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Tiling Deficient Staircase Regions with L-trominoes

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Introduction

A polyomino is a shape made by connecting a certain number of 1×1 squares, which are joined edge to edge. Polyominoes are generalizations of dominoes and seem to have been introduced by Golomb [**1**], who considered several questions concerning the tileability of certain square regions by different polyominoes. Since then, tiling problems related to polyominoes have played in important role in recreational mathematics, and we refer the reader, for example, to the monographs [**2**, **3**] for a collection and overview of many results on this topic.

In this note, we will focus on tiling problems related to a special kind of polyominoes, namely the L-trominoes, by which we mean any of the four shapes made of three 1×1 squares given in Figure 1.

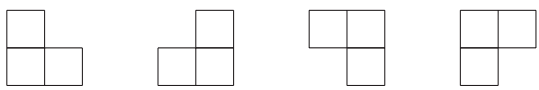

Figure 1 The four L-trominoes.

We mention that there exists a further kind of tromino, the I-tromino (or straight tromino), which is composed of three aligned squares. These trominoes will not be considered here and consequently, by a tromino we will always mean an L-tromino and tiling will always refer to tiling by L-trominoes.

In general, in order for a region to admit a tiling by L-trominoes, it is, of course necessary that the region is composed of 1×1 squares, whose total number is a multiple of three. An obvious, but useful observation (that we shall repeatedly make use of) is that a 2×3 rectangle (and hence also a 3×2 rectangle), as well as the "big L-tromino" can be tiled, as can be seen in Figure 2.

We also state a result by Chu and Johnsonbaugh [**4**], who considered the problem of tiling squares and rectangles by L-trominoes, and proved that any $n \times m$ rectangle with $3|nm$ can be tiled, except when one dimension is three and the other is odd.

For $n \in \mathbb{N}$, we denote by $S(n)$ the staircase region with height n, that is the staircase region with n steps, given in Figure 3.

We shall be concerned with the question of tiling staircase regions by L-trominoes. The region $S(n)$ is composed of $\frac{n(n+1)}{2}$ squares, which is divisible by three only in case

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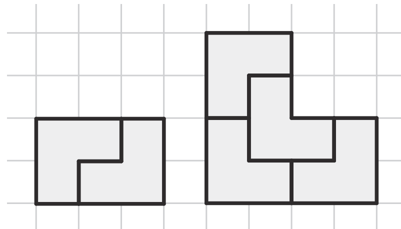

Figure 2 Tiling of a 2×3 rectangle and a big L-tromino.

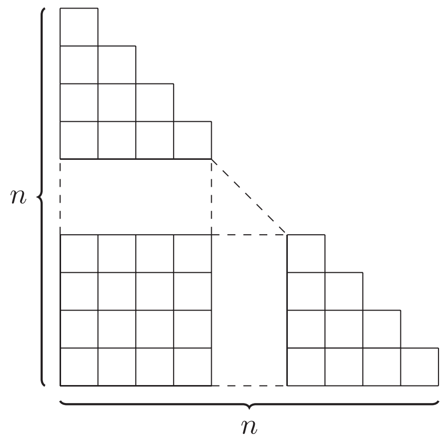

Figure 3 The staircase region $S(n)$.

$n \equiv 0, 2, 3, 5 \mod 6$. Hence, only for such n is a tiling of $S(n)$ might be possible. It is rather straightforward to check that $S(2)$, $S(6)$, $S(9)$, and $S(11)$ admit a tiling by L-trominoes (see Figure 4), while $S(3)$ and $S(5)$ do not.

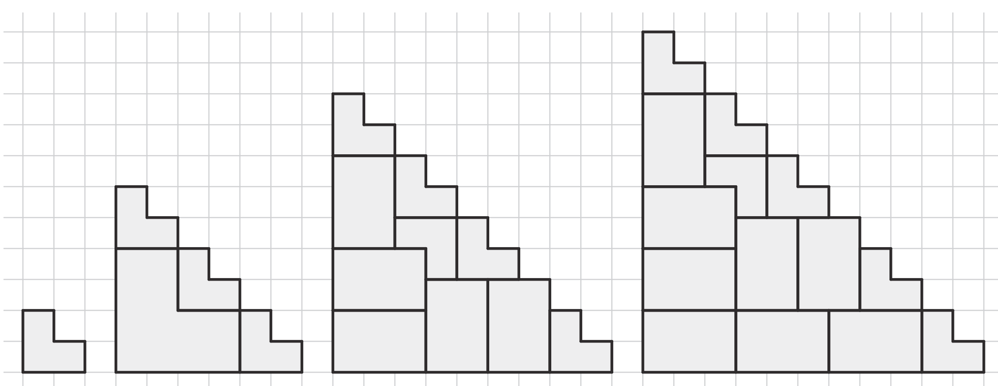

Figure 4 Tilings of $S(2)$, $S(6)$, $S(9)$ and $S(11)$.

