diffusion $\sigma$ that is Lipschitz in space and an initial distribution exhibiting exponential moments. For the drift $b$, the authors consider several cases, in particular the one where the interaction is given by some integral kernel $b(t,x,\mu) = \int_{\mathbb{R}^d} \tilde{b}(t,x,y)\mathrm{d}\mu(y)$, also called the Vlasov case.

**Kernel Estimators for Empirical Measure and Drift**

Let $K\colon \mathbb{R}^d \to \mathbb{R}$ be a kernel of order $l \in \mathbb{N}$ and write $K_h = h^{-d}K(\frac{\cdot}{h})$ for any bandwidth $h > 0$. We define a pointwise kernel estimator of $\mu_{t_0}(x_0)$ for any $t_0 \in [0,T]$ and $x \in \mathbb{R}^d$ by

$$\hat{\mu}_h^N(t_0,x_0) = K_h * \mu_{t_0}^N(x_0).$$

The data-driven Goldenshluger-Lepski estimator is given by $\hat{\mu}_{\mathrm{GL}}^N(t_0,x_0) = \hat{\mu}_{\hat{h}}^N(t_0,x_0)$, where $\hat{h}$ is selected using the Goldenshluger-Lepski algorithm.

Let us next construct an estimator for $b$. The idea is to define a kernel estimator $\hat{\pi}_{\mathbf{h}}^N$ for $\pi(t,x) = b(t,x,\mu_t)\mu_t(x)$ and then define a quotient estimator using $\hat{\pi}_{\mathbf{h}}^N$ and $\hat{\mu}_h^N$. Consider the measure $\pi^N$ on $[0,T]\times\mathbb{R}^d$ defined by

$$\int_{[0,T]\times\mathbb{R}^d} f(t,x)\,\mathrm{d}\pi^N(t,x) = \frac{1}{N}\sum_{i=1}^N \int_0^T f(t,X_t^{i,N})\,\mathrm{d}X_t^{i,N}.$$

Let $H\colon [0,T] \to \mathbb{R}$ be another kernel of order $l$. For $\mathbf{h} = (h_1,h_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, define $\mathbf{K}_{\mathbf{h}}(t,x) = H_{h_1}(t)K_{h_2}(x)$ with the kernel function $K$ as above. We obtain a kernel estimator for $\pi$ by taking the space-time convolution of $\pi^N$ with the tensorised kernel function $\mathbf{K}$:

$$\hat{\pi}_{\mathbf{h}}^N(t_0,x_0) = \mathbf{K}_{\mathbf{h}} * \pi^N(t_0,x_0).$$

Finally, we arrive at an estimator for $b$ by taking the quotient of the two estimators and adding a threshold parameter $\omega > 0$ to prevent a blow up.

$$\hat{b}_{h,\mathbf{h},\omega}^N(t_0,x_0) = \frac{\hat{\pi}_{\mathbf{h}}^N(t_0,x_0)}{\hat{\mu}_h^N(t_0,x_0)\vee\omega}\ .$$

The Goldenshluger-Lepski estimator $\hat{b}_{\mathrm{GL}}^N(t_0,x_0)$ can be defined using $\hat{h}$ as above, and another estimator $\hat{\mathbf{h}}$, which is also selected using the Goldenshluger-Lepski algorithm. It is shown that the risks of $\hat{\mu}_h^N(t_0,x_0)$ and $\hat{b}_{h,\mathbf{h},\omega}^N(t_0,x_0)$ can be naturally decomposed into a bias and variance term. Furthermore, following the Goldenshluger-Lepski method, $\hat{\mu}_{\mathrm{GL}}^N(t_0,x_0)$ and $\hat{b}_{\mathrm{GL}}^N(t_0,x_0)$ are shown achieve the optimal bias-variance trade-off in terms of bandwidth selection, see [DH22, Theorem 7 and Theorem 9].

**Optimality in Anisotropic Hölder Classes**

Let $\mathscr{P}$ be the family of admissible triplets $(b,\sigma,\mu_0)$ satisfying some assumptions regarding existence and uniqueness of solutions, in particular that $(t,x)\mapsto\mu_t(x)$ is the weak solution of the nonlinear Fokker-Planck equation

$$\begin{cases}\partial_t\mu = \frac{1}{2}\sum_{j,k=1}^d \partial_{jk}^2\sigma\sigma^\top\mu - \nabla(b\mu). \\ \mu(t=0) = \mu_0.\end{cases} \tag{1.7}$$

Denote $S\colon \mathscr{P} \to L^1(\mathbb{R}^d)$, $(b,\sigma,\mu_0)\mapsto(\mu_t)_{t\in[0,T]}$ as the solution map of (1.7). Consider the anisotropic Hölder class $\mathscr{H}^{\alpha,\beta}$ with $\alpha$-Hölder regularity in time and $\beta$-Hölder regularity in space. For the exact definition, we refer to [DH22, Definition 11]. The function class with respect to which optimality of $\hat{\mu}_{\mathrm{GL}}^N(t_0,x_0)$ is proved is given by

$$\mathscr{S}_L^{\alpha,\beta} = \left\{(b,c,\mu_0)\in\mathscr{P}\colon\ \|S(b,c,\mu_0)\|_{\mathscr{H}^{\alpha,\beta}} \le L\right\}.$$

In this function class, the convergence rate of the Goldenshluger-Lepski estimator is shown to be optimal up to a logarithmic factor, provided the estimation kernel is chosen to be of higher order than the space-Hölder regularity of $\mu$.

**Theorem 1.10** ([DH22], Theorem 14). *Let $\alpha, \beta, L > 0$. For any $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$ it is*

$$\sup_{(b,c,\mu_0)\in\mathscr{S}_L^{\alpha,\beta}} \mathbb{E}\left[\left|\hat{\mu}_{\mathrm{GL}}^N(t_0, x_0) - \mu_{t_0}(x_0)\right|^2\right]^{\frac{1}{2}} \lesssim \left(\frac{\log(N)}{N}\right)^{\frac{\beta\wedge l}{2\beta\wedge l+d}}.$$

*Moreover,*

$$\inf_{\hat{\mu}^N} \sup_{(b,c,\mu_0)\in\mathscr{S}_L^{\alpha,\beta}} \mathbb{E}\left[\left|\hat{\mu}^N - \mu_{t_0}(x_0)\right|^2\right]^{\frac{1}{2}} \gtrsim N^{-\frac{\beta}{2\beta+d}},$$

*where the infimum is taken over all estimators constructed using $\mu_{t_0}^N$.*

The lower bound of the estimation problem is computed using a two-point lower bound argument with LeCam's lemma, whereas the upper bound follows from classical arguments in the Goldenshluger-Lepski methodology. A similar result also holds for the drift estimator $\hat{b}_{\mathrm{GL}}^N$, see [DH22, Theorem 15]. There the rate also depends on the regularity in time $\alpha$.

### Consistency of an Interaction Estimator

The introduced kernel estimators can also be employed for the estimation of the interaction function $\varphi$ in the case of a time-homogeneous drift given by $b(x, \mu) = G(x) + \varphi * \mu(x)$, whereby the function $G$ is regarded as a nuisance parameter. In order to express $\varphi$ in terms of only $(\mu_t^N)_{t\in[0,T]}$ and $b$, the authors introduce an integral operator which removes the dependence of $b$ on $G$. For a family of measures $(\mu_t)_{t\in[0,T]}$, the operator $\mathscr{L} : L^1([0, T], \mathbb{C}) \to \mathbb{R}$ is defined in such a way that, for any $x \in \mathbb{R}^d$,

$$\mathscr{L} b(x, \mu_\cdot) = \varphi * \mathscr{L} \mu_\cdot(x).$$

On the left hand side, $\mathscr{L}$ acts on the function $t \mapsto b(x, \mu_t)$ for a fixed $x \in \mathbb{R}^d$. Using the Fourier transform to invert the convolution, we have the relation

$$\mathscr{F}(\varphi) = \frac{\mathscr{F}\left(\mathscr{L} b(\cdot, \mu_\cdot)\right)}{\mathscr{F}\left(\mathscr{L} \mu_\cdot\right)}.$$

An estimator $\hat{\varphi}^N$ is then obtained by replacing $b$ and $\mu_\cdot$ with their respective estimators and adding truncations to ensure the estimator is well-defined. For details, we refer to [DH22, Section 5]. The resulting estimator is shown to be consistent with respect to the $L^2(\mathbb{R}^d)$-norm in [DH22, Theorem 17]. The statement is purely qualitative; in order to derive a convergence rate additional assumptions on the distributions $(\mu_t)_{t\in[0,T]}$ would be necessary.

## 1.3.3 Adaptive Projection Estimators

[CG23] extended the study of [Bis11] by studying nonparametric estimation of the functions $\alpha, \beta \in L^2([0, T], \mathbb{R})$ in the following one-dimensional interacting particle system.

$$\begin{cases} \mathrm{d}X_t^{i,N} = \left(\alpha(t)X_t^{i,N} + \beta(t)(X_t^{i,N} - \bar{X}_t^{(N)})\right)\mathrm{d}t + \mathrm{d}B_t^i, \\ (X_0^{i,N})_{1\le i\le N} \sim \mu_0^{\otimes N}. \end{cases}$$

This model was first studied in [Kas90] as an example in the case of constant coefficients, and in [Bis11] the joint estimation of $(\alpha, \beta) \in L^2([0, T], \mathbb{R}^2)$ was considered. In [CG23], the problem of separately estimating $\alpha$ and $\beta$ is studied, given continuous observations $(X_t^{(N)})_{t\in[0,T]}$ and letting $N \to \infty$. The dependence of the log-likelihood on the parameters can be decoupled, motivating two contrast functionals for the estimation of $\alpha$ and $\gamma := \alpha - \beta$ which are minimised separately using projection estimators. We note that this methodology is very similar to the first work of this thesis, the main difference being that some computations can be done explicitly due to the dependence on the functions $\alpha$ and $\gamma$. Denoting the processes of the empirical mean and empirical variance as $\bar{X}^{(N)}$ and $V^{(N)}$ respectively, the authors consider the functionals

$$U_{N,1}(f) = \int_0^T f(t)^2 \bar{X}_t^{(N)} \mathrm{d}t - 2\int_0^T f(t)\bar{X}_t^{(N)} \mathrm{d}\bar{X}_t^{(N)}$$
$$U_{N,2}(f) = \int_0^T f(t)^2 V_t^{(N)} \mathrm{d}t - \frac{2}{N}\int_0^T f(t)\sum_{i=1}^N (X_t^{i,N} - \bar{X}_t^{(N)})\,\mathrm{d}X_t^{i,N}.$$

These two terms indeed decompose the log-likelihood into two parts which only depend on $\alpha$ and $\gamma$ respectively, since it is

$$U_{N,1}(\alpha) + U_{N,2}(\gamma) = L_N((\alpha,\gamma)).$$

for any $\alpha, \gamma \in L^2([0,T],\mathbb{R})$. This decoupling of the likelihood allows for separate estimation of $\alpha$ and $\gamma$. Let $(S_{1,m})_{m\in\mathbb{N}}$ and $(S_{2,p})_{p\in\mathbb{N}}$ be two sequences of increasing subspaces of $L^2([0,T],\mathbb{R})$. The projection estimators $\hat{\alpha}_m$ and $\hat{\gamma}_p$ are then defined as

$$\hat{\alpha}_m = \underset{f\in S_{1,m}}{\operatorname{argmin}}\, U_{N,1}(f) \quad \text{and} \quad \hat{\gamma}_p = \underset{f\in S_{2,p}}{\operatorname{argmin}}\, U_{N,2}(f).$$

We will next outline the procedure and results for $\hat{\alpha}_m$ and note that the study of $\hat{\gamma}_p$ is treated in a similar way. Computing the estimator is the same as solving a certain inverse problem: Let $(e_m)_{m\in\mathbb{N}}$ be an orthonormal basis of $L^2([0,T],\mathbb{R})$ and $S_{1,m} = \operatorname{span}(e_1,\dots,e_m)$. Then $\hat{\alpha}_m = \sum_{j=1}^m c_{\alpha,j} e_j$, and the coefficients are given by $c_{\alpha,j} = \left(\hat{\Psi}_m^{-1}\hat{Z}_m\right)_j$, where

$$(\hat{\Psi}_m)_{j,k} = \int_0^T e_j(t)e_k(t)\left(\bar{X}_t^{(N)}\right)^2 \mathrm{d}t, \quad (\hat{Z}_m)_j = \int_0^T e_j(t)\bar{X}_t^{(N)} \mathrm{d}\bar{X}_t^{(N)}.$$

The study of $\hat{\alpha}_m$ revolves around the behaviour of the matrix $\hat{\Psi}_m$, as well as a deterministic counterpart $\Psi_m \approx \mathbb{E}\left[\hat{\Psi}_m\right]$. As is standard for these kinds of projection estimators (see [CG19; CG20]), the spectral norms $\left\|\hat{\Psi}_m^{-1}\right\|_{\mathrm{op}}$ and $\left\|\Psi_m^{-1}\right\|_{\mathrm{op}}$ are increasing as $m \to \infty$, such that there needs to be some restriction on the choices of $m$ in order to guarantee the stability of the estimation procedure, namely, for some $c > 0$,

$$L_m(\left\|\Psi_m^{-1}\right\|_{\mathrm{op}} \vee 1) \leq \frac{cN}{2\log N}, \qquad L_m := \sum_{j=1}^m \left\|e_j\right\|_\infty^2. \tag{1.8}$$

Similarly, the estimator needs to be restricted to the event $\Lambda_m = \left\{L_m(\left\|\hat{\Psi}_m^{-1}\right\|_{\mathrm{op}} \vee 1) \leq \frac{cN}{\log N}\right\}$, meaning that we consider the truncation $\tilde{\alpha}_m = \hat{\alpha}_m \mathbf{1}_{\Lambda_m}$. To study the risk of $\tilde{\alpha}_m$, there are two norms which arise naturally which involve the empirical mean. For this, the authors first show in [CG23, Proposition 1] that $\bar{X}^{(N)}$ solves a SDE with a diffusion term that vanishes as $N \to \infty$. As such, there exists a deterministic processes $(x_t)_{t\in[0,T]}$ that is the mean-field limit of $\bar{X}^{(N)}$, that is

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \left|\bar{X}_t^{(N)} - x_t\right| = 0 \quad \text{a.s.} \tag{1.9}$$

Thus, the natural norms in which to measure the risk of $\hat{\alpha}_m$ are given by an empirical norm $\|\cdot\|_{\bar{X}^{(N)}}$ and its deterministic counterpart $\|\cdot\|_x$, defined as

$$\|f\|_{\bar{X}^{(N)}} = \int_0^T f(t)^2\left(\bar{X}_t^{(N)}\right)^2 \mathrm{d}t, \quad \|f\|_x = \int_0^T f(t)^2 x_t^2 \mathrm{d}t.$$

The probability that these norms are equivalent on $L^2([0,T],\mathbb{R})$ approaches 1 with arbitrary polynomial speed as $N\to\infty$, see [CG23, Proposition 3]. Following the same ideas, the risk of $\hat{\gamma}_p$ can be naturally measured with respect to two norms that depend on the empirical variance and its mean-field limit in the same sense as in (1.9). Similarly to $\bar{X}^{(N)}$, the empirical variance is the solution of a SDE that only depends on $\gamma$ and vanishing noise.

The risk of the projection estimator with respect to the two introduced norms is as follows.

**Theorem 1.11** ([CG23], Proposition 8). *Suppose $m$ is such that (1.8) is satisfied. Then it is*

$$\mathbb{E}\left[\|\tilde{\alpha}_m-\alpha\|^2_{\bar{X}^{(N)}}\right]\leq\left(1+\frac{C}{N}\right)\left(\inf_{f\in S_m}\|f-\alpha\|_x^2+2\frac{m}{N}\right)+\frac{1}{N},$$

*as well as*

$$\mathbb{E}\left[\|\tilde{\alpha}_m-\alpha\|^2_{x}\right]\leq\left(1+\frac{C}{N}\right)\left(\inf_{f\in S_m}\|f-\alpha\|_x^2+2\frac{m}{N}\right)+\frac{1}{N},$$

*where $C$ depends on $T$, $\alpha$, and the first two moments of $X_0^i$.*

This statement is proved by splitting the risk into events on which the estimator is well-behaved and where it is not, the probability of the latter can be handled using concentration inequalities for random matrices involving $\hat{\Psi}_m$. To illustrate the result, the authors note that this convergence rate results in an optimal estimator with respect to the Sobolev-Laguerre basis.

An adaptive estimation method is given by using a data-driven choice of $m$, where the added penalty reflects the bias-variance trade off of the risk. Define

$$\hat{\mathscr{M}}_N=\left\{m\leq N: L_m(\left\|\hat{\Psi}_m^{-1}\right\|_{\mathrm{op}}\vee 1)\leq\frac{cN}{\log N}\right\},\qquad \hat{m}=\operatorname*{argmin}_{m\in\hat{\mathscr{M}}_N}U_{N,1}(\alpha_m)+\kappa_1\frac{m}{N}.$$

We recall that $\hat{\mathscr{M}}_N$ is the set of projections dimensions such that the growth condition $\Lambda_m$ holds. As a result, the estimator minimises the risk in $\mathscr{M}_N$, which is the set of $m\leq N$ such that the empirical counterpart of $\Lambda_m$ (as stated in (1.8)) holds.

**Theorem 1.12** ([CG23], Theorem 1). *There exists a numerical constant $\kappa_0$ such that for any $\kappa_1\geq\kappa_0$,*

$$\mathbb{E}\left[\|\hat{\alpha}_{\hat{m}}-\alpha\|^2_{\bar{X}^{(N)}}\right]\lesssim\inf_{m\in\mathscr{M}_N}\left(1+\frac{C}{N}\right)\left(\inf_{f\in S_m}\|f-\alpha\|_x^2+\kappa_1\frac{m}{N}\right)+\frac{1}{N},$$

*where $C$ is the same constant as in [CG23, Proposition 8].*

### 1.3.4 Semiparametric Estimation in the Ergodic Case

In [BPP23], a semiparametric estimation procedure in the joint asymptotics of $N,T\to\infty$ has been studied. The system is given by one-dimensional particles

$$\begin{cases}\mathrm{d}X_t^{i,N}=\varphi*\mu_t^N(X_t^{i,N})\mathrm{d}t+\mathrm{d}B_t^i,\\ (X_0^{i,N})_{1\leq i\leq N}\sim\mu_0^{\otimes N}.\end{cases}$$

Following [CGM08], the interaction function is assumed to be locally Lipschitz with polynomial growth, and it is the derivative of an interaction potential $W:\mathbb{R}\to\mathbb{R}$ that is strictly convex with convexity constant $\lambda>0$. This ensures well-posedness of the system as well as the existence and uniqueness of an invariant measure $\pi$, and a propagation of chaos statement that is uniform in time. For the statistical estimation, the interaction potential is assumed to be of the form

$$W(x)=\sum_{j=1}^{J}a_jx^{2j}+\beta(x),$$

where $a_j \geq 0$ are the parameters and $\beta$ the unknown function to be estimated. The authors also allow the potential to include trigonometric contributions, however we omit these terms here as they do not affect the estimation procedure. A key observation [BPP23, Lemma 3.1] is that the convolution of the polynomial part of $W$ with $\pi$ preserves the polynomial structure: it is $(\sum_{j=1}^J a_j(\cdot)^{2j}) * \pi(x) = \sum_{j=1}^J \alpha_j x^{2j}$, where the parameters $\alpha = (\alpha_j)_j$ are obtained through a linear transformation of $(a_j)_j$. Therefore, the authors construct an estimator for $\alpha$ that is motivated by the quantities

$$l(x) = \frac{\pi'(x)}{\pi(x)} \qquad \text{and} \qquad l(x,\alpha) = -\sum_{j=1}^J 2j\alpha_j x^{2j-1}.$$

An empirical version of the log-density $l$ is given by $l_{N,T} = \frac{\pi'_{N,T}}{\pi_{N,T}} \mathbf{1}_{\{\pi_{N,T}>\delta\}}$, where $\pi_{N,T}$ is a regularisation of $\pi$ using a kernel function, and $\delta > 0$ is a threshold to prevent blow-up. Using a weight function $w_U$, the estimator of the parameters $\alpha$ is defined as

$$\alpha_{N,T} = \underset{\alpha\in\mathbb{R}^J}{\operatorname{argmin}} \int_{\mathbb{R}} \left(l(x,\alpha) - l_{N,T}(x)\right)^2 w_U(x)\mathrm{d}x.$$

The estimation procedure for $\beta'$ is motivated by the relation

$$\beta' * \pi(x) = l(x,\alpha) - l(x).$$

The left hand side of this expression is estimated by $\Psi_{N,T} = l(\cdot, \alpha_{N,T}) - l_{N,T}$, such that we obtain an estimator for $\beta'$, denoted as $\beta'_{N,T}$, through deconvolution via Fourier transforms

$$\mathscr{F}(\beta'_{N,T}) = -\frac{\mathscr{F}(\Psi_{N,T})}{\mathscr{F}(\pi_{N,T})} \mathbf{1}_{\{\mathscr{F}(\pi_{N,T})>\omega\}},$$

where $\omega > 0$ is a threshold to prevent blow-up of the estimator. The constructed estimator converges with logarithmic speed if the decay of $\beta'$ is assumed to be polynomial.

**Theorem 1.13** ([BPP23], Proposition 4.7). *Assume that $\beta'$ is an entire function of first order and type less than some $\vartheta > 0$. Furthermore, assume that*

$$\lim_{x\to\infty} x^{p-\frac{1}{2}} \int_x^\infty |\beta'(u)| \,\mathrm{d}u + x^p \left( \int_x^\infty |\beta'(u)|^2 \,\mathrm{d}u \right)^{\frac{1}{2}} < \infty.$$

*Then*

$$\mathbb{E}\left[ \left\| \beta'_{N,T} - \beta \right\|^2_{L^2(\mathbb{R})} \right] \lesssim \omega^{-2} (\log N_T)^{-p/J}.$$

This logarithmic rate is shown to be minimax optimal in a certain class of functions $\mathscr{F} = \mathscr{F}_{p,C,(C_i)_i,\lambda}$ given by

$$\mathscr{F} = \left\{ f \in C_b^2(\mathbb{R}) \colon \|f\|_\infty \leq C_0, \left\|f'\right\|_\infty \leq C_1, \inf_{x\in\mathbb{R}} f''(x) \geq -C_2, \limsup_{x\to\infty} x^{2p} \int_x^\infty f'(y)^2 \mathrm{d}y \leq C \right\},$$

where the constants $C, C_i$ satisfy some relations. Using the two hypothesis method (see for example [Tsy08]), the logarithmic rate is retrieved the setting of i.i.d. observations of the limiting process.

**Theorem 1.14** ([BPP23], Theorem 5.1). *Let $p > \frac{1}{2}$. Then there exists a constant $c_0$ such that*

$$\inf_{\hat{\beta}'} \sup_{\beta\in\mathscr{F}} \mathbb{P}_\beta^{\otimes N} \left( \left\| \hat{\beta}' - \beta' \right\|^2_{L^2(\mathbb{R})} > c_0 (\log N)^{-p/J} \right).$$

The logarithmic rate is reminiscent of the rates in the classical deconvolution problem, where the role of the added noise is played by the invariant measure $\pi$, and the conditions on the Fourier transform of the data distribution found in deconvolution problems correspond to the conditions on the tail behaviour of $\beta'$ imposed above.

### 1.3.5 Estimation of Friction and Interaction using Method of Moments and Sieves

Motivated by the method of moment estimators introduced in [PZ22a] for parametric estimation, the work [CGL24] considers a nonparametric estimation method that combines a method of moment procedure with projection estimation. Consider ergodic McKean-Vlasov equations of the type

$$\begin{cases} \mathrm{d}X_t = -(b(X_t) + \varphi * \pi(X_t))\,\mathrm{d}t + \sigma \mathrm{d}B_t, \\ X_0 \sim \pi. \end{cases} \tag{1.10}$$

Under some conditions on the friction term $b$ and the interaction function $\varphi$, the process $(X_t)_{t\geq 0}$ is stationary with $\mathrm{Law}(X_t) = \pi$ for all $t \geq 0$. The aim is to estimate the two functions $(b, \varphi)$, given observations $(X_t)_{t\in[0,T]}$ as $T \to \infty$. The authors consider a (joint) projection estimator as follows. Let $(S_{1,m})_{m\in\mathbb{N}}$, $(S_{2,p})_{p\in\mathbb{N}}$ be two sequences of increasing subspaces of $L^2(\mathbb{R})$, given by $S_{1,m} = \mathrm{span}(e_{1,1}, \dots, e_{1,m})$ and $S_{2,p} = \mathrm{span}(e_{2,1}, \dots, e_{2,p})$. The procedure is motivated by the minimisation objective

$$(b_m^V, \varphi_p^V) = \operatorname*{argmin}_{(f_1,f_2)\in S_{1,m}\times S_{2,p}} \|(f_1 - b, f_2 - \varphi)\|_V .$$

Here $\|\cdot\|_V$ is the natural (semi-)norm associated to (1.10), given by

$$\|(f_1, f_2)\|_V^2 = \int_{\mathbb{R}} (f_1(x) + f_2 * \pi(x))^2 \mathrm{d}\pi(x),$$

whenever this expression is well-defined for a pair of functions $(f_1, f_2)$. Let $(\mathbf{b}, \mathbf{c})_{m,p}$ denote the coefficients of $(b_m^V, \varphi_p^V)$ with respect to $(e_{1,j}, e_{2,k})_{1\leq j\leq m, 1\leq k\leq p}$. The quantities in the resulting inverse problem $\mathbf{V}_{m,p}(\mathbf{b}, \mathbf{c})_{m,p} = \mathbf{Z}_{m,p}$ are approximated using the aforementioned method of moments. For the method to work, it is assumed that four trajectories following (1.10) are observed, such that the approximations $\hat{\mathbf{V}}_{m,p}$ and $\hat{\mathbf{Z}}_{m,p}$ can be constructed using two given paths each. The necessity of two paths is again due to the involvement of the unknown distribution. The estimator is given by the minimiser of the approximate inverse problem, with a truncation to ensure stability, that is

$$\widetilde{(b,\varphi)}_{m,p} = \hat{\mathbf{V}}_{m,p}^{-1} \hat{\mathbf{Z}}_{m,p} \mathbf{1}_{\left\{\left\|\hat{\mathbf{V}}_{m,p}^{-1}\right\|_{\mathrm{op}} \leq c\right\}}.$$

The risk is shown to naturally decompose into a bias and a variance term. As with the sieve estimator presented in [CG23], the main idea is to split the risk into parts where it is easily controlled, and parts whose probabilities vanish sufficiently fast, and the latter statement is shown using concentration inequalities for random matrices. To state the result, we define $L_{1,m} = \sum_{j=1}^m \left\|e_{1,j}\right\|_\infty^2$ and $L_{2,p} = \sum_{j=1}^p \left\|e_{2,j}\right\|_\infty^2$.

**Theorem 1.15** ([CGL24], Theorem 1). *Assume that $L_{1,m} + L_{2,p} \leq T$. Then*

$$\mathbb{E}\left[\left\|\widetilde{(b,\varphi)}_{m,p} - (b,\varphi)\right\|_V^2\right] \leq \inf_{(f_1,f_2)\in S_{1,m}\times S_{2,p}} \|(f_1 - b, f_2 - \varphi)\|_V^2 + C\frac{L_{1,m} + L_{2,p}}{T}.$$

Lastly, the authors show that the bias and variance term are sensible on a heuristic level by considering the Hermite basis on $L^2(\mathbb{R})$. Since $b$ and $\varphi$ are assumed to grow polynomially in order for the invariant measure to be well-defined, both functions are not square integrable, however the example still provides some intuition on the validity of their risk bounds.

## 1.4 Organisation of the Thesis

The remainder of this thesis, presenting the main findings during the PhD, is split into two parts. The first part, Chapter 2, also available as [BPZ24], focuses on projection estimators of the interaction function $\varphi$ in interacting particle systems of the form

$$\begin{cases} \mathrm{d}X_t^{i,N} = (\varphi * \mu_t^N)(X_t^{i,N})\,\mathrm{d}t + \sigma \mathrm{d}B_t^i, \\ (X_0^{i,N}) \sim \mu_0^{\otimes N}. \end{cases}$$

Our estimator $\varphi_N$ is defined through risk minimisation on sieves $(S_N)_{N\in\mathbb{N}}$ with respect to a loss function $\|\cdot\|_*$ that is natural for the particle systems considered. We present two methodologies, depending on the function class in question. The first case encompasses compact spaces where the supremum norm is uniformly bounded by some constant $K_\varphi$ that is assumed to be known. In this setup, we can employ techniques from the study of $U$-statistics to bound the risk term using Dudley integrals of metric spaces, denoted by $\mathsf{DI}(X,\rho,\psi)$ and defined in (2.5). Our main result reads as follows.

**Theorem 2.3.** *For any $q \geq 2$, it is*

$$\mathbb{E}\big[\|\varphi_N-\varphi\|_*^q\big]^{1/q} \leq \inf_{f\in S_N}\|f-\varphi\|_* + C\Bigg(\frac{K_\varphi\,\mathsf{DI}(S_N(2K_\varphi),\|\cdot\|_\infty,\psi_{e,2})}{\sqrt{N}} + \frac{p^2K_\varphi\,\mathsf{DI}(S_N(2K_\varphi),\|\cdot\|_\infty,\psi_{p/2})}{N} + \frac{p^3K_\varphi\,\mathsf{DI}(S_N(2K_\varphi),\|\cdot\|_\infty,\psi_{p/3})}{N^{3/2}}\Bigg).$$

The risk naturally decomposes into a bias term and a variance term, where the latter is achieved by using concentration inequalities for $U$-statistics. Furthermore, we present function spaces for which our achieved bound is polynomial in $N$ and almost minimax optimal. For $A, D > 0$, define

$$S(A,D) := \left\{ x \mapsto \frac{1}{\sqrt{A}}\sum_{k=0}^{D} c_k \exp(i\pi kx/A) : \quad \sum_{k=0}^{D}|c_k| \leq \sqrt{A}K_\varphi \right\}.$$

In these function classes, our estimator achieves a parametric convergence rate up to a logarithmic factor with respect to the loss $\|\cdot\|_*$, assuming the initial distribution is a centred Gaussian with variance $\zeta > 0$.

**Theorem 2.6.** *Set with arbitrary small $\delta > 0$,*

$$A_N := \sqrt{2(\zeta^2+T)\log(N)}, \qquad D_N := \sqrt{(2+\delta)\frac{\zeta^2+T}{\zeta^2}\log(N)}.$$

*Then, it is*

$$\mathbb{E}\big[\|\varphi_N-\varphi\|_*^2\big]^{1/2} \lesssim \sqrt{\frac{\exp(K_\varphi^2T)}{N}} + K_\varphi\sqrt{\frac{\log(N)}{N}}.$$

We also are able to present risk bounds in general vector spaces of functions, removing the strong assumption on uniform boundedness. The study of the projection estimator in this setup is closely related to [CG19; CG20; CG23] and heavily relies on understanding and controlling the behaviour of a change of base matrix that is associated with transforming an orthonormal basis in $S_N$ into an orthonormal system with respect to the inner product induced by $\|\cdot\|_\star$. We obtain uniform risk bounds in certain function spaces as follows.

**Theorem 2.9.** *It is*

$$\sup_{\varphi} \mathbb{E}\left[\|\widehat{\varphi}_N - \varphi\|_*^2\right] \leq (1+o(1)) \sup_{\varphi} \inf_{f \in S_N} \|f-\varphi\|_*^2$$
$$+ CN^{-1/2}(1+L_N) \sup_{\varphi} \|\varphi_* - \varphi\|_{**}^2 + \frac{CD_N}{NT},$$

*where the supremum is taken over the class of functions* $\Phi$ *defined as*

$$\Phi := \left\{\varphi : \mathbb{R} \to \mathbb{R} : \ \|\varphi\|_\infty \leq K_1, \ K_2 \leq \|\varphi\|_{\text{Lip}} \leq K_3\right\}$$

*for fixed constants* $K_1, K_2, K_3 > 0$.

The quantity $L_N$ depends on the choice of basis vectors and can be controlled in many examples. The functional $\|\cdot\|_{**}$ is closely related to the original loss $\|\cdot\|_*$, and we present an example in which the term can be controlled in Proposition 2.12.

Finally, we investigate the minimax optimality of this estimation problem. The parametric rate obtained in compact function classes is shown to be optimal in Theorem 2.13, using the two hypothesis method. Furthermore, we are able to generalise the optimality result in [BPP23]. In this work, the authors proved that in a semiparametric estimation setting, the optimal rate of convergence is logarithmic, given observations of the McKean-Vlasov equation at fixed point in time. We extend this result in Theorem 2.14 by proving that, given continuous observations of the same processes, the optimal result is still logarithmic.

In Chapter 3, available as [Amo+24], we study the estimation of particle systems with both a confinement term and an interaction term of the form

$$\begin{cases} dX_t^{i,N} = -(V'(X_t^{i,N}) + W' * \mu_t^N(X_t^{i,N}))dt + dB_t^i, \\ (X_0^{i,N})_{1\leq i\leq N} \sim \mu_0^{\otimes N}. \end{cases}$$

The assumptions imposed on $V$ and $W$ are such that existence and uniqueness of an invariant measure $\Pi$ are guaranteed, and that the empirical measure converges to $\Pi$ with a certain rate $N_T$ as $N, T \to \infty$ with respect to the Wasserstein-1 distance. Furthermore, we are able to show a uniform propagation of chaos result in $L^{2p}(\mathbb{R})$ for any $p \in \mathbb{N}$, which differs from the existing literature in that the convexity assumption on $W$ is weakened.

**Proposition 3.6.** *Let* $p \geq 1$. *There exists a constant* $c > 0$ *such that for any* $N \in \mathbb{N}$ *and* $1 \leq i \leq N$, *it is*

$$\sup_{t\geq 0} \mathbb{E}\left[|X_t^{i,N} - \overline{X}_t^i|^{2p}\right] \leq cN^{-p}.$$

In the above statement, $\overline{X}_t^i$ are $N$ i.i.d. copies of the associated mean-field equation. The framework of this chapter is similar to [BPP23] and we seek to answer some questions that naturally arise from their results. We study a deconvolution-type estimator for the interaction function $W'$ which is similar to the aforementioned work, however due to the more complicated potentials in play, the resulting estimator requires a more delicate study. As proved in [BPP23, Proposition 4.7], polynomial decay assumptions on the potential $W$ lead to logarithmic convergence rates of the estimator $W'_{N,T}$. Thus a natural question to ask is whether an exponential decay assumption can lead to a polynomial rate. Our main result is that we are able to answer this question positively in

**Theorem 3.11.** *Let* $\varepsilon \in (0,1)$ *and* $a \in \mathbb{R}$. *There exist* $c, \gamma > 0$ *such that*

$$\mathbb{E}\left[\int_{\mathbb{R}} |W'_{N,T}(y) - W'(y)|^2 \, \mathrm{d}y\right]^{\frac{1}{2}} \leq c \exp\left(\frac{a\epsilon}{2}(c \log N_T)^{\frac{1}{2}}\right)(\log N_T)^{\frac{1}{4}} N_T^{-\frac{\gamma}{2}}.$$

This statement hinges on the knowledge of the roots of $\mathscr{F}(\Pi)$, which is the Fourier transform of the invariant measure. More precisely, we have the following

**Assumption 7.** For $a \in \mathbb{R}$, define the set $\mathscr{L}_a = \{y + ia \colon y \in \mathbb{R}\} \subset \mathbb{C}$. There exists a $\bar{a} \geq 0$ such that

$$\int_{\mathscr{L}_{\bar{a}}} \left| \frac{\mathscr{F}\left(W'\right)(z)}{\mathscr{F}(\Pi)(z)} \right|^2 \mathrm{d}z < \infty.$$

This condition is notably difficult to verify, deserving its own section dedicated towards its analysis. Utilising Hadamard's factorisation theorem, we find lower bounds for $\mathscr{F}(\Pi)$, which together with the exponential decay assumption on $W'$ guarantees the existence of the above integral. We formalise this in Corollary 3.20. Lastly, we present an example wherein $\mathscr{F}(\Pi)$ indeed exhibits a polynomial decay, provided the confinement potential is not smooth, but is only continuously differentiable up to some $J \in \mathbb{N}$.

