Optimal behavior under irreversibility risk and distance to the irreversibility thresholds in local pollution

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Abstract

We study optimal behavior under irreversible pollution risk for a general class of models with irreversible local pollution. Irreversibility comes from the decay rate of pollution dropping to zero above a pollution level featuring non-convexity. In addition, the economy can instantaneously move from a reversible to an irreversible pollution mode, following a Poisson process. First, we are able to prove for the general class of models that for any value of the Poisson probability, the optimal emission policy leads to more pollution with the irreversibility risk than without in a neighborhood of the irreversibility pollution threshold. Indeed, we prove that the extent of uncertainty (as captured by the Poisson arrival rate) is second-order in this neighborhood (in the sense of Taylor expansions). Second, to study analytically the robustness of the latter result at any pollution level, we consider the case of linear-quadratic objective functions which we solve in closed form. We find that the general local result does not necessarily hold if actual pollution is far enough from the irreversibility threshold.

Keywords: Irreversible pollution, uncertainty, piecewise deterministic, optimal behavior under risk, avoidability of the irreversible regime

JEL classification: Q52, C61, D81

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1 Introduction

Optimal pollution control is a quite old stream in economics, including economic theory (see for example Keeler *et al.*, 1972). Along with the ongoing ecological transition and climate change, the latter area has recently become increasingly central in many economic research agendas. Of course, the existential nature of the global warming problem is one of the key motivations behind this research upsurge.[1] However, many other related or unrelated topics are currently intensively explored reflecting the large diversity of issues involved in the ecological transition.

Some of these topics are not strictly speaking new but they have recently regained interest as they turn to be very relevant for the current ecological/climate problems. One of these reminiscent topics is irreversibility of pollution. The main idea behind irreversibility is that continuous accumulation of pollutants (local soil, water or air pollution or global warming due to GHG overaccumulation in the atmosphere) could at a certain point in time reach a threshold level such that beyond this level, key regulating and vital mechanisms become permanently partially or totally defective. In the case of global warming, this translates into an upward destabilizing temperature path and other subsequent potentially permanent damages.[2] In the case of local pollution, an early theoretical contribution is due to Forster (1975): in his framework, the decay rate of pollution (or Nature's ability to absorb pollution) declines sharply above a certain level of pollution. Numerous examples of possible irreversibility features in local pollution have been put forward since then. A significant part is actually connected to the literature of tipping points in dynamic systems as in the so-called shallow lake problem (see for example, Maler, 2000): the emergence of tipping points in the lake ecosystem dynamics follows small variations in phosphorus loads, ultimately leading to significant losses in ecosystem services.[3] This work is concerned with irreversible **local** pollution in the sense of Forster (1975).

A key question turns out to (legitimately) be to which extent irreversible

[1] Climate change has been recently proclamated as *the ultimate challenge for economists*, by Nordhaus (2019) in his Nobel lecture.

[2] For example, there exists a growing evidence on the weakening of the AMOC (Atlantic Meridian Overturning Circulation). See Bonnet et al (2021).

[3] The irreversible pollution literature also significantly intersects with the broader stream on regime shifts, see Boucekkine et al. (2013) for an application including regime shifts of the irreversible pollution type.

environmental changes can be avoided. In a seminal paper, Tahvonen and Withagen (1996) have demonstrated in a quite general convex-concave optimal pollution control problem that avoiding the threat of irreversible pollution thresholds is not granted. In their model, similarly to Forster (1975), irreversibility occurs if pollution reaches a critical level above which the decay rate of pollution drops permanently to zero. Tahvonen and Withagen (TW hereafter) prove that even a benevolent central planner cannot always avoid crossing the critical pollution level, which formulated into a game-theoretic frame means that even full cooperation among players cannot always prevent this unpleasant outcome (see the dynamic game extension of TW in Boucekkine et al, 2023). Moreover, even though optimal maintenance/abatement are introduced for pollution control, optimal paths with irreversible pollution paths may still emerge under some non-extreme parametric conditions (Prieur, 2009).

Another aspect appears to be potentially important for the ultimate impact of irreversible pollution: uncertainty. Indeed, uncertainty surrounding this event can have multiple sources, the most obvious being that the inherent irreversibility threshold levels are not known with certainty. Of course, uncertainty is not only on the level of the irreversibility thresholds, it also bears on the extent of damages caused when the irreversibility thresholds are reached. There is a strong research line which associates irreversibility levels crossing with catastrophic damage, therefore entailing a catastrophic risk component. Clarke and Reed (1994) and Tsur and Zemel (1998) are two representatives of this research. In the former, the value function associated with the underlying optimal control problem is zero at the threshold irreversibility level while it is minus infinity in the latter. This goes with the idea that as the irreversibility levels are reached, this will come with catastrophic events (such as extreme climate consequences). It's unclear whether these scenarios are the most relevant, especially when it comes to local pollution which is the object of our study: alternatively one could assume that reaching the irreversible regime will produce a permanent sharp drop in the value function, not necessarily to minus infinity or to zero. This is the case in Le Kama et al., 2014, within a different theoretical framework that however keeps some analogies with TW. This is also the game-theoretic set-up of Wirl (2008) where an Ito-process traces the accumulation of pollution and differentiates between reversible and irreversible emissions.[4]

[4]Many other insightful stochastic models with irreversibility or related features (such

Putting technicalities apart, the essential issue tackled in this literature is whether the irreversibility risk with the associated sharp drop in environmental quality, be it catastrophic or not, will induce more conservative or more aggressive behavior, for example in terms of pollutants' emissions. Intuitively, one would think that subject to the irreversibility risk, the economic agents would prefer to behave in a more conservative way and pollute less. Indeed, a key point is whether the irreversibility risk depends or not on pollution, and if so, how. This question is central in the catastrophic risk literature. An excellent paper illustrating this point in the case of catastrophic pollution risks (due to global warming) is due to van der Ploeg (2014) who characterizes the optimal pigouvian taxation depending on the shape of the hazard function (that's on how the irreversibility risk depends on pollution). Broadly speaking, if the latter risk is *strongly enough* increasing (Resp. decreasing) with the pollution stock, optimal behavior is likely to be more (Resp. less) conservative and one would gets lower (Resp. larger) pollution and consumption at the steady state in the presence of risk. A second important point is stressed in this literature following Clarke and Reed (1994): mere dependence of the risk on pollution is not enough to generate the conservative behavior under the irreversibility risk. Another key aspect is the **the avoidability of the irreversibility regime**. Even though the hazard function is strongly increasing with pollution, economic agents may not refrain from polluting if they are convinced that the irreversible regime is necessarily to come.

In this paper, we shall study this avoidability argument on a general class of stochastic local pollution problems building on Forster and TW. In particular, we shall explore whether the distance to the irreversibility thresholds inherent in these local pollution problems is a relevant determinant of the (un)avoidability question outlined above. We incidentally move away from the catastrophic risk modelling, which is certainly much more appropriate to study the global warming problem, among other global pollution problems, but has the disadvantage to restrict the analysis around the steady states for tractability reasons. In the case of local pollution problems, the appraisal of the avoidability argument is natural through the distance to the irreversibility pollution thresholds but it requires some tractability in deriving optimal short-term dynamics, in particular when initial pollution is far

as tipping points or regime shifts problems) can be found in the economic literature. See in particular Bretschger and Vinagradova (2019) or Diekert (2017).

from the irreversibility threshold. However, the question of how contemporaneous or short-term optimal behavior is shaped by the irreversibility risk is actually interesting in itself. Indeed, it might be equally relevant to explore optimal behaviour at any given level of pollution in the presence or absence of the latter risk even if (and especially if) the *event does not occur* to use the terminology of the related literature (i.e. even if the bad shock does not occur).

In this paper, we propose a fully analytical approach to the latter questions. To this end, we consider a piecewise deterministic extension of the general class of convex-concave models with irreversible (local) pollution proposed by TW: the stochastic extension has two modes, a reversible vs an irreversible pollution mode, and the probability to move from the former to the latter is a Poisson process, the irreversible mode being an absorbing state. Compared to van der Ploeg (2014) and other related papers surveyed above, we therefore consider that the hazard function is constant in order to have a chance to bring out analytical results. As alluded to above, the law of motion of pollution is of the Forster type: the pollution decay rate drops to possibly zero above a certain pollution level, featuring the non-convex problem which makes it nontrivial to solve. To deal properly with this non-convexity, we apply a dynamic programming approach as in the deterministic problem studied in Boucekkine et al. (2023). We partly rely on Dockner et al. (2000) to deal with the stochastic extension examined here.

We start with the general piecewise deterministic class of TW convex-concave models with irreversible pollution. Within this general framework, we characterize the optimal pollution emission policy in the neighborhood of the irreversibility threshold, that's close enough to this threshold from the right. We are able to obtain a general result according to which optimal emissions are higher under the irreversibility risk than without for any level of risk (that's for any value of the Poisson arrival rate) in this neighborhood. This is not so counterintuitive if one keeps in mind the rationale already outlined by several authors: the 'avoidability' of the irreversible regime argument. We show, among others, that the Poisson arrival rate is a second-order determinant of the optimal pollution policies (in a precise sense to be clarified in Section 3) in the neighborhood of the irreversibility threshold, which explains part of the general result we get locally.

While the general class of models allows for analytical results locally, it's no longer the case globally, which is hardly surprising given the convex-concave nature of the problem. We therefore pick a model in the general class which does admit full analytical solution globally: we consider a linear-quadratic objective function which permits the derivation of a comprehensive analytical characterization of the optimal policy paths, which is actually still very far from trivial given the convex-concave features deriving from irreversibility. This in turn allows us to tackle the main research question and deeply explore the contemporaneous or short-term optimal behavior in terms of the irreversibility risk. While confirming the general local result (around the irreversibility threshold), we find that the latter no longer holds in several specifications when the pollution level is far enough from the irreversibility threshold.

The rest of the paper is organized as following. Section 2 describes the general class of piecewise-deterministic convex-concave models that we solve. Section 3 proves the general local result on optimal emission policy in the neighborhood of the irreversibility threshold. Section 4 provides with the global solution to our optimal control problem when the objective function is linear-quadratic, and establishes a general characterization of irreversibility thresholds reachability conditions for given risk. Section 5 uses the solution of Section 4 to study "globally" the impact of uncertainty on optimal emission, in particular when actual pollution is far from the irreversibility thresholds. Section 6 concludes.

2 The general class of stochastic irreversible pollution models

Following TW, we investigate a situation where the decision maker faces an irreversible local pollution problem. For simplicity, the pollution emission, $y(t)$, is used to measure the output level, which is standard in this literature. The objective of the decision maker is to maximize the welfare function:

$$\max_{y(\cdot)} W = \int_0^{+\infty} (U(y) - D(z))e^{-rt}dt, \tag{1}$$

where r is time preference, $z(t)$ is accumulated pollution, $U(y)$ is the utility from enjoying final output generated with pollution $y(t)$, and $D(z)$ is the damage function due to the aggregate pollution stock z.

Pollution stock $z(t)$ may decay at rate $\frac{\delta(z)}{z}$, where $\delta(z)$ is the so-called decay function (following the terminology of Forster, 1975). However, the decay rate drops irreversibly to zero as $z(t)$ reaches a threshold value $\bar{z}$. After the drop, no decay is possible. In other words, the pollution accumulation is given by the following:

$$\dot{z} = \begin{cases} y - \delta(z) & \text{if } z < \bar{z}, \\ y & \text{if } z \geq \bar{z}, \end{cases} , \quad z(0) = z_0 \text{ given.} \tag{2}$$

The following assumptions characterize the general class of models we consider.

