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Tara TRAUTHWEIN

QUANTITATIVE CLTS ON THE POISSON SPACE VIA p-POINCARÉ INEQUALITIES

Dissertation defence committee

Dr Giovanni Peccati, dissertation supervisor
Professor, Université du Luxembourg

Dr Ivan Nourdin, Chairman
Professor, Université du Luxembourg

Dr Günter Last, Vice-Chairman
Professor, Karlsruher Institut für Technologie

Dr Matthias Schulte,
Professor, Technische Universität Hamburg

Dr Joseph Yukich,
Professor, Lehigh University

Quantitative CLTs on the Poisson space

via p-Poincaré inequalities

Tara Trauthwein

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Luxembourg National Research Fund

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Jury

Prof. Dr. Giovanni Peccati
Supervisor

Prof. Dr. Ivan Nourdin
President of the jury

Prof. Dr. Günter Last
External jury member

Prof. Dr. Matthias Schulte
External jury member

Prof. Dr. Joseph Yukich
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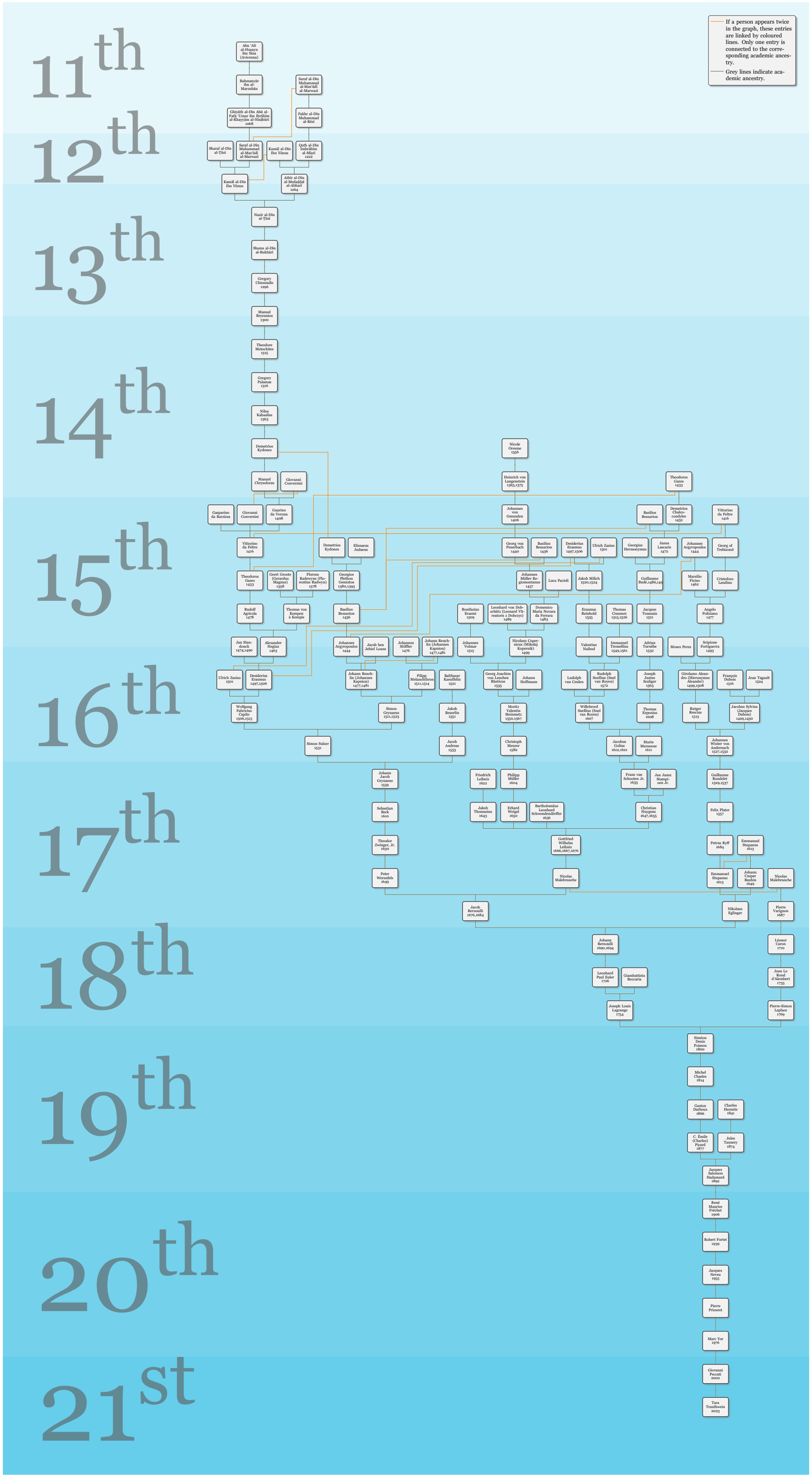

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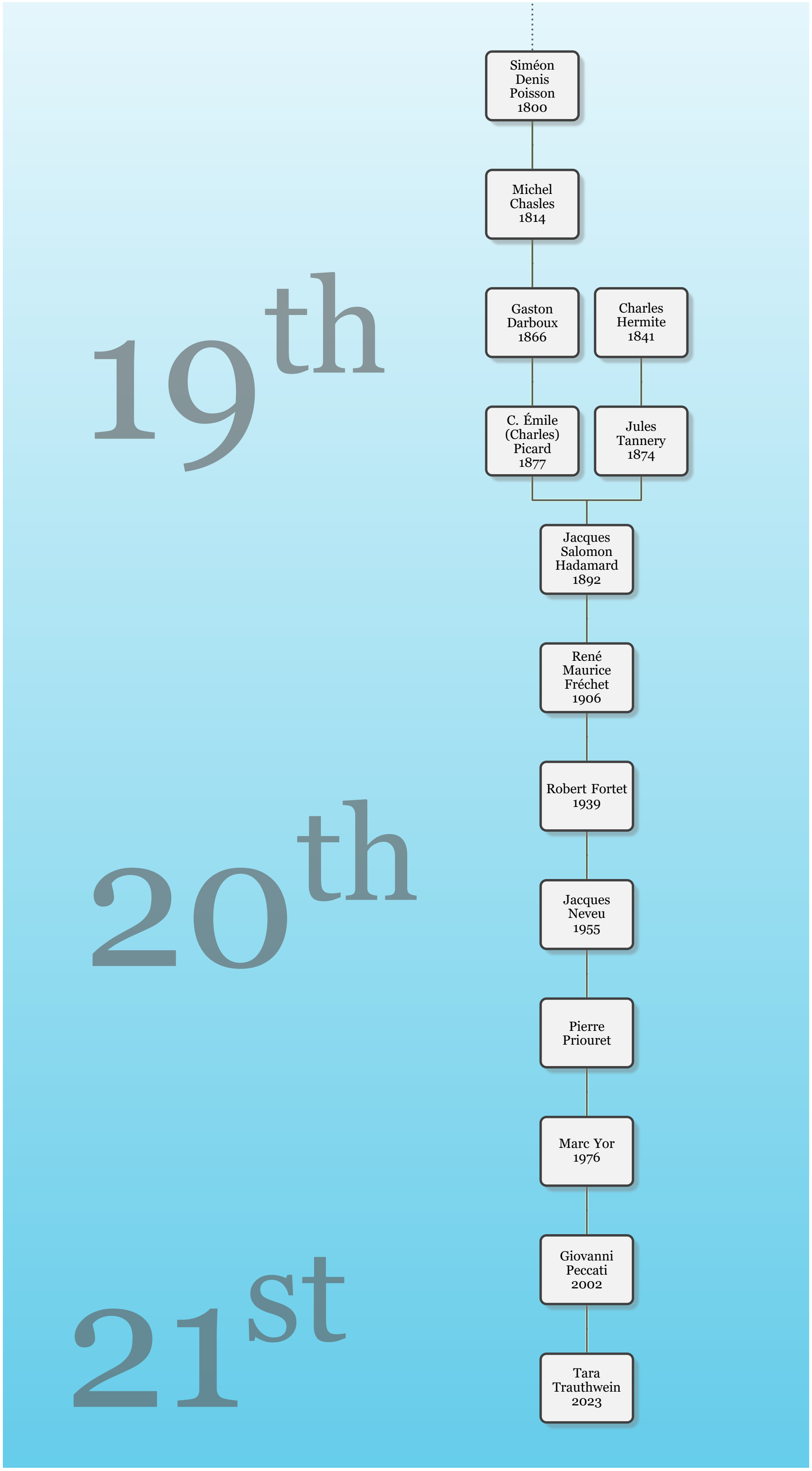
Grey lines indicate academic ancestry.



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Michel Chasles 1814
Gaston Darboux 1866
Charles Hermite 1841
C. Émile (Charles) Picard 1877
Jules Tannery 1874
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Robert Fortet 1939
Jacques Neveu 1955
Pierre Priouret
Marc Yor 1976
Giovanni Peccati 2002
Tara Trauthwein 2023

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Overview

This thesis is based on the preprint [Tra22] and on ongoing work, which is presented in Section 4.2 and Chapter 7. The goal of the present work is to establish a new collection of probabilistic inequalities, yielding quantitative central limit theorems (CLTs) for sequences of functionals of Poisson measures under minimal moment assumptions.

Poisson measures are ubiquitous in modelling — already used in 1898 by Bortkievicz to model the distribution of death by horse-kick in the Prussian army [Bor98, p.23-25], these objects have been applied to represent anything between the busy periods in a telephone queue [Goo86] to the distribution of photon background noise [GF96].

Given a space and an intensity measure, the Poisson measure allows us to sample a random collection of points in this space, with many points in areas of high density and fewer points in areas of lower density. A Poisson functional, then, is any real quantity evaluated on these collections of points. This can be for example some property of a graph having the points as its vertices: the length of the minimal spanning tree, the number of triangles in a k-Nearest Neighbour Graph, the number of connected components in a Gilbert Graph, etc. In terms of applications, these quantities can represent the total cost of establishing a telecommunication network, or the number of size-3 cliques in a social network.

When the network to be studied is large, exact computations of some quantity of interest F become difficult. Instead, one turns to estimating the fluctuations of F around its mean. If these fluctuations are normally distributed and we know something about the asymptotic variance of F, then we can adequately predict the behaviour of F when the network is large enough. In order to show asymptotic normality, we need a central limit theorem:

$$\frac{F_n - \mathbb{E}F_n}{\sqrt{\mathrm{Var}(F_n)}} \overset{n\to\infty}{\longrightarrow} \mathcal{N}(0,1),$$

where n denotes the size of the network to be studied and the convergence is in distribution. Knowing that a certain quantity converges to a normal distribution might sometimes not be enough - we want to know how fast this convergence takes place and how large the error is when we approximate the fluctuations around the mean by a normal distribution. This is where **quantitative central limit theorems** come in, yielding bounds on the distance between the fluctuations of a functional F_n and a standard normal variable N. A typical quantitative central limit theorem takes the form:

$$d\left(\frac{F_n - \mathbb{E}F_n}{\sqrt{\operatorname{Var}(F_n)}}, N\right) < cn^{-1/2},$$

where c is an absolute constant.

A very special type of functionals are those that are in some sense locally determined. As a simple example, let us consider the **Gilbert Graph** of distance parameter 1. This graph has Poisson points as vertices, and is such that two points are connected if their distance is less than 1. Suppose you might want to know the number of edges in this graph, and call this quantity F. An easy way to calculate F is to take the sum over all points x, count how many other points are within distance 1, and divide by 2. If two points x and y are at distance 2 or more, they do not share any neighbours and have no influence on each other. In a sense, F is thus the sum of local contributions, since any point x only counts what happens within distance 1 of itself.

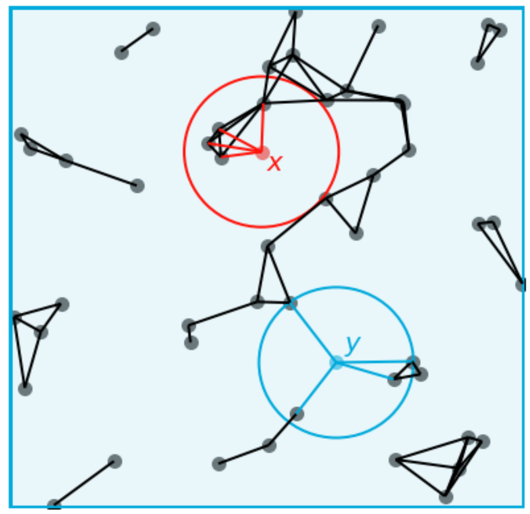


Figure 1: Gilbert Graph with two additional points x and y, far enough apart not to influence each other

To study this type of behaviour in more generality, we introduce the concept of an **add-one cost operator** $\mathrm{D}_x F$. This is the amount of change happening to the functional when we add the point x to the collection of points. In our example, this is just the number of points within distance 1 of x. If the quantity $\mathrm{D}_x F$ is in a certain sense locally determined, we call F **stabilising**, since enlarging the space around the point x eventually won't change $\mathrm{D}_x F$, as this quantity is determined within a small ball around x.