A full solution to the question of which $S(n)$ can be tiled is then obtained using the following observation.

Proposition 1. *Suppose that $n \geq 2$ and that a tiling for $S(n)$ exists. Then there exists a tiling for $S(n+6)$.*

Proof. For the proof, we note that $S(n+6)$ can be decomposed into $S(n)$, a $6 \times n$ rectangle and $S(6)$, as can be seen in Figure 5.

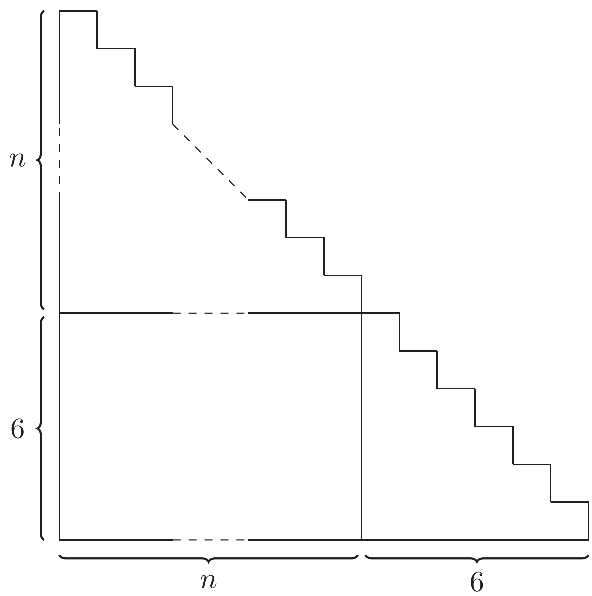

Figure 5 Decomposition of $S(n+6)$.

Now, by assumption, $S(n)$ can be tiled, and the same holds for the $6 \times n$ rectangle (a $6 \times n$ rectangle can be tiled for every $n \geq 2$). As mentioned above, $S(6)$ can also be tiled, so that we finally obtain the tileability of $S(n+6)$. ■

Combining this proposition with what we stated before, we obtain the following result, which can also be found in [5].

Proposition 2. *A tiling of* $S(n)$ *is possible if and only if* $n \equiv 0, 2, 3, 5 \mod 6$ *and* $n \notin \{3, 5\}$.

In what follows, we shall consider the following problem, which seems to not have been investigated before. Suppose that $n \geq 2$ and $n \equiv 1, 4 \mod 6$. It is easily seen that the number of 1×1 squares that compose $S(n)$ is then given by $3k + 1$ for some $k \in \mathbb{N}$. Hence, a tiling by trominoes is impossible, but the question arises if in this case $S(n)$ can be tiled if we remove one 1×1 square, and if so, if the position of the removed square plays a role. We will give a complete answer to this question in the following section and fully characterize the squares that can be removed from $S(n)$ in order for a tiling to exist. In particular, we will show that these squares follow a general pattern, with the exception of the case $n = 7$.

We mention that the corresponding problem of tiling squares from which a 1×1 square has been removed (so-called deficient squares) by L-trominoes was first investigated by Golomb [1], who proved that such squares can always be tiled, independent of the position of the removed 1×1 square, if their side length is a power of two. The case of general deficient squares and deficient rectangles was later considered by Chu and Johnsonbaugh [6] and Ash and Golomb [7] (see also [8]).

Tiling deficient staircase regions

In the following, we will call a staircase region with a missing 1×1 square a deficient staircase region and we will denote by $S_d(n)$ a deficient staircase region of height n. When considering deficient staircase regions, a useful observation is that $S(n)$ has a symmetry axis (namely the line passing through the bottom left and top right corner of the first square in the bottom row); hence, if a tiling of an $S_d(n)$ region exists, if a square above (or below) the symmetry axis has been removed from $S(n)$, then the same holds if the corresponding square below (or above) the axis is removed. We further mention that Proposition 1 obviously also holds for deficient staircase regions, that is if $S_d(n)$ is tileable, then the same holds for $S_d(n+6)$, since an $S_d(n)$ region can be extended to an $S_d(n+6)$ region in the same way as $S(n)$ can be extended to $S(n+6)$.

The cases $n = 4$ and $n = 7$ We start by considering tilings of the deficient staircase regions $S_d(4)$ and $S_d(7)$.

Proposition 3. *The deficient staircase regions $S_d(4)$ and $S_d(7)$ can be tiled if and only if the removed 1×1 square is one of the white squares in Figure 6.*

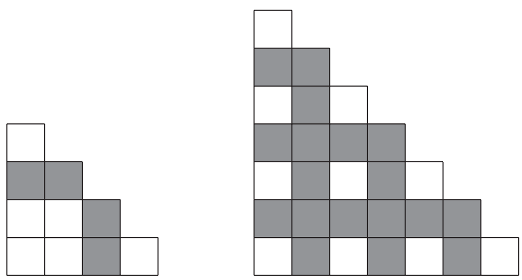

Figure 6 The regions $S(4)$ and $S(7)$. White squares can be removed to obtain tileable $S_d(4)$ and $S_d(7)$ regions, gray squares cannot be removed.

