Cardinalities and the continuum hypothesis

Infinite sets have various sizes, that are called cardinalities.

The smallest possible size for an infinite set is the one of the natural numbers. Infinite sets with that size are called "countable". Infinite sets are countable if you can label each element with the natural numbers (such that all elements have a label and such that distinct elements have distinct labels).
For example, it is clear that the negative integers

-1,-2,-3,-4,…

can be assigned such labels. It might be less clear that this also holds for the integers, but this is possible by ordering them as follows:

0,1,-1,2,-2,3,-3,….

It is even less clear that the rational numbers are countable. Let's see why the positive rational numbers are countable, because then we might resolve to a trick as above to include all rational numbers. We can order positive reduced fractions by first listing those with numerator and denominator bounded by 1 (only the number 1), those with numerator and denominator bounded by 2 (1,2,1/2), those with numerator bounded by 3 and so on. So the positive rational numbers are countable because we may order them as follows:

1,2,1/2,3,1/3,2/3,4,1/4,3/4,…

Comparing sizes is very important, and it might not be intuitive at first. Two sets have the same size if we can put their elements in a correspondence. If we can use the elements of a set A to label the elements of a set B, then the size of A is at least the size of B.
It can be shown that a set and its power set (note: the power set of a set S is the set whose elements are the subsets of S) have different sizes, the power set having a larger size. We deduce that the real numbers have a larger size with respect to the natural numbers (note: there is also a direct accessible argument to prove this, that is called Cantor's diagonal argument). Indeed, we can encode a subset of the natural numbers as a sequence of 1's and 0's (where 1 means "take this element" and 0 means the contrary). So the subsets of the natural numbers correspond to distinct real numbers from 0 to 1, by looking at the sequence of digits after the comma. For example 0,10101010… are the odd natural numbers and 0,010101010…. are the even natural numbers.

By taking power sets of power sets we can produce larger and larger sizes. However, a natural question remains: are there intermediate sizes between the one of the natural numbers and the one of the real numbers? Supposing that there are no intermediate sizes is what is called "the continuum hypothesis". This property cannot be proven or disproven with the usual axioms for the numbers. Since this is truly hard to grasp, we resolve to a metaphor. Say that the usual axioms of the numbers describe various aspects of a company that produces T-shirts. You know that they produce the size Small (natural numbers) and you know that they produce the size Large (real numbers). However, from the description it is not clear at all whether the company produces the size Medium (namely, a size larger than the one of the natural numbers and smaller than the one of the real numbers). They could produce the Medium size and they could not produce it, no option can be excluded. Clarifying this matter is impossible with the given description, one would need a more accurate description (namely, more axioms for the numbers).

Aside: Ordinal numbers take into account not only the size of sets, but also how they can be ordered. A nice metaphor for them (that is described in detail in the book *Beyond Infinity* by Eugenia Cheng) is "infinite queues". For example, adding a last guest to an infinite (countable) queue is not the same as an infinite queue because in the latter there is no last person. And adding an infinite queue to an infinite queue of priority guests is not the same as an infinite queue because the first non-priority guest cannot clearly identify the last guest before them (despite not being the first in line).