The add-one cost can be iterated, namely we look at the change happening to $\mathrm{D}_x F$ when adding a point y. This is called the **iterated add-one cost** and denoted by $\mathrm{D}^{(2)}_{x,y} F$. Since $\mathrm{D}_x F$ is locally determined, if y is quite far from x, it will not have an influence on $\mathrm{D}_x F$ and thus $\mathrm{D}^{(2)}_{x,y} F = 0$. In our example, if y is outside of $B(x,1)$, then the change to F when adding x is still equal to the number of points inside $B(x,1)$ and thus $\mathrm{D}^{(2)}_{x,y} F = 0$. If however $y \in B(x,1)$, then the change when adding x is given by $(\#B(x,1)+1)$ (where $\#B(x,1)$ is the number of points within distance 1 of x), since there is now a new point y inside the ball. In this case $\mathrm{D}^{(2)}_{x,y} F = (\#B(x,1)+1) - \#B(x,1) = 1$.

Stabilising (and translation invariant) functionals are interesting because they are in a sense akin to sums of weakly dependent, identically distributed random variables. A stabilising functional F consists of the sum over all points of each small local contribution of a point, and two contributions are close to independent if the corresponding points are far apart. This means that if a functional is 'sufficiently stabilising', its fluctuations should be normal just like those of a sum of i.i.d. random variables in the standard CLT. The theory of stabilisation started with groundbreaking work by Kesten and Lee [KL96], and was developed further in the seminal contributions by Penrose and Yukich (e.g. [PY01, PY03, Pen05, PY05]).

In this work, we use a method that allows to prove quantitative central limit theorems just by estimating moments of the first and second add-one costs $\mathrm{D}_x F$ and $\mathrm{D}^{(2)}_{x,y} F$. Once these bounds are given and a suitable lower bound for the variance has been found, a central limit theorem automatically follows.

To illustrate this with our example, consider a Poisson measure with unit intensity on the ball $B(0,t) \subset \mathbb{R}^d$ and define the functional F_t as the number of edges in the Gilbert Graph of parameter 1. Then one can show, using the methods in this work, that

$$d\left(\frac{F_t - \mathbb{E}F_t}{\text{Var}(F_t)}, N\right) < ct^{-d/2},$$

for the so-called Wasserstein and Kolmogorov distances, thereby giving a quantitative CLT.

In many cases, the stabilizing behaviour is not as nice as in our illustrating example. It is possible to have a heavy-tailed distance distribution of how large the influence of a point is, or even a distribution that depends on the point and cannot be uniformly estimated. The present work is tailor-made for such cases. In fact, in previous works (see e.g. [LPS16]), the moment assumptions on the add-one costs asked for a $4+\epsilon$ moment, whereas we only ask for a $2+\epsilon$ moment. These optimal moment conditions have far-reaching consequences for applications.

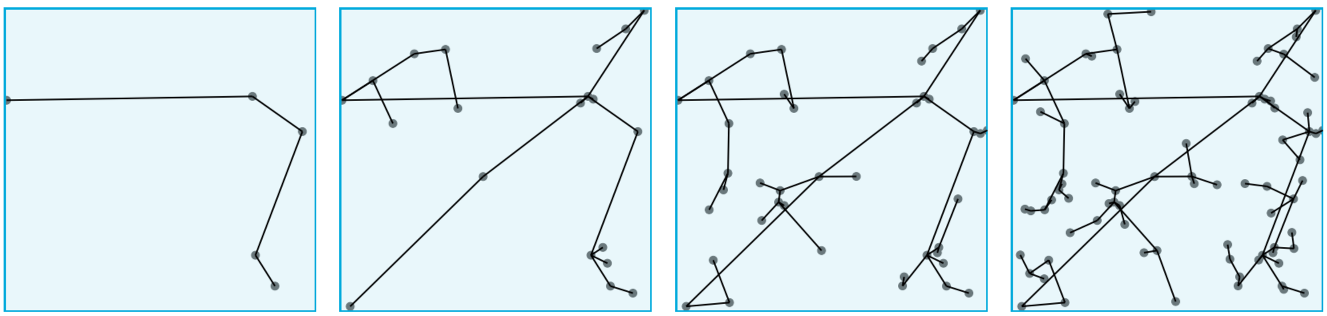


Figure 2: A sequentially growing ONNG

In the scope of this thesis, we look at various graphs built on Poisson measures, and the functionals of interest are sums of power-weighted edge-lengths. The most prominent example of these is the **Online Nearest Neighbour Graph** (ONNG). It arises as a limit case of several models, among them the FKP-model [FKP02] for the internet, and it is a simple model for a network growing in time. Consider a collection of points where the points arrive sequentially. Every time a point arrives, it connects to the nearest neighbour which is already there. In this way, early points have very long edges, because they have few neighbours to chose from, while late points have short edges, since they have a lot of potential neighbours. If the functional we consider is the sum of all edge-lengths, this means that the contribution of each point (i.e. the distance to its closest neighbour) depends on its arrival time. The add-one costs are likewise not uniformly bounded and depend heavily on the time of insertion of an additional point x. In the paper [Wad09], the author conjectured a CLT for the fluctuations of this functional, however no solution has been proposed since. With the help of the minimal moment conditions developed in Theorems 4.2 and 4.3 below, we derive a quantitative CLT for the ONNG in Theorem 6.1.

We should also point out that applications are not limited to graphs — the results of the author's preprint [Tra22] have been applied in [BZ23, BXZ23] in order to derive central limit theorems for solutions to stochastic differential equations driven by Lévy white noise.

In Section 4.2, we study multivariate extensions of the method explained above. Indeed, it might be the case that one needs results about the joint behaviour of several functionals of the same point set. Again, we derive quantitative central limit theorems under minimal moment assumptions on first and second add-one costs of the components of the multivariate functional in question. As applications, we study two families of multivariate functionals of the Gilbert Graph.

The underlying theory leading to our minimal moment conditions discussed above consists of new abstract functional inequalities which we believe to be of independent interest. We show new bounds for moments of Poisson functionals, and for functions of so-called Skorohod integrals. The method used is a novel version of Itô formula for Poisson processes with anticipative integrands.

Plan of the Thesis

We start this thesis with an introduction split into two parts. The first part, Chapter 1, is all about background material — here we talk about the history of the subject area, existing results, while giving some examples. In the second part of the introduction, Chapter 2, we give an overview of our main results, together with a discussion of the surrounding literature. Within the introduction, you will find 'Roadmaps' directing you to related parts of the thesis, be it for further background, more precise statements of results, or proofs. You might also encounter 'Tangents' — these are infoboxes about somewhat related, but not directly relevant material, which seems nonetheless of interest to the author.

In Chapter 3, we explain the framework and give some notation which will become necessary for the later statements of main results.

The main body of the thesis starts in Chapter 4, where we give in all details our new bounds on various probability distances, with the univariate case (Wasserstein and Kolmogorov distances) being treated in Section 4.1 and the multivariate case (d_2 and d_3 distances) being exhibited in Section 4.2. In Chapter 5 we present our novel results on Poisson stochastic calculus (see Section 5.1) and functional inequalities (see Section 5.2). New CLTs for a collection of univariate functionals can be found in Chapter 6, with the Online Nearest Neighbour Graph, the Gilbert Graph, the k-Nearest Neighbour Graphs and the Radial Spanning Tree each heading their respective section. We present multivariate applications in Chapter 7, where we inspect different functionals of Gilbert Graphs.

Chapter 8 collects all relevant results on Poisson Malliavin calculus, while the final Chapter 9 contains all the proofs.

1 Introduction: Background

In this chapter, we provide an overview of historical results, existing methods and research related to the results exhibited in this thesis. The impatient reader may also skip ahead directly to the main contributions of the author, presented in Chapter 2. We will start with the fundamental idea of the classical Central Limit Theorem, move on to second-order Poincaré inequalities and talk about the Malliavin-Stein method in both Gaussian and Poisson space, while discussing examples all along.

1.1 Central Limits and the Berry-Esseen Theorem

One of the most classical results in probability theory is the Central Limit Theorem.

Central Limit Theorem [Bil95, Thm. 27.1]

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. (independent identically distributed) random variables such that $\mathbb{E}X_i = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Define the sum S_n as

$$S_n := \sum_{i=1}^{n} X_i.$$

Then

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{n\to\infty} \mathcal{N}(0,1),$$

where the convergence is in distribution and $\mathcal{N}(0,1)$ denotes the standard normal law.

This result is universal, in the sense that it holds regardlessly of the original distribution of the X_i, which is one of its strengths. It explains why so many quantities observed in nature and society tend to follow a bell-shaped normal distribution and it is of paramount importance in statistics, probability, modelling and many other fields.

Tangent 1.1: The origins of the CLT

According to [CGS11], the origins of the Central Limit Theorem stem from work by De Moivre in 1733. De Moivre was at the time out of an academic job and instead earned money as a consultant for probability and gambling. The problem he was faced with was the following: given a sequence $X_1, X_2, \ldots, X_n$ of independent Bernoulli trials with success probability p, what is the probability of having fewer than m successes? This translates to calculating $\mathbb{P}(S_n \leq m)$, where $S_n = X_1 + \ldots + X_n$. The success probability can be given in closed form, namely it is equal to

$$\mathbb{P}(S_n \leq m) = \sum_{i=1}^{n} \binom{n}{k} p^i (1-p)^{n-i}. \tag{1.1}$$

This expression is however hard to evaluate, even when n is small. De Moivre noticed that it can be approximated quite well by the probability $\mathbb{P}(Z_n \leq m)$, where Z_n is a normally distributed random variable with mean np and variance $np(1-p)$.

A question the Central Limit Theorem does not answer is this: how large does the sample size have to be until the approximation by a normal law is good? This is where the Berry-Esseen theorem comes in. It was discovered independently by Andrew C. Berry in 1941 and Carl-Gustav Esseen in 1942 [Ber41, Ess42] and it provides a quantitative bound on the distance between the normalised mean and a standard Gaussian, under some moment assumptions on the random variables.

Berry-Esseen Theorem [Pet75, Thm. V.4]

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that $\mathbb{E}X_1 = \mu$ and $\mathrm{Var}(X_1) = \sigma^2$. Assume that $\mathbb{E}|X_1|^3 =: \rho < \infty$ and define the sum S_n as before. There is a universal constant $C > 0$ such that

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq x \right) - \mathbb{P}(N \leq x) \right| \leq \frac{C\rho}{2\sigma^3\sqrt{n}},$$

where $N \sim \mathcal{N}(0,1)$ is a standard Gaussian.

The precise value of the constant C is not known. The best currently known upper bound is 0.4748 (see [She14]), which is an improvement on Esseen's original bound of 7.59. A lower bound of 0.4097 has been established by Esseen in 1956 (see [Ess56]).

The distance we have used here is the so-called **Kolmogorov distance**, defined as

$$d_K(F,G) := \sup_{x \in \mathbb{R}} |\mathbb{P}(F \leq x) - \mathbb{P}(G \leq x)| \tag{1.2}$$

for random variables F, G. Convergence in Kolmogorov distance is equivalent to convergence in supremum norm of the cumulative distribution functions, and is, in general, stronger than convergence in distribution[1]. As such, it is one of the most natural distance functions to look at.

[1]Convergence in d_K distance is *equivalent* to convergence in distribution whenever the limit random variable G is such that $x \mapsto \mathbb{P}(G \leq x)$ is continuous, see [NP12, Prop. C.3.2].

The classical CLT and the Berry-Esseen theorem have been studied and generalised in many ways. One of the most well-known extensions is the Lindeberg condition for sequences of independent random variables with possibly different distributions. It can be shown that this condition is sufficient for a CLT to hold and necessary under some weak additional assumptions (see [Bil95, Thm. 27.2]).

For sequences of weakly dependent random variables, no general result exists. Many different results cover various cases, for instance the study of mixing coefficients (see e.g. [Bil95, Thm. 27.4] or [Bra07]). The quantities studied in our work can be interpreted in some way as sums of weakly dependent variables, and we are interested in deriving quantitative CLTs for them. The approach we choose, and which has proven successful in countless earlier works, is Stein's method.

1.2 Stein's Method

Stein's Method for normal approximations is a fundamental tool for the study of quantitative CLTs. The starting point is the following observation made by Stein in 1972 and developed further in 1986, see [Ste72, Ste86]:

A real-valued random variable N is standard Gaussian if and only if

$$\mathbb{E}f(N)N = \mathbb{E}f'(N) \tag{1.3}$$

for a suitably large collection of functions $f : \mathbb{R} \to \mathbb{R}$.

Adequate collections of functions are for example the set of absolutely continuous functions for which the above expressions are well-defined. Heuristically, it should now be true that if $\mathbb{E}f(Z)Z - \mathbb{E}f'(Z)$ is close to zero for some real-valued variable Z and functions f in some large set, the distribution of Z should be 'close to Gaussian' in a certain sense to be defined.

