Generalized associativity

The addition of natural numbers is associative, so we can simply write the sum of n natural numbers without parenthesis:

$$a_1 + a_2 + \cdots + a_n$$

To compute this sum we have to compute $n-1$ sums of two numbers. Since addition is also commutative, we can permute the given numbers at leisure. Or, sparing us the permutation, we can start by picking any two of the numbers and replacing them by their sum. Repeating this procedure $n-1$ times allows us to compute the sum.

If a binary operation $*$ is associative but not necessarily commutative, to compute

$$a_1 * a_2 * \cdots * a_n$$

we may start with any of the *operation signs*, meaning that we could start by performing the operation between two *neighbouring* elements and replacing them with the result R of their operation:

$$a_1 * \cdots * (a_m * a_{m+1}) * \cdots * a_n = a_1 * \cdots * a_{m-1} * R * a_{m+2} \cdots * a_n$$

Proceeding in this way allows us, in $n-1$ steps, to compute the final result. The fact that the ordering of the chosen operations does not matter for the final result is the so-called *generalized associativity*.

Generalized associativity is a consequence of associativity that is often described as a rule concerning any possible *parenthesization*. What is, mathematically speaking, a choice of parenthesis for $a_1 * a_2 * \cdots * a_n$? We need $n-1$ left (respectively, right) brackets placed before (respectively, after) an element in the formula, in such a way that at any position, counting from left to right, the number of right brackets does not exceed the number of left brackets. Moreover, we exclude that a single element is enclosed between brackets. We should also exclude as invalid the following kind of parenthesization:

$$(1 + (2 + 3) + 4)$$

It's then easier to define parenthesization with a strong induction, namely we can group the first m elements together and the remaining elements together (for some m that is strictly between 1 and n):

$$(a_1 * \cdots * a_m) * (a_{m+1} * \cdots * a_n)$$

Like this we reduce to handling fewer elements, for which we assume to know what parenthesizations are.

The point of parenthesizations is giving us an ordering in which to perform the displayed operations: one can actually *prove* that a different choice of parenthesization corresponds to a different ordering. Assuming this as a fact, as there are $n-1$ operations, we have at most $(n-1)!$ possible parenthesizations. Unfortunately, a parenthesization gives us a unique ordering only if, for example, we rely on the convention that we prioritise operations on the left, as the following example shows:

$$(1 + 2) + (3 + 4)$$

The correct combinatorial concept to describe parenthesization is then allowing for an ordering of the operations that allows for *ex aequo.* In this case, ranked first would be all the operations between two elements that are enclosed by most inner parenthesis.

Let’s make an intermezzo. The sum of real (or complex) numbers is associative, and the same holds for the multiplication. We can replace the numbers by polynomials with real (or complex) coefficients, or by square matrices of a given size (with real or complex entries), and again we have associativity. The composition of functions from a set to itself is associative. We are much used to associativity. However, elementary operations that are not associative are the subtraction

$$7 - (5 - 1) \neq (7 - 5) - 1$$

and the division

$$32 : (8 : 4) \neq (32 : 8) : 4$$

We can also consider more fancy operations, for example the winning element in Rock Paper Scissors is not associative (recall that Rock wins over Scissors, Paper over Rock, and Scissors over Paper):

$$(R * P) * S = P * S = S \qquad R * (P * S) = R * S = R$$

Back to parenthesizations: they are very many, for example there are 14 of them for 5 elements. We can depict parenthesizations in the so-called *associahedron,* which is a convex polytope (in a nutshell, we can place them at vertices of a graph, an edge being one application of the associativity rule that connects two parenthesizations)

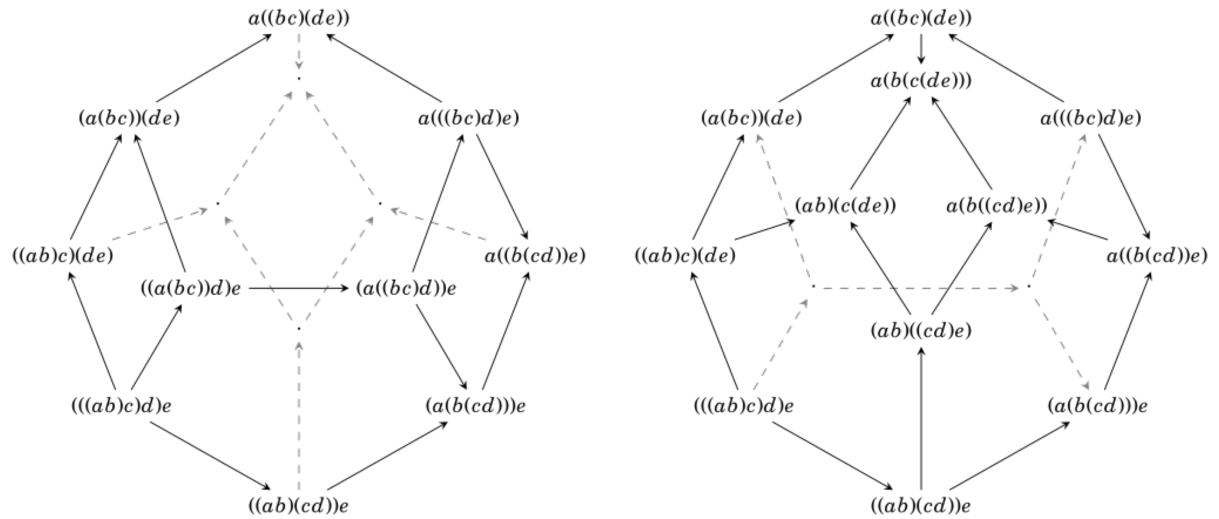

Figure: The associahedron for 5 elements (a polyhedron, here front and the back view).

To show that, for an associative binary operation, any possible parenthesization leads to the same final result (namely, to prove the generalized associativity) we need a general proof. The associativity rule may need to be applied many times, but fortunately there exists a simple (but not obvious) proof with strong induction. In fact, also for this proof, it would be easier to rephrase generalized associativity as an ordering of the operations to be performed between neighbouring elements.

We conclude with a side remark: the property that two powers —with a positive and integer exponent — of one same element commute (even if the binary operation is not commutative) is a direct consequence of associativity:

$$\underbrace{(a * \cdots * a)}_{n} * \underbrace{(a * \cdots * a)}_{m} = \underbrace{a * \cdots * a}_{n+m=m+n} = \underbrace{(a * \cdots * a)}_{m} * \underbrace{(a * \cdots * a)}_{n}$$