Assumption 1 *U, D, and δ are $\mathbb{R}_+ \mapsto \mathbb{R}_+$ functions and satisfy the following conditions.*

1. *$\delta(z) > 0$ for $z < \bar{z}$; $\delta(\bar{z}) = 0$ and $\delta'(\bar{z}) < 0$.*
2. *$U''(y) < 0$ and $U'(y) < \infty$ for all $y \geq 0$; there exists $\bar{y} > 0$ such that $U'(\bar{y}) = 0$.*
3. *$D''(z) > 0$ for all $z \geq 0$; $D(0) = D'(0) = 0$ and $\lim_{z\to\infty} D'(z) = \infty$.*

These assumptions follow essentially TW. Some comments are in order here. First of all, note that we do not assume any particular path for the decay function, $\delta(z)$, except positivity and the local assumption that it's strictly decreasing in the neighborhood of the threshold $\bar{z}$. These conditions are checked by the decay function of Forster (1975) but contrary to Forster, we don't assume strict concavity of the decay function. Lighter forms of irreversibility could be considered, for example assuming that $\delta(\bar{z}) \geq 0$, not necessarily zero, but this will not change the main qualitative results of this paper. The other assumptions on the concavity of the utility function $U(.)$ and the convexity of the damage function $D(.)$ are standard. It's also common to assume that the utility function reaches a finite maximum (at $\bar{y}$ in our notations) in the local pollution problems like ours (see TW for more details).[5] Assuming strict monotonicity of $U(.)$ like in the social welfare maximization problem of Forster (1975) (so with $\bar{y} = \infty$) would generate additional issues (like multiple steady states and the like).[6] Finally notice that

[5]In the simplest justification, if the decision maker is a firm choosing its level of output given the production cost, the existence of a finite output value maximizing utility of the firm is a typical outcome, see again TW.

[6]That's precisely why Forster (1975) ended up considering an alternative maximization with a bliss utility level in the social welfare function to get rid of multiplicity.

by assuming that the damage function goes to infinity when the pollution stock increases indefinitely will necessarily imply a finite pollution level at the optimum (if any), which implies that emissions must go to zero at this optimum. Note however that optimal emissions may still be strictly positive at the irreversibility threshold and beyond before declining to zero.

We now introduce the stochastic ingredient of our problem by opening an exogenous avenue to irrversibility. We assume that, while $z(t)$ has not reached $\bar{z}$ the decay can still drop to zero due an exogenous shock such as a major ecological accident or a climate singularity bringing the economy instantaneously to the irreversible pollution regime. Quite naturally, we model this occurrence as a piecewise deterministic process (see Davis, 1984, or Dockner et al., 2000): there are two modes, with and without pollution decay, denoted by $m = 1$, the reversible regime, and 0, the irreversible regime, respectively. The jump from mode 1 to 0 occurs at the constant rate

$$\lambda = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr \{ m(t + \Delta t) = 0 | m(t) = 1 \}.$$

In other words, the probability of the mode change during the interval $(t, t + \Delta t]$, given that the mode at t is 1, is proportional to Δt, that is, the arrival of the irreversible regime follows Poisson process with intensity parameter $\lambda \geq 0$. Obviously, when $\lambda = 0$, no regime change happens as long as $z(t) < \bar{z}$.

As outlined in the Introduction, we consider the Poisson case as it goes with a constant hazard rate, which leaves a chance to derive the full optimal paths for certain specifications of the general models, in particular when the welfare function is linear-quadratic. This is enough to make analytically our point on the crucial importance of the avoidability of the irreversible regime (as captured by the distance of actual pollution to the irreversibility threshold) in the shape of optimal economic behavior under risk. This does not mean that the dependence of the hazard rates upon pollution is not a relevant feature of the irreversibility problem (again see van der Ploeg, 2014), it simply renders analytical study outside the steady states hardly feasible.

Along with the piecewise-deterministic stochastic of our model, the optimal control problem can be decomposed into two (connected) sub-problems, or using a more proper terminology, into Periods I and II, corresponding to modes (or regimes) 1 and 0, respectively. Precisely, the per-Period state dynamics are given by:

Period I: $t < T$**,** where T is the time of mode switching (either by spontaneous jump or as $z(t)$ reaches $\bar{z}$). During this period the state is governed by

$$\dot{z} = y - \delta(z) \qquad \text{for } 0 < t < T; \qquad z(0) = z_0, \tag{3}$$

Period II: $t \geq T$**.** During this period the state is governed by

$$\dot{z} = y \qquad \text{for } t > T; \;\; z(T) = z\left(T^{-}\right). \tag{4}$$

We now get to the optimization part (dynamic programming).

Optimal control of the piecewise deterministic process The optimal control of a piecewise deterministic process is far from a new topic in optimization theory as outlined above. Here, we mostly rely on the dynamic programming approach developed in Dockner et al (2000). Let $V_m(z)$ denote the value function in mode m for $m = 0, 1$.

In mode 0 there is no possibility of mode change. The optimal control problem is determined by the system dynamic equation (4) and utility and damage functions $U(y)$ and $D(z)$. The usual dynamic programming method leads to the HJB equation

$$rV_0(z) = \max_{y \geq 0} \{U(y) - D(z) + V_0'(z)\, y\} = \max_{y \geq 0} \{U(y) + yV_0'(z)\} - D(z). \tag{5}$$

Since mode 0 is an absorbing state, the HJB above is the standard deterministic one which only depends on the value function $V_0(z)$. This is not the case in mode 1.

Indeed in mode 1, there is a probability that the mode changes at a given Poisson probability λ. Accordingly, the corresponding HJB equation should be written as (see Theorem 8.1 in Dockner et al. 2000)

$$rV_1(z) = \max_{y \geq 0} \{U(y) - D(z) + V_1'(z)[y - \delta(z)] + \lambda [V_0(z) - V_1(z)]\}.$$

A new term emerges compared to mode 0: the change in value induced by a possible Poisson jump to mode 0 enters the HJB as $\lambda [(V_0(z) - V_1(z))]$, which is nonzero as long as the jump risk is nonzero ($\lambda > 0$). Notice that this mode HJB now includes the two value functions $V_m(z)$, $m = 0, 1$, which features the intertemporal (or inter-period) nature of the problem. The HJB above can be rewritten into a more practical form:

$$(r + \lambda) V_1 (z) = \max_{y \geq 0} \{U (y) + yV_1' (z)\} - \delta (z) V_1' (z) - D (z) + \lambda V_0 (z) . \quad (6)$$

Obviously, $r + \lambda$ plays the role of an effective discount rate.

3 A general local result on optimal emissions under irreversibility risk

In this section, we establish one of two main results of this paper: we will show that for the general class of models and for any level of uncertainty (that's for any value of λ), optimal (polluting) behaviour is less conservative under the irreversibility risk ($\lambda > 0$) than without ($\lambda = 0$) for an actual pollution level close enough to the threshold level, $\bar{z}$. More concretely, we examine how the irreversibility risk affects $y_1 (z)$, the pollution rate in Period I, compared to $y_1^d (z)$ which denotes the optimal pollution rate in Mode 1 with $\lambda = 0$.

To establish these properties, a careful analysis of the HJB equations and a few intermediate results are needed. We develop some of the essential steps in this Section for transparency, the heaviest mathematical developments are reported in the Appendix. We first derive the value function and the pollution rate in Mode 0 which is needed in this section.

3.1 Value function and emission rate in Mode 0

The general model in Mode 0 has been well-studied (see, e.g. Tahvonen and Withagen, 1996). We summarized the results briefly.

Let $y (t)$ be the optimal emission rate. Then the optimal trajectory $(z, y) : \mathbb{R}_+ \mapsto \mathbb{R}_+^2$ solves the system of autonomous equations

$$\dot{z} = y, \qquad \dot{y} = -\frac{D' (z) - rU' (y)}{U'' (y)} \qquad \text{if } y \geq 0. \quad (7)$$

By Assumption 1(items 2, 3), $U' (0) > U' (\bar{y}) = 0$, $D' (0) = 0$ and $D' (z)$ is increasing. Thus the equation

$$D' (z) = rU' (0)$$

has a unique positive solution, denoted by $\bar{z}_0$. It follows that $D'(z) < rU'(0)$ for any $z < \bar{z}_0$. Hence, by (7), $\dot{y} < 0$ and $y > 0$ for $z < \bar{z}_0$. As a result, $(\bar{z}_0, 0)$ is an equilibrium of (7). The costate $V_0'(z)$ and the value function $V_0(z)$ are found by

$$V_0'(z) = -U'(y), \qquad V_0(z) = \frac{1}{r}\left[U(y) - D(z) + yV_0'(z)\right] \tag{8}$$

if $0 \le z \le \bar{z}_0$. For $z > \bar{z}_0$, $y = 0$, hence by (5),

$$V_0(z) = \frac{1}{r}\left[U(0) - D(z)\right].$$

We prove the following result which is instrumental in the derivation of the main properties of this Section.

Proposition 1 *The pollution rate in Mode 0 never exceeds $\bar{y}$. i.e., $0 \le y_0(t) < \bar{y}$ for all $t \ge 0$.*

A proof is given in Appendix.

3.2 Value function and emission rate in Mode 1

In Mode 1 from (6) we derive

$$U'(y) + V_1'(z) = 0 \qquad \text{if } y \ge 0,$$

and by differentiating the two sides of (6) with respect to z,

$$(r+\lambda)V_1'(z) = (y-\delta)V_1''(z) - \delta'(z)V_1' - D'(z) + \lambda V_0'(z)$$

The above two equations lead to

$$\begin{aligned} \dot{z} &= y - \delta(z), \\ \dot{y} &= \frac{1}{U''(y)}\left[(r+\lambda+\delta'(z))U'(y) - D'(z) + \lambda V_0'(z)\right] \end{aligned} \tag{9}$$

for $y \ge 0$, $z < \bar{z}$. The second equation of 9 is worth studying. It delivers the optimal emission rule (as in any Ramsey model with pollution control) in terms of the fundamentals of the model (e.g. the relative risk aversion $-\frac{yU''(y)}{U'(y)}$ can be trivially shown up). The stochastic (piecewise deterministic) nature of the models is reflected in two terms, the original (endogenous) utility discount term, $r+\lambda+\delta'(z)$, which will be interpreted here below, and of course the expected payoff if a jump to the irreversible regime exogenously happens through the term $\lambda V_0'(z)$.

Let us examine more closely the discount rate. In addition to the pure rate of preference for the present, r, two terms are added. The first one is standard in Poisson-based models: as the arrival probability of the "bad" regime (here the irreversible pollution regime), the present is even more preferred, so to speak. The second additional term is more interesting. If $\delta'(z) > 0$, which is assumed for example by Forster (1975) for low enough pollution levels, then the preference for the present is augmented with the (expected) increase in the decay rate leading to an optimal increase in current emissions *ceteris paribus* as the capacity of pollution absorption is raised. The reverse happens when $\delta'(z) < 0$, which necessarily happens in the neighborhood of the irreversibility threshold, $\bar{z}$, according to Assumption 1-item 1 (and to Forster, 1975), that is when normally pollution is high enough, in such a case the preference for the present goes down because pollution absorption capacity will drop. If the drop in absorption capacity is sharp enough, say if $\delta'(\bar{z}) \ll 0$, then the modified discount rate may become negative, which makes the overall picture extremely complicated. We shall consider hereafter a local approach around the irreversibility threshold to extract a general result for the class of models considered under Assumption 1.

Before, we need some intermediate steps. The costate $V_1'(z)$ and the value functions are given by

$$\begin{aligned} V_1'(z) &= -U'(y), \\ V_1(z) &= \frac{1}{r+\lambda}\left[U(y) + (y-\delta(z))\, V_1'(z) - D(z) + \lambda V_0(z)\right]. \end{aligned}$$

There are potentially two cases, either the pollution stock reaches $\bar{z}$ or it does not. In the former case there is $T > 0$ such that $z(T) = \bar{z}$. In the latter case

$$\lim_{t\to\infty} z(t) \le \bar{z}. \tag{10}$$

The decision maker chooses the one which yields higher total welfare. The next proposition shows that if $\bar{z} < \bar{z}_0$ then the decision maker always chooses to let the threshold be crossed if the pollution stock is sufficiently high.

Proposition 2 *Let Assumption 1 hold. If $\bar{z} < \bar{z}_0$ then there is $\varepsilon > 0$ such that $\bar{z}$ is reached in finite time for any initial pollution z_0 on the interval $\bar{z} - \varepsilon \leq z_0 < \bar{z}$.*

A proof is given in Appendix. The Proposition above highlights a very important property of our general class of irreversible pollution models. One may think that in order to stay in the reversible regime, the optimal emission must be $y_1^*(z) = 0$ when $z \to \bar{z}^-$; thus no emission is the optimal choice. This is not the case here: in the absence of catastrophic risk bringing welfare to minus infinity or zero as explained in the Introduction, crossing the irreversible threshold is an option, and indeed the Proposition above shows that the optimal emission should be $y_1^*(\bar{z}) > 0$. Considering the extent of damages from pollution accumulation and gains from emissions, it is indeed more beneficial to keep polluting when approaching the threshold. This is entirely consistent with the works of TW and Le Kama *et al.* (2014), among others, who deal with similar local irreversible pollution problems.