This heuristic argument can be made precise when considering a distance between the distributions of a random variable F and a standard Gaussian N. Let us define the distance $d_{\mathcal{H}}$ as

$$d_{\mathcal{H}}(F, N) := \sup_{h \in \mathcal{H}} |\mathbb{E}h(F) - \mathbb{E}h(N)| , \tag{1.4}$$

where $\mathcal{H}$ is a sufficiently large collection of functions depending on the choice of distance. To achieve the Kolmogorov distance, for instance, one can choose $\mathcal{H} = \{\mathbb{1}_{(-\infty,x]} : x \in \mathbb{R}\}$. For each $h \in \mathcal{H}$, we introduce a differential equation:

$$f'(x) = xf(x) + h(x) - \mathbb{E}h(N). \tag{1.5}$$

If we call f_h the **canonical solution**[2] to this equation, we can rewrite $d_{\mathcal{H}}(F, N)$ as

$$d_{\mathcal{H}}(F, N) = \sup_{h \in \mathcal{H}} |\mathbb{E}f_h(F)F - \mathbb{E}f_h'(F)|. \tag{1.6}$$

Note that there is no N any more on the right hand side – the fact that we are comparing F to a standard Gaussian is encoded in the solution f_h and in particular in the choice of functional

[2] The family of solutions to (1.5) is given by $x \mapsto ce^{x^2/2} + e^{x^2/2} \int_{-\infty}^{x} (h(y) - \mathbb{E}[h(N)])\, e^{-y^2/2} dy$, with $c \in \mathbb{R}$. The canonical solution corresponds to the choice of $c = 0$.

equation (1.5). Depending on the choice of set $\mathcal{H}$, the solutions to the equation (1.5) usually verify useful differentiability and boundedness properties, which simplify finding a bound on $d_{\mathcal{H}}(F, N)$.

Stein's Method has proven to be extremely powerful. There are scores and scores of papers, surveys and books that have been written about the topic — for a selection, see [BC05, CGS11, Ros11, NP12, Cha14], and [Swa23] for a webpage keeping book about the various publications connected to Stein's method. A crucial step after having done the transformation (1.6) is to bound the new right hand side of (1.6). Many different methods have been developed, applicable in different contexts: exchangeable pairs, size-bias coupling, dependency neighbourhoods, zero-bias coupling, and others. Our method of choice is the Malliavin-Stein method, see Section 1.4 and thereafter.

Tangent 1.2: Stein's Method for non-Normal Approximations

Stein's method has been used not only for Gaussian approximation, but for a large number of different distributions. The most prominent variant of Stein's method is perhaps the Chen-Stein method [Che75], developed by Louis Chen after finishing his PhD under the supervision of Stein, shortly after the publication of [Ste72]. Here the limit distribution is the Poisson distribution and the characterising property is as follows: a random variable Z is Poisson distributed with parameter $\lambda > 0$ if and only if

$$\mathbb{E}Zf(Z) = \lambda\mathbb{E}f(Z+1) \tag{1.7}$$

for a sufficiently large collection of functions f. Applications are numerous, see for example [AGG90, SCM05, Ros11]. Beyond this, several general methods exist for finding new Stein operators for distributions which have not been studied yet, e.g. the density approach, the generator method and coupling equations (see [Rei05], [CGS11, Chapter 13] and [LRS17]).

1.3 Second-Order Poincaré Inequalities

Our goal in this work is to derive so-called **second-order Poincaré inequalities**. The term was coined by Chatterjee in [Cha09] as an analogy to the well-known Chernoff-Nash-Poincaré inequality (see [Che81, Nas58]) on the Gaussian space: if $X = (X_1, \ldots, X_d)$ is a standard Gaussian vector with identity covariance matrix, and $f : \mathbb{R}^d \to \mathbb{R}$ a smooth function, then

$$\mathrm{Var}(f(X)) \leq \mathbb{E}\|\nabla f(X)\|^2. \tag{1.8}$$

This inequality bounds 'first-order fluctuations' of $f(X)$ in terms of first-order derivatives of f. Motivated by problems in random matrix theory, Chatterjee asked if it was possible, using second-order derivatives of f, to show that $f(X)$ was approximately Gaussian, i.e. giving a statement about 'second-order fluctuations' of $f(X)$. Using Stein's method, he provided the following result in [Cha09]:

Theorem 1.3 [Cha09, Thm.2.2]

Let $X = (X_1, \ldots, X_d)$ be a standard Gaussian vector and $f : \mathbb{R}^d \to \mathbb{R}$ a twice continuously differentiable function. Then

$$d_{TV}(f(X), Z) \leq \frac{2\sqrt{5}}{\mathrm{Var}(f(X))} \cdot \left(\mathbb{E}\|\nabla f(X)\|^4\right)^{1/4} \cdot \left(\mathbb{E}\|Hess(f(X))\|_{op}^4\right)^{1/4}, \tag{1.9}$$

where $\|.\|_{op}$ denotes the operator norm and Z is a normal random variable with the same mean and variance as $f(X)$.

In the previous statement, the distance d_{TV} is the so-called **total variation distance**, defined as

$$d_{\mathrm{TV}}(F, G) = \sup_{A \subset \mathbb{R}} \left|\mathbb{P}(F \in A) - \mathbb{P}(G \in A)\right|, \tag{1.10}$$

where the supremum is taken over all Borel-measurable subsets $A \subset \mathbb{R}$.

What one should note is that the right hand side of (1.9) contains only:

- the variance of $f(X)$;
- a first-order difference operator of $f(X)$, namely $\|\nabla f(X)\|$;
- a second-order difference operator of $f(X)$ in the form of $\|\mathrm{Hess}(f(X))\|_{op}$.

These three quantities are enough to prove convergence to normality of $f(X)$. This is a recurring theme with second-order Poincaré inequalities — they usually only require knowledge of the variance, a first and a second-order difference quantity of the random variable to be approximated, in order to achieve quantitative central limit results.

1.4 Malliavin-Stein Method on the Gaussian Space

The first combination of Stein's method with Malliavin calculus appeared in 2009 in a paper by Nourdin and Peccati [NP09]. Here, the authors established probabilistic bounds for elements of the Wiener chaos by bounding the right hand side of (1.6) using Malliavin operators and relations.

Malliavin calculus is a calculus of variations for random objects. Originally introduced in a Gaussian setting by Paul Malliavin [Mal78], versions of it now exist for various types of distributions, e.g. for Gaussian processes [Nua06, NP12] and Poisson spaces [NV90, Pri94], Rademacher random variables [PS02, Pri08], Lévy processes [NS00] and sequences of independent random variables [DH19]. The essential common points of all these different frameworks are

- the existence of a difference operator D, the Malliavin derivative;
- an integration by parts formula, e.g.

$$\mathbb{E}\langle DF, G\rangle = \mathbb{E}F\delta(G), \tag{1.11}$$

  where δ is most commonly called the Skorohod integral.

For a comparison between different types of Malliavin calculus, we refer to the PhD thesis [Hal20].

In the paper [NP09], the authors combined Malliavin calculus with Stein's method in order to prove the first quantitative equivalent to the **fourth moment theorem** (see [NP05] for the first occurrence of the fourth moment theorem). The inequality in [NP09] was further refined in [NPR09] and we state it in its final form taken from [NP12]:

Theorem 1.4 [NP12, Thm. 5.2.6]

Let $q \geq 2$ be an integer, and let $F = I_q(f)$ have the form of a multiple Wiener integral of order q such that $\mathbb{E}F^2 = 1$. Then, for $N \sim \mathcal{N}(0,1)$,

$$d_{TV}(F, N) \leq 2\sqrt{\mathbb{E}F^4 - 3}. \tag{1.12}$$

The functional F here is an element of the q^{th} Wiener chaos – we refer to the monograph [NP12] for a self-contained and detailed introduction to the Malliavin-Stein method on the Gaussian space.

The combination of Malliavin Calculus with Stein's method proved to be very fruitful and has led to an entirely new direction of research and an enormous amount of publications, not only in the Gaussian space, but in many other domains as well. We refer to [Nou22] for an updated list of well over 500 related contributions and [APY21, Dec22] for recent surveys.

The paper by Nourdin, Peccati and Reinert [NPR09] took the method one step further and showed a result that holds for general functionals of isonormal Gaussian processes.

Theorem 1.5 [NPR09, Thm. 1.1]

Let X be an isonormal Gaussian process over some real separable Hilbert space $\mathbf{H}$ and let $F = F(X)$ be a twice differentiable (in a Malliavin sense) functional. Then

$$d_W(F, Z) \leq \frac{\sqrt{10}}{2\,\text{Var}(F)} \mathbb{E}\left[\|\mathrm{D}^{(2)} F\|_{op}^4\right]^{1/4} \mathbb{E}\left[\|\mathrm{D}\, F\|_{\mathbf{H}}^4\right]^{1/4}, \tag{1.13}$$

where Z is a Gaussian with the same mean and variance as F, the norm $\|.\|_{\mathbf{H}}$ denotes the norm in the Hilbert space $\mathbf{H}$ and D (resp. $\mathrm{D}^{(2)}$) is the first (resp. second) Malliavin derivative operator.

Here d_W is the **Wasserstein distance**, defined as

$$d_W(F, G) = \sup_{h \text{ 1-Lipschitz}} |\mathbb{E}h(F) - \mathbb{E}h(G)|, \tag{1.14}$$

where the supremum is taken over all 1-Lipschitz continuous functions $h : \mathbb{R} \to \mathbb{R}$.

This result is an infinite-dimensional extension of Theorem 1.3 and very powerful in its generality. It suffices to control the variance and the first and second difference operators of a functional F in order to derive a central limit theorem. In practice, it is often very convenient to bound the first and second order difference operators. However, the quantity $\|\mathrm{D}^{(2)} F\|_{op}$ is an operator norm and in some applications hard to estimate, with the resulting speeds of convergence often not being optimal. A further work by Vidotto removes this obstacle:

Theorem 1.6 [Vid20, Thm. 1.3]

Let $\mathbf{H} = L^2(A, \mathcal{A}, \mu)$*, where* $(A, \mathcal{A})$ *is a Polish space endowed with its Borel* σ*-field and* μ *is a positive,* σ*-finite, non-atomic measure. Let* X *be an isonormal Gaussian process over* $\mathbf{H}$ *and* $F = F(X)$ *a twice Malliavin-differentiable functional such that* $\mathbb{E}F = 0$*. Then*

$$d_{TV}(F, Z) \leq \frac{2\sqrt{3}}{\mathrm{Var}(F)} \left(\int_{A \times A} \mathbb{E}\left[\left(\left(\mathrm{D}^{(2)} F \otimes_1 \mathrm{D}^{(2)} F \right)(x, y) \right)^2 \right]^{1/2} \mathbb{E}\left[(\mathrm{D}\, F(x)\, \mathrm{D}\, F(y))^2 \right]^{1/2} \mu(dx)\mu(dy) \right)^{1/2}, \quad (1.15)$$

where Z *is a centred Gaussian with the same variance as* F*.*

Here the difference operators appear as pointwise functions and can be bounded immediately. The proof strategies in [Vid20] were inspired by an adaptation of the Malliavin-Stein method to the Poisson space, which we will explain in the next section.

1.5 Malliavin-Stein Method on the Poisson Space

On the Poisson space, we are interested in a Poisson random measure η on some space $\mathbb{X}$, with some σ-finite intensity measure λ. On such a measure space $(\mathbb{X}, \lambda)$, the random point process η can be a.s. identified with a random collection of points, which allows for a very concrete geometric interpretation (see [LP18]).

The Poisson measure holds a very important position in applications, e.g. in telecom networks and queuing theory [BB09a, BB09b, DM12], in astronomy [GF96, Hil12], in continuum percolation [Gri99, BR06], for Lévy processes [BNRM01], in statistical estimation [JY11], etc.

Poisson functionals play an important role in **Stochastic Geometry**, e.g. as functionals of graphs built on Poisson input [Gil61, Pen03, PW08a, ST17], of Voronoi tessellations [Møl94, CCE20, PS22, GKT23], of Poisson polyhedra or convex bodies [HHRT15, GT21] and more.

In order to apply the Malliavin-Stein method on the Poisson space, one needs a Malliavin calculus on the Poisson space. Different versions have been proposed, compare e.g. [NV90] and [Pri94]. The version we are using is close to the one of [NV90] and a comprehensive introduction can be found e.g. in [PR16]. In this version, the Malliavin derivative is a simple **add-one cost operator**:

$$\mathrm{D}_x F(\eta) := F(\eta \cup \{x\}) - F(\eta), \quad (1.16)$$

for any measurable functional F of a Poisson measure η. This object has a very geometric interpretation, since it is the difference of the functional F when we add an additional point x. See Example 1.7 for an illustration.

Example 1.7: Add-one costs for the Radial Spanning Tree

Consider a Poisson measure η of Lebesgue intensity on a square centred at $(0, 0)$. Each point connects to its nearest neighbour among all those points with smaller norm, including the point $(0, 0)$. The ensuing graph is called the **Radial Spanning Tree** (see [BB07, ST17] and Section 6.4). Assume F is a functional of this graph, e.g. the sum of all edge-lengths.

