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Fifty years of mathematical growth theory: Classical topics and new trends [1]

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Abstract

We present an overview of selected contributions of the *Journal of Mathematical Economics*' authors to growth theory in the last half century. We start with the classical optimal growth theory within a benchmark multisector model and outline the successive developments in the analysis of this model, including the turnpike theory. Different refinements of the benchmark are considered along the way. We then survey the abundant literature on endogenous fluctuations in two-sector models. We conclude with two strong trends in the recent growth literature: *green* growth and infinite-dimensional growth models.

Keywords: Growth theory, multisector models, turnpike theory, *green* growth, infinite-dimensional growth models, optimization

JEL classification: C60, C61, O41

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1 Introduction

Growth theory is one of the most visited research areas in the history of the *Journal of Mathematical Economics* since its creation in 1974. The first growth paper was published in the journal's first volume, second issue under the title "The modified golden rule of a multisector economy" by Peleg and Ryder [104]. The research questions tackled in this paper were recurrent subject of study at that time: the design of modified golden rules (in the sense of Nikaido [97]) and the subsequent inherent turnpike properties. Since then, the *JME* has published dozens of papers on a large variety of growth theory topics.[2] The objective of this anniversary survey is obviously not to provide with a very close overview of the cumulated growth literature published in the *JME* over the last 50 years. This is out of the scope of this survey. We shall rather review in a quite compact way some of the most explored growth theory sub-areas with a few representative articles for each sub-area, to give an immediate idea of the contributions of the journal to the development of growth theory since its creation.

A natural starting point is the theory of optimal growth which has been a dominant area for decades, a large majority of papers published in the *JME* takes this avenue (as in other theory journals). Several papers more specifically focusing on equilibrium dynamics have been of course published, see for example, Le Van *et al.* [84].[3] The integration of turnpike theory and equilibrium theory has been also central in a number of important *JME* papers, a remarkable one due to Bewley [26]. With all these observations made, it is difficult to not start our overview with a benchmark multisector optimal growth model. We shall elaborate on it with special attention to the classical existence of optimal steady states and turnpike problems, later developing some of the salient refinements and extensions for two popular classes models, the Ramsey models, and the two-sector models. The latter have been an inexaustible source of research questions, in particular on (optimal and equilibrium) endogenous fluctuations in a large variety of models from the Robinson-Solow-Srinavasan (RSS) model (see Deng, Khan and Mitra [48]) to the Lucas model (see Brito and Venditti [40]). We close the review of these classical research streams by a brief account of the *JME* literature on stochastic growth. We further document two emerging research areas in the Journal: research on growth models with environmental aspects, labelled as *green growth* models, and on a more methodological ground, the upsurge of infinite-dimensional growth models recently used to account for continuous time, age and spatial structures in the growth processes. The latter is a clear

[2] An elementary literature search in the website of the journal with the keywords "growth theory" plus "capital accumulation" returns 107 published papers.

[3] Many of these contributions typically consider settings with heterogenous agents and/or with endogenous growth. See d'Albis and Le Van[3] for an exemple for the latter area.

example of economic theory research facilitated by the development of new mathematical tools (such as dynamic programming in infinite-dimensional functional spaces by Bensoussan *et al.* [25], applied in the solution of vintage models, see for example Fabbri and Gozzi [56]).

By making these editorial choices (which are essentially motivated by the time span of this survey), we have necessarily minimized the space that other recently active publication areas would have deserved.[4] Also to minimize space devoted to technicalities, we will stick to discrete time modelling, which covers a large majority of economic growth articles published in theory journals. We only use continuous time/age/space modelling in the last section devoted to infinite-dimensional growth models (by construction). Furthermore, we will not spend space on deep technicalities in that we will not describe formally the mathematical tools employed in the papers surveyed.

The paper is structured as follows. Section 2 develops a generic multisector optimal growth models and provide with the most important (known) results concerning existence of optimal steady states and turnpike properties. Section 3 specializes in the class of Ramsey models, briefly developing some of the most salient refinements of the model (heterogenity of agents, endogenous disocunting,...). Section 4 is a compact exposition of the stochastic growth contributions of the *JME* authors while Section 5 is a more comprehensive survey of the rich literature on endogenous fluctuations in two-sector models. Finally Section 6 and 7 report on two more recent trends, green growth and infinite-dimensional growth models.

2 A generic multisector optimal growth model

2.1 Temporary equilibrium and the social production function

Standard multi-sector models of optimal growth generally describe an economy composed of a pure consumption good y_0 and n capital goods y_j, $j = 1, \dots, n$. The labor supply is assumed to be inelastic. Total labor is normalised to 1 and each good is produced with a standard constant returns to scale technology:

$$\begin{aligned} y_0 &= f^0(k_{10}, \dots, k_{n0}, l_0), \\ y_j &= f^j(k_{1j}, \dots, k_{nj}, l_j) \end{aligned}$$

[4]For example, we have not devoted a full section to endogenous growth theory, it is referred to along the way despite becoming a definitely more active publication growth theory area in the last decade, see Etro [55]. Also the thin *JME* finance-growth stream has been put aside despite a few recent remarkable contributions such as Miao and Wang [91], we however discuss the implications of borrowing/debt constraints in Section 3.

with k_{ij} the amount of capital good i used in the production of good j, $\sum_{j=0}^{n} k_{ij} \leq k_i$, k_i being the total stock of capital i, and l_j the amount of labor used in the production of good j with $\sum_{j=0}^{n} l_j \leq 1$.

Assumption 1. *Each production function $f^j : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$, $j = 0, 1, \ldots, n$, is C^2, increasing in each argument, concave and homogeneous of degree one.*

At any given date t, assuming that the capital stocks accumulated up to that date and the levels of production of the investment goods are known, i.e. for any given $(k, y) = (k_1, \ldots, k_n, y_1, \ldots, y_n)$, the question is to determine the *temporary equilibrium*, i.e. the best allocations of capital and labor between the $n+1$ sectors in order to obtain the highest level of production of the consumption good y_0. For any given $(k, y) = (k_1, \ldots, k_n, y_1, \ldots, y_n)$, we then solve the following maximization problem:

$$\begin{aligned} T(k,y) = \quad & \max_{k_{ij}, l_j} \quad f^0(k_{10}, \ldots, k_{n0}, l_0) \\ & s.t. \quad y_j \leq f^j(k_{1j}, \ldots, k_{nj}, l_j), \quad j = 1, \ldots, n, \\ & \qquad \sum_{i=0}^{n} k_{ji} \leq k_j, \qquad\qquad\qquad j = 1, \ldots, n, \\ & \qquad \sum_{i=0}^{n} l_i \leq 1, \\ & \qquad k_{ij}, l_j \geq 0, \qquad\qquad\qquad i = 1, \ldots, n, \;\; j = 0, \ldots, n. \end{aligned} \tag{1}$$

The value function $T(k, y)$ is called the *social production function*. Under Assumption 1, basic properties of $T(k, y)$ have been established by Benhabib et Nishimura [22]:

Lemma 1. *Under Assumption 1, $T(k, y)$ is C^2, increasing in k, decreasing in y, and non-strictly concave with $T_{11}(k, y)$ in (k, y) a negative definite matrix.*

From the first order conditions corresponding to the optimization problem (1), it is easy show that the first derivatives of the social production function give the rental rates of capital, $w_i(k, y)$, and the prices of the investment goods, $q_i(k, y)$, all in terms of the price of the consumption good which is chosen as the numéraire:

$$\begin{aligned} \frac{\partial T}{\partial k_i}(k,y) &= w_i(k,y), \quad i = 1, \ldots, n, \\ \frac{\partial T}{\partial y_i}(k,y) &= -q_i(k,y), \quad i = 1, \ldots, n. \end{aligned} \tag{2}$$

Let us define the technical coefficients of capital and labor in each sector:

$$a_{0j}(k,y) = \tfrac{l_j}{y_j} \text{ and } a_{ij}(k,y) = \tfrac{k_{ij}}{y_j}, \text{ for } i = 1, \ldots, n, \;\; j = 0, \ldots, n \tag{3}$$

where a_{0j} measures the quantity of labor used to produce one unit of good $j = 0, ..., n$ and a_{ij} measures the quantity of capital $i = 1, ..., n$ used to produce one unit of good $j = 0, ..., n$. Denoting

$$T_{11}(k,y) = \begin{pmatrix} \frac{\partial^2 T}{\partial k_1^2} & \cdots & \frac{\partial^2 T}{\partial k_1 \partial k_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial k_n \partial k_1} & \cdots & \frac{\partial^2 T}{\partial k_n^2} \end{pmatrix}, \; T_{12}(k,y) = \begin{pmatrix} \frac{\partial^2 T}{\partial k_1 \partial y_1} & \cdots & \frac{\partial^2 T}{\partial k_1 \partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial k_n \partial y_1} & \cdots & \frac{\partial^2 T}{\partial k_n \partial y_n} \end{pmatrix},$$

it is then easy to show that

$$\begin{aligned} T_{12}(k,y) &= -T_{11}(k,y)B \\ &\equiv -T_{11}(k,y) \begin{pmatrix} b_{110} & \cdots & b_{1n0} \\ \vdots & \ddots & \vdots \\ b_{n10} & \cdots & b_{nn0} \end{pmatrix} \begin{pmatrix} a_{01} & & 0 \\ & \ddots & \\ 0 & & a_{0n} \end{pmatrix} \end{aligned} \tag{4}$$

where

$$b_{ji0} = \frac{a_{ji}}{a_{0i}} - \frac{a_{j0}}{a_{00}} \tag{5}$$

is the relative intensity difference in capital j between the sector of capital i and the consumption good sector. Similarly, we get

$$T_{22}(k,y) = \begin{pmatrix} \frac{\partial^2 T}{\partial y_1^2} & \cdots & \frac{\partial^2 T}{\partial y_1 \partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial y_n \partial y_1} & \cdots & \frac{\partial^2 T}{\partial y_n^2} \end{pmatrix} = B^t T_{11}(k,y)B.$$

From Lemma 1 we know that $T_{22}(k,y)$ is a quasi-negative definite matrix. However, no such immediate property is available for B. Depending on the restrictions imposed on the capital intensity differences (5), the matrix B could be either a quasi-negative or quasi-positive definite matrix.

2.2 Two popular cases

The vast majority of papers dealing with optimal growth typically tackle two cases: the one-sector case ($n = 0$), when there is a single good produced which is either consumed or invested, and the two-sector case where the consumption and capital goods are produced in two different sectors. Section 3 and Section 4 will deal more specifically with the former while Section 5 studies the latter. The one-sector model is a degenerate case (see below) that covers the so-called Ramsey class of models, which has been the object of numerous refinements. Studies on two-sector models have flourished in the economic literature since the 60s with the seminal works of Uzawa [119], Robinson [107], Solow [111] or Srinavasan [114]. We shall summarize some of the main contributions of the *Journal of Mathematical Economics* authors to the analysis of the two model classes. We start giving here below some of the immediate technological properties of these models, namely through matrix B defined above.

The two-sector case In the particular case of a two-sector model with $n = 1$ that will be discussed later, the matrix B reduces to a scalar b such that

$$b(k, y) \equiv a_{01}(k, y) \left(\frac{a_{11}(k,y)}{a_{01}(k,y)} - \frac{a_{10}(k,y)}{a_{00}(k,y)} \right). \quad (6)$$

The sign of $b(k, y)$ is therefore positive (negative) if and only if the investment good sector is relatively more (less) capital-intensive than the consumption good sector. It follows from (4) that the sign of the cross derivative $T_{12}(k, y) = -T_{11}(k, y)b(k, y)$ is given by the sign of the capital intensity difference $b(k, y)$.