Now, we come to the general local result announced in the Introduction: the prospect of spontaneous drop of decay leads to higher the emission rate under Poisson process than under certainty, if the pollution stock is sufficiently high, that is if it's close enough to the irreversibility threshold. We demonstrate this idea step by step and unveil the mechanism along the way.

Let $y_m(\bar{z})$ be the emission rate at $\bar{z}$ for $m = 0, 1$. Since $\bar{z} < \bar{z}_0$, it follows that $\bar{y}_0(\bar{z}) > 0$. Hence, the first order condition in Mode 0 yields

$$U'(y_0(\bar{z})) + V_0'(\bar{z}) = 0.$$

Also, since $\bar{z}$ is reached in finite time, by Proposition 2, it follows that $y_1(\bar{z}) > 0$ and satisfies the first order condition in Mode 1:

$$U'(y_1(\bar{z})) + V_1'(\bar{z}) = 0.$$

We first show that:

Lemma 1 *The optimal emission in Mode 0 and 1 check $y_1(\bar{z}) = y_0(\bar{z}) > 0$ for any $\lambda \geq 0$.*

The proof is straightforward and given in the Appendix. We next expand the functions $\delta(z)$, $D(z)$ and $y_1(z)$ around the threshold $\bar{z}$ as following:

$$\begin{aligned}
\delta(z) &= \delta_1(z-\bar{z}) + \frac{\delta_2}{2}(z-\bar{z})^2 + o\left(|z-\bar{z}|^2\right), \\
D(z) &= D_0 + D_1(z-\bar{z}) + \frac{D_2}{2}(z-\bar{z})^2 + o\left(|z-\bar{z}|^2\right), \\
y_m(z) &= A + B_m(z-\bar{z}) + \frac{C_m}{2}(z-\bar{z})^2 + o\left((z-\bar{z})^2\right) \qquad \text{for } m = 0, 1,
\end{aligned} \tag{11}$$

where $\delta_j = \delta^{(j)}(\bar{z})$, $D_j = D^{(j)}(\bar{z})$, j=0, 1, 2, are the $j-$th derivatives of function $\delta(z)$ and $D(z)$, respectively, and evaluated at $\bar{z}$, with $\delta_0 = \delta(\bar{z}) = 0$.

A, B_m and C_m are Taylor expansion coefficients of $y_m(z)$ evaluated at $\bar{z}$, where $m = 0, 1$. By Lemma 1, $A = y_1(\bar{z}) = y_0(\bar{z})$ is independent of m and λ. In the following, we make the following crucial claim:

Lemma 2 *Let Assumption 1 hold and suppose $\bar{z} < \bar{z}_0$. Then B_1 is independent of λ and $B_1 > B_0$, and C_1 is increasing in λ.*

Since the proof of this lemma yields important information for the understanding of the mechanisms involved, we keep it in the main text. Differentiating the equations (5) and (6) with respect to z, we obtain

$$\begin{aligned}
rV_0'(z) &= U'(y_0(z))\,y_0'(z) + y_0'(z)\,V_0'(z) + y_0(z)\,V_0''(z) - D'(z), \\
(r+\lambda)\,V_1'(z) &= U'(y_1(z))\,y_1'(z) + y_1'(z)\,V_1'(z) + y_1(z)\,V_1''(z) \\
&\quad -\delta'(z)\,V_1'(z) - \delta(z)\,V_1''(z) - D'(z) + \lambda V_0'(z).
\end{aligned}$$

Using the identities

$$U'(y_m(z)) + V_m'(z) = 0 \qquad \text{for } m = 0, 1,$$

we can write the equations as

$$\begin{aligned}
&rU'(y_0(z)) = y_0(z)\,[U'(y_0(z))]_z + D'(z), \\
&(r+\lambda+\delta'(z))\,U'(y_1(z)) = (y_1(z) - \delta(z))\,[U'(y_1(z))]_z \\
&\qquad\qquad + D'(z) + \lambda U'(y_0(z)).
\end{aligned} \tag{12}$$

By differentiability of the utility function (to the third order), we can further get

$$
\begin{aligned}
U'\left(y_m(z)\right) &= U'\left(y_m(\bar{z})\right) + U''\left(y_m(\bar{z})\right)\left(y_m(z) - y_m(\bar{z})\right) \\
&\quad + \frac{U'''\left(y_m(\bar{z})\right)}{2}\left(y_m(z) - y_m(\bar{z})\right)^2 + o\left(\left|y_m(z) - y_m(\bar{z})\right|^2\right) \\
&= U'(A) + U''(A)\left[B_m(z-\bar{z}) + \frac{C_m}{2}(z-\bar{z})^2\right] \\
&\quad + \frac{U'''\left(y_m(\bar{z})\right)}{2}\left[B_m(z-\bar{z})\right]^2 + o\left(|z-\bar{z}|^2\right).
\end{aligned}
$$

Hence,

$$
\left[U'\left(y_m(z)\right)\right]_z = U''(A)\left[B_m + C_m(z-\bar{z})\right] + U'''(A)\, B_m^2 (z-\bar{z}) + o\left(|z-\bar{z}|\right). \tag{13}
$$

We now compare the coefficients of powers of $z-\bar{z}$ for identification. Substituting (11) and (13) into (12), we find

$$
\begin{aligned}
& rU'(A) = AU''(A)\, B_0 + D_1, \\
& (r+\delta_1)\, U'(A) = AU''(A)\, B_1 + D_1,
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
& rU''(A)\, B_0 = A\left[U''(A)\, C_0 + U'''(A)\, B_0^2\right] + B_0^2 U''(A) + D_2, \\
& (r+\lambda+\delta_1)\, U''(A)\, B_1 + \delta_2 U'(A) \\
& = A\left[U''(A)\, C_1 + U'''(A)\, B_1^2\right] + B_1^2 U''(A) - \delta_1 B_1 U''(A) \\
& \qquad + D_2 + \lambda U''(A)\, B_0.
\end{aligned} \tag{15}
$$

(14) implies,

$$
B_0 = \frac{rU'(A) - D_1}{AU''(A)} = \frac{r}{A}\frac{U'(A)}{U''(A)} - \frac{D_1}{AU''(A)} \tag{16}
$$

and

$$
B_1 = \frac{(r+\delta_1)\, U'(A) - D_1}{AU''(A)} = \frac{(r+\delta_1)}{A}\frac{U'(A)}{U''(A)} - \frac{D_1}{AU''(A)}. \tag{17}
$$

It follows that B_1 is independent of λ. In addition, since $U''(A) < 0$, $\delta_1 < 0$, and by Proposition 1,

$$
U'(A) = U'\left(y_0(\bar{z})\right) > U'(\bar{y}) = 0,
$$

it follows that $B_1 > B_0$ as claimed in the Lemma above.

To prove the assertion regarding C_1, we use (15) to derive

$$C_0 = \frac{B_0}{A}\left[r - B_0\right] - \frac{D_2}{AU''(A)} - \frac{U'''(A)}{U''(A)}B_0^2 \tag{18}$$

and

$$C_1 = \frac{B_1}{A}\left[r + 2\delta_1 - B_1\right] + \frac{\delta_2 U'(A) - D_2}{AU''(A)} - \frac{U'''(A)}{U''(A)}B_1^2 + \frac{\lambda}{A}\left(B_1 - B_0\right). \tag{19}$$

It is clear that since $B_1 > B_0$, C_1 is increasing in λ. This establishes the last very important property claimed in the Lemma.

It's now time to draw the main implications of the results above regarding in particular optimal emissions in the reversible regime in the neighborhood of the irreversibility threshold. We summarize them in the following Proposition which is actually a corollary of the properties established in the Lemmas demonstrated above.

Proposition 3 *Let Assumption 1 hold. Also assume that $\delta'(\bar{z}) < 0$, that $\bar{z} < \bar{z}_0$, and that $U'''(y)$ exists for any $y > 0$. Then for any $\lambda > 0$ there is an $\varepsilon > 0$ such that if λ is increased slightly, $y_1(z)$ is also increased on the interval $\bar{z} - \varepsilon < z < \bar{z}$. In particular, $y_1(z) > y_1^d(z)$ on the interval for any $\lambda > 0$.*

The Proposition characterizes comprehensively optimal behavior before and after crossing the irreversibility threshold. In particular Part 1 delivers that in our general class of models, optimal emissions are always larger than without the irreversibility risk for pollution levels close enough to the threshold. Indeed by Lemma 2, B_1 is independent of λ, $B_1 > B_0$ and C_1 is increasing in λ. Thus, if λ is increased slightly,

$$y_1(z) = A + B_1(z - \bar{z}) + \frac{C_1}{2}(z - \bar{z})^2 + o\left(|z - \bar{z}|^2\right)$$

is increased in a left neighborhood of $\bar{z}$. In particular, $y_1(z) > y_1^d(z)$ on such a neighborhood ($y_1^d(z)$ corresponding to $\lambda = 0$). The rationale behind has already been pointed out by Clarke and Reed (1994) in its generic form: avoidability. As the economy becomes close enough to the irreversibility threshold, so that the irreversibility regime sounds as unavoidable. The unavoidability of the irreversible regime imposing itself at these levels of pollution, at whatever the extent of uncertainty, the the central planner taking into account the pollution benefit/cost tradeoff and the law of motion of pollution, will result clearly less pro-environmental. Nevertheless, the extent of uncertainty (as measured by λ here) is second-order in this neighborhood (in the sense of the second-order term of the Taylor-expansions performed above).

4 The case of linear-quadratic objective functions

In this Section, we provide with a comprehensive analysis of the solutions to the HJB equation (22) to characterize optimal behavior more globally. This will allow us to explore analytically the reachability of the irreversibility threshold for any level of risk (or value of λ). The latter is fully addressed in Proposition 6 below. The proof of the proposition together with the characterization of the optimal decisions into the form of policy functions (which is itself required for the proof of Proposition 6) is quite long, it's given in the Appendix. The next section will use the policy function analysis in the proof of Proposition 6 to illustrate numerically the variety of optimal behavior under risk depending notably on the actual value of pollution (not necessarily close to $\bar{z}$).

For analytical tractability, we take linear-quadratic functional forms :

$$U(y) = ay - \frac{y^2}{2}, \qquad D(z) = \frac{cz^2}{2}. \tag{20}$$

Let's concentrate here on the reachability of the threshold for any level of uncertainty $\lambda \geq 0$ in the absence of any instantaneous Poisson jump from mode 1 to mode 0. We define two types of reachability as follows.

HJB equations in final form We now use the precise functional specifications given in the beginning of this Section, to write down the HJB equations into the final forms we will handle in our analytical part. Using (20), we find

$$\begin{aligned} \max_{y\geq 0}\{U(y) + yV_0'(z)\} &= \max_{y\geq 0}\left\{ay - \frac{y^2}{2} + yV_m'(z)\right\} \\ &= \begin{cases} (a + V_m'(z))/2 & \text{if } a + V_m'(z) > 0, \\ 0 & \text{if } a + V_m'(z) \leq 0, \end{cases} \end{aligned}$$

for $m = 0, 1$. Therefore, the HJB equations take the form

$$2rV_0(z) = \begin{cases} (a + V_0'(z))^2 - cz^2 & \text{if } a + V_0'(z) \geq 0, \\ -cz^2 & \text{if } a + V_0'(z) < 0, \end{cases} \qquad \text{for } z > 0, \tag{21}$$

and

$$2(r + \lambda) V_1(z) = \begin{cases} \begin{array}{c} (a + V_1'(z))^2 - 2\delta(z) V_1'(z) \\ -cz^2 + 2\lambda V_0(z) \end{array} & \text{if } a + V_1'(z) \geq 0, \\ -2\delta(z) V_1'(z) - cz^2 + 2\lambda V_0(z) & \text{if } a + V_1'(z) < 0, \end{cases} \tag{22}$$

for $0 < z < \bar{z}$. Note that for $z \geq \bar{z}$ there is no Mode 1. Hence, $V_1(z) = V_0(z)$ for $z \geq \bar{z}$. In particular,

$$V_1(\bar{z}) = V_0(\bar{z}). \tag{23}$$

Note that in any mode the emission rate y_m^* that maximizes $U(y) + yV_m'(z)$ is

$$y_m^*(z) = \max\{a + V_m'(z), 0\} \qquad \text{for } m = 0, 1. \tag{24}$$

The corresponding net pollution emission rates ("pollution rates" thereafter) are

$$f_0(z) = y_0^*(z), \qquad f_1(z) = y_1^*(z) - \delta(z). \tag{25}$$

4.1 Value function and emission rate in Mode 0

We first find the value function, $V_0(z)$, using (21).