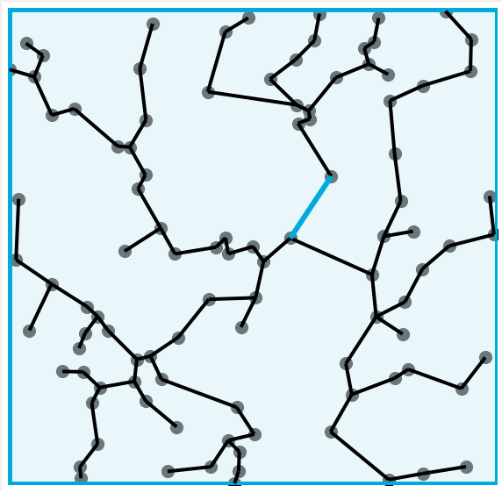


Figure 1.1: Radial Spanning Tree on the original point set

Figure 1.2: Radial Spanning Tree with an additional red point **x**

Consider now Figures 1.1 and 1.2. In Figure 1.1, the Radial Spanning Tree is constructed on the point set η. In Figure 1.2, we have introduced a new point **x** to the point set and the graph looks slightly different: two new red edges have been added, and the blue edge from Figure 1.1 has disappeared. This is due to two changes that happen: the point **x** connects to its nearest radial neighbour, and the outer point of the blue edge is closer to **x** than to its nearest neighbour in η, hence at the addition of **x**, it now connects to **x** instead of its previous neighbour.
For the sum of all edge-lengths F, this means the following: the add-one cost of F is the change induced to F by the addition of the point **x**, namely

$$\mathrm{D}_{\mathbf{x}} F = F(\eta \cup \{\mathbf{x}\}) - F(\eta) = \sum \text{length}(\text{—}) - \sum \text{length}(\text{—}). \tag{1.17}$$

Hence $\mathrm{D}_{\mathbf{x}} F$ is the sum of the lengths of the new red edges, minus the lengths of the lost blue edges.

The first application of the Malliavin-Stein method to the Poisson space can be found in [PSTU10]. The authors derived a second-order Poincaré inequality for Poisson functionals, where the bound involves only quantities of the type $\mathrm{D}\, F$ and $\mathrm{D}\,\mathrm{L}^{-1} F$.

Theorem 1.8 [PSTU10, Thm. 3.1]

Let F be a Poisson functional such that $\mathbb{E}F = 0$, $\mathbb{E}F^2 = 1$ and $\mathrm{D}\, F$ is square-integrable. Then

$$d_W(F, N) \leq \mathbb{E}\left|1 - \int_{\mathbb{X}} \mathrm{D}_x F \cdot (-\mathrm{D}_x \mathrm{L}^{-1} F)\lambda(dx)\right| + \mathbb{E}\int_{\mathbb{X}} (\mathrm{D}_x F)^2 \cdot \left|\mathrm{D}_x \mathrm{L}^{-1} F\right| \lambda(dx), \tag{1.18}$$

where $N \sim \mathcal{N}(0,1)$ is a standard Gaussian.

The operator L^{-1} is the inverse of the Ornstein-Uhlenbeck generator, a Malliavin operator (see (8.21)). This result, like the previous ones on second-order Poincaré inequalities, requires knowledge of the variance (to rescale the functional), and of two difference operators. It opened the door towards a study of quantitative central limit theorems on the Poisson space, for functionals whose add-one costs were tractable and was extended soon after in [ET14, LPS16, Sch16, LRPY22]. The bound (1.18) still contains terms depending on L^{-1} however, which implies there are quantities that are not always easily computable. This problem was addressed in [LPS16], when the authors combined the above bound with the Poisson Poincaré inequality and a Mehler-type formula for L^{-1}. The Poincaré inequality is the equivalent to the Chernoff-Nash-Poincaré inequality on the Gaussian space:

$$\mathrm{Var}(F) \leq \mathbb{E} \int_{\mathbb{X}} (\mathrm{D}_x F)^2 \lambda(dx), \tag{1.19}$$

whenever $F = F(\eta)$ is a square-integrable Poisson functional. The resulting bound in [LPS16] contains only moments and products of the first and second-order add-one costs $\mathrm{D} F$ and $\mathrm{D}^{(2)} F$, where the operator $\mathrm{D}^{(2)}$ is defined as

$$\mathrm{D}^{(2)}_{x,y} F = \mathrm{D}_x F(\eta \cup \{y\}) - \mathrm{D}_y F(\eta). \tag{1.20}$$

Theorem 1.9 [LPS16, Thm. 1.1]

Let $F = F(\eta)$ be a Poisson functional such that $\mathbb{E}F = 0$, $\mathbb{E}F^2 = 1$ and $\mathrm{D} F$ is square-integrable. Then

$$d_W(F, N) \leq \gamma_1 + \gamma_2 + \gamma_3, \tag{1.21}$$

where

$$\gamma_1 := 2 \left(\int \left[\mathbb{E} \, (\mathrm{D}_x F)^2 \, (\mathrm{D}_y F)^2 \right]^{1/2} \mathbb{E} \left[\left(\mathrm{D}^{(2)}_{x,z} F \right)^2 \left(\mathrm{D}^{(2)}_{y,z} F \right)^2 \right]^{1/2} \lambda^{(3)}(dx, dy, dz) \right)^{1/2}$$

$$\gamma_2 := \left(\int \mathbb{E} \left(\mathrm{D}^{(2)}_{x,z} F \right)^2 \left(\mathrm{D}^{(2)}_{y,z} F \right)^2 \lambda^{(3)}(dx, dy, dz) \right)^{1/2}$$

$$\gamma_3 := \int \mathbb{E} | \mathrm{D}_x F |^3 \lambda(dx).$$

A similar bound exists for the Kolmogorov distance $d_K(F, N)$ (see [LPS16, Thm. 1.2]). Related bounds can be found in [ET14, Sch16, LRPY22]. While this bound might not seem tractable at first glance, the add-one cost operators are often easy to estimate and well-behaved, and they have a geometric interpretation in the Poisson space, as illustrated in Example 1.7. Applications of this bound have been numerous (see e.g. [FY20, Gry20, TL20, ABD22, AGY22, OBA22] for a selection of just the most recent applications). Theorem 1.9 seems particularly amenable to functionals exhibiting a type of stabilisation, a concept which will be made precise in the next section.

Tangent 1.10: The Fourth Moment Theorem on the Poisson Space

One of the consequences of the new methods developed in [PSTU10, LPS16] and others was the establishment of a fourth moment bound akin to Theorem 1.4 for the Poisson space. Such a bound had been elusive for years — yet in the paper [DP18], the authors achieve a quantitative central limit theorem for elements of the Poisson-Wiener chaos, which requires only the convergence of the fourth moment.

Theorem 1.11 [DP18, Thm. 1.3]. *Let $q \geq 1$ and let $F = I_q(f)$ be a multiple Wiener-Itô integral with respect to the compensated Poisson measure $\hat{\eta}$. Assume $\mathbb{E}F^2 = 1$. Under some further integrability assumptions, we have*

$$d_W(F, N) \leq \left(\sqrt{\frac{2}{\pi}} + 2 \right) \sqrt{\mathbb{E}F^4 - 3}, \tag{1.22}$$

where $N \sim \mathcal{N}(0, 1)$ is a standard Gaussian.

For an explanation what the Wiener-Itô integrals with respect to a compensated Poisson measure are, we refer to [Las16]. Note that, in this case, we always have $\mathbb{E}F^4 > 3$. This implies that no element of the q^{th} Wiener chaos with respect to a Poisson measure can be Gaussian.

1.6 Stabilisation

Bounds of the type shown in Theorem 1.9 are highly flexible and particularly adapted for dealing with functionals displaying a form of **geometric stabilisation**. The first contribution on this topic was the paper [KL96], where the idea of geometric stabilisation was developed to show a central limit theorem for sums of weighted edge-lengths for the minimal spanning tree. The concept was expanded in [Lee97, Lee99] by Lee, and substantially generalised in the works [PY01, PY03, Pen05, PY05] by Penrose and Yukich. Precise definitions differ, but the idea of strong stabilisation is the following: assume we work on $\mathbb{R}^d$ and have a functional $F = F(\chi)$, where $\chi \subset \mathbb{R}^d$ are locally finite subsets. We call F **strongly stabilising** (see e.g. [PY01, Def. 2.1]) if there exists for every $x \in \mathbb{R}^d$ an a.s. finite radius $R(x, \chi) > 0$ such that

$$\mathrm{D}_x F(\chi) = \mathrm{D}_x F(\chi \cap B(x, R(x, \chi))), \tag{1.23}$$

where $B(x, r)$ denotes the ball of centre x and radius r. In essence, this means that the change induced by adding the point x to the point configuration χ is limited to the inside of a ball, i.e. the change is local.

Example 1.12: Radius of Stabilisation for the Radial Spanning Tree

We return to the example of the radial spanning tree exhibited in Example 1.7. In Figure 1.3, red edges are the edges added when adding the red point **x**, the dotted blue edge is an edge that is lost when adding **x**. The disc in a darker shade of blue represents the ball of stabilisation radius around **x** — it is the ball inside which all changes take place.
The radius of stabilisation in this case can be taken to be the maximum of:

- the distance from **x** to its radial nearest neighbour;
- the distance from **x** to any point it is a radial neighbour of;
- the distance from **x** to any point that lost a neighbour because **x** is closer.

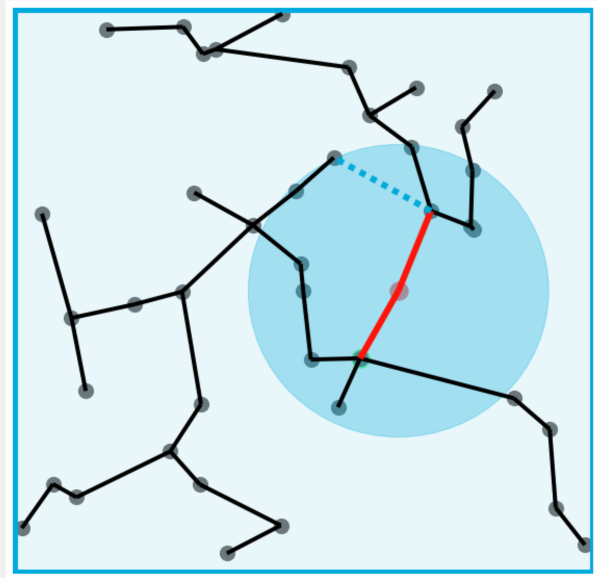


Figure 1.3: A radius of stabilisation for the red point **x**

The existence of a radius of stabilisation can have two important consequences:

1. Assume that a strongly stabilising functional of interest F_t depends on the restriction of an $\mathbb{R}^d$-Poisson measure η to a set H_t of size growing with $t \geq 1$. Heuristically speaking, for each $x \in \mathbb{X}$, we can choose t large enough such that the ball $B(x, R(x, \eta))$ will be inside H_t, and hence any change induced by x is independent of the size of the surrounding set. To make this precise, some further technical assumptions are needed. When these are fulfilled, then one has
$$\mathrm{D}_x F_t \overset{t\to\infty}{\longrightarrow} \Delta_x \quad \text{a.s.}, \tag{1.24}$$
where Δ_x is some limiting random variable. For the bound (1.21), this usually means that moments of $\mathrm{D}_x F_t$ can be bounded independently of t. Functionals satisfying only (1.24) are called **weakly stabilising** (see e.g. [PY01, Def. 3.1]).

2. Given a radius of stabilisation, one can find conditions under which $\mathrm{D}^{(2)}_{x,y} F = 0$. Assume that $|x-y| > R(x,\eta)$ implies $R(x, \eta \cup \{y\}) = R(x,\eta)$, i.e. that the addition of a point y far enough from x does not change the radius of stabilisation of x. Then $|x-y| > R(x,\eta)$ implies that $\mathrm{D}^{(2)}_{x,y} F = 0$. Indeed, one has
$$\begin{aligned}\mathrm{D}^{(2)}_{x,y} F &= \mathrm{D}_x F(\eta \cup \{y\}) - \mathrm{D}_x F(\eta)\\ &= \mathrm{D}_x F((\eta\cup\{y\}) \cap B(x, R(x, \eta\cup\{y\}))) - \mathrm{D}_x F(\eta \cap B(x, R(x,\eta)))\\ &= \mathrm{D}_x F((\eta\cup\{y\}) \cap B(x, R(x,\eta))) - \mathrm{D}_x F(\eta \cap B(x, R(x,\eta)))\\ &= \mathrm{D}_x F(\eta \cap B(x, R(x,\eta))) - \mathrm{D}_x F(\eta \cap B(x, R(x,\eta)))\\ &= 0\end{aligned}$$
For the bound (1.21), this observation implies that in many cases, $\mathrm{D}^{(2)}_{x,y} F = 0$ whenever x and y are far apart.

Within the concept of strong stabilisation, one speaks of **exponential stabilisation** if the radius of stabilisation has exponentially decreasing tails, i.e. if

$$\mathbb{P}(R(x,\eta) > u) \leq \exp(-c\,u^a), \tag{1.25}$$

for some $c, a > 0$ (see [Sch09, Section 4.2.1]). If in addition the first add-one cost $\mathrm{D}_x F$ is bounded in terms of $R(x,\eta)$, then $\mathrm{D}_x F$ has moments of all order and the bound (1.21) collapses nicely.

Assumptions on stabilisation and on moments of the first add-one cost are enough to show a central limit theorem.

Theorem 1.13 [PY01, Thm. 2.1]

Let η be a Poisson functional on $\mathbb{R}^d$ of Lebesgue intensity and let H be a translation invariant, strongly stabilizing functional s.t. the 4^{th} moment of its add-one cost is uniformly bounded. For $n \geq 1$, let $B_n \subset \mathbb{R}^d$ form a (sufficiently nice) sequence of subsets of volume n.
Then there is a constant $\sigma \geq 0$ such that as $n \to \infty$,

$$n^{-1}\,\mathrm{Var}\left(H(\eta_{|B_n})\right) \to \sigma^2 \tag{1.26}$$

and

$$n^{-1/2}\left(H(\eta_{|B_n}) - \mathbb{E}H(\eta_{|B_n})\right) \xrightarrow{d} \mathcal{N}(0,\sigma^2). \tag{1.27}$$

For a precise definition of the above assumptions on H and B_n, we refer to [PY01]. The paper [Pen05] provides a multivariate extension of this result.