Proof. We start with the case $n = 4$. First of all, it is clear that removing a gray square leaves a region that can not be tiled. That a tiling is possible after removing one of the white squares follows from Figure 7.

Indeed, the left picture shows that the top square can be removed (and hence, by symmetry, the same holds for the square in the bottom right corner), while the right picture shows that any of the four squares in the bottom left corner can be removed (a deficient 2×2 square can always be tiled, independent of the removed square).

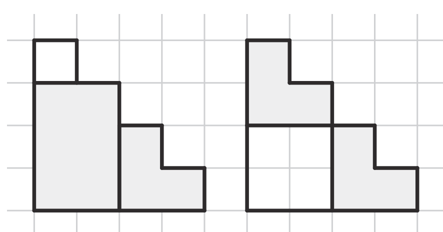

Figure 7 Tilings showing that removing a white square leads to a tileable $S_d(4)$ region.

For the case $n = 7$, the necessity follows from the fact that $S(7)$ is composed of 28 squares, hence 9 trominoes are needed to tile $S_d(7)$. However, an L-tromino can cover at most one of the white squares in the picture at right in Figure 6, and since there are 10 such squares, removing a gray square clearly leaves a region that is not tileable.

That removing a white square is sufficient for a tiling to exist follows from Figure 8 and symmetry considerations. ■

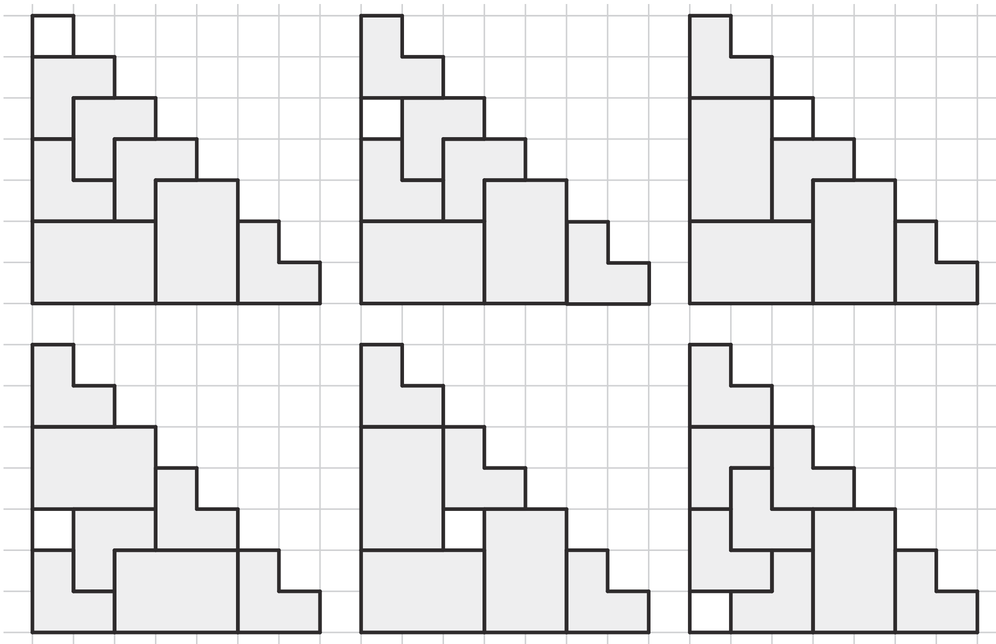

Figure 8 Tilings showing that removing a white square leads to a tileable $S_d(7)$ region.

The cases $n = 10$ and $n = 13$ Using the results obtained for $n = 4$ and $n = 7$, we will now treat the cases $n = 10$ and $n = 13$. A similar technique will later be used to obtain a result for the general case.

Proposition 4. *The deficient staircase regions $S_d(10)$ and $S_d(13)$ can be tiled if and only if the removed 1×1 square is one of the white squares in Figure 9.*

Proof. In both cases, the necessity is easily seen, for, removing one of the two gray squares in the second top row clearly leaves a region that can not be tiled by L-trominoes. Suppose now that the single gray square in the fourth top row has been removed. Since there is only one possibility to cover the square at the very top by an L-tromino, we are then left with a region whose top part is given in Figure 10. There is now only one possibility to cover the square above the gray square with a tromino, however, it is obvious that after having placed this tromino, the remaining region can not be tiled. Hence, none of the three gray squares in the top part of $S(10)$ and $S(13)$ in Figure 9 can be removed, and by symmetry, the same then holds for the three gray squares in the bottom right part.

We will now show that $S_d(10)$ can be tiled if any of the white squares has been removed. The idea is to extend tilings of $S_d(4)$ to tilings of $S_d(10)$. Recall that we know from the case $n = 4$ that any of the $S_d(4)$ regions in Figure 11 can be tiled, and consider the regions in Figure 12. These regions, together with an $S(6)$ region, can be used to extend the regions in Figure 11 to $S(10)$.