The Ramsey case It is also worthwhile to discuss the degenerate case of the model where the aggregate level of output is decomposed into consumption and investment, that is when $n = 0$. Denoting Y_t, K_t, L_t and C_t the aggregate output, aggregate capital, aggregate labor and aggregate consumption, aggregate net investment is therefore $I_t = K_{t+1} - K_t$. Let us then define output per capita $y_t = Y_t/L_t$, capital per capita $k_t = K_t/L_t$, net investment per capita $i_t = I_t/L_t$ and consumption per capita $c_t = C_t/L_t$. Assuming that capital depreciates at rate $\mu \in [0, 1]$ each period, and that labor grows at an exogenous growth rate $g \geq 0$, we get

$$i_t = (1 + g)k_{t+1} - k_t$$

and consumption can be expressed as a *social production function* which links the current capital stock to investment such that

$$c_t = f(k_t) - i_t - \mu k_t = f(k_t) + (1 - \mu)k_t - (1 + g)k_{t+1} \equiv T(k_t, k_{t+1}), \quad (7)$$

where $f(k)$ satisfies the following standard properties:

Assumption 2. *$f(k)$ is C^2 and such that $\forall k > 0$, $f'(k) > 0$, $f''(k) < 0$. Moreover, $f(0) = 0$, $f'(0) = +\infty$ and $f'(+\infty) = 0$.*

Obviously $T(k_t, k_{t+1})$ is non-strictly concave in (k_t, k_{t+1}) with $T_{11}(k_t, k_{t+1}) = f''(k) < 0$ and $T_{22}(k_t, k_{t+1}) = 0$. Moreover, we easily get that the cross derivative satisfies

$$T_{12}(k_t, k_{t+1}) \equiv \frac{\partial^2 T(k_t,k_{t+1})}{\partial k_t \partial k_{t+1}} = 0$$

which means that the elasticity of substitution between the current capital stock and investment is infinite. We could also recover this result from the equation (6) giving the capital intensity difference assuming that the two sectors are identical and thus produce the same unique good with $a_{11} = a_{10}$, $a_{01} = a_{00}$ and $b(k, y) = 0$.

2.3 Multisector optimal growth models: Set-up and remarkable properties

We consider an economy populated by a number N_t of identical infinitely-lived agents. To be consistent with the previous section, we assume that the total population also grows at the rate $g \geq 0$, i.e. $N_{t+1} = (1+g)N_t$. The representative consumer offers a unit of labor (inelastic) in each period and draws utility from his consumption from the following function:

Assumption 3. *$u(c)$ is C^2 and such that $\forall c > 0$, $u'(c) > 0$, $u''(c) < 0$. Moreover, $u(0) = 0$, $u'(0) = +\infty$ and $u'(+\infty) = 0$.*

The equilibrium on the labor market implies that $N_t = L_t$. The maximisation program of the representative agent is therefore:

$$\begin{array}{lll} \max\limits_{\{y_{jt}\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{+\infty} \delta^t u(T(k_t, y_t)) & \\ s.t. & (1+g)k_{jt+1} = y_{jt} + (1-\mu)k_{jt}, & j = 1, \dots, n, \\ & k_{j0} \text{ given} & j = 1, \dots, n \end{array} \tag{8}$$

with $k_t = (k_{1t}, \dots, k_{nt})$ and $y_t = (y_{1t}, \dots, y_{nt})$, where $\delta \in (0,1)$ is the discount factor that describes the rate of preference for present of the representative consumer. The rate of depreciation $\mu_j \in [0,1]$ of capital $j = 1, \dots, n$ allows to define the following matrix

$$1 - \mu = \begin{pmatrix} 1-\mu_1 & & 0 \\ & \ddots & \\ 0 & & 1-\mu_n \end{pmatrix}$$

In the literature, it is standard to reformulate the intertemporal optimization program of the representative agent under a reduced form which is more general. From the fundamental ingredients of the model, in particular the social production function $T(k, y)$ which provides the optimal output of the consumption good and thus the consumption level as a function of (k, y), we may indeed define the *indirect utility function*:

$$V(k_t, k_{t+1}) = u(T(k_t, (1+g)k_{t+1} - (1-\mu)k_t)).$$

Under Assumptions 1 and 3, $V(x, y)$ is increasing in x, decreasing in y and strictly concave. The maximisation program (8) may then be written as follows

$$\begin{array}{ll} \max\limits_{\{k_t\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}) \\ s.t. & (k_t, k_{t+1}) \in \mathcal{D}, \\ & k_0 \text{ given} \end{array} \tag{9}$$

with the set of admissible paths

$$\mathcal{D} = \left\{ (k_t, k_{t+1}) \in \mathbb{R}^{2n}_+ / \frac{(1-\mu)}{1+g} k_t \leq k_{t+1} \leq \frac{h(k_t) + (1-\mu)k_t}{1+g} \right\}$$

and $h(k_t)$ such that $T(k_t, h(k_t)) = 0$. The first order conditions for an interior maximum are given by the Euler equations

$$\frac{\partial V}{k_{jt+1}}(k_t, k_{t+1}) + \delta \frac{\partial V}{k_{jt+1}}(k_{t+1}, k_{t+2}) = 0, \quad j = 1, \ldots, n \tag{10}$$

which represent a set of n second-order non-linear implicit difference equations. We also need to satisfy the n transversality conditions

$$\lim_{t \to +\infty} \delta^t k_{jt} \frac{\partial V}{k_{jt}}(k_t, k_{t+1}) = 0, \quad j = 1, \ldots, n. \tag{11}$$

Any capital path $\{k_t\}_{t=0}^{+\infty} = \{(k_{1t}, \ldots, k_{nt})\}_{t=0}^{+\infty}$ that satisfies conditions (10)-(11) is an *intertemporal optimal equilibrium.*

The turnpike property An optimal steady state (OSS) is a stationary optimal path $k_t = k^*_\delta = (k^*_{1\delta}, \ldots, k^*_{n\delta})$ for all $t \geq 0$ solution of the Euler equations (10). It can be proved that (see e.g. Scheinkman [108])

Proposition 1. *Under Assumptions 1 - 3, there exists $\epsilon > 0$ such that if $1 \geq \delta > 1 - \epsilon$, there exists only one OSS k^*_δ with $c^*_{0\delta} = T(k^*_\delta, (\mu + g)k^*_\delta)$.*

The OSS is usually called *the turnpike* in optimal growth theory and corresponds to the *Modified Golden Rule.* And the *turnpike property* is associated to the stability property of the OSS which guarantees that the optimal path is converging toward the OSS. It is well-known since the contribution of Scheinkman [108] that if the Hessian matrix of the indirect utility function $V(k_t, k_{t+1})$ evaluated at the steady state is negative-definite, the turnpike property holds. Actually, Scheinkman [108] first proves that the optimal paths visit neighborhoods of the OSS and second that the OSS is locally stable. The following neighborhood turnpike theorem is then obtained (see Scheinkman [108]):

Theorem 1. *Under Assumptions 1 - 3, there exists $\epsilon > 0$ such that if $1 \geq \delta > \delta^* \equiv 1 - \epsilon$, there is a unique optimal path which converges toward the unique OSS k^*_δ.*[5]

[5] A stronger conclusion proving a uniform neighborhood turnpike theorem has been established by Guerrero-Luchtenberg [65] under stronger assumptions. The uniform neighborhood turnpike property allows to derive that chaotic dynamic vanishes as the discount factor tends to one.

While some global stability conditions of the OSS have been provided, the literature has mainly focused on the cases where the stability of the OSS is lost. Indeed, it can be shown for instance that when δ crosses δ^* from above, the turnpike property may be lost and there may exist some endogenous fluctuations.

Haurie [69] defines the optimal control over an infinite time horizon, and investigates the existence and the asymptotic behaviour of optimal trajectories for a class of convex systems which encompasses many continuous time economic growth models. An important finding of this paper is that the turnpike property and the nice asymptotic behaviour of optimal trajectories provide the needed conditions and lead to fairly general conditions of existence of a solution to the problem. An additional contribution is that the general results do not depend upon the dimension of the state space of the system, thus, there is no need for the phase diagram analysis.

Hori [70] analyzes the asymptotic properties of rolling plans in a multisector growth model with time-independent preferences and technology. Assuming that the model under consideration has a unique turnpike towards which all the finite optimal programs bend. Furthermore, if plans are constantly revised with a fixed but sufficiently long planning horizon, the resulting growth path converges to a neighborhood of the turnpike.[6]

Finally, general comparative statics have been established in the multisector case by Amir [8]. Using lattice programming, he develops sufficient conditions for the value function to be monotone and supermodular, and for the optimal policies to be monotone in the state and in other parameters. Interestingly enough, he finds that the comparative statics with respect to the structural parameters of the model (like the discount rate) are based much more restrictive conditions than those for the state variables, which derive quite naturally from the one-sector Ramsey case, studied in detail by Takekuma [118].[7]

Due to lack of space, we overlook here part of the very rich technical literature which has been developed in the multisector case to deal with existence and uniqueness of optimal solutions to problem (8). In particular, we do not touch the literature around the applicability of the contraction mapping theorems and alternatives (depending in particular on the boundedness of the utility functions). The reader can refer to Le Van and Morhaim [83] for more details, a representative *JME* contribution to this important literature being Ha-Huy and Thien Tran [66], among others. We now specialize in the traditional one-sector Ramsey model.

[6] The multisector linear case is deeply explored in Freni *et al.* [59].

[7] Another important contribution in the one-sector Ramsey case is due to Amir *et al.* [9] who provide comparative dynamics results.

3 The Ramsey model: Extensions and refinements

A huge literature has been devoted to the Ramsey model either weakening or relaxing the classical assumptions (such like Assumptions 1, 2 or 3), or exploring the implications of even more structural changes such as non-additively separable social welfare functions, heterogenous agents, debt constraints or endogenous discounting. We shall summarize some of the *JME* output on these different issues, starting with non-convex Ramsey models.[8]

3.1 Non-convex/non-differentiable problems

The seminal paper of optimal growth with convex-concave production functions is of course Dechert and Nishimura [45], generalizing Skiba's earlier work [110]. Basically, this seminal general analysis of the convex-concave production case delivered two important results, one about the characterization of optimal paths (monotonicity and convergence to steady states), and a more intriguing finding on the emergence of poverty traps, which can be in turn related to history dependence. The latter is intricately related to the existence of a threshold capital stock (usually called *DNS point* for Dechert-Nishimura-Skiba) such that any trajectory starting below it goes to zero (the reverse holding if the economy starts above, ultimately converging to a nonzero stationary state).

It is out of the scope of this survey to review the abundant literature following Dechert and Nishimura's paper, in particular on the history dependence outcome. We shall single out two *JME* contributions to this line of research, namely Akao *et al.* [2] and Hartl and Kort [68]. To our knowledge, the former is the first paper rigorously deriving comparative statics for the threshold capital stock. A key economic parameter is the discount rate: more patient countries are supposed to be more prone to investment, therefore potentially better equipped to avoid poverty traps. The first question addressed by the authors is therefore whether the threshold capital stock goes down with the discount rate (inversely related to the impatience rate in their model). A second even more important question is whether this parametric dependence is continuous or not. Concerning the first question, Akao *et al.* do find that more patient countries (with larger discount rates) are associated with smaller threshold capital stocks, which indeed make them less likely to get into poverty traps. More interestingly (and much more involved technically speaking), the authors also prove continuity of the DNS point with respect to the discount rate. This is an important result as it rules out the possibility to have sudden regime changes (such as a growing economy suddenly dooming to shrink).

[8] We do not detail here some of the refinements such as those related to the endogenization of labor supply. See Goenka and Nguyen [62] and Le Van *et al.* [84] for example.