# Chapter 2

# Risk Minimisation of a Projection Estimator

## 2.1 Introduction

We present our work [BPZ24]. This chapter is dedicated to the study of $N$-dimensional interacting particle systems described by the equation

$$dX_t^{i,N} = (\varphi * \mu_t^N)(X_t^{i,N})\,dt + \sigma dB_t^i, \quad i = 1, \dots, N, \tag{2.1}$$

where $t \in [0, T]$, $\varphi : \mathbb{R} \to \mathbb{R}$ is the interaction potential and $(B^i)_{i=1}^N$ are independent one-dimensional Brownian motions. Here, $\mu_t^N$ stands for the empirical measure of the particle system at time $t$, given by

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}}, \tag{2.2}$$

and $\varphi * \mu(x) := \int_{\mathbb{R}} \varphi(x - y)\,d\mu(y)$. We make the assumption that we observe $N$ paths $(X_t^{i,N}, t \in [0, T], i = 1, \dots, N)$ and aim to estimate the unknown interaction function $\varphi$ as $N \to \infty$ with $T > 0$ being fixed. It is noteworthy that, given complete path observations, the diffusion coefficient $\sigma$ can be considered to be known.

Interacting particle systems of the type (2.1) play an important role in probability theory and applications. Initially introduced by McKean in his pioneering work [McK66], these systems served as models for plasma dynamics. In more recent years, diffusion-type interacting particle systems have found wide-ranging applications in finance [Car+19; Dje+22; FS13; GSS20; GH11], social science [Cha+17], neuroscience [Bal+12], and population dynamics [ME99], among others. From a probabilistic perspective, the particle system outlined in (2.1) is inherently linked to its corresponding mean-field limiting equation—a one-dimensional McKean-Vlasov stochastic differential equation

$$\begin{cases} dX_t = (\varphi * \mu_t)(X_t)\,dt + \sigma dB_t, \\ \mathrm{Law}(X_t) = \mu_t, \end{cases} \tag{2.3}$$

where $B$ is a 1-dimensional Brownian motion. Under suitable assumptions on the function $\varphi$, a well-known phenomenon known as *propagation of chaos* emerges. This phenomenon indicates that, as $N \to \infty$, the empirical probability measure $\mu_t^N$ weakly converges to $\mu_t$ for any $t > 0$, and the McKean-Vlasov stochastic differential equation (2.3) characterises the asymptotic trajectory of an individual particle. Classical treatments of propagation of chaos and McKean-Vlasov SDEs can be found in the monographs [Kol10; Szn91]. Furthermore, the measure $\mu_t$ is absolutely continuous with respect to the Lebesgue measure for all $t > 0$, assuming certain regularity conditions on $\varphi$. Moreover, the density of $\mu_t$, denoted again by $\mu_t$ with a slight abuse of notation, satisfies the granular media equation

$$\partial_t \mu_t = \partial_x \left\{ (\sigma^2/2)\partial_x \mu_t + (\varphi * \mu_t)\mu_t \right\}. \tag{2.4}$$

The realm of statistical inference for interacting particle systems remains a less explored domain. While a considerable number of recent articles delve into parametric estimation for particle systems and McKean-Vlasov stochastic differential equations under diverse model and sampling assumptions (see, for instance, [Amo+23; Bis11; Che21; CG23; DH23; GL21a; GL21b; Kas90; LQ22; Sha+23; Wen+16]), non- and semiparametric estimation of the interaction potential $\varphi$ presents a notably more intricate challenge. This complexity arises from the convolution with measures the $\mu_t^N$ or $\mu_t$ and has only been the subject of investigation in three recent papers. Specifically, the articles [BPP23] and [Amo+24] explore the semiparametric estimation of $\varphi$ given observations $X_T^{i,N}, i = 1, \ldots, N$, with $N, T \to \infty$. The methodology heavily relies on the existence of the invariant density, necessitating the asymptotic regime $T \to \infty$, and employs the deconvolution method. Another relevant work, [DH22], focuses on the same observation scheme as the current paper, presenting a nonparametric approach for the drift function $\varphi * \mu_t$ but demonstrating consistency only for an estimator of the interaction function $\varphi$.

This work aims to investigate nonparametric estimation of the function $\varphi$ using the sampling scheme $X_t^{i,N}, t \in [0, T], i = 1, \ldots, N$, with $N \to \infty$ and $T$ held fixed. We propose two interconnected methods, both based on empirical risk minimisation over sieves $(S_m)_{m\geq 1}$, employing an unconventional underlying loss function $\|\cdot\|_*$ that depends on the unknown measure $(\mu_t)_{t\in[0,T]}$. Our first approach addresses the scenario where the functional spaces $(S_m)_{m\geq 1}$ are compact with respect to $\|\cdot\|_\infty$. In this case, the analysis of the rate of the proposed estimator heavily relies on empirical process theory for (degenerate) $U$-statistics. The second approach investigates a more traditional setting, where $(S_m)_{m\geq 1}$ are finite-dimensional vector spaces. Here, we require a further truncation of the empirical risk minimiser, akin to the approach introduced in [CG20] for classical stochastic differential equations, where the coefficients are independent of the laws $(\mu_t)_{t\in[0,T]}$. We derive convergence rates with respect to the loss function $\|\cdot\|_*$, involving an examination of stochastic and approximation errors. We will demonstrate that the resulting convergence rate is often $\sqrt{\log(N)/N}$ in the first scenario, which essentially matches the minimax lower bound. However, for the second method, the convergence rate decreases due to the truncation procedure. Additionally, we will establish that the minimax lower bound for the traditional $L^2(\mathbb{R})$-loss is logarithmic, consistent with the findings of [BPP23].

The remainder of this chapter is structured as follows. Section 2.2 concentrates on a risk minimisation approach over compact functional classes. This section elucidates the core concepts, establishes convergence rates in terms of metric entropy, and analyses the approximation error associated with the estimation method. In Section 2.3, we introduce an alternative approach that involves risk minimisation over vector spaces coupled with an additional dimension truncation. Uniform convergence rates for the resulting estimator are presented. Section 2.4 delves into an exploration of minimax lower bounds. Specifically, we demonstrate that the estimator developed in Section 2.2 attains essentially optimal rates with respect to the $\|\cdot\|_*$-norm. Additionally, we establish a logarithmic lower bound when considering the traditional $L^2$-loss. For the detailed proofs of the main results, we refer to Section 2.5.

**Notation**

All random variables and stochastic processes are defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Throughout, positive constants are represented by $C$ (or $C_p$ if dependent on an external parameter $p$), though their values may vary across lines. We use the notation $C^k(\mathbb{R})$ (resp. $C_c^k(\mathbb{R})$) to denote the space of $k$ times continuously differentiable functions (resp. the space of $k$ times continuously differentiable functions with compact support). The sup-norm of a function $f : \mathbb{R} \to \mathbb{R}$ is denoted as $\|f\|_\infty$ or $\|f\|_Q := \sup_{x\in Q} |f(x)|$. When referring to a probability measure or density $\mu$, $\widehat{\mu}$ denotes its Fourier transform:

$$\widehat{\mu}(z) = \int_{\mathbb{R}} \exp(izx)\mu(x)\,\mathrm{d}x.$$

Given a matrix $A \in \mathbb{R}^{k\times k}$, we denote by $\|A\|_{\mathrm{op}}$ the operator norm of $A$; we write $A \succcurlyeq B$ if $A-B$ is a positive semidefinite matrix. For two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we write $a_n \lesssim b_n$ if there exists a constant $C > 0$ with $a_n \leq C b_n$ for all $n \in \mathbb{N}$. The Orlicz norm of a real valued random variable $\eta$ with respect to a nondecreasing, convex function $\psi$ on $\mathbb{R}$ with $\psi(0) = 0$ is defined by

$$\|\eta\|_\psi := \inf\left\{t > 0 : \mathbb{E}\left[\psi\big(|\eta|/t\big)\right] \leq 2\right\}.$$

When $\psi_p(x) := x^p$ with $p \geq 1$ the corresponding Orlicz norm is (up to a constant) the $L^p$-norm $\|\eta\|_p = (\mathbb{E}[|\eta|^p])^{1/p}$. We say that $\eta$ is *sub-Gaussian* if $\|\eta\|_{\psi_{e,2}} < \infty$ with $\psi_{e,2} := \exp(x^2) - 1$. In particular, this implies that for some constants $C, c > 0$,

$$\mathbb{P}(|\eta| \geq t) \leq 2\exp\left(-\frac{ct^2}{\|\eta\|^2_{\psi_{e,2}}}\right) \quad \text{and} \quad \mathbb{E}[|\eta|^p]^{1/p} \leq C\sqrt{p}\|\eta\|_{\psi_{e,2}}, \quad p \geq 1.$$

Consider a real valued random process $(X_t)_{t\in\mathscr{T}}$ on a metric space $(\mathscr{T}, \mathsf{d})$. We say that the process has *sub-Gaussian increments* if there exists $C > 0$ such that

$$\|X_t - X_s\|_{\psi_{e,2}} \leq C\mathsf{d}(t,s), \quad \forall t,s \in \mathscr{T}.$$

Let $(\mathsf{Y}, \rho)$ be a metric space and $\mathsf{X} \subseteq \mathsf{Y}$. For $\varepsilon > 0$, we denote by $\mathscr{N}(\mathsf{X}, \rho, \varepsilon)$ the covering number of the set $\mathsf{X}$ with respect to the metric $\rho$, that is, the smallest cardinality of a set (or net) of $\varepsilon$-balls in the metric $\rho$ that covers $\mathsf{X}$. Then $\log \mathscr{N}(\mathsf{X}, \rho, \varepsilon)$ is called the *metric entropy* of $\mathsf{X}$ and

$$\mathsf{DI}(\mathsf{X}, \rho, \psi) := \int_0^{\mathrm{diam}(\mathsf{X})} \psi^{-1}\big(\mathscr{N}(\mathsf{X}, \rho, u)\big)\,\mathrm{d}u \tag{2.5}$$

with $\mathrm{diam}(\mathsf{X}) := \max_{x,x'\in\mathsf{X}} \rho(x,x')$, is called the Dudley integral with respect to $\psi$. For example, if $|\mathsf{X}| < \infty$ and $\rho(x,x') = 1_{\{x\neq x'\}}$ we get $\mathsf{DI}(\mathsf{X}, \rho, \psi_{e,2}) = \sqrt{\log|\mathsf{X}|}$, where $|\mathsf{X}|$ is the cardinality of $\mathsf{X}$

## 2.2 Risk Minimisation over Compact Functional Classes

We start by introducing assumptions on the underlying interacting particle system that insure propagation of chaos. We assume that the initial law of the model is given by $\mathrm{Law}(X_0^{1,N}, \ldots, X_0^{N,N}) = \mu_0^{\otimes N}$, where $\mu_0^{\otimes N}$ denotes the $N$-fold product measure of some probability measure $\mu_0$ with $\int x^k \mu_0(dx) < \infty$ for all $k \in \mathbb{N}$. Furthermore, we assume the following condition.

**Assumption 2.** The interaction function $\varphi : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous and bounded, that is,

$$|\varphi(x)-\varphi(y)| \le L_\varphi |x-y|, \quad |\varphi(x)| \le K_\varphi, \quad x,y \in \mathbb{R}$$

for some finite $L_\varphi, K_\varphi > 0$.

This condition guarantees the validity of propagation of chaos in the sense that, for all $t \in [0,T]$, the random measure $\mu_t^N$ converges to the deterministic measure $\mu_t$ in law (see e.g. [CD22b, Theorem 3.1]).

### 2.2.1 Construction of the Estimator and Main Results

We initiate our exposition by outlining the fundamental principles of our estimation methodology. To begin, we consider a sequence of spaces $(S_m)_{m\ge 1}$ (sieves). The crucial idea of our approach lies in the following minimisation strategy:

$$\min_{f \in S_N} \frac{1}{NT} \sum_{i=1}^N \int_0^T \left(f * \mu_t^N(X_t^{i,N}) - \varphi * \mu_t^N(X_t^{i,N})\right)^2 \mathrm{d}t. \tag{2.6}$$

However, the above risk function cannot be directly computed from the data since the interaction function $\varphi$ is unknown. We derive an empirical (noisy) version of the minimisation problem by omitting the irrelevant term $\varphi * \mu_t^N(X_t^{i,N})^2$ in the integrand and minimising the resulting quantity:

$$\gamma_N(f) := \frac{1}{NT} \sum_{i=1}^N \left( \int_0^T f * \mu_t^N(X_t^{i,N})^2 \,\mathrm{d}t - 2 \int_0^T f * \mu_t^N(X_t^{i,N}) \,\mathrm{d}X_t^{i,N} \right)$$

over $S_N$. For further analysis, we introduce the following bilinear forms:

$$\langle f, g \rangle_N := \frac{1}{NT} \sum_{i=1}^N \int_0^T (f * \mu_t^N)(X_t^{i,N})(g * \mu_t^N)(X_t^{i,N}) \,\mathrm{d}t,$$

$$\langle f, g \rangle_* := \frac{1}{T} \int_0^T \int_{\mathbb{R}} (f * \mu_t)(x)(g * \mu_t)(x)\mu_t(x) \,\mathrm{d}x \,\mathrm{d}t.$$

We set $\|f\|_N^2 := \langle f, f \rangle_N$ and $\|f\|_*^2 := \langle f, f \rangle_*$. With these notations at hand, we deduce the identity

$$\|f - \varphi\|_N^2 = \gamma_N(f) + \nu_N(f) + \|\varphi\|_N^2 \tag{2.7}$$

where $\nu_N(f)$ is the martingale term defined via

$$\nu_N(f) := \frac{2\sigma}{NT} \sum_{i=1}^N \int_0^T (f * \mu_t^N)(X_t^{i,N}) \,\mathrm{d}B_t^i.$$

Finally, our estimator $\varphi_N$ is defined as follows:

$$\varphi_N := \mathrm{argmin}_{f \in S_N} \gamma_N(f), \qquad \varphi_*^N := \mathrm{argmin}_{f \in S_N} \|f - \varphi\|_*^2. \tag{2.8}$$

The identity (2.7) means that by minimising $\gamma_N(f)$ we minimise (up to a martingale term $\nu_N(f)$) $\|f - \varphi\|_N$ which is close to $\|f - \varphi\|_*$ for $N$ large enough.

Before we proceed with the asymptotic analysis, we give some remarks about a rather unconventional loss function $\|\cdot\|_*$ and its relation to the classical $L^2(\mathbb{R})$-norm.

**Remark 2.1.** The upcoming analysis will reveal that the risk $\|\cdot\|_*$ is the intrinsic norm for evaluating error bounds. Unlike conventional literature on empirical risk minimisation, it is crucial to note that this norm is elusive since the laws $\mu_t$ remain unobserved. This presents one of the principal mathematical challenges in our statistical analysis.

**Remark 2.2.** The norm $\|\cdot\|_*$ can be bounded from above by the $L^2(\mathbb{R})$-norm. Indeed, if we assume that the densities $(\mu_t)_{t\in[0,T]}$ satisfy $\int_0^T \|\mu_t\|_\infty \,\mathrm{d}t < \infty$, we conclude by Young's convolution inequality that

$$\|f\|_*^2 \leq \frac{1}{T}\int_0^T \|\mu_t\|_\infty \|f * \mu_t\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}t \leq \|f\|_{L^2(\mathbb{R})}^2 \frac{1}{T}\int_0^T \|\mu_t\|_\infty \,\mathrm{d}t.$$

Nonetheless, these two norms are far from being equivalent. To illustrate this point, we assume, for simplicity, that $\mu_t = \mu$ represents the density of the standard normal distribution for all $t$. Now, consider the function $f(x) = 1_{[2M,3M]}(x)$ for some $M > 0$. In this scenario, it is evident that $\|f\|_{L^2(\mathbb{R})}^2 = M$. On the other hand, there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned}\|f\|_*^2 &= \int_{[-M,M]} (f*\mu)^2(x)\mu(x)\,\mathrm{d}x + \int_{[-M,M]^c} (f*\mu)^2(x)\mu(x)\,\mathrm{d}x \\ &\leq c_1 \exp(-c_2 M^2).\end{aligned}$$

In particular, $\|f\|_{L^2(\mathbb{R})} \to \infty$ while $\|f\|_* \to 0$ as $M \to \infty$ at an exponential rate.

To formulate our main statements, we assume that the spaces $(S_m)_{m\geq 1}$ are compact with respect to $\|\cdot\|_\infty$ with $\|f\|_\infty \leq K_\varphi$ for $f \in S_m$. In certain cases, we employ the notation $S_m(K_\varphi)$ to explicitly express the dependence of the spaces on the parameter $K_\varphi$. Additionally, we assume the existence of spaces $S_m(2K_\varphi)$ which are also compact with respect to $\|\cdot\|_\infty$ with $\|f\|_\infty \leq 2K_\varphi$ and

$$f, g \in S_m(K_\varphi) \implies f \pm g \in S_m(2K_\varphi).$$

Subsequently, we establish the bounds for the error term $\|\varphi_N - \varphi\|_*$.

**Theorem 2.3.** *It holds under Assumption 2,*

$$\begin{aligned}\mathbb{E}\big[\|\varphi_N - \varphi\|_*^q\big]^{1/q} \leq \|\varphi_*^N - \varphi\|_* + C\Bigg(&\frac{K_\varphi \,\mathrm{DI}(S_N(2K_\varphi), \|\cdot\|_\infty, \psi_{e,2})}{\sqrt{N}} \\ &+ \frac{p^2 K_\varphi \,\mathrm{DI}(S_N(2K_\varphi), \|\cdot\|_\infty, \psi_{p/2})}{N} + \frac{p^3 K_\varphi \,\mathrm{DI}(S_N(2K_\varphi), \|\cdot\|_\infty, \psi_{p/3})}{N^{3/2}}\Bigg)\end{aligned}$$

*for any $p > 2$, $2 \leq q \leq p$, where $C > 0$ is an absolute constant and $\varphi^*$ stands for the best approximation of $\varphi$ as defined in (2.8).*

Let us outline the key concepts behind the proof of Theorem 2.3. We commence with the inequality:

$$\|\varphi_N - \varphi\|_*^2 \leq \|\varphi_*^N - \varphi\|_*^2 + 2\left(\sup_{f\in S_N} \left|\|f-\varphi\|_N^2 - \|f-\varphi\|_*^2\right| + \sup_{f\in S_N} |\nu_N(f)|\right). \tag{2.9}$$

This inequality decomposes the estimation error into an approximation error $\|\varphi_*^N - \varphi\|_*^2$ and stochastic errors related to the last two terms. To handle the stochastic errors, we initially consider i.i.d. observations from the McKean-Vlasov SDE (2.3). Remarkably, the term $\|f - \varphi\|_N^2 - \|f-\varphi\|_*^2$ turns out to be a linear combination of (degenerate) U-statistics, while $\nu_N(f)$ represents a martingale, as noted earlier. Concentration inequalities are applied to both terms, leading to the moment bound in Theorem 2.3 via the metric entropy. In the final step of the proof, we employ a change of measure device initially proposed in [DH22] to transfer the statement from i.i.d. observations of the McKean-Vlasov SDE (2.3) to observations of the original particle system (2.1). In case we have a bound on the covering number of the space $S_N$, we immediately obtain the following corollary.

**Corollary 2.4.** *Suppose that $\mathcal{N}\big(S_N(2K_\varphi), \|\cdot\|_\infty, \varepsilon\big) \lesssim \varepsilon^{-D_N}$ for all $N$ and an increasing sequence of positive numbers $D_N$ satisfying $D_N^2 N^{-1/2} \to 0$ as $N \to \infty$. Then it holds*

$$\mathbb{E}\big[\|\varphi_N - \varphi\|_*^q\big]^{1/q} \leq \|\varphi_*^N - \varphi\|_* + CK_\varphi \sqrt{\frac{D_N}{N}}.$$

Our focus now shifts to examining the approximation error $\|\varphi_*^N - \varphi\|_*$, which will be addressed in the subsequent analysis.

### 2.2.2 Approximation Error

In this segment, we delve into an examination of the approximation error outlined in Theorem 2.3. The strong contractivity inherent in the norm $\|\cdot\|_*$ introduces a surprising characteristic wherein the error $\|\varphi_*^N - \varphi\|_*$ often exhibits exponential decay in the dimensionality of the space $S_N$. In the following, we adopt the following assumption regarding the initial distribution:

**Assumption 3.** Assume that the initial distribution is normal, that is,

$$\mu_0(x) = \frac{1}{\sqrt{2\pi\zeta^2}} \exp(-x^2/(2\zeta^2))$$

for some $\zeta > 0$.

We also consider a condition on the interaction function $\varphi$:

**Assumption 4.** There is a monotone increasing sequence $(\alpha_n)$ with

$$\alpha_n \to \infty, \quad \alpha_{n+1}/\alpha_n \to 1, \quad n \to \infty$$

such that the following representation holds

$$(\mu_t * \varphi)(x) = \frac{1}{\sqrt{\alpha_n}} \sum_{k=0}^{\infty} c_k \exp(i\pi kx/\alpha_n)\widehat{\mu}_t(-k/\alpha_n), \quad x \in [-\alpha_n, \alpha_n],$$

with coefficients $c_k = c_k(\alpha_n)$ satisfying $\sum_{k=1}^{\infty} |c_k| \lesssim K_\varphi \sqrt{\alpha_n}$.

**Remark 2.5.** Let us examine Assumption 4. This assumption holds for any function $\varphi$ which admits a Fourier series representation of the form

$$\varphi(x) = \frac{1}{\sqrt{\alpha}} \sum_{k=0}^{\infty} a_k \exp(i\pi kx/\alpha), \quad x \in \mathbb{R}, \tag{2.10}$$

with absolutely summable coefficients $(a_k)_{k\geq 0}$ satisfying $\sum_k |a_k| \leq \sqrt{\alpha} K_\varphi$ for some $\alpha > 0$. These types of periodic potentials appear in McKean-Vlasov equations arising in physics, see e.g. the Kuramoto-Shinomoto-Sakaguchi model in [Fra05]. Indeed, for such functions we readily deduce the identity

$$(\mu_t * \varphi)(x) = \frac{1}{\sqrt{\alpha}} \sum_{k=0}^{\infty} a_k \exp(i\pi kx/\alpha)\widehat{\mu}_t(-k/\alpha).$$

Taking for instance $\alpha_n = \alpha n$ and comparing the coefficients, we obtain the statement of Assumption 4 with $c_{kn} = a_k\sqrt{n}$ and $c_l = 0$ for $l \notin \{kn:\ k \geq 0\}$.

Next, for any $A > 1$ and $D \in \mathbb{N}$, we consider the functional space

$$S(A,D) := \left\{ x \mapsto \frac{1}{\sqrt{A}} \sum_{k=0}^{D} c_k \exp(i\pi k x/A) : \quad \sum_{k=0}^{D} |c_k| \leq \sqrt{A} K_\varphi \right\}.$$

The main result of this subsection is the following theorem.

**Theorem 2.6.** *Suppose that Assumption 3 and Assumption 4 hold. Set with arbitrary small* $\delta > 0$,

$$A_N := \sqrt{2(\zeta^2 + T)\log(N)}, \qquad D_N := \sqrt{(2+\delta)\frac{\zeta^2 + T}{\zeta^2}\log(N)}$$

*and define* $n_N := \min\{n : \alpha_n \geq A_N\}$, $N \in \mathbb{N}$. *Then we obtain*

$$\inf_{g \in S_N = S(\alpha_{n_N}, D_N)} \|g - \varphi\|_*^2 \lesssim \frac{\exp(K_\varphi^2 T)}{N}.$$

*As a consequence, we deduce that*

$$\mathbb{E}\big[\|\varphi_N - \varphi\|_*^2\big]^{1/2} \lesssim \sqrt{\frac{\exp(K_\varphi^2 T)}{N}} + K_\varphi \sqrt{\frac{\log(N)}{N}}.$$

As a consequence of Theorem 2.6, the resulting convergence rate is established as $\sqrt{\log(N)/N}$. We will demonstrate in Section 2.4 that this rate is essentially optimal. However, it is important to note, as highlighted in Remark 2.2, that this does not necessarily imply a polynomial convergence rate with respect to the classical $L^2(\mathbb{R})$-norm. In fact, we will establish in Section 2.4 that the minimax lower bound for the $L^2(\mathbb{R})$-norm is logarithmic, consistent with the findings of [BPP23].

**Remark 2.7.** Theorem 2.6 suggests that the approximation error decays exponentially when dealing with functions of the form described in (2.10). However, this phenomenon is not exclusive to the Fourier basis. Remarkably, a similar result can be observed for the polynomial basis. The following explanation sheds light on the fast rates of approximation in the $\|\cdot\|_*$-norm. Suppose that $\varphi \in L^1(\mathbb{R})$. Then we have by the Parseval identity

$$(\varphi * \mu_t)(x+iy) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(iux - uy)\widehat{\varphi}(u)\widehat{\mu}_t(-u)\,\mathrm{d}u.$$

Under Assumption 3, $|\widehat{\mu}_t(u)| \leq \exp(-u^2\zeta^2/2)$ (see (2.38)) and hence

$$\begin{aligned} |(\varphi * \mu_t)(x+iy)| &\leq \frac{\|\varphi\|_{L^1}}{2\pi}\int \exp(-uy)|\widehat{\mu}_t(u)|\,\mathrm{d}u \\ &\leq \sqrt{\frac{1}{2\pi\zeta^2}}\|\varphi\|_{L^1}\exp\big(y^2/(2\zeta^2)\big). \end{aligned}$$

Thus

$$\max_{|z|=r} |(\varphi * \mu_t)(z)| \leq \sqrt{\frac{1}{2\pi\zeta^2}}\|\varphi\|_{L^1}\exp\big(r^2/(2\zeta^2)\big), \quad r > 0.$$

As a result, the function $\varphi * \mu_t$ is entire of order 2 and finite type. Furthermore, we deduce the power series expansion

$$(\varphi * \mu_t)(z) = \sum_{n=0}^{\infty} c_n z^n$$

with $c_n \leq \left(e/\zeta^2 n\right)^{n/2}$. This implies the inequality

$$\sup_{x\in[-A,A]} \left|(\varphi * \mu_t)(x) - \sum_{n=0}^{D} c_n x^n\right| \leq \varepsilon_{D,A} = \sum_{n=D+1}^{\infty} \left(\frac{eA^2}{\zeta^2 n}\right)^{n/2}$$

Note that

$$\varepsilon_{D,A} \leq 2e^{-(1+D)/2}$$

provided $(1+D)^{-1}A^2 \leq \zeta^2/e^2$. Let $T_K[f] = 1_{\{|f(x)|\leq K\}} f(x) + K\mathrm{sign}(f(x))$ be a truncation operator at level $K$. Then it holds that

$$\inf_{\psi\in\mathrm{span}\{z^n, n\leq D\}} \frac{1}{T}\int_0^T \int \left((\mu_t * \varphi)(x) - T_{K_\varphi}[\psi * \mu_t](x)\right)^2 \mu_t(x)\,\mathrm{d}x\mathrm{d}t \leq \varepsilon_{D,A}^2 + R_T(A)$$

with

$$R_T(A) = \frac{4K_\varphi^2}{T}\int_0^T \int_{|x|>A} \mu_t(x)\,\mathrm{d}x\mathrm{d}t.$$

Here we used the fact that for any $\psi \in \mathrm{span}\{z^n : n \leq D\}$, it is $\psi * \mu_t \in \mathrm{span}\{z^n : n \leq D\}$. Similarly to the case of Fourier basis (see the proof of Theorem 2.6), we get

$$R_T(A) \leq 8K_\varphi^2 \exp\left(K_\varphi^2 T/2\right) \exp\left(-\frac{A^2}{2(\zeta^2 + T)}\right).$$

If we choose $A_N = \sqrt{2(\zeta^2+T)\log(N)}$ and $D_N = \frac{2e^2(\zeta^2+T)}{\zeta^2}\log(N)$, then

$$\inf_{\varphi\in\mathrm{span}\{z^n:\, n\leq D_N\}} \frac{1}{T}\int_0^T \int \left((\mu_t * \varphi)(x) - T_{K_\varphi}[\psi * \mu_t](x)\right)^2 \mu_t(x)\,\mathrm{d}x\mathrm{d}t \lesssim \frac{\exp\left(K_\varphi^2 T/2\right)}{N}.$$

In other words, we obtain an exponential decay of the approximation error for the polynomial basis as well.

## 2.3 Risk Minimisation over Vector Spaces

While the empirical minimisation approach presented in the preceding section serves as a valuable estimation method, it is not without its limitations. Firstly, the construction of the estimator $\varphi_N$ in (2.8) inherently assumes knowledge of the constant $K_\varphi$. Secondly, a significant drawback lies in the necessity of the compactness property of functional spaces $S_m$ with respect to the supremum norm, a condition that is notably stringent.

### 2.3.1 Setting and Construction of the Estimator

This section focuses on a more traditional setting of risk minimisation within vector spaces. We assume that the finite-dimensional spaces $S_N$ are equipped with an inner product $\langle\cdot,\cdot\rangle$, and $(e_1,\ldots,e_{D_N})$ represents an orthonormal basis of $S_N$ satisfying $\|e_j\|_\infty < \infty$ for all $j = 1,\ldots,D_N$.

Due to the vector space structure, the estimator $\varphi_N$ introduced at (2.8) can be computed explicitly as

$$\varphi_N = \sum_{j=1}^{D_N} (\theta_N)_j \, e_j, \tag{2.11}$$

where $\theta_N = \Psi_N^{-1} Z_N$ and the random vectors $\Psi_N$, $Z_N$, taking values in $\mathbb{R}^{D_N \times D_N}$ and $\mathbb{R}^{D_N}$ respectively, are given by

$$(\Psi_N)_{jk} = \langle e_j, e_k \rangle_N,$$

$$(Z_N)_j = \frac{1}{NT} \sum_{i=1}^{N} \int_0^T e_j * \mu_t^N(X_t^{i,N}) \, \mathrm{d}X_t^{i,N}. \tag{2.12}$$

There is a theoretical counterpart to the empirical matrix $\Psi_N$. Indeed, we have that $\mathbb{E}[\Psi_N] \approx \Psi$ where the matrix $\Psi$ is given by the formula

$$(\Psi)_{jk} = \langle e_j, e_k \rangle_* . \tag{2.13}$$

Obviously, both matrices are positive semidefinite by construction. In the next step, we will regularise the estimator $\varphi_N$. For a given sequence $D_N$, the operator norm $\|\Psi_N^{-1}\|_{\mathrm{op}}$ may become too large compared to the sample size $N$ and we would like to avoid such situations. For this purpose, we restrict the growth of $\|\Psi^{-1}\|_{\mathrm{op}}$.

**Assumption 5.** Let $\eta \geq 5$ be a given number. We assume that $\Psi_N$ is invertible almost surely and the sequence $D_N$ is such that the following growth condition is satisfied:

$$L_N^2 \|\Psi^{-1}\|_{\mathrm{op}}^2 \leq \frac{c_{\eta,T} NT}{4 \log(NT)},$$

where $L_N := \sum_{j=1}^{D_N} \|e_j\|_\infty^2$ and $c_{\eta,T} := (72\eta T)^{-1}$.

We now consider the regularised version of the initial estimator $\varphi_N$ introduced at (2.8):

$$\widehat{\varphi}_N = \varphi_N 1_{\left\{L_N^2 \|\Psi_N^{-1}\|_{\mathrm{op}}^2 \leq \frac{c_{\eta,T} NT}{\log(NT)}\right\}} \tag{2.14}$$

An analogous methodology was suggested in [CG20] within the framework of classical stochastic differential equations. Nevertheless, the probabilistic analysis of $\widehat{\varphi}_N$ becomes notably more intricate due to the sophisticated structure of the model. Moreover, unlike the analysis in [CG20], the theoretical quantity $\|\Psi^{-1}\|_{\mathrm{op}}^2$ in Assumption 5 proves challenging to control due to the absence of information about $\mu_t$. We will discuss this condition in detail in Section 2.3.3.

### 2.3.2 Main Results

This section is devoted to the asymptotic analysis of the estimator $\widehat{\varphi}_N$. We start by introducing two random sets, which will be crucial for our proofs. We define

$$\Lambda_N := \left\{ L_N^2 \|\Psi_N^{-1}\|_{\mathrm{op}}^2 \leq \frac{c_{\eta,T} NT}{\log(NT)} \right\}, \qquad \Omega_N := \left\{ \left| \frac{\|f\|_N^2}{\|f\|_*^2} - 1 \right| \leq \frac{1}{2} \; \forall f \in S_N \right\}. \tag{2.15}$$

We recall that $\Lambda_N$ reflects the cut-off introduced in (2.14). On the other hand, on $\Omega_N$ the norms $\|\cdot\|_N$ and $\|\cdot\|_*$ are equivalent, i.e.

$$\frac{1}{2}\|f\|_*^2 \leq \|f\|_N^2 1_{\Omega_N} \leq \frac{3}{2}\|f\|_*^2.$$

Our first theoretical statement shows that, under Assumption 5, $\mathbb{P}(\Lambda_N)$ and $\mathbb{P}(\Omega_N)$ approach 1 as $N \to \infty$.

**Proposition 2.8.** *Suppose that Assumption 2 and Assumption 5 are satisfied.*

*(i) There exists a $n \in \mathbb{N}$ such that for all $N \geq n$:*

$$\mathbb{P}\left(\left\|\Psi^{-1/2}\Psi_N\Psi^{-1/2} - I_{D_N}\right\|_{\text{op}} > \frac{1}{2}\right) \leq D_N^{1/4} \exp\left(-\frac{c_{\eta,T}\eta NT}{16L_N^2\|\Psi^{-1}\|_{\text{op}}^2}\right),$$

*where $I_{D_N}$ denotes the identity matrix in $\mathbb{R}^{D_N \times D_N}$.*

*(ii) It holds that*

$$\mathbb{P}(\Lambda_N^c) \leq \mathbb{P}(\Omega_N^c) \leq (NT)^{-\eta+1}.$$

The concentration bound in Proposition 2.8(i) is key in understanding the asymptotic behaviour of the estimator $\widehat{\varphi}_N$. The main result of this section is the following theorem.

**Theorem 2.9.** *Suppose that Assumption 2 and Assumption 5 are satisfied. Then we obtain uniform bounds*

$$\sup_{\varphi} \mathbb{E}\left[\|\widehat{\varphi}_N - \varphi\|_N^2\right] \leq \sup_{\varphi} \mathbb{E}\left[\inf_{f \in S_N} \|f - \varphi\|_N^2\right] + \frac{CD_N}{NT}, \tag{2.16}$$

*as well as*

$$\begin{aligned}\sup_{\varphi} \mathbb{E}\left[\|\widehat{\varphi}_N - \varphi\|_*^2\right] &\leq (1 + o(1)) \sup_{\varphi} \inf_{f \in S_N} \|f - \varphi\|_*^2 \\ &\quad + CN^{-1/2}(1 + L_N) \sup_{\varphi} \|\varphi_*^N - \varphi\|_{**}^2 + \frac{CD_N}{NT},\end{aligned} \tag{2.17}$$

*where the supremum is taken over the class of functions $\Phi$ defined as*

$$\Phi := \left\{\varphi : \mathbb{R} \to \mathbb{R} : \ \|\varphi\|_\infty \leq K_1, \ K_2 \leq \|\varphi\|_{\text{Lip}} \leq K_3\right\}$$

*for fixed constants $K_1, K_2, K_3 > 0$. Here the norm $\|f\|_{**}$ is defined as*

$$\|f\|_{**}^2 := \frac{1}{T}\int_0^T \int_{\mathbb{R}} f^2 * \mu_t(x)\mu_t(x)\,\mathrm{d}x\mathrm{d}t.$$

We remark that $\|f\|_* \leq \|f\|_{**}$. However, it does not hold a priori that $\|\varphi_*^N - \varphi\|_{**} \to 0$ as $N \to \infty$. We will discuss this term in the next subsection.