Note that the right-hand side of (21) is piecewise quadratic in z and V_0'. We seek the solution V_0 to be piecewise quadratic. By (21),

$$V_0(z) = -\frac{cz^2}{2r} \qquad \text{if } a + V_0'(z) < 0.$$

Since $V_0'(z) = -cz/r$, it follows that $a + V_0'(z) < 0$ if and only if

$$z > \frac{ar}{c} = \bar{z}_0.$$

It can be seen from (4) and (24) that $\bar{z}_0$ is the steady state in mode 0. This steady state $\bar{z}_0$ will play a role in our results here below, especially in the analysis of global and local behavior in Section 5. Notice it has a simple and economically meaningful structure: it's proportional to productivity (parameter a) and inversely proportional to the pollution cost (parameter c), the proportionality factor being the discount rate, r: the larger this factor, the bigger the impact of productivity and pollution cost on $\bar{z}_0$.

For $z \leq \bar{z}_0$ we assume

$$V_0(z) = \frac{A_0}{2} z^2 + B_0 z + C_0.$$

Appendix A.4 provides detailed calculation of the value function and demonstrates the following optimal emission in Mode 0.

Proposition 4 *Under linear-quadratic functional forms given above, in Mode 0, the optimal pollution emission is given by*

$$f_0(z) = \max\{a + V_0'(z), 0\} = \begin{cases} h_0(z - \bar{z}_0) & \text{if } z \leq \bar{z}_0, \\ 0 & \text{if } z > \bar{z}_0, \end{cases} \tag{26}$$

where

$$h_0 = \frac{r - \sqrt{r^2 + 4c}}{2} \equiv A_0 \tag{27}$$

and $\bar{z}_0 = \frac{ar}{c}$ *is the long-run steady state of pollution accumulation.*

The computations from above proposition allow to obtain

$$0 \leq f_0(z) < a,$$

which is consistent with Proposition 1. In short, production (and consumption) are strictly bounded by the productivity level of the economy, a. This might seem as an automatic implication of the linear-quadratic utility function, it's not. It depends primarily on the structure of our problem. Since $h_0 < 0$, by (31)

$$0 \leq f_0(z) \leq -h_0 \bar{z}_0.$$

By the definitions of h_0 and $\bar{z}_0$ in (31) and (33),

$$-h_0 \bar{z}_0 = \frac{\sqrt{r^2 + 4c} - r}{2} \frac{ar}{c} = \frac{2ra}{\sqrt{r^2 + 4c} + r} < a.$$

We next derive the pollution rates f_1 and f_1^d in Mode 1, which is definitely much more complicated.

4.2 Pollution rates in Mode 1 near $\bar{z}$

From Lemma 1 and Proposition 3 we immediately have.

Proposition 5 *Suppose $\bar{z} < \bar{z}_0$. Then for any $\lambda > 0$ there is an $\varepsilon > 0$ such that $f_1(z) \geq f_1^d(z)$ if $\bar{z} - \varepsilon < z < \bar{z}$.*

The main conclusion to draw from Proposition 5 is that at **whatever the level of the irreversibility risk as captured by** λ, the optimal polluting behavior is more aggressive under the irreversibility risk than without as soon as the actual pollution stock gets close enough to the threshold $\bar{z}$. The last section of this paper will show that this is indeed a local property which need not hold far from the irreversibility threshold.

4.3 Pollution rate in Mode 1 far from $\bar{z}$

Definition 1 *The threshold $\bar{z}$ is called globally reachable if any pollution stock $z(t)$ with the initial value $z(0) < \bar{z}$ grows across $\bar{z}$ in finite time. The threshold $\bar{z}$ is called locally reachable if there is a stable steady state in mode 1, denoted $\bar{z}_1$, and a critical value, $\bar{z}_1^*$ that satisfy $0 \leq \bar{z}_1 < \bar{z}_1^* < \bar{z}$, such that the pollution stock $z(t)$ converge to $\bar{z}_1$ if $z(0) < \bar{z}_1^*$ and $z(t)$ grows across $\bar{z}$ in finite time if $z(0) > \bar{z}_1^*$.*

The next proposition gives a necessary and sufficient condition for the threshold $\bar{z}$ to be reachable, and shows that it is globally reachable if it is sufficiently small.

To prove this result, we need to specify more precisely the decay function, we impose the following Assumption

($\mathbf{A}_2$) $\delta(z)$ is a linear function

$$\delta(z) = \alpha - \beta z.$$

and $\delta(\bar{z}) = 0$.

Clearly $\bar{z} = \alpha/\beta$ under this condition.

Proposition 6 *Suppose (A_2) holds. Then $\bar{z}$ is (globally or locally) reachable if and only if $\bar{z} < \bar{z}_0$. Furthermore, there is $\hat{z} \in (0, \bar{z}_0]$ such that $\bar{z}$ is globally reachable if and only if $\bar{z} < \hat{z}$.*

The proof is in the Appendix. From the proof one can see that for $z < \bar{z}$ and in a neighborhood of $\bar{z}$, the decision maker has two options, one is to produce more, consume more, and as a result, pollute more. The other is the opposite. In the case where $\bar{z}$ is globally reachable, the first option always yields a higher total benefit. On the other hand, if $\bar{z}$ is locally reachable, then there is a pollution level $\bar{z}_1^* < \bar{z}$, such that if $z < \bar{z}_1^*$ the decision maker would be better off restricting production to lower the pollution level, but if $z > \bar{z}_1^*$ the decision maker would rather consume more and thus pollute more. As we shall see in our numerical exercises below, the pollution rate, f_1, may be low enough or even negative for $z < \bar{z}_1^*$ leading to asymptotic convergence to the steady state value in mode 1 (provided it's lower than $\bar{z}_1^*$).

Remark 1 *Of course, the threshold value $\bar{z}_1^*$ is endogenously determined, and therefore it may depend on all the parameters of the optimal control problem. Unfortunately, we cannot obtain this threshold in closed form (see the proof of Proposition 6 in the Appendix, Case 2-b). In the next section we show via numerical exercises that it does depend on the pollution cost parameter, c, on the irreversibility threshold, $\bar{z}$, and on the level of uncertainty, λ. In particular, we show that $\bar{z}_1^*$ need not be monotonic in λ for given c. See Example 2 in Section 5.*

Referring to Definition 1, the presence of $\hat{z}$ ensures that under mode 1, either a stable steady state, $\overline{z}_1$, does not exist, or the initial pollution level surpasses $\overline{z}_1$, indicating $z(0) > \bar{z}_1^*$. Consequently, pollution accumulation invariably increases until it exceeds the threshold, $\bar{z}$. Put differently, in a more robust ecological system where the threshold level is high, reaching the irreversible regime can be prevented if the reversible stable steady state in mode 1 has not been already significantly exceeded. However, in a fragile ecological system characterized by a low irreversible threshold or when the pollution level is already near the threshold, irrespective of uncertainties, transitioning into the irreversible regime becomes inevitable. This final statement further elucidates the local behavior of the pollution rate in Proposition 5 - the inevitability of crossing the threshold. In such a scenario, the irreversible stable steady state $\bar{z}_0$ acts as an attractor.

We can now examine the impact of uncertainty on optimal emission rates for all $z \in (0, \bar{z}]$, complementing our local analysis (around the threshold $\bar{z}$) in Section 3. This will be done in the next Section.

5 Irreversibility, uncertainty level and the optimal emission rate

In the following in order to use the analytical characterization of the optimal solutions obtained and used in the proof of Proposition 6 (see the Appendix), we keep on relying on Assumption (A). We compare the optimal emission rates for different levels of pollution unit costs (parameter c), irreversibility thresholds ($\bar{z}$) and uncertainty (parameter λ), keeping the following parameters fixed:

$$r = 0.2, \qquad \beta = 0.1, \qquad a = 18.$$

These parameter values are used in the numerical example of Tahvonen and Withagen (1996). In the first example, we use $c = 0.002$ and $\bar{z}$ taking values of 100, 180, and 300 respectively. In the second example we use $c = 0.02$ and $\bar{z} = 100$, 120, and 140 respectively. Also, in each example, we compare the optimal pollution rates $f_1(z)$ with zero, small and large values of λ. Recall that function $f_1(z)$ are optimal feedback functions in the sense of dynamic programming. Specifically, we use $\lambda = 0, 0.1$ and 1.0, respectively in each numerical illustration. Also recall that $\lambda = 0$ corresponds to the absence of the irreversibility risk: with the notations of Section 3, we have in such a case, $f_1(.) \equiv f_1^d(.)$.

Example 1 *-small damage with $c = 0.002$. In this case $\bar{z}_0 = 1800$. For $\bar{z} = 100$, the threshold $\bar{z}$ is globally reachable for all three values of λ. For $\bar{z} = 180$, the threshold is globally reachable for $\lambda = 0$ and it is locally reachable for $\lambda = 0.1$ and 1.0, with $\bar{z}_1^* = 4.8$ and 8.3, respectively. In the latter case, for any initial pollution level below $\bar{z}_1^*$, $z(t)$ decreases to 0 as $t \to \infty$. The graphs are shown in Fig. 1.*
For $\bar{z} = 300$, the threshold is locally reachable for the three values of λ, with $\bar{z}_1^* = 150$ for $\lambda = 0$, $\bar{z}_1^* = 145.6$ for $\lambda = 0.1$ and $\bar{z}_1^* = 142.4$ for $\lambda = 1.0$. In all cases, for any initial pollution below $\bar{z}_1^*$, $z(t) \to 0$ as $t \to \infty$. The graphs are shown in Fig. 2.

In Fig. 1 and Fig. 2, one first important conclusion is that the value of the irreversibility threshold is crucial in the ranking of the optimal feedback functions for the three different Poisson rate values. In Fig. 1, when $\bar{z} =$

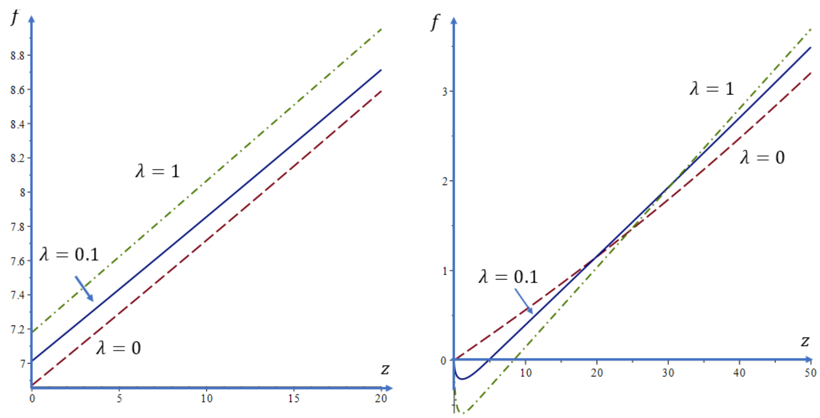

Figure 1: Small damage with $c = 0.002$ and $\bar{z} = 100$ (left) and $\bar{z} = 180$ (right).

100, the optimal emission rate is the highest at the largest Poisson rate, the lowest corresponding to the case of zero irreversibility risk, for any value of the pollution stock z. This seems to extend the local result established in Proposition 5 globally. However, when $\bar{z} = 180$, one can clearly see in the graphic at the right side of Fig. 1 that while the latter picture holds when z is large enough (consistently with Proposition 5), it's no longer the case when z is low enough: in this z-values interval (say z below the threshold value $z_1^* = 8.3$ corresponding to $\lambda = 1$), the optimal emission rate is the highest at the lowest Poisson rate (that's the zero risk case), the lowest corresponding to the highest irreversibility risk. Of course, this ranking is reversed as the pollution level increases enough in accordance with Proposition 5. The very same outcome arises from Fig. 2 where the irreversibility threshold is even higher ($\bar{z} = 300$), and also in the exercises conducted in Example 2 with a much higher pollution unit cost (See Fig. 3 and 4).

