The decimal representation of a fraction

A rational number can be expressed uniquely as a fraction where numerator and denominator are coprime integers and the denominator is strictly positive.

If the minimal denominator divides a power of 10, then the decimal representation of the number is finite (terminating decimal), namely the period is 0. Else, we have nevertheless a periodical representation: the digits eventually repeat periodically (only the periods with all digits 9 are not accepted, because in this case there is an alternative representation with period 0).

For later use, if we work with a period, then the minimal period can easily be determined (the length of the minimal period divides the length of any period; the minimal period can be determined by spotting repetitions inside the period). For example, having period 123123 easily allows to say that the minimal period is 123.

From the decimal representation to the fraction we must evaluate an infinite sum. Luckily, all boils down to the formula for the *geometric series*. One has to pay attention to the fact that, if the digits in the period are 012 (and we tend to read this as the integer number 12) then this period has 3 digits so the ratio in the geometric series is 1/1000.

Converting a fraction into the decimal representation means performing the long division numerator : denominator. In fact, arithmetical considerations allow us to stop after computing finitely many decimal digits. Notice that knowing the first digits of a number after the comma, say 0,121212…, does not allow to conclude that we know the number (this could be for example the number 0,121212333… where all the missing digits are 3). While performing the long division *numerator : denominator* (removing the sign to the rational number) it's possible to notice a repetition pattern and stop at the appropriate point. However, performing the long division could take long… luckily, there are alternatives way to proceed.

Modular arithmetic comes to our rescue because it tells us that **the denominator of the given fraction divides the product of a power of 10 times a power of 10 minus 1**

$$10^A(10^N - 1)\,.$$

The smallest integer A that does the trick can be determined by inspecting the largest power of 2 and the largest power of 5 that divide the denominator (and taking the maximum of the two exponents).
The smallest integer N that does the trick is more complicated, and in fact it is not even clear a priori that such an integer should exist. We may consider the denominator deprived of the prime factors 2 and 5, and we call this integer m. If $m = 1$, then we can take $N = 1$. Else, we consider the residue class of 10 modulo m. Since 10 and m are coprime, Fermat's Little Theorem tells us that there is some power of 10 that is congruent to 1 modulo m. In other words, m divides an integer of the form $10^N - 1$. By that result, an integer N that does the trick is $m(m-1)$ and we may also take $N = \varphi(m)$, where $\varphi(m)$ is the number of integers from 1 to m that are coprime to m (after having determined the prime factorisation of m, the *Euler's totient function* $\varphi(m)$ can easily be computed by combining the following information: for a prime power p^e we have

$\varphi(p^e) = p^{e-1}(p-1)$; for coprime integers a and b we have $\varphi(ab) = \varphi(a) \cdot \varphi(b)$. For example, we have $\varphi(80) = \varphi(2^4)\varphi(5) = 2^3(2-1)(5-1) = 32$.)
The smallest positive integer N such that m divides $10^N - 1$ is called the *multiplicative order* of 10 modulo m. It is a divisor of $\varphi(m)$ and it can be determined by trial and error. For example, for $m = 11$ we look for a divisor of $\varphi(11) = 10$ and luckily we may take $N = 2$ because 11 divides $10^2 - 1 = 99$. However, as 7 fails to divide 99 and 999, then the smallest possible N for $m = 7$ is $\varphi(7) = 6$, and in fact 7 divides $999999 = 10^6 - 1$.

Provided we inspected the minimal denominator, A is precisely the amount of digits that we must take before the period, while N is precisely the length of the minimal period. In particular, it suffices to determine $A + N$ digits after the comma. If we work with a multiple of N instead, then we still obtain a periodical representation and we need one more step to obtain the canonical one with the minimal period. Should we have worked with an unnecessarily large A (for example because the given fraction was not reduced), we need an additional step to make the period start as soon as possible, cyclically permuting the digits under the period.

Coming back to the problem of determining the decimal representation of a rational number given as a fraction: we could patiently perform the long division between the numerator (without sign) and the denominator. Or we could alternatively reason as follows:

- If the rational number is negative, we need to figure out the decimal representation for its absolute value. We then suppose that the rational number is positive.
- If the number is larger than one, then we can easily remove its integer part (determined by performing a division with remainder) which constitutes the digits before the comma. In short, we now reduced to a rational number strictly between 0 and 1.
- Write the given rational number between 0 and 1 as $\dfrac{x}{10^A(10^N-1)}$. By performing the division with remainder $x : (10^N - 1)$ we can write the number as
$$\frac{q}{10^A} + \frac{1}{10^A}\frac{r}{10^N-1}$$

The digits of q, possibly with 0 digits at the beginning as to obtain precisely A digits, are the digits for the decimal representation after the comma and before the period. The digits of r, possibly with 0 digits at the beginning as to obtain precisely N digits, are the digits of the period for the decimal representation.
This can easily be observed with the geometric series because a purely periodical number between 0 and 1 with N digits in the period and such those digits represent the integer r is precisely the rational number $\dfrac{r}{10^N-1}$.

To conclude, the decimal representation of a rational number embeds nice results of arithmetics. In particular, such results are needed when a simple calculator is used to perform the long division (because such device only computes the first decimal digits and one needs to argument that enough digits have been computed).

An interesting exercise for talented pupils is adapting the above results and algorithm to a numeral system in a basis different from 10.