INTERPRETING SETS OF NATURAL NUMBERS AS REAL NUMBERS

A bridge between two distinct mathematical objects allows to carry notions from one object to the other, thus gaining a new perspective. We explore a correspondence between real numbers and sets of positive integers.

Consider the positive integers $1, 2, 3, \ldots$ A subset of the positive integers can be described with a sequence of 0's and 1's. Indeed, we can ask whether the number 1 is in the set, whether the number 2 is in the set, and so on, recording 1 for an affirmative answer and 0 for a negative answer. For example, the set of the even positive integers corresponds to the sequence

$$0\,1\,0\,1\,0\,1\,0\,1\ldots$$

while the set of prime numbers corresponds to the more mysterious sequence

$$0\,1\,1\,0\,1\,0\,1\,0\ldots$$

In turn, we can associate to such a sequence some real number between 0 and 1 (written in the binary system), by considering the sequence of digits after the point. For example, thanks to the so-called geometric series we can evaluate

$$(0.010101\ldots)_2 = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \frac{1}{4^n} + \cdots = \frac{1}{3}$$

while the real number corresponding to the set of prime numbers is

$$(0.01101010\ldots)_2 = \frac{1}{4} + \frac{1}{8} + \frac{1}{32} + \cdots$$

Remember from the decimal system that we can write a number with period 9 as a number with period 0. In the binary system, we may similarly write a number with period 1 as a number with period 0. With the above correspondence, the finite sets of positive integers become the real numbers with period 0. So, to ensure that different sets correspond to different real numbers, we exclude the finite sets. Among the infinite sets, there are the sets with finite complement (namely, those sets containing all but finitely many positive integers) and they correspond to the real numbers with period 1. For example, the set of all positive integers is

$$(0.11111\ldots)_2 = 1\,.$$

We then have a correspondence that maps all infinite subsets of the positive integers to all real numbers from 0 (excluded) to 1 (included). We have special real numbers corresponding to special sets, for example the real number corresponding to the set of prime numbers. Conversely, we have special sets corresponding to special real numbers, for example the set corresponding to the fractional part of π.

What does it mean that two sets are close? Two real numbers (from 0 to 1) are close when their first few digits after the comma are the same. The two corresponding sets then have the same initial elements. For example, the difference between two real numbers being at most $(0.00011111\ldots)_2 = \frac{1}{8}$ means that the two corresponding sets contain in particular the same integers from 1 to 3.

What does it mean that a set is rational? Suppose that an infinite set corresponds to a rational number. Thus, the digits of the number eventually repeat periodically. For example, $(0.\overline{01})_2$ is the set of even positive integers $2, 4, 6, \ldots$ which form in particular an arithmetic progression. In fact, the purely periodic numbers correspond to sets that are a disjoint union of arithmetic progressions, one arithmetic progression for each digit 1 in the period. For example, the number

$$(0.\overline{110})_2 = (0.\overline{100})_2 + (0.\overline{010})_2$$

corresponds to the set containing the integers $1, 4, 7 \ldots$ and $2, 5, 8, \ldots$ More generally, if a number is periodic (and not necessarily purely periodic), then its corresponding set is, up to finitely many initial elements, the disjoint union of arithmetic progressions.

Normal numbers. There is a notion of regularity for the subsets of the positive integers, namely some of them admit the so-called *natural density*. In particular, a finite union of arithmetic progressions admits a natural density. The corresponding real numbers are *simply normal numbers in base* 2, and they include the rational numbers. A stronger notion is the one of *normal numbers*: most numbers are normal, which means that most sets are very regular concerning gaps.

Functions over sets. We can consider functions that are defined over the real numbers from 0 to 1. Thanks to the above correspondence, they are also defined over the infinite subsets of the positive integers. An *increasing function* implies, roughly speaking, that the impact on the function value of the initial elements of the set is very significant. A *continuous function* means, roughly speaking, that the function values do not differ much for two sets that have the same few initial elements. One example of continuous function on sets is the following: consider a sequence a_n of non-negative real numbers such that their sum is finite (say, $a_n = \frac{1}{n^2}$). Then we can map an infinite set S of positive integers to the sum of those terms a_n such that n belongs to S. On the other hand, a function that is not continuous is the function that maps a set S to 1 (respectively, to 0) if the set S admits (respectively, does not admit) a natural density, because the tail of the set matters much for this property.

Can you think of further notions for sets that can be translated to notions for real numbers or conversely?