Our numerical exercises deliver more findings. One interesting outcome is that for given pollution unit cost, and for large enough irreversibility threshold values, $\bar{z}$, the latter may only be locally reachable for certain Poisson rate values: for $\lambda = 0.1, 1$ in Fig. 1, and for all λ values in Fig. 2 when the irreversibility threshold is very large. Notice that in these cases the endogenous threshold z_1^* is increasing in λ for $\bar{z} = 180$ but it's decreasing in λ when $\bar{z} = 300$ (of course at the discrete λ-values considered). It's neither monotonic as we will see in the Example 2 below and associated Figures 3

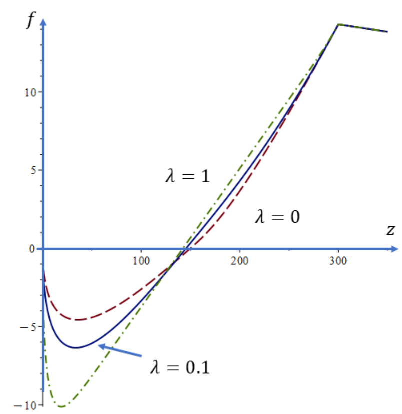

Figure 2: Small damage with $c = 0.002$ and $\bar{z} = 300$.

and 4, when the unit pollution cost is much higher.

Another remarkable property we can see in Fig. 1 (right one), and Fig. 2 is the possible non-monotonicity of the policy functions, $f_1(.)$. This is also apparent in the case of large pollution cost (Fig. 3 and 4). One frequent case of non-monotonicity is when reachability is only local but this is not a sufficient condition for non-monotonicity (see Example 2 below, Fig. 4 for $\lambda = 1$). Indeed, when reachability is local and the actual pollution stock is below the endogenous threshold, $\bar{z}_1^*$, the optimal net emission rates may even be negative when the actual pollution stock is low enough, leading the pollution stock to converge to zero asymptotically, corresponding to the steady state in mode 1, that's $\bar{z}_1 = 0$ (Example 1, Figure 2). Figure 3 documents similar cases when the pollution cost is much larger: in these cases, when pollution is below $\bar{z}_1^*$, the optimal net emission rate may decrease sharply initially for certain values of the Poisson rate, without being negative, then converging to a strictly positive steady state in mode 1, $\bar{z}_1 > 0$.

Example 2 *-large damage with $c = 0.02$. In this case $\bar{z}_0 = 180$. For $\bar{z} = 100$, the threshold is globally reachable for all three values of λ. For $\bar{z} = 120$, it is globally reachable with $\lambda = 1.0$, but is locally reachable with $\lambda = 0$ and 0.1. In the latter cases, $\bar{z}_1^* = 113.5$ for $\lambda = 0$ and $\bar{z}_1^* = 108.6$ for $\lambda = 0.1$. In addition, for any initial pollution below $\bar{z}_1^*$ the pollution stock converges to the steady state $\bar{z}_1 = 60$ for $\lambda = 0$ and $\bar{z}_1 = 98$ for $\lambda = 0.1$. The pollution rates are shown in Fig. 3. For $\bar{z} = 140$, the threshold is locally reachable for all three values of λ, with $\bar{z}_1^* = 138.1$ for $\lambda = 0$, $\bar{z}_1^* = 138.7$ for $\lambda = 0.1$ and*

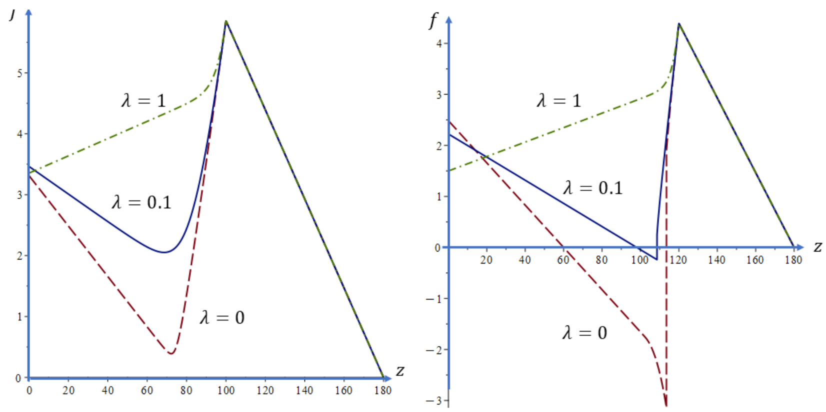

Figure 3: Large damage with $c = 0.02$ and $\bar{z} = 100$ (left) and $\bar{z} = 120$ (right).

$\bar{z}_1^* = 25.2$ for $\lambda = 1.0$. For an initial value of z below $\bar{z}_1^*$, $z(t)$ converges to $\bar{z}_1 = 40$ for $\lambda = 0$, $\bar{z}_1 = 43.4$ for $\lambda = 0.1$, and $\bar{z}_1 = 0$ for $\lambda = 1.0$. The graphs are shown in Fig. 4.

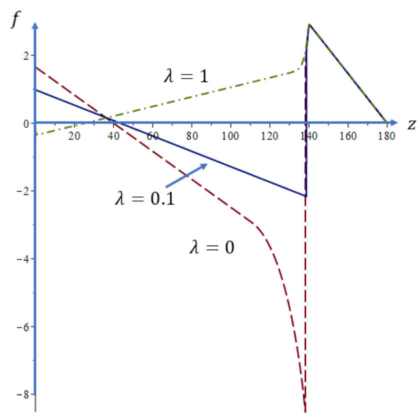

Figure 4: Large damage with $c = 0.02$ and $\bar{z} = 140$.

6 Concluding remarks

In this paper, we have study optimal pollution emissions decision for a general class of models with irreversible local pollution *à la* Tahvonen and Withagen (1996) with an additional driver to irreversibility: an exogenous Poisson process with constant arrival probability. We prove a striking general local result (more aggressive emission behavior under irrversibility risk than without) and we study its robustness relying on a fully tractable particular model in the general class. Indeed with a linear-quadratic objective function, and while keeping a general non-convex specification of pollution decay, we have been able to show that the general local result may not hold whatever the level of pollution: possible reversals of the general local result may arise when actual pollution is far enough from this threshold. These reversals depends also on the other parameters of the model, for example the unit pollution cost and, more crucially, on the value of the irreversibility threshold itself.

All the results are generated analytically with the exception of the numerical illustrations of the last section (which do derive directly from closed-form solutions). While the Poisson arrival rate does not depend on pollution, the analytical case we have constructed permits to highlight complementary highly relevant aspects which cannot be obtained from a steady state-based approach. Also the possibility to study in depth the implications of sudden changes in the value of the Poisson arrival rate does somehow mimick the situations where, for many obvious reasons, the irreversibility risk may rise or decrease sharply. Of course, this is not equivalent to endogenizing the risk but it helps estimating analytically the consequences of exogenously moving risk.

References

[1] Bonnet, R., Swingedouw, D., Gastineau, G. et al (2021). Increased risk of near term global warming due to a recent AMOC weakening. *Nature Communications*, 12, 6108 (2021). `https://doi.org/10.1038/s41467-021-26370-0`

[2] Boucekkine, R., W. Ruan and B. Zou (2023). The irreversible pollution game. *Journal of Environmental Economics and Management*, 120, Article 102841.

[3] Boucekkine, R., A. Pommeret and F. Prieur (2013). Technological vs. ecological switch and the Environmental Kuznets Curve. *American Journal of Agricultural Economics*, 95, 252-260.

[4] Bretschger, L., and Vinogradova (2019). Best policy response to envi-

ronmental shocks: Applying a stochastic framework. *Journal of Environmental Economics and Management*, 97, 23-41.

[5] Clarke, H., and W. Reed (1994). Consumption/pollution tradeoofs in an environemental vulnerable to pollution-related catastrophic collpase. *Journal of Economic Dynamics and Control*, 18, 991-1010.

[6] Davis, M. H. A. (1984). Piecewise-deterministic Markov processes: a general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society (B)*, 46, 353-388.

[7] Diekert, F. (2017). Threatening Thresholds? The effect of disastrous regime shifts on the non-cooperative use of environmental goods and service. *Journal of Public Economics*, 147, 30-49.

[8] Dockner, E. J., S. Jϕrgensen, N. V. Long and G. Sorger (2000). *Differential Games in Economics and Management Science*, Cambridge, Cambridge University Press.

[9] Dong, S., M. Baringer, and G. Goni (2019). Slow Down of the Gulf Stream during 1993-2016. *Scientific Reports*, 9, 6672. `https://doi.org/10.1038/s41598-019-42820-8`

[10] Forster, B. (1975). Optimal pollution control with a nonconstatnt exponential rate of decay. *Journal of Environmental Economics and Management*, 2, 1-6.

[11] Keeler, E., M. Spence and R. Zeckhauser (1972). The optimal control of pollution. *Journal of Economic Theory*, 4(1), 19-34.

[12] Le Kama A., A. Pommeret and F. Prieur (2014). Optimal emission policy under the risk of irreversible pollution. *Journal of Public Economic Theory*, 16(6), 959-980.

[13] Maler, K.G. (2000). Development, ecological resources and their management: A study of complex dynamic systems. *European Economic Review*, 44(4-6), 645-665.

[14] Nordhaus, W. (2019). Climate change: The ultimate challenge for economics. *American Economic Review*, 109(6), 1991-2014.

[15] Prieur, F. (2009). The environmental Kuznets curve in a world of irreversibility. *Economic Theory*, 40, 57-90.

[16] Tahvonen, O. and C. Withagen (1996). Optimality of irreversible pollution accumulation. *Journal of Economic Dynamics and Control*, 20, 1775-1795.

[17] Tsur, Y., and A. Zemel (1998). Pollution control in an uncertain environment. *Journal of Economic Dynamics and Control*, 22, 967-975.

[18] van der Ploeg, F. (2014). Abrupt positive feedback and the social cost of carbon. *European Economic Review*, 67, 28-41.

[19] Wirl, F. (2008). Tragedy of the commons in a stochastic game of a stock externality. *Journal of Public Economic Theory*, 10, 99-124.

A Appendix

A.1 Proof of Proposition 1

By (A_1)-2, 3, $U'(y)$ is decreasing in y and $D'(z)$ is increasing in z. By definition, $D'(\bar{z}_0) = rU'(\bar{y}_0)$, by $D'(z(t)) < rU'(y(t))$ for any solution $(z(t), y(t))$ of (7) that satisfies $z(t) < \bar{z}_0$. It follows that

$$rU'(\bar{y}_0) < D'(z(t)) < rU'(y(t)).$$

This leads to $y(t) < \bar{y}_0$. Furthermore, since

$$rU(\bar{y}_0) = D'(\bar{z}_0) > 0 = rU'(\bar{y}),$$

it follows that $\bar{y}_0 < \bar{y}$.

Since $y = 0$ whenever $z \geq \bar{z}_0$, we also have $y < \bar{y}$.

This completes the proof.

A.2 Proof of Proposition 2

We compare the value function V_1 at $\bar{z}$ in two scenarios, one with $\bar{z}$ reached and the other with $\bar{z}$ not reached. In the first case, by the continuity of the value function, $V_1(\bar{z}) = V_0(\bar{z})$. In the second case, starting from an initial pollution z_0 near $\bar{z}$, $z(t)$ is decreasing in t and

$$0 \leq \lim_{t \to \infty} z(t) < \bar{z}. \tag{28}$$

The trajectory $(z(t), y(t))$ either stays in the region $R = \{(z, y) : y > \delta(z)\}$ for $t > T$ for some $T > 0$ or there is $\hat{t} > 0$ such that $y(t) = 0$ for $t \geq \hat{t}$. In the former case, since $\delta(\bar{z}) = 0$, it follows that $0 \geq y(t) \to 0$ as $t \to \infty$. Taking the limit $t \to \infty$ in the HJB equation (6) which can be written as

$$(r + \lambda) V_1(z(t)) = U(y(t)) - [y(t) - \delta(z(t))] U'(y(t)) - D(z(t)) + \lambda V_0(z(t)),$$

we obtain

$$(r + \lambda) V_1(\bar{z}) = U(0) - D(\bar{z}) + \lambda V_0(\bar{z}). \tag{29}$$

In the latter case, (6) takes the form

$$(r + \lambda) V_1(z) = U(0) - \delta(z) V_1'(z) - D(z) + \lambda V_0(z) \qquad \text{if } \hat{z} < z < \bar{z}$$

where $\hat{z} = z\left(\hat{t}\right)$. The linear differential equation has the solution

$$V_1(z) = \frac{V_1(\hat{z}) \mu(\hat{z})}{\mu(z)} + \frac{1}{\mu(z)} \int_{\hat{z}}^{z} \frac{\mu(\xi)}{\delta(\xi)} [U(0) - D(\xi) + \lambda V_0(\xi)] \, d\xi$$

where

$$\mu(z) = e^{\int^z \frac{r+\lambda}{\delta(\xi)} d\xi}.$$

Since $\delta(\xi) = \delta_1(\xi - \bar{z}) + o(|\xi - \bar{z}|)$ as $\xi \to \bar{z}$ and $\delta_1 = \delta'(\bar{z}) < 0$, it follows that

$$\mu(z) \sim |z - \bar{z}|^{(r+\lambda)/\delta_1} \to \infty$$

Hence,

$$\begin{aligned}
\lim_{z \to \bar{z}} V_1(z) &= \lim_{z \to \bar{z}} \frac{1}{\mu(z)} \int_{\hat{z}}^{z} \frac{\mu(\xi)}{\delta(\xi)} [U(0) - D(\xi) + \lambda V_0(\xi)] \, d\xi \\
&= \lim_{z \to \bar{z}} \frac{\mu(z)}{\mu'(z) \delta(z)} [U(0) - D(z) + \lambda V_0(z)] \\
&= \frac{1}{r + \lambda} [U(0) - D(\bar{z}) + \lambda V_0(\bar{z})].
\end{aligned}$$

Hence, (29) again holds.