### 2.3.3 Checking Assumption 4 and Bounding the Norm $\|\varphi_*^N - \varphi\|_{**}$

The primary challenge in implementing the theoretical findings from the preceding section lies in selecting an appropriate sequence of dimensions $D_N$, which ensures the fulfilment of Assumption 4. This requirement is essentially equivalent to determining an upper bound for the quantity $\|\Psi^{-1}\|_{\text{op}}$. In this context, we introduce an approach to establish an upper bound on the operator norm $\|\Psi^{-1}\|_{\text{op}}$, guided by a Gaussian-type condition on the densities $\mu_t$. We introduce the following condition on $(\mu_t)_{t \in [0,T]}$:

**Assumption 6.** We suppose that the densities $\mu_t$ are symmetric and there exist numbers $c_1, c_2 > 0$ and functions $g_1$, $g_2$ such that $g_2 \in L^2([0,T])$, $g_1 g_2^2 \in L^1([0,T])$, and

$$\mu_t(x) \geq g_1(t)\exp(-x^2/c_1) \qquad \text{and} \qquad \widehat{\mu}_t(x) \geq g_2(t)\exp(-x^2/c_2).$$

**Remark 2.10.** The lower bound for the densities $(\mu_t)_{t\in[0,T]}$ can be deduced via [QZ02, Theorem 1]. Indeed, under Assumption 3, it holds that

$$\begin{aligned}\mu_t(x) &\geq \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}\mu_0(y)\int_{\frac{|x-y|}{\sqrt{t}}}^{\infty} z\exp\left(-\frac{(z+K_\varphi\sqrt{t})^2}{2}\right)\mathrm{d}z\mathrm{d}y\\ &\geq \frac{\exp\left(-K_\varphi^2 t\right)}{2\pi\sqrt{t\zeta^2}}\int_{\mathbb{R}}\exp\left(-\frac{y^2}{2\zeta^2}\right)\exp\left(-\frac{(x-y)^2}{t}\right)\mathrm{d}y\\ &= \frac{\exp\left(-K_\varphi^2 t\right)}{\sqrt{2\pi(\zeta^2+t/2)}}\exp\left(-\frac{x^2}{2\zeta^2+t}\right).\end{aligned}$$

Obtaining lower bounds for the Fourier transforms $(\widehat{\mu}_t)_{t\in[0,T]}$ is a more delicate problem, see Chapter 3.

We consider the vector space $S_N$ generated by functions $(e_k)_{1\leq k\leq D_N}$ with

$$e_k(x) = (2A_N)^{-1/2}\exp(i\pi kx/A_N), \qquad x\in\mathbb{R},$$

which is an orthonormal system in $L^2([-A_N, A_N])$. For this choice of basis we obtain the identity

$$\Psi_{kl} = \langle e_k, e_l\rangle_* = \frac{1}{2A_N T}\int_0^T \widehat{\mu}_t(-k/A_N)\widehat{\mu}_t(l/A_N)\widehat{\mu}_t((l-k)/A_N)dt. \tag{2.18}$$

We deduce the following result.

**Proposition 2.11.** *Suppose that Assumption 6 holds. Assume that $D_N = a_D\log(N)$ and $A_N = a_A\sqrt{\log(N)}$ for some $a_D, a_A > 0$. Then it holds that*

$$\|\Psi^{-1}\|_{\mathrm{op}} \leq v_T N^{\frac{2a_D^2}{c_2 a_A^2}+\frac{a_A^2}{c_1}}(1+o(1)),$$

*where* $v_T := 2\sqrt{2\pi}T\left(\int_0^T g_1(t)g_2(t)^2dt\right)^{-1}$.

In the next step we will study the norm $\|\varphi_*^N-\varphi\|_{**}$. For this purpose, we assume that $\varphi$ has the representation

$$\varphi(x) = \sum_{k=1}^{\infty} c_k e_k(x)$$

with coefficients $c_k = c_k(N)$ satisfying the condition $\sum_{k=1}^\infty |c_k| < \infty$.

**Proposition 2.12.** *Suppose that Assumption 3 is satisfied. Then it holds that*

$$\|\varphi_*^N-\varphi\|_{**}^2 \lesssim \left(1 + A_N^{-1}D_N^4\|\Psi^{-1}\|_{\mathrm{op}}^2\exp\left(-\frac{3D_N^2\zeta^2}{2A_N^2}\right)\right)\left(\sum_{k=D_N+1}^{\infty}|c_k|\right)^2.$$

Now, we can combine the statements of Theorem 2.9, Proposition 2.11 and Proposition 2.12, to derive the convergence rate for $\|\widehat{\varphi}_N-\varphi\|_*$. For clarity, we focus on the ergodic scenario where $\mu_t = \mu_0$ and $\mu_0$ satisfies Assumption 3. In this context, the constants $c_1$ and $c_2$ from Assumption 6 are explicitly given by

$$c_1 = 2\zeta^2, \qquad c_2 = \frac{2}{\zeta^2}.$$

As $\|e_j\|_\infty^2 = A_N^{-1}$ and $D_N = a_D \log(N)$, $A_N = a_A\sqrt{\log(N)}$, Proposition 2.11 implies the following bound:

$$L_N^2 \|\Psi^{-1}\|_{\mathrm{op}}^2 \lesssim \frac{a_D^2}{a_A^2} \log(N) N^{\frac{2a_D^2\zeta^2}{a_A^2} + \frac{a_A^2}{\zeta^2}}.$$

Hence, for Assumption 5 to hold true, the constants $a_A$ and $a_D$ need to fulfil

$$\frac{2a_D^2\zeta^2}{a_A^2} + \frac{a_A^2}{\zeta^2} < 1.$$

On the other hand, exploring the proof of Theorem 2.6, the approximation error is obtained as

$$\|\varphi_*^N - \varphi\|_*^2 \lesssim N^{-\frac{a_D^2\zeta^2}{a_A^2}} + N^{-\frac{a_A^2}{2\zeta^2}}.$$

Consequently, if we choose $a_D^2\zeta^2/a_A^2 = a_A^2/(2\zeta^2) = (1-\epsilon)/4$ for a small $\epsilon > 0$, we finally conclude that

$$\mathbb{E}\left[\|\widehat{\varphi}_N - \varphi\|_*^2\right] \lesssim N^{-1/4+\epsilon} \tag{2.19}$$

provided the condition $(\sum_{k=D_N+1}^{\infty} |c_k|)^2 \le CN^{-z}$ with $z > 5/8$ holds. The substantial decrease in the convergence rate compared to Theorem 2.6 directly stems from the stringent constraint imposed by Assumption 5. Still this is the first result in the literature regarding the convergence rates of the linear-type estimates within the context of McKean-Vlasov SDEs.

## 2.4 Lower bounds

This section is dedicated to deriving minimax lower bounds. We start by establishing a lower bound for the previously examined estimation problem concerning the norm $\|\cdot\|_*$. To achieve this, we focus on a simplified scenario involving i.i.d. observations drawn from a McKean-Vlasov SDE:

$$\begin{cases} dX_t = (\varphi * \mu_t)(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \\ \mathrm{Law}(X_t) = \mu_t, \end{cases}$$

and denote by $\mathbb{P}_\varphi$ the associated probability measure. We assume that $X_0^\varphi = x \in \mathbb{R}$ for all $\varphi$. The ensuing theorem establishes a minimax lower bound with respect to the norm $\|\cdot\|_*$.

**Theorem 2.13.** *There exists a constant $c > 0$ such that, for every $N \in \mathbb{N}$,*

$$\inf_{\widehat{\varphi}_N} \sup_{\varphi} \mathbb{P}_\varphi^{\bigotimes N}\left(\|\widehat{\varphi}_N - \varphi\|_* > cN^{-1/2}\right) > 0, \tag{2.20}$$

*where $\mathbb{P}_\varphi^{\bigotimes N}$ is the $N$-fold product measure, the supremum is taken over all functions $\varphi$ satisfying Assumption 4 and $\|\varphi\|_\infty \le (32T)^{-1/2}$, and the infimum is taken over all estimators of $\varphi$ retrieved from $N$ observations of $(X_t)_{t\in[0,T]}$.*

*Proof.* Consider two functions $\varphi_0, \varphi_1 \in S$ and denote by $\mu_t^0$ and $\mu_t^1$ the corresponding marginal densities. The Kullback-Leibler divergence $\mathrm{KL}(\mathbb{P}_{\varphi_0}, \mathbb{P}_{\varphi_1})$ between the probability measures $\mathbb{P}_{\varphi_0}$ and $\mathbb{P}_{\varphi_1}$ can be explicitly computed as

$$\mathrm{KL}(\mathbb{P}_{\varphi_0}, \mathbb{P}_{\varphi_1}) = \frac{1}{2}\int_0^T \int_{\mathbb{R}} \left(\varphi_0 * \mu_t^0(x) - \varphi_1 * \mu_t^1(x)\right)^2 \mu_t^0(x)\,\mathrm{d}x\mathrm{d}t.$$

As a consequence we obtain the inequality

$$\mathrm{KL}(\mathbb{P}_{\varphi_0},\mathbb{P}_{\varphi_1}) \le T\|\varphi_0-\varphi_1\|_*^2 + \|\varphi_1\|_\infty^2 \int_0^T \left[\int_{\mathbb{R}} \left|\mu_t^0(x)-\mu_t^1(x)\right| \mathrm{d}x\right]^2 \mathrm{d}t,$$

where the $\|\cdot\|_*$-norm is computed with respect to $\mu_t^0$. Furthermore, using Theorem 1.1 of [BRS16], we derive

$$\begin{aligned}&\int_0^T \left[\int_{\mathbb{R}} \left|\mu_t^0(x)-\mu_t^1(x)\right| \mathrm{d}x\right]^2 \mathrm{d}t\\ &\le 4\int_0^T \left[\int_0^t \int (\varphi_0*\mu_s^0(x)-\varphi_1*\mu_s^1(x))^2\mu_s^0(x)\,\mathrm{d}x\mathrm{d}s\right]\mathrm{d}t\\ &\le 8T\,\mathrm{KL}(\mathbb{P}_{\varphi_0},\mathbb{P}_{\varphi_1}).\end{aligned}$$

Since $\|\varphi_1\|_\infty^2 \le (16T)^{-1}$ we finally deduce that

$$\mathrm{KL}(\mathbb{P}_{\varphi_0},\mathbb{P}_{\varphi_1}) \le 2T\|\varphi_0-\varphi_1\|_*^2.$$

Applying the two hypotheses method described in [Tsy08, Theorem 2.2], we obtain the statement of the theorem. □

We will now consider the $L^2(\mathbb{R})$-norm instead of $\|\cdot\|_*$ and compare the results to [BPP23]. We recall that logarithmic rates have been obtained in the $L^2(\mathbb{R})$-norm in [BPP23, Theorem 5.1] for a similar estimation problem given the observation scheme $(X_T^{i,N})_{1\le i\le N}$ with $N,T\to\infty$. It turns out that these rates cannot be improved to polynomial ones in their framework, and the same holds in our case (note however that polynomial rates may appear under a different set of conditions as demonstrated in Chapter 3). To see this, we again consider simplified setting of i.i.d. observations drawn from McKean-Vlasov SDE (2.3). Furthermore, we consider a similar model as presented in [BPP23]: Let $W:\mathbb{R}\to\mathbb{R}$ be an even and strictly convex interaction potential of the form

$$W(x) = p(x) + \beta(x), \qquad p(x) = \sum_{j=1}^J a_j x^{2j},$$

where $(a_j)_{j=1,\dots,J}$ are known constants and $\beta \in C_c^2(\mathbb{R})$. The potential $W$ determines the interaction function $\varphi$ via $\varphi = W'$ and we are interested in the estimation of the non-parametric part $\beta$ in the case of stationarity. The invariant law $\mu$ solves the elliptic nonlinear Fokker-Planck equation

$$\left[\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}(\varphi*\mu)\right]\mu = 0, \tag{2.21}$$

meaning that $\mu$ is given by the implicit equation

$$\mu(x) = c_\mu \exp\left(-W*\mu(x)\right), \tag{2.22}$$

where $c_\mu$ is a normalising constant such that $\mu$ is a probability density.

To introduce the appropriate class of functions, we consider $C_c^2(\mathbb{R})$-wavelets. More specifically, we let the mother wavelet be a symmetric function $\psi \in C_c^2(\mathbb{R})$ with $\mathrm{supp}(\psi) = [-1,-1/2] \cup [1/2,1]$, $\|\psi\|_{L^2(\mathbb{R})} = 1$ and, for $n,k\in\mathbb{Z}$, define

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k).$$

The set $(\psi_{n,k})_{n,k\in\mathbb{Z}}$ forms a basis of $L^2(\mathbb{R})$. Now, for $\alpha,\ K>0$, we consider the functional space

$$\mathscr{F}_K^\alpha = \left\{\beta' :\ \beta \in C_c^2(\mathbb{R}),\ \sum_{n\in\mathbb{Z}}\sum_{|k|>m} \left\langle \beta', \psi_{n,k}\right\rangle^2_{L^2(\mathbb{R})} \le K m^{-\alpha} \quad \forall m\in\mathbb{N}\right\}.$$

For the estimation of the nonparametric part $\beta'$ of $\varphi$ based on i.i.d. observations of (2.3) we obtain the following lower bound.

**Theorem 2.14.** *Assume that the coefficients $(a_j)_{1\le j\le J}$ and the function $\beta$ satisfy the conditions of [BPP23, Theorem 5.1]. Then there exists a constant $c>0$ such that, for every $N\in\mathbb{N}$,*

$$\inf_{\widehat{\beta}'_N}\sup_{\beta'\in\mathscr{F}_K^\alpha} \mathbb{P}_\varphi^{\bigotimes N}\left(\|\widehat{\beta}'_N-\beta'\|^2_{L^2(\mathbb{R})} > c\,(\log N)^{-\frac{\alpha}{2J}}\right) > 0, \tag{2.23}$$

*where $\mathbb{P}_\varphi^{\bigotimes N}$ is the $N$-fold product measure and the infimum is taken over all estimators of $\beta'$ retrieved from $N$ observations of $(X_t)_{t\in[0,T]}$.*

*Proof.* We will use the two hypotheses method described in [Tsy08, Theorem 2.2]. We will find two interaction forces $\varphi_0,\varphi_1\in\mathscr{F}_K^\alpha$ such that, for some constant $c>0$,

$$\|\varphi_0-\varphi_1\|^2_{L^2(\mathbb{R})} = c\,(\log N)^{-\frac{\alpha}{2J}} \quad \text{and} \quad \mathrm{KL}\left(\mathbb{P}_{\varphi_0}^{\bigotimes N},\mathbb{P}_{\varphi_1}^{\bigotimes N}\right)\lesssim 1.$$

The Radon-Nikodym density of $\mathrm{d}\mathbb{P}_{\varphi_1}/\mathrm{d}\mathbb{P}_{\varphi_0}$ can be computed using Girsanov's theorem: denoting the stationary marginals as $\mu_i$, $i=0,1$, we have for $X\in C([0,T];\mathbb{R})$

$$\begin{aligned}\log\left(\frac{\mathrm{d}\mathbb{P}_{\varphi_1}}{\mathrm{d}\mathbb{P}_{\varphi_0}}(X)\right) &= \int_0^T \varphi_1*\mu_1(X_t)-\varphi_0*\mu_0(X_t)\mathrm{d}X_t\\ &\quad -\frac{1}{2}\int_0^T \varphi_1*\mu_1(X_t)^2-\varphi_0*\mu_0(X_t)^2\mathrm{d}t,\end{aligned}$$

such that

$$\begin{aligned}\mathrm{KL}\left(\mathbb{P}_{\varphi_0},\mathbb{P}_{\varphi_1}\right) &= \int_{\mathbb{R}}\log\left(\frac{\mathrm{d}\mathbb{P}_{\varphi_1}}{\mathrm{d}\mathbb{P}_{\varphi_0}}(x)\right)\mu_1(x)\mathrm{d}x\\ &= \frac{T}{2}\int_{\mathbb{R}}\left(\varphi_1*\mu_1(x)-\varphi_0*\mu_0(x)\right)^2\mu_1(x)\mathrm{d}x.\end{aligned}$$

In a similar manner as in [BPP23, Theorem 5.1], we can show that

$$\mathrm{KL}\left(\mathbb{P}_{\varphi_1},\mathbb{P}_{\varphi_0}\right)\le\gamma^2\mathrm{KL}\left(\mathbb{P}_{\varphi_1},\mathbb{P}_{\varphi_0}\right)+\frac{T}{2}\int_{\mathbb{R}}(\beta'_0-\beta'_1)*\mu_0(x)^2\mu_1(x)\mathrm{d}x,$$

with $\gamma\in(0,1)$. So, it remains to show the estimate

$$\int_{\mathbb{R}}(\beta'_0-\beta'_1)*\mu_0(x)^2\mu_1(x)\mathrm{d}x\lesssim\frac{1}{N} \tag{2.24}$$

for a proper choice of $\varphi_0,\varphi_1$. Now we will choose the two hypotheses. Let $\rho>0$, $M\in\mathbb{N}$ be rescaling parameters to be determined later. We define

$$f=\rho\sum_{|k|=M}^{2M}\psi_{0,k}.$$

Note that our choice of basis elements implies that $\operatorname{supp} f = [-2M-1, -M-1/2] \cup [M+1/2, 2M+1]$ and $\|f\|^2_{L^2(\mathbb{R})} = \rho^2(M+1)$. We will consider the hypotheses $\varphi_0 \in \mathscr{F}^\alpha_K$ and $\varphi_1 = \varphi_0 + f$. In order to guarantee $\varphi_1 \in \mathscr{F}^\alpha_K$, we observe that

$$\rho^2 \sum_{n\in\mathbb{Z}} \sum_{|k_1|>m} \sum_{|k_2|=M}^{2M} \langle \psi_{n,k_1}, \psi_{0,k_2} \rangle_2^2 \begin{cases} \lesssim \|f\|_2^2 = \rho^2 M, & m \le 2M, \\ = 0, & m > 2M. \end{cases}$$

Hence, in order for $\varphi_1 \in \mathscr{F}^\alpha_K$ to hold, we set

$$\rho^2 = O(M^{-\alpha-1}).$$

This means we choose

$$\|\varphi_0 - \varphi_1\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2(\mathbb{R})} = O(M^{-\alpha}).$$

We will now verify that (2.24) holds. The main ingredient is the observation that, for $i = 0, 1$,

$$\mu_i(x) \lesssim \exp\left(-a_J x^{2J}\right), \tag{2.25}$$

which in particular implies $\|\mu_i\|_{L^2(\mathbb{R})} < \infty$. By symmetry of $\mu_i$ and $f$, it suffices to show

$$\int_0^\infty f * \mu_0(x)^2 \mu_1(x) \mathrm{d}x \lesssim \frac{1}{N}. \tag{2.26}$$

Let us split the integral at some $0 < k < M$ which we determine later. The tail behaviour is governed by (2.25): Using Cauchy-Schwarz, we have $\|f * \mu_0\|^2_\infty \le \|f\|^2_{L^2(\mathbb{R})} \|\mu_0\|^2_{L^2(\mathbb{R})} \lesssim \rho^2 M^2$, such that

$$\int_k^\infty f * \mu_0(x)^2 \mu_1(x) \mathrm{d}x \lesssim \rho^2 M \frac{\exp\left(-a_J k^{2J}\right)}{k^{2J-1}}.$$

For the remaining integral, we have

$$\int_0^k f * \mu_0(x)^2 \, \mathrm{d}\mu_1(x) \lesssim \sup_{x\in[0,k]} f * \mu_0(x)^2.$$

We deduce that

$$\begin{aligned} \sup_{x\in[0,k]} f * \mu_0(x)^2 \le \sup_{x\in[0,k]} \int_{-\infty}^{x-M} f(x-y)\mu_0(y)\mathrm{d}y \le \|f\|^2_{L^2(\mathbb{R})} \int_{-\infty}^{k-M} \mu_0(y)^2 \mathrm{d}y \\ \lesssim \rho^2 M \frac{\exp\left(-a_J (M-k)^{2J}\right)}{(M-k)^{2J-1}}. \end{aligned}$$

From this, we see that by choosing $k = M/2$, the entire integral is bounded by

$$\int_0^\infty f * \mu_0(x)^2 \mu_1(x) \mathrm{d}x \lesssim \exp\left(-a_J M^{2J}\right).$$

Choosing $M = c\,(\log N)^{\frac{1}{2J}}$ with some constant $c > 0$ gives (2.26), which in turn gives us the rate

$$\|\varphi_0 - \varphi_1\|^2_{L^2(\mathbb{R})} = C M^{-\alpha} = C\,(\log N)^{-\frac{\alpha}{2J}},$$

which implies the claim of the theorem. □

## 2.5 Proofs

This section is devoted to proofs of the main results. We will often transfer probabilistic results from i.i.d. observations of the McKean-Vlasov SDE displayed in (2.3) to observations of the original particle system at (2.1) via a change of measure device established in [DH22]. For this purpose we consider two $N$-dimensional processes given by the weak solutions of

$$\begin{aligned} d\overline{X}_t^i &= (\varphi * \mu_t)(\overline{X}_t^i) + \sigma dB_t^i \ , \quad 1 \le i \le N, \\ dX_t^i &= (\varphi * \mu_t^N)(X_t^i) + \sigma dB_t^i \ , \quad 1 \le i \le N \end{aligned}$$

having the same initial value, and denote by $\overline{\mathbb{P}}^N$ and $\mathbb{P}^N$, respectively, the associated probability measures.

### 2.5.1 Proof of Theorem 2.3

Before we begin the proof, we briefly recall a result related to $U$-statistics.

#### Maximal Inequalities for $U$-statistics

Let $\mathcal{G}$ be a class of real-valued functions on $\mathbb{R}^K$ and $(U_N(g) : g \in \mathcal{G})$ be a $U$-process of the form

$$U_N(g) = \frac{1}{(N)_K} \sum_{i_1,\ldots,i_K=1}^{N} g(X_{i_1}, \ldots, X_{i_K})$$

where $X_1, \ldots, X_N$ is a sequence of i.i.d. random variables with common distribution $\mathbb{P}$ and $(N)_K = N(N-1) \cdot \ldots \cdot (N-K+1)$. If for each $g \in \mathcal{G}$,

$$\mathbb{E}_{X\sim\mathbb{P}}[g(x_1, \ldots, x_{i-1}, X, x_{i+1}, \ldots, x_K)] = 0, \quad i = 1, \ldots, K,$$

for arbitrary $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_K \in \mathbb{R}$, then the process $(U_N(g), g \in \mathcal{G})$ is called degenerate.

**Theorem 2.15.** *Suppose that $\mathcal{G}$ is a class of uniformly bounded functions with $\sup_{g\in\mathcal{G}} \|g\|_\infty \le G$ and the $U$-process $(U_N(g), g \in \mathcal{G})$ is degenerate. Then it holds*

$$\left\{\mathbb{E} \sup_{g\in\mathcal{G}} \left|N^{K/2} U_N(g)\right|^q\right\}^{1/q} \lesssim G p^K \, \mathsf{DI}(\mathcal{G}, \|\cdot\|_\infty, \psi_{p/K})$$

*for any $p \ge q$.*

*Proof.* We can apply the maximal inequality from [She94, Section 5] and note that for uniformly bounded classes $\mathcal{G}$

$$d_{U_N} := \left[U_N\left(|f-g|^2\right)\right]^{1/2} \le \|f-g\|_\infty$$

as well as $U_N\left(|g|^2\right) \le G^2$ with probability 1 for all $f, g \in \mathcal{G}$. □

Let us now come to the proof of Theorem 2.3. We consider the setting of i.i.d. observations drawn from the mean field limit (2.3), that is we work under $\overline{\mathbb{P}}^N$. We recall the inequality

$$\|\varphi_N - \varphi\|_*^2 \le \|\varphi_*^N - \varphi\|_*^2 + 2\left(\sup_{f\in S_N} \left|\|f-\varphi\|_N^2 - \|f-\varphi\|_*^2\right| + \sup_{f\in S_N} |\nu_N(f)|\right), \tag{2.27}$$

where $\varphi_*^N$ has been introduced in (2.8). Next, we set $\nu_N(f) =: 2M_N(f)/\sqrt{NT}$ where for all $f \in S$, $M_N(f)$ is a martingale with quadratic variation

$$\frac{\sigma^2}{NT}\sum_{i=1}^{N}\int_0^T [(f*\mu_t^N)(X_t^{i,N})]^2\,\mathrm{d}t \le \sigma^2\|f\|_\infty^2.$$

Hence, we deduce by the Bernstein inequality for continuous martingales,

$$\overline{\mathbb{P}}^N(|M_N(f)-M_N(g)|\ge a)\le 2\exp\big(-a^2/(\sigma^2\|f-g\|_\infty^2)\big)$$

for any $a>0$ and $f,g\in S_N$. Since for all $q\ge 1$, the $L^q(\Omega)$-norm is dominated by $\|\cdot\|_{\psi_{e,2}}$, an application of the general maximal inequality [Kos08, Theorem 8.4] implies

$$\Big\|\sup_{f\in S_N}|M_N(f)-M_N(f_0)|\Big\|_{L^q(\Omega)} \lesssim \mathsf{DI}(S_N,\|\cdot\|_\infty,\psi_{e,2}) \tag{2.28}$$

for any $f_0\in S_N$. Consider now the difference $\|g\|_N^2-\|g\|_*^2$ for a function $g\in S_N(2K_\varphi)$. We obtain the decomposition

$$\|g\|_N^2-\|g\|_*^2 = U_{1,N}(g)+U_{2,N}(g)+U_{3,N}(g)+O(N^{-1}) \tag{2.29}$$

where $\Delta_t(x,y) := g(x-y)-(g*\mu_t)(x)$, and

$$U_{1,N}(g) := \frac{2}{T}\int_0^T\int_{\mathbb{R}}\left(\frac{1}{N}\sum_{j=1}^{N}\Delta_t(x,X_t^{j,N})\right)(g*\mu_t)(x)\mu_t(x)\,\mathrm{d}x\mathrm{d}t$$
$$+\left(\frac{1}{NT}\sum_{i=1}^{N}\int_0^T(g*\mu_t)(X_t^{i,N})^2\,\mathrm{d}t-\frac{1}{T}\int_0^T\int_{\mathbb{R}}(g*\mu_t)(x)^2\mu_t(x)\,\mathrm{d}x\mathrm{d}t\right),$$

$$U_{2,N}(g) := \frac{1}{N(N-1)(N-2)}\sum_{i=1}^{N}\sum_{j,k\ne i,j\ne k}\frac{1}{T}\int_0^T\int_{\mathbb{R}}\Delta_t(x,X_t^{j,N})\Delta_t(x,X_t^{k,N})\mu_t(x)\,\mathrm{d}x\mathrm{d}t$$
$$+\frac{2}{NT}\sum_{i=1}^{N}\int_0^T\frac{1}{N-1}\sum_{j\ne i}\Delta_t(X_t^{i,N},X_t^{j,N})(g*\mu_t)(X_t^{i,N})\,\mathrm{d}t$$
$$-\frac{2}{NT}\sum_{i=1}^{N}\int_0^T\frac{1}{N-1}\sum_{j\ne i}\int_{\mathbb{R}}\Delta_t(x,X_t^{j,N})(g*\mu_t)(x)\mu_t(x)\,\mathrm{d}x\mathrm{d}t,$$

$$U_{3,N}(g) := \frac{1}{N(N-1)(N-2)}\sum_{i=1}^{N}\sum_{j,k\ne i,j\ne k}\frac{1}{T}\int_0^T\Delta_t(X_t^{i,N},X_t^{j,N})\Delta_t(X_t^{i,N},X_t^{k,N})\,\mathrm{d}t$$
$$-\frac{1}{N(N-1)(N-2)}\sum_{i=1}^{N}\sum_{j,k\ne i,j\ne k}\frac{1}{T}\int_0^T\int_{\mathbb{R}}\Delta_t(x,X_t^{j,N})\Delta_t(x,X_t^{k,N})\mu_t(x)\,\mathrm{d}x\mathrm{d}t.$$

The key observation is that each $U_{r,N}$ is a degenerate $U$-statistic of order $r$ for $r=1,2,3$, while $U_{1,N}$ is the leading term. First, turn to $U_{1,N}$ and note that

$$U_{1,N}(g)=\frac{1}{N}\sum_{j=1}^{N}\big(Z_j(g)-\mathbb{E}_{\overline{\mathbb{P}}^N}Z_j(g)\big)$$

with

$$Z_j(g) = \frac{1}{T}\int_0^T \left(2\int_{\mathbb{R}} \Delta_t(x, X_t^{j,N})(g*\mu_t)(x)\mu_t(x)\,\mathrm{d}x\right) + (g*\mu_t)(X_t^{i,N})^2\,\mathrm{d}t.$$

It holds that

$$|Z_j(g)| \le 8K_\varphi^2, \quad |Z_j(g) - Z_j(g')| \le 4K_\varphi \|g - g'\|_\infty,$$

with probability 1 for $g, g' \in S_N(2K_\varphi)$. Hence

$$\|Z_j(g) - Z_j(g')\|_{\psi_{e,2}} \lesssim K_\varphi \|g - g'\|_\infty.$$

This implies that the process $\widetilde{U}_{1,N} = \sqrt{N}U_{1,N}$ has sub-Gaussian increments and

$$\|\widetilde{U}_{1,N}(g) - \widetilde{U}_{1,N}(g')\|_{\psi_{e,2}} \lesssim K_\varphi \|g - g'\|_\infty.$$

Fix some $g_0 \in S_N(2K_\varphi)$. By the triangle inequality,

$$\sup_{g\in S_N(2K_\varphi)} |\widetilde{U}_{1,N}(g)| \le \sup_{g,g'\in S_N(2K_\varphi)} |\widetilde{U}_{1,N}(g) - \widetilde{U}_{1,N}(g')| + |\widetilde{U}_{1,N}(g_0)|.$$

By the Dudley integral inequality, see [Ver18, Theorem 8.1.6], for any $\delta \in (0,1)$, we conclude that

$$\sup_{g,g'\in S_N(2K_\varphi)} |\widetilde{U}_{1,N}(g) - \widetilde{U}_{1,N}(g')| \lesssim K_\varphi\left[\mathsf{DI}(S_N, \|\cdot\|_\infty, \psi_{e,2}) + \sqrt{\log(2/\delta)}\right]$$

holds with probability at least $1-\delta$. Applying Hoeffding's inequality, see e.g. [Ver18, Theorem 2.6.2.], we have for any $\delta \in (0,1)$,

$$|\widetilde{U}_{1,N}(g_0)| \lesssim K_\varphi \sqrt{\log(1/\delta)},$$

with probability at least $1-\delta$. In other words, for any $u > 0$,

$$\overline{\mathbb{P}}^N\left(\sup_{g\in S_N(2K_\varphi)} |\widetilde{U}_{1,N}(g)| > u\right) \le \exp\left(-\frac{(u - K_\varphi \mathsf{DI}(S_N(2K_\varphi), \|\cdot\|_\infty, \psi_{e,2}))^2}{2K_\varphi^2}\right).$$

Putting things together we conclude that

$$\mathbb{E}_{\overline{\mathbb{P}}^N}\Big[\sup_{g\in S_N(2K_\varphi)} |U_{1,N}(g)|^q\Big]^{1/q} \lesssim \frac{qK_\varphi\,\mathsf{DI}(S_N(2K_\varphi), \|\cdot\|_\infty, \psi_{e,2})}{\sqrt{N}}. \tag{2.30}$$

Furthermore, due to Theorem 2.15,

$$\mathbb{E}_{\overline{\mathbb{P}}^N}\Big[\sup_{g\in S_N(2K_\varphi)} \big|NU_{2,N}(g)\big|^q\Big]^{1/q} \lesssim p^2 K_\varphi\,\mathsf{DI}(S_N(2K_\varphi), \|\cdot\|_\infty, \psi_{p/2}) \tag{2.31}$$

and

$$\mathbb{E}_{\overline{\mathbb{P}}^N}\Big[\sup_{g\in S_N(2K_\varphi)} \big|N^{3/2}U_{3,N}(g)\big|^q\Big]^{1/q} \lesssim p^3 K_\varphi\,\mathsf{DI}(S_N(2K_\varphi), \|\cdot\|_\infty, \psi_{p/3}) \tag{2.32}$$

for any $p > 2$ and $q \le p$. This implies the statement of Theorem 2.3 in the setting of i.i.d. observations drawn from the McKean-Vlasov SDE (2.3).

Now, we will transfer the result from i.i.d. observations to observations of the original particle system at (2.1) via a change of measure device established in [DH22]. Recalling the definition of probability measures $\overline{\mathbb{P}}^N$ and $\mathbb{P}^N$, we deduce the identity

$$\frac{\mathrm{d}\mathbb{P}^N}{\mathrm{d}\overline{\mathbb{P}}^N} = \mathscr{E}_T(\overline{M}^N)$$

where $\mathscr{E}_t(\overline{M}^N) = \exp\big(\overline{M}_t^N - \frac{1}{2}\big\langle \overline{M}^N \big\rangle_t\big)$ and

$$\overline{M}_t = \sum_{i=1}^{N} \int_0^t \sigma^{-1} \left[\varphi * (\mu_t^N - \mu_t)(\overline{X}_t^i)\right] \mathrm{d}W_t^i$$

is a $\overline{\mathbb{P}}^N$-local martingale. According to [DH22, Proposition 19], we obtain the bound

$$\sup_{N\geq 1} \sup_{t\in[0,T-\delta]} \mathbb{E}_{\overline{\mathbb{P}}^N}\left[\exp\left(\tau\left(\left\langle \overline{M}^N\right\rangle_{t+\delta} - \left\langle \overline{M}^N\right\rangle_t\right)\right)\right] \leq C_{\delta,\tau} \tag{2.33}$$

for every $\tau > 0$, $0 \leq \delta \leq \delta_0$ and some constant $C_{\delta,\tau} > 0$. Let $\xi$ be any $\mathscr{F}_T$-measurable random variable. Then by the Hölder inequality

$$\mathbb{E}_{\mathbb{P}^N}[|\xi|^r] = \mathbb{E}_{\overline{\mathbb{P}}^N}\left[|\xi|^r \mathscr{E}_T(\overline{M}^N)\right] \leq \left\{\mathbb{E}_{\overline{\mathbb{P}}^N}[|\xi|^{2r}]\right\}^{1/2} \left\{\mathbb{E}_{\overline{\mathbb{P}}^N}[\mathscr{E}_T^2(\overline{M}^N)]\right\}^{1/2}. \tag{2.34}$$

Next, fix a grid $0 = t_0 < t_1 < \ldots < t_K = T$ with $|t_{k+1} - t_k| \leq \delta$ and decompose

$$\mathbb{E}_{\overline{\mathbb{P}}^N}\left[\mathscr{E}_T^2(\overline{M}^N)\right] = \mathbb{E}_{\overline{\mathbb{P}}^N} \prod_{k=1}^{K} \exp\Big(2(\overline{M}_{t_k}^N - \overline{M}_{t_{k-1}}^N) + \left\langle \overline{M}^N \right\rangle_{t_{k-1}} - \left\langle \overline{M}^N \right\rangle_{t_k}\Big) \tag{2.35}$$

$$= \mathbb{E}_{\overline{\mathbb{P}}^N} \prod_{k=1}^{K} \mathscr{E}_{t_k}(\overline{M}_{t_k}^N - \overline{M}_{t_{k-1}}^N) \exp\Big(\left\langle \overline{M}_{t_k}^N - \overline{M}_{t_{k-1}}^N \right\rangle \tag{2.36}$$

$$+ \left\langle \overline{M}^N \right\rangle_{t_{k-1}} - \left\langle \overline{M}^N \right\rangle_{t_k}\Big)$$

By using the martingale property of $\mathscr{E}_{t_k}$, repeatedly applying the Hölder inequality and using (2.33), we deduce the inequality

$$\mathbb{E}_{\overline{\mathbb{P}}^N}[\mathscr{E}_T^2(\overline{M}^N)] \leq C_{\delta,T}.$$

As a consequence the bounds (2.28)-(2.32) remain valid under the probability measure $\mathbb{P}^N$. This completes the proof.