A closely related form of stabilisation is the so-called **score-stabilisation**, where the functionals to be studied are sums of 'score functions', which exhibit a similar kind of stabilising behaviour. Early works on score stabilisation include [PY05, PW08b] and recent works are given by [LRSY19, SY23]. A selection of other works relating to stabilisation is given by [LPS16, SY19, Tri19, CRX21, LRPY22, SBP22, CX23].

In this work, we will occasionally reference stabilisation, or radii of stabilisation, but we will not assume any stabilising properties for our results. Instead, our bounds rely on the closely related properties of the first and second add-one cost operators in order to derive CLTs.

1.7 Badly Behaved Examples

In this section, we will talk about three examples that exhibit stabilisation, but are not as nicely behaved as the ones mentioned in Section 1.6. Each of these examples is treated in detail in this work, see Chapters 6 and 7.

The first example concerns the very well-known **Gilbert Graph**, or **random geometric graph**. In this graph, a threshold $\epsilon > 0$ is fixed and two points connect to each other if their distance is less than ϵ, see Figure 1.4 for an illustration. The functional we look at is as follows: Construct the Gilbert Graph on a Poisson measure with Lebesgue intensity inside some fixed cube in d dimensions and let $F^{(\alpha)}$ be the sum of edge-lengths to the power $-\alpha$, where $\alpha > 0$, i.e.

$$F^{(\alpha)} = \sum_{\text{edges } e} |e|^{-\alpha}. \tag{1.28}$$

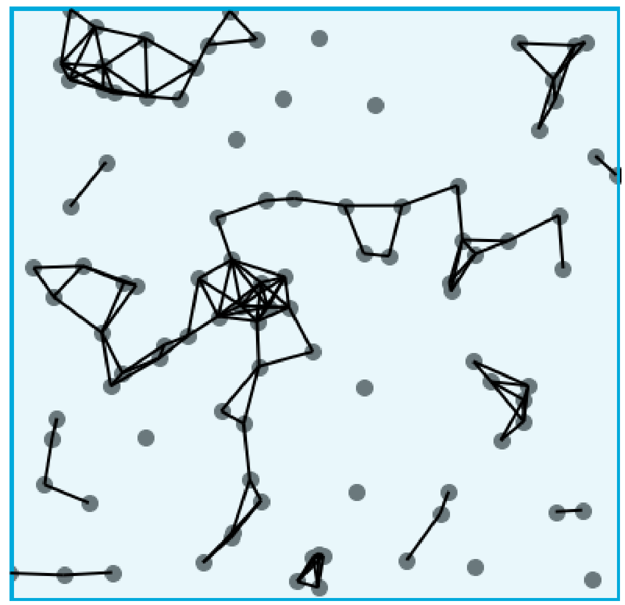


Figure 1.4: An illustration of a Gilbert Graph

This functional is clearly exponentially stabilising — when adding a point x, the only change will be an addition of edges to all points within range ϵ of x. The radius is thus deterministically given by ϵ.

If the ball of radius ϵ around x is not empty, then the point x connects at least to the point closest to it, thus

$$\mathrm{D}_x F^{(\alpha)} \geq e(x, \eta)^{-\alpha}, \tag{1.29}$$

where $e(x, \eta)$ is the distance from x to its closest neighbour. The quantity $e(x, \eta)^{-\alpha}$ has a density roughly equivalent to $\exp(-s^{-d/\alpha})s^{-d/\alpha-1}$ (see Figure 1.5), implying that it only has moments up to order d/α. This also means that $\mathrm{D}_x F^{(\alpha)}$ only has finite moments up to order d/α, and hence results like the ones in [LPS16] are not always applicable because they require moments of order at least 4.

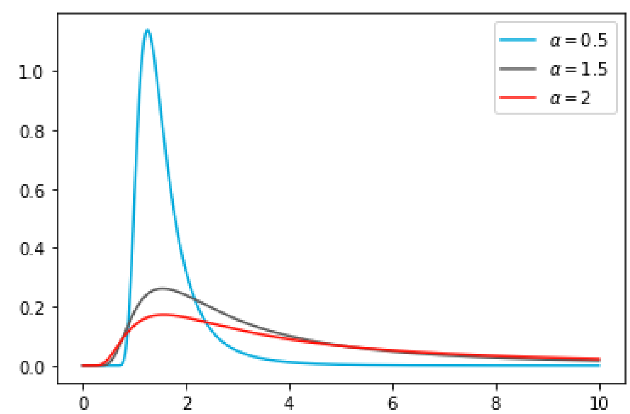


Figure 1.5: The density of $e(x, \eta)^{-\alpha}$ for different values of α in dimension $d = 2$

In our second example, we look at the same functional, but for the **Radial Spanning Tree**. Construct the Radial Spanning Tree on a Poisson measure of Lebesgue intensity inside some fixed d-dimensional cube centred at 0. Again, we look at

$$F^{(\alpha)} = \sum_{\text{edges } e} |e|^{-\alpha}, \tag{1.30}$$

where $\alpha > 0$.

It can be shown that this functional is exponentially stabilising, see the discussion in Examples 1.7 and 1.12 for an illustration and c.f. also the proof of Lemma 4.2 in [ST17]. However, the amount of change introduced to the functional when adding a point x is at least as big as the length of the edge from x to its radial nearest neighbour, to the power $-\alpha$. In formulas:

$$\mathrm{D}_x F^{(\alpha)} \geq g(x,\eta)^{-\alpha}, \qquad (1.31)$$

where $g(x,\eta)$ is the length of the edge from x to its radial nearest neighbour (see Section 9.9).

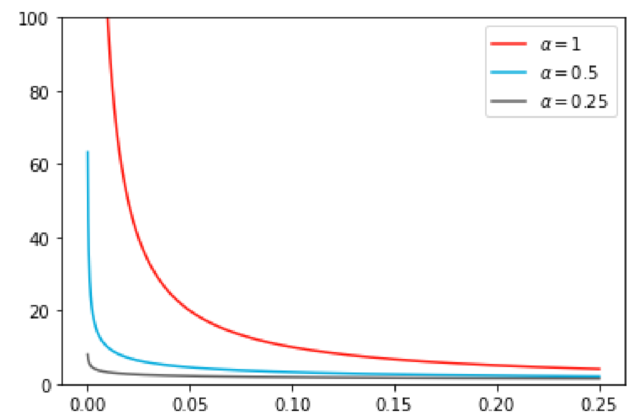


Figure 1.6: The function $u \mapsto u^{-\alpha}$ for different values of α

Note that the length $g(x,\eta)$ is at most $|x|$, since the point x can always connect to the origin. Hence

$$|\mathrm{D}_x F| \geq |x|^{-\alpha}. \qquad (1.32)$$

This implies that moments of $\mathrm{D}_x F$ cannot be uniformly bounded. Moreover, the larger α is, the more pronounced is the effect of adding points close to the origin, c.f. Figure 1.6. This behaviour does not occur for functionals

$$F^{(\alpha)} = \sum_{\text{edges } e} |e|^{\alpha} \qquad (1.33)$$

with positive exponents, which have been treated in [ST17], where the authors use the results from [LPS16].

The third example we discuss is the **Online Nearest Neighbour Graph** (ONNG). For this graph, points carry a time mark in $[0,1]$ and each point connects to its nearest neighbour in $\mathbb{R}^d$ among those points with a smaller arrival time, see Figure 1.7 for an illustration. Points with small arrival times tend to have very long edges, since there are few other points to connect to: given an arrival time of s, the expected length of an edge is proportional to $s^{-1/d}$.

This graph admits a radius of stabilisation (see the discussion in Section 9.5.1), but it is not exponentially stabilising. For a point (x,s) with spatial coordinate x and arrival time s, the tails of the radius of stabilisation are given by

$$\mathbb{P}(R(x,s,\eta) > u) < C\,\exp(-csu^d), \qquad (1.34)$$

Figure 1.7: An illustration of an Online Nearest Neighbour Graph

implying that there is no uniform bound on the radius over all points (x, s). The smaller the arrival time is, the larger is the radius of stabilisation, c.f. Figure 1.8 for an illustration of the density.

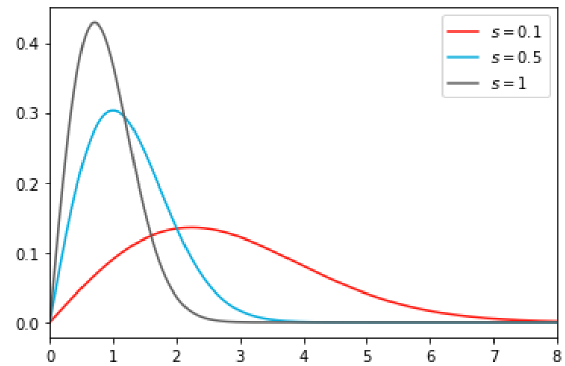


Figure 1.8: The density of the radius of stabilisation for different arrival times

Not only does the radius grow inversely to the arrival time, but the amount of change induced by the addition of a point **x** at time s also increases when s is small. Compare for example the illustrations 1.9 and 1.10: the spatial coordinate of the point is the same, however in Figure 1.9 the point **x** has been inserted very early on and a lot of points now connect to **x** instead of their previous neighbour, whereas in Figure 1.10 the point **x** has a very late arrival time and the only change is an addition from **x** to its nearest neighbour.

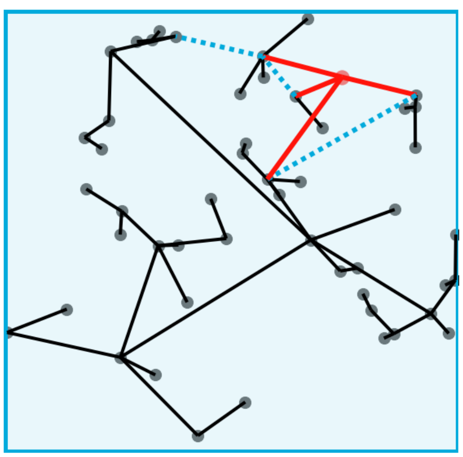


Figure 1.9: Change induced by adding **x** at a small time

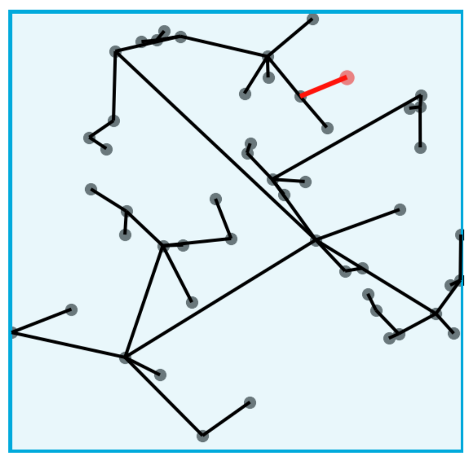


Figure 1.10: Change induced by adding **x** at a large time

For each of these examples, the methods developed in [LPS16], and similar approaches in [LRSY19, SY19, LRPY22, SBP22] do not suffice to show a central limit theorem, because they typically require one to uniformly bound over $\mathbb{X}$ the moments of order $(4 + \epsilon)$ (with $\epsilon > 0$) of $\mathrm{D}\, F$ and $\mathrm{D}^{(2)}\, F$. For examples such as those discussed in this section, these conditions are too strong. For the ONNG, a central limit theorem was conjectured in [Wad09] to hold for certain sums of power-weighted edge-lengths, and to the best of our knowledge, our work [Tra22] is the first to close this conjecture. We discuss these applications and the methods we used in more detail in Sections 2.1 and 2.2 of the introduction and in Chapters 6 and 7 of this work.

Roadmap

- For a discussion of the Gilbert Graph in a univariate setting, see Section 2.2.2 of the introduction and Section 6.2 of the later work; for a multivariate setting, see Section 2.2.3 and Chapter 7;
- the Radial Spanning Tree is treated in Section 2.2.5 of the introduction and Section 6.4 of the main body;
- we talk more about the ONNG in Sections 2.2.1 and 6.1;
- a further example of this type is a functional of the k-Nearest Neighbour Graph, which we address in Sections 2.2.4 and 6.3.

1.8 Multivariate Methods

When extending bounds like the ones given in Theorem 1.9 to a multivariate setting, one often needs stronger differentiability assumptions. The multivariate analogue of Stein's equation (1.5) is given by

$$g(x) - \mathbb{E}g(X) = \langle x, \nabla f(x) \rangle_{\mathbb{R}^m} - \langle C, \text{Hess } f(x) \rangle_{H.S.}, \tag{1.35}$$

where C must be positive-definite and any solution f must be twice differentiable. Here Hess denotes the Hessian and $\langle ., . \rangle_{H.S.}$ the Hilbert-Schmidt inner product. Comparing (1.35) in dimension 1 to (1.5), the multivariate version requires a higher degree of differentiability.