Hartl and Kort [68] propose a different mechanism through which history dependence emerges in a partial equilibrium capital accumulation problem. Rather than relying on a convex-concave production function, they study a concave capital accumulation problem which entails a pointwise non-differentiability. While the revenue function is increasing and concave in the stock of capital, it has a kink: the first order derivative jumps upwards at a given capital level. This assumption may be justified on different grounds at the firm level as explained by the authors, quite differently from the typical justification of convex-concave production functions in the tradition of DNS (increasing returns at the macro level). The main outcome of this work is the emergence of multiple equilibria, and the (numerical) identification of DNS points with different optimal paths (and steady states) occuring depending on the relative position of the initial capital stocks with respect to the DNS points.[9]

3.2 Borrowing and debt constraints

The aggregate model has also been studied considering different types of constraints affecting the behavior of agents. Building on the seminal contribution of Becker [21], borrowing constraints have been introduced in models with heterogeneous agents characterized by different discount rates (see also Section 3.5). Assuming that there are $J > 1$ households indexed by $j = 1, \dots, J$, Becker shows that if they are not allowed to borrow against their future wage incomes, i.e. $k_{jt} \geq 0$ for $j = 1, \dots, J$, the equilibrium is characterized by some wealth inequality with the most patient agents holding the whole amount of capital.[10] Becker *et al.* [20] have relaxed the no borrowing condition by allowing the households to borrow against their future wage incomes for an exogenous maximum number $N \in \mathbb{N}$ of periods before debt must be repaid. Now at time t, each household can borrow against the wage earned from time $t + 1$ to $t + N$, i.e., a household could have negative savings at any time t which is bounded below by the present value of the prevailing wage in time period $t + 1$ to $t + N$. Let us indeed denote the present value of future wages over the period from $t+1$ to $t+N$

$$A(t+1, N) = \frac{w_{t+1}}{(1+r_{t+1})} + \cdots + \frac{w_{t+N}}{(1+r_{t+1})\dots(1+r_{t+N})}$$

[9] In another *JME* contribution, Banerjee and Mitra [17] consider production functions which are both non-differentiable and non-concave to explore the robustness of the equivalence between the set of utilitarian maximal programs and the set of "weakly maximal" programs from any positive initial stock, first pointed out by Brock [42], in such a non-smooth context.

[10] Similar positivity constraints have also been introduced in models with a representative agents to study the impact of irreversible investment. Mitra and Ray [94] for instance provide a full characterization of efficient and optimal programs under the constraint that depreciated capital stock cannot be used for present consumption.

A household could then have negative savings at any time t which is bounded below by the per capita discounted wealth $A(t+1,N)/J$. Hence, for $j = 1,\dots,J$,[11] Becker *et al.* [20] assume that

$$s_t^j + \frac{w_{t+1}}{(1+r_{t+1})J} + \cdots + \frac{w_{t+N}}{(1+r_{t+1})\dots(1+r_{t+N})J} \geq 0.$$

Under such a liberal borrowing constraint, they prove that there exists a unique stationary Ramsey equilibrium where all households except the most patient one are indebted and all their wage incomes are spent for the payment of their debts. The impatient households never own capital and in every period they consume by borrowing against the next period wage income. As in the no-borrowing economy considered by Becker [21], the most patient household owns all capital and all debts of the other households. However, the output is distributed among the households in a somewhat different way: under a liberal borrowing constraint, the steady state consumption of the impatient households is a decreasing function of N, and equivalently the patient household's consumption rises with N and approaches the aggregate consumption as N tends to infinity. As a consequence, the steady state wealth inequality increases as the credit regime is liberalized.[12]

Aggregate Ramsey models are also used to study with the interplay between debt, capital and dynamics. Nishimura *et al.* [99] consider a model in which a constant level of public spending is financed through debt and distortionary taxation on income and debt earnings. To avoid insolvency of public debt, they assume a debt constraint defined as a constant ratio of debt over GDP. This ratio is considered as a policy parameter fixed by the government. The budget constraint of the representative household and the government are respectively given by

$$\begin{aligned} c_t + k_{t+1} + b_{t+1} &= k_t[(1-\tau_t)r_t + 1 - \mu] + (1-\tau_t)\bar{r}_t b_t + (1-\tau_t)w_t, \\ G + \bar{r}_t b_t &= \tau_t(y_t + \bar{r}_t b_t) + b_{t+1}, \end{aligned}$$

with b_t the amount of debt issued by the government at date t, r_t the real interest rate on physical capital, $\bar{r}_t$ the return of government bonds and $\tau_t \in (0,1)$ the tax rate on income and bonds return. Moreover, debt is assumed to be a fixed proportion of GDP, namely $b_t = \alpha y_t$ with $\alpha \geq 0$, in accordance for instance with the Maastricht criteria. It follows that the tax rate adjusts at each period to fulfill the intertemporal budget constraint of the government. Since capital and government bonds are perfect substitutable assets, we obviously get the equality $(1-\tau_t)r_t + 1 - \mu = (1-\tau_t)\bar{r}_t$.

Nishimura *et al.* [99] show that such a non-linear tax rule can be a source of macroeconomic instability related to self-fulfilling expectations on

[11] Borissov and Dubey [28] have considered the particular case $N = 1$.

[12] Becker *et al.* [20] also show that if the equilibrium path converges to the unique steady state, then the turnpike property holds and the equilibrium is also efficient. They also provide an example in which period-two equilibrium cycles can exist.

the future income tax rate. Indeed, in the economy with debt, if agents expect an increase of the future tax rate, they will invest less, implying a lower income in the future. According to the debt constraint, debt emission should be lower, and therefore the income tax rate has to increase today to satisfy the government intertemporal budget constraint. Self-fulfilling equilibria then generate sunspots and endogenous fluctuations (see Section 5.2). It is also shown that this mechanism is promoted by a larger ratio of debt over GDP.

3.3 Alternative intertemporal utility functions

Most of optimal growth models use additively separable social welfare functions (or individual intertemporal utility functions). This assumption has been discussed for a long time. One of the earliest *JME* contributions to this line of research is due to Epstein [53] who constructs the so-called *implicitly additive* utility functions. The main idea comes as follows. As any optimization problem, optimal growth problems properties depend on their sets of constraints and on indifferences curves at the optimum. What if one assesses the validity of the turnpike properties of the optimal growth models when the intertemporal utility function is **not** additive, but has the "*property that each of its indifference sets is also an indifference set for some other utility function which is additive and has a constant rate of discount*"? In the view of Epstein, while several studies have already explored at that time the robustness of the turnpike property even with respect to the specification of the intertemporal utility functions, it is of utmost importance to determine which properties of preferences are responsible for this robustness to hold if so. These implicit additive intertemporal utility functions are later characterized and the robustness analysis of the inherent turnpike property performed and established.

Two other sets of *JME* contributions are more represented in this area of research, one of course dealing with recursive utility, and the other related to intergenerational justice and sustainability issues. In the first set, one can cite the notable paper of Dana and Le Van [44]. The key methodological contribution of the authors is to establish that the problem of Pareto optima characterization for a stationary intertemporal economy where agents have recursive utilities can be transformed into a generalized McKenzie problem with recursive criterion. In a sense, Dana and Le Van extend a well known property in the case of additively separable utilities (with appropriate weights). As in their generalized McKenzie problem where Bellman and Euler equations still hold, they are able to provide turnpike theorems for the stationary recursive preferences case. Admittedly, McKenzie's framing is quite good for that (McKenzie [90]).

As a representative of the second set of contributions, Banerjee [16]'s work on the so-called Suppes-Sen grading principle (i.e, equal treatment for all generations and Pareto efficiency) is worth mentioning. A highly interesting research question is the relationship between this principle and the so-called Brundtland sustainability criterion which stipulates that sustainable policies are those which meet the needs of current generations without compromising the ability of the future generations to meet theirs. This intuitive question has been treated by several authors, notably Asheim *et al.* [11] and Dubey and Mitra [52]. Incidentally, both criteria imply that time discounting inherent in the neoclassical growth model is ethically unacceptable. Banerjee [16] studies the implications of the Suppes-Sen grading principle in the context of the undiscounted neoclassical growth model (of course under the overtaking optimality criterion due to the absence of discounting). He demonstrates the existence of cyclical consumption paths that are maximal according to the Suppes-Sen grading principle. This finding implies that using an equitable quasi-order (*i.e* reflexive and transitive relation) in evaluating social states such as the Suppes-Sen principle does not guarantee Bruntland sustainability contrary to pre-existing studies building on different growth frameworks (such as Asheim *et al.* [11]).

3.4 Endogenous discounting

Endogenous discounting is one of the trickiest extensions of the Ramsey model as it can potentially break down the concavity of the associated optimal growth model. Several conceptual approaches to endogenizing discounting have been taken in the literature. We single out here three taken from the *JME* literature.

The first approach is suitable in a context where the social status is relevant, which is obviously the case in the study of inequalities. In such a context, it seems reasonable to assume that households with a higher consumption (or income) discount future less, since they can afford to defer consumption of additional income and wealth. This ultimately paves the way to model the discount rate as a function of consumption (income). This is the key specification in Iwasa and Zhao [74], following one strand of the endogenous time preference literature (see for example Lucas and Stokey [86]). Precisely, Iwasa and Zhao define the discount rate as a function of the present level of consumption, $\delta = \delta(c_t)$, where the discounting factor is decreasing and convex in terms of consumption: $\delta'(c_t) < 0 < \delta''(c_t)$, which covers the reasonable rationale invoked just above. In a neoclassical growth model with this type of endogenous discounting, several lessons can be drawn. Consider the simple case of a society with only two types of household: rich and poor households. If the individuals in this society are characterized by a decreasing marginal impatience, a drop in the share of rich households raises poor households' income and consumption, contrary to the benchmark case with constant marginal impatience. Furthermore, inequality exhibits an inverted-U shape as more people become rich. A notable policy implication of the model is that a tax on capital income reduces poor households' income when the fraction of the rich is sufficiently small.

The above specification of the discount rate usually ensures stable optimal capital sequences that converge to a unique steady state independently of the initial conditions. Nonetheless, there is another strand of literature (see Becker and Mulligan [27] for example) that builds in contrast on the postulate the agents get more patient as they grow richer, which in turn would impose that the discount rate depends upon the stock of wealth (and not on flow variables like before). Following this strand of literature, Erol *et al.* [54] define the discount rate as

Assumption 4.

$$\delta^t = \Pi_{s=1}^t \, \beta(k_s),$$

where function β continuous, differentiable, strictly increasing and satisfies $\sup\limits_{k>0} \beta(k) = \beta_m < 1$ *and* $\sup\limits_{k>0} \beta'(k) < +\infty$*.*

Then the following properties along optimal path can be proved, including multiplicity (of steady states):

Proposition 2. *Suppose Assumptions 2, 3 and 4 hold. Then*

- *there exists an optimal path k_t and the associated optimal consumption path c_t is given by $c_t = f(k_t) - k_{t+1}$, $\forall t$;*
- *if $k_0 > 0$, then every solution (k, c) to the optimal growth model satisfies $k_t > 0$ and $c_t > 0$, $\forall t$;*
- *the optimal path k starting from k_0 is monotonic;*
- *there exists an $\epsilon > 0$ such that if $\sup_{k>0} f'(k) < (1-\epsilon)/\beta_m$, any optimal path converges to zero;*
- *if $k_0 > 0$, $\inf\limits_{k>0} \beta(k) = \underline{\beta}$ and $f'(0) > 1/\underline{\beta}$ and suppose there are exactly two optimal steady states: $k_l < k_h$, then there exists $k_c \in [k_l, k_h]$ such that any optimal path k starting from k_0 converges to k_l if $k_0 < k_c$ and converges to k_h if $k_0 > k_c$.*

Last but not least, the recent *green* growth literature emphasizes that the time discount rate may also depend upon individuals' perceptions of key environmental indicators. In this spirit, Schumacher and Zou [109] specify the time preference rate as a function of environmental quality. More precisely, the authors study how *discrete* changes in the pro-environmental preferences of individuals (either due to exogenous shocks or to active government campaining) can affect economic and ecological dynamics. To this end they introduce the idea of threshold preferences to investigate the impact of discrete changes to preferences on the trade-off between consumption and the environment. They do that that in a two-period overlapping generation where the environmental quality dynamics entirely depend on the the actions taken by the individuals, their consumption and abatement efforts. The key preference parameter is the relative preference of the generations for the environment over consumption: this parameter takes two different values (say high and low) depending on the position of current environmental quality relative to a given threshold value. If a (young) individual faces a level of environmental quality below the threshold, their preferences will be more directed towards environmental quality. The theoretical findings (and subsequent policy implications) are quite interesting: for low (high) thresholds, environmental quality converges to a low (high) steady state, while for intermediate levels it converges to a stable p-cycle, with environmental quality being asymptotically bounded below and above by the low and high steady state.