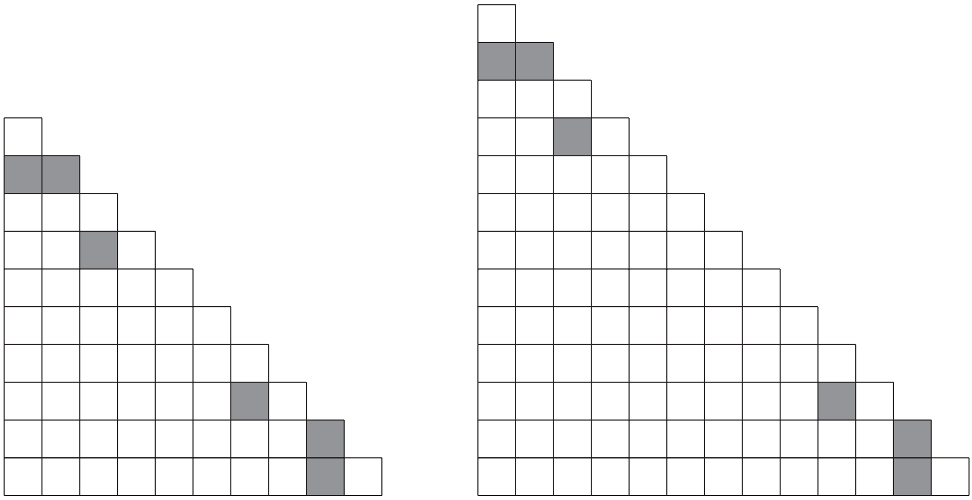

Figure 9 The regions $S(10)$ and $S(13)$. White squares can be removed to obtain tileable $S_d(10)$ and $S_d(13)$ regions, gray squares cannot be removed.

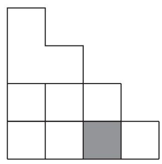

Figure 10 Top part of $S_d(10)$ and $S_d(13)$, in case the removed square is the gray square in the fourth top row.

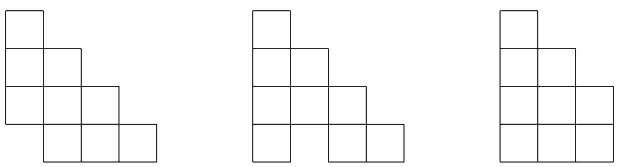

Figure 11 Tileable $S_d(4)$ regions.

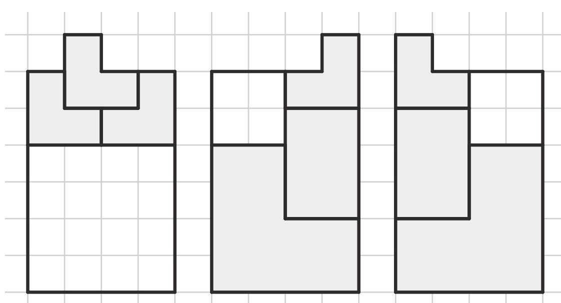

Figure 12 Tilings used to extend tilings of $S_d(4)$ to tilings of $S_d(10)$.

Now, using the fact that a deficient 2×2 and 4×4 square can always be tiled by L-trominoes (independent of the position of the removed square, see, e.g., [1]), we can infer from Figure 12 and the finding that $S(6)$ can be tiled, as well as from Proposition 3 and symmetry considerations, that the deficient staircase region $S_d(10)$ we obtain after one of the white squares in Figure 13 is removed, can be tiled. However, Figure 14 and symmetry considerations show that the blue squares can also be removed for a tiling to exist, so that we finally obtain that a tiling of $S_d(10)$ is possible, except if one of the six gray squares in the left picture in Figure 9 is removed.

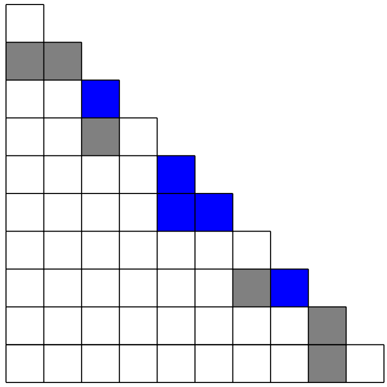

Figure 13 The region $S(10)$. Our general considerations show that removing a white square leads to a tileable $S_d(10)$ region, blue squares remain to be checked.