Finally, since $U'' < 0$, it follows that

$$[U(y) - yU'(y)]' = -yU''(y) > 0 \qquad \text{for } y > 0.$$

Hence, by (8,

$$V_0(\bar{z}) = \frac{1}{r}\left[U(\bar{y}_0) - D(\bar{z}) - \bar{y}_0 U'(\bar{y}_0)\right] > \frac{1}{r}\left[U(0) - D(\bar{z})\right].$$

Therefore, by (29) $V_1(\bar{z}) < V_0(\bar{z})$ if (28) holds. This shows that the decision maker would choose the irreversible optimal trajectory if z_0 is near $\bar{z}$.

The proof is complete.

A.3 Proof of Lemma 1

By (5) and (6),

$$\begin{aligned} rV_m(\bar{z}) &= U(y_m(\bar{z})) + y_m(\bar{z}) V_m'(\bar{z}) - D(\bar{z}) \\ &= U(y_m(\bar{z})) - y_m(\bar{z}) U'(y_m(\bar{z})) - D(\bar{z}) \qquad \text{for } m = 0, 1. \end{aligned}$$

Since $\bar{z}$ is reached in finite time, $V_1(\bar{z}) = V_0(\bar{z})$. Hence, the above equation leads to

$$U(y_0(\bar{z})) - y_0(\bar{z}) U'(y_0(\bar{z})) = U(y_1(\bar{z})) - y_1(\bar{z}) U'(y_1(\bar{z})).$$

By (A_1)-2, $U''(y) < 0$. Hence, the function $y \mapsto U(y) - yU'(y)$ has the derivative $-yU''(y) > 0$ for $y > 0$. This means $U(y) - yU'(y)$ is one-to-one. Therefore, $y_1(\bar{z}) = y_0(\bar{z})$.

This completes the proof.

A.4 Proof Proposition 4

Substituting the right-hand side into (21) and comparing coefficients, we find

$$rA_0 = A_0^2 - c, \quad rB_0 = A_0(B_0 + a), \quad 2rC_0 = (B_0 + a)^2. \tag{30}$$

The quadratic equation for A_0 has two roots, one negative and the other positive. We use the negative one since V_0 is decreasing and concave. Thus

$$A_0 = \frac{r - \sqrt{r^2 + 4c}}{2} \equiv h_0 \tag{31}$$

and consequently,

$$B_0 = \frac{h_0 a}{r - h_0}, \quad C_0 = \frac{(B_0 + a)^2}{2r}. \tag{32}$$

As a result,

$$a + V_0'(z) = h_0 z + \frac{h_0 a}{r - h_0} + a = h_0 z + \frac{ar}{r - h_0}.$$

Using (31) we find

$$\frac{ar}{h_0 (h_0 - r)} = \frac{ar}{c} \equiv \bar{z}_0. \tag{33}$$

Hence,

$$V_0'(z) + a = h_0 \left(z - \frac{ar}{c}\right) = h_0 (z - \bar{z}_0).$$

Since $h_0 < 0$, the above quantity is positive if and only if $z < \bar{z}_0$. Hence

$$V_0(z) = \begin{cases} \frac{1}{2r} \left[h_0^2 (z - \bar{z}_0)^2 - cz^2\right] & \text{for } z \leq \bar{z}_0, \\ -\frac{c}{2r} z^2 & \text{for } z > \bar{z}_0. \end{cases} \tag{34}$$

By differentiation,

$$V_0' = \begin{cases} \frac{1}{r} \left[h_0^2 (z - \bar{z}_0) - cz\right] & \text{for } z \leq \bar{z}_0, \\ -\frac{c}{r} z & \text{for } z > \bar{z}_0. \end{cases}$$

By the definitions of h_0 and $\bar{z}_0$ in (31) and (33), respectively, one can derive

$$\frac{1}{r} \left[h_0^2 - c\right] = h_0, \qquad \frac{c}{r} \bar{z}_0 = a.$$

Hence,

$$V_0'(z) = \frac{1}{r} \left[h_0^2 (z - \bar{z}_0) - c (z - \bar{z}_0)\right] - \frac{c}{r} \bar{z}_0 = h_0 (z - \bar{z}_0) - a$$

for $z < \bar{z}_0$. That completes the proof.

A.5 Proof of Proposition 6

We first prove that the threshold, $\bar{z}$, is reachable if and only if $\bar{z} < \bar{z}_0$.

Suppose $\bar{z} \geq \bar{z}_0$. Then, by (23) and (34),

$$V_1(\bar{z}) = V_0(\bar{z}) = -\frac{c\bar{z}^2}{2r}.$$

Hence, by (22) and Assumption (A),

$$f_1(\bar{z}) = a + V_1'(\bar{z}) = 0. \tag{35}$$

Differentiate the both sides of (22) with respect to z, we obtain

$$f_1(z) f_1'(z) = (r+\lambda) f_1(z) + (r+\delta'(z))(\delta(z)-a) + cz + \lambda[\delta(z) - V_0'(z) - a].$$

This equation is equivalent to the dynamical system

$$\begin{aligned} \dot{x} &= (r+\lambda)x + (r+\delta'(z)+\lambda)\delta(z) - a(r+\delta'(z)) + cz - \lambda[V_0'(z)+a], \\ \dot{z} &= x, \end{aligned} \tag{36}$$

where $x = f_1(z)$ and the differentiation is with respect to t. If $\bar{z}$ is reached in finite time, by (35), the trajectory passes through the point $(0, \bar{z})$ at certain time $\bar{t}$. Hence, $(0, \bar{z})$ cannot be an equilibrium of the dynamical system and so, $\dot{x}(\bar{t}) < 0$. From the first equation in (36), we find

$$-a(r+\delta'(\bar{z})) + c\bar{z} - \lambda[V_0'(\bar{z}) + a] < 0.$$

However, since $\bar{z} \geq \bar{z}_0$, it follows that $V_0'(\bar{z}) + a \leq 0$. Also, since $\delta(z) \geq \delta(\bar{z}) = 0$ for $z < \bar{z}$, it follows that $\delta'(\bar{z}) \leq 0$. The above inequality implies

$$-ar + c\bar{z} < a\delta'(\bar{z}) + \lambda[V_0'(\bar{z}) + a] \leq 0.$$

Using $ar = c\bar{z}_0$, we obtain $\bar{z} < \bar{z}_0$. This contradicts the assumption of $\bar{z} \geq \bar{z}_0$. Hence, $\bar{z}$ cannot be reached in finite time.

Suppose $\bar{z} < \bar{z}_0$. By (26), $f_0(z) > 0$. Let $\phi_1(z)$ be defined by

$$\phi_1(z) = V_1'(z) + a - \delta(z). \tag{37}$$

Then, ϕ_1 exists at least locally for z near $\bar{z}$. Furthermore, $\phi_1(\bar{z}) = f_0(\bar{z}) > 0$. Hence, $\phi_1(z) > 0$ for z near $\bar{z}$. With $\phi_1(z)$ solved, one finds the value function $V_1(z)$ by (22), which leads to

$$\begin{aligned} V_1(z) = \tfrac{1}{2(r+\lambda)} \{ &[\max\{\phi_1(z), -\delta(z)\} - \delta(z)](\phi_1(z) + \delta(z)) \\ &+ 2a\delta(z) - cz^2 + 2\lambda V_0(z)\}. \end{aligned} \tag{38}$$

Therefore, the optimal strategy leads to z reaching $\bar{z}$ in finite time.

We now prove the second part of the proposition. That is, there is $\hat{z} \in (0, \bar{z}_0]$ such that $\bar{z}$ is globally reachable if and only if $\bar{z} < \hat{z}$. For this purpose, we derive the solution of the optimal control problem.

Differentiate the both sides of (22) with respect to z. We obtain

$$\phi_1(z)\,\phi_1'(z) = (r+\lambda)\,\phi_1(z) + (r+\delta'(z))\,(\delta(z)-a) + cz + \lambda\left[\delta(z) - f_0(z)\right]$$

if $\phi_1(z) > -\delta(z)$ and

$$\begin{aligned} -\delta(z)\,\phi_1'(z) &= (r+\lambda)\,\phi_1(z) + (r+\delta'(z))\,(\delta(z)-a) + cz + \lambda\left[\delta(z) - f_0(z)\right] \\ &\quad +\delta'(z)\,(\phi_1(z)+\delta(z)) \end{aligned}$$

if $\phi_1(z) \leq -\delta(z)$. The two cases can be combined into

$$\begin{aligned} \max\{\phi_1(z), -\delta(z)\}\,\phi_1'(z) &= (r+\lambda)\,\phi_1(z) + (r+\delta'(z))\,(\delta(z)-a) + cz \\ &\quad +\lambda\left[\delta(z) - f_0(z)\right] + \delta'(z)\min\{\phi_1(z)+\delta(z), 0\}\,. \end{aligned} \tag{39}$$

By (22) and (23),

$$\phi_1(\bar{z}) = f_0(\bar{z})\,. \tag{40}$$

Notice that any solution ϕ_1 of (39)-(40) defines a value function V_1 by (38). In general, at any pollution level, z, the decision maker has two choices, either to produce more and pollute more, or produce less and pollute less. We use $\tilde{f}_1$ to denote the solution ϕ_1 of (39)-(40) which is positive near $\bar{z}$, and use $\hat{f}_1$ to denote the solution which is negative near $\bar{z}$. Substituting $\tilde{f}_1(z)$ and $\hat{f}_1(z)$ for $\phi_1(z)$ in (38), we obtain respective functions $\hat{V}_1(z)$ and $\tilde{V}_1(z)$. Clearly the decision maker chooses the strategy based on the larger solution. Hence

$$V_1(z) = \max\left\{\tilde{V}_1(z), \hat{V}_1(z)\right\} \qquad \text{for } z \in (0, \bar{z})\,. \tag{41}$$

At points where only $\tilde{f}_1$ or $\hat{f}_1$ exists, there is no ambiguity in the definition of V_1. If at a point z^* the decision maker switches from one strategy to the other, it is necessary that $\tilde{V}_1(z^*) = \hat{V}_1(z^*)$ holds. By (38), $\tilde{f}_1(z^*)$ and $\hat{f}_1(z^*)$ are related by

$$\hat{f}_1(z^*) = \begin{cases} -\tilde{f}_1(z^*) & \text{if } \tilde{f}_1(z^*) < \delta(z^*)\,, \\ -\tilde{f}_1(z^*)^2 / \left[2\delta(z^*)\right] - \delta(z^*)/2 & \text{if } \tilde{f}_1(z^*) \geq \delta(z^*)\,. \end{cases} \tag{42}$$

To construct a solution to (41), we first construct solutions $\tilde{f}_1$ and $\hat{f}_1$ of Eq. (39) on the interval $(0, \bar{z})$, then compare the corresponding functions $\tilde{V}_1$ and $\hat{V}_1$. This is achieved by converting Eq. (39) into dynamical systems.