### 2.5.2 Proof of Theorem 2.6

Note that $\alpha_{n_N - 1} < A_N \leq \alpha_{n_N}$ and hence $A_N/\alpha_{n_N} \to 1$ as $N \to \infty$. Under Assumption 4 considered for the sequence $\alpha_{n_N}$, we obtain the inequality

$$\inf_{g\in S_N} \frac{1}{T}\int_0^T \int_{\mathbb{R}} ((\varphi * \mu_t)(x) - (g * \mu_t)(x))^2 \mu_t(x)\,\mathrm{d}x\mathrm{d}t$$

$$\leq \frac{1}{\alpha_{n_N} T}\int_0^T \|\mu_t\|_{[-\alpha_{n_N},\alpha_{n_N}]} \int_{-\alpha_{n_N}}^{\alpha_{n_N}} \left|\sum_{k=D_N+1}^{\infty} c_k \exp(i\pi kx/\alpha_{n_N})\widehat{\mu}_t(-k/\alpha_{n_N})\right|^2 \mathrm{d}x\mathrm{d}t + R(\alpha_{n_N})$$

$$\leq \frac{1}{\alpha_{n_N} T}\int_0^T \|\mu_t\|_{[-\alpha_{n_N},\alpha_{n_N}]} \sum_{k=D_N+1}^{\infty} \left(|c_k||\widehat{\mu}_t(-k/\alpha_{n_N})|\right)^2 \mathrm{d}t + R(\alpha_{n_N}),$$

where

$$R(A) \leq \frac{4K_\varphi^2}{T} \int_0^T \int_{|x|>A} \mu_t(x)\,\mathrm{d}x\mathrm{d}t.$$

In the next step we would like to obtain estimates for the marginal densities $\mu_t$. For this purpose we consider the SDE

$$d\widetilde{X}_t = (\varphi * \mu_t)(\widetilde{X}_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \quad \widetilde{X}_t \sim \mu_0 \tag{2.37}$$

which is obtained by freezing the densities $\mu_t$ in the McKean-Vlasov equation (2.3). Note that, under Assumption 2, (2.37) is the standard SDE with bounded and Lipschitz continuous drift function $\varphi * \mu_t$. By applying [TT20, Theorem 3.1] to (2.3) and using our Assumption 3, we deduce that

$$\begin{aligned}
\mu_t(x) &\leq \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \mu_0(y) \int_{\frac{|x-y|}{\sqrt{t}}}^{\infty} z \exp\left(-\frac{(z-K_\varphi\sqrt{t})^2}{2}\right) \mathrm{d}z\mathrm{d}y \\
&\leq \frac{\exp\left(K_\varphi^2 T/2\right)}{2\pi\sqrt{t\zeta^2}} \int_{\mathbb{R}} \exp\left(-\frac{y^2}{2\zeta^2}\right) \exp\left(-\frac{(x-y)^2}{2t}\right) \mathrm{d}y \\
&= \frac{\exp\left(K_\varphi^2 T/2\right)}{\sqrt{2\pi(\zeta^2+t)}} \exp\left(-\frac{x^2}{2(\zeta^2+t)}\right).
\end{aligned}$$

Therefore, we get

$$R(A) \leq 8K_\varphi^2 \exp\left(K_\varphi^2 T/2\right) \exp\left(-\frac{A^2}{2(\zeta^2+T)}\right).$$

Next, note that (2.3) can be written as

$$X_t = X_0 + \int_0^t (\varphi * \mu_s)(X_s)\,\mathrm{d}s + \sigma B_t$$

meaning that $\widehat{\mu}_t(u) = \widehat{\mu}_0(u)\psi_t(u)$ where $\psi_t$ is the characteristic function of the random variable $\int_0^t (\varphi * \mu_s)(X_s)\,\mathrm{d}s + \sigma B_t$. Thus, under Assumption 3, we conclude that

$$|\widehat{\mu}_t(u)| \leq \exp(-u^2\zeta^2/2). \tag{2.38}$$

Furthermore, by [TT20, Remark 3.3], $\|\mu_t\|_\infty \leq 2\|\mu_0\|_\infty + K_\varphi$. As a consequence we derive

$$\begin{aligned}
\inf_{g \in S_N} \|g-\varphi\|_*^2 \lesssim \frac{K_\varphi}{A_N} \sum_{k=D_N+1}^{\infty} |c_k|^2 \exp\left(-\frac{k^2\zeta^2}{A_N^2}\right) \\
+ \exp\left(K_\varphi^2 T/2\right) \exp\left(-\frac{A_N^2}{2(\zeta^2+T)}\right)
\end{aligned}$$

Hence under our choice of $A_N$ and $D_N$ we obtain that

$$\inf_{g \in S_N} \|g-\varphi\|_*^2 \lesssim \frac{\exp(K_\varphi^2 T/2)}{N}.$$

This completes the proof of Theorem 2.6.

### 2.5.3 Proof of Proposition 2.8

Before we begin the proof of Proposition 2.8, we present some preliminary results.

#### Some Properties of $\Psi$

We begin by pointing out that $\Psi$ is invertible. Indeed, this immediately follows from the representation $\Psi = (\langle e_i, e_j \rangle_*)_{1\le i,j\le D_N}$, since $(e_j)_{1\le j\le D_N}$ is an orthonormal system. The invertibility of the matrix $\Psi$ implies that we can transform $(e_j)_{1\le j\le D_N}$ into an orthonormal system with respect to $\langle\cdot,\cdot\rangle_*$.

**Lemma 2.16.** *The collection $\bar{e} = (\bar{e}_1, \dots, \bar{e}_{D_N})^\top$ given by*

$$\bar{e} = \Psi^{-1/2} e, \tag{2.39}$$

*where $e = (e_1, \dots, e_{D_N})^\top$, is an orthonormal system with respect to $\langle\cdot,\cdot\rangle_*$.*

*Proof.* Denoting the Kronecker delta as $\delta_{ij}$, we have

$$\langle \bar{e}_i, \bar{e}_j \rangle_* = \langle (\Psi^{-1/2}e)_i, (\Psi^{-1/2}e)_j \rangle_* = \left(\Psi^{-1/2}\Psi\Psi^{-1/2}\right)_{ij} = \delta_{ij}. \tag{2.40}$$

□

#### Matrix Concentration Inequality for Independent Observations

First of all, we recall a result from [PMT16], which provides a concentration inequality for operator norms of self-adjoint matrices.

**Theorem 2.17** ([PMT16], Corollary 6.1)**.** *Let $(X_i)_{1\le i\le N}$ be a sequence of independent random variables taking values in a Polish space $\mathcal{X}$, and let $H$ be a function from $\mathcal{X}^N$ into the space of $m\times m$ real self-adjoint matrices $\mathbb{H}^m$. Assume that there exists a sequence $(A_i)_{1\le i\le N} \subset \mathbb{H}^m$ that satisfies*

$$A_i^2 \succcurlyeq \left(H(x_1,\dots,x_{l-1},x_l,x_{l+1}\dots,x_N) - H(x_1,\dots,x_{l-1},x_l',x_{l+1}\dots,x_N)\right)^2$$

*for each index $1 \le i \le N$, where $x_l$ and $x_l'$ range over all possible values of $X_l$. Define $\nu^2 = \left\|\sum_{i=1}^N A_i^2\right\|_{\mathrm{op}}$. Then it holds that*

$$\mathbb{P}\left(\|H(X_1,\dots,X_N) - \mathbb{E}H(X_1,\dots,X_N)\|_{\mathrm{op}} > t\right) \le m \exp\left(-\frac{t}{2\nu^2}\right).$$

We begin by considering the random matrix $\overline{\Psi}_N$ which has been constructed from i.i.d. observations $\overline{X}^{i,N}$, $i = 1,\dots,N$, drawn from the mean field limit (2.3) instead of mutually dependent particles (thus, we work under $\overline{\mathbb{P}}^N$). That is,

$$(\overline{\Psi}_N)_{jk} = \frac{1}{NT}\sum_{i=1}^N \int_0^T e_j * \mu_t^N(\overline{X}_t^i) e_k * \mu_t^N(\overline{X}_t^i) \mathrm{d}t$$

for $1 \le j,k \le D_N$. As a slight abuse of notation, we denote the corresponding bilinear form as $\langle\cdot,\cdot\rangle_N$. In what follows, we prove that Proposition 2.8 holds for $\overline{\Psi}_N$ and then transition to the particle system in the next section.

*Proof of Proposition 2.8 in the independent case.* (i) Here we would like to prove the inequality

$$\overline{\mathbb{P}}^N\left(\left\|\Psi^{-1/2}\overline{\Psi}_N\Psi^{-1/2}-\Psi^{-1/2}\mathbb{E}\overline{\Psi}_N\Psi^{-1/2}\right\|_{\text{op}}\geq\frac{1}{2}\right)\leq D_N\exp\left(-\frac{c_{\eta,T}\eta NT}{4L_N^2\left\|\Psi^{-1}\right\|_{\text{op}}^2}\right). \tag{2.41}$$

via Theorem 2.17. Recall that by Lemma 2.16

$$\left(\Psi^{-1/2}\overline{\Psi}_N\Psi^{-1/2}\right)_{jk}=\left\langle\bar{e}_j,\bar{e}_k\right\rangle_N,$$

where $\bar{e}=\Psi^{-1/2}e$. We consider the space $\mathscr{X}=C([0,T],\mathbb{R})$ and the map $H$ given by

$$\begin{aligned}
&H\colon (C([0,T],\mathbb{R}))^N\to\mathbb{H}^{D_N},\\
&(x_1,\dots,x_N)\mapsto\left(\frac{1}{NT}\sum_{i=1}^N\int_0^T\bar{e}_j*\delta_{x(t)}(x_i(t))\bar{e}_k*\delta_{x(t)}(x_i(t))\mathrm{d}t\right)_{1\leq j,k\leq D_N}
\end{aligned}$$

where $\delta_{x(t)}=\frac{1}{N}\sum_{j=1}^N\delta_{x_j(t)}$. We have that

$$\begin{aligned}
&\Psi^{-1/2}\overline{\Psi}_N\Psi^{-1/2}-\Psi^{-1/2}\mathbb{E}_{\overline{\mathbb{P}}^N}\left[\overline{\Psi}_N\right]\Psi^{-1/2}\\
&=H\left(\overline{X}^1,\dots,\overline{X}^N\right)-\mathbb{E}_{\overline{\mathbb{P}}^N}\left[H\left(\overline{X}^1,\dots,\overline{X}^N\right)\right].
\end{aligned}$$

For $x=(x_1,\dots,x_N)\in(C([0,T],\mathbb{R}))^N$ we denote by $\tilde{x}$ the same vector where the $l$-th entry is replaced by $x_l{}'$. We need to study $\Delta H_{jk}:=(H(x)-H(\tilde{x}))_{jk}$, which is given by

$$\begin{aligned}
\Delta H_{jk}=\frac{1}{TN^3}\sum_{i,r_1,r_2=1}^N\int_0^T&\bar{e}_j(x_i(t)-x_{r_1}(t))\bar{e}_k(x_i(t)-x_{r_2}(t))\\
&-\bar{e}_j(\tilde{x}_i(t)-\tilde{x}_{r_1}(t))\bar{e}_k(\tilde{x}_i(t)-\tilde{x}_{r_2}(t))dt.
\end{aligned}$$

If either none or all of the indices are equal to $l$, the difference vanishes. Therefore, there are at most $3N(N-1)$ non-zero terms. Consequently, for any $1\leq l\leq N$, we can bound the difference as

$$|\Delta H_{jk}|\leq\frac{6}{N}\left\|\bar{e}_j\right\|_\infty\left\|\bar{e}_k\right\|_\infty.$$

Moving on to the spectral norm of $\Delta H$, we have for any $y\in\mathbb{R}^{D_N}$:

$$\begin{aligned}
y^\top\Delta H^2y&=\sum_{j,k=1}^{D_N}y_jy_k\sum_{l=1}^{D_N}\Delta H_{jl}\Delta H_{kl}\leq\frac{36}{N^2}\sum_{j,k=1}^{D_N}|y_jy_k|\sum_{l=1}^{D_N}\left\|\bar{e}_j\right\|_\infty\left\|\bar{e}_k\right\|_\infty\left\|\bar{e}_l\right\|_\infty^2\\
&\leq\frac{36}{N^2}\left(\sum_{j=1}^{D_N}\left\|\bar{e}_j\right\|_\infty^2\right)^2\|y\|^2\leq\frac{36}{N^2}L_N^2\left\|\Psi^{-1}\right\|_{\text{op}}^2\|y\|^2.
\end{aligned}$$

This means that

$$A_l^2:=\frac{36}{N^2}L_N^2\left\|\Psi^{-1}\right\|_{\text{op}}^2I_{D_N}$$

satisfies the condition $A_l^2\succcurlyeq\Delta H^2$ for all $1\leq l\leq N$. We also define $\nu^2:=\frac{36}{N}L_N^2\left\|\Psi^{-1}\right\|_{\text{op}}^2$. Now we apply the concentration inequality of Theorem 2.17 and deduce the statement (2.41), which completes this part of the proof.

(ii) To finish the proof of Proposition 2.8(i) in the independent case we observe that

$$\left\|\Psi^{-1/2}\overline{\Psi}_N\Psi^{-1/2} - I_{D_N}\right\|_{\mathrm{op}} \le \left\|\Psi^{-1/2}\overline{\Psi}_N\Psi^{-1/2} - \Psi^{-1/2}\mathbb{E}_{\overline{\mathbb{P}}^N}\left[\overline{\Psi}_N\right]\Psi^{-1/2}\right\|_{\mathrm{op}} + \left\|\Psi^{-1/2}\mathbb{E}_{\overline{\mathbb{P}}^N}\left[\overline{\Psi}_N\right]\Psi^{-1/2} - I_{D_N}\right\|_{\mathrm{op}}.$$

Thus we only need to show the convergence

$$\left\|\Psi^{-1/2}\mathbb{E}_{\overline{\mathbb{P}}^N}\left[\overline{\Psi}_N\right]\Psi^{-1/2} - I_{D_N}\right\|_{\mathrm{op}} \to 0 \qquad \text{as } N \to \infty. \tag{2.42}$$

Since the particles are exchangeable and independent, we have

$$\begin{aligned}
\mathbb{E}_{\overline{\mathbb{P}}^N}\left[(\overline{\Psi}_N)_{jk}\right] &= \frac{1}{TN^3}\sum_{i,r_1,r_2=1}^{N}\int_0^T \mathbb{E}_{\overline{\mathbb{P}}^N}\left[e_j(\overline{X}_t^i - \overline{X}_t^{r_1})e_k(\overline{X}_t^i - \overline{X}_t^{r_2})\right]dt \\
&= \frac{1}{TN^2}\int_0^T \Big((N-1)^2\mathbb{E}_{\overline{\mathbb{P}}^N}\left[e_j * \mu_t(\overline{X}_t^1)e_k * \mu_t(\overline{X}_t^1)\right] + e_j(0)e_k(0) \\
&\quad + (N-1)\Big(\mathbb{E}_{\overline{\mathbb{P}}^N}\left[e_j * \mu_t(\overline{X}_t^1)\right]e_k(0) + \mathbb{E}_{\overline{\mathbb{P}}^N}\left[e_k * \mu_t(\overline{X}_t^1)\right]e_j(0)\Big)\Big)dt \\
&= \frac{(N-1)^2}{N^2}\Psi_{jk} + \frac{e_j(0)e_k(0)}{N^2} \\
&\quad + \frac{N-1}{TN^2}\int_0^T \Big(\mathbb{E}_{\overline{\mathbb{P}}^N}\left[e_j * \mu_t(\overline{X}_t^1)\right]e_k(0) + \mathbb{E}_{\overline{\mathbb{P}}^N}\left[e_k * \mu_t(\overline{X}_t^1)\right]e_j(0)\Big)dt.
\end{aligned}$$

Hence, we deduce that

$$\left|\mathbb{E}_{\overline{\mathbb{P}}^N}\left[(\overline{\Psi}_N)_{jk}\right] - \Psi_{jk}\right| \lesssim \frac{1}{N}\left((I_{D_N})_{jk} + \|e_j\|_\infty \|e_k\|_\infty\right).$$

Now, the spectral norm can be bounded through Assumption 5:

$$\begin{aligned}
\left\|I_{D_N} - \Psi^{-1/2}\mathbb{E}\overline{\Psi}_N\Psi^{-1/2}\right\|_{\mathrm{op}} &\lesssim \frac{1}{N}\left(1 + \left\|\Psi^{-1}\right\|_{\mathrm{op}}\left(\sum_{j,k=1}^{D_N}\|e_j\|_\infty^2\|e_k\|_\infty^2\right)^{1/2}\right) \\
&\lesssim \frac{L_N\left\|\Psi^{-1}\right\|_{\mathrm{op}}}{N} \\
&\lesssim \frac{1}{\sqrt{NT}\log(NT)}.
\end{aligned} \tag{2.43}$$

This gives the claim (2.42) and thus the validity of Proposition 2.8(i) in the independent case.
(iii) On $\Lambda_N^c$, the following inequalities hold:

$$\begin{aligned}
c_{\eta,T}\frac{NT}{\log(NT)} &\le L_N^2\left(\left\|\Psi^{-1}\right\|_{\mathrm{op}} + \left\|\Psi^{-1} - \overline{\Psi}_N^{-1}\right\|_{\mathrm{op}}\right)^2 \\
&\le 2L_N^2\left\|\Psi^{-1}\right\|_{\mathrm{op}}^2 + 2L_N^2\left\|\Psi^{-1} - \overline{\Psi}_N^{-1}\right\|_{\mathrm{op}}^2.
\end{aligned}$$

So, if Assumption 5 holds, we deduce that

$$\left\|\Psi^{-1} - \overline{\Psi}_N^{-1}\right\|_{\mathrm{op}}^2 \ge \frac{c_{\eta,T}}{4L_N^2}\frac{NT}{\log(NT)} \ge \left\|\Psi^{-1}\right\|_{\mathrm{op}}^2,$$

and [CG19, Proposition 4 (ii)] yields the claim. □

### From Independent Observations to the Particle System

To transfer the results of Proposition 2.8 from the independent case to the particle system introduced in (2.1), we use an elegant transformation of measure argument investigated in [DH22]. We recall the probability measures $\mathbb{P}^N$ and $\overline{\mathbb{P}}^N$ that have been introduced in Section 2.5.1. According to [DH22, Theorem 18, eq. (35)], there exists a $k \in \mathbb{N}$ such that

$$\mathbb{P}^N(A) \leq C_{\varphi}^{k(k+1)/8} \overline{\mathbb{P}}^N(A)^{k/4} \quad \forall A \in \mathscr{F}_T, \tag{2.44}$$

where $C_\varphi$ is a constant independent of $N$ and bounded by

$$C_\varphi \leq 1 + \sum_{p=1}^{\infty} (\delta a_\varphi)^p \, \|\varphi\|_{\text{Lip}}^{2p}, \tag{2.45}$$

where $a_\varphi$ depends exponentially on $\|\varphi\|_{\text{Lip}}$ and $\delta$ needs to be chosen. Clearly,

$$\delta < \left(2a_\varphi \, \|\varphi\|_{\text{Lip}}^2\right)^{-1}$$

gives us an upper bound for $C_\varphi$, which is uniform over the class $\Phi$. Therefore, we have

$$\begin{aligned} \mathbb{P}^N\left(\left\|\Psi^{-1/2}\Psi_N\Psi^{-1/2} - I_{D_N}\right\|_{\text{op}} > \frac{1}{2}\right) &\lesssim \overline{\mathbb{P}}^N\left(\left\|\Psi^{-1/2}\overline{\Psi}_N\Psi^{-1/2} - I_{D_N}\right\|_{\text{op}} > \frac{1}{2}\right)^{k/4} \\ &\lesssim D_N^{1/4} \exp\left(-\frac{c_{\eta,T}}{16L_N^2 \, \|\Psi^{-1}\|_{\text{op}}^2} \eta N T\right), \end{aligned}$$

meaning that Proposition 2.8 is true for the particle system (2.1). This completes the proof of Proposition 2.8.

## 2.5.4 Proof of Theorem 2.9

We begin by considering the $\mathbb{R}^{D_N}$-valued random vector $E_N = Z_N - \langle e, \varphi \rangle_N$, that is

$$(E_N)_j = \frac{\sigma}{NT} \sum_{i=1}^{N} \int_0^T e_j * \mu_t^N(X_t^{i,N}) \, \mathrm{d}B_t^i.$$

By the Itô isometry,

$$\mathbb{E}\left[(E_N)_j (E_N)_k\right] = \frac{\sigma^2}{NT} \Psi_{jk} + O(N^{-2}). \tag{2.46}$$

**Lemma 2.18.** *For $E_N$ defined as above, we have*

$$\mathbb{E}\left[\|E_N\|^4\right] \lesssim \frac{D_N L_N^2}{(NT)^2}.$$

*Proof.* Using Jensen's inequality, the Burkholder-Davis-Gundy inequality, and independence of the $B^i$, we get

$$\begin{aligned}\mathbb{E}\left[\|E_N\|^4\right] &\le \sigma^4 D_N \sum_{j=1}^{D_N} \mathbb{E}\left[(E_N)_j^4\right] \lesssim \sigma^4 D_N \sum_{j=1}^{D_N} \mathbb{E}\left[\left(\frac{1}{(NT)^2}\sum_{i=1}^N \int_0^T \left(e_j * \mu_t^N(X_t^{i,N})\right)^2 \mathrm{d}t\right)^2\right] \\ &\le \frac{\sigma^4 D_N}{N^2T^3} \sum_{j=1}^{D_N}\sum_{i=1}^N \int_0^T \mathbb{E}\left[\left(e_j * \mu_t^N(X_t^{i,N})\right)^4\right]\mathrm{d}t \\ &\le \frac{\sigma^4 D_N}{(NT)^2}\sum_{j=1}^{D_N} \|e_j\|_\infty^4 \lesssim \frac{D_N L_N^2}{(NT)^2},\end{aligned}$$

which is the claim. □

For the asymptotic behaviour in $\|\cdot\|_*$ and $\|\cdot\|_N$, we consider orthogonal projections and their error with respect to our introduced norms. We begin with some intermediate estimates.

**Proposition 2.19.** *Let $\Pi^N : L^2(\|\cdot\|_*) \to S_N$ (resp. $\Pi^*$) be the orthogonal projection with respect to $\|\cdot\|_N$ (resp. $\|\cdot\|_*$) and recall that $\Pi^*\varphi = \varphi_*^N$. Suppose that Assumption 5 is satisfied. Then we have*

*(i)* $\mathbb{E}\left[\left\|\varphi_N - \Pi^N\varphi\right\|_N^2 1_{\Lambda_N\cap\Omega_N}\right] \lesssim \frac{D_N}{NT}$.

*(ii)* $\mathbb{E}\left[\left\|\varphi_N - \Pi^N\varphi\right\|_N^2 1_{\Lambda_N\cap\Omega_N^c}\right] \lesssim \frac{1}{NT}$.

*(iii)* $\mathbb{E}\left[\left\|\Pi^N\left(\varphi - \varphi_*^N\right)\right\|_*^2 1_{\Omega_N}\right] \lesssim N^{-1/2}(1+L_N)\|\varphi-\varphi_*^N\|_{**}^2 + O(N^{-1}) + o\left(\|\varphi-\varphi_*^N\|_*^2\right)$.

**Remark 2.20.** Depending on the choice of $\eta$, (ii) actually decays faster than stated. However, as (i) is the slowest term in Theorem 2.9, a convergence rate of $(NT)^{-1}$ is sufficient. We will see in our proofs that this requires $\eta \ge 3$, see (2.55). Considering we are analysing the projection error in its respective norm, the proof of this lemma hardly differs from that of [CG20, Lemma 6.3].

*Proof of Proposition 2.19.* (i) As $\Pi^N\varphi - \varphi = \operatorname{argmin}_{f\in S_N}\|f-\varphi\|_N$, a straightforward computation gives

$$\Pi^N\varphi = \sum_{j=1}^{D_N}(\widehat{a}_N)_j e_j, \quad \widehat{a}_N = \Psi_N^{-1}\langle e,\varphi\rangle_N\,. \tag{2.47}$$

By definition of $E_N$, $\theta_N - \widehat{a}_N = \Psi_N^{-1}E_N$. Hence,

$$\left\|\varphi_N - \Pi^N\varphi\right\|_N^2 = \left\|\sum_{j=1}^{D_N}\left(\Psi_N^{-1}E_N\right)_j e_j\right\|_N^2 = E_N^\top\Psi_N^{-1}E_N. \tag{2.48}$$

On $\Omega_N$, all eigenvalues of $\Psi^{-1/2}\Psi_N\Psi^{-1/2}$ are in $[\frac{1}{2},\frac{3}{2}]$, which implies that

$$\left\|\Psi^{1/2}\Psi_N^{-1}\Psi^{1/2}\right\|_{\mathrm{op}} \le 2, \tag{2.49}$$

such that $E_N^\top\Psi_N^{-1}E_N 1_{\Omega_N} \le 2E_N^\top\Psi^{-1}E_N$. Together with (2.46) and Lemma 2.18, we have deduced by Assumption 5:

$$\mathbb{E}\left[\left\|\varphi_N-\Pi^N\varphi\right\|_N^2 1_{\Lambda_N\cap\Omega_N}\right]\leq 2\mathbb{E}\left[E_N^\top\Psi^{-1}E_N\right]\lesssim\sum_{j,k=1}^{D_N}\left(\Psi^{-1}\right)_{jk}\left(\frac{1}{NT}\Psi_{jk}+\frac{1}{N^2}\right)$$
$$\lesssim\frac{D_N}{NT}+\frac{D_N\left\|\Psi^{-1}\right\|_{\mathrm{op}}}{N^2}\lesssim\frac{D_N}{NT}+\frac{D_N}{L_N\sqrt{N^3\log(NT)}}$$
$$\lesssim\frac{D_N}{NT}.$$

(ii) By Proposition 2.8, (2.48) and Lemma 2.18, we have

$$\mathbb{E}\left[\left\|\varphi_N-\Pi^N\varphi\right\|_N^2 1_{\Lambda_N\cap\Omega_N^c}\right]\leq\mathbb{E}\left[\left\|\Psi_N^{-1}\right\|_{\mathrm{op}}^2 1_{\Lambda_N}\left\|E_N\right\|^4\right]^{1/2}\mathbb{P}\left(\Omega_N^c\right)^{1/2}$$
$$\leq\sqrt{D_N}(NT)^{-(\eta/2+1)}.$$

(iii) We define $g=\varphi-\varphi_*^N$ and write the projection of $\Pi^N g$ according to (2.47) as an inner product. Using Lemma 2.16, Parseval's identity, and (2.49) we deduce the following inequalities on $\Omega_N$:

$$\left\|\Pi^N g\right\|_*^2=\left\|e^\top\Psi_N^{-1}\left\langle e,g\right\rangle_N\right\|_*^2 1_{\Omega_N}=\left\|\bar{e}^\top\Psi^{1/2}\Psi_N^{-1}\Psi^{1/2}\right\|\left\langle\bar{e},g\right\rangle_*^2$$
$$=\left\|\Psi^{1/2}\Psi_N^{-1}\Psi^{1/2}\left\langle\bar{e},g\right\rangle_N\right\|^2\leq 4\left\|\left\langle\bar{e},g\right\rangle_N\right\|^2$$
$$\leq 4\left\|\Psi^{-1}\right\|_{\mathrm{op}}\sum_{j=1}^{D_N}\left\langle e_j,g\right\rangle_N^2.$$

Due to the identity $g=\varphi-\varphi_*^N$, we have that $\left\langle e_j,g\right\rangle_*=0$ for all $j=1,\dots,D_N$. Hence,

$$\sum_{j=1}^{D_N}\mathbb{E}\left[\left\langle e_j,g\right\rangle_N^2\right]=\sum_{j=1}^{D_N}\mathbb{E}\left[\left(\left\langle e_j,g\right\rangle_N-\left\langle e_j,g\right\rangle_*\right)^2\right].\qquad(2.50)$$

For the latter term we want to use the decomposition (2.29). However, we note that $\|g\|_\infty$ is not necessarily uniformly bounded in $N$, and thus we need a precise estimate of this norm. We recall that $\|\varphi\|_*\leq\|\varphi\|_\infty\leq K_1$ for all $\varphi\in\Phi$. Also note the identity

$$\varphi_*^N=\sum_{j=1}^{D_N}\left\langle\bar{e}_j,\varphi\right\rangle_*\bar{e}_j.$$

Consequently, we deduce that

$$\left\|\varphi_*^N\right\|_\infty\leq\sum_{j=1}^{D_N}\left|\left\langle\bar{e}_j,\varphi\right\rangle_*\right|\cdot\|\bar{e}_j\|_\infty\leq\left(\sum_{j=1}^{D_N}\left\langle\bar{e}_j,\varphi\right\rangle_*^2\right)^{1/2}\left(\sum_{j=1}^{D_N}\|\bar{e}_j\|_\infty^2\right)^{1/2}$$
$$\leq L_N^{1/2}\left\|\Psi^{-1}\right\|_{\mathrm{op}}^{1/2}\left(\sum_{j=1}^{D_N}\left\langle\bar{e}_j,\varphi\right\rangle_*^2\right)^{1/2}.$$

On the other hand, we have

$$\left\|\varphi_*^N\right\|_*^2=\sum_{j=1}^{D_N}\left\langle\bar{e}_j,\varphi\right\rangle_*^2\leq\|\varphi\|_*^2\leq K_1^2.$$

Hence, we obtain the inequality

$$\|g\|_\infty \lesssim \left(1 + L_N^{1/2} \left\|\Psi^{-1}\right\|_{\mathrm{op}}^{1/2}\right). \tag{2.51}$$

In the next step, we use the decomposition (2.29) and the polarisation identity $\langle f, g\rangle_* = (\|f+g\|_*^2 - \|f-g\|_*^2)/4$ to get

$$\left\langle e_j, g\right\rangle_N - \left\langle e_j, g\right\rangle_* = U_{1,N}(g, e_j) + U_{2,N}(g, e_j) + U_{3,N}(g, e_j) + O\left(N^{-1}\|g\|_\infty \|e_j\|_\infty\right), \tag{2.52}$$

where $U_{k,N}(g, e_j) := (U_{k,N}(g + e_j) - U_{k,N}(g - e_j))/4$ for $k = 1, 2, 3$. Now, we need to estimate the variance of the terms $U_{k,N}(g, e_j)$. For this purpose we will use the same change of measure device as proposed in Section 2.5.1. In particular, due to the Cauchy-Schwarz inequality (2.34), it suffices to find an appropriate bound for the fourth moment of $U_{k,N}(g, e_j)$, $k = 1, 2, 3$, under the probability measure $\overline{\mathbb{P}}^N$.

We start with the U-statistics $U_{2,N}(g, e_j)$ and $U_{3,N}(g, e_j)$. Replacing $g$ (resp. $e_j$) by $g/\|g\|_\infty$ (resp. by $e_j/\|e_j\|_\infty$), we conclude as in Section 2.5.1:

$$\begin{aligned} \mathbb{E}_{\overline{\mathbb{P}}^N}\left[U_{2,N}(g, e_j)^4\right]^{1/2} &\lesssim \|g\|_\infty^2 \|e_j\|_\infty^2 N^{-2}, \\ \mathbb{E}_{\overline{\mathbb{P}}^N}\left[U_{3,N}(g, e_j)^4\right]^{1/2} &\lesssim \|g\|_\infty^2 \|e_j\|_\infty^2 N^{-3}. \end{aligned} \tag{2.53}$$

Next, we treat the term $U_{1,N}(g, e_j)$. Under $\overline{\mathbb{P}}^N$ we deal with i.i.d. random variables. Observe that for a statistic $W_N := N^{-1}\sum_{i=1}^N Z_i$, where $Z_i$'s are i.i.d. random variables with $\mathbb{E}[Z_1] = 0$ and $\mathbb{E}[Z_1^4] < \infty$, we have that $\mathbb{E}[W_N^4] = 3N^{-2}\mathbb{E}[Z_1^2]^2 + N^{-3}\mathbb{E}[Z_1^4]$. Using this observation and applying the Hölder inequality, a straightforward computation shows that

$$\mathbb{E}_{\overline{\mathbb{P}}^N}\left[U_{1,N}(g, e_j)^4\right]^{1/2} \lesssim \|e_j\|_\infty^2 \left(\|g\|_*^2 + \|g\|_{**}^2\right)\left(N^{-1} + N^{-3/2}\|e_j\|_\infty^2 \|g\|_\infty^2\right). \tag{2.54}$$

Now, we put everything together. According to (2.34), we deduce that

$$\mathbb{E}_{\mathbb{P}^N}\left[U_{k,N}(g, e_j)^2\right] \le C\mathbb{E}_{\overline{\mathbb{P}}^N}\left[U_{1,N}(g, e_j)^4\right]^{1/2} \qquad \text{for } k = 1, 2, 3.$$

Plugging the inequalities (2.53) and (2.54) into (2.50), and taking into account the bound in (2.51), we conclude that

$$\begin{aligned} \mathbb{E}\left[\left\|\Pi^N\left(\varphi - \varphi_*^N\right)\right\|_*^2 1_{\Omega_N}\right] \lesssim &\left\|\Psi^{-1}\right\|_{\mathrm{op}}^2 L_N^2 \left(N^{-2} + N^{-3}\right) \\ &+ N^{-1}\left\|\Psi^{-1}\right\|_{\mathrm{op}} L_N \left(\|g\|_*^2 + \|g\|_{**}^2\right) \\ &+ N^{-3/2}\left\|\Psi^{-1}\right\|_{\mathrm{op}}^2 L_N^3 \left(\|g\|_*^2 + \|g\|_{**}^2\right). \end{aligned}$$

Recalling Assumption 5, we obtain the statement of the proposition. □

Now, we proceed with the proof of Theorem 2.9. Let us start with the asymptotic result in the empirical norm. We split

$$\begin{aligned} \|\widehat{\varphi}_N - \varphi\|_N^2 &= \inf_{f \in S_N} \|f - \varphi\|_N^2 + \left\|\widehat{\varphi}_N - \Pi^N \varphi\right\|_N^2 \\ &= \inf_{f \in S_N} \|f - \varphi\|_N^2 + \left\|\varphi_N - \Pi^N \varphi\right\|_N^2 \left(1_{\Lambda_N \cap \Omega_N} + 1_{\Lambda_N \cap \Omega_N^c}\right) + \|\varphi\|_N^2 1_{\Lambda_N^c} \\ &=: \inf_{f \in S_N} \|f - \varphi\|_N^2 + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3\,. \end{aligned}$$

Recall that $\mathbb{E}[\mathrm{I}_1]$ and $\mathbb{E}[\mathrm{I}_2]$ are treated in Proposition 2.19. Furthermore, we have $\mathbb{E}\mathrm{I}_3 \lesssim 1/N$ due to Proposition 2.8. This gives us

$$\mathbb{E}\left[\|\widehat{\varphi}_N - \varphi\|_N^2\right] \lesssim \mathbb{E}\left[\inf_{f\in S_N} \|f-\varphi\|_N^2\right] + \frac{D_N}{NT}.$$

Taking the supremum over $\varphi \in \Phi$ gives the first claim.