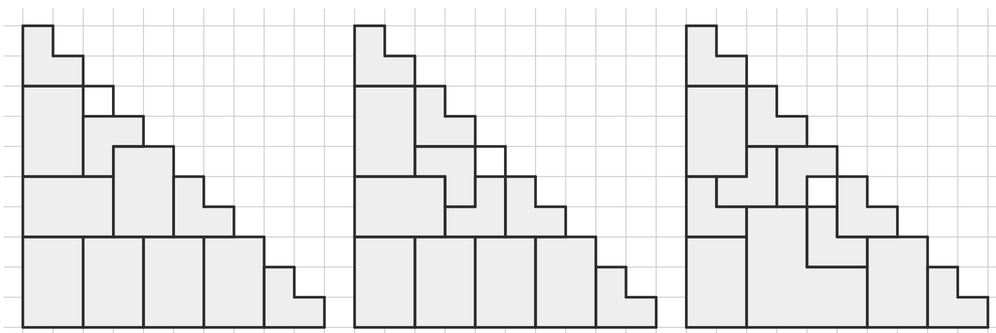

Figure 14 Tilings showing that the blue squares in Figure 13 can be removed.

We now consider the case $n = 13$. To show that any of the white squares may be removed, we proceed as before. We first recall the previously shown fact that $S_d(7)$ can be tiled, if it results from $S(7)$ by removing either the first, third, fifth or seventh square in the bottom row. We then infer from Figure 15 and the tileability of $S(6)$, as well as from Proposition 3 and symmetry considerations, that the $S_d(13)$ region we obtain after one of the white squares in Figure 16 is removed, can be tiled.

However, noting that $S_d(4)$ can be extended to $S_d(13)$ by adding an 9×4 rectangle to the bottom and $S(9)$ to the bottom right, which both can be tiled, we infer that a tiling of the $S_d(13)$ region that results from the removal of one of the red squares in Figure 16 is also possible. Finally, that a blue square can also be removed follows from Figure 17.

Note that these tilings can be extended to tilings of $S_d(13)$ by adding a 5×6 (first two cases) or a 4×6 (last two cases) rectangle to the bottom, and an $S_d(7)$ region (obtained by removing the top square from $S(7)$) to the bottom right, which can all be tiled. Hence, the only squares that can not be removed are the six gray squares in the right picture in Figure 9. ■

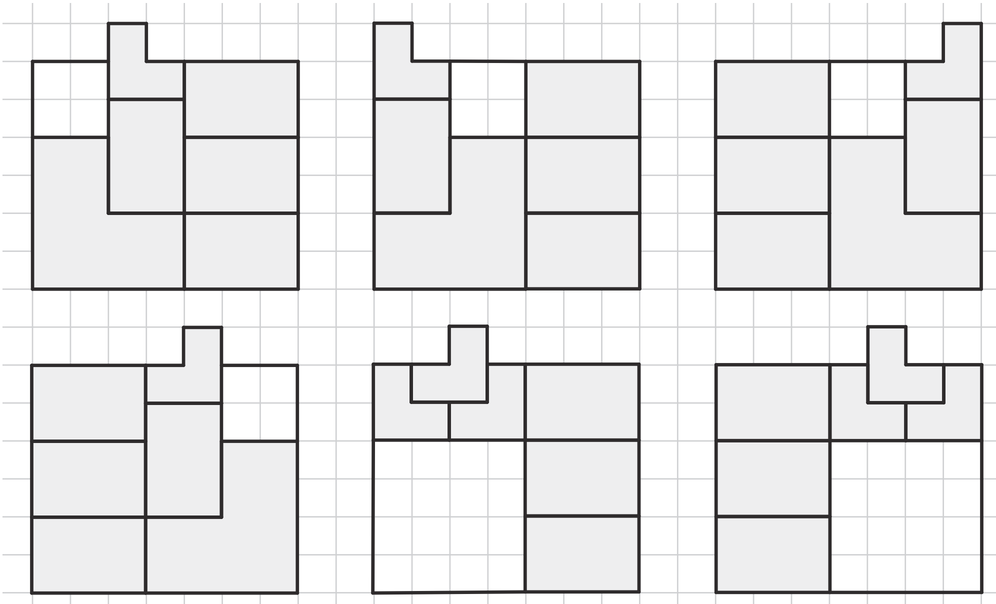

Figure 15 Tilings used to extend tilings of $S_d(7)$ to tilings of $S_d(13)$.

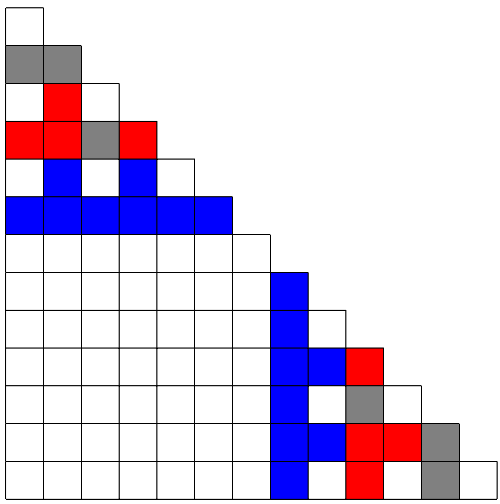

Figure 16 The region $S(13)$. Our general considerations show that removing a white square leads to a tileable $S_d(13)$ region, blue and red squares remain to be checked.