3.5 Heterogenous agents

Not surprisingly, as a journal which has a long standing interest in general equilibrium theory, growth settings with heterogenous agents are not scarce in the *JME*, an obvious example being Bewley [26]. We single out here two contributions, one building somehow on Bewely's initial works, and a more recent one with a large set of heterogeneities including notably endgenous labor supply, a hardly visited avenue in this research strand. Both contributions discusses however similar questions, the typical ones addresses in optimal growth models with heterogenous discount rates (following Becker [21]) as we will see.

Jensen [75] studies unbounded growth with heterogenous consumers where heterogeneity originates in the discounting rates and initial distribution of incomes as well as from the utility functions. One main difference with respect to the seminal contribution of Bewley [26] is that the set of feasible consumption plans under the setting of Jensen is not assumed to be uniformly bounded over time: the production technology is not restricted by decreasing returns. Furthermore, while all consumers hold homogenous instant utility functions,[13] heterogeneity in the homogeneity degree rules out the existence of a representative agent with homogenous preferences, therefore leading to depart from the standard endogenous growth setting *à la* Alvarez and Stokey [7]. One of the very nice contributions of this paper is

[13]The typical assumption to formalize this goes as follows:

Assumption 5. *For every consumer i, there exists a positive, affine transformation of instant utility u^i, which is homogenous of degree $\alpha_i < 1$.*

indeed to integrate the general equilibrium frame devised by Bewley [21] and increasing returns. Jensen shows, among several other interesting results, that with unbounded growth the long-run distribution of income is endogenous in the sense that different initial distributions of income generally lead to different balanced growth equilibria with different associated long-run distributions of income. That is not the case with bounded growth models *à la* Bewley: in such a case the consumers with the highest discount factors eventually end up holding all wealth, making the long-run distribution of income ultimately exogenous.

Bosi and Seegmuller [31] consider heterogenous agents and endogenous labor supply at the same time, the latter being rather hardly studied in this stream of literature (see the seminal paper of Becker [21]). In addition, the two previous features are combined with a rather rich set of heterogeneities: indeed, heterogeneity shows up not only in capital endowment, but also in time preference and intra-temporal preferences in terms of consumption and labor. Differently from the endogenous time preference discussed in the last subsection, the $n+1$ infinite-lived agents are ranked according to their (exogenous) time preference in a decreasing ranking order:

$$0 \leq \delta_n \leq \delta_{n-1} \leq \cdots \leq \delta_1 < \delta_0 < 1.$$

Obviously, agent 0 is the most patient one and is strictly more patient than the others to guarantee the heterogeneity. Bosi and Seegmuller also introduce borrowing constraints which prevent the impatient agents from consuming more today and working hard tomorrow to refund the debt as in Le Van *et al.* [84] optimal growth model, where impatient agents experience vanishing consumption bundles and leisure time in the long run.

The key contribution of the paper results however from the introduction of endogenous labor supply in a setting with heterogenous agents. Part of the findings obtained at equilibrium still echoes known properties obtained in the tradition heterogenous agents literature *à la* Becker where the most patient household (the capitalist) owns the whole capital stock at a steady state, whereas the others consume their per-period labor income. Key new results derive from the addition of the endogenous labor supply channel. First, two types of steady states emerge depending on the marginal utility from consumption and leisure: one where the most patient agent works and one where he does not. Second, at a steady state, two social classes emerge: the capitalist, agent 0, who smooths his consumption across time and invests, and n workers who consume at each period their whole labor earnings, without investing.

4 Stochastic growth

To take account of the role of uncertainty, several papers in the economics/mathematics literature have considered economic growth problems where the state/control variables are stochastic processes and the state equation is a stochastic difference equation or a stochastic differential equation (SDE). For the discrete time case one can see, e.g., the book [1, Chapter 17] while, for the continuous time case, we recall the books [89, Chapter 3] and [96, Chapter 9]. Notable contributions on this subject have been published in JME, on various related issues. Following the structure of the present survey, we concentrate ourselves on the ones which seem more significant for their impact from the mathematical/economic viewpoint, in the discrete time case. In particular, we look at the existence of optimal strategies, at alternative approaches to solve the related Bellman equations, at the properties of long-run invariant distributions.

4.1 Existence of optimal strategies

The first point is the existence of optimal strategies. This issue is studied, in the discrete time case, in the book [117] and in the JME paper [43]. In particular the last one concerns also infinite horizon cases in presence of imperfect information and nonconvexities. Note first that existence theorems in a stochastic control environment are more difficult than in the deterministic case, as the randomness makes the control strategies dependent not only on time, but also on the stochastic variable (typically denoted by ω). This means that, with respect to the deterministic case, existence theorems are technically more involved and need a bigger amount of nontrivial functional analysis. In particular we go deeper into the framework and the results of Chichilnisky [43]. In such paper the returns are stochastic and their law is described by Probability Distribution Functions (PDF) which are taken to belong to a suitable family of weighted Banach space named H_1^β.[14] The admissible strategies are time dependent, hence they are functions which associate to every $t \in \mathbb{N}$ an element of H_1^β. In such a set of sequences Chichilnisky [43] considers two types of norms: an l_2 weighted norm and a sup norm. The sets where such norms are finite are called respectively Γ and Ω. Under suitable and reasonable growth assumptions on the admissible strategies and utility functions, and on the PDF, the set of admissible strategies (which we call ψ) is shown to be bounded in the Ω sup norm. This, in turn, implies compactness of ψ in the Γ norm, using arguments which are somehow related to the ones of Sobolev embeddings.

Various extensions of such type of arguments are possible and have been

[14] H_1^β are Banach spaces isomorphic to L_1 with the finite measure induced by the density function $e^{-\beta x}$ on $\mathbb{R}_+$, $\beta \in (0\ \ 1)$ being the time discount rate of the corresponding Ramsey problem.

done in the literature, even recently. On the other hand, there are interesting cases (see, e.g. the case of CIES utility $u(c) = (1-\sigma)^{-1}[c^{1-\sigma}-1]$ when $\sigma > 1$) where the assumptions needed to apply such types of existence theorems are not verified, hence one has to resort to different ideas to prove existence, e.g. through the sufficient conditions associated to the Bellman equation. Next Section is devoted to such issue.

4.2 Dynamic programming in unbounded cases

In Ma *et al.* [87] the authors proposed a new approach to solving dynamic programs with unbounded rewards, based on Q-transforms which have their root in the machine learning algorithms. To give rough explanation, the author take a simple Bellman Equation of the following type

$$V(x) = \sup_{c \in C}\{u(c) + \beta \mathbb{E} V(f(x, c, \eta))\}$$

where x is the state variable, c the control variable, X, C are the state and control space, V is the value function, f is the state dynamics, η is a random variable, $\mathbb{E}$ is the expectation. Here the unknown is V. The author propose to change variable, calling

$$g(x, c) = \beta \mathbb{E} V(f(x, c, \eta))$$

and rewriting the BE taking g as unknown. In such a way, some cases where the utility is unbounded (like the previously mentioned case $u(c) = (1-\sigma)^{-1}[c^{1-\sigma}-1]$ when $\sigma > 1$)) are transformed in somehow easier problems where existence theorems for the BE and the consequent sufficient optimality conditions can be proved, then extending the set of treatable cases.

4.3 Invariant Distributions

Another important issue treated in the JME growth papers is the existence of invariant distributions and their stability properties. This is somehow the analogous of the study of steady states and their stability in the deterministic case. As for existence of optimal strategies, the stochastic case is technically more difficult to deal with, but it displays some interesting features that it is worth to mention.

- In Nishimura *et al.* [98] (see also Gong *et al.* [63]) the authors prove that, in a simple growth model setting with nonconvex technology, only two alternatives arise: either there exists a unique globally stable invariant distribution or the economy collapses. This is different from the analogous deterministic case as the authors clarify: "*Nonconvex technology introduces the possibility that many optimal policies exist for the one economy. For these models it has been shown (Dechert and*

Nishimura [45], Lemma 6) that different optimal trajectories can have very different dynamics, even from the same initial condition. Indeed, there may be two optimal policies such that the optimal path generated by the first sustains a nontrivial long run equilibrium, whereas that generated by the second leads to economic collapse. For our stochastic model this is not possible."

This phenomenon arises also in many others cases. Somehow the presence of disturbances, under certain conditions, "regularizes the system" in various ways: selecting one solution among many possible ones in the analogous deterministic model (see e.g. Nishimura *et al.* [98]); making (in continuous time) the solution of the associated closed loop equation or HJB equation more smooth (see e.g. Flandoli *et al.* [57]); stabilizing unstable equilibria as in Arnold *et al.* [10]; and other similar features.

- In Mitra and Privileggi [92] the authors show that in a simple stochastic growth model, the long run behavior of optimal paths has an attractor given by the Cantor set, hence the invariant distribution is singular with respect to the Lebesgue measure. This fact shows how very complex dynamics can arise in simple stochastic models, a well known fact that show how careful one has to be in designing and interpreting such types of mathematical models.
- In Zhang [124] (see also Kamihigashi [78] and Stachurski [115]) various results on existence, uniqueness and stability of invariant distributions are proved, also in cases which display non convexity and/or non compactness. Such results are technically involved and, as the authors explain in their conclusions, establish a frontier out of which one cannot expect results of this type.

5 Endogenous fluctuations in two-sector models

As outlined in the Introduction, two-sector models are a very important and classical part of growth theory. The complexity of the two-sector models even in the special RSS case (see Khan and Mitra [80] and Khan and Zaslavski [79]) has stimulated a very large literature, a substantial part has been published in the *JME*.[15] Since multisector models are not systemati-

[15]Of course, the main reason behind the complexity comes from the fact that these models can be hardly solved analytically. This is a particularly acute problem in the case of two-sector endogenous growth models in which balanced growth paths levels are indeterminate by construction. In some special cases such as the Lucas-Uzawa model in continuous time, one can solve in closed-form the optimal dynamics using Gaussian hypergeometric functions, therefore overcoming the latter indeterminacy problem. See Boucekkine and Ruiz-Tamarit [37] and Boucekkine *et al.* [35].

cally characterized by some stability of the OSS, the question then immediately arises as to what kind of dynamic behavior might emerge when the stability conditions are not satisfied. With the revival of nonlinear dynamical systems theory in the 1970s, a great deal of work has focused on the analysis of endogenous fluctuations in optimal growth models. The aim is to show that, despite the strong assumptions of pure and perfect competition and perfect forecasts, the behavior of economic agents can be the source of macroeconomic fluctuations.

Based on these type of results, most economists have then reached the conclusion that macroeconomic fluctuations cannot be explained solely by stochastic shocks affecting technology or preferences, as asserted by the Real Business Cycle Theory. But still explaining actual business cycles by regular fluctuations under some perfect competition and perfect foresight assumptions may not be satisfactory. We need also to study the existence of business cycles explained by the expectations of economic agents in frameworks where imperfections may exist. Along these lines, a large literature has studied fluctuations generated by the volatility of individuals' beliefs. Considering market imperfections such as external effects in production has led to the concept of sunspot equilibria, along which business cycle fluctuations are driven by self-fulfilling beliefs.