For $\tilde{f}_1$, since it is positive in a neighborhood of $\bar{z}$, Eq. (39) is equivalent to the dynamical system

$$\begin{aligned} \dot{x} &= (r+\lambda)\,x + (r-\beta+\lambda)\,\delta(z) + (\beta-r)\,a + cz - \lambda f_0(z)\,, \\ \dot{z} &= x, \end{aligned} \tag{43}$$

where $x(t) = \tilde{f}_1(z(t))$. Note that the right-hand sides of the equations in (43) is linear in x and z, and can be written as

$$\dot{x} = (r+\lambda)\,x + B_1 z + C_1$$

where

$$\begin{aligned} B_1 &= \beta(\beta-r-\lambda) + c - \lambda h_0, \\ C_1 &= (r-\beta+\lambda)\,\alpha + (\beta-r)\,a + \lambda h_0 \bar{z}_0. \end{aligned} \tag{44}$$

There is an equilibrium at $(\bar{z}_1, 0)$ where

$$\bar{z}_1 = -C_1/B_1. \tag{45}$$

The Jacobian matrix takes the form

$$J = \begin{pmatrix} r+\lambda & B_1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues are

$$\begin{aligned} h_1 &= \tfrac{1}{2}\left[r+\lambda-\sqrt{(r+\lambda)^2+4B_1}\right], \\ h_2 &= \tfrac{1}{2}\left[r+\lambda+\sqrt{(r+\lambda)^2+4B_1}\right] \end{aligned} \tag{46}$$

Depending on whether $B_1 > 0$ or $B_1 < 0$, the equilibrium is a saddle point or a repeller.

As for $\hat{f}_1$, we first define the zx-plane regions

$$R_1 = \{(z,x) : x > -\delta(z)\}\,, \qquad R_2 = \{(z,x) : x \leq \delta(z)\}\,.$$

Eq. (39) is equivalent to the dynamical system (43) in R_1 and

$$\begin{aligned} \dot{x} &= (r+\lambda-\beta)\,x + B_1 z + C_1 - \beta\delta(z)\,, \\ \dot{z} &= -\delta(z) \end{aligned} \tag{47}$$

in R_2, where $x(t) = \hat{f}_1(z(t))$. Observe that by the assumption $\delta(\bar{z}) = 0$ and relation (42), there is no solution $\hat{f}_1$ that satisfies $\hat{V}_1(\bar{z}) = V_0(\bar{z})$. Indeed, since $\tilde{f}_1(\bar{z}) = y_1^*(\bar{z}) > 0$, by (42),

$$\hat{f}_1(\bar{z}) = -\frac{\tilde{f}_1(\bar{z})^2}{2\delta(\bar{z})} - \frac{\delta(\bar{z})}{2}$$

which does not exist. We also notice from (42) that $\hat{f}_1(\bar{z}) \to -\infty$ as $z \to \bar{z}$. Furthermore, we observe that if $\hat{f}_1(0)$ exists and $\hat{f}_1(0) < 0$, then $z(t) = 0$ for all $t > 0$. This is possible only if $\hat{f}_1(0) = 0$. Hence the solution curve $\left(z, \hat{f}_1(z)\right)$ starts from the point $(0, 0)$. If $\hat{f}_1(0)$ exists and $\hat{f}_1(0) > 0$, then $z(t)$ is increasing as long as $\hat{f}_1(z) > 0$. Hence, $\hat{f}_1(z)$ can only vanish at the steady state $\bar{z}_1$. This can only happen if $(\bar{z}_1, 0)$ is a saddle point and $\left(z, \hat{f}_1(z)\right)$ is on its stable manifold.

There are two cases, $B_1 > 0$ and $B_1 \leq 0$. In each case there are three subcases: (a) $\bar{z}_1 \leq 0$, (b) $0 < \bar{z}_1 \leq \bar{z}$ and (c) $\bar{z}_1 > \bar{z}$. In each subcase we show the existence of a positive $\hat{z}$ such that $\bar{z}$ is globally reachable if and only if $\bar{z} < \hat{z}$.

Case 1. $B_1 > 0$**.**

In this case $(0, \bar{z}_1)$ is a saddle point. In addition, $\langle h_i, 1 \rangle$ is an eigenvector of the Jacobian J corresponding to the eigenvalue h_i. Hence, the stable manifold at the equilibrium has the negative slope and the unstable one has the positive slope.

Case 1-a. $\bar{z}_1 \leq 0$**.** Since $(\bar{z}_1, 0)$ is a saddle point, there is a trajectory passing through $(0, 0)$. Then, $\hat{f}_1(z)$ is defined by the part of this trajectory below the z-axis. The part of this trajectory above the z-axis intersects the vertical line $z = \bar{z}$. Let $\bar{f}_1 > 0$ be the x-coordinate of this intersection. Then, the trajectory starting at the point $(\bar{z}, f_0(\bar{z}))$ lies above the z-axis for all $z \in [0, \bar{z}]$ if

$$f_0(\bar{z}) = h_0(\bar{z} - \bar{z}_0) \geq \bar{f}_1, \tag{48}$$

and if the above inequality does not hold, the trajectory starting at $(\bar{z}, f_0(\bar{z}))$ intersects the z-axis within the interval $(0, \bar{z})$. In the latter case, we let $\bar{z}_1^*$ be a solution to the equation (42). Note that at the zero of $\tilde{f}$ the right-hand side of the above equation vanishes while the left-hand side is negative. We show that for $z < \bar{z}$ and is sufficiently close to $\bar{z}$, the right-hand side approaches

$-\infty$ faster than the left-hand side. This would imply the existence of solution (42). Note that $\tilde{f}_1(\bar{z}) = f_0(\bar{z}) > 0$ and

$$\delta(z) = \beta(\bar{z} - z).$$

Hence

$$\tilde{f}_1(z)^2 / [2\delta(z)] + \delta(z)/2 = O\left(|\bar{z} - z|^{-1}\right). \tag{49}$$

To estimate $\hat{f}_1$ as $z \to \bar{z}^-$, we solve (39) which which takes the form

$$\begin{aligned} -\delta(z)\hat{f}_1'(z) &= (r+\lambda)\hat{f}_1(z) + (r-\beta)(\delta(z) - a) + cz \\ &\quad + \lambda[\delta(z) - f_0(z)] - \beta\left[\hat{f}_1(z) + \delta(z)\right]. \end{aligned}$$

This is a linear equation. Multiplying the integrating factor

$$\mu(z) = (\bar{z} - z)^{1-(r+\lambda)/\beta}$$

to the both sides of the equation and integrate from $\bar{z}$ to z, we obtain

$$\hat{f}_1(z)(\bar{z} - z)^{1-\frac{r+\lambda}{\beta}} = \int_{\bar{z}}^{z} (\bar{z} - s)^{1-\frac{r+\lambda}{\beta}} [2\beta - r - \lambda - (\beta - r)a - cs + \lambda f_0(s)]\, ds. \tag{50}$$

If $2\beta > r + \lambda$, then the right-hand side approaches zero as $z \to \bar{z}$. Thus, by l'Hôpital's rule,

$$\begin{aligned} \lim_{z \to \bar{z}^-} \frac{\hat{f}_1(z)}{\bar{z} - z} &= \lim_{z \to \bar{z}^-} \frac{\int_{\bar{z}}^{z} (\bar{z} - s)^{1-\frac{r+\lambda}{\beta}} [2\beta - r - \lambda - (\beta - r)a - cs + \lambda f_0(s)]\, ds}{(\bar{z} - z)^{2-(r+\lambda)/\beta}} \\ &= -\frac{2\beta - r - \lambda - (\beta - r)a - c\bar{z} + \lambda f_0(\bar{z})}{2 - (r+\lambda)/\beta}. \end{aligned}$$

This implies that

$$\hat{f}_1(z) = O(|z - \bar{z}|).$$

If $2\beta < r + \lambda$, the right-hand side of (50) diverges. We have

$$\begin{aligned} \lim_{z \to \bar{z}} \hat{f}_1(z)(\bar{z} - z) &= \lim_{z \to \bar{z}^-} \frac{\int_{\bar{z}}^{z} (\bar{z} - s)^{1-\frac{r+\lambda}{\beta}} [2\beta - r - \lambda - (\beta - r)a - cs + \lambda f_0(s)]\, ds}{(\bar{z} - z)^{-(r+\lambda)/\beta}} \\ &= \lim_{z \to \bar{z}^-} (\bar{z} - z)^2 [-2\beta + r + \lambda + (\beta - r)a + cz - \lambda f_0(z)] = 0. \end{aligned}$$

Hence,

$$\hat{f}_1(z) = o\left(|\bar{z} - z|^{-1}\right).$$

In any case, in view of (49),

$$\hat{f}_1(z) > -\frac{\tilde{f}_1(z)^2}{2\delta(z)} - \frac{\delta(z)}{2}$$

if $z < \bar{z}$ and is sufficiently close to $\bar{z}$. Therefore, there is at least one solution $\bar{z}_1^*$. Let

$$\tilde{V}_1(z) = \frac{1}{2(r+\lambda)}\left\{\tilde{f}_1(z)^2 - \delta(z)^2 + 2a\delta(z) - cz^2 + 2\lambda V_0(z)\right\} \qquad \text{for } 0 \le z \le \bar{z} \tag{51}$$

and

$$\hat{V}_1(z) = \frac{1}{2(r+\lambda)} \cdot \begin{cases} \hat{f}_1(z)^2 - \delta(z)^2 + 2a\delta(z) - cz^2 + 2\lambda V_0(z) & \text{if } \hat{f}_1(z) > -\delta(z), \\ 2\delta(z)\left[a - \hat{f}_1(z) - \delta(z)\right] - cz^2 + 2\lambda V_0(z) & \text{if } \hat{f}_1(z) \le -\delta(z). \end{cases} \tag{52}$$

Then there is at least one point at which $\tilde{V}_1(z) = \hat{V}_1(z)$. Define $V_1(z)$ by (41), we obtain a continuous value function. (See Fig. 5.) As a conclusion,

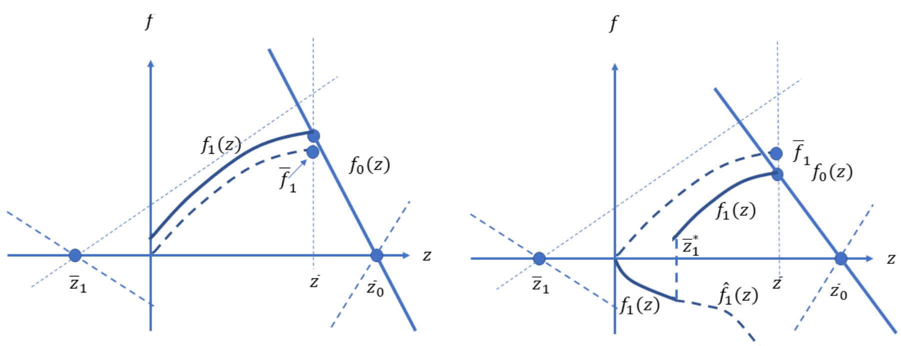

Figure 5: Case 1-a with $f_0(\bar{z}) \ge \bar{f}_1$ (left) and $f_0(\bar{z}) < \bar{f}_1$ (right).

$\bar{z}$ is globally reachable if and only if (48) holds. Thus,

$$\hat{z} = \bar{z}_0 + \frac{\bar{f}_1}{h_0} \tag{53}$$

in this case.

Case 1-b. $0 < \bar{z}_1 \le \bar{z}$**.** In this case the trajectory that passes through $(0,0)$ stays in the region where $z \le 0$. Hence, $\hat{f}_1(0) > 0$. Therefore, $\left(z, \hat{f}_1(z)\right)$ is on the stable manifold of the equilibrium $(\bar{z}_1, 0)$. This implies that

$$\hat{f}_1(z) = h_1(z - \bar{z}_1) \qquad \text{if } h_1(z - \bar{z}_1) > -\delta(z).$$

For z that satisfies

$$h_1(z - \bar{z}_1) \le -\delta(z),$$

we solve (39) for $z > z'$ with the initial condition

$$\hat{f}_1(z') = -\delta(z'),$$

where

$$z' = \bar{z}_1 - \frac{\delta(z')}{h_1}.$$

Let

$$\bar{f}_1 = h_2(\bar{z} - \bar{z}_1).$$

Then the trajectory starting at the point $(\bar{z}, f_0(\bar{z}))$ lies above the z-axis if (48) holds, and it intersects the z-axis on the interval $(0, \bar{z})$ if the reversed inequality holds. In the latter case the intersection point is greater than $\bar{z}_1$. (See Fig. 6.)