The general case We will now consider the general case and thus obtain a complete answer to the problem of tiling deficient staircase regions.

Theorem 1. *Let $n \in \mathbb{N}$ be given with $n \geq 10$ and $n \equiv 1$ or $4 \mod 6$. Then $S_d(n)$ can be tiled if and only if the removed 1×1 square is one of the white squares in Figure 18.*

The necessity of the condition follows exactly as for the cases $n = 10$ or $n = 13$. Hence, we need to prove that the condition is sufficient. This will be an immediate consequence of Proposition 4 and the following result.

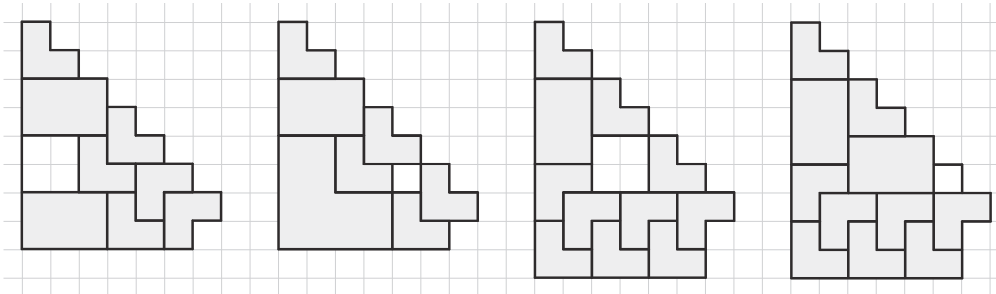

Figure 17 Tilings showing that the blue squares in Figure 16 can be removed.

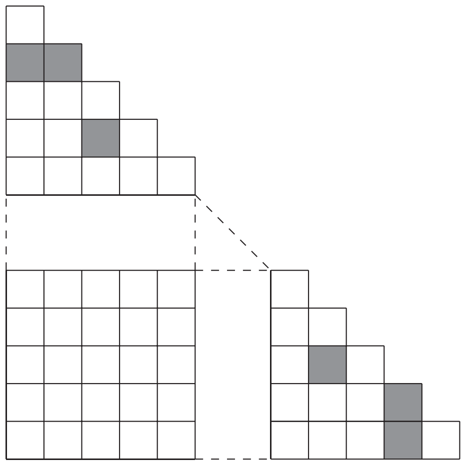

Figure 18 The region $S(n)$ with $n \geq 10$ and $n \equiv 1, 4 \mod 6$. White squares can be removed to obtain tileable $S_d(n)$ regions, gray squares cannot be removed.

Proposition 5. *Let $n \in \mathbb{N}$ be given with $n \geq 10$ and $n \equiv 1$ or $4 \mod 6$. Suppose that $S_d(n)$ can be tiled if and only if the removed 1×1 square is one of the white squares in Figure 18. Then the same holds for $S_d(n+6)$.*

Proof. We apply a similar reasoning as above and extend tilings of $S_d(n)$ to tilings of $S_d(n+6)$ by adding a $6 \times n$ rectangle and $S(6)$, which can both be tiled. From the assumption and symmetry considerations, we then immediately obtain the tileability of the $S_d(n+6)$ region that results from removing one of the white squares in Figure 19.

Now, we first note that since $n \geq 10$, we can decompose the $6 \times n$ rectangle we added at the bottom into a 6×6 square and a $6 \times (n-6)$ rectangle, which can both be tiled individually (i.e., without an overlapping tromino). We can further suppose that the left part of the $6 \times n$ rectangle is occupied by the 6×6 square. By assumption, $S_d(n)$ admits a tiling if it is obtained from $S(n)$ after removing any of the first six 1×1 squares in the bottom row. We can then infer from Figure 20 and symmetry considerations that there also exists a tiling of the $S_d(n+6)$ region that results from removing any of the blue 1×1 squares in the bottom left 6×6 square in Figure 19.

It remains to show that the yellow and the two red squares can also be removed. We first note that if $n \geq 13$, and hence $n+6 \geq 19$, this follows immediately from symmetry considerations, since the squares that are symmetric to these three squares are white squares and hence can be removed. In case $n = 10$ (that is $n+6 = 16$), we see that the yellow square can be removed, as the square that is symmetric to it