5.1 Optimal endogenous fluctuations

In a two-sector economy, linearizing the corresponding Euler equation derived from (10) around the OSS k^*_δ leads to the following degree-two polynomial

$$P(\lambda) = \lambda^2 \delta V^*_{12} + \lambda(\delta V^*_{11} + V^*_{22}) + V^*_{12} = 0 \tag{12}$$

where $V^*_{ij} = V_{ij}(k^*_\delta, k^*_\delta)$ corresponds to the second order derivatives of the indirect utility function evaluated at the steady state. The two-sector formulation is characterized by a strong property. Under Assumptions 1 and 3, it can be easily shown that the characteristic roots solutions of (12) are necessarily real and that their sign is given by the sign of the cross derivative V^*_{12}. Moreover, if λ is a characteristic root then $1/(\delta\lambda)$ is also a characteristic root.[16]

Let us define the following elasticities:
- the *elasticity of intertemporal substitution in consumption*

$$\epsilon_c = -\left(u''(c^*_\delta)c^*_\delta/u'(c^*_\delta)\right)^{-1} > 0, \tag{13}$$

which measures the increase of future consumption necessary to compensate a 1% decrease of current consumption,

[16] Of course complex roots can occur if $n \geq 2$, i.e. there are at least 2 investment goods beside the consumption good.

- the *elasticity of the consumption good output* $y_0 = T(k, y)$ *with respect to the capital stock*

$$\varepsilon_{ck} = T_1^*(k_\delta^*, (\mu + g)k_\delta^*)k_\delta^*/T^*(k_\delta^*, (\mu + g)k_\delta^*) > 0, \tag{14}$$

which measures the increase of y_0 when k increases by 1%, and

- the *elasticity of the rental rate of capital* w_1 *with respect to the capital stock*

$$\varepsilon_{w_1k} = -T_{11}^*(k_\delta^*, (\mu + g)k_\delta^*)k_\delta^*/T_1^*(k_\delta^*, (\mu + g)k_\delta^*) > 0, \tag{15}$$

which measures the decrease of w_1 when k increases by 1%.

The stability properties of the OSS crucially depend on the sign of the cross derivative V_{12}^* as given by:

$$V_{12}^* = u''(c_\delta^*)c_\delta^*T_{11}^*(1 + g)\left[\frac{\varepsilon_{ck}}{\varepsilon_{w_1k}}\frac{(1+g)\theta^2}{\delta} + \epsilon_c b[1 + (1 - \mu)b]\right] \tag{16}$$

which depends on the sign of the capital intensity difference b. If $b > 0$, i.e. the investment good sector is more capital intensive than the consumption good sector, then $V_{12}^* > 0$ and the characteristic roots are positive, ruling out any endogenous fluctuations. A more general result can be proved (see Bosi *et al.* [29]):

Turnpike Theorem. *Under Assumptions 1 and 3, if* $b \geq 0$ *then for any given initial capital stock* k_0 *and any given* $\delta \in (0, 1)$*, there exists a unique optimal path which converges monotonically toward the OSS* k_δ^*.

This Turnpike Theorem also contains the case of the aggregate model corresponding to $b = 0$.[17]

On the contrary, if $b < 0$, i.e. the consumption good sector is more capital intensive than the investment good sector, with $b \in (-1/(1 - \mu), 0)$, then Bosi *et al.* [29] prove that there exist some values of the elasticity of intertemporal substitution in consumption leading to the existence of endogenous fluctuations through the occurence of period-two cycles.[18]

Theorem 2. *Under Assumptions 1 and 3, consider the bounds*

$$\underline{\epsilon}_c \equiv -\frac{\varepsilon_{ck}(1+g)\theta^2}{\varepsilon_{w_1k}\delta b[1+(1-\mu)b]} < \bar{\epsilon}_c \equiv -\frac{\varepsilon_{ck}}{\varepsilon_{w_1k}}\frac{2(1+\delta)(1+g)^2\theta^2}{\delta[1+(2-\mu+g)b][\delta+b[1+g+(1-\mu)\delta]]}.$$

Then for any given initial capital stock k_0 *and any given* $\delta \in (0, 1)$*, there exists a unique optimal path which converges with oscillations towards the*

[17] It is however shown in Iwaza and Sorger [73] that in a discrete-time version of the neoclassical one-sector growth model with elastic labour supply, periodic solutions may occur only if leisure is not a normal good.

[18] Actually, Bosi *et al.* [29] consider the more general formulation with endogenous labor and provide some additional conditions for the existence of endogenous fluctuations which are related to the wage elasticity of the labor supply.

OSS k^*_δ if one of the following two sets of conditions is satisfied:

i) $b \in \left(-\frac{1}{1-\mu}, -\frac{1}{2-\mu+g}\right] \cup \left[-\frac{\delta}{1+g+\delta(1-\mu)}, 0\right)$ and $\epsilon_c > \underline{\epsilon}_c$;

ii) $b \in \left(-\frac{1}{2-\mu+g}, -\frac{\delta}{1+g+\delta(1-\mu)}\right)$ and $\epsilon_c \in (\underline{\epsilon}_c, \bar{\epsilon}_c)$.

In case ii), $\bar{\epsilon}_c$ is a flip bifurcation value and two configurations may appear:

a) for any ϵ_c in a right-neighborhood of $\bar{\epsilon}_c$, k^*_δ is locally unstable and there exists a unique optimal path which converges toward a period-two cycle;

b) for any ϵ_c in a left-neighborhood of $\bar{\epsilon}_c$, the period-two cycle is locally unstable while k^*_δ is a saddle-point. Then there exists a unique optimal path which converges with oscillations toward the OSS k^*_δ provided k_0 belongs to a neighborhood of k^*_δ.[19] *In such a case, as the elasticity of intertemporal substitution in consumption ϵ_c is equal to $+\infty$, they prove the existence of a bifurcation value for the discount factor δ.*[20]

Remark 1. *The capital intensity difference b is related to the derivatives $dw_1/dq_1 = 1/b$ and $dy_1/dk = 1/b$, which are known in the international trade theory as respectively the* Stolper-Samuelson *and* Rybczinski effects. *In a perfect competitive framework, there is indeed a standard duality between Rybczinski and Stolper-Samuelson affects.*

When $b < 0$, the consumption good sector is more capital intensive than the investment good sector. Let us then consider from the OSS an instantaneous increase of the capital stock k_t. This generates two opposing forces:

- Since the consumption good is more capital intensive, the trade-off in the production process is in favor of the consumption good. Following the Rybczinsky effect, the output of the consumption good y_{0t} more than proportionaly increases while the output of the investment good y_{1t} decreases. As a result the productive investment and thus the next period capital stock k_{t+1} decrease.

- At the next period, the decrease k_{t+1} now implies a trade-off in favor of the investment good which is less capital intensive. Again through the Rybczinsky effect, we get a more than proportional increase of the investment good output y_{1t+1} and a decrease of the consumption good output. Productive investment and thus period $t + 2$ capital stock k_{t+2} increase.

As a whole, we have a purely technological mechanism that can explain fluctuations of capital stock and output in each sector. Of course fluctuations of capital stock imply fluctuations of the rental rate of capital w_{1t}

[19] The existence of period-two cycles has initially been proved by Benhabib et Nishimura [23] under the assumption of a linear utility function $u(c) = c$.

[20] Benhabib and Nishimura [22] also prove the existence of endogenous cycles through a Hopf bifurcation in a continuous-time $n + 1$-sector optimal growth model where n needs to be larger or equal to 2 to allow for the existence of complex roots and instability of the steady state.

and, through the Stolper-Samuelson effect, fluctuations of the price of the investment good q_{1t}.[21]

The assumption $b < 0$ is crucial,[22] but it is not sufficient. The technological mechanism also depends on the depreciation rate of capital μ which has to be large enough to ensure that the variation of the investment good output is not enough to compensate for the decrease of the capital stock. More precisely, let us consider the capital accumulation equation $(1+g)k_{t+1} = y_{1t} + (1-\mu)k_t$. Total differenciation gives from the Rybczinsky effect

$$\frac{dk_{t+1}}{dk_t} = \frac{1}{1+g}\left[\frac{dy_{1t}}{dk_t} + (1-\mu)\right] = \frac{1}{1+g}\left[b^{-1} + (1-\mu)\right]$$

The existence of fluctuations then requires $dk_{t+1}/dk_t < 0$ which implies $b \in (-1/(1-\mu), 0)$.

The last ingredient to explain the existence of optimal endogenous fluctuations is coming from the intertemporal preferences. Indeed the representative agent needs to accept fluctuations of consumption. He must then be able to compensate the decrease of utility associated to a loss of consumption today by an equivalent increase of utility tomorrow derived from an increase of future consumption. Such a compensation can be obtained under two conditions:

- a large enough elasticity of intertemporal substitution in consumption ϵ_c,

- a discount factor δ sufficiently lower than 1. A sufficient degree of myopia is indeed necessary in order to rule out intertemporal trade-off based on price fluctuations. He must indeed be unable to delay consumption decisions according to the marginal rate of transformation between consumption and investment, in which case he would be able to speculate against fluctuations, thereby eliminating them.

Extensions to more complex dynamics Many contributions have also refered to the extensive literature on chaotic dynamics that was developed by mathematicians from the 1970s onwards. One major contribution has been provided by Boldrin and Montrucchio [27]. They show that any given dynamical system, possibly characterized by complex chaotic paths, can be a solution of an optimal growth model provided the discount factor δ is

[21] Optimal growth models can also be built from standard overlapping generations models under the assumption of rational altruism *à la* Barro [18]. Based on similar technological mechanism, considering a two-sector overlapping generations model with altruistic agents, Pelgrin and Venditti [105] provide a long-run cycle perspective to explain the behavior of the annual flow of inheritance.

[22] If $b > 0$, an instantaneous increase of the capital stock k_t from the OSS implies through the Rybczinsky effect an increase of the investment good output y_t and of the next period capital stock k_{t+1}, which implies a new increase of y_{1t+1} and k_{t+2}. We then obtain an unstable dynamics which cannot be compatible with the transversality condition.

close enough to zero. This conclusion has been clarified exploring more in details the values of the discount factor compatible with specific dynamical properties of the optimal path.[23]

5.2 Sunspot fluctuations

Introducing externalities in production, it has been shown by Nishimura and Venditti [100] that endogenous fluctuations can be also obtained from self-fulfilling expectations driven by sunspot equilibria.[24] Considering in a two-sector model that returns to scale are constant ex post, i.e. at the equilibrium, they prove that a simple modification of the condition on the capital intensity can yield to the local indeterminacy of the equilibrium implying the existence of sunspot fluctuations.

Local indeterminacy is obtained when for a given initial stock of capital, there exist many equilibrium paths converging toward the steady state. As Woodford [121] shows, local indeterminacy is a sufficient condition for the existence of sunspot equilibria, i.e. expectation-driven business cycles fluctuations. In such a case indeed, when expectations are subject to shocks, the agents can coordinate on different equilibrium paths implying that expectations shocks translate into business-cycle fluctuations.

The production functions are now explicitly formulated assuming a Cobb-Douglas specification:[25]

$$y_0 = k_0^{\alpha_1} l_0^{\alpha_2} e_0, \quad y = k_1^{\beta_1} l_1^{\beta_2} e_1 \tag{17}$$

with $\alpha_i, \beta_i > 0$. The externalities e_j are assumed to be sector specific

$$e_0 = \bar{k}_0^{a_1} \bar{l}_0^{a_2}, \quad e_1 = \bar{k}_1^{b_1} \bar{l}_1^{b_2} \tag{18}$$

with $a_i, b_i > 0$ and where $\bar{k}_i, \bar{l}_i$ are the average levels of capital and labor in each sector. At the equilibrium, all firms in each sector being identical, we have $\bar{k}_i = k_i$ and $\bar{l}_i = l_i$, and we can define *social production functions*

$$y_0 = k_0^{\hat{\alpha}_1} l_0^{\hat{\alpha}_2}, \quad y = k_1^{\hat{\beta}_1} l_1^{\hat{\beta}_2} \tag{19}$$

with $\hat{\alpha}_i = \alpha_i + a_i$ and $\hat{\beta}_i = \beta_i + b_i$. We then assume that the returns to scale are constant at the social level, i.e. $\hat{\alpha}_1 + \hat{\alpha}_2 = 1$ and $\hat{\beta}_1 + \hat{\beta}_2 = 1$, and thus decreasing at the private level.[26]

[23] Raines and Stockman [106] show that chaotic and cyclic equilibria are possible and that this behavior is not dependent on the steady state being "locally" a saddle, sink or source. Sorger [112] shows that there is a strong connection between the discount factor and the possible divergence of two different optimal paths. This conclusion has been extended by Montruccchio and Sorger [95] (see also Sorger [113]).

