Exponentiation as a complex operation

Pupils familiarise themselves with the exponential function $x \mapsto e^x$ defined over the real numbers. They also study complex numbers, and then they should be made aware of some dangers concerning the exponentiation for complex numbers.

Probably the first non-real exponentiation that the pupils are confronted with is

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

where θ can be any real number. This defines exponentiation (for base e) in the case where the exponent is purely imaginary. Because the trigonometric functions $\cos$ and $\sin$ are periodic, one can easily deduce that the exponentiation with purely imaginary exponent, namely the function defined over the real numbers

$$x \mapsto e^{ix}$$

is periodic with minimal period 2π. This periodicity says for example that

$$1 = e^{0i} = e^{2\pi ki}$$

holds for every integer k.
One may define the exponentiation of a complex number as follows

$$z \mapsto e^z$$

where, if $z = a + ib$ (a and b being real numbers) we have

$$e^{a+ib} = e^a\cos(b) + ie^a\sin(b)\ .$$

The formula

$$e^{z+w} = e^z e^w$$

holds for all complex numbers z and w.

If we want to define the logarithm as the inverse of the exponential function we have an issue because this periodic function, being not injective, is not a bijection and hence cannot be inverted. We can accept that the logarithm takes multiple values and therefore also define exponentiation with any complex basis that is non-zero:

$$z^w = e^{w\log(z)}$$

Consider that, despite the ambiguity of $\log(z)$ being only with multiples of $2\pi i$, the different values of z^w may fail to have the same argument (in case the real part of w is not an integer).

For example, one could pick as value for the complex logarithm its principal value, so that $log(i) = \frac{\pi}{2}i$, and then we have $i^i = e^{i\log(i)} = e^{-\pi/2}$. However, as multiple values are possible for the logarithm, then multiple values are also possible for i^i.

Even if one makes a considered choice for the logarithms, still there are some dangers to be considered:
— We have the following congruence, and in general not the equality:

$$\log(z^w) = w\log(z) \bmod (2\pi i)$$

— We don't have the identity $e^{xy} = (e^x)^y$.

Reference: Wikipedia contributors. "Exponentiation." *Wikipedia, The Free Encyclopedia*. Wikipedia, The Free Encyclopedia, 6 Mar. 2024. Web. 13 Mar. 2024.