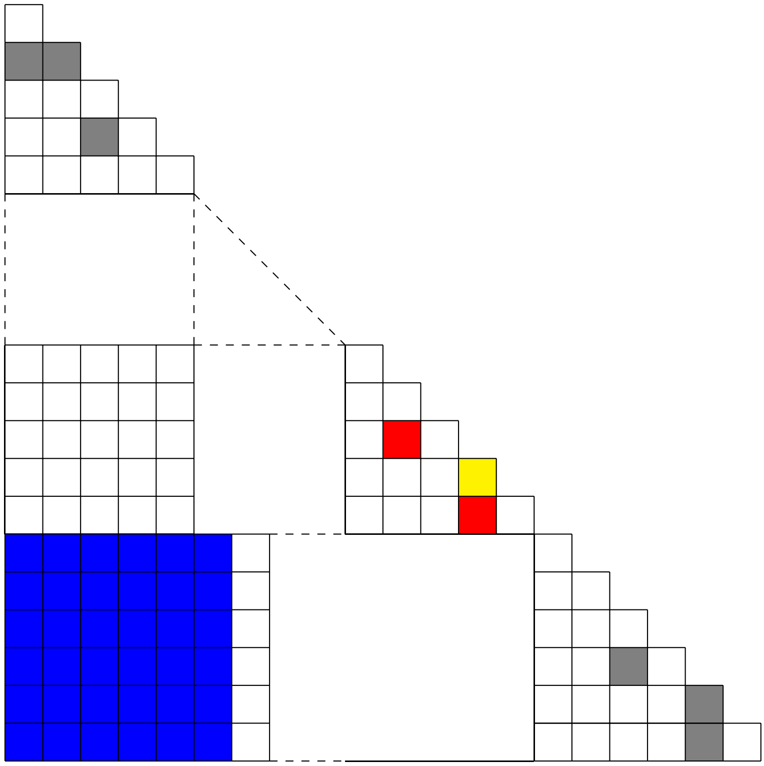

Figure 19 The region $S(n+6)$ with $n \geq 10$ and $n \equiv 1, 4 \mod 6$. Our initial considerations show that removing a white square leads to a tileable $S_d(n+6)$ region, blue, red and yellow squares remain to be checked.

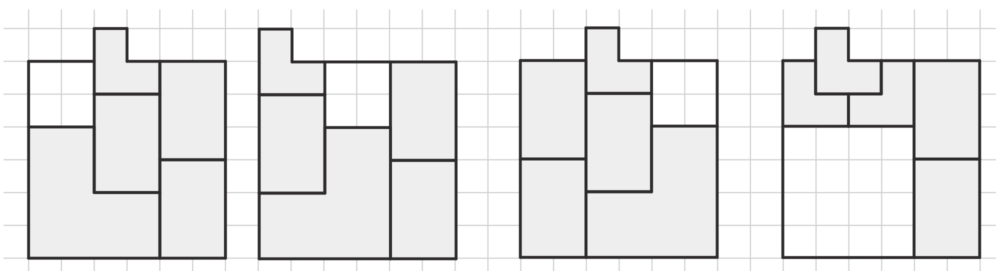

Figure 20 Tilings showing that the blue squares in Figure 19 can be removed.

is white. However, in this case it is not clear if the two red squares can be removed, since they are symmetric to each other. Hence, we need to check if the $S_d(16)$ region we obtain by removing one of these two squares can be tiled. Figure 21 shows that this is indeed the case (note that this tiling can be extended to a tiling of $S_d(16)$ by adding a 5×9 rectangle to the bottom, which can be tiled according to a result by Chu and Johnsonbaugh [4], and an $S_d(7)$ region (obtained by removing the top square from $S(7)$) to the bottom right, which can also be tiled), which finishes the proof. ∎

Remark. *Note that even if in the case $n = 4$, there are only four "forbidden" squares, this case still fits into the general pattern given in Theorem 1, as the six forbidden squares we obtain in the general case reduce to four squares for $n = 4$. The case*

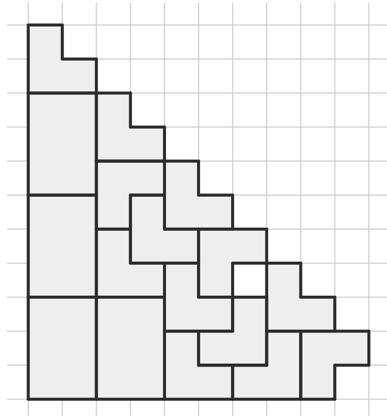

Figure 21 Tiling showing that the red squares in Figure 19 can be removed in case $n = 10$.

$n = 7$, on the other hand, is truly exceptional in the sense that it is the only case where besides the six squares from the general case, additional forbidden squares exist.

Consequences for other regions

The results from the previous sections can be used to infer the tileability of regions that are composed of several staircase regions. We give only a few examples here and invite the reader to investigate further possibilities. The first region we consider is given in Figure 22.

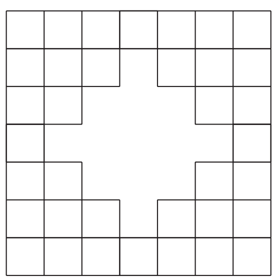

Figure 22 A region that can be decomposed into four tileable $S_d(4)$ regions.

It is easily seen that this region can be constructed by combining four $S_d(4)$ regions that are obtained by removing the top 1×1 square from $S(4)$. Hence, by the results from the previous section, it can be tiled by L-trominoes. In fact, every such region having a height of $2n - 1$ for $n \geq 2$ and $n \equiv 1, 4 \mod 6$ can be tiled.