[24] The first proof of this result has been provided by Benhabib et Nishimura [24] in a continuous-time model.

[25] Nishimura and Venditti [100] provide more general results with CES technologies, then discussing the role of the capital-labor elasticity of substitution.

[26] Positive profits of firms can be for instance justified by the existence of a fixed factor such as land.

Profit maximisation in each sector allows in this new framework to define two different capital intensities driving the *Stolper-Samuelson* and *Rybczinski effects*. Defining the input coefficients at the private level

$$a_{00} = \frac{\alpha_2}{w_0},\ a_{10} = \frac{\alpha_1}{w_1},\ a_{01} = \frac{\beta_2 q_1}{w_0} \text{ and } a_{11} = \frac{\beta_1 q_1}{w_1} \tag{20}$$

and the input coefficients at the social level

$$\hat{a}_{00} = \frac{\hat{\alpha}_2}{w_0},\ \hat{a}_{10} = \frac{\hat{\alpha}_1}{w_1},\ \hat{a}_{01} = \frac{\hat{\beta}_2 q_1}{w_0} \text{ and } \hat{a}_{11} = \frac{\hat{\beta}_1 q_1}{w_1}, \tag{21}$$

the *Rybczinski effect* is indeed driven by the following capital intensity difference at the private level

$$b \equiv a_{01}\left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}}\right) = \frac{\alpha_1}{w_1}\left(\frac{\beta_1}{\beta_2} - \frac{\alpha_1}{\alpha_2}\right), \tag{22}$$

while the *Stolper-Samuelson effect* is driven by the following capital intensity at the social level

$$\hat{b} \equiv \hat{a}_{01}\left(\frac{\hat{a}_{11}}{\hat{a}_{01}} - \frac{\hat{a}_{10}}{\hat{a}_{00}}\right) = \frac{\hat{\alpha}_1}{w_1}\left(\frac{\hat{\beta}_1}{\hat{\beta}_2} - \frac{\hat{\alpha}_1}{\hat{\alpha}_2}\right). \tag{23}$$

It can then be shown that $dy/dk = 1/b$ and $dw_1/dq_1 = 1/\hat{b}$. When there is no externality, i.e. $a_i = b_i = 0$, we obviously get the standard duality result with $b = \hat{b}$ and thus $dy/dk = dw_1/dq_1$. But in presence of externalities, as they are considered as given by the entrepreneurs when they maximize profit, the optimal demand functions for production factors are determined by the input coefficients at the private level and thus the Rybczinski effect driving the impact of the capital stock on the investment good output depends on these coefficients. However, the externalities do affect the actual level of production of each good, and therefore the quantities of goods placed on the market. The equilibrium prices of these goods are therefore directly affected by the presence of externalities, which explains why the Stolper-Samuelson effect driving the impact of the price level on the rental rate of capital depends on input coefficients at the social level.

In this context with externalities, we can also define a social production function similar to (1) providing the optimal level of the consumption good output which now also depends on e_0 and e_1, namely:

$$y_{0t} = T(k_t, y_t, e_{1t}, e_{2t}) \tag{24}$$

and which satisfies as before

$$w_1 = T_1(k, y, e_0, e_1) \text{ and } q_1 = -T_2(k, y, e_0, e_1) \tag{25}$$

where $T_1 = \frac{\partial T}{\partial k}$ and $T_2 = \frac{\partial T}{\partial y}$.

Assuming constant population and complete depreciation of capital in one period,[27] the intertemporal optimization program is given by

[27] Full depreciation is introduced in order to simplify the analysis and to focus on the role of preferences.

$$\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \frac{T(k_t, y_t, e_{0t}, e_{1t})}{1-\sigma}^{1-\sigma}$$
$$s.t. \quad k_{t+1} = y_t,$$
$$k_0, (e_{0t})_{t=0}^{+\infty}, (e_{1t})_{t=0}^{+\infty} \quad given$$

where $\sigma \geq 0$ is the inverse of the elasticity of intertemporal substitution in consumption, i.e. $\epsilon_c = 1/\sigma$. The Euler equation is then

$$q_{1t} c_t^{-\sigma} = \delta w_{1t+1} c_{t+1}^{-\sigma}. \tag{26}$$

From the input demand functions together with the external effects (18) considered at the equilibrium we may define the equilibrium factors demand fonctions $\hat{k}_i = \hat{k}_i(k_t, k_{t+1})$, $\hat{l}_i = \hat{l}_i(k_t, k_{t+1})$ so that $\hat{e}_1 = \hat{e}_1(k_t, k_{t+1}) = \hat{k}_0^{a_1} \hat{l}_0^{a_2}$ and $\hat{e}_2 = \hat{e}_2(k_t, k_{t+1}) = \hat{k}_1^{b_1} \hat{l}_1^{b_2}$.[28] From (25) prices now satisfy

$$\begin{aligned} w_1(k_t, k_{t+1},) &= T_1(k_t, k_{t+1}, \hat{e}_1(k_t, k_{t+1}), \hat{e}_2(k_t, k_{t+1})), \\ q_1(k_t, k_{t+1}) &= -T_2(k_t, k_{t+1}, \hat{e}_1(k_t, k_{t+1}), \hat{e}_2(k_t, k_{t+1})) \end{aligned} \tag{27}$$

and the consumption level at time t is given by

$$c(k_t, k_{t+1}) = T(k_t, k_{t+1}, \hat{e}_1(k_t, k_{t+1}), \hat{e}_2(k_t, k_{t+1})). \tag{28}$$

We then get equations (26) evaluated at $\hat{e}_c$ and $\hat{e}_y$:

$$q_1(k_t, k_{t+1}) c(k_t, k_{t+1})^{-\sigma} = \delta w_1(k_{t+1}, k_{t+2}) c(k_{t+1}, k_{t+2})^{-\sigma}. \tag{29}$$

Any solution $\{k_t\}_{t=0}^{+\infty}$ which also satisfies the transversality condition

$$\lim_{t \to +\infty} \delta^t c(k_t, k_{t+1})^{-\sigma} q_1(k_t, k_{t+1}) k_{t+1} = 0$$

is called an equilibrium path.

With such a Cobb-Douglas formulation, the existence of a unique steady state k^* such that $q_1(k^*, k^*) = \delta w_1(k^*, k^*)$ is easily proved. The local stability properties of the steady state are as previously derived from the characteristic polynomial associated to the linearization of equation (29) around k^*. We now introduce the following standard definition.

Definition 1. *A steady state k^* is called locally indeterminate if there exists $\epsilon > 0$ such that from any k_0 belonging to $(k^* - \epsilon, k^* + \epsilon)$ there are infinitely many equilibrium paths converging to the steady state.*

If both roots of the characteristic polynomial have modulus less than one then the steady state is locally indeterminate. If a steady state is not locally indeterminate, then we call it locally determinate.

Echoing the conditions for the existence of optimal period-two cycles previously stated, local indeterminacy cannot occur if the investment good

[28]These functions are obtained as solutions of a fixed point problem. See d'Albis and Le Van [3] for a similar analysis in an endogenous growth model.

sector is capital intensive at the private level. Considering separately the cases of linear ($\sigma = 0$) and nonlinear ($\sigma > 0$) utility function, Nishimura and Venditti [100] prove indeed the following results:[29]

Theorem 3. *1- When $\sigma = 0$, the steady state is locally indeterminate if and only if the consumption good sector is capital intensive at the private level with $\alpha_1/\alpha_2 - \beta_1/\beta_2 > 1/(\delta\beta_2)$, and $\hat{\beta}_1 > \hat{\alpha}_1 - \hat{\alpha}_2$.*

2- When $\sigma > 0$, let $\alpha_1/\alpha_2 - \beta_1/\beta_2 > 1/(\delta\beta_2)$, and $\hat{\beta}_1 > \hat{\alpha}_1 - \hat{\alpha}_2$. Then there exists $\bar{\sigma} \in (0, +\infty)$ such that the steady state is locally indeterminate when $\sigma \in [0, \bar{\sigma})$, saddle-point stable when $\sigma > \bar{\sigma}$, and undergoes a flip bifurcation when $\sigma = \bar{\sigma}$. More precisely, we can have (1) a locally unstable two-period cycle with a stable steady state inside on the left of $\bar{\sigma}$ and a locally unstable steady on the right of $\bar{\sigma}$ without cycles around (subcritical flip bifurcation) or (2) a locally stable steady state on the left of $\bar{\sigma}$ without cycles around and a locally stable two-period cycle with an unstable steady state inside on the right of $\bar{\sigma}$ (supercritical flip bifurcation).[30]

It is worthwhile to notice that condition $\hat{\beta}_1 > \hat{\alpha}_1 - \hat{\alpha}_2$ is always satisfied *i)* if the consumption good is labor intensive from the social perspective, i.e. $\hat{\beta}_1 > \hat{\alpha}_1$, but can be satisfied *ii)* if the consumption is also capital intensive from the social perspective with $\hat{\alpha}_2 > \hat{\alpha}_1 - \hat{\beta}_1 > 0$.

Benhabib and Nishimura [24] have conducted a similar analysis in a two-sector Cobb-Douglas economy in continuous time with a linear utility function. They prove that if the consumption good is capital intensive from the private perspective, but labor intensive from the social perspective, then the steady state is indeterminate.[31] This corresponds to case *i*) but we need additional restriction on the factor intensity difference at the private level. Moreover in the discrete time model, even if the consumption good is capital intensive from both private and social perspectives, indeterminacy can also take place as in *ii*).

To understand the intuition for these results we need to refer to Rybczynski and Stolper-Samuelson effects as given respectively by $dy/dk = 1/b$ and $dw_1/dq_1 = 1/\hat{b}$. We can easily show when $\sigma = 0$ that the two characteristic roots satisfy $x_1 = dy/dk$ and $x_2 = (dw_1/dq_1)^{-1}$.

Starting from an equilibrium path, let the agents believe that given k_t there is another equilibrium in which the shadow price of investment q_{1t} is higher than its current value. From $q_{1t} = -T_2(k_t, y_t, e_{0t}, e_{1t})$, this results

[29] Nishimura and Venditti [100] also provide more general results with endogenous labor, then discussing the role of the elasticity of the labor supply.

[30] See also Drugeon [50] for similar results with general technologies and utility function.

[31] Dufourt *et al.* [51] consider standard a one sector model with endogenous labor, productive externalities and variable capacity utilization, and provide similar conclusions. They show that, when labor is infinitely elastic, local indeterminacy occurs through flip and Hopf bifurcations for a large set of values for the elasticity of intertemporal substitution in consumption, the degree of increasing returns to scale and the elasticity of capital-labor substitution.

in an increase of y_t and therefore an increase of k_{t+1} since $y_t = k_{t+1}$. The question is to know whether or not this new sequence can be an equilibrium.