In the work [PZ10], the authors provide bounds akin to Theorem 1.8 on

$$d_{\mathcal{H}}(F, X) = \sup_{h \in \mathcal{H}} |\mathbb{E}h(F) - \mathbb{E}h(X)|, \tag{1.36}$$

where F is a multivariate Poisson functional and $X \sim \mathcal{N}(0, C)$ a centred Gaussian vector with covariance matrix C. They treat two distances: the d_2 distance where the functions h are required to be 1-Lipschitz and have a Hessian with bounded operator norm, and the d_3 distance, which requires bounded first and second derivatives. For the d_2 distance, they use the multivariate Malliavin-Stein method (see also [CM08, RR09, NPR10b, SY19]) and for the d_3 distance, they use the so-called 'smart paths' interpolation method (see [Tal03, Cha08, NPR10a]). While the Malliavin-Stein method works well for $\mathcal{C}^2$-functions h, it requires the matrix C to be positive-definite, which can be avoided when using the smart paths method at the cost of having $\mathcal{C}^3$ functions.

The multivariate extension of Theorem 1.9 was given in [SY19], where the authors Schulte and Yukich go one step further: they provide a bound for the convex distance, where the functions h in (1.36) are indicator functions of convex sets. The bound itself and the proof of it are much more involved than the corresponding ones for the smooth distances d_2 and d_3, yet the authors achieve the same speed of convergence in applications. Their results can be summarised as follows.

Theorem 1.14 [SY19, Thms. 1.1 & 1.2]

Let $F = (F_1, \dots, F_m)$ be a vector of Poisson functionals such that $\mathbb{E}F_i = 0$ and $\mathrm{D}\, F_i$ is square-integrable for every i. Let $X \sim \mathcal{N}(0, C)$ be a centred Gaussian vector with covariance matrix $C = (\sigma_{ij})_{1 \le i,j \le m}$. Then

$$d_3(F, X) \le \frac{m}{2} \sum_{i,j=1}^{m} |\sigma_{ij} - \mathrm{Cov}(F_i, F_j)| + m\gamma_1 + \frac{m}{2}\gamma_2 + \frac{m^2}{4}\gamma_3, \tag{1.37}$$

where

$$\gamma_1 := \left(\sum_{i,j=1}^{m} \int \left[\mathbb{E}\,(\mathrm{D}_x\, F_i)^2\,(\mathrm{D}_y\, F_i)^2\right]^{1/2} \mathbb{E}\left[\left(\mathrm{D}^{(2)}_{x,z}\, F_j\right)^2 \left(\mathrm{D}^{(2)}_{y,z}\, F_j\right)^2\right]^{1/2} \lambda^{(3)}(dx, dy, dz) \right)^{1/2}$$

$$\gamma_2 := \left(\sum_{i,j=1}^{m} \int \mathbb{E}\left[\left(\mathrm{D}^{(2)}_{x,z}\, F_i\right)^2 \left(\mathrm{D}^{(2)}_{y,z}\, F_i\right)^2\right]^{1/2} \mathbb{E}\left[\left(\mathrm{D}^{(2)}_{x,z}\, F_j\right)^2 \left(\mathrm{D}^{(2)}_{y,z}\, F_j\right)^2\right]^{1/2} \lambda^{(3)}(dx, dy, dz) \right)^{1/2}$$

$$\gamma_3 := \sum_{i=1}^{m} \int \mathbb{E}|\,\mathrm{D}_x\, F_i|^3 \lambda(dx).$$

If C is positive definite, then

$$\begin{aligned} d_2(F, X) \le{}& \|C^{-1}\|_{op}\|C\|_{op}^{1/2} \sum_{i,j=1}^{m} |\sigma_{ij} - \mathrm{Cov}(F_i, F_j)| \\ &+ 2\|C^{-1}\|_{op}\|C\|_{op}^{1/2}\gamma_1 + \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\gamma_2 + \frac{\sqrt{2\pi}m^2}{8}\|C^{-1}\|_{op}^{3/2}\|C\|_{op}\gamma_3 \end{aligned} \tag{1.38}$$

and

$$\begin{aligned} d_{convex}(F, X) \le{}& 941 m^5 \max\left\{\|C^{-1/2}\|_{op}, \|C^{-1/2}\|_{op}^3\right\} \\ &+ \max\left\{ \sum_{i,j=1}^{m} |\sigma_{ij} - \mathrm{Cov}(F_i, F_j)|\,, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \right\}, \end{aligned} \tag{1.39}$$

where $C^{1/2}$ is such that $C = C^{1/2}C^{1/2}$ and $\gamma_4, \gamma_5, \gamma_6$ are similar but more involved terms.

These bounds are very powerful, because they only require knowledge of

- the covariances $\mathrm{Cov}\,(F_i, F_j)$,
- and the add-one costs $\mathrm{D}\, F_i$ and $\mathrm{D}^{(2)}\, F_i$.

One can thus derive joint convergence results where the dependence between components is taken into account only in the covariance matrix. As pointed out in Section 1.7, these bounds typically require however moments of the add-one costs of order $4 + \epsilon$, a problem which we will address in Section 2.1.2 of the introduction and Section 4.2 of the later work.

The background literature on multivariate CLTs is rather large. Qualitative results for Poisson functionals can be found e.g. in [Pen05] , while quantitative results are discussed for example in [PZ10, BP14, LPST14, SY19, LRPY22]. We refer to [CGS11, Chapter 6] for a general overview, as well as [Xia11, Rai19] for results on sums of random vectors and discussions of literature. The paper [HLS16] deals with the Boolean model, while [KT17] discusses Rademacher functionals. [NPY22] uses the results from [SY19] to develop improved bounds in a Gaussian framework. The methods used are also quite varied – see [RR94, Fan16] for dependency methods, [FR15] for Stein couplings, [RR09] for exchangeable pairs. We also refer to the literature discussed in the papers [PZ10, SY19, NPY22].

2 Introduction: Main Contributions

In this chapter we present our main findings over the course of the PhD. At the time of writing, all results pertaining to the univariate case can be found in the preprint [Tra22], whereas the work presented in Sections 2.1.2 and 2.2.3 is work in progress and not publicly available yet. The results presented in this introduction are a summary of our main contributions — more detailed versions of all the results presented in this chapter can be found in Chapters 4–7.

2.1 Second-Order p-Poincaré Inequalities

2.1.1 The Univariate Case

The goal of developing second-order p-Poincaré inequalities is to achieve inequalities under minimal moment assumptions on the add-one costs $\mathrm{D}\, F$ and $\mathrm{D}^{(2)}\, F$ (see (1.16)), for the Wasserstein distance d_W and the Kolmogorov distance d_K (see (1.14) and (1.2)).

For $p, q \in [1, 2]$, we define the terms $\gamma_1, \dots, \gamma_7$ as follows:

$$\gamma_1 := 4 \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \mathbb{E}\left[|\, \mathrm{D}_y\, F|^{2p} \right]^{\frac{1}{2p}} \cdot \mathbb{E}\left[|\, \mathrm{D}^{(2)}_{x,y}\, F|^{2p} \right]^{\frac{1}{2p}} \lambda(dy) \right)^p \lambda(dx) \right)^{1/p}$$

$$\gamma_2 := 2 \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \mathbb{E}\left[|\, \mathrm{D}^{(2)}_{x,y}\, F|^{2p} \right]^{1/p} \lambda(dy) \right)^p \lambda(dx) \right)^{1/p}$$

$$\gamma_3 := 2 \int_{\mathbb{X}} \mathbb{E}\, |\mathrm{D}_y\, F|^{q+1}\, \lambda(dy)$$

and

$$\gamma_4 := \left(4\int_{\mathbb{X}} \mathbb{E}\left[|\operatorname{D}_y F|^{2p}\right]\lambda(dy)\right)^{1/p}$$

$$\gamma_5 := \left(4p\int_{\mathbb{X}}\int_{\mathbb{X}} \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F|^{2p}\right]\lambda(dy)\lambda(dx)\right)^{1/p}$$

$$\gamma_6 := \left(2^{2+p}p\int_{\mathbb{X}}\int_{\mathbb{X}} \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F|^{2p}\right]^{1/2}\cdot \mathbb{E}\left[|\operatorname{D}_x F|^{2p}\right]^{1/2}\lambda(dy)\lambda(dx)\right)^{1/p}$$

$$\gamma_7 := \left(8p\int_{\mathbb{X}}\int_{\mathbb{X}} \left(\mathbb{E}|\operatorname{D}^{(2)}_{x,y} F|^{2p}\right)^{\frac{1}{2p}}\cdot \left(\mathbb{E}|\operatorname{D}_x F|^{2p}\right)^{\frac{1}{2p}}\cdot\left(\mathbb{E}\,|\operatorname{D}_y F|^{2p}\right)^{1-1/p}\lambda(dy)\lambda(dx)\right)^{1/p}.$$

One of our main results is the following:

Theorem 2.1 (see also Theorem 4.4)

Let η be a $(\mathbb{X},\lambda)$-Poisson measure and let $F = F(\eta)$ be a Poisson functional such that $\mathbb{E}F = 0$, $\mathbb{E}F^2 = 1$ and $\operatorname{D} F$ is square-integrable. Then for any $p, q \in [1,2]$, one has

$$d_W(F,N) \leq \gamma_1 + \gamma_2 + \gamma_3 \tag{2.1}$$

and

$$d_K(F,N) \leq \sqrt{\tfrac{\pi}{2}}\gamma_1 + \sqrt{\tfrac{\pi}{2}}\gamma_2 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7. \tag{2.2}$$

When choosing $p = q = 2$, we recover bounds very similar to the ones in [LPS16], presented in Theorem 1.9. However, in our result one has the freedom to choose p and q such that the moments of the add-one costs can be of order $2+\epsilon$ for a fixed arbitrary $\epsilon \in (0,2]$ (resp. $\epsilon \in (0,1]$ for γ_3). This allows for a fine-tuning of the moment conditions, making applications possible that do not satisfy a $4+\epsilon$ condition, and also those that have non-uniform bounds whose fourth power is not integrable. We illustrate this with a simple example.

Example 2.2: Berry-Esseen under weak moment assumptions

An extension of the classical Berry-Esseen theorem (presented in Section 1.1) is given in [Pet75, Theorem 6, p. 115] and can be stated as follows:
Let ν be a centred probability measure on $\mathbb{R}$ such that

$$c := \int_{\mathbb{R}} |u|^{2+\epsilon}\,\nu(du) < \infty \tag{2.3}$$

for some $\epsilon \in (0,1]$. Let $X_1, X_2, \ldots$ be i.i.d. random variables distributed according to ν and set $\sigma^2 := \mathbb{E}X_1^2 = \int_{\mathbb{R}} u^2\,\nu(du)$. Define the sum

$$G_n := \sum_{i=1}^n X_i. \tag{2.4}$$

Then by [Pet75, Theorem 6, p. 115], it holds that

$$d_K\left(\frac{G_n}{\sqrt{n}\sigma}, N\right) \leq \frac{c}{\sigma^{2+\epsilon}} n^{-\epsilon/2}, \tag{2.5}$$

where $N \sim \mathcal{N}(0,1)$ is a standard Gaussian.
Take η to be a Poisson measure on $\mathbb{R} \times [0,\infty)$ with intensity $\nu(du) \otimes ds$. Let $T > 0$ and define

$$F_T := \int_0^T \int_{\mathbb{R}} u\, \eta(du, ds). \tag{2.6}$$

Then F_T is equal in law to $G_{N(T)}$, where $N(T)$ is a Poisson distributed random variable with parameter T independent of the sequence $(X_i)_{i\geq 1}$.
Theorem 2.1 implies

$$d_W\left(\frac{F_T}{\sqrt{T}\sigma}, N\right) \leq \frac{2c}{\sigma^{2+\epsilon}} T^{-\epsilon/2} \tag{2.7}$$

and

$$d_K\left(\frac{F_T}{\sqrt{T}\sigma}, N\right) \leq \frac{(4c)^{\frac{1}{1+\epsilon/2}}}{\sigma^2} T^{-\frac{\epsilon/2}{1+\epsilon/2}} \tag{2.8}$$

for the Wasserstein and Kolmogorov distances respectively. The speed of convergence in Wasserstein distance corresponds exactly to the one given by Petrov. For the Kolmogorov distance we find a slightly slower speed, which is however still converging much faster than the square root of the Wasserstein distance, which is implied by the classic estimate $d_K(., N) \leq 2\sqrt{d_W(., N)}$ (see e.g. [NP12, Remark C.2.2]).

The bound in Theorem 2.1 is sufficient for showing central limit theorems for all the badly behaved examples discussed in Section 1.7, except in the critical case for the Online Nearest Neighbour Graph, which occurs when $\alpha = \frac{d}{2}$. Here, we need to refine the bound even further and we do so by introducing a time component. Let η be a Poisson measure on $\mathbb{X} \times [0,1]$ with intensity $\lambda \otimes ds$ and F a functional of η. Then, a statement analogous to Theorem 2.1 holds, but where we consider moments of the conditional expectations of the add-one costs, as can be seen in the statement below. A bound of this type is new and the distinction between moments of $\mathrm{D}\, F$ and moments of conditional expectations of $\mathrm{D}\, F$ is crucial for the critical case of the ONNG, as discussed in Section 9.5.