For the asymptotic results in $\|\cdot\|_*$, we split

$$\begin{aligned}\|\widehat{\varphi}_N - \varphi\|_*^2 &= \|\widehat{\varphi}_N - \varphi\|_*^2 \left(1_{\Lambda_N\cap\Omega_N} + 1_{\Lambda_N\cap\Omega_N^c}\right) + \|\varphi\|_*^2 1_{\Lambda_N^c}\\ &=: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3\,.\end{aligned}$$

For $\mathrm{II}_1$, we set $g = \varphi - \varphi_*^N$. Note that $\Pi^N\Pi^* = \Pi^*$, therefore

$$\Pi^N\varphi - \varphi = \Pi^N(g + \Pi^*\varphi) - \varphi = \Pi^N g - g.$$

So we split the terms in $\mathrm{II}_1$ into

$$\begin{aligned}\|\widehat{\varphi}_N - \varphi\|_*^2 &\le \left\|\widehat{\varphi}_N - \Pi^N\varphi\right\|_*^2 + \left\|\Pi^N\varphi - \varphi\right\|_*^2\\ &\le \left\|\widehat{\varphi}_N - \Pi^N\varphi\right\|_*^2 + \left\|\Pi^N g\right\|_*^2 + \|g\|_*^2\,.\end{aligned}$$

Since $\|\cdot\|_N$ and $\|\cdot\|_*$ are equivalent on $\Omega_N$, we have

$$\mathbb{E}\left[\left\|\widehat{\varphi}_N - \Pi^N\varphi\right\|_*^2 1_{\Lambda_N\cap\Omega_N}\right] \le 2\mathbb{E}\left[\left\|\widehat{\varphi}_N - \Pi^N\varphi\right\|_N^2 1_{\Lambda_N\cap\Omega_N}\right] \lesssim \frac{D_N}{NT}.$$

By Proposition 2.19 (iii), we have

$$\mathbb{E}\left[\left\|\Pi^N g\right\|_*^2 1_{\Lambda_N\cap\Omega_N}\right] + \|g\|_*^2 \lesssim (1+o(1))\,\|g\|_*^2 + N^{-1/2}(1+L_N)\,\|g\|_{**}^2 + O(N^{-1}).$$

Moving on to $\mathrm{II}_2$, we have

$$\mathbb{E}\left[\|\widehat{\varphi}_N - \varphi\|_*^2 1_{\Lambda_N\cap\Omega_N^c}\right] \le \mathbb{E}\left[\|\varphi_N\|_*^2 1_{\Lambda_N\cap\Omega_N^c}\right] + \|\varphi\|_*^2\,\mathbb{P}\left(\Omega_N^c\right).$$

Recalling the definition of $\varphi_N$ given in (2.11) and that $(Z_N)_j = (E_N)_j + \left\langle e_j, \varphi\right\rangle_N$, we have

$$\begin{aligned}\|\varphi_N\|_*^2 &= \left(\Psi_N^{-1}Z_N\right)^\top \Psi\Psi_N^{-1}Z_N \le \left\|\Psi_N^{-1}\right\|_{\mathrm{op}}^2 \|\Psi\|_{\mathrm{op}}\,\|Z_N\|^2\\ &= \left\|\Psi_N^{-1}\right\|_{\mathrm{op}}^2 \|\Psi\|_{\mathrm{op}}\left(\left\|\langle e,\varphi\rangle_N\right\|^2 + \|E_N\|^2\right).\end{aligned}$$

Since $\varphi$ is bounded,

$$\begin{aligned}\sum_{j=1}^{D_N}\mathbb{E}\left[\left\langle e_j,\varphi\right\rangle_N^4\right] &\le \frac{1}{(NT)^2}\sum_{j=1}^{D_N}\int_0^T \mathbb{E}\left[e_j * \mu_t^N(X_t^{1,N})^4 \varphi * \mu_t^N(X_t^{1,N})^4\right]\mathrm{d}t\\ &\le \sum_{j=1}^{D_N}\frac{\left\|e_j\right\|_\infty^4}{(NT)^2}\int_0^T \mathbb{E}\left[\varphi * \mu_t^N(X_t^{1,N})^4\right]\mathrm{d}t \le \frac{L_N^2\,\|\varphi\|_\infty^4}{(NT)^2},\end{aligned}$$

such that

$$\begin{aligned}\mathbb{E}\left[\left(\left\|\langle e,\varphi\rangle_N\right\|^2 + \|E_N\|^2\right)1_{\Lambda_N\cap\Omega_N^c}\right] &\le \left(\mathbb{E}\left[\left\|\langle e,\varphi\rangle_N\right\|^4\right]^{1/2} + \mathbb{E}\left[\|E_N\|^4\right]^{1/2}\right)\mathbb{P}\left(\Omega_N^c\right)^{1/2}\\ &\le \left(\frac{\|\varphi\|_\infty^2 L_N}{NT} + \frac{\sqrt{D_N}L_N}{NT}\right)(NT)^{-\frac{\eta-1}{2}}\\ &\le \left(\frac{\|\varphi\|_\infty^2}{\sqrt{D_N}} + 1\right)L_N(NT)^{-\frac{\eta}{2}}.\end{aligned}$$

Finally, with Assumption 5 and Lemma 2.18, and the fact that $\|\Psi\|_{\mathrm{op}} \le L_N$, we obtain

$$\begin{aligned}\mathbb{E}\left[\|\varphi_N\|_*^2 1_{\Lambda_N\cap\Omega_N^c}\right] &\le \mathbb{E}\left[\left\|\Psi_N^{-1}\right\|_{\mathrm{op}}^2 \|\Psi\|_{\mathrm{op}}\left(\left\|\langle e,\varphi\rangle_N\right\|^2 + \|E_N\|^2\right) 1_{\Lambda_N\cap\Omega_N^c}\right] \\ &\le c_{\eta,T}\left(\frac{\|\varphi\|_\infty^2}{\sqrt{D_N}}+1\right)(NT)^{-(\eta-1)/2}.\end{aligned} \tag{2.55}$$

Last but not least, we have that

$$\mathbb{E}[\mathrm{II}_3] \lesssim \mathbb{P}(\Lambda_N^c)$$

which completes the proof due to Proposition 2.8.

### 2.5.5 Proof of Proposition 2.11

Let us denote by $\lambda_{\min}(Q)$ the smallest eigenvalue of a symmetric matrix $Q\in\mathbb{R}^{D_N\times D_N}$. Since

$$\lambda_{\min}(Q_1Q_2) \ge \lambda_{\min}(Q_1)\lambda_{\min}(Q_2) \qquad \text{and}$$

$$\lambda_{\min}\left(T^{-1}\int_0^T Q_t dt\right) \ge T^{-1}\int_0^T \lambda_{\min}(Q_t)dt,$$

for all positive semi-definite matrices $Q_1, Q_2, (Q_t)_{t\ge 0}$, we obtain via (2.18):

$$\begin{aligned}\lambda_{\min}(\Psi) &\ge \frac{1}{2A_N T}\int_0^T \min_{k=1,\dots,D_N}(\widehat{\mu}_t(k/A_N))^2\lambda_{\min}(R_t)dt \\ &\ge \frac{\exp\left(-2D_N^2/(c_2A_N^2)\right)}{2A_N T}\int_0^T g_2^2(t)\lambda_{\min}(R_t)dt\end{aligned}$$

where $R_t\in\mathbb{R}^{D_N\times D_N}$ is a Toeplitz matrix given as $R_t^{kl} = \widehat{\mu}_t((l-k)/A_N)$. To deduce a lower bound for $\lambda_{\min}(R_t)$, we introduce the discrete-time Fourier transform

$$F(\omega) := \sum_{n\in\mathbb{Z}}\widehat{\mu}_t(n/A_N)\exp(-i\omega n), \qquad \omega\in[-\pi,\pi].$$

A well-known result states that

$$\lambda_{\min}(R_t) \ge \min_{\omega\in[-\pi,\pi]} F(\omega),$$

and hence we need to compute $F(\omega)$. For this purpose, we will apply the Poisson summation formula. Define the function $f(x) := \widehat{\mu}_t(x/A_N)\exp(-i\omega x)$ for $x\in\mathbb{R}$ and $\mathscr{F}f(z) := \int_{\mathbb{R}} f(x)\exp(-i2\pi z x)dx$. The Poisson summation formula states that

$$F(\omega) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}}\mathscr{F}f(n).$$

By the Fourier inversion formula we obtain the identity

$$\mathscr{F}f(n) = \int_{\mathbb{R}}\widehat{\mu}_t(x/A_N)\exp\left(-ix(\omega+2\pi n)\right)dx = \frac{A_N}{\sqrt{2\pi}}\mu_t\left(A_N(\omega+2\pi n)/\pi\right).$$

Due to Assumption 6, we deduce that

$$\begin{aligned}F(\omega) &\ge \frac{A_N g_1(t)}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\exp\left(-\frac{A_N^2(\omega+2\pi n)^2}{c_1\pi^2}\right) \\ &\ge \frac{A_N g_1(t)}{\sqrt{2\pi}}\exp\left(-\frac{A_N^2}{c_1}\right)(1+o(1)).\end{aligned}$$

Putting things together, we conclude that

$$\lambda_{\min}(\Psi) \geq \frac{\exp\left(-2D_N^2/(c_2 A_N^2)\right)\exp\left(-A_N^2/c_1\right)}{2\sqrt{2\pi}T} \int_0^T g_1(t)g_2(t)^2 dt.$$

This implies the statement of Proposition 2.11.

### 2.5.6 Proof of Proposition 2.12

Consider the decomposition

$$\varphi = \sum_{j=1}^{\infty} c_j e_j = \sum_{j=1}^{D_N} c_j e_j + \sum_{j=D_N+1}^{\infty} c_j e_j =: \varphi_1 + \varphi_2.$$

In this scenario it holds that

$$\varphi - \Pi^* \varphi = \varphi_2 - \Pi^* \varphi_2.$$

Furthermore, it is

$$\|\varphi_2\|_{**}^2 \leq \|\varphi_2\|_\infty^2 \lesssim A_N^{-1} \left( \sum_{j=D_N+1}^{\infty} |c_j| \right)^2.$$

To estimate the term $\|\Pi^* \varphi_2\|_{**}^2$, we introduce the decomposition

$$\Pi^* \varphi_2 = \sum_{j=1}^{D_N} \left\langle \bar{e}_j, \varphi_2 \right\rangle_* \bar{e}_j.$$

Now, we deduce that

$$\|\Pi^* \varphi_2\|_{**}^2 \lesssim \|\Pi^* \varphi_2\|_\infty^2.$$

We obtain the inequality

$$\|\Pi^* \varphi_2\|_\infty \leq \sum_{j=1}^{D_N} |\left\langle \bar{e}_j, \varphi_2 \right\rangle_*| \cdot \|\bar{e}_j\|_\infty.$$

In the next step we will find a bound for the term $|\left\langle \bar{e}_j, \varphi_2 \right\rangle_*|$. First, we note

$$\left\langle \bar{e}_j, \varphi_2 \right\rangle_* = \sum_{k=1}^{D_N} \sum_{l=D_N+1} \Psi_{jk}^{-1/2} c_l \left\langle e_k, e_l \right\rangle_*.$$

We already know that

$$\langle e_k, e_l \rangle_* = \frac{1}{2A_N T} \int_0^T \widehat{\mu}_t(-k/A_N)\widehat{\mu}_t(l/A_N)\widehat{\mu}_t((l-k)/A_N)\mathrm{d}t.$$

Due to Assumption 3, the leading order of the above term is achieved for $l = D_n + 1$ and $k = D_n/2$. Hence, we conclude that

$$|\left\langle \bar{e}_j, \varphi_2 \right\rangle_*| \lesssim D_N \|\Psi^{-1}\|_{\mathrm{op}}^{1/2} \exp\left(-\frac{3D_N^2 \zeta^2}{4A_N^2}\right) \sum_{j=D_N+1}^{\infty} |c_j|.$$

Consequently, we obtain the inequality

$$\|\Pi^* \varphi_2\|_\infty \lesssim A_N^{-1/2} D_N^2 \|\Psi^{-1}\|_{\mathrm{op}} \exp\left(-\frac{3D_N^2 \zeta^2}{4A_N^2}\right) \sum_{j=D_N+1}^{\infty} |c_j|,$$

which completes the proof.

# Chapter 3

# Polynomial Rates through Deconvolution Estimation

The foundation of stochastic systems involving interacting particles and the development of nonlinear Markov processes, initially introduced by McKean in the 1960s [McK66], can be traced back to their roots in statistical physics, particularly within the domain of plasma physics. Over subsequent decades, the significance of these systems in probability theory has steadily grown. This area has witnessed the development of fundamental probabilistic tools, including propagation of chaos, geometric inequalities, and concentration inequalities. Pioneering contributions from researchers such as [Mél96; Mal01; CGM08; Szn91] have played a crucial role in shaping this field.

However, formulating a modern statistical inference program for these systems remained challenging until the early 2000s, with few exceptions, such as Kasonga's early paper [Kas90]. Several factors contributed to this challenge. Firstly, the advanced probabilistic tools required for estimation were still under development. Secondly, the microscopic particle systems originating from statistical physics were not naturally observable, making the motivation for statistical inference less apparent. This situation began to change around the 2010s with the widespread adoption of these models in various fields where data became observable and collectable. Applications expanded into diverse fields, including the social sciences (e.g., opinion dynamics [Cha+17] and cooperative behaviours [CFT12]), mathematical biology (e.g., structured models in population dynamics [ME99] and neuroscience [Bal+12]), and finance (e.g., the study of systemic risk [FS13] and smile calibration [GH11]). Mean-field games have emerged as a new frontier for statistical developments, as evident in the references [Car+19; Dje+22; GSS20]. This transition has led to a growing need for a systematic statistical inference program, which constitutes our primary focus. Recently, this interest has manifested in two primary directions. On one front, statistical investigations are rooted in the direct observation of large interacting particle systems, as evidenced in works [Amo+23; Che21; CG23; DH22; DH23; PZ22b]. On the other front, statistical inference revolves around the observation of the mean-field limit, the McKean-Vlasov process, as exemplified in [GL21b; GL21a; Sha+23]. Concerning stationary McKean-Vlasov SDEs, the literature is relatively sparse. To the best of our knowledge, only a handful of references exist, including [PZ22a] and [GL23a], which focus on the special McKean-Vlasov model without a potential term. In [GL23b], a more general model is explored.

This chapter, based on [Amo+24], focuses on statistical inference for an interacting particle system described by the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X_t^{i,N} = -(V'(X_t^{i,N}) + \frac{1}{2}W' * \mu_t^N(X_t^{i,N}))\mathrm{d}t + \mathrm{d}B_t^i, \\ (X_0^{i,N}) \sim \mu_0^{\otimes N}, \end{cases} \tag{3.1}$$

where the processes $B^i := (B_t^i)_{t\geq 0}$ are independent standard Brownian motions. The function $V$ is referred to as the *confinement potential*, while $W$ (or $W'$) is the *interaction potential* (or interaction function respectively). Our primary objective is to estimate the interaction function $W'$ based on observations $X_T^{1,N}, \dots, X_T^{N,N}$ of the particles, which are solutions of the system (3.1). Our approach hinges on the analysis of the associated inverse problem concerning the underlying stationary Fokker-Planck equation, relying on various results related to the probabilistic properties of the model.

Our research is closely connected to a recent study [BPP23] that discusses estimation of the interaction function with a specific semiparametric structure. The authors in [BPP23] develop an estimation procedure, demonstrating convergence rates that critically depend on the tail behaviour of the nonparametric part of the interaction function $W$. Specifically, assuming a polynomial decay of the tails, they establish logarithmic convergence rates, proven to be optimal in that context. This naturally raises the question of whether polynomial rates can be achieved under a different set of conditions. This work aims to address this question, and our key finding is that, by assuming exponential decay of the interaction function $W$, we can introduce an estimator that achieves polynomial convergence rates, as demonstrated in Theorem 3.11.

Compared to the framework proposed in [BPP23], our model features some distinctions. The smoothness of the confinement potential $V$ emerges as a crucial factor influencing the achieved convergence rate. Additionally, the regularity of the invariant density $\pi$ of the associated McKean-Vlasov equation and the analysis of its Fourier transform $\mathscr{F}(\pi)$ are vital for establishing the asymptotic properties of our estimator. Notably, a lower bound on $\mathscr{F}(\pi)$ is required, presenting one of the primary challenges in this work. We address this challenge using Hadamard factorisation, leading to the desired lower bound under mild assumptions on the model. Furthermore, we provide an example demonstrating how a non-smooth confinement potential $V$ results in the Fourier transform $\mathscr{F}(\pi)$ exhibiting a polynomial decay. As another interesting situation, we consider the case of non-smooth potential $W$ and show that even in this case the Fourier transform of $\pi$ decays exponentially fast. Further technical tools include the extension of the Kantorovich-Rubinstein dual theorem to functions lacking Lipschitz continuity, presented in Theorem 3.4, and an extension of the uniform propagation of chaos in the $L^{2p}$-norm without a convexity assumption for $W$, as demonstrated in Proposition 3.6.

The structure of this chapter is as follows. In Section 3.1, we introduce the model assumptions. This section also offers a concise overview of interacting particle systems, and the relevant tools about Fourier and Laplace transforms. Crucially, we present key probabilistic results that lay the groundwork for our main findings. Section 3.2 is dedicated to formulating our primary statistical problem and the associated estimation procedure. Additionally, we establish upper bounds on the $L^2$ risk of the proposed drift estimator. The proof of our main results is provided in Section 3.3, where we delve into the detailed verification of our key findings. Finally, in Section 3.4, we explore the sufficient conditions necessary to ensure that the transforms satisfy the requirements for our main results. All remaining proofs are collected in Section 3.5.

**Notation**

All random variables and stochastic processes are defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Throughout the chapter, we use the symbol $c$ to represent positive constants, although these constants may vary from one line to another. For any function $f : \mathbb{R} \to \mathbb{R}$, we denote its supremum as $\|f\|_\infty := \sup_{y\in\mathbb{R}} |f(y)|$. The notation $x_n \lesssim y_n$ signifies the existence of a constant $c > 0$, independent of $n$, such that $x_n \leq c y_n$. The derivatives of a function $f$ are denoted as $f', f'', \ldots$, or $f^{(k)}$, $k \geq 1$. For a complex number $z \in \mathbb{C}$, we denote its complex conjugate, real part, and imaginary part as $\overline{z}$, $\mathrm{Re}(z)$, and $\mathrm{Im}(z)$, respectively. For $a \in \mathbb{R}$, we use the notation

$$\mathscr{L}_a := \{y + ia : y \in \mathbb{R}\}.$$

## 3.1 Model and Assumptions

We start our analysis by introducing a set of assumptions on the confinement potential $V$ and the interaction potential $W$. It will become evident that the smoothness of $V$ plays a pivotal role in our asymptotic analysis. Our investigation encompasses two distinct scenarios: Either the confinement potential of smoothness takes the form $V(x) = \alpha x^2/2$ for a positive constant $\alpha$, or the confinement potential is expressed as $V(x) = \alpha x^2/2 + \widetilde{V}(x)$, where $\alpha > 0$ and $\widetilde{V}$ is a known function characterised by non-smooth features, as explained below. We assume that $V$ and $W$ satisfy the following hypothesis:

**Assumption 7.** The potentials $W : \mathbb{R} \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$ are such that

- The interaction potential $W \in C^2(\mathbb{R})$ is even with bounded derivatives $W' \in L^1(\mathbb{R})$ and $W''$ such that $\inf_{x\in\mathbb{R}} W''(x) = -C_W$ for some $C_W > 0$.
- The confinement potential $V$ is given by $V(x) = \frac{\alpha}{2}x^2 + \widetilde{V}(x)$, with $\alpha + \inf_{x\in\mathbb{R}} \widetilde{V}''(x) = C_V > 0$, where $C_V$ and $C_W$ satisfy the relation $C_V - C_W > 0$. $\widetilde{V}$ is given by either

  A1. $\widetilde{V} = 0$, or

  A2. $\widetilde{V}$ is even, and there exists $J \in \mathbb{N}$, $J \geq 2$ such that $\widetilde{V} \in C^J(\mathbb{R})$ and $\widetilde{V} \notin C^{J+1}(\mathbb{R})$. Furthermore, for each $2 \leq j \leq J$, $\sup_{x\in\mathbb{R}} |\widetilde{V}^{(j)}(x)| = \tilde{c}_j < \infty$.

Additionally, the initial distribution admits a density, which (by abuse of notation) we also denote as $\mu_0$, satisfying

$$\int_{\mathbb{R}} \exp(cx)\mu_0(x)\,\mathrm{d}x < \infty, \quad \forall c \in \mathbb{R}, \qquad \int_{\mathbb{R}} \log(\mu_0(x))\mu_0(x)\,\mathrm{d}x < \infty. \tag{3.2}$$

**Remark 3.1.** We emphasise that we do not assume convexity of the interaction potential $W$, as is commonly done in most works. Instead, our assumption is that $W''$ is bounded below by a constant $-C_W < 0$. While the geometric convergence of the distribution of the system (3.1) to the invariant distribution for $t \to \infty$ now follows from [CMV03, Theorem 2.1], Proposition 3.6 establishes a uniform (in time) propagation of chaos under the above assumptions. Lastly, note also that A1 can be seen as a special case of A2, where $J = \infty$.

### 3.1.1 Probabilistic Results

The mean field equation associated to the interacting particle system introduced in (3.1) is given by the 1-dimensional McKean-Vlasov equation

$$\begin{cases} \mathrm{d}\overline{X}_t = -\left(V'(\overline{X}_t) + \frac{1}{2}W' * \mu_t(\overline{X}_t)\right)\mathrm{d}t + \mathrm{d}B_t, \\ \mathrm{Law}(X_t) = \mu_t. \end{cases} \tag{3.3}$$

Under Assumption 7, existence and uniqueness of strong solutions of (3.1) and (3.3) follow as in [Ben+98; CMV03; Mal03]. Additionally, the measure $\mu_t$ possesses a smooth Lebesgue density and the McKean-Vlasov equation admits an invariant density $\pi$ solving the stationary Fokker-Planck equation

$$\frac{1}{2}\pi'' = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(V' + \frac{1}{2}W' * \pi\right)\pi\right), \tag{3.4}$$

which means that $\pi$ is given by

$$\pi(x) = \frac{1}{Z_\pi}\exp\left(-2V(x) - W * \pi(x)\right), \quad x \in \mathbb{R}, \tag{3.5}$$

with a normalising constant

$$Z_\pi = \int_{\mathbb{R}} \exp\left(-2V(x) - W * \pi(x)\right)\mathrm{d}x < \infty.$$

The invariant density can be upper and lower bounded under our assumptions, according to the following lemma. Its proof can be found in Section 3.5.

**Lemma 3.2.** *Suppose that Assumption 7 holds true. Then, for any $x \in \mathbb{R}$, there exist two constants $c_1, c_2 > 0$ such that*

$$c_1\exp(-\widetilde{C}x^2) \le \pi(x) \le c_2\exp(-C_V x^2),$$

*with $\widetilde{C} := \alpha + \tilde{c}_2$ and $C_V$ and $\tilde{c}_2$ are as in Assumption 7. Furthermore, it holds that*

$$|\pi^{(n)}(x)| \le c(1+|x|)^n\exp(-C_V x^2), \quad 0 \le n \le J.$$

A crucial tool in our estimation procedure consists in a result which combines uniform in time propagation of chaos for (3.1) with convergence to the equilibrium of (3.3). In order to state it we start by introducing the Wasserstein metric. The *Wasserstein p-distance* between two measures $\mu$, $\nu$ on $\mathbb{R}$ is defined by

$$W_p(\mu,\nu) := \left(\inf_{X\sim\mu, Y\sim\nu}\mathbb{E}[|X-Y|^p]\right)^{\frac{1}{p}},$$

where the infimum is taken over all the possible couplings $(X, Y)$ of random variables $X$ and $Y$ with respective laws $\mu$ and $\nu$. The following result combines point (iv) of [CMV03, Theorem 2.1] with [Mal03, Theorem 5.1] or [CGM08, Theorem 3.1], adapted to the current framework, see also our Proposition 3.6 below.

**Theorem 3.3.** *Let $\overline{X}^i$, $1 \le i \le N$, be i.i.d. copies of the process $\overline{X}$ defined in (3.3) so that every $\overline{X}^i$ is driven by the same Brownian motion as the i-th particle of the system (3.1) and they are equal at time 0. Denote by*

$$\Pi_{N,T} = N^{-1}\sum_{i=1}^{N}\delta_{X_T^{i,N}}$$

*the empirical distribution of the particle system $X_T^{i,N}$ for $1 \le i \le N$, and by $\Pi$ the law associated to the invariant density $\pi$. Under Assumption 7 there exists a constant $c > 0$ independent of $N$ and $T$ such that*

$$\sup_{t \ge 0} \mathbb{E}\left[ |X_t^{i,N} - \overline{X}_t^{i,N}|^2 \right] \le cN^{-1}$$

*and*

$$\mathbb{E}[W_1(\Pi_{N,T}, \Pi)^2] \le c\left(N^{-1} + \exp(-\lambda T)\right) =: cN_T^{-1},$$

*where $\lambda = C_V - C_W > 0$, and $C_V$ and $C_W$ have been introduced in Assumption 7.*

The aforementioned bound asserts that the invariant distribution $\Pi$ of the mean field equation can be accurately approximated by the empirical measure $\Pi_{N,T}$, while providing an associated error bound for this approximation. In the upcoming estimation procedure all convergence rates will be measured in terms of $N_T$. In the following, we will present and prove a similar result, but from a dual perspective, involving the Laplace transforms of $\Pi_{N,T}$ and $\Pi$.

We opt to replace the use of the Fourier transform, as seen in [BPP23], with the Laplace transform. This choice is made with a similar intent as the authors of [BG21], allowing for greater flexibility and the inclusion of diverse scenarios.

Before we proceed further, let us now introduce some notation and properties concerning the Laplace transform. For any locally integrable function $\phi$, we define the bilateral Laplace transform as follows:

$$\widehat{\phi}(z) := \int_{\mathbb{R}} \phi(t) \exp(-zt)\, dt. \tag{3.6}$$

The Laplace transform $\widehat{\phi}(z)$ is an analytic function within the convergence region $\Sigma_\phi$, which typically takes the form of a vertical strip in the complex plane:

$$\Sigma_\phi := \left\{ z \in \mathbb{C} \,:\, x_\phi^- \le \mathrm{Re}(z) \le x_\phi^+ \right\}$$

for some $x_\phi^-$, $x_\phi^+$ such that $-\infty < x_\phi^- \le x_\phi^+ < \infty$. The convergence region $\Sigma_\phi$ can degenerate to a vertical line in the complex plane, in that case it is $\Sigma_\phi := \left\{ z \in \mathbb{C} \,:\, \mathrm{Re}(z) = x_\phi \right\}$, with $x_\phi \in \mathbb{R}$. If $\phi$ is a probability density, then the imaginary axis always belongs to the convergence region $\Sigma_\phi$. In this case the Fourier transform of $\phi$

$$\widehat{\phi}(-iy) = \mathscr{F}(\phi)(y) := \int_{\mathbb{R}} \phi(t) \exp(iyt)\, dt, \qquad y \in \mathbb{R},$$

is the characteristic function of $\phi$. This degenerate case pertains to distributions whose characteristic function lacks the ability to be analytically extended to a strip surrounding the imaginary axis on the complex plane.

We are now ready to state a propagation of chaos type result for the Fourier transforms that is reminiscent of Theorem 3.3 by means of the Kantorovich-Rubinstein theorem. In particular, we can demonstrate that the transform of $\Pi$ can also be effectively approximated by the transform of the empirical measure $\Pi_{N,T}$. The proof can be found in Section 3.3.

**Theorem 3.4.** *Under Assumption 7, there exists a constant $c > 0$ such that, for any $z \in \mathscr{L}_{\pm a}$,*

$$\mathbb{E}[|\mathscr{F}(\Pi)(z) - \mathscr{F}(\Pi_{N,T})(z)|^2] \le c|z|^2 N_T^{-1}.$$

**Remark 3.5.** Note that for $z \in \mathbb{R}$, this theorem is a direct implication of the Kantorovich-Rubinstein dual formulation, applicable to 1-Lipschitz functions. However, when dealing with

$z \in \mathscr{L}_{\pm a}$, the function $\exp(iz)$ no longer possesses Lipschitz continuity, thereby rendering the use of the Kantorovich-Rubinstein dual formulation unfeasible. This motivates us to establish this analogous formulation.

Some crucial results will be instrumental in the derivation of the theorem above. Specifically, we present an extension of the propagation of chaos theorem, as stated in Proposition 3.6 below, in the $L^{2p}$ norm, for $p \geq 1$. For the detailed proof of this result, we refer to Section 3.5.

**Proposition 3.6.** *Under Assumption 7 for any $p \geq 1$ there exists a constant $c > 0$ such that for any $N \in \mathbb{N}$ and $1 \leq i \leq N$, it is*

$$\sup_{t \geq 0} \mathbb{E}\left[ |X_t^{i,N} - \overline{X}_t^{i,N}|^{2p} \right] \leq cN^{-p}.$$

## 3.2 Statistical Framework and Main Results

### 3.2.1 The Estimation Procedure

Suppose we observe the data $X_T^{1,N}, \dots, X_T^{N,N}$ in the asymptotic framework where $N, T \to \infty$. Our goal is to estimate the interaction function $W'$. In particular, we will propose an estimator $W'_{N,T}$ and study its performance by considering the associated mean integrated squared error, aiming to achieve polynomial convergence rates.

The estimation procedure is semiparametric in the sense that it consists of four different steps, involving both parameter estimation and nonparametric estimation techniques.

1. The first step consists in the estimation of the derivative of the log-density which we denote as $l(y)$:

$$l(y) := (\log \pi)'(y) = \frac{\pi'(y)}{\pi(y)}, \qquad y \in \mathbb{R}.$$

This will be achieved by the introduction of kernel estimators for both $\pi$ and $\pi'$. Let $K$ be a smooth kernel of order $m \geq 2$, that is

$$\int_{\mathbb{R}} K(x)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}} x^j K(x)\,\mathrm{d}x = 0 \quad j = 0, \dots, m-1, \qquad \int_{\mathbb{R}} x^m K(x)\,\mathrm{d}x \neq 0.$$

It is worth noting the choice of the kernel order, denoted by $m$, is flexible and can be determined by the statistician. As we will see later on, the choice of $m$ is determined by the regularity of $V$: a smooth confinement potential allows us to choose an arbitrary $m \in \mathbb{N}$. On the other hand, if $V$ is non-smooth as described in A2 of Assumption 7, we need to additionally assume that $m \leq J$. Indeed, this is a standard restriction on the order of the kernel when estimating non-smooth functions. Let us also introduce the bandwidths $h_i := h_{i,N,T}$, $i = 0, 1$, which satisfy $h_i \to 0$ for $N, T \to \infty$ and the notation $K_h(x) = \frac{1}{h} K(\frac{x}{h})$. Then, we can define the kernel estimators $\pi_{N,T}$ and $\pi'_{N,T}$ for $\pi$ and $\pi'$, respectively, as

$$\pi_{N,T}(y) := \frac{1}{N} \sum_{i=1}^{N} K_{h_0}(y - X_T^{i,N}), \qquad \pi'_{N,T}(y) := \frac{1}{Nh_1} \sum_{i=1}^{N} K'_{h_1}(y - X_T^{i,N}).$$

An estimator for the derivative of the log-density $l(y)$ is then given by

$$l_{N,T}(y) := \frac{\pi'_{N,T}(y)}{\pi_{N,T}(y)} 1_{\{\pi_{N,T}(y) > \delta\}},$$

where $\delta = \delta_{N,T} \to 0$ as $N, T \to \infty$.

2. In the second step, we estimate the parameter $\alpha > 0$ appearing in the confinement potential. This is based on the identity

$$l(y) = -2\alpha y - 2\widetilde{V}'(y) - W' * \pi(y)$$

and on a contrast function method. Indeed, since $W' \in L^1(\mathbb{R})$, we know that $\left|W' * \pi(y)\right| \to 0$ as $|y| \to \infty$, which allows us to construct a minimal contrast estimator for $\alpha$. In particular, for any $\epsilon \in (0,1)$ arbitrarily small, we can introduce an integrable weight function $w$ with support on $[\epsilon, 1]$ and a parameter $U = U_{N,T}$, which satisfies $U \to \infty$ for $N, T \to \infty$. Then, we can define the estimator $\alpha_{N,T}$ for $\alpha$ as

$$\alpha_{N,T} := \underset{\alpha\in\mathbb{R}}{\operatorname{argmin}} \int_{\mathbb{R}} \left(l_{N,T}(y) + 2\alpha y + 2\widetilde{V}'(y)\right)^2 w_U(y)\,\mathrm{d}y.$$

3. Using the results in the previous step we can construct an estimator $\Psi_{N,T}$ of $\Psi := -W' * \pi$. Given the estimators $l_{N,T}$ and $\alpha_{N,T}$ constructed above, we define

$$\Psi_{N,T}(y) := \left(l_{N,T}(y) + 2\alpha_{N,T}\, y + 2\widetilde{V}'(y)\right) \mathbb{1}_{\{|y|\le\epsilon U\}}, \qquad y \in \mathbb{R}.$$

4. The last step consists in applying the deconvolution and the inverse Laplace transform to obtain an estimator for $W'$. Note that because we want to consider values of $\mathscr{F}(\Pi)$ on a domain where the function is analytic (e.g. to avoid zeros), we can not use the standard regularisation techniques of deconvolution problems consisting of cutting off the estimates outside bounded intervals on the real line, see [Joh09]. Instead, we are going to employ a Tikhonov-type regularisation. More specifically, for some $a \ge 0$, choose a sequence of entire functions $\rho_{N,T}$ such that for $z \in \mathscr{L}_{\pm a}$, $\left|\mathscr{F}\left(\Pi_{N,T}\right)(z) + \rho_{N,T}(z)\right| \ge \varepsilon_{N,T} > 0$, where $\varepsilon_{N,T} \to 0$ for $N, T \to \infty$. Then, we define the estimator $W'_{N,T}$ via

$$\mathscr{F}\left(W'_{N,T}\right)(z) := -\frac{\mathscr{F}\left(\Psi_{N,T}\right)(z)}{\mathscr{F}\left(\Pi_{N,T}\right)(z) + \rho_{N,T}(z)}.$$

Observe that the right hand side is well-defined since $\Psi_{N,T} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Additionally, for all $z \in \mathscr{L}_{\pm a}$, $|\mathscr{F}(W'_{N,T})(z)| \le \varepsilon_{N,T}^{-1} \left|\mathscr{F}(\Psi_{N,T})(z)\right|$, such that $W'_{N,T}$, as the inverse Fourier transform of the right hand side, is well-defined.

It is natural to draw a comparison between our proposed estimation approach with the one presented in [BPP23], particularly in scenarios where the interacting drift exhibits polynomial tails. Although the overall steps in the estimation process share similarities, our context introduces several novel considerations.

Our estimation procedure can be divided into two parts, depending on whether we are addressing the first case with infinite smoothness, where $\widetilde{V} = 0$, or the less smooth case where $\widetilde{V}$ adheres to condition A2. The model distinction arises from the absence of the potential function $V(x)$ in [BPP23]. Instead, the interaction potential in [BPP23] comprises two components: the potential, encompassing trigonometric and polynomial functions, and the non-parametric component of $W$. Despite this difference, it does not significantly impact the estimation procedure. The parametric component in [BPP23] plays a role similar to the confinement potential in our context, and both are estimated through a contrast function, yielding comparable results in steps 1 and 2 for both cases.

The deviation becomes evident in step 3, where the constraint on exponential tails of $W'$ results in polynomial convergence rates (see Theorem 3.8), a contrast to the logarithmic convergence rates for $\Psi$ in [BPP23] due to polynomial tails of $W'$.