Let us first deal with the case *i)*. The consumption good being capital intensive from the private perspective, the Rybczynski effect implies a decline in the output y_{t+1} which may offset the initial rise of y_t, i.e. $|dy/dk| < 1$, if $\alpha_1/\alpha_2 - \beta_1/\beta_2 > 1/(\delta\beta_2)$. From $q_{1t} = -T_2(k_t, y_t, e_{0t}, e_{1t})$, the decrease of y_{t+1} implies a decline of q_{1t+1}. Since the investment good is capital intensive at the social level, then given k_{t+1}, from the Stolper-Samuelson effect, the decrease of q_{1t+1} implies a more than proportional decline in the rental rate w_{1t+1}, i.e. $dw_1/dq_1 > 1$. As a result this new path converges towards the steady state and the transversality condition therefore holds. This implies that the initial expectations are self-fulfilling and the steady state is locally indeterminate.[32]

Consider now the case *ii)* in which the investment good is also labor intensive at the social level. From the Euler equation $q_{1t} = \delta w_{1t+1}$, the initial rise of q_{1t} leads to a more than proportional increase of the rental rate of capital w_{1t+1} since $\delta < 1$. The Stolper-Samuelson effect implies therefore a decrease in the price of the investment good q_{1t+1}, which is less intensive in the use of capital inputs. This decrease may be more than proportional, i.e. $|(dw_1/dq_1)^{-1}| < 1$, and it may offset the initial rise. In this case again the initial expectations are self-fulfilling and the steady state is locally indeterminate.

When $\sigma > 0$, expectations-driven fluctuations can be sustained if the elasticity of intertemporal substitution in consumption $\epsilon_c = 1/\sigma$ is large enough allowing the representative household to compensate current decrease of consumption by future increase. Moreover, when the elasticity of intertemporal substitution is decreased and crosses $\bar{\epsilon}_c = 1/\bar{\sigma}$, the steady state becomes saddle-point stable through a flip bifurcation.

Extensions to other frameworks : Local indeterminacy and sunspot fluctuations in two-sector models has been exhibited in a variety of frameworks. Ghiglino [60] consider two-sector growth models with technological externalities and many trading countries. Bosi *et al.* [30] explore the occurrence of local indeterminacy in a two-sector monetary economy with a general MIUF model. Brito and Venditti [40] study the existence of local indeterminacy in an extended Lucas [85] model of endogenous growth.

6 Green growth

An increasing number of contributions is being devoted to theoretical normative and positive issues arising in the ongoing debate on sustainable growth.

[32] Similar conclusions can also be obtained in two-sector overlapping generations models. See e.g. Nourry and Venditti [101].

Differently from the research questions addressed within the traditional endogenous growth framework (still active, see for example Ha-Huy and Thien Tran [66] for a very recent contribution), this new literature intersects with the environmental literature and questions the long term viability of growth regime in the presence of environmental external effects (pollution and global warming) and availability of natural resources and backstop technologies. In response to the Meadows report published in 1972, a number of economic growth models incorporating natural resources as input into otherwise neoclassical production functions have been elaborated such like the famous DHSS (Dasgupta-Heal-Solow-Stiglitz) model, see for example Stiglitz [116]. Needless to say, this topic has been also occasionally explored in the *JME* decades ago (see below). The research questions around sustainable development and environmental quality have more recently come into account.

Natural resources, growth or extinction. One interesting contribution, though not connected to the debate around the DHSS model, is the early paper by van Geldrop *et al.* [58] on the existence of general equilibria in economies with natural exhaustible resources and an infinite horizon. It incidentally includes an interesting technical innovation with respect to the typical proof strategy of general equilibrium with infinite-dimensional commodity spaces which does not apply in the case of exhaustible resources.

More recent papers with exhaustible resources (physical and human) have taken another direction with respect to the traditional DHSS-like literature: instead of identifying (mainly technological) conditions that would eventually help ensuring long-term growth, they aim at characterizing the optimal occurrence of **extinction**. A very interesting contribution due to Mitra and Roy [93] clarifies the conditions under which random shocks (technological or ecological for example) may lead to the collapse of the whole macroeconomy. The paper builds on the idea that a key condition for sustainability is that stocks of (including replenishable) physical and natural capital are not depleted to zero over time. The rationale behind is a kind of radical irreversibility, extinction being an absorbing state possibly leading to poverty traps and the like. Mitra and Roy consider a stationary Markovian structure of the shocks hitting the stock of assets, say y_t, with the transition law:

$$y_{t+1} = G(y_t, r_{t+1}),$$

where (r_t) is a sequence of i.i.d. random variables with bounded support and G is a time-invariant transition function mapping the current stock and the realization of the random shock to the next period's stock. The dynamic (non-concave) optimization of the stock under quite weak assumptions on the transition function G delivers some important results on the occurence of optimal extinction. In particular, almost sure extinction may occur from all initial stocks.[33] More macroeconomic models have buit on this idea of shocks accelerating the occurrence of poverty traps can be find in the more recent *JME* literature, see for example Askan Mavi [12].

A very different optimal extinction problem anchored in the literature of vintage human capital is due to Boucekkine *et al.* [33]. The problem posed is infinite-dimensional, it is described with more details below in Section 7. The problem is an optimal population size problem but with the additional condition that individuals born can only live $T < \infty$ periods during which they produce and consume, assuming same productivity for all individuals of all generations. In the terminology of vintage capital modelling, this is a *onehoss shay* vintage human capital growth model, the simplest possible. Procreation is the unique way to transfer resources forward in time. The social welfare functions considered are standard utilitarian ranging from Benthamite to Millian through intermediate social choice criteria with impure intergenerational altruism. When altruism is impure, egalitarianism is impossible in the context of a growing economy. Either in the Benthamite or impure altruism cases, it is shown that procreation is never optimal for small enough life spans, leading to finite time extinction and maximal consumption for all existing individuals.

Connected to the normative discussion above, several insightful papers published in the *JME* have indeed discussed the impact of future uncertainty in the allocation of exhaustible resources across generations although not always anchored in growth settings. A particularly intriguing contribution is due to Llavador *et al.* [82]. The authors compare the optimal allocations of two types of planners (or *Ethical Observers* in their terminology), one of the Rawlsian type (thus more prone to sustainable allocation across generations) and the second utilitarian. In the basic cake eating problem, the two planners surprisingly choose the same allocation. When moving to the classical setting of the Solow growth model where generations are also connected by their saving decisions, the results are even more intricate. In particular, the authors show that, in contrast to the utilitarian planner, it is optimal for the Rawlsian "*to ignore the uncertainty concerning the possible disappearance of the human species in the future*"! See Llavador *et al.* [82] for more details.

Pollution, environmental quality and growth . This research stream is more recent in the *JME*, becoming definitely active in the last decade. One natural basic question is the pollution control problem in a neoclassical growth set-up with overlapping generations. A recent contribution to this research line is due to Goenka *et al.* [61]. In this model, only the young agents work, their probability to survive depending on income (thus on the stock

[33] The authors emphasize the crucial role played by the nature of the transition function under the worst realization of the i.i.d. random shock in the rich set of extinction results generated. See Mitra and Roy [93] for more details.

of capital) and (negatively) on pollution (which comes as a side-product of production as usual). Pollution control operates to taxes on the labor income of the young individuals to finance pollution abatement. Optimal tax (second-best) plans are more precisely characterized under time consistency. Two highly nontrivial sets of results are obtained. First of all, the optimal tax is zero at low levels of capital, becoming only weakly increasing function of the capital stock above a threshold value. Second, because of the non-homogeneity of the tax function and other general equilibrium effects, additional steady states, stability reversals and oscillations may emerge.[34]

Even more recently, pollution control has been studied in fully specified spatiotemporal models of pollution diffusion. An early and highly significant contribution to this research stream is due to La Torre *et al.* [81]. Using a linear-quadratic model in continuous space with transboundary pollution through a partial differential equation, a so-called diffusion equation, they study to which extent transboundary pollution and the inherent externalities lead to inefficiencies. They first show that if the initial pollution distribution is spatially homogeneous then the local (ignoring transboundary externalities) and global solutions will coincide and thus no efficiency loss will arise from transboundary externalities. Corrective environmental policy is needed if the initial pollution distribution is uneven, and it is possible to precisely design it at any location and any time in the linear-quadratic frame designed by the authors. The result hold true in different setting, with and without capital accumulation.

Along the same lines, Boucekkine *et al.* [32] have developed a novel non linear-quadratic model with transboundary pollution and a very rich set of spatial heterogeneities (technological, ecological and cultural in terms of environmental awareness, among others), still allowing for an analytical approach carefully exploiting the infinite-dimensional structure of the problem, which derives from the assumed diffusion equation like in La Torre *et al.* This feature is discussed in more details in the next Section.

7 Infinite-dimensional economic growth models: Continuous time, age and spatial structures

In the two recent decades, several classes of infinite-dimensional economic growth models have emerged in the economic literature mostly via **continuous** structures in time (delay, memory effects), in age (vintage human or physical vintage models) or in space (typically via diffusion problems across

[34]Another complementary line of research, recently emerging in the *JME* concerns the study of optimal transition to "green" economic regimes, see Orlov and Rovenskaya [102], which typically uses multi-stage optimal control techniques.

space). While none of these classes of models is truly new in economics,[35] the possibility to handle them (to a certain extent) analytically has only been opened to economists in the last decades (late 90s precisely). This was indeed made possible by the development of more powerful mathematical tools, widely applicable to these problems (even when they are embedded into optimal control settings), and allowing to incorporate a rich set of essential heterogeneities which have been for a long time overlooked for technical problems.

For example, as it will be clearer below, the advances in dynamic programming in infinite-dimensional spaces (see Bensoussan *et al.* [25] for example) have enabled to consider and to tackle the Hamilton Jacobi Bellman equations, resulting from the optimization problems posed, on appropriate functional spaces. Furthermore, with some additional linear specification(s), such like the use of AK production functions, the latter techniques have enabled to construct explicit solutions, uncovering new channels to understand for instance economic fluctuations, or consumption smoothing. Last but not least, the inclusion of non-local equations has made it possible to model the interactions between heterogeneous economic agents.

7.1 Delay and memory

The revival of delay and memory-based models owes a lot to the development of mathematical theories and tools to explore in depth delay equations (ssee for example, Hale and Lunel [67] or Dieckmann *et al.* [49]), and equations with delay and advance (also called mixed-delay differential equations, see Mallet-Parret and Verduyn-Lunel [88]) and their control (Vinter and Kwong [120], Delfour and Mitter [47], Bensoussan *et al.* [25]). Of course the concrete implementation of these new techniques and theories has not been always trivial (due to the typical specifications in economic growth theory involving predetermined and non-predetermined variable, plus other specific constraints) and has indeed occasionnaly required several tricky adaptations and extensions. We shall review here below some of these advances in a particular research area, the optimal growth models with delayed external effects.

