Quadratic equations: from trivial to impossible

What is the hardest quadratic equation you can conceive?

Consider a quadratic equation with real coefficients, aiming to determine the real solutions. Mathematically speaking, the equation $x^2 = 0$ is probably the easiest one: the only real number whose square equals zero is, clearly, zero. However, after presenting the quadratic formula, this is not the easiest equation: two coefficients are hidden (as they are zero), there is only one solution, and actually the number 0 is always a bit suspect.

Some equations have intuitive solutions, and in particular pupils should be confronted with equations like $(x-1)(x-2)=0$, where some might wrongly believe that a polynomial multiplication is required. Clearly, a teacher should present all basic sources of mistakes: the equation being not monic, the equation not being written in the canonical form (e.g. terms not in the expected order). In any case, the quadratic formula easily allows to solve, for example, the equation $x^2 + x - \pi = 0$ and in general the formula is valid for all possible real coefficients. Are we able to solve all quadratic equations?

Let's remember our limits. A natural number could have in principle more digits than the number of atoms in the universe… However, for a given quadratic equation, it has been possible to write down or express the coefficients. And the solutions will be a small algebraic expression in the given coefficients.

Warning: solving a quadratic equation means determining whether the discriminant is non-negative and, in the affirmative, writing the quadratic formula in terms of the given coefficients. Are you able to determine the sign of the discriminant in the equation $x^2 + 10^{99}x + 99! = 0$? In this example the computer still can, but computers also have their mathematical limits. The above issue is due to very large numbers, however there is also the problem of numbers close to zero (which provides a nice approximation exercise for pupils): for example, can you solve the equation $x^2 + 2x + \sqrt{17} - \pi = 0$?

The teacher can spice-up the topic of quadratic equations by embedding some algebraic calculations because of parametric coefficients, for example $x^2 + (a-2)x - a = 0$ (here the constant term is $-a$: pupils should be able to change "names" in the quadratic formula without making mistakes).
To solve parametric quadratic equations, one may need for example to solve a quadratic inequality in the parameter, as for the equation $x^2 + \lambda x + (\lambda + 1) = 0$, and there could be a case distinction concerning the number of solutions depending on the parameter…

To conclude with a happy ending, notice that all of the above-mentioned issues for quadratic equations disappear when working with the complex numbers. Indeed, we are then allowed to take the quadratic formula without reflection and, by simply plugging in the given coefficients, we have written down the complex solutions.

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