The divergence continues into the fourth and final step, where the estimation procedure takes on entirely different forms. The primary challenge lies in analysing the joint decay of the transforms of $W'$ and $\pi$, introducing the condition $\int_{\mathscr{L}_a} |\frac{\mathscr{F}(W')(z)}{\mathscr{F}(\Pi)(z)}|^2 \,\mathrm{d}z < \infty$. Analysing such a condition proves to be a complex task. Notably, the analysis hinges on studying the zeros of the transform of $\pi$, leading us to use the Laplace transform instead of the Fourier transform. It is worth noting that selecting $a = 0$ in $\mathscr{L}_a$ allows obtaining the Fourier transform from the Laplace transform. A comprehensive explanation regarding the fulfillment of the mentioned constraint can be found in Section 3.4.

### 3.2.2 Main Results: Convergence Rates

Let us start with the first step, which consists in the estimation of $l$. We remark again that the order of the kernel can be chosen arbitrarily in the case of $\widetilde{V} = 0$, whereas for $\widetilde{V}$ as in Assumption 7, A2, we require the condition $m \leq J$. In the sequel the bandwidths $h_0$ and $h_1$ for $\pi_{N,T}$ and $\pi'_{N,T}$, respectively, are chosen as

$$h_0 := N_T^{-\frac{1}{2(m+1)}} \quad \text{and} \quad h_1 := N_T^{-\frac{1}{2(m+2)}}.$$

Finally, the threshold parameter is chosen as

$$\delta := \frac{c_1}{2} \exp(-\widetilde{C}U^2),$$

where $c_1$, $\widetilde{C}$ are the constants appearing in Lemma 3.2. We remark that the definitions for $h_0$ and $h_1$ are the same as in [BPP23] while the choice of $\delta$ is due to our lower bound on $\pi$ as presented in Lemma 3.2.

**Proposition 3.7.** *Let $h_0$, $h_1$ and $\delta$ be as above and let $U \geq 1$. Assume that Assumption 7 holds. Then it is*

$$\sup_{|x| \leq U} \mathbb{E}\left[|l_{N,T}(x) - l(x)|^2\right]^{\frac{1}{2}} \lesssim \exp(\widetilde{C}U^2)\left(N_T^{-\frac{m}{2(m+2)}} + UN_T^{-\frac{m}{2(m+1)}}\right).$$

We now proceed to estimate $\alpha$ in step 2, employing the estimator $\alpha_{N,T}$. In our current context, this step is less challenging than it was in the previous work cited as [BPP23], thanks to the specific model under consideration. In particular, the estimation of $\alpha$ essentially involves simplifying step 2 from [BPP23] to the scalar case, where an additional potential $\widetilde{V}$ has been introduced. However, this potential is already known, and it can be chosen in a way such that it does not contribute to the convergence rate at this stage of the analysis.

**Theorem 3.8.** *Let $U \geq 1$ and recall that $m$ is the order of the kernel $K$. If Assumption 7 holds then, for any $\epsilon \in (0,1)$,*

$$\mathbb{E}\left[\int_{\mathbb{R}} |\Psi_{N,T}(y) - \Psi(y)|^2 \,\mathrm{d}y\right]^{\frac{1}{2}} \lesssim \exp(\widetilde{C}U^2)U^{\frac{1}{2}}\left(N_T^{-\frac{m}{2(m+2)}} + UN_T^{-\frac{m}{2(m+1)}}\right) + \frac{2\exp(-C_V(\epsilon\frac{U}{2})^2)}{\epsilon U} + \left(\int_{|y|>\frac{\epsilon U}{2}} |W'(y)|^2 \,\mathrm{d}y\right)^{\frac{1}{2}}. \tag{3.7}$$

The dependence of the convergence rate for $\Psi_{N,T}$ on the tail behaviour of the function $W'$ is evident in Theorem 3.8. In particular, when the tails of $W'$ exhibit exponential decay, the subsequent corollary, which directly follows from the previous result, provides a precise bound.

**Corollary 3.9.** *In the setting of the previous theorem, let $p > C_V$ and assume that*

$$\limsup_{x\to\infty} \exp(2px^2) \int_{|y|>x} |W'(y)|^2 \mathrm{d}y < \infty. \tag{3.8}$$

*Then, for any $\epsilon \in (0,1)$, choosing $U^2 = c_u \log(N_T)$, where*

$$c_u = \frac{m}{2(m+2)} \frac{1}{(\widetilde{C} + C_V \frac{\epsilon^2}{4})}, \tag{3.9}$$

*gives*

$$\mathbb{E}\left[\int_{\mathbb{R}} |\Psi_{N,T}(y) - \Psi(y)|^2 \,\mathrm{d}y\right]^{\frac{1}{2}} \lesssim (\log N_T)^{\frac{1}{4}} N_T^{-\gamma}, \tag{3.10}$$

*where*

$$\gamma = \frac{m}{2(m+2)} \frac{1}{1 + \frac{4\widetilde{C}}{\epsilon^2 C_V}}. \tag{3.11}$$

**Remark 3.10.** One might question why the convergence rate above depends on the auxiliary parameter $\epsilon \in (0,1)$ introduced in the second step of our estimation procedure. By following the proof of Theorem 3.8, it is easy to verify that when the value of $\alpha$ is known and does not require pre-estimation, the results still hold with $\epsilon = 1$. However, when estimating the parameter $\alpha$, we lose the option of setting $\epsilon = 1$ and can only use $\epsilon \in (0,1)$, which slightly affects our convergence rate. The optimal choice is to take $\epsilon$ as close to 1 as possible, resulting in

$$\gamma = \frac{m}{2(m+2)} \frac{1}{1 + \frac{4\widetilde{C}}{C_V}} - \tilde{\epsilon}$$

for any arbitrarily small $\tilde{\epsilon} > 0$.

The estimation of $\Psi$ leads us to the estimation of $W'$ as stated in the following theorem. In this context, an assumption regarding the decay of the transforms of $\pi$ and $W'$ will be crucial.

**Assumption 8.** Recall that $\mathscr{L}_a = \{y + ia : y \in \mathbb{R}\}$ for some $a \geq 0$. There exists an $\overline{a} \geq 0$ such that

$$\int_{\mathscr{L}_{\pm\overline{a}}} \left| \frac{\mathscr{F}\left(W'\right)(z)}{\mathscr{F}(\Pi)(z)} \right|^2 \mathrm{d}z < \infty. \tag{3.12}$$

We will further analyse this assumption in Section 3.4. Therein, we will employ tools from complex analysis to establish a lower bound on $\mathscr{F}(\Pi)$, allowing us to verify that Assumption 8 holds true in specific situations. Additionally, we will conclude said section by providing an example, illustrating how a non-smooth confinement potential implies polynomial decay of the transform of the invariant density. This example assists us in verifying the validity of Assumption 8 as mentioned earlier.

In the sequel, $\varepsilon_{N,T}$ is chosen as

$$\varepsilon_{N,T} = \exp\left(\frac{a\epsilon}{2}(c_u \log N_T)^{\frac{1}{2}}\right)(\log N_T)^{\frac{1}{4}} N_T^{-\frac{\gamma}{2}}.$$

From a quick look at Theorem 3.11, it is easy to see that this constitutes the final convergence rate for the estimation of $W'$.

**Theorem 3.11.** *Suppose that Assumption 7 and Assumption 8 hold true for some $a \geq 0$. Then*

$$\mathbb{E}\left[\int_{\mathbb{R}} |W'_{N,T}(y) - W'(y)|^2 \,\mathrm{d}y\right]^{\frac{1}{2}} \leq c \exp\left(\frac{a\epsilon}{2}(c_u \log N_T)^{\frac{1}{2}}\right)(\log N_T)^{\frac{1}{4}} N_T^{-\frac{\gamma}{2}},$$

*where $c_u$ is as in (3.9) and $\gamma$ as in (3.11).*

**Remark 3.12.** Observe that when Assumption 8 holds for $a = 0$, the convergence rate found in Theorem 3.11 is $(\log N_T)^{\frac{1}{4}} N_T^{-\frac{\gamma}{2}}$. Thanks to Hadamard's representation theorem, we can obtain such a result if $\mathscr{F}(\Pi)$ does not have zeros on the real line, see Theorem 3.19 below. In Theorem 3.11, we consider a more general case, where $\mathscr{F}(\Pi)$ is allowed to have real zeros. The crucial condition is indeed that there exists at least one line in $\mathbb{C}$ that is parallel to the real axis on which the function does not have any zeros. Let us stress that this condition is much weaker than requiring no zeros of $\mathscr{F}(\Pi)$ on the real axis since there is a large class of densities with Fourier transforms vanishing only on the real line, see, e.g. [Bru50]. We refer to Section 3.4 for more details.

**Remark 3.13.** Even when Assumption 8 holds for $a \neq 0$, the observed convergence rate, as determined in the previous theorem, remains polynomial. This outcome arises from the dominance of the polynomial term over the exponential term in the presented upper bound.

**Remark 3.14.** It is important to highlight that the convergence rate outlined in Theorem 3.11 is contingent on the smoothness of the confinement potential $\widetilde{V}$. Specifically, in the scenario of smooth potentials, we have the flexibility to choose the order of the kernel $m$ to be arbitrarily large. Additionally, it is worth noting that the convergence rate $\gamma$ is $1/(2(1+4\widetilde{C}/C_V))$ at best, which we would obtain if the choice of $\epsilon = 1$ were admissable. In contrast, when dealing with a confinement potential of smoothness $J$, we encounter the constraint $m \leq J$. In this case, the optimal choice is to set $m = J$, leading to $\gamma$ being close to $J/(2(J+2)(1+4\widetilde{C}/C_V))$.

## 3.3 Proof of the Main Results

Let us introduce some notation and properties of the Laplace transform, as defined in (3.6), that will be useful in the following section. The inverse Laplace transform is given by the following formula

$$\phi(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \widehat{\phi}(z)\exp(zt)\mathrm{d}z = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\phi}(x+iy)\exp((x+iy)t)\,\mathrm{d}y,$$

with $x \in (x_\phi^-, x_\phi^+)$. The bilateral Laplace transform is unique in the sense that if $\widehat{\phi}_1$ and $\widehat{\phi}_2$ are such that $\widehat{\phi}_1(z) = \widehat{\phi}_2(z)$ in a common strip of convergence $\mathrm{Re}(z) \in (x_{\phi_1}^-, x_{\phi_1}^+) \cap (x_{\phi_2}^-, x_{\phi_2}^+)$, then $\phi_1(t) = \phi_2(t)$ for almost all $t \in \mathbb{R}$ (see [Wid41, Theorem 6b]).

Moreover, the following identities will prove to be useful. Let $\phi$ be a locally integrable function and $a > 0$. By Parseval's theorem, (see [Doe74, Theorem 31.7])

$$\begin{aligned}\|\phi\|_2^2 &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \overline{\mathscr{F}(\phi)(-i\overline{s})}\mathscr{F}(\phi)(is)\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathscr{F}(\phi)(-ia-y)}\mathscr{F}(\phi)(ia-y)\,\mathrm{d}y.\end{aligned}$$

Then,

$$\|\phi\|_2^2 \lesssim \int_{-\infty}^{\infty} |\overline{\mathscr{F}(\phi)(-ia-y)}\mathscr{F}(\phi)(ia-y)|\,\mathrm{d}y \lesssim \Delta_- + \Delta_+,$$

where

$$\Delta_- = \int_{-\infty}^{\infty} |\overline{\mathscr{F}(\phi)(-ia-y)}|^2 \,\mathrm{d}y = \int_{\mathscr{L}_{-a}} |\mathscr{F}(\phi)(z)|^2 \,\mathrm{d}z,$$
$$\Delta_+ = \int_{-\infty}^{\infty} |\mathscr{F}(\phi)(ia-y)|^2 \,\mathrm{d}y = \int_{\mathscr{L}_a} |\mathscr{F}(\phi)(z)|^2 \,\mathrm{d}z,$$

and $\mathscr{L}_a = \{y + ia : y \in \mathbb{R}\}$.

### 3.3.1 Proof of Theorem 3.4

As it will be helpful for the proof of Theorem 3.4, we now state a lemma establishing that finite exponential moments of $\overline{X}_0^1 = X_0^{1,N} = X_0^i$ imply finite exponential moments of $\overline{X}_t^1$ and $X_t^{1,N}$ for all $t \geq 0$. Its proof can be found in Section 3.5.3. Remark that the sub-Gaussianity implies the required finite exponential moments (3.2). Indeed, a centered random variable $Z$ is sub-Gaussian if there exists a $K > 0$ such that $\mathbb{E}[\exp(tZ)] \leq \exp(K^2t^2/2)$ for all $t \in \mathbb{R}$. For sufficient conditions ensuring sub-Gaussianity see [DH22, Definition 30] and references therein.

**Lemma 3.15.** *Under Assumption 7, we have*

$$\sup_{t\geq 0} \mathbb{E}\left[\exp(\pm c\overline{X}_t) + \exp(\pm cX_t^{i,N})\right] < \infty$$

*uniformly in $i$ and $N$. Additionally, for any $k \geq 1$,*

$$\sup_{t\geq 0} \mathbb{E}\left[|\overline{X}_t|^k + |X_t^{i,N}|^k\right] < \infty \tag{3.13}$$

*uniformly in $i$ and $N$.*

We start proving Theorem 3.4 by introducing some notation. Let $\overline{\Pi}_t$ be the law of $\overline{X}_t$, $\Pi$ the law with density $\pi$ and $\Pi_T^N = \Pi_{N,T}$ with

$$\mathscr{F}(\Pi_T^N)(z) = \frac{1}{N}\sum_{j=1}^N \exp(izX_T^{j,N}), \qquad \mathscr{F}(\overline{\Pi}_T^N)(z) = \frac{1}{N}\sum_{j=1}^N \exp(iz\overline{X}_T^j),$$
$$\mathscr{F}(\overline{\Pi}_T)(z) = \mathbb{E}[\exp(iz\overline{X}_T)], \qquad \mathscr{F}(\Pi)(z) = \mathscr{F}(\pi)(z) = \int_{-\infty}^{\infty} \exp(izx)\pi(x)\mathrm{d}x.$$

In order to prove the result we now consider the following decomposition:

$$\mathbb{E}[|\mathscr{F}(\Pi)(z) - \mathscr{F}(\Pi_T^N)(z)|^2] \lesssim \sum_{i=1}^3 \delta_i(z),$$

where

$$\delta_1(z) = |\mathscr{F}(\Pi)(z) - \mathscr{F}(\overline{\Pi}_T)(z)|^2, \qquad \delta_2(z) = \mathbb{E}\left[|\mathscr{F}(\overline{\Pi}_T)(z) - \mathscr{F}(\overline{\Pi}_T^N)(z)|^2\right],$$

and

$$\begin{aligned}
\delta_3(z) &= \mathbb{E}\left[|\mathscr{F}(\overline{\Pi}_T^N)(z) - \mathscr{F}(\Pi_T^N)(z)|^2\right] \\
&\leq \left(\frac{1}{N}\sum_{j=1}^N \mathbb{E}\left[|\exp(iz\overline{X}_T^j) - \exp(izX_T^{j,N})|^2\right]^{\frac{1}{2}}\right)^2 \\
&= \mathbb{E}\left[|\exp(iz\overline{X}_T^1) - \exp(izX_T^{1,N})|^2\right].
\end{aligned}$$

For $x = x_1 + ix_2$ and $y = y_1 + iy_2$, we have that

$$\begin{aligned}|\exp(x+y)-\exp(y)| &= |\exp(x)-1|\exp(y_1)\\ &\le (|\exp(ix_2)-1|\exp(x_1)+|\exp(x_1)-1|)\exp(y_1)\\ &\le (|x_2|\exp(x_1)+|x_1|(\exp(x_1)+1))\exp(y_1)\\ &\le \sqrt{2}|x|(\exp(x_1+y_1)+\exp(y_1)) \qquad (3.14)\end{aligned}$$

which implies, for any $z \in \mathscr{L}_a$,

$$\delta_3(z) \le 2|z|^2\mathbb{E}\left[|\overline{X}_T^1 - X_T^{1,N}|^2(\exp(-a\overline{X}_T^1)+\exp(-aX_T^{1,N}))^2\right].$$

Using the Cauchy-Schwarz inequality, we get

$$\delta_3(z) \lesssim |z|^2\mathbb{E}\left[|\overline{X}_T^1 - X_T^{1,N}|^4\right]^{\frac{1}{2}},$$

because the exponential moments of $\overline{X}_T^1$ and $X_T^{1,N}$ are uniformly bounded in $T, N$ according to Lemma 3.15. By Proposition 3.6,

$$\mathbb{E}\left[|\overline{X}_T^1 - X_T^{1,N}|^4\right] \lesssim \frac{1}{N^2},$$

which yields

$$\delta_3(z) \lesssim \frac{|z|^2}{N}. \qquad (3.15)$$

Next, consider $\delta_2(z)$. Since $\overline{\Pi}_T^N$ is based on i.i.d. $\overline{X}_T^1, \dots, \overline{X}_T^N$ with common law $\overline{\Pi}_T$, we have that

$$\delta_2(z) = \frac{1}{N}\mathbb{E}\left[|\mathbb{E}[\exp(iz\overline{X}_T^1)] - \exp(iz\overline{X}_T^1)|^2\right] \lesssim \frac{1}{N}, \qquad (3.16)$$

where exponential moments of $\overline{X}_T^1$ are uniformly bounded in $T$ by Lemma 3.15. Finally, let us deal with $\delta_1(z)$. We introduce a random vector $(\overline{X}_T, X)$ with $\mathrm{Law}(\overline{X}_T) = \overline{\Pi}_T$, $\mathrm{Law}(X) = \Pi$, and $\mathbb{E}[|\overline{X}_T - X|^2] = W_2^2(\overline{\Pi}_T, \Pi)$. Then, using again (3.14), it is

$$\begin{aligned}\delta_1(z)^{\frac{1}{2}} &= |\mathbb{E}[\exp(iz\overline{X}_T)] - \mathbb{E}[\exp(izX)]|\\ &\le \mathbb{E}[|\exp(iz\overline{X}_T) - \exp(izX)|]\\ &\le \sqrt{2}|z|\mathbb{E}[|\overline{X}_T - X|(\exp(-a\overline{X}_T)+\exp(-aX))].\end{aligned}$$

Using the Cauchy-Schwarz inequality we obtain

$$\mathbb{E}[|\overline{X}_T - \overline{X}|(\exp(-a\overline{X}_T)+\exp(-aX))] \lesssim W_2(\overline{\Pi}_T, \Pi),$$

since the moments and the exponential moments of $\overline{X}_T$, $X$ are uniformly bouned in $T$ by Lemma 3.15 and the fact that $\mathrm{Law}(X) = \Pi$ is the invariant law of the McKean-Vlasov SDE. [Mal03, Theorem 1.4] ensures that there exist two constants $c$ and $\lambda := C_V - C_W > 0$ such that

$$W_2(\overline{\Pi}_T, \Pi) \le c\exp(-\lambda T).$$

It yields that

$$\delta_1(z) \lesssim |z|^2\exp(-2\lambda T). \qquad (3.17)$$

From (3.15), (3.16) and (3.17) we get, for any $z \in \mathscr{L}_a$,

$$\mathbb{E}[|\mathscr{F}(\Pi)(z) - \mathscr{F}(\Pi_T^N)(z)|^2] \lesssim |z|^2\left(\frac{1}{N}+\exp(-\lambda T)\right) = \frac{|z|^2}{N_T}.$$

Our reasoning easily extends to $z \in \mathscr{L}_{-a}$, thereby concluding the proof of the theorem.

### 3.3.2 Proof of Proposition 3.7

The proof of Proposition 3.7 follows closely the proof of [BPP23, Theorem 4.1]. Recall that $l_{N,T}$ is a kernel estimator of

$$l(x) = \frac{\pi'(x)}{\pi(x)} = -2V'(x) - W' * \pi(x). \tag{3.18}$$

Let us decompose its error into the sum

$$|l_{N,T}(x) - l(x)| = |l(x)| 1_{\{\pi_{N,T}(x) \le \delta\}} + r(x) \tag{3.19}$$

where

$$\begin{aligned} r(x) &:= |l_{N,T}(x) - l(x)| 1_{\{\pi_{N,T}(x) > \delta\}} \\ &\le \delta^{-1} \Big( |\pi'_{N,T}(x) - \pi'(x)| + |l(x)||\pi_{N,T}(x) - \pi(x)| \Big). \end{aligned} \tag{3.20}$$

Under our assumptions, we have that $|V'(x)| \le c(1+|x|+|\widetilde{V}'(x)|) \le c(1+|x|)$, where we have used that $\widetilde{V}'(0) = 0$ and $\widetilde{V}''$ is bounded. Moreover, $|W' * \pi(x)| \le \|W'\|_\infty \|\pi\|_1 = \|W'\|_\infty < \infty$, which imply

$$\sup_{|x| \le U} |l(x)| \lesssim U.$$

Next, Lemma 3.2 gives the lower bound $\pi(x) \ge c_1 \exp(-\widetilde{C}|x|^2)$, which in turn implies $\pi(x) \ge 2\delta$ for all $|x| \le U$. It follows that for all $|x| \le U$,

$$\begin{aligned} \mathbb{P}(\pi_{N,T}(x) \le \delta) &= \mathbb{P}(\pi(x) - \pi_{N,T}(x) \ge \pi(x) - \delta) \\ &\le \mathbb{P}(\|\pi - \pi_{N,T}\|_\infty \ge \delta) \le \delta^{-2} \mathbb{E}\big[\|\pi - \pi_{N,T}\|_\infty^2\big]. \end{aligned}$$

Finally, consider

$$\pi_{N,T}(x) - \pi(x) = r_0(x) + r_1(x)$$

with

$$r_0(x) = K_{h_0} * (\Pi_{N,T}(x) - \Pi)(x), \quad r_1(x) = K_{h_0} * \Pi(x) - \pi(x),$$

where recall $K_{h_0}(x) = h_0^{-1} K(h_0^{-1} x)$ is a scaled kernel. We get

$$|r_0(x)| \le c h_0^{-1} W_1(\Pi_{N,T}, \Pi)$$

by applying the Kantorovich-Rubinstein theorem, moreover, $\mathbb{E}[W_1^2(\Pi_{N,T}, \Pi)] \le c N_T^{-1}$ because of Theorem 3.3. After substitution, we have that

$$r_1(x) = \int_{\mathbb{R}} (\pi(x + h_0 y) - \pi(x)) K(y) \, \mathrm{d}y,$$

where by the Taylor theorem

$$\pi(x + y h_0) = \pi(x) + \sum_{i=1}^{m-1} \frac{\pi^{(k)}(x)}{k!} (y h_0)^k + \frac{\pi^{(m)}(x + \tau y h_0)}{m!} (y h_0)^m$$

for some $0 \le \tau \le 1$. Recall that $K$ has order $m$. Moreover, the bound in Lemma 3.2 ensures that $|\pi^{(m)}(x)| \le c$, hence,

$$|r_1(x)| \le c h_0^m$$

uniformly in $x \in \mathbb{R}$. Our choice of $h_0$ yields

$$\mathbb{E}\left[\|\pi_{N,T} - \pi\|_\infty^2\right] \lesssim N_T^{-\frac{m}{m+1}}$$

and similarly, that of $h_1$ yields

$$\mathbb{E}\left[\|\pi'_{N,T} - \pi'\|_\infty^2\right] \lesssim N_T^{-\frac{m}{m+2}}.$$

Using these bounds in (3.19), (3.20), we obtain

$$\sup_{|x|\le U} \mathbb{E}\left[|l_{N,T}(x) - l(x)|^2\right]^{\frac{1}{2}} \lesssim \exp(\widetilde{C}U^2)\left(N_T^{-\frac{m}{2(m+2)}} + UN_T^{-\frac{m}{2(m+1)}}\right),$$

which concludes the proof.

### 3.3.3 Proof of Theorem 3.8

Recall that

$$\Psi_{N,T}(x) = \left(l_{N,T}(x) + 2\alpha_{N,T}x + 2\widetilde{V}'(x)\right)1_{\{|x|\le \epsilon U\}},$$
$$\Psi(x) = l(x) + 2\alpha x + 2\widetilde{V}'(x) = -W' * \pi(x).$$

We decompose the mean integrated squared error of $\Psi_{N,T}$ into

$$\mathbb{E}\left[\int_{\mathbb{R}} \left(\Psi_{N,T}(x) - \Psi(x)\right)^2 \mathrm{d}x\right]^{\frac{1}{2}} = \left(\int_{|x|>\epsilon U} (\Psi(x))^2 \mathrm{d}x\right)^{\frac{1}{2}} + I, \tag{3.21}$$

where

$$I := \left(\int_{|x|\le \epsilon U} \mathbb{E}\left[(\Psi_{N,T}(x) - \Psi(x))^2\right] \mathrm{d}x\right)^{\frac{1}{2}}.$$

Applying Minkowski's inequality, we get $I \le I_1 + I_2$, where

$$I_1 := \left(\int_{|x|\le \epsilon U} \mathbb{E}\left[(l_{N,T}(x) - l(x))^2\right] \mathrm{d}x\right)^{\frac{1}{2}} \lesssim U^{\frac{1}{2}} \sup_{|x|\le U} \mathbb{E}\left[(l_{N,T}(x) - l(x))^2\right]^{\frac{1}{2}},$$
$$I_2 := \left(\int_{|x|\le \epsilon U} \mathbb{E}\left[(2\alpha_{N,T}x - 2\alpha x)^2\right] \mathrm{d}x\right)^{\frac{1}{2}} \lesssim U^{\frac{3}{2}} \mathbb{E}\left[(\alpha_{N,T} - \alpha)^2\right]^{\frac{1}{2}}.$$

Next, consider the mean squared error of

$$\alpha_{N,T} := \operatorname*{argmin}_{\alpha\in\mathbb{R}} \int_{\mathbb{R}} \left(l_{N,T}(x) + 2\alpha x + 2\widetilde{V}'(x)\right)^2 w_U(x)\mathrm{d}x,$$

where $w_U(\cdot) := w(\cdot/U)/U$. The estimator $\alpha_{N,T}$ and $\alpha$ can be computed explicitly via

$$\alpha_{N,T} = -\frac{1}{2C_2U^2} \int_{\mathbb{R}} \left(l_{N,T}(x) + 2\widetilde{V}'(x)\right) x w_U(x)\mathrm{d}x,$$
$$\alpha = -\frac{1}{2C_2U^2} \int_{\mathbb{R}} \left(l(x) + 2\widetilde{V}'(x) + W' * \pi(x)\right) x w_U(x)\mathrm{d}x,$$

where $C_2 := \int_{\mathbb{R}} x^2 w(x)\mathrm{d}x$. Since the support of $w_U$ is $[\epsilon U, U]$, Jensen's inequality can be applied to get

$$(\alpha_{N,T} - \alpha)^2 \le \frac{1}{4C_2^2U^4} \int_{\mathbb{R}} \left(l_{N,T}(x) - l(x) - W' * \pi(x)\right)^2 x^2 w_U(x)\mathrm{d}x.$$

By Minkowski's inequality, we obtain

$$\mathbb{E}\left[(\alpha_{N,T}-\alpha)^2\right]^{\frac{1}{2}} \leq \frac{1}{2C_2U^2}(J_1+J_2),$$

where

$$J_1 := \left(\int_{\mathbb{R}} \mathbb{E}\left[(l_{N,T}(x)-l(x))^2\right]x^2 w_U(x)\mathrm{d}x\right)^{\frac{1}{2}} \leq C_2^{\frac{1}{2}} U \sup_{|x|\leq U} \mathbb{E}\left[(l_{N,T}(x)-l(x))^2\right]^{\frac{1}{2}},$$

$$J_2 := \left(\int_{\mathbb{R}} (W' * \pi(x))^2 x^2 w_U(x)\mathrm{d}x\right)^{\frac{1}{2}} \leq \frac{C_\infty U^{\frac{1}{2}}}{2}\left(\int_{|x|>\epsilon U} (W' * \pi(x))^2 \mathrm{d}x\right)^{\frac{1}{2}},$$

with $C_\infty := \sup_{x\in\mathbb{R}} x^2 w(x)$. Using the above bounds in (3.21) we conclude that

$$\mathbb{E}\left[\int_{\mathbb{R}} (\Psi_{N,T}(x)-\Psi(x))^2\,\mathrm{d}x\right]^{\frac{1}{2}} \lesssim U^{\frac{1}{2}} \sup_{|x|\leq U} \mathbb{E}\left[(l_{N,T}(x)-l(x))^2\right]^{\frac{1}{2}} + \left(\int_{|x|>\epsilon U} (W' * \pi(x))^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

The first term on the right hand side has been studied in Proposition 3.7, and we are therefore left to study the second term. It is

$$\left(\int_{|x|>\epsilon U} (W' * \pi(x))^2\,\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_{|x|>\epsilon U} \left(\int_{\mathbb{R}} W'(x-y)\pi(y)\,\mathrm{d}y\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

By Minkowski's inequality this is bounded by

$$\int_{|y|\leq\frac{\epsilon}{2}U}\left(\int_{|x|>\epsilon U} |W'(x-y)|^2\mathrm{d}x\right)^{\frac{1}{2}} \pi(y)\,\mathrm{d}y + \int_{|y|>\frac{\epsilon}{2}U}\left(\int_{|x|>\epsilon U} |W'(x-y)|^2\mathrm{d}x\right)^{\frac{1}{2}} \pi(y)\,\mathrm{d}y. \tag{3.22}$$

Then, we apply a change of variables $x-y := \tilde{x}$, observing that $|x| > \epsilon U$ and $|y| \leq \epsilon U/2$ imply $|x-y| > \epsilon U/2$. For the second integral we enlarge the domain of integration to $\mathbb{R}$. It follows that (3.22) is upper bounded by

$$\left(\int_{|x|>\frac{\epsilon}{2}U} |W'(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \int_{\mathbb{R}} \pi(y)\,\mathrm{d}y + \left(\int_{|y|>\frac{\epsilon}{2}U} \pi(y)\,\mathrm{d}y\right)\left(\int_{\mathbb{R}} |W'(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Thanks to Lemma 3.2, we know that

$$\int_{\frac{\epsilon}{2}U}^{\infty} \pi(x)\,\mathrm{d}x \lesssim \int_{\frac{\epsilon}{2}U}^{\infty} \overline{\pi}(x)\,\mathrm{d}x,$$

with $\overline{\pi}(x) = c_2\exp(-C_V x^2)$ satisfying $\overline{\pi}'(x) = -2c_2 C_V x\overline{\pi}(x)$. Hence, we can write

$$\int_u^{\infty} \overline{\pi}(x)\,\mathrm{d}x \leq \frac{1}{u}\int_u^{\infty} x\overline{\pi}(x)\,\mathrm{d}x = \frac{\overline{\pi}(u)}{2c_2 C_V u}.$$

It implies that

$$\int_{\epsilon\frac{U}{2}}^{\infty} \pi(x)\,\mathrm{d}x \lesssim \frac{2\exp(-C_V(\frac{\epsilon}{2}U)^2)}{\epsilon U} \tag{3.23}$$

Then, the boundedness of $\int_{\mathbb{R}} \pi(y)\,\mathrm{d}y$ is straightforward, while $\int_{\mathbb{R}} |W'(x)|^2\,\mathrm{d}x$ is bounded as $W' \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. It follows

$$\left(\int_{|x|>\epsilon U} (W' * \pi(x))^2\,\mathrm{d}x\right)^{\frac{1}{2}} \lesssim \left(\int_{|x|>\frac{\epsilon}{2}U} |W'(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \frac{2\exp(-C_V(\frac{\epsilon}{2}U)^2)}{\epsilon U}, \tag{3.24}$$

as we wanted.

### 3.3.4 Proof of Corollary 3.9

Corollary 3.9 is a consequence of Theorem 3.8 and of the exponential decay of the tails of $W'$. The choice of the threshold

$$U^2 = c_u \log(N_T) \tag{3.25}$$

gives

$$\left(\int_{|y|>\frac{\epsilon}{2}U} |W'(y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \lesssim \exp\left(-\frac{p\epsilon^2 U^2}{4}\right) = N_T^{-\frac{p\epsilon^2 c_u}{4}}.$$

Together with Theorem 3.8 it implies that

$$\mathbb{E}\left[\int_{\mathbb{R}} |\Psi_{N,T}(y) - \Psi(y)|^2\,\mathrm{d}y\right]^{\frac{1}{2}} \lesssim (\log N_T)^{\frac{1}{4}} N_T^{\widetilde{C}c_u}\left(N_T^{-\frac{m}{2(m+2)}} + (\log N_T)^{\frac{1}{2}} N_T^{-\frac{m}{2(m+1)}}\right)$$
$$+ N_T^{-C_V\frac{\epsilon^2}{4}c_u} + N_T^{-p\frac{\epsilon^2}{4}c_u}.$$

Recall that $p > C_V$. Then, we can choose $c_u$ in order to obtain the balance between the remaining two terms above:

$$c_u = \frac{m}{2(m+2)}\frac{1}{(\widetilde{C} + C_V\frac{\epsilon^2}{4})}. \tag{3.26}$$

Then, the convergence rate is

$$(\log N_T)^{\frac{1}{4}} N_T^{-\frac{m}{2(m+2)}\frac{C_V\frac{\epsilon^2}{4}}{(\widetilde{C}+C_V\frac{\epsilon^2}{4})}}$$

as claimed.