As a further example, we consider the regions in Figure 23, which can be decomposed into two $S_d(7)$ regions (obtained by removing the first square in the bottom row from $S(7)$), and an $S(6)$ and such an $S_d(7)$ region, respectively (as indicated in the figure). In both cases, a tiling is possible according to our results.

Our final example is the so-called Aztec diamond region given in Figure 24. Note that its height is always even, and if its height is n, then it can be decomposed into four $S\left(\frac{n}{2}\right)$ regions. Hence, if $\frac{n}{2} \equiv 0, 2, 3, 5 \mod 6$ and $\frac{n}{2} \notin \{3, 5\}$, an Aztec diamond

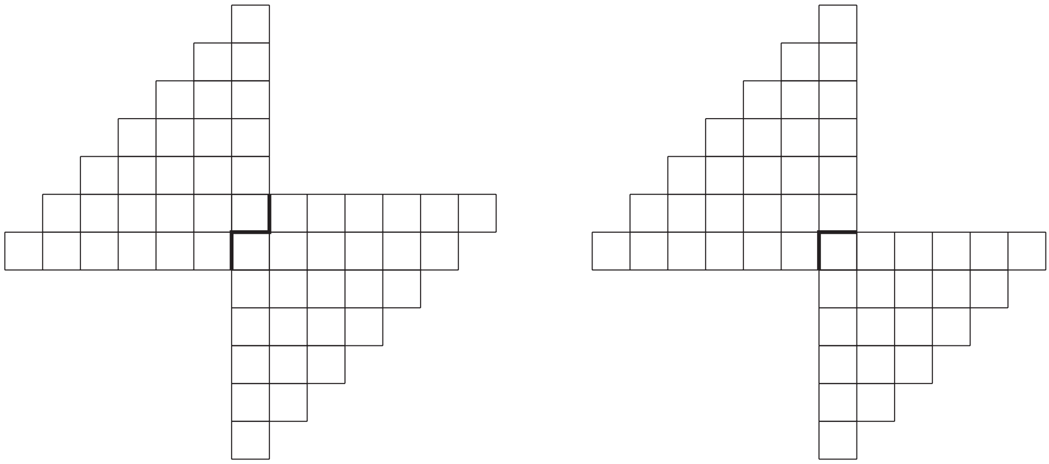

Figure 23 Regions that can be decomposed into two tileable $S_d(7)$ regions (left), and into an $S(6)$ and a tileable $S_d(7)$ region (right).

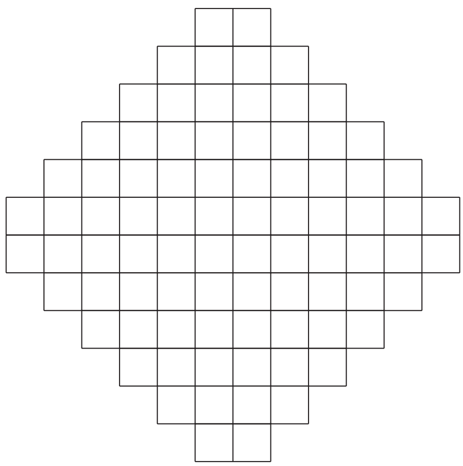

Figure 24 An Aztec diamond.

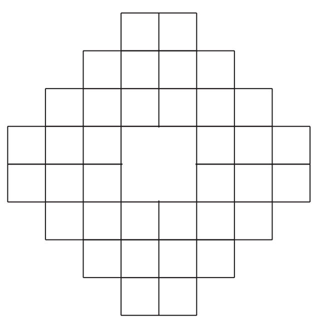

Figure 25 A region that can be decomposed into four tileable $S_d(4)$ regions.

can be tiled by L-trominoes by Proposition 2. On the other hand, if $n \geq 2$ and $\frac{n}{2} \equiv 1$ or 4 mod 6, a tiling is not possible, since the total number of 1×1 squares is not a multiple of three. However, we can infer from the results in the previous section that in this case a tiling of the "deficient Aztec diamond" is possible, at least if one of the four center squares are removed. As an example, we consider the case $n = 8$, hence $\frac{n}{2} = 4$. We know that the $S_d(4)$ region that results from removing the first square in the bottom row from $S(4)$ can be tiled, hence, the region in Figure 25, which is composed of four such $S_d(4)$ regions, admits a tiling.

It is then clear that the deficient Aztec diamond of height $n = 8$, where one of the four 1×1 squares in the center has been removed, can be tiled by L-trominoes.

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Summary. We consider the problem of tiling staircase regions with L-trominoes. In particular, in case such a tiling is not possible, we investigate if the situation changes if we remove one 1×1 square from the staircase region and if the position of the removed square plays a role. We give a complete solution to this question. Moreover, we use our results to infer the tileability of some regions that are composed of several staircase regions.

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