Theorem 2.3 (see also Theorem 4.3)

Let η be a $(\mathbb{X} \times [0,1], \lambda \otimes ds)$-Poisson-measure and let F be a functional of η such that $\mathbb{E}F = 0$ and $\mathbb{E}F^2 = 1$. Let $p \in [1,2]$. Then

$$d_W(F, N) \leq 2\left(\int_{\mathbb{X}}\int_0^1 \left(\int_{\mathbb{X}}\int_s^1 \mathbb{E}\left[\mathbb{E}\left[|\,\mathrm{D}^{(2)}_{(y,u),(x,s)} F|\,\middle|\,\eta_{|\mathbb{X}\times[0,s)}\right]^{2p}\right]^{1/p} dydu\right)^p dxds\right)^{1/p} + \textit{similar terms}. \tag{2.9}$$

These bounds have minimal moment assumptions, in a sense that we will explain now. Bounds like (2.9) or Theorem 1.9 are trade-offs between integrals of moments of add-one costs, i.e. expressions like the ones on the RHS of (2.9), and the variance of F. In [LP11b, Thm. 1.5], Last and Penrose showed that

$$\mathrm{Var}(F) = \mathbb{E} \int_{\mathbb{X}} \int_0^1 \mathbb{E}[\mathrm{D}_{(x,t)}\, F|\eta_{|\mathbb{X}\times[0,t)}]^2 \lambda(dx)dt, \tag{2.10}$$

meaning that the variance is itself an integral of an add-one cost. Taking $p = 1$ in our bounds from Theorems 2.1 or 2.3 would roughly mean that the integrals of moments of the add-one costs would be of the same order as the variance, and one can show that under some conditions on F, the bounds cannot converge to zero when $p = 1$. Taking $p = 1 + \epsilon$ instead gives us thus the minimally possible moment condition under which we can still derive a central limit theorem.

Remark 2.4 (Discussion of literature). As already pointed out, our results in this section are a substantial extension of [LPS16]. The bounds given in [LPS16, Theorems 1.1 and 1.2] contain moments of first and second order add-one costs with exponent 4 (or even $4 + \epsilon$, see [LPS16, Proposition 1.4]). While this is a very powerful tool for showing asymptotically Gaussian behaviour, a finite 4th moment is too strong a condition for some applications, most notably for the ONNG discussed in Sections 2.2.1 and 6.1. Our Theorem 2.1 reduces this condition to finite $2p$ moments, where $p \in (1, 2]$, while retaining similar bounds in the case $p = 2$. In particular, [LPS16, Theorem 6.1 and Proposition 1.4] follow from our Theorem 2.1. A precise discussion of similarities and differences with the results of [LPS16] can be found in Section 4.1.

Other results making use of minimal $2 + \epsilon$ moment conditions in a similar framework are [PY05, Tri19], where the authors use exponentially stabilising score functions and additional assumptions on weak stabilisation respectively to derive qualitative central limit theorems.

The proof of Theorems 2.1 and 2.3 uses the Malliavin-Stein method and we refer to Chapter 1 for a discussion of the history and related results. Bounds related to our Theorems 2.1 and 2.3 can be found in [ET14, Sch16, LPS16, LRPY22].

Applications of our bounds can be found in Section 2.2 of the introduction and in Chapter 6. The applications we look at only pertain to random graphs, but we stress that our bounds are very general. They have been applied e.g. in the preprints [BZ23, BXZ23] to achieve central limit theorems for solutions of stochastic differential equations driven by Lévy white noise.

Roadmap

- Rigorous statements of the results presented here are located in Chapter 4;
- Proofs can be found in Section 9.3;
- Further discussions of related literature can be read about in Sections 1.3 and 1.5.
- Applications are presented in Section 2.2 of the introduction and Chapter 6 of the main text.

2.1.2 The Multivariate Case

We provide multivariate analogues to Theorem 2.1. For $p, q \in [1, 2]$, define the terms $\zeta_1, \ldots, \zeta_4$ by

$$\zeta_1 := \sum_{i,j=1}^{m} |C_{ij} - \mathrm{Cov}(F_i, F_j)|$$

$$\zeta_2 := 2^{2/p-1} \sum_{i,j=1}^{m} \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \mathbb{E}\left[|\, \mathrm{D}^{(2)}_{x,y} F_i|^{2p} \right]^{1/2p} \mathbb{E}\left[|\, \mathrm{D}^{(2)}_{x,y} F_j|^{2p} \right]^{1/2p} \lambda(dx) \right)^{p} \lambda(dy) \right)^{1/p}$$

$$\zeta_3 := 2^{2/p} \sum_{i,j=1}^{m} \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \mathbb{E}\left[|\, \mathrm{D}_{x} F_i|^{2p} \right]^{1/2p} \mathbb{E}\left[|\, \mathrm{D}^{(2)}_{x,y} F_j|^{2p} \right]^{1/2p} \lambda(dx) \right)^{p} \lambda(dy) \right)^{1/p}$$

$$\zeta_4 := m^{q-1} \sum_{i,j=1}^{m} \int_{\mathbb{X}} \mathbb{E}\left[|\, \mathrm{D}_{x} F_i|^{q+1} \right]^{1/(q+1)} \mathbb{E}\left[|\, \mathrm{D}_{x} F_j|^{q+1} \right]^{1-1/(q+1)} \lambda(dx).$$

Then the following bound holds.

Theorem 2.5 (see also Theorem 4.6)

Let η be a $(\mathbb{X}, \lambda)$-Poisson measure. Let $m \geq 1$ and let $F = (F_1, \dots, F_m)$ be an $\mathbb{R}^m$-valued random vector such that F_i and $\mathrm{D}\, F_i$ are square-integrable and satisfy $\mathbb{E}F_i = 0$. Let $C = (C_{ij})_{1 \leq i,j \leq m}$ be a symmetric positive-semidefinite matrix and let $X \sim \mathcal{N}(0, C)$ be a multivariate Gaussian with covariance matrix C. Then for all $p, q \in [1, 2]$,

$$d_3(F, X) \leq \tfrac{1}{2}(\zeta_1 + \zeta_2 + \zeta_3) + \zeta_4. \tag{2.11}$$

If moreover the matrix C is positive-definite, then for all $p, q \in [1, 2]$,

$$d_2(F, X) \leq \|C^{-1}\|_{op} \|C\|_{op}^{1/2} (\zeta_1 + \zeta_2 + \zeta_3) + \left(2\|C^{-1}\|_{op} \|C\|_{op}^{1/2} \vee \tfrac{\sqrt{2\pi}}{8} \|C^{-1}\|_{op}^{3/2} \|C\|_{op} \right) \zeta_4. \tag{2.12}$$

The proof of this result relies on [PZ10], where both the multivariate Stein's method and a smart paths interpolation method are used, see Section 1.8.

We provide bounds in the d_2 and the d_3 distance, both of which are known as 'smooth distances' because they require a high degree of differentiability of the test functions. An adaptation to minimal moment assumptions of the bound on the convex distance given in [SY19, Thm. 1.2] is work in progress.

For a discussion of the surrounding literature, we refer to Section 1.8.

As applications of this bound, we look at two different multivariate functionals of the Gilbert Graph, see Section 2.2.3 and Chapter 7.

Roadmap

- Rigorous statements of the results presented here are located in Section 4.2;
- Proofs can be found in Section 9.4;
- Further discussions of related literature can be read about in Section 1.8.
- Applications are presented in Section 2.2.3 of the introduction and Chapter 7 of the main text.

2.2 Applications

In Chapters 6 and 7, we apply Theorems 2.1, 2.3 and 2.5 to edge-statistics of four types of random graphs: for the Online Nearest Neighbour Graph, we look at functionals of the type

$$F^{(\alpha)} = \sum_{\text{edges } e} |e|^\alpha \tag{2.13}$$

in the exponent range $\alpha \in \left(0, \frac{d}{2}\right]$, thereby closing a conjecture made in [Wad09]; for the Gilbert Graph, we study univariate functionals of the type (2.13) in the range $\alpha \in \left(-\frac{d}{2}, \infty\right)$ (extending existing results from [RST17]) and multivariate functionals where we vary α on one hand and the domain in which the edges are allowed to be on the other hand; we also deal with the k-Nearest Neighbour Graph and the Radial Spanning Tree for a general class of decreasing functions $\phi : (0, \infty) \to (0, \infty)$ applied to the edge-lengths. A detailed discussion of the literature will be given separately for each graph below. A graphic representation of each graph in dimension 3 can be found at the end of this chapter in Figure 2.5.

In all our applications, the speeds we find are the same in the Wasserstein and Kolmogorov distances (and d_2, d_3 distances respectively). Roughly speaking, a $2p$-moment bound leads to a speed of convergence of $t^{d(1/p-1)}$, where t^d is the order of the variance and $p \in (1, 2]$. If $p = 2$, we recover the speed of order 'square root of the variance', which is often presumed to be optimal and has in some contexts been shown to be optimal. If however $p < 2$, the resulting speed is slower. Comparing with Example 2.2 and setting $2p = 2 + \epsilon$, it corresponds to $t^{-d\frac{\epsilon/2}{1+\epsilon/2}}$ in both Wasserstein and Kolmogorov distances. Whether or not this speed is optimal remains an open question.

2.2.1 The Online Nearest Neighbour Graph

Given a Poisson measure η on $\mathbb{R}^d \times [0, 1]$ with Lebesgue intensity, we have a collection of points with space coordinates in $\mathbb{R}^d$ and time coordinates in $[0, 1]$. We construct our functional of interest by choosing a convex body $H \subset \mathbb{R}^d$ and making it grow via multiplication with a positive parameter $t \geq 1$. Construct the ONNG G_t as follows:

- Vertices are given by points of η whose space coordinates are inside tH;

- each point of η within tH connects to its nearest neighbour among those points of η (within tH) with smaller time mark.

Our functional of interest $F_t^{(\alpha)}$ is now given by

$$F_t^{(\alpha)} := \sum_{\text{edges } e \text{ in } G_t} |e|^{\alpha}, \tag{2.14}$$

for $\alpha > 0$.

The graph G_t is known as **Online Nearest Neighbour Graph**. It is a simple model for a network growing in time, inspired by the modelling of real life networks (see [DM02, New03]). Already mentioned in [Ste89], it appeared in [BBB+07] as a limit case of the FKP model (see [FKP02]) of the internet graph. It is also a special case of geometric preferential attachment models as mentioned in [MS02, JW15] and an element of the class of minimal directed spanning trees (see [PW09]). Various properties of the ONNG have been studied, among them laws of large numbers in [Wad07], an analysis of the ONNG on the real line in [PW08a] and upper bounds for edge-lengths and variances in [Wad09]. We refer to [PW09] for a survey up to the year 2009. Graph theoretic properties have also been studied in [LM21, Cas23] under the name of **Nearest Neighbour Tree**, and colourings associated with the ONNG (called **Poisson rain**) were investigated in [Ald18, BBCS23] as well as a related voter model in [Pre09].

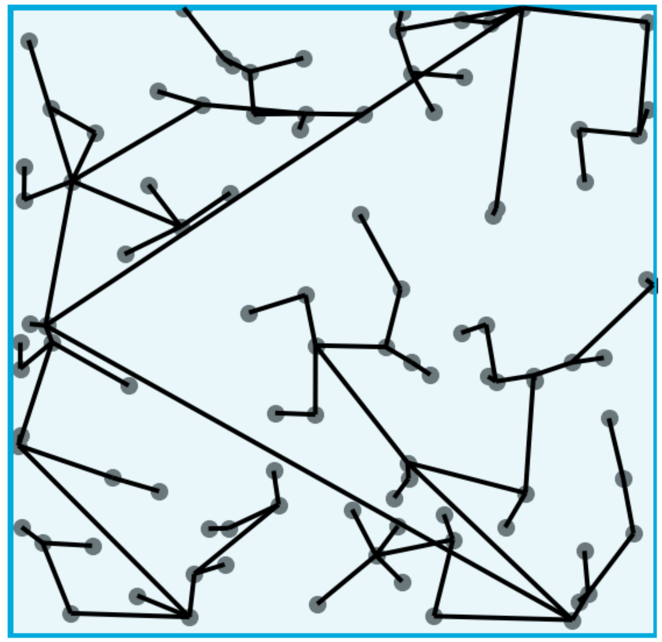


Figure 2.1: An illustration of an Online Nearest Neighbour Graph

Central limit theorems for the functional $F_t^{(\alpha)}$ were only known in the exponent range $\alpha \in (0, d/4)$, by a result in [Pen05], with a quantitative counterpart being provided in [LRPY22]. The results in [Pen05, LRPY22] hold for multivariate functionals which are slightly more general than $F_t^{(\alpha)}$, however in this context, we limit ourselves to the univariate case with edge-lengths of power $\alpha \geq 0$.