### 3.3.5 Proof of Theorem 3.11

Let $\mathscr{L}_a := \{y + ia : y \in \mathbb{R}\}$ for some $a \geq 0$. Assume $\mathrm{Re}(\mathscr{F}(\Pi)(s)) > 0$ for all $s \in \mathscr{L}_a$. Recall that we defined a sequence of entire functions $\rho_{N,T}(s) := \mathscr{F}(\Pi)(s) - \mathscr{F}(\Pi_{N,T})(s) + \varepsilon_{N,T}$, $s \in \mathbb{C}$, for some $\varepsilon_{N,T} > 0$. Note that for all $s \in \mathscr{L}_a$

$$|\mathscr{F}(\Pi_{N,T})(s) + \rho_{N,T}(s)| = |\mathscr{F}(\Pi)(s) + \varepsilon_{N,T}| \geq \varepsilon_{N,T} > 0 \tag{3.27}$$

and $W'_{N,T}$ is defined via

$$\mathscr{F}(W'_{N,T})(s) := -\frac{\mathscr{F}(\Psi_{N,T})(s)}{\mathscr{F}(\Pi_{N,T})(s) + \rho_{N,T}(s)}.$$

Let $\delta_{N,T} := W'_{N,T} - W'$. Plancherel's theorem gives

$$\int_{-\infty}^{\infty} |\delta_{N,T}(x)|^2\,\mathrm{d}x = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \overline{\mathscr{F}(\delta_{N,T})(i(-\bar{s}))}\mathscr{F}(\delta_{N,T})(is)\mathrm{d}s$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathscr{F}(\delta_{N,T})(y-ia)}\mathscr{F}(\delta_{N,T})(y+ia)\,\mathrm{d}y \lesssim \Delta_{-a} + \Delta_a, \tag{3.28}$$

where

$$\Delta_{\pm a} := \int_{-\infty}^{\infty} |\mathscr{F}(\delta_{N,T})(y \pm ia)|^2 \,\mathrm{d}y = \int_{\mathscr{L}_{\pm a}} |\mathscr{F}(\delta_{N,T})(s)|^2 \,\mathrm{d}s.$$

It suffices to consider $\Delta_a$, because the analysis of $\Delta_{-a}$ follows a similar route. Rewrite

$$\mathscr{F}(\delta_{N,T})(s) = -\frac{\mathscr{F}(\Psi_{N,T})(s) - \mathscr{F}(\Psi)(s) + \varepsilon_{N,T}\mathscr{F}(W')(s)}{\mathscr{F}(\Pi_{N,T})(s) + \rho_{N,T}(s)}.$$

Let us deal with the denominator by using (3.27) and $\mathrm{Re}(\mathscr{F}(\Pi)(s)) + \varepsilon_{N,T} > \mathrm{Re}(\mathscr{F}(\Pi)(s)) > 0$ for all $s \in \mathscr{L}_a$. We get

$$\Delta_a \lesssim \frac{1}{\varepsilon_{N,T}^2}\Delta_{a,1} + \varepsilon_{N,T}^2 \int_{\mathscr{L}_a} \left|\frac{\mathscr{F}(W')(s)}{\mathscr{F}(\Pi)(s)}\right|^2 \mathrm{d}s, \tag{3.29}$$

where

$$\Delta_{a,1} := \int_{\mathscr{L}_a} |\mathscr{F}(\Psi_{N,T})(s) - \mathscr{F}(\Psi)(s)|^2 \,\mathrm{d}s.$$

$$\mathbb{E}[\Delta_{a,1}]^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} \exp(-2ax)\mathbb{E}\left[|\Psi_{N,T}(x) - \Psi(x)|^2\right] \mathrm{d}x\right)^{\frac{1}{2}} \leq D_1 + D_2, \tag{3.30}$$

where

$$D_1 := \left(\int_{|x|>\epsilon U} \exp(-2ax)|\Psi(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}},$$
$$D_2 := (2\epsilon U)^{\frac{1}{2}} \exp(a\epsilon U) \sup_{|x|\leq \epsilon U} \mathbb{E}\left[|\Psi_{N,T}(x) - \Psi(x)|^2\right]^{\frac{1}{2}}.$$

We start handling $D_1$, while the analysis on $D_2$ heavily relies on the bounds gathered in previous steps. We recall that

$$\Psi(x) = -W' * \pi(x) = -\int_{\mathbb{R}} W'(x-y)\pi(y)\,\mathrm{d}y.$$

By the Minkowski inequality,

$$D_1 \leq \int_{\mathbb{R}} \left(\int_{|x|>\epsilon U} \exp(-2ax)W'(x-y)^2 \mathrm{d}x\right)^{\frac{1}{2}} \pi(y)\,\mathrm{d}y = I_1 + I_2,$$

where after a change of variable the right hand side has been decomposed into

$$I_1 := \int_{|y+c_0|\leq \frac{\epsilon}{2}U} \left(\int_{|x+y|>\epsilon U} \exp(-2ax)W'(x)^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \exp(-ay)\pi(y)\,\mathrm{d}y,$$
$$I_2 := \int_{|y+c_0|> \frac{\epsilon}{2}U} \left(\int_{|x+y|>\epsilon U} \exp(-2ax)W'(x)^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \exp(-ay)\pi(y)\,\mathrm{d}y,$$

with $c_0 := \frac{a}{2C_V}$. Note that $|y + c_0| \leq \frac{\epsilon}{2}U$ and $|x + y| > \epsilon U$ imply $|x - c_0| > \frac{\epsilon}{2}U$ in the inner integral in $I_1$. Let us also enlarge the domain of integration to $\mathbb{R}$ in the outer and inner integrals in $I_1$ and $I_2$ respectively. Then

$$I_1 + I_2 \leq J_1 \int_{\mathbb{R}} \exp(-ay)\pi(y)\,\mathrm{d}y + J_2 \left(\int_{\mathbb{R}} \exp(-2ax)W'(x)^2 \,\mathrm{d}x\right)^{\frac{1}{2}}, \tag{3.31}$$

where

$$J_1 := \left( \int_{|x-c_0|>\frac{\epsilon}{2}U} \exp(-2ax) W'(x)^2 \, \mathrm{d}x \right)^{\frac{1}{2}}, \qquad J_2 := \int_{|y+c_0|>\frac{\epsilon}{2}U} \exp(-ay)\pi(y) \, \mathrm{d}y.$$

The upper bound on $\pi$ in Lemma 3.2 implies that the first integral on the right hand side of (3.31) is finite, furthermore,

$$\begin{aligned} J_2 &\lesssim \int_{|y+c_0|>\frac{\epsilon}{2}U} \exp(-ay - C_V y^2) \, \mathrm{d}y \\ &\approx \int_{|y|>\frac{\epsilon}{2}U} \exp(-C_V y^2) \, \mathrm{d}y \lesssim \frac{\exp(-C_V(\frac{\epsilon}{2}U)^2)}{\frac{\epsilon}{2}U}. \end{aligned}$$

Now consider $J_1$ and the second integral on the right hand side of (3.31), where $W' \in L^2(\mathbb{R})$ is odd and satisfies the assumption (3.8) for $p > C_V$. This means that there exists a $c > 0$ such that

$$\int_x^\infty W'(u)^2 \, \mathrm{d}u \leq c \exp(-2px^2)$$

for all $x \geq 1$, that is

$$\int_1^\infty \chi(u) W'(u)^2 \, \mathrm{d}u \leq c \int_1^\infty \chi(u) \, \mathrm{d}F(u),$$

where $F(u) := 1 - \exp(-2pu^2)$ and $\chi(u) := 1(u \geq x)$, $u \in \mathbb{R}$, for all $x \geq 1$. By monotone approximation the above inequality remains valid for all non-negative non-decreasing functions $\chi : [1, \infty) \to [0, \infty)$, for example, $\chi(u) = \exp(2au)$. We get that the second integral of $W' \in L^2(\mathbb{R})$ on the right hand side of (3.31) is finite. Moreover,

$$\begin{aligned} (J_1)^2 &\lesssim \int_{\frac{\epsilon}{2}U-c_0}^\infty \exp(2ax) W'(x)^2 \, \mathrm{d}x \\ &\lesssim \int_{\frac{\epsilon}{2}U-c_0}^\infty \exp(2ax) \, \mathrm{d}F(x) = \int_{\frac{\epsilon}{2}U-c_0}^\infty \exp(2ax - 2px^2)(4px) \, \mathrm{d}x \\ &\lesssim \exp\left(-2p\left(\frac{\epsilon}{2}U - c_1\right)^2\right), \end{aligned}$$

where $c_1 = c_0 + \frac{a}{2p}$. Since $p > C_V$, it follows that

$$(J_1)^2 \lesssim \frac{\exp(-2C_V(\frac{\epsilon}{2}U)^2)}{(\frac{\epsilon}{2}U)^2}.$$

We conclude that

$$D_1 \lesssim \frac{\exp(-C_V(\frac{\epsilon}{2}U)^2)}{\frac{\epsilon}{2}U}. \tag{3.32}$$

Regarding $D_2$, we recall that from the definition of $\Psi_{N,T}(y)$ and $\Psi(y)$ we deduce

$$D_2 \lesssim (2\epsilon U)^{\frac{1}{2}} \exp(a\epsilon U)(R_1 + R_2),$$

where

$$R_2 := \sup_{|y|\leq U} \mathbb{E}\left[|l_{N,T}(y) - l(y)|^2\right]^{\frac{1}{2}},$$

$$R_1 := \epsilon U \mathbb{E}\left[\left(\alpha_{N,T} - \alpha\right)^2\right]^{\frac{1}{2}} \lesssim R_2 + U^{-\frac{1}{2}} \left( \int_{|x|>\epsilon U} \Psi(x)^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

The integral in the upper bound on $R_1$ coincides with $D_1$ when $a = 0$ hence satisfies (3.32), whereas the upper bound on $R_2$ follows from Proposition 3.7. We get the upper bound

$$D_2 \lesssim \exp(a\epsilon U)\left((\epsilon U)^{\frac{1}{2}} \exp(\widetilde{C}U^2)\left(N_T^{-\frac{m}{2(m+2)}} + UN_T^{-\frac{m}{2(m+1)}}\right) + \frac{\exp(-C_V(\frac{\epsilon}{2}U)^2)}{\frac{\epsilon}{2}U}\right),$$

which also works for $\mathbb{E}[\Delta_{a,1}]^{\frac{1}{2}}$. The choice $U^2 = c_u \log N_T$ for $c_u$ as in (3.26) gives us

$$\mathbb{E}[\Delta_{a,1}]^{\frac{1}{2}} \lesssim \exp\left(a\epsilon(c_u \log N_T)^{\frac{1}{2}}\right)(\log N_T)^{\frac{1}{4}} N_T^{-\gamma} =: \lambda_{N,T} \tag{3.33}$$

for $\gamma$ as in (3.11). Finally, from (3.28), (3.29) and (3.33) it follows that

$$\mathbb{E}\left[\int_{-\infty}^{\infty} \left|\delta_{N,T}(y)\right|^2 \mathrm{d}y\right]^{\frac{1}{2}} \lesssim \frac{\lambda_{N,T}}{\varepsilon_{N,T}} + \varepsilon_{N,T},$$

where note that $\varepsilon_{N,T}$ can be chosen such that $\varepsilon_{N,T} := \lambda_{N,T}^{\frac{1}{2}} \to 0$ for $N, T \to \infty$. It yields

$$\mathbb{E}\left[\int_{-\infty}^{\infty} \left|\delta_{N,T}(y)\right|^2 \mathrm{d}y\right]^{\frac{1}{2}} \lesssim \lambda_{N,T}^{\frac{1}{2}}$$

as required.

## 3.4 On the Fourier Transform of the Invariant Measure

We have observed that Assumption 8 is crucial in order to obtain the polynomial convergence rate stated in Theorem 3.11. However, this assumption may appear somewhat unclear at first. The objective of this section is to investigate sufficient conditions that guarantee the validity of Assumption 8. Given our focus on the super-smooth case, it seems natural to require that the transform of the function $W'$ we aim to estimate decays exponentially fast. Verifying the condition (3.12) entails seeking a lower bound for the transform of $\pi$, which will be the main objective of this section. We will thus begin by examining the properties of the Fourier transform of $\pi$, with the intention of studying the set of zeros and finding a lower bound outside that set. Recall that the Fourier transform of the density is (by definition) the transform of its associated measure, that is

$$\mathscr{F}(\Pi) = \mathscr{F}(\pi) = \int_{\mathbb{R}} \exp(izx)\pi(x)\,\mathrm{d}x.$$

We will use tools from complex analysis to study $\mathscr{F}(\pi)$ as a function defined on $\mathbb{C}$. More specifically, our aim is to represent the entire function $\mathscr{F}(\pi)$ via Hadamard's factorisation theorem. For definitions of the terminology used in this work, we refer to [Hol73]. We begin by deriving an upper bound of the order of $\mathscr{F}(\pi)$, which stems from the fact that $\pi$ has Gaussian tails.

**Theorem 3.16.** *Let Assumption 7 hold and $\pi$ be as in (3.5). The map*

$$\mathscr{F}(\pi)\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto \int_{\mathbb{R}} \exp(izx)\pi(x)\mathrm{d}x \tag{3.34}$$

*is an entire function which coincides with the characteristic function of $\pi$ on $\mathbb{R}$. Moreover, the order of $\mathscr{F}(\pi)$ does not exceed 2.*

*Proof.* The integral in (3.34) exists and defines a continuous function at any $z = a + ib$ with $a, b \in \mathbb{R}$ since $|\mathscr{F}(\pi)(a+ib)| \leq \int_{\mathbb{R}} \exp(-bx)\pi(x)\mathrm{d}x < \infty$ by using $\pi(x) \lesssim \exp(-C_V x^2)$ and $\|W' * \pi\|_\infty \leq \|W'\|_\infty < \infty$.

Furthermore, we have $\int_C \mathscr{F}(\pi)(z)\,\mathrm{d}z = 0$ for any closed contour $C$ which follows from the Cauchy theorem applied to the function $z \mapsto \exp(izx)$. By Morera's theorem, $\mathscr{F}(\pi)$ is an entire function. To determine the order of $\mathscr{F}(\pi)$, we use the inequality

$$|zx| \leq \frac{1}{2}\left(\frac{1}{c}|z|^2 + c\,|x|^2\right) \quad \forall z, x \in \mathbb{C},\ c > 0,$$

such that, for any $z \in \mathbb{C}$,

$$\begin{aligned}|\mathscr{F}(\pi)(z)| &\leq \int_{\mathbb{R}} \exp(|\operatorname{Im}(z)x|)\pi(x)\,\mathrm{d}x \\ &\leq \exp\left(\frac{\operatorname{Im}(z)^2}{2c}\right)\int_{\mathbb{R}} \exp\left(\frac{cx^2}{2}\right)\pi(x)\,\mathrm{d}x.\end{aligned}$$

Recall that according to Lemma 3.2, $\pi(x) \lesssim \exp(-C_V x^2)$. Hence, the above integral is finite if $x \mapsto \exp(-(C_V - c/2)x^2)$ is integrable on $\mathbb{R}$, which is the case for $C_V > c/2$. As a result, the order of $\mathscr{F}(\pi)$ does not exceed 2. □

Now that we have that $\mathscr{F}(\pi)$ is an entire function of finite order, we will find an expression for the function using Hadamard's factorisation theorem. For this, we denote the zeros of $\mathscr{F}(\pi)$ as $(a_j)_{j\in\mathbb{N}}$ and order them by increasing modulus. Note that the zeros are symmetric around the imaginary axis, in other words, if $a_j$ is a zero of $\mathscr{F}(\pi)$, then so is its negative conjugate $-\overline{a}_j$. Moreover, $\mathscr{F}(\pi)$ has no zeros on the imaginary axis because $\mathscr{F}(\pi)(ia) > 0$ for all $a \in \mathbb{R}$, see [LO77, Corollary 1 to Theorem 2.3.2]. Let us introduce the critical exponent of convergence $\rho_1$ of the sequence $(a_j)_{j\in\mathbb{N}}$:

$$\rho_1 = \inf\left\{r > 0 : \sum_{j\in\mathbb{N}} \frac{1}{|a_j|^r} < \infty\right\}.$$

We denote the order of $\mathscr{F}(\pi)$ as $\rho$ and make the following

**Assumption 9.** $\mathscr{F}(\pi)$ satisfies either $\rho_1 < \rho$ or $\rho_1 = \rho < 2$.

We will use the Hadamard canonical factors

$$E_d(z) = \begin{cases} 1 - z, & d = 0, \\ (1-z)\exp(z), & d = 1, \end{cases}$$

defined for $z \in \mathbb{C}$, to study the infinite product representation of $\mathscr{F}(\pi)$.

**Theorem 3.17.** *Suppose that Assumption 7 and Assumption 9 hold and let $\pi$ be as in (3.5) and $\mathscr{F}(\pi)$ as in (3.34). Then, there exist $p_1 \in \mathbb{R}$ and $p_2 \geq 0$ such that for all $z \in \mathbb{C}$,*

$$\mathscr{F}(\pi)(z) = \exp(-p_2 z^2 + ip_1 z)\prod_{j\in\mathbb{N}} E_1\left(\frac{z}{a_j}\right). \tag{3.35}$$

*Proof.* Firstly, we consider the case $\rho_1 < \rho$. Then $\rho$ is either 2 or 1 by [Hol73, Lemma 4.10.1]. The representation in (3.35) follows from [LO77, Remark, page 42]. We note that if $\mathscr{F}(\pi)$ is of order $\rho = 2$, then $p_2 > 0$ because $p_2 = 0$ would lead to the contradiction $\rho \leq \max(1, \rho_1)$ by

[LO77, Theorems 1.2.5, 1.2.7, 1.2.8]. However, if $\mathscr{F}(\pi)$ is of order $\rho = 1$ then its representation may be reduced to

$$\mathscr{F}(\pi)(z) = \exp(i\widetilde{p}_1 z)\prod_j E_0\left(\frac{z}{a_j}\right), \qquad \widetilde{p}_1 := p_1 + \sum_j \operatorname{Im}\left(\frac{1}{a_j}\right). \tag{3.36}$$

Next, we turn to the case $\rho_1 = \rho < 2$. According to Hadamard's factorisation theorem,

$$\mathscr{F}(\pi)(z) = \exp(q_1 z + q_0)\prod_j E_1\left(\frac{z}{a_j}\right)$$

for some $q_1, q_0 \in \mathbb{C}$. We note that $q_0 = 0$ since $\mathscr{F}(\pi)(0) = 1$. In order to say something about $q_1$, we note that $X$ with a probability density function $\pi$ satisfies $\mathbb{E}[|X|] < \infty$. Moreover, $\mathbb{E}[X] = 0$, whence

$$0 = \operatorname{Re}\left(\frac{\mathscr{F}(\pi)'(0)}{\mathscr{F}(\pi)(0)}\right) = \operatorname{Re}(\log \mathscr{F}(\pi)(z))'|_{z=0} = \operatorname{Re}\left(q_1 + \sum_j \frac{z}{a_j(z-a_j)}\right)\Bigg|_{z=0} = \operatorname{Re}(q_1).$$

□

**Remark 3.18.** We have $p_2 > 0$ only if $\rho_1 < \rho = 2$. If $\mathscr{F}(\pi)$ has no zeros then $\pi$ must be a density of a normal distribution with mean zero by [LO77, Corollary to Theorem 2.5.1], since $\rho$ is finite.

The following step consists in bounding the infinite product in (3.35). If $\lfloor \rho_1 \rfloor = 0$, the term $\prod_{j\in\mathbb{N}} \exp(z/a_j)$ is well-defined, and the representation becomes

$$\mathscr{F}(\pi)(z) = \exp(-p_2 z^2 + i\widetilde{p}_1 z)\prod_{j\in\mathbb{N}} E_0\left(\frac{z}{a_j}\right),$$

where $\widetilde{p}_1$ is the same as in (3.36). Let us now use [Dup17, Lemma 4.12]. For all $z \in \mathbb{C}$ outside $\cup_j B_{\epsilon_j}(a_j)$, where $\epsilon_j = 1/|a_j|^{\rho_1+\varepsilon}$ and $\varepsilon > 0$, $\rho_1 + \varepsilon \le 2$, we get

$$\left|\prod_{j\in\mathbb{N}} E_{\lfloor \rho_1 \rfloor}\left(\frac{z}{a_j}\right)\right| \gtrsim \exp(-c_\pi |z|^{\rho_1+\varepsilon}).$$

This leads us to the following theorem.

**Theorem 3.19.** *Let Assumption 7 and Assumption 9 hold. Then there exist $c_\pi > 0$ and a family of positive numbers $(\epsilon_j)_{j\in\mathbb{N}}$ such that, for any $z \in \mathbb{C}$ outside $\cup_{j\ge1} B_{\epsilon_j}(a_j)$, it holds that*

$$|\mathscr{F}(\pi)(z)| \gtrsim \exp(-c_\pi |z|^2). \tag{3.37}$$

Due to Theorem 3.19 we can explicitly describe a scenario where condition (3.12) is satisfied, assuming that we are in the super-smooth case where the tails of the transform of the interaction function are exponential.

Let us introduce slowly varying functions $l\colon \mathbb{R}^+ \to \mathbb{R}^+$ (see also [Sen76]). These are positive and measurable functions such that, for each $\lambda > 0$, the following holds as $t \to \infty$: $\frac{l(\lambda t)}{l(t)} \to 1$.

**Assumption 10.** Let $(a_j)_{j\in\mathbb{N}}$, $c_\pi$, and $(\epsilon_j)_{j\in\mathbb{N}}$ be as in Theorem 3.19. Assume that $\cup_{j\geq 1} B_{\epsilon_j}(a_j) \cap \mathscr{L}_0 = \emptyset$. Furthermore, there is a slowly varying function $l$ such that $\liminf_{z\to\infty} l(z) > c_\pi$ for which the following holds true on $\mathscr{L}_0$:

$$|\mathscr{F}(W')(z)| = O(\exp(-|z|^2 l(|z|))) \quad \text{as } |z| \to \infty. \tag{3.38}$$

Note that in the aforementioned assumption, we assert that the transform of $W'$ exhibits a decay that is almost Gaussian. This observation aligns with our expectations, given the nature of the model under consideration, where the confinement potential is driven by $x^2$.

It is noteworthy that [Sed08, Theorem 5], specifically in the scenario where $\bar{a} = 0$, yields both necessary and sufficient conditions for the existence of non-trivial functions exhibiting an almost Gaussian nature, accompanied by an almost Gaussian Fourier transform as defined in (3.38). Furthermore, according to [Har33, Theorem 1], if both $W'$ and its Fourier transform follow the asymptotic behaviour $O(|z|^s \exp(-\frac{1}{2}z^2))$ as $|z| \to \infty$, then both can be expressed as a finite linear combination of Hermite functions. This provides concrete examples that satisfy our assumptions.

**Corollary 3.20.** *Let Assumption 7, Assumption 9, and Assumption 10 hold. Then, we have*

$$\int_{\mathscr{L}_0} \left| \frac{\mathscr{F}\left(W'\right)(z)}{\mathscr{F}(\pi)(z)} \right|^2 \mathrm{d}z < \infty.$$

*Proof.* The corollary is a straightforward consequence of Theorem 3.19 and Assumption 10. We have indeed

$$\int_{\mathscr{L}_0} \left| \frac{\mathscr{F}\left(W'\right)(z)}{\mathscr{F}(\pi)(z)} \right|^2 \mathrm{d}z \leq \int_{\mathscr{L}_0} \exp(-|z|^2 l(|z|) + c_\pi |z|^2)\, \mathrm{d}z,$$

which is bounded due to Assumption 10. □

We can conclude by noting that, under the assumptions of Corollary 3.20, Assumption 8 is clearly satisfied. This implies that we can achieve a polynomial convergence rate for the estimation of $W'$, as stated in Theorem 3.11. To underline these findings, we conclude this section by providing two examples.

**Example 3.21.** Let us demonstrate how a non-smooth confinement potential leads to an invariant density with a Fourier transform exhibiting polynomial decay. We have that

$$\pi(x) = \frac{1}{Z_\pi} \exp\left(-2V(x) - W * \pi(x)\right),$$

where $V(x) = (\alpha/2)x^2 + \widetilde{V}(x)$ and $W(x)$ satisfy Assumption 7. In addition, assume that $W$, $\widetilde{V}$ are smooth on $\mathbb{R}$, $\mathbb{R}\setminus\{0\}$ respectively and there exist $\widetilde{V}^{(J+1)}(0^\pm) := \lim_{x\to 0^\pm} \widetilde{V}^{(J+1)}(x) < \infty$ such that

$$\widetilde{V}^{(J+1)}(0^+) \neq \widetilde{V}^{(J+1)}(0^-).$$

Furthermore, $\widetilde{V}^{(j)}$, $2 \leq j \leq J+2$, are bounded on $\mathbb{R}\setminus\{0\}$. Therefore, $\pi$ is smooth on $\mathbb{R}\setminus\{0\}$ and there exist $\pi^{(J+1)}(0^+) \neq \pi^{(J+1)}(0^-)$. Then, iteratively integrating by parts we obtain

$$\mathscr{F}(\pi)(z) = \frac{1}{(-iz)^{(J+1)}}(I_-(z) + I_+(z)),$$

where

$$I_-(z) := \int_{-\infty}^{0} \exp(izx)\pi^{(J+1)}(x)\,dx, \qquad I_+(z) := \int_{0}^{\infty} \exp(izx)\pi^{(J+1)}(x)\,dx.$$

If we integrate by parts once more, we obtain

$$I_+(z) = \frac{1}{iz}\left(-\pi^{(J+1)}(0^+) - \int_0^\infty \exp(izx)\pi^{(J+2)}(x)\,dx\right) \tag{3.39}$$

since $\pi^{(J+1)}(x) \to 0$ as $x \to \infty$ and $\pi^{(J+2)}$ is integrable, which in turn follow using the same arguments as in the proof of Lemma 3.2. By Riemann's lemma, the last integral in (3.39) tends to zero and so

$$I_+(z) := -\frac{1}{iz}\pi^{(J+1)}(0^+) + o\left(\frac{1}{z}\right)$$

as $z \to \infty$. Clearly an analogous reasoning applies to $I_-(z)$. It yields that for all large enough $z$,

$$|\mathscr{F}(\pi)(z)| \geq \frac{c}{|z|^{J+2}}.$$

Hence, the Fourier transform of $\pi$ has a polynomial decay, as claimed.

**Example 3.22.** Consider a Burgers' type ODE of the form

$$\pi_{xx} + (((W' + x) * \pi)\pi)_x = 0$$

with $W'(x) = \alpha\delta_0$ for some $\alpha \in (0,1)$. Even though $W \notin C^2(\mathbb{R})$ in this case, it is still instructive to study this situation. In particular, this example shows that for non-smooth potentials $W$, the Fourier transform of $\pi$ can decay rather fast. Let us introduce $\varphi(x) := \alpha\delta_0 + x$. Then we have

$$((W' + x) * \pi)(x) = \int_{\mathbb{R}} \varphi(x-y)\pi(y)\,dy = \alpha\pi(x) + (x - \mu)$$

provided $\int_{\mathbb{R}} \pi(y)\,dy = 1$ and $\int_{\mathbb{R}} y\pi(y)\,dy = \mu$. Then

$$\pi_x + (x-\mu)\pi + \alpha\pi^2 = 0.$$

Applying the transformation

$$\pi = y'/(\alpha y),$$

we obtain

$$(y'/y)^2 + (y''y - (y')^2)/y^2 + (x-\mu)y'/y = 0$$

or

$$y'' + (x-\mu)y' = 0.$$

The general solution of this equation is given by

$$y(x) = c_1 + c_2 \int_{-\infty}^{x} \exp(-(s-\mu)^2/2)\,ds, \quad c_1, c_2 \in \mathbb{R}.$$

Hence, we deduce the identity

$$\pi(x) = \frac{\exp(-(x-\mu)^2/2)}{c + \alpha\int_{-\infty}^{x} \exp(-(s-\mu)^2/2)\,ds}$$

with the constant $c > 0$ chosen in such a way that $\int_{\mathbb{R}} \pi(x)\,\mathrm{d}x = 1$. The Fourier transform of $\pi$ can be written as

$$\mathscr{F}(\pi)(\omega) = \int_{\mathbb{R}} \frac{\exp(i\omega x - (x-\mu)^2/2)}{F(x-\mu)}\,\mathrm{d}x = \exp(i\omega\mu)\int_{\mathbb{R}} \frac{\exp(S(\omega;z))}{F(z)}\,\mathrm{d}z$$

where $S(\omega;z) = i\omega z - z^2/2$ and

$$\begin{aligned} F(z) &= c + \alpha\int_{-\infty}^{z} \exp(-s^2/2)\,\mathrm{d}s \\ &= c + \alpha\sqrt{\pi/2} + \alpha\int_0^z \exp(-s^2/2)\,\mathrm{d}s, \end{aligned}$$

having used that $\int_{-\infty}^{z} \exp(-s^2/2)\,\mathrm{d}s + \int_z^0 \exp(-s^2/2)\,\mathrm{d}s = \sqrt{\pi/2}$. We have the identity

$$\int_0^{iw} \exp(-t^2/2)\,\mathrm{d}t = i\sqrt{2}\exp(w^2/2)D_+(w/\sqrt{2})$$

where $D_+$ is the Dawson function satisfying

$$D_+(x) = \frac{1}{2x} + \frac{1}{4x^3} + O\left(\frac{1}{x^5}\right), \quad x \to \infty.$$

Hence, we conclude that

$$\int_0^{iw} \exp(-t^2/2)\,\mathrm{d}t = -\frac{\exp(w^2/2)}{iw}\left[1 + O\left(\frac{1}{w^2}\right)\right], \quad w \to \infty.$$

The same asymptotic holds for $\mathrm{Re}(w)/\mathrm{Im}(w) \to \infty$. We can therefore write

$$F(iw) = c + \alpha\sqrt{\frac{\pi}{2}} - \frac{\alpha}{iw}\exp(w^2/2)\left[1 + O\left(\frac{1}{w^2}\right)\right], \quad w \to \infty.$$

Furthermore, note that the solution of the equation

$$\frac{\exp(w^2/2)}{iw} = q$$

has the form

$$w = \sqrt{2\log(q) + 2\log(i\sqrt{2\log(q)})} + O(\log^{-1}(q))$$

for $q$ large enough. Then the function $F(z)$ has a zero at the point $z_0 = iw_0$ with

$$\begin{aligned} w_0 = &\sqrt{2\log((c+\alpha\sqrt{\pi/2})/\alpha) + 2\log(i\sqrt{2\log((c+\alpha\sqrt{\pi/2})/\alpha)})} \\ &+ O(\log^{-1}((c+\alpha\sqrt{\pi/2})/\alpha)) \end{aligned}$$

for $\alpha$ small enough (note that $c \to \sqrt{2\pi}$ for $\alpha \to 0$). Observe that we can write

$$\int_{\mathbb{R}} \frac{\exp(S(\omega;z))}{F(z)}\,\mathrm{d}z = I(w) + \int_{2iw_0-\infty}^{2iw_0+\infty} \frac{\exp(S(\omega;z))}{F(z)}\,\mathrm{d}z, \tag{3.40}$$

where clearly

$$I(w) = \int_{\mathbb{R}} \frac{\exp(S(\omega;z))}{F(z)}\,\mathrm{d}z - \int_{2iw_0-\infty}^{2iw_0+\infty} \frac{\exp(S(\omega;z))}{F(z)}\,\mathrm{d}z.$$

For $I(w)$ we can use Cauchy's integral theorem. Indeed, it can be considered as an integral over a closed contour of $\exp(-z^2/2) \to 0$ as $|\mathrm{Re}(z)| \to \infty$. On the strip $\{0 < \mathrm{Im}(z) < 2\mathrm{Re}(w_0)\}$, the function $\exp(S(\omega; z))/F(z)$ has a simple pole in $z_0 = iw_0$ with residue

$$\lim_{z\to z_0}(z-z_0)\frac{\exp(S(\omega;z))}{F(z)} = \lim_{z\to z_0}\frac{\exp(S(\omega;z))}{F'(z)} = \frac{\exp(-\omega w_0 + w_0^2/2)}{\alpha\exp(w_0^2/2)} = \alpha^{-1}\exp(-\omega w_0).$$

Recall that if a function $f$ is analytic inside a contour $C$ except at some isolated singularities $a_1, \dots, a_n$, then Cauchy's residue theorem ensures that $\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^n \mathrm{Res}(f, a_k)$. Hence, we obtain

$$I(\omega) = 2\pi i \alpha^{-1}\exp(-\omega w_0).$$

Besides that, the contribution of the second integral in (3.40) is negligible as

$$\int_{2iw_0-\infty}^{2iw_0+\infty}\frac{\exp(S(\omega;z))}{F(z)}\,\mathrm{d}z = \int_{-\infty}^{\infty}\frac{\exp(S(\omega;z+2iw_0))}{F(z+2iw_0)}\,\mathrm{d}z = O(\exp(-2\omega w_0))$$

since $\inf_{z\in\mathbb{R}}|F(z+2iw_0)| > 0$. Thus, we deduce that

$$\mathscr{F}(\pi)(\omega) = \frac{2\pi i}{\alpha}\exp(-\omega w_0 + i\omega\mu) + O(\exp(-2\omega w_0)).$$

## 3.5 Proofs of Technical Results

This section is devoted to proofs of our technical lemmas. We start by providing the proof of Lemma 3.2.

### 3.5.1 Proof of Lemma 3.2

We can write the invariant density $\pi(x)$ in the following equivalent ways:

$$\pi(x) = \frac{1}{Z_\pi}\exp(-2V(x) - W * \pi(x)) \tag{3.41}$$

$$= \frac{1}{Z_0}\exp(-2V_0(x) - W_0 * \pi(x)), \tag{3.42}$$

where $Z_\pi, Z_0$ are the normalising constants and

$$V_0(x) := \int_0^x V'(u)\,\mathrm{d}u = V(x) - V(0),$$
$$W_0(x) := \int_0^x W'(u)\,\mathrm{d}u = W(x) - W(0),$$

satisfy $W_0(0) = V_0(0) = 0$ and $W_0'(x) = W'(x)$, $V_0'(x) = V'(x)$ for all $x \in \mathbb{R}$. Note

$$W_0 * \pi(x) = W * \pi(x) - W(0)$$

and

$$W_0 * \pi(x) - W_0 * \pi(0) = \int_0^x W' * \pi(u)\,\mathrm{d}u = W * \pi(x) - W * \pi(0).$$

To obtain the bounds for $\pi$ let us use the representation (3.42). According to Assumption 7 we have $\|W'\|_1 < \infty$. We deduce that for all $x \in \mathbb{R}$,

$$|W_0(x)| \le \int_0^{|x|}|W'(u)|du \le \|W'\|_1,$$

hence,

$$|W_0 * \pi(x)| \le \|W_0\|_\infty \|\pi\|_1 \le \|W'\|_1.$$

We get that

$$\pi(x) \le \frac{1}{Z_0} \exp(-2V_0(x) + \|W'\|_1) \approx \exp(-2V_0(x)) \tag{3.43}$$

and

$$\pi(x) \ge \frac{1}{Z_0} \exp(-2V_0(x) - \|W'\|_1) \approx \exp(-2V_0(x)).$$

As we have assumed that there exists a $C_V > 0$ such that $V'' \ge C_V$ we obtain, for all $x \ge 0$,

$$\begin{aligned} V_0(x) &= \int_0^x (V'(u) - V'(0))\, \mathrm{d}u + V'(0)x \\ &\ge \frac{C_V}{2} x^2 + V'(0)x \end{aligned}$$

and

$$\begin{aligned} V_0(-x) &= \int_{-x}^0 V'(u)\, \mathrm{d}u \\ &= \int_{-x}^0 (V'(0) - V'(u))\, \mathrm{d}u - V'(0)x \\ &\ge \frac{C_V}{2} x^2 - V'(0)x. \end{aligned}$$

In conclusion, we deduce that

$$V_0(x) \ge \frac{C_V}{2} x^2 + V'(0)x. \tag{3.44}$$

Recall that $V'(0) = 0$ under both A1 and A2 as $\widetilde{V} = 0$ or $\widetilde{V} \in C^2(\mathbb{R})$ is assumed to be even. The proof about the upper bound of $\pi$ is therefore concluded. For the lower bound, we have

$$\pi(x) \ge c \exp(-\alpha x^2 - 2\widetilde{V}(x)). \tag{3.45}$$

Observe that, similarly as above, we can take advantage of the fact that $\tilde{V}'' \le \tilde{c}_2$ to obtain, for all $x \ge 0$,

$$\widetilde{V}(x) - \widetilde{V}(0) \int_0^x (\widetilde{V}'(u) - \widetilde{V}'(0))\, \mathrm{d}u + \widetilde{V}'(0)x \le \tilde{c}_2 \int_0^x u\, \mathrm{d}u = \frac{\tilde{c}_2}{2} x^2,$$

having also used that $\widetilde{V}'(0) = 0$. An analogous reasoning holds true for $x \le 0$. It implies that, for any $x \in \mathbb{R}$, $\widetilde{V}(x) \le \widetilde{V}(0) + \frac{\tilde{c}_2}{2} x^2$. Replacing it in (3.45) we obtain

$$\pi(x) \ge c \exp(-\alpha x^2 - \tilde{c}_2 x^2 - 2\widetilde{V}(0)) = c \exp(-\widetilde{C} x^2)$$

with $\widetilde{C} = \alpha + \tilde{c}_2$, as we wanted.

Let us move to the proof of the upper bound for the derivatives of $\pi$. More specifically, we want to prove by induction that for every $n \in \mathbb{N}$, $n \le J$, there exists a $c > 0$ such that

$$|\pi^{(n)}(x)| \le c(1 + |x|)^n \pi(x). \tag{3.46}$$

Let us begin with the base case $n = 1$. Then $\pi = -\varphi'\pi$, where $\varphi' := 2V' + W' * \pi$. Note that $|\varphi'(x)| \leq c(|x| + 1)$, where $|V'(x)| \leq c|x|$ follows from $\|V''\|_\infty < \infty$ and $V'(0) = 0$. Now assume the claim (3.46) holds for some $n < J$. Then the $(n+1)$-th derivative of $\pi$ is

$$\pi^{(n+1)} = (\pi')^{(n)} = -\sum_{k=0}^{n} \binom{n}{k} \varphi^{(k+1)} \pi^{(n-k)}. \tag{3.47}$$

The inductive hypothesis ensures that $|\pi^{(n-k)}(x)| \leq c(1 + |x|)^{(n-k)}\pi(x)$. Moreover, we have that $|\varphi^{(k+1)}(x)| \leq c(1+|x|)$ since $\|W' * \pi^{(k)}\|_\infty \leq \|W'\|_1 \|\pi^{(k)}\|_\infty < \infty$, whereas $|V^{(k+1)}(x)| \leq c(1 + |x|)$ follows from $\|V^{(j)}\|_\infty \leq c$, $2 \leq j \leq J$. We conclude by replacing it in (3.47), which yields

$$|\pi^{(n+1)}(x)| \leq c \sum_{k=0}^{n} (1 + |x|)(1 + |x|)^{(n-k)} \pi(x) \leq c(1 + |x|)^{n+1} \pi(x)$$

as we wanted, for $n + 1 \leq J$.