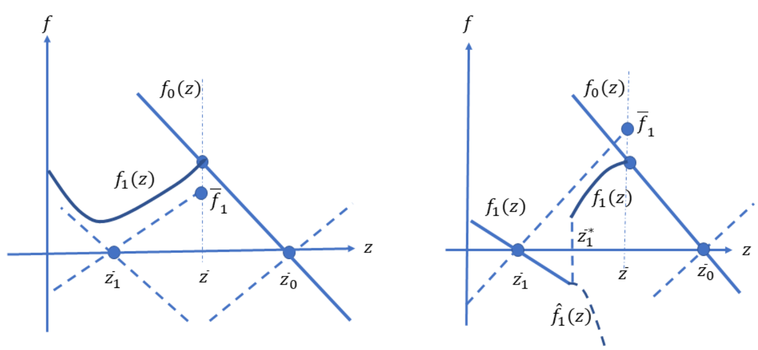

Figure 6: Case 1-b with $f_0(\bar{z}) \ge \bar{f}_1$ (left) and $f_0(\bar{z}) < \bar{f}_1$ (right).

From the above discussion we see that $\bar{z}$ is globally reachable if $\bar{z} < \hat{z}$ where $\hat{z}$ is given by (53).

Case 1-c. $\bar{z}_1 > \bar{z}$**.** In this case any trajectory starting at $(0,0)$ does not enter the region $z > 0$, and any trajectory below the z-axis for $z \in (0, \bar{z})$ does not intersect the z-axis on this interval. Thus $\hat{f}_1(z)$ is undefined for the entire interval $[0, \bar{z}]$. For any positive value of $f_0(\bar{z})$, $\tilde{f}_1(z)$ is defined for all $z \in [0, \bar{z}]$. Thus only $\tilde{V}_1(z)$ exists. (See Fig. 7.)

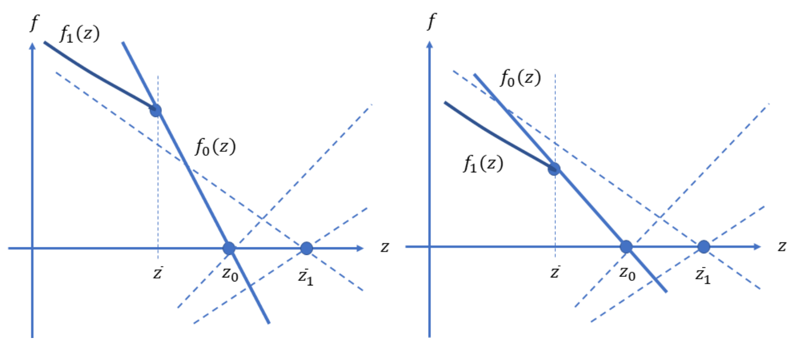

Figure 7: Case 1-c with $f_0(\bar{z}) \geq h_1(\bar{z} - \bar{z}_1)$ (left) and $f_0(\bar{z}) < h_1(\bar{z} - \bar{z}_1)$ (right).

Since any $\bar{z} < \bar{z}_0$ is globally reachable, it suffices to choose $\hat{z} = \bar{z}_0$.

Case 2. $B_1 \leq 0$**.**

In this case $(0, \bar{z}_1)$ is a repeller and both unstable manifolds have position slopes in the fz-plane. As $z \to \bar{z}_1$, points $\left(z, \tilde{f}_1(z)\right) \to (\bar{z}_1, 0)$ along the unstable manifold $x = Y_1(z - \bar{z}_1)$. There are three subcases.

Case 2-a. $\bar{z}_1 \leq 0$**.** In this case the trajectory that passes through $(0,0)$ does not enter the region below the z-axis for $z > 0$. Hence, $\hat{f}_1(z)$ is not defined for any $z \in (0, \bar{z})$. On the other hand, $\tilde{f}_1(z)$ is defined for all $z \in [0, \bar{z}]$ with any value of $f_0(\bar{z})$. (See Fig. 8.)

In this case we again see that $\hat{z} = \bar{z}_0$ since $\bar{z}$ is globally reachable for any $\bar{z} < \bar{z}_0$.

Case 2-b. $0 < \bar{z}_1 \leq \bar{z}$**.** In this case the trajectory passing through $(0,0)$ enter into the region $z > 0$ both above and below the z-axis. The one below the z-axis joins the equilibrium $(0, \bar{z}_1)$. Hence, $\hat{f}_1(z) < 0$ for $0 < z < \bar{z}_1$. The one above the z-axis intersects the vertical line $z = \bar{z}$ at a point, denoted

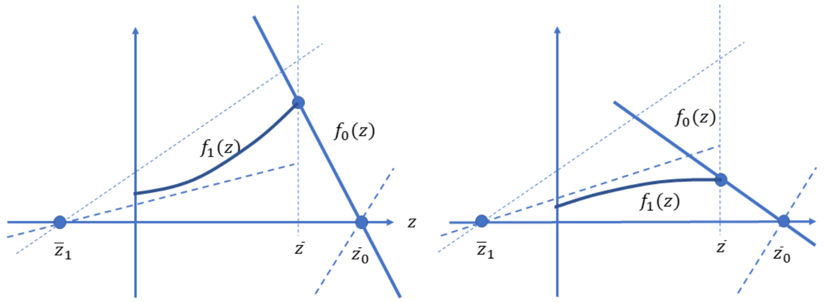

Figure 8: Case 2-a with $f_0(\bar{z}) \geq h_1(\bar{z}-\bar{z}_1)$ (left) and $f_0(\bar{z}) < h_1(\bar{z}-\bar{z}_1)$ (right).

by $\bar{f}_1$. In the case where $f_0(\bar{z}) \geq \bar{f}_1$, $\tilde{f}_1(z)$ is defined and positive for all $z \in [0, \bar{z}]$. See the left graph in Fig. 9). On the other hand, if

$$h_2(\bar{z}-\bar{z}_1) < f_0(\bar{z}) < \bar{f}_1,$$

$\tilde{f}_1(z') = 0$ at some $0 < z' < \bar{z}_1$. It is clear that

$$\tilde{f}_1(\bar{z}_1) > 0 = \hat{f}_1(\bar{z}_1), \quad \tilde{f}_1(z') = 0 > \hat{f}_1(z').$$

By (51)-(52), the first chained inequality of the above implies that

$$\tilde{V}_1(\bar{z}_1) = \frac{1}{2(r+\lambda)}\left\{-\delta(\bar{z})^2 + 2a\delta(\bar{z}) - c\bar{z}^2 + 2\lambda V_0(\bar{z})\right\} < \hat{V}_1(\bar{z}_1),$$

and the second implies that

$$\begin{aligned}\tilde{V}_1(z') &= \frac{1}{2(r+\lambda)}\left\{\tilde{f}_1(z') - \delta(z')^2 + 2a\delta(z') - c(z')^2 + 2\lambda V_0(z')\right\} \\ &> \frac{1}{2(r+\lambda)}\left\{2\delta(z')\left[a-\delta(z')\right] - c(z')^2 + 2\lambda V_0(z')\right\} = \hat{V}_1(z').\end{aligned}$$

Hence, there is a point $\bar{z}_1^*$ such that $\hat{V}_1(\bar{z}_1^*) = \tilde{V}_1(\bar{z}_1^*)$. Define $V_1(z)$ by (41) and

$$f_1(z) = \begin{cases} \hat{f}_1(z) & \text{if } z < \bar{z}_1^*, \\ \tilde{f}_1(z) & \text{if } z > \bar{z}_1^*. \end{cases} \tag{54}$$

We obtain a continuous value function. See the right graph in Fig. 9.

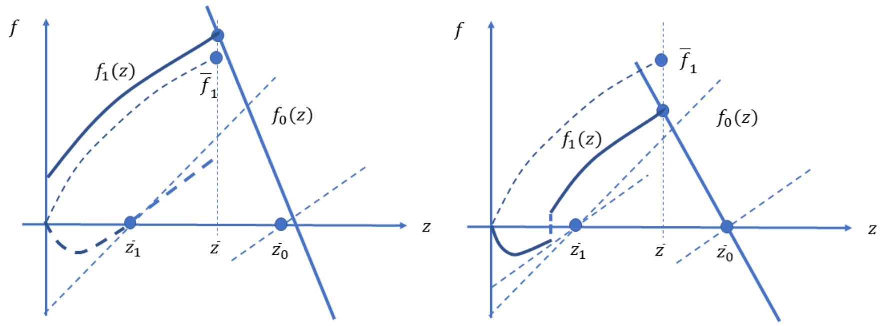

Figure 9: Case 2-b with $f_0\,(\bar{z}) \geq \bar{f}_1$ (left) and $h_2\,(\bar{z}-\bar{z}_1) \leq f_0\,(\bar{z}) < \bar{f}_1$ (right).

In the case where

$$f_0\,(\bar{z}) \leq h_2\,(\bar{z}-\bar{z}_1)\,,$$

The trajectory that passes through $(\bar{z}, f_0\,(\bar{z}))$ approaches $(\bar{z}_1, 0)$ along the unstable manifold $(z, h_1\,(z-\bar{z}))$. Thus, $\tilde{V}_1\,(z)$ is defined for $z > \bar{z}_1$ and $\hat{V}_1\,(z)$ is defined for $z < \bar{z}_1$. We define

$$V_1\,(z) = \begin{cases} \hat{V}_1\,(z) & \text{for } z < \bar{z}_1, \\ \tilde{V}_1\,(z) & \text{for } z > \bar{z}_1, \end{cases}$$

and define $f_1\,(z)$ similarly. The graph of $f_1\,(z)$ is shown in Fig. 10.

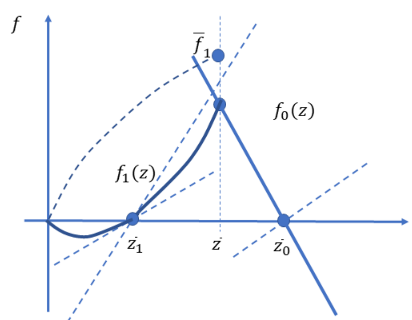

Figure 10: Case 2-b with $f_0\,(\bar{z}) < h_2\,(\bar{z}-\bar{z}_1)$.

As can be seen, $\bar{z}$ is globally reachable if and only if $\bar{z} < \hat{z}$ with $\hat{z}$ defined by (53).

Case 2-c. $\bar{z}_1 > \bar{z}$**.** Similar to Case 2-b, $\tilde{f}(z) \geq 0$ for all $z \in (0, \bar{z})$ if $f_0(\bar{z}) \geq \bar{f}_1$ and $\tilde{f}(z) = 0$ on the interval $(0, \bar{z})$. In the former case, $V_1(z) = \tilde{V}_1(z)$ on $[0, \bar{z}]$. In the latter case, there is a point $\bar{z}_1^*$ such that $\hat{V}_1(\bar{z}_1^*) = \tilde{V}_1(\bar{z}_1^*)$. We define $V_1(z)$ and $f_1(z)$ by (41) and (54), respectively. See Fig. 11.

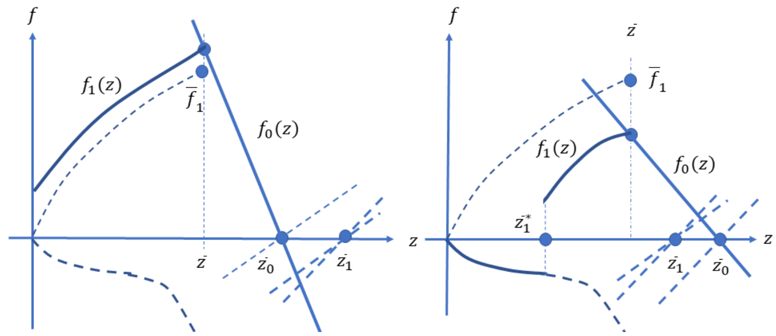

Figure 11: Case 2-c with $f_0(\bar{z}) \geq \bar{f}_1$ (left) and $f_0(\bar{z}) < \bar{f}_1$ (right).

It is clear that $\bar{z}$ is globally reachable if and only if $\bar{z} < \hat{z}$ with $\hat{z}$ given by (53).

This completes the proof.