THE WALLET PROBLEM

Consider coins with values 1 and 10. If you have seven coins of value 1, then you have exact change for any value from 1 to 7. If you have seven coins of value 1 and two coins of value 10, then you have exact change for the value 27 but you don't have exact change, for example, for the value 8.

- Suppose you want to have exact change for any value from 1 to 99. You can freely choose coins of value 1 and 10. Which coins do you choose so that the total number of coins you have is as small as possible?
- *With nine coins of value* 1 *and nine coins of value* 10 *one has exact change for all values from* 1 *to* 99*. One needs nine coins of value* 1 *to have exact change for the value* 9*, and one needs nine coins of value* 10 *(having at most eight coins of value* 10*, one would need at least* 19 *coins of value* 1*).*
- Consider the same question (with coins of value 1 and 10) where you want to have exact change for the values from 1 to 100.
- *One could choose ten coins of value* 1 *and nine coins of value* 10*. Alternatively, one could choose nine coins of value* 1 *and ten coins of value* 10*. Clearly, with these choices, one has exact changes for all values from* 1 *to* 100*. Similarly to the previous question, one needs at least nine coins of value* 1 *and nine coins of value* 10*. These coins have a total value of* 99*, so an additional coin is needed to have exact change for* 100*.*
- Consider the same question (with coins of value 1 and 10) where you want to have exact change for the values from 1 to some given positive integer N.
- *If* $N < 10$*, then one takes* N *coins of value* 1*. Now suppose that* $N \geqslant 10$ *and call* Q *the quotient of* N *after division by* 10*. If* N *is not a multiple of* 10*, then (similarly to the example of* 99*) one takes nine coins of value* 1 *and* Q *coins of value* 10*. If* N *is a multiple of* 10*, then (similarly to the example of* 100*) one may take nine coins of value* 1 *and* Q *coins of value* 10 *or ten coins of value* 1 *and* $Q-1$ *coins of value* 10*.*
- As a challenge, consider the same question (with coins of value 1 and 10 and 100) where you want to have exact change for the values from 1 to some given positive integer N.
- *If* $N < 10$*, then one takes* N *coins of value* 1*. If* $10 \leqslant N < 100$*, then one takes nine coins of value* 1 *and sufficiently many coins of value* 10 *(the number being the quotient of* N *after division by* 10*). Finally, suppose that* $N \geqslant 100$ *and call* Q *(respectively,* R*) the quotient (respectively, remainder) of* N *after division by* 100*. Similarly to the previous examples, one must take at least nine coins of value* 1 *and nine coins of value* 10 *and* $Q-1$ *coins of value* 100*. And then it suffices to take one additional coin: one may take one coin of value of* 100*; one may take one coin of value* 10 *if* $R < 10$*; one may take one coin of value* 1 *if* $R = 0$*.*

The general formulation of the Wallet Problem

Let E and N be positive integers. There are coins with values the powers of 10 from 1 to 10^E. You want to have exact change for any value from 1 to N. You can freely choose the coins (the values that you want, and how many times you want). Which coins do you choose so that the total number of coins you have is as small as possible?

Solution to the Wallet Problem

Answer. *Without loss of generality we may suppose that $10^E \leqslant N$ (because one does not need coins with value larger than N). Then call Q and R the quotient and remainder of N after division by 10^E. We choose nine coins of each value from 1 to 10^{E-1} and $Q-1$ coins with value 10^E, and one additional coin where the value can be 10^V for any $0 \leqslant V \leqslant E$ such that $R < 10^V$.*

Solution. *We first prove that, with the above choice, one has exact change for all values up to N.* Having nine coins of all values from 1 to 10^{E-1} allows to have exact change for any value from 1 to $10^E - 1$. We conclude because, using the additional coins, one is left to pay an amount of

$$N - (Q-1)10^E - 10^V = 10^E + R - 10^V < 10^E \,.$$

We need to choose at least nine coins of each value from 1 to 10^{E-1}. Consider a choice of coins that allows to pay all values up to N. To pay the value 9, we need at least nine coins of value 1. Now, by induction, we suppose to have at least nine coins of values up to 10^{C-1} (with $0 \leqslant C-1 \leqslant E-2$) and show that we need at least nine coins of value 10^C. We need exact change for $10^{C+1}-1$ and this value can be paid only with coins of values up to 10^C. By using nine coins of all values up to 10^{C-1} we are left to pay the amount

$$(10^{C+1}-1) - 9(1 + 10 + \cdots 10^{C-1}) = 10^{C+1} - 1 - (10^C - 1) = 9 \cdot 10^C \,.$$

If we have at most eight coins of value 10^C, then we need at least ten additional coins of smaller values. We could replace those with one coin of value 10^C, resulting in having fewer coins (we still have a solution is because the other coins allow to have exact change for any value less than 10^C).
We need to choose, additionally, at least $Q-1$ coins of value 10^E. This is obvious if $Q = 1$, so suppose that $Q \geqslant 2$. Recall that Q additional coins suffice. So suppose to choose Q additional coins, with at most $Q-2$ coins of value 10^E. It is not possible to pay the value N because the total value of the coins is at most

$$9(1 + \cdots + 10^{E-1}) + (Q-2)10^E + 2 \cdot 10^{E-1} =$$
$$(10^E - 1) + Q10^E - 2 \cdot 10^E + 2 \cdot 10^{E-1} < Q10^E - 1 < N \,.$$

We need to choose one further coin of value larger than R. The previously chosen coins have total value

$$9(1 + \cdots + 10^{E-1}) + (Q-1)10^E = (10^E - 1) + Q10^E - 10^E = N - R - 1$$

so to be able to pay N, one needs a further coin with a value that is larger than R.

Further questions

Remark that, in the above solution for $10^E \leqslant N$, the number of coins that we choose is $9E + Q$ and their total value is $N - R + 10^V - 1$. Also remark that, if E would be zero, then there would only be coins of value 1 and hence one would simply take N coins of value 1.

One may replace the powers of 10 by the powers of some fixed integer $b \geqslant 2$. The solution is analogous, namely the number 10 must be replaced by b (and the number 9 must be replaced by $b-1$).

One may introduce further coin values. For example, beyond the powers of 10 one may take the values $5, 50, 500, \ldots$.