### 3.5.2 Proof of Proposition 3.6

For convenience, we omit the dependency on $N$ in our notation $X_t^{i,N}$. We will prove the claim by applying a Grönwall-type argument to the function

$$y(t) := \mathbb{E}[|X_t^i - \overline{X}_t^i|^{2p}].$$

We define

$$\Pi_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i}, \qquad \overline{\Pi}_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\overline{X}_t^i}, \qquad \overline{\Pi}_t := \mathrm{Law}(\overline{X}_t).$$

Since $(X_t^i)_{t\geq 0}$ and $(\overline{X}_t^i)_{t\geq 0}$ start at $X_0^i = \overline{X}_0^i$ and are driven by the same Brownian motion $(B_t^i)_{t\geq 0}$, it holds that

$$X_t^i - \overline{X}_t^i = -\int_0^t V'(X_s^i) - V'(\overline{X}_s^i) + \frac{1}{2} W' * \Pi_s^N(X_s^i) - \frac{1}{2} W' * \overline{\Pi}_s(\overline{X}_s^i) \, \mathrm{d}s.$$

Applying Itô's formula, summing over $i = 1, \ldots, N$ and dividing by $N$, yields

$$\frac{1}{N} \sum_{i=1}^{N} |X_t^i - \overline{X}_t^i|^{2p} = -\frac{2p}{N} \sum_{i=1}^{N} \int_0^t \left( A_i(s) + \frac{1}{2} B_i(s) + \frac{1}{2} C_i(s) \right) \mathrm{d}s,$$

where

$$\begin{aligned}
A_i(s) &:= (X_s^i - \overline{X}_s^i)^{2p-1} (V'(X_s^i) - V'(\overline{X}_s^i)), \\
B_i(s) &:= (X_s^i - \overline{X}_s^i)^{2p-1} (W' * \Pi_s^N(X_s^i) - W' * \overline{\Pi}_s^N(\overline{X}_s^i)), \\
C_i(s) &:= (X_s^i - \overline{X}_s^i)^{2p-1} (W' * \overline{\Pi}_s^N(\overline{X}_s^i) - W' * \overline{\Pi}_s(\overline{X}_s^i)).
\end{aligned}$$

Taking the expectation and derivative gives

$$y'(t) = -\frac{p}{N} \sum_{i=1}^{N} \mathbb{E}[2A_i(t) + B_i(t) + C_i(t)].$$

Using the assumption $V'' \geq C_V > 0$ and the mean value theorem gives

$$-\mathbb{E}[A_i(t)] \leq -C_V \mathbb{E}[|X_t^i - \overline{X}_t^i|^{2p}].$$

The analysis of $B_i(t)$ makes use of the symmetry of $W$ and the exchangeability of $(X_t^i, \overline{X}_t^i)$, $i = 1, \ldots, N$. Indeed, we obtain

$$-\mathbb{E}[B_i(t)] = -\frac{1}{N} \sum_{j=1}^{N} \mathbb{E}[B_{ij}(t)],$$

where

$$\begin{aligned}\mathbb{E}[B_{ij}(t)] &= \mathbb{E}[(X_t^i - \overline{X}_t^i)^{2p-1}(W'(X_t^i - X_t^j) - W'(\overline{X}_t^i - \overline{X}_t^j))] \\ &= \frac{1}{2}\mathbb{E}[((X_t^i - \overline{X}_t^i)^{2p-1} - (X_t^j - \overline{X}_t^j)^{2p-1})(W'(X_t^i - X_t^j) - W'(\overline{X}_t^i - \overline{X}_t^j))].\end{aligned}$$

By the mean value theorem, the assumption $-W'' \leq C_W$ gives

$$\begin{aligned}-\mathbb{E}[B_{ij}(t)] &\leq \frac{C_W}{2}\mathbb{E}[((X_t^i - \overline{X}_t^i)^{2p-1} - (X_t^j - \overline{X}_t^j)^{2p-1})((X_t^i - X_t^j) - (\overline{X}_t^i - \overline{X}_t^j))] \\ &\leq 2C_W \mathbb{E}[|X_t^i - \overline{X}_t^i|^{2p}],\end{aligned}$$

hence,

$$-\mathbb{E}[B_i(t)] \leq 2C_W \mathbb{E}[|X_t^i - \overline{X}_t^i|^{2p}].$$

Hölder's inequality for $C_i(t)$ implies

$$-\mathbb{E}[C_i(t)] \leq \mathbb{E}[|X_t^i - \overline{X}_t^i|^{2p}]^{\frac{2p-1}{2p}} R_i(t)^{\frac{1}{2p}},$$

where

$$\begin{aligned}R_i(t) :=& \mathbb{E}[|W' * \overline{\Pi}_t^N(\overline{X}_t^i) - W' * \overline{\Pi}_t(\overline{X}_t^i)|^{2p}] \\ =& \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N} W'(\overline{X}_t^i - \overline{X}_t^j) - W' * \overline{\Pi}_t(\overline{X}_t^i)\right|^{2p}\right].\end{aligned}$$

Expanding this term, we deduce that

$$R_i(t) = \frac{1}{N^{2p}} \sum_{j_1,\ldots,j_{2p}=1}^{N} \mathbb{E}\left[\prod_{k=1}^{2p} \mathbb{E}[\, W'(\overline{X}_t^i - \overline{X}_t^{j_k}) - W' * \overline{\Pi}_t(\overline{X}_t^i) \Big| \overline{X}_t^i]\right],$$

because of the independence of $\overline{X}_t^i$, $i = 1, \ldots, N$. The key remark to study this expectation is that for $i \neq j$,

$$\mathbb{E}[W'(\overline{X}_t^i - \overline{X}_t^j) - W' * \overline{\Pi}_t(\overline{X}_t^i)|\overline{X}_t^i] = 0$$

since Law$(\overline{X}_t^j) = \overline{\Pi}_t$. We observe that if there exists $k$ such that $j_k \neq j_{\tilde{k}}$ $\forall \tilde{k} \in \{1, \ldots, 2p\} \setminus k$, then the $2p$-fold product vanishes. In other words, in order for a term to contribute to the sum in $R_i(t)$, for every index $j_k$ there must be another $j_{\tilde{k}}$ such that $j_k = j_{\tilde{k}}$. We can have at most $N^p$ of these combinations of indices. We recall we have assumed $\|W'\|_\infty < \infty$, which implies $\mathbb{E}[|W'(\overline{X}_t^i - \overline{X}_t^j) - W' * \overline{\Pi}_t(\overline{X}_t^i)|^k] < \infty$ for any $k \in \mathbb{N}$. Therefore, we have

$$R_i(t)^{\frac{1}{2p}} \lesssim N^{-\frac{1}{2}}.$$

In conclusion, by exchangeability,

$$-\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[C_i(t)] \lesssim \frac{1}{\sqrt{N}}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[|X_t^i-\overline{X}_t^i|^{2p}]\right)^{\frac{1}{2p}}.$$

Finally, we obtain

$$y'(t) \le -2(C_V - C_W)y(t) + \frac{c}{\sqrt{N}},$$

and since $y(0) = 0$ and $C_V - C_W > 0$, the conclusion follows by integrating this Grönwall-like differential inequality, which provides $y(t) \le c/\sqrt{N}$ uniformly in $t$.

### 3.5.3 Proof of Lemma 3.15

Observe that for $c > 0$ and any random variable $X$,

$$0 < \frac{1}{2}\left(\mathbb{E}[\exp(cX)] + \mathbb{E}[\exp(-cX)]\right) = 1 + \sum_{k=1}^{\infty}\frac{c^{2k}}{(2k)!}\mathbb{E}[X^{2k}], \tag{3.48}$$

hence, it suffices to study the asymptotic growth rate of its even moments in order to show $\mathbb{E}[\exp(\pm cX)] < \infty$.

Let us start by looking at $\mathbb{E}[\exp(\pm cX_t^{i,N})]$, which leads us to study $\mathbb{E}[(X_t^{i,N})^{2k}]$ for $k \in \mathbb{N}$. For convenience, we omit the dependency on $N$ in our notation and recall that the particles follow the system of SDE's:

$$X_t^i = X_0^i + \int_0^t -V'(X_s^i) - \frac{1}{2N}\sum_{j=1}^{N}W'(X_s^i - X_s^j)\,\mathrm{d}s + B_t^i, \quad i = 1,\dots,N.$$

Applying the Itô lemma gives

$$(X_t^i)^{2k} = (X_0^i)^{2k} + \int_0^t -2k(X_s^i)^{2k-1}(V'(X_s^i) + \frac{1}{2N}\sum_{j=1}^{N}W'(X_s^i - X_s^j)) + \frac{1}{2}(2k)(2k-1)(X_s^i)^{2k-2}\mathrm{d}s$$
$$+ 2k\int_0^t (X_s^i)^{2k-1}\mathrm{d}B_s^i.$$

Taking the expectation and then differentiating,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[(X_t^i)^{2k}] = -2k\mathbb{E}[(X_t^i)^{2k-1}V'(X_t^i)] - \frac{k}{N}\sum_{j=1}^{N}\mathbb{E}[(X_t^i)^{2k-1}W'(X_t^i - X_t^j)]$$
$$+ k(2k-1)\mathbb{E}[(X_t^i)^{2k-2}],$$

where

$$-\mathbb{E}[(X_t^i)^{2k-1}V'(X_t^i)] = -\mathbb{E}[(X_t^i)^{2k-1}(V'(X_t^i) - V'(0))] - \mathbb{E}[(X_t^i)^{2k-1}V'(0)]$$
$$\le -C_V\mathbb{E}[|X_t^i|^{2k}] + |V'(0)|\mathbb{E}[|X_t^i|^{2k-1}], \tag{3.49}$$

follows from the strong convexity of $V$, and $\|W'\|_\infty < \infty$ implies

$$-\mathbb{E}[(X_t^i)^{2k-1}W'(X_t^i - X_t^j)] \le \|W'\|_\infty\mathbb{E}[|X_t^i|^{2k-1}].$$

We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|X_t^i|^{2k}] \leq -2kC_V\mathbb{E}[|X_t^i|^{2k}] + 2k|V'(0)|\mathbb{E}[|X_t^i|^{2k-1}] + k\|W'\|_\infty\mathbb{E}[|X_t^i|^{2k-1}] \\ + k(2k-1)\mathbb{E}[|X_t^i|^{2k-2}], \tag{3.50}$$

where

$$\mathbb{E}[|X_t^i|^{2k-l}] = \mathbb{E}[|X_t^i|^{2k-l}(1(|X_t^i| \leq C) + 1(|X_t^i| > C))] \\ \leq C^{2k-l} + C^{-l}\mathbb{E}[|X_t^i|^{2k}], \quad l = 1, 2,$$

for all $C > 0$. Denote $m_t(2k) := \mathbb{E}[|X_t^i|^{2k}]$. Then it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}m_t(2k) \leq -2kC_V m_t(2k) + k(2|V'(0)| + \|W'\|_\infty)(C^{2k-1} + C^{-1}m_t(2k)) \\ + k(2k-1)(C^{2k-2} + C^{-2}m_t(2k)) \\ = -Ak(m_t(2k) - B),$$

where

$$A := 2C_V - (2|V'(0)| + \|W'\|_\infty)C^{-1} - (2k-1)C^{-2}, \\ B := A^{-1}((2|V'(0)| + \|W'\|_\infty)C^{2k-1} + (2k-1)C^{2k-2})$$

do not depend on $t$. For some fixed $\varepsilon \in (0,1)$, set

$$C = C(k) := \left(\frac{(2k-1)}{2C_V\varepsilon}\right)^{\frac{1}{2}}, \quad k \in \mathbb{N},$$

so that

$$A = A(k) \sim 2C_V(1-\varepsilon) > 0, \qquad B = B(k) \sim \frac{2C_V\varepsilon C(k)^{2k}}{A(k)},$$

where $\sim$ denotes asymptotic equality as $k \to \infty$. The Grönwall inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}m_t(2k) = \frac{\mathrm{d}}{\mathrm{d}t}(m_t(2k) - B(k)) \leq -A(k)k(m_t(2k) - B(k))$$

gives

$$m_t(2k) \leq B(k) + (m_0(2k) - B(k))e^{-A(k)kt},$$

which in turn implies that, for all large enough $k$, uniformly in $t$,

$$m_t(2k) \leq \max(m_0(2k), B(k)).$$

Based on (3.48) and the subsequent analysis, we conclude that $\sup_{t\geq 0}\mathbb{E}[\exp(\pm cX_t^i)]$ is bounded as long as

$$\sup_{t\geq 0}\sum_{k=1}^{\infty}\frac{c^{2k}}{(2k)!}m_t(2k) \lesssim \sum_{k=1}^{\infty}\frac{c^{2k}}{(2k)!}(m_0(2k) + B(k))$$

is also bounded. We observe that $\sum_{k=1}^{\infty} c^{2k} m_0(2k)/(2k)!$ converges because of (3.48) and our assumption $\mathbb{E}[\exp(\pm c X_0^i)] < \infty$. To test the convergence of the series $\sum_{k=1}^{\infty} c^{2k} B(k)/(2k)!$ we use the ratio test also known as d'Alembert's criterion:

$$\begin{aligned}\frac{c^{2k+2}B(k+1)/(2k+2)!}{c^{2k}B(k)/(2k)!} &\sim \frac{c^2 C(k+1)^{2k+2}}{C(k)^{2k}(2k+1)(2k+2)}\\ &= \frac{c^2(2k+1)^{k+1}}{(2C_V\varepsilon)(2k-1)^k(2k+1)(2k+2)} \to 0\end{aligned}$$

as $k \to \infty$. The result for the exponential moments of $X_t^i$ follows.

One can follow the preceding proof to get $\sup_{t\geq 0} \mathbb{E}[\exp(\pm c\overline{X}_t)] < \infty$ from $\mathbb{E}[\exp(\pm c\overline{X}_0)] < \infty$, where recall $(\overline{X}_t)_{t\geq 0}$ is a solution of

$$\overline{X}_t = \overline{X}_0 - \int_0^t V'(\overline{X}_s) + \frac{1}{2} W' * \overline{\Pi}_s(\overline{X}_s)\,\mathrm{d}s + B_t$$

with $\overline{\Pi}_t := \mathrm{Law}(\overline{X}_t)$. Indeed, Itô's lemma for $(\overline{X}_t)^{2k}$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[(\overline{X}_t)^{2k}] = &-2k\mathbb{E}[(\overline{X}_t)^{2k-1}(V'(\overline{X}_t) + \frac{1}{2}W' * \overline{\Pi}_t(\overline{X}_t)]\\ &+ k(2k-1)\mathbb{E}[(\overline{X}_t)^{2(k-1)}],\end{aligned}$$

where, in the same manner as in (3.49), we have

$$-\mathbb{E}[(\overline{X}_t)^{2k-1}V'(\overline{X}_t)] \leq -C_V\mathbb{E}[|\overline{X}_i|^{2k}] + |V'(0)|\mathbb{E}[|\overline{X}_t^i|^{2k-1}],$$

and since $\|W'\|_\infty < \infty$, we get

$$-\mathbb{E}[(\overline{X}_t)^{2k-1}W' * \overline{\Pi}_t(\overline{X}_t)] \leq \|W'\|_\infty \mathbb{E}[|\overline{X}_t|^{2k-1}].$$

Thus, we obtain the inequality (3.50) for $\mathbb{E}[|\overline{X}_t|^{2k}]$ instead of $\mathbb{E}[|X_t^i|^{2k}]$. The rest of the proof remains the same.

# References


[AG93] Miguel A. Arcones and Evarist Giné. “Limit theorems for $U$-Processes”. In: *The Annals of Probability* 21.3 (1993), pp. 1494–1542. DOI: 10.1214/aop/1176989128.

[Amo+23] Chiara Amorino et al. “Parameter estimation of discretely observed interacting particle systems”. In: *Stochastic Processes and their Applications* 163 (2023), pp. 350–386. DOI: 10.1016/j.spa.2023.06.011.

[Amo+24] Chiara Amorino et al. *Polynomial rates via deconvolution for nonparametric estimation in McKean-Vlasov SDEs*. 2024. arXiv: 2401.04667.

[AN20] Kweku Abraham and Richard Nickl. “On statistical Calderón problems”. In: *Mathematical Statistics and Learning* 2.3 (2020), pp. 165–216. DOI: 10.4171/MSL/14.

[Bal+12] Javier Baladron et al. “Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons”. In: *The Journal of Mathematical Neuroscience* 2.10 (2012). DOI: 10.1186/2190-8567-2-10.

[Ben+98] Said Benachour et al. “Nonlinear self-stabilizing processes – I Existence, invariant probability, propagation of chaos”. In: *Stochastic Processes and their Applications* 75.2 (1998), pp. 173–201. DOI: 10.1016/S0304-4149(98)00018-0.

[BG21] Denis Belomestny and Alexander Goldenshluger. “Density deconvolution under general assumptions on the distribution of measurement errors”. In: *The Annals of Statistics* 49.2 (2021), pp. 615–649. DOI: 10.1214/20-AOS1969.

[BGG13] François Bolley, Ivan Gentil, and Arnaud Guillin. “Uniform convergence to equilibrium for granular media”. In: *Archive for Rational Mathematics and Analysis* 208 (2013), pp. 429–445. DOI: 10.1007/s00205-012-0599-z.

[Bis11] Jaya P. N. Bishwal. “Estimation in interacting diffusions: continuous and discrete Sampling”. In: *Applied Mathematics* 2.9 (2011), pp. 1154–1158. DOI: 10.4236/am.2011.29160.

[BPP23] Denis Belomestny, Vytautė Pilipauskaitė, and Mark Podolskij. “Semiparametric estimation of McKean–Vlasov SDEs”. In: *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques* 59.1 (2023), pp. 79–96. DOI: 10.1214/22-AIHP1261.

[BPZ24] Denis Belomestny, Mark Podolskij, and Shi-Yuan Zhou. *On nonparametric estimation of the interaction function in particle system models*. 2024. arXiv: 2402.14419.

[BR18] Viorel Barbu and Michael Röckner. “Probabilistic representation for solutions to nonlinear Fokker-Planck equations”. In: *SIAM Journal on Mathematical Analysis* 50.4 (2018), pp. 4246–4260. DOI: 10.1137/17M1162780.

[BR20] Viorel Barbu and Michael Röckner. “From nonlinear Fokker–Planck equations to solutions of distribution dependent SDE”. In: *The Annals of Probability* 48.4 (2020), pp. 1902–920. DOI: 10.1214/19-AOP1410.

[BRS16] Vladimir I. Bogachev, Michael Röckner, and Stanislav V. Shaposhnikov. “Distances between transition probabilities of diffusions and applications to nonlinear Fokker-Planck-Kolmogorov equations”. In: *Journal of Functional Analysis* 271.5 (2016), pp. 1262–1300. DOI: 10.1016/j.jfa.2016.05.016.

[Bru50] Nicolaas Govert de Bruijn. “The roots of trigonometric integrals”. In: *Duke Mathematical Journal* 17.3 (1950), pp. 197–226. DOI: 10.1215/S0012-7094-50-01720-0.

[Car+19] Pierre Cardaliaguet et al. *The Master Equation and the Convergence Problem in Mean Field Games*. Princeton University Press, 2019. ISBN: 9780691193717. DOI: 10.1515/9780691193717.

[CD22a] Louis-Pierre Chaintron and Antoine Diez. “Propagation of chaos: A review of models, methods and applications. I. Models and methods”. In: *Kinetic and Related Models* 15.6 (2022), pp. 895–1015. DOI: 10.3934/krm.2022017.

[CD22b] Louis-Pierre Chaintron and Antoine Diez. “Propagation of chaos: A review of models, methods and applications. II. Applications”. In: *Kinetic and Related Models* 15.6 (2022), pp. 1017–1173. DOI: 10.3934/krm.2022018.

[CFT12] Claudio Canuto, Fabio Fagnani, and Paolo Tilli. “An eulerian approach to the analysis of Krause’s consensus models”. In: *SIAM Journal on Control and Optimization* 50.1 (2012), pp. 243–265. DOI: 10.1137/100793177.

[CG19] Fabienne Comte and Valentine Genon-Catalot. “Regression function estimation as a partly inverse problem”. In: *Annals of the Institute of Statistical Mathematics* 72 (2019), pp. 1023–1054. DOI: 10.1007/s10463-019-00718-2.

[CG20] Fabienne Comte and Valentine Genon-Catalot. “Nonparametric drift estimation for i.i.d. paths of stochastic differential equations”. In: *The Annals of Statistics* 48.6 (2020), pp. 3336–3365. DOI: 10.1214/19-AOS1933.

[CG23] Fabienne Comte and Valentine Genon-Catalot. “Nonparametric adaptive estimation for interacting particle systems”. In: *Scandinavian Journal of Statistics* 50.4 (2023), pp. 1716–1755. DOI: 10.1111/sjos.12661.

[CGL24] Fabienne Comte, Valentine Genon-Catalot, and Catherine Larédo. “Nonparametric moment method for McKean-Vlasov stochastic differential equations”. Preprint. Feb. 2024. URL: https://hal.science/hal-04460327.

[CGM08] Patrick Cattiaux, Arnaud Guillin, and Florent Malrieu. “Probabilistic approach for granular media equations in the non-uniformly convex case”. In: *Probability Theory and Related Fields* 140 (2008), pp. 19–40. DOI: 10.1007/s00440-007-0056-3.

[Cha+17] Bernard Chazelle et al. "Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics". In: *Journal of Differential Equations* 263.1 (2017), pp. 365–397. DOI: https://doi.org/10.1016/j.jde.2017.02.036.

[Che21] Xiaohui Chen. "Maximum likelihood estimation of potential energy in interacting particle systems from single-trajectory data". In: *Electronic Communications in Probability* 26 (2021), pp. 1–13. DOI: 10.1214/21-ECP416.

[CMV03] José A. Carillo, Robert J. McCann, and Cédric Villani. "Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates". In: *Revista Matemática Iberoamericana* 19.3 (2003), pp. 971–1018. DOI: 10.4171/RMI/376.

[DH22] Laetitia Della Maestra and Marc Hoffmann. "Nonparametric estimation for interacting particle systems: McKean–Vlasov models". In: *Probability Theory and Related Fields* 182 (2022), pp. 551–613. DOI: 10.1007/s00440-021-01044-6.

[DH23] Laetitia Della Maestra and Marc Hoffmann. "The LAN property for McKean–Vlasov models in a mean-field regime". In: *Stochastic Processes and their Applications* 155 (2023), pp. 109–146. DOI: 10.1016/j.spa.2022.10.002.

[Dir15] Sjoerd Dirksen. "Tail bounds via generic chaining". In: *Electronic Journal of Probability* 20 (2015), pp. 1–29. DOI: 10.1214/EJP.v20-3760.

[Dje+22] Boualem Djehiche et al. "Optimal portfolio choice with path dependent benchmarked labor income: a mean field model". In: *Stochastic Processes and their Applications* 145 (2022), pp. 48–85. DOI: 10.1016/j.spa.2021.11.010.

[Doe74] Gustav Doetsch. *Introduction to the Theory and Application of the Laplace Transformation*. Springer Berlin Heidelberg, 1974. ISBN: 978-3-642-65692-7. DOI: 10.1007/978-3-642-65690-3.

[Dup17] Taylor Dupuy. "Hadamard's theorem and entire functions of finite order". 2017. URL: https://tdupu.github.io/complexspring2017/hadamard.pdf.

[Fou23] Nicolas Fournier. "Convergence of the empirical measure in expected Wasserstein distance: non-asymptotic explicit bounds in $\mathbb{R}^d$". In: *European Series in Applied and Industrial Mathematics: Probability and Statistics* 27 (2023), pp. 749–775. DOI: 10.1051/ps/2023011.

[Fra05] Till D. Frank. *Nonlinear Fokker-Planck Equations*. Springer Berlin Heidelberg, 2005. ISBN: 978-3-642-05954-4. DOI: 10.1007/b137680.

[FS13] Jean-Pierre Fouque and Li-Hsien Sun. "Systemic Risk Illustrated". In: *Handbook on Systemic Risk*. Cambridge University Press, 2013, pp. 444–452. DOI: 10.1017/CBO9781139151184.023.

[Gär88] Jürgen Gärtner. "On the McKean-Vlasov limit for interacting diffusions". In: *Mathematische Nachrichten* 137.1 (1988), pp. 197–248. DOI: 10.1002/mana.19881370116.

[GH11] Julien Guyon and Pierre Henry-Labordère. "The smile calibration problem solved". In: *Structured Markets Equity Derivatives Journal* (2011). DOI: 10.2139/ssrn.1885032.

[GHM22] Lucio Galeati, Fabian A. Harang, and Avi Mayorcas. “Distribution dependent SDEs driven by additive continuous noise”. In: *Electronic Journal of Probability* 27 (2022), pp. 1–38. DOI: 10.1214/22-EJP756.

[GL21a] Valentine Genon-Catalot and Catherine Larédo. “Parametric inference for small variance and long time horizon McKean-Vlasov diffusion models”. In: *Electronic Journal of Statistics* 15.2 (2021), pp. 5811–5854. DOI: 10.1214/21-EJS1922.

[GL21b] Valentine Genon-Catalot and Catherine Larédo. “Probabilistic properties and parametric inference of small variance nonlinear self-stabilizing stochastic differential equations”. In: *Stochastic Processes and their Applications* 142 (2021), pp. 513–548. DOI: 10.1016/j.spa.2021.09.002.

[GL23a] Valentine Genon-Catalot and Catherine Larédo. “Inference for ergodic McKean-Vlasov stochastic differential equations with polynomial interactions”. Preprint. Feb. 2023. URL: https://cnrs.hal.science/hal-03866218.

[GL23b] Valentine Genon-Catalot and Catherine Larédo. “Parametric inference for ergodic McKean-Vlasov stochastic differential equations”. Preprint. Apr. 2023. URL: https://hal.science/hal-04071936.

[GSS20] Kay Giesecke, Gustavo Schwenkler, and Justin A. Sirignano. “Inference for large financial systems”. In: *Mathematical Finance* 30.1 (2020), pp. 3–46. DOI: 10.1111/mafi.12222.

[Gyö+02] László Györfi et al. *A Distribution-Free Theory of Nonparametric Regression*. Springer Series in Statistics. Springer New York, 2002. ISBN: 9780387954417. DOI: 10.1007/b97848.

[Har33] Godfrey Harold Hardy. “A theorem concerning Fourier transforms”. In: *Journal of the London Mathematical Society* s1-8.3 (1933), pp. 227–231. DOI: 10.1112/jlms/s1-8.3.227.

[HO16] Marc Hoffmann and Adélaïde Olivier. “Nonparametric estimation of the division rate of an age dependent branching process”. In: *Stochastic Processes and their Applications* 126.5 (2016), pp. 1433–1471. DOI: https://doi.org/10.1016/j.spa.2015.11.009.

[Hoe94] Wassily Hoeffding. “Probability Inequalities for Sums of Bounded Random Variables”. In: *The Collected Works of Wassily Hoeffding*. Springer New York, 1994, pp. 409–426. ISBN: 978-1-4612-0865-5. DOI: 10.1007/978-1-4612-0865-5_26.

[Hol73] A. S. B. Holland. *Introduction to the Theory of Entire Functions*. Acadamic Press, 1973. ISBN: 978-0-12-352750-9.

[HRW21] Xing Huang, Panpan Ren, and Feng-Yu Wang. “Distribution dependent stochastic differential equations”. In: *Frontiers of Mathematics in China* 16 (2021), pp. 257–301. DOI: 10.1007/s11464-021-0920-y.

[HZ21] Xiaodan Huo and Yufei Zhan. “A note on the entire functions: theorems, properties and examples”. In: *Journal of Physics: Conference Series* 2012.1 (2021), p. 012058. DOI: 10.1088/1742-6596/2012/1/012058.

[IH81] Il’dar A. Ibragimov and Rafail Z. Has’minskii. *Statistical Estimation*. Springer New York, 1981. ISBN: 978-1-4899-0027-2. DOI: 10.1007/978-1-4899-0027-2.

[JM98] Benjamin Jourdain and Sylvie Méléard. “Propagation of chaos and fluctuations for a moderate model with smooth initial data”. In: *Annales de l’Institut Henri Poincare (B) Probability and Statistics* 34.6 (1998), pp. 727–766. DOI: 10.1016/S0246-0203(99)80002-8.

[Joh09] Jan Johannes. “Deconvolution with unknown error distribution”. In: *The Annals of Statistics* 37.5A (2009), pp. 2301–2323. DOI: 10.1214/08-AOS652.

[Kam82] I. P. Kamynin. “Generalization of the theorem of Marcinkiewicz on entire characteristic functions of probability distributions”. In: *Journal of Soviet Mathematics* 20 (1982), pp. 2175–2180. DOI: 10.1007/BF01239994.

[Kas90] Raphael A. Kasonga. “Maximum likelihood theory for large interacting systems”. In: *SIAM Journal on Applied Mathematics* 50.3 (1990). DOI: 10.1137/0150050.

[Kol10] Vassili N. Kolokoltsov. *Nonlinear Markov Processes and Kinetic Equations*. Cambridge Tracts in Mathematics. Cambridge University Press, 2010. DOI: 10.1017/CBO9780511760303.

[Kos08] Michael R. Kosorok. *Introduction to Empirical Processes and Semiparametric Inference*. Springer Series in Statistics. Springer New York, 2008. ISBN: 9780387749778. DOI: 10.1007/978-0-387-74978-5.

[Lac] Daniel Lacker. “Mean field games and interacting particle systems”. URL: http://www.columbia.edu/~dl3133/MFGSpring2018.pdf.

[LO77] Yuri V. Linnik and Iossif V. Ostrovskii. *Decomposition of Random Variables and Vectors*. Translations of Mathematical Monographs. American Mathematical Society, 1977. ISBN: 978-0-8218-4646-9. DOI: 10.1090/mmono/048.

[LQ22] Meiqi Liu and Huijie Qiao. “Parameter estimation of path-dependent McKean-Vlasov stochastic differential equations”. In: *Acta Mathematica Scientia* (2022), pp. 876–886. DOI: 10.1007/s10473-022-0304-8.

[Maï+22] Mylène Maïda et al. “Statistical deconvolution of the free Fokker-Planck equation at fixed time”. In: *Bernoulli* 28.2 (2022), pp. 771–802. DOI: 10.3150/21-BEJ1366.

[Mal01] Florent Malrieu. “Logarithmic Sobolev inequalities for some nonlinear PDE’s”. In: *Stochastic Processes and their Applications* 95.1 (2001), pp. 109–132. DOI: 10.1016/S0304-4149(01)00095-3.

[Mal03] Florent Malrieu. “Convergence to equilibrium for granular media equations and their Euler schemes”. In: *The Annals of Applied Probability* 13.2 (2003), pp. 540–560. DOI: 10.1214/aoap/1050689593.

[McK66] Henry P. McKean. "A class of Markov processes associated with nonlinear parabolic equations". In: *Proceedings of the National Academy of Sciences of the United States of America* 56.6 (1966), pp. 1907–1911. DOI: 10.1073/pnas.56.6.1907.

[ME99] Alex Mogilner and Leah Edelstein-Keshet. "A non-local model for a swarm". In: *Journal of Mathematical Biology* 38 (1999), pp. 534–570. DOI: 10.1007/S002850050158.

[Mél96] Sylvie Méléard. "Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models". In: *Probabilistic Models for Nonlinear Partial Differential Equations*. Springer Berlin Heidelberg, 1996, pp. 42–95. ISBN: 978-3-540-68513-5. DOI: 10.1007/BFb0093177.

[MNP21] François Monard, Richard Nickl, and Gabriel P. Paternain. "Consistent inversion of noisy non-Abelian x-ray transforms". In: *Communications on Pure and Applied Mathematics* 74.5 (2021), pp. 1045–1099. DOI: https://doi.org/10.1002/cpa.21942.

[Nic20] Richard Nickl. "Bernstein–von Mises theorems for statistical inverse problems I: Schrödinger equation". In: *Journal of the European Mathematical Society* 22.8 (2020), pp. 2697–2750. DOI: 10.4171/JEMS/975.

[PMT16] Daniel Paulin, Lester Mackey, and Joel A. Tropp. "Efron–Stein inequalities for random matrices". In: *The Annals of Probability* 44.5 (2016). DOI: 10.1214/15-AOP1054.

[PZ22a] Grigorios A. Pavliotis and Andrea Zanoni. "A method of moments estimator for interacting particle systems and their mean field limit". Dec. 2022.

[PZ22b] Grigorios A. Pavliotis and Andrea Zanoni. "Eigenfunction martingale estimators for interacting particle systems and their mean field limit". In: *SIAM Journal on Applied Dynamical Systems* 21.4 (2022), pp. 2338–2370. DOI: 10.1137/21M1464348.

[QZ02] Zhongmin Qian and Weian Zheng. "Sharp bounds for transition probability densities of a class of diffusions". In: *Comptes Rendus Mathematique* 335.11 (2002), pp. 953–957. DOI: 10.1016/S1631-073X(02)02579-7.

[Sed08] Anatolii M. Sedletskii. "Classes of entire functions that are rapidly decreasing on the real axis: theory and applications". In: *Sbornik: Mathematics* 199.1 (2008). DOI: 10.1070/SM2008V199N01ABEH003913.

[Sen76] Eugene Seneta. *Regularly Varying Functions*. Springer Berlin Heidelberg, 1976. ISBN: 978-3-540-07618-6. DOI: 10.1007/BFb0079658.

[Sha+23] Louis Sharrock et al. "Online parameter estimation for the McKean–Vlasov stochastic differential equation". In: *Stochastic Processes and their Applications* 162 (2023), pp. 481–546. DOI: 10.1016/j.spa.2023.05.002.

[She94] Robert P. Sherman. "Maximal inequalities for degenerate $U$-processes with applications to optimization estimators". In: *The Annals of Statistics* 22.1 (1994), pp. 439–459. DOI: 10.1214/aos/1176325377.

[Szn91] Alain-Sol Sznitman. "Topics in propagation of chaos". In: *Ecole d'Eté de Probabilités de Saint-Flour XIX*. Springer Berlin Heidelberg, 1991, pp. 165–251. ISBN: 978-3-540-53841-7. DOI: 10.1007/BFb0085169.

[Tsy08] Alexandre B. Tsybakov. *Introduction to Nonparametric Estimation*. Springer Series in Statistics. Springer New York, 2008. ISBN: 978-0-387-79051-0. DOI: 10.1007/b13794.

[TT20] Denis Talay and Milica Tomašević. "A new McKean–Vlasov stochastic interpretation of the parabolic–parabolic Keller–Segel model: The one-dimensional case". In: *Bernoulli* 26.2 (2020), pp. 1323–1353. DOI: 10.3150/19-BEJ1158.

[Ver18] Roman Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2018. ISBN: 9781108231596. DOI: 10.1017/9781108231596.

[Wen+16] Jianghui Wen et al. "Maximum likelihood estimation of McKean–Vlasov stochastic differential equation and its application". In: *Applied Mathematics and Computation* 274 (2016), pp. 237–246. DOI: 10.1016/j.amc.2015.11.019.

[Wid41] David V. Widder. *The Laplace Transform*. Princeton University Press, 1941. ISBN: 9780691653693.