When a delayed term is introduced as an externality (for example to model learning-by-doing as in d'Albis *et al.* [4]), the resulting dynamics yield a delay differential equation. When the externality shows up in the production function, the dynamics of the capital write as follows

$$\dot{k}\,(t) = f\,(k\,(t)\,, e\,(t)) - \delta k\,(t) - c\,(t)\,,$$

[35] For example, vintage capital models were popular in the 50s-60s, see Johansen [76], and spatiotemporal models have been warmly recommended to study the optimal location of economic activities by Isard and Liossatos [72] in the late 70s.

where δ is the capital depreciation rate, while the external effects, $e(t)$, may be driven by cumulative gross investment

$$e(t) = \int_{t-\tau}^{t} g(t-s)\, k(s)\, ds.$$

This formulation encompass many configurations. For instance, for $g(t-s) = \delta_\tau(t-s)$, where δ_τ is a Dirac Function such that $\delta_\tau(s) = 1$ if $s = \tau$ and 0 otherwise, then the externality only depends on capital at delayed time $t - \tau$. Initial conditions of the model are then given by $k_0(s)$, for $s \in [-\tau, 0]$. Intertemporal maximization program is

$$\max_c \int_0^\infty e^{-\rho t} u(c(t))\, dt.$$

Pontryagin Maximum Principal yields a two dimensional system of delay differential equations, and one non-predetermined (forward) variable, consumption. The dynamics can be indeed written as follows.

$$\begin{aligned} \dot{k}(t) &= f(k(t), e(t)) - \delta k(t) - c(t), \\ \dot{c}(t) &= -\frac{u'(c)}{u''(c)} \left[f'_k(k(t), e(t)) - (\delta + \rho) \right]. \end{aligned}$$

Let us assume that the system admits a unique steady state k^* solving $f'_k\left(k^*, k^* \int_0^\tau g(s)\, ds\right) = \delta + \rho$, and $c^* = f(k^*, e^*) - \delta k^*$. Characteristic equation in the neighborhood of the steady state is given as $\Delta(\lambda) = 0$, where $\Delta(\lambda)$ is given as follows.

$$\Delta(\lambda) = \det \begin{bmatrix} \lambda - f'_k(k^*, e^*) - f'_e(k^*, e^*) \int_0^\tau g(s)\, e^{-\lambda s} ds - \delta & 1 \\ -\frac{u'(c)}{u''(c)} \left[f'_{kk}(k^*, e^*) + f'_{ke}(k^*, e^*) \int_0^\tau g(s)\, e^{-\lambda s} ds \right] & \lambda \end{bmatrix}.$$

Considering solutions λ to the characteristic equation, d'Albis et al., 2014 [5] provides conditions of existence, uniqueness/indetermination of such equations. With specific production function, CIES utility function and an extended form of the externality, d'Albis *et al.* [4] prove that a slight memory effect characterizing the learning-by-doing process is enough to generate business cycle fluctuations through a Hopf bifurcation leading to stable periodic orbits. When habit formation is externalized, delayed (typically consumption) variables are then control variables (i.e. habit formation (Augeraud and Bambi [13]). The dynamics are still governed by delay differential equations but with a different structure: the dynamic system has now two backward variables, plus a forward variable. The comparison with the infinite memory habit formation model is very well developed in the paper cited above, leading to several enlightening results.[36]

[36] Delays also show up in other interesting contexts such that internal habit formation models (see Augeraud *et al.* [14], for which the delay appears both in the objective function and in the constraint) or, by construction, in time-to-build models. Another interesting context is continuous time overlapping generations model with realistic demography, andh optimal schooling and/or pension timing decisions as in d'Albis and Augeraud [6].

7.2 Age/time structures

Vintage capital models As mentioned above, the vintage capital growth models have been at the heart of growth theory since the 1960s, mostly within Solow-type of models, that is without intertemporal optimization. In the late 1990s, thanks to the optimal control theory breakthrough documented above, vintage capital theory addresses the question of optimal replacement of obsolete capital within optimal growth frames. A representative problem of those can be written as follows (see Boucekkine *et al.* [38]

$$
\begin{aligned}
& \max_c \int_0^\infty e^{-\rho t} u\left(c\left(t\right)\right) dt, \\
k\left(t\right) &= \int_{t-\tau(t)}^{t} i\left(s\right) ds, \\
c\left(t\right) &= Ak^{\alpha}\left(t\right) - i\left(t\right), \\
k_0\left(s\right), \text{ given for } s &\in \left[-\tau(0), 0\right].
\end{aligned}
$$

$\tau(t)$ is the age of the oldest capital good still in use at time t, it can be also interpreted as the scrapping time (of obsolete capital goods). The production function can be either neoclassical ($\alpha < 1$) or AK ($\alpha = 1$). Most of the modern vintage capital theory use linear specifications, along with the vintage literature of the 60s which typically builds on Leontief production functions. A notable (and excellent) exception is Jovanovic and Yatsenko [77] who apply the Pontryagin maximum principle directly on the delayed integral equations above.[37] Also it is worth noting that the scrapping time is often taken constant with very few exceptions in growth theory (e.g. Boucekkine *et al.* [38]). If not, enough linearity is introduced in the structure of the model (in preferences in the case of [38]) to allow for *ad hoc* (and still nontrivial) analytical approaches.[38]

The optimal (endogenous) growth AK case is typically studied with constant capital life time (that is assuming $\tau(t)$ is constant), the most comprehensive analysis being provided by Fabbri and Gozzi [56], using dynaming programing in infinite-dimensional functional (auxiliary) spaces *à la* Bensoussan *et al.* [25]. Fabbri and Gozzi provide with the corresponding closed-loop policy functions. It should be first noticed that in this model, delay appears in the control variable. One of the technical issue of this approach, is then to define the structural state (i.e. the new state variable), on the Hilbert space $M^2 = \mathbb{R} \times L^2\left([-\tau, 0] \times \mathbb{R}\right)$, and to rewrite the original state dynamics as a (linear) ODE on M^2 . Then the Hamilton-Jacobi-Bellman

[37]See also Boucekkine *et al.* [36] for an application of the same technique to *green* growth.

[38]See the earlier book by Hritonenko and Yatsenko [71] for a much richer set of applications in engineering and economics where capital goods and technologies' lifetime is a control variable.

equation is derived and solved for the value function of the problem in that functional space, before recovering the optimal solutions in the initial space. AK vintage capital growth models are ultimately shown to admit a well-behaved closed-loop solution when capital lifetime is large enough while generating (generally oscillatory) optimal transition dynamics, contrary to the AK model with homogenous infinite-lived capital which deliver no transition.

Demography and age structure Infinite-dimensional optimal growth problems also arise when demography is taken seriously enough, for example when we depart from the *eternal youth* assumptions, typical in the Yaari-Blanchard structure. Since human capital is embodied in humans, one can reproduce the vintage capital structure for human capital too. An early contribution to this area is due to de la Croix and Licandro [46]. In the same vein, one can consider a replication of the AK model described above when physical K is replaced by (skilled) labor N, assuming a constant finite human lifetime. This is done in Boucekkine *et al.* [33], already considered with the lens of optimal extinction under scarcity of human capital. While in the AK model, the control variable is investment in physical capital, the control turns out to be the optimal procreation rate, n, in the AN model, therefore intersecting with the classical optimal population size problem. The corresponding optimal control problem writes as follows.

$$\begin{aligned} & \max_{n,c} \int_0^{\infty} e^{-\rho t} u\left(c\right) N\left(t\right)^{\gamma} dt, \\ N\left(t\right) &= \int_{t-T}^{t} n\left(s\right) ds, \\ AN(t) &= N(t)c(t) + bn(t), \\ n\left(t\right) &\geq 0, \end{aligned}$$

together with initial condition $n_0\left(s\right), s \in [-T, 0)$ given, T being the constant human lifetime, b the unit cost of procreation and A the productivity of human capital, assumed constant over time. Parameter γ is one of the key parameter of this model, as it represents the degree of altruism toward future generations. Optimal solutions are obtained through infinite dimensional dynamic programming exactly as in Fabbri and Gozzi [56].

Several results can be drawn from this canonical model in addition to those related to population ethics under scarcity outlined in Section 6. The optimal fertility rate depends on the social welfare function, and more precisely on the degree of altruism, γ. In the Benthamite case ($\gamma = 1$), optimal consumption per capita and fertility rates are constant and independent of the initial procreation profile when growth is optimal. However, under intermediate altruism, the optimal dynamics of consumption per capita and fertility rates adjust to the initial data, and therefore, they are not constant in general. Thus the optimal fertility rate and per capita consumption are non-constant in the impure altruism case ($0 < \gamma < 1$), contrary to the benchmark case with infinite human lives. Enforcing intergenerational egalitarianism of any sort can therefore only be socially optimal in the case of the Benthamite planner, which goes at odds with one of main *repugnant conclusions* known in population ethics (see Parfit [103]). A key rationale behind the result is the role of balanced endogenous growth which emerges optimally from $t = 0$ in the Benthamite case when the human lifetime, T, is large enough.

7.3 Spatial growth models

Optimal spatiotemporal growth models are the next class of infinite-dimensional models which have been favored lately by the development of new adapted (or adaptable) mathematical tools. This is fortunate since many important questions relating spatiotemporal growth to optimal location of economic activity and income distribution across space can be now treated properly, even allowing for the incorporation of a rich set of heterogeneities across space. Typically, these models involve spatio-temporal dynamics governed by a diffusion equation or via non-local equations that enable to model the interactions in the neighborhood of localities. We shall concentrate here on physical mobility of capital across regions, the same tools can be used to study the diffusion of transboundary pollution across an heterogenous continuous space as alluded to in Section 6 above.

Formally, the law of motion of capital is given as follows (see Brito [39]).

$$\frac{\partial k(t,x)}{\partial t} = f(k(t,x)) - \delta k(t,x) - c(t,x) - \tau(t,x),$$

where $(t,x) \in \mathbb{R} \times \Omega$, where $\tau(t,x)$ is the net trade balance.

$$\tau(t,x) = -\int_S J(t,x)\, dS = -\int_V div(J(t,x))\, dV,$$

for $V \subset \mathbb{R}^d$, a small volume with $\partial V = S$.

The literature distinguishes two distinct mobility laws. First, following Brito [39]),

$$J(t,x) = \frac{\partial k(t,x)}{\partial x}.$$

Then, letting the volume tends to 0, then $\tau(t,x) = \Delta k(t,s)$. For 1-dimensional problem, $\tau(t,x) = \frac{\partial^2 k(t,x)}{\partial x^2}$.

A second approach has been proposed by Xepapadeas and Yannacopoulos [122]. It builds on the assumption of marginal-productivity-driven

(MPD) capital flows, which is itself based on the idea that it is the economic distance (and not the geographical distance) which matters, and this economic distance is therefore a very relevant proxy for transportation costs. According to this assumption, capital moves toward locations where the marginal productivity of capital is relatively higher than the productivity at the location of origin. After defining the rate J of net flow of capital as

$$J(t,.) = v(t,.) \odot k(t,.),$$

where $\odot$ is element-wise multiplication, and

$$v(t,x) = B(x)\,\Psi(k(t,x))\,\nabla_x\left(\frac{\partial y}{\partial k}(t,x)\right),$$

with $y(t,x) = A(x)\,f(k(t,x))$.

Xepapadeas and Yannacopoulos [122] analyze the spatial Solow model using this approach. The nonlinear diffusion term in the spatial growth model affects the capital accumulation equation by introducing a spatially varying diffusion coefficient that depends on the capital stock and the rate of change of marginal productivity of capital. This Solow spatial growth model with nonlinear diffusion generates spatial distributions for per capita capital and GDP that are characterized by large and persistent spatial nonhomogeneities.[39]

Laplacian formulation of net flow is more widely considered (see for example Brito [39], Ballestra [15] and Boucekkine *et al.* [34]). In such a case, the state equation is given as a parabolic partial differential equation. While Boucekkine *et al.* [34] use infinite-dimensional programming dynamic to solve the optimal spatiotemporal AK model completely in line with the methodology already described above on vintage capital models, Ballestra [15] use a Pontryagin maximum principle method after identifying the suitable transversality conditions needed to ensure uniqueness of the optimal solution. Spatial Ramsey model with AK technology, diffusive capital and CIES utility function on a circle domain exhibit optimal consumption level that does not depend on location. One of the main result of this model, is that despite the use of an AK technology, the optimal spatio-temporal dynamics lead to equalize the capital level across locations. In other words, inequalities that may exist in the initial capital distribution do not persist in the long run provided the central planner is averse to inequality across space and across time. This result echoes the results obtained on other infinite-dimensional optimal AK (or AN) models surveyed throughout this section. That should not come as a surprise: these models have a similar structures to which one can apply basically the same analytical methods.

[39] A Ramsey model using the same spatial framework is studied in Xepapadeas and Yannacopoulos [123].

Stochastic models In some recent papers (see, e.g. Gozzi and Leocata [64]) several authors have analyzed stochastic growth models where the state/control variables such as capital and consumption depend not only on time t, but also on space x, providing a first extension of previous results found in the deterministic model of spatial growth (see e.g. Boucekkine [32, 34] or Brock and Xepapadeas [41]). From the technical point of view we observe that the guess for the value function is completely analogous to that of the corresponding deterministic case, based on the homogeneity of the problem. On the other hand, the method for finding the optimal feedback controls is different due to the difficulties brought by the stochastic term; in particular we need to use a different approach for different values of the elasticity of the utility functions considered (see in particular Gozzi and Leocata [64]).

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