Theorem 2.6 [Pen05, Thm. 3.6] & [LRPY22, Thm. 2.2]

For $0 \leq \alpha < \frac{d}{4}$, *there is a constant* $\sigma_{\alpha,d} > 0$ *such that as* $t \to \infty$,

$$t^{-d}\operatorname{Var}\left(F_t^{(\alpha)}\right) \to \sigma_{\alpha,d} \quad \text{and} \quad \frac{F_t^{(\alpha)} - \mathbb{E}F_t^{(\alpha)}}{t^{d/2}} \xrightarrow{d} \mathcal{N}(0, \sigma_{\alpha,d}). \tag{2.15}$$

Moreover,

$$\max\{d_W\left(F_t^{(\alpha)}, N\right), d_K\left(F_t^{(\alpha)}, N\right)\} \leq ct^{d(p-4/8)}. \tag{2.16}$$

In [Wad09], it was conjectured that a result akin to Theorem 2.6 holds in the exponent range $\alpha \in [\frac{d}{4}, \frac{d}{2}]$. In particular, the conjecture stated the following:

Conjecture 2.7 [Wad09, Conjectures 2.1 & 2.2]

For $\frac{d}{4} \le \alpha < \frac{d}{2}$, there is a constant $\sigma_{\alpha,d} > 0$ such that as $t \to \infty$,

$$t^{-d}\,\mathrm{Var}\left(F_t^{(\alpha)}\right) \to \sigma_{\alpha,d} \quad \textit{and} \quad \frac{F_t^{(\alpha)} - \mathbb{E}F_t^{(\alpha)}}{t^{d/2}} \xrightarrow{d} \mathcal{N}(0, \sigma_{\alpha,d}). \tag{2.17}$$

Moreover, for $\alpha = \frac{d}{2}$, there is a constant $\sigma_d > 0$ such that as $t \to \infty$,

$$t^{-d}\log(t)^{-1}\,\mathrm{Var}\left(F_t^{(\alpha)}\right) \to \sigma_d \quad \textit{and} \quad \frac{F_t^{(\alpha)} - \mathbb{E}F_t^{(\alpha)}}{(t^d\log(t))^{1/2}} \xrightarrow{d} \mathcal{N}(0, \sigma_{d/2,d}). \tag{2.18}$$

Using the bounds given in Theorems 2.1 and 2.3, we can close this conjecture in the entire range $\alpha \in [\frac{d}{4}, \frac{d}{2}]$, as well as provide speeds of convergence in the range $\alpha \in (0, \frac{d}{4})$ which are faster than the ones given in [LRPY22]. The case $\alpha > \frac{d}{2}$ is discussed in [PW08a] (especially for $d = 1$) and in [Wad09], where it is shown that a limit exists in this case, but is non-Gaussian for $\alpha > d$.

Theorem 2.8 (see also Theorem 6.1)

For $0 < \alpha < \frac{d}{2}$, and for every $1 < p < \frac{d}{2\alpha}$ such that $p \le 2$, there is a constant $c_1 > 0$ such that for all $t \ge 1$ large enough

$$\max\left\{ d_W\left(\frac{F_t^{(\alpha)} - \mathbb{E}F_t^{(\alpha)}}{\sqrt{\mathrm{Var}\left(F_t^{(\alpha)}\right)}}, N\right), d_K\left(\frac{F_t^{(\alpha)} - \mathbb{E}F_t^{(\alpha)}}{\sqrt{\mathrm{Var}\left(F_t^{(\alpha)}\right)}}, N\right)\right\} \le c_1 t^{-d\left(1-\frac{1}{p}\right)}, \tag{2.19}$$

where N denotes a standard normal random variable. Moreover, there are constants $c_2, C_2 > 0$ such that for all $t \ge 1$ large enough

$$c_2 t^d < \mathrm{Var}(F_t^{(\alpha)}) < C_2 t^d. \tag{2.20}$$

For $\alpha = \frac{d}{2}$, there is a constant $c_3 > 0$ such that for all $t \ge 1$ large enough

$$d_W\left(\frac{F_t^{(d/2)} - \mathbb{E}F_t^{(d/2)}}{\sqrt{\mathrm{Var}\left(F_t^{(d/2)}\right)}}, N\right) \le c_3 \log(t)^{-1}. \tag{2.21}$$

Moreover, there are constants $c_4, C_4 > 0$ such that for all $t \ge 1$ large enough

$$c_4 t^d \log(t^d) < \mathrm{Var}(F_t^{(d/2)}) < C_4 t^d \log(t^d). \tag{2.22}$$

The constants $c_1, c_2, C_2, c_3, c_4, C_4$ may depend on $H,\ \alpha,\ d$ and p.

Note that in the exponent range $\alpha \in (0, \frac{d}{4})$, one can choose $p = 2$ and recover a speed of convergence of $t^{-d/2}$, which corresponds to the presumably optimal speed of square root of the order of the variance.

Roadmap

- For a fully detailed presentation, go to Section 6.1;
- for proofs, look into Section 9.5;
- for a discussion why this graph is special, check out Section 1.7.

2.2.2 The Gilbert Graph: Univariate case

For this application, we consider a convex body $W \subset \mathbb{R}^d$ and a Poisson measure η^t on W of intensity measure $t\,dx$. Take a sequence $(\epsilon_t)_{t>0}$ of positive real numbers s.t. $\epsilon_t \to 0$ as $t \to \infty$. Then the **Gilbert Graph** G_t is constructed by connecting two points x and y if $|x - y| < \epsilon_t$.

For $\alpha \in \mathbb{R}$, we define

$$L_t^{(\alpha)} := \sum_{\text{edges } e \text{ in } G_t} |e|^\alpha, \tag{2.23}$$

where $|e|$ denotes the length of an edge. Define

$$\hat{L}_t^{(\alpha)} := \frac{L_t^{(\alpha)} - \mathbb{E} L_t^{(\alpha)}}{\sqrt{\mathrm{Var}\left(L_t^{(\alpha)}\right)}}. \tag{2.24}$$

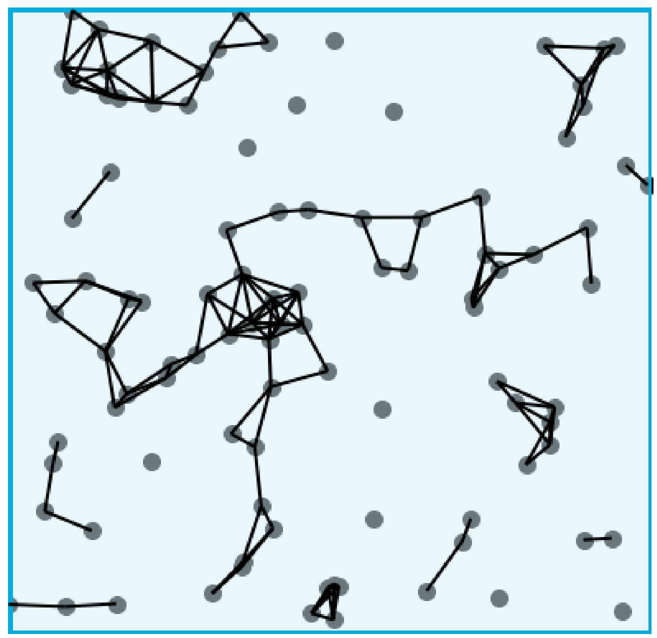


Figure 2.2: An illustration of a Gilbert Graph

The first mention of the Gilbert Graph was by Gilbert in [Gil61], in dimension $d = 2$. It has been treated in many works under various names: geometric or proximity graph, interval graph (when $d = 1$) or disk graph (when $d = 2$). The book [Pen03] provides a vast background and literature review and we also refer to [LRP13a, LRP13b] for central limit theorems of generalisations of the Gilbert Graph and [RS13] for a quantitative CLT on a sum of weighted edge-lengths. For a comprehensive overview of the Gilbert Graph in the context of U-statistics, see [LRR16], especially Section 4.3. See also [McD03, Mü08, HM09, BP14, DST16, GT20]. In [RST17], the authors give a complete picture of the asymptotic behaviour of $\hat{L}_t^{(\alpha)}$ for $\alpha \in \mathbb{R}$. In particular, they show that for $\alpha > -\frac{d}{2}$, the quantity $\hat{L}_t^{(\alpha)}$ converges in distribution to a standard Gaussian as $t \to \infty$, provided that $t^2 \epsilon_t^2 \to \infty$. They also give a quantitative bound on the speed of convergence in Kolmogorov distance in the case $\alpha > -\frac{d}{4}$. As an application of our estimates, we recover this speed of convergence below and extend to the case $-\frac{d}{2} < \alpha \leq -\frac{d}{4}$. The authors of [RST17] show that CLTs hold also for $-d < \alpha \leq -\frac{d}{2}$ with different rescalings; however, establishing corresponding speeds of convergence in this range is still an open problem.

Theorem 2.9 (see also Theorem 6.3)

Let $\alpha > -\frac{d}{2}$ and assume that $t^2\epsilon_t^d \to \infty$ as $t \to \infty$. Then for $t \geq 1$ large enough

- *if $\alpha > -\frac{d}{4}$, there is a constant $c_1 > 0$ such that*

$$\max\left\{d_W\left(\hat{L}_t^{(\alpha)}, N\right), d_K\left(\hat{L}_t^{(\alpha)}, N\right)\right\} \leq c_1\left(t^{-1/2} \vee (t^2\epsilon_t^d)^{-1/2}\right). \tag{2.25}$$

- *if $-\frac{d}{2} < \alpha \leq -\frac{d}{4}$, then for any $1 < p < -\frac{d}{2\alpha}$, there is a constant $c_2 > 0$ such that*

$$\max\left\{d_W\left(\hat{L}_t^{(\alpha)}, N\right), d_K\left(\hat{L}_t^{(\alpha)}, N\right)\right\} \leq c_2\left(t^{-1+1/p} \vee (t^2\epsilon_t^d)^{-1+1/p}\right). \tag{2.26}$$

Roadmap

- A detailed discussion of the Gilbert Graph can be found in Section 6.2;
- proofs are located in Section 9.6;
- an explanation what makes this functional hard to deal with is given in Section 1.7.

2.2.3 The Gilbert Graph: Multivariate case

We also investigate two multivariate functionals of the Gilbert Graph, by use of Theorem 2.5. For both, we consider the functional $L_t^{(\alpha)}$, but once for a vector with different values of α and once for the same α but on different underlying domains W.

Given real parameters $\alpha_1, \dots, \alpha_m$ such that $\alpha_i + \alpha_j > -\frac{d}{2}$ and a convex body $W \subset \mathbb{R}^d$, define the vector

$$\tilde{L}_t := \left(\tilde{L}_t^{(\alpha_1)}, \dots, \tilde{L}_t^{(\alpha_m)}\right), \tag{2.27}$$

where $\tilde{L}_t^{(\alpha_i)}$ are centred and suitably rescaled versions of $L_t^{(\alpha_i)}$. A qualitative central limit theorem for $\tilde{L}_t$ has been shown in [RST17, Thm. 5.2], as well as the convergence of the covariance matrix to a matrix $C = (C_{ij})_{1\leq i,j\leq m}$. We give a quantitative analogue to this result, of a speed which is presumably optimal in the case where $\min\{\alpha_1, \dots, \alpha_m\} > -\frac{d}{4}$. For a discussion of further related literature, see Section 1.8.

Theorem 2.10 (see also Theorem 7.1)

Assume that $t^2\epsilon_t^d \to \infty$ as $t \to \infty$. Let $X \sim \mathcal{N}(0, C)$ be a centred Gaussian with covariance matrix C. Then for $t \geq 1$ large enough

- *if $\alpha_1, ..., \alpha_m > -\frac{d}{4}$, there is a constant $c_1 > 0$ such that*

$$d_3(\tilde{L}_t, X) \leq c_1\left(\epsilon_t + \max_{1\leq i,j\leq m}\left|\mathrm{Cov}\left(\tilde{L}_t^{(\alpha_i)}, \tilde{L}_t^{(\alpha_j)}\right) - C_{ij}\right| + \left(t^{-1/2} \vee (t^2\epsilon_t^d)^{-1/2}\right)\right). \tag{2.28}$$

- *if $-\frac{d}{2} < \min\{\alpha_1, ..., \alpha_m\} \leq -\frac{d}{4}$, then for any $1 < p < -\frac{d}{2}\min\{\alpha_1, ..., \alpha_m\}^{-1}$, there is a* *constant* $c_2 > 0$ *such that*

$$d_3(\tilde{L}_t, X) \leq c_2 \left(\epsilon_t + \max_{1 \leq i,j \leq m} \left| \operatorname{Cov}\left(\tilde{L}_t^{(\alpha_i)}, \tilde{L}_t^{(\alpha_j)}\right) - C_{ij} \right| + \left(t^{-1+1/p} \vee (t^2 \epsilon_t^d)^{-1+1/p}\right) \right). \tag{2.29}$$

If $\lim_{t \to \infty} t\epsilon_t^d < \infty$*, then the bounds* (7.6) *and* (7.7) *apply to* $d_2(\tilde{L}_t, X)$ *as well for different constants* $c_1, c_2 > 0$.

We also study multivariate functionals where we fix an exponent $\alpha > -\frac{d}{2}$ and vary the underlying domain W. Let η^t be a Poisson measure of intensity $t\,dx$ on $\mathbb{R}^d$ and let $W_1, \ldots, W_m \subset \mathbb{R}^d$ be convex bodies. Construct the Gilbert Graph $G_t^{(i)}$ on η^t restricted to W_i and let $F_t^{(i)}$ be the sum of α-powered edge-lengths, i.e.

$$F_t^{(i)} = \sum_{\text{edges } e \text{ in } G_t^{(i)}} |e|^\alpha. \tag{2.30}$$

Setting

$$\tilde{F}_t := (\tilde{F}_t^{(1)}, \ldots, \tilde{F}_t^{(m)}), \tag{2.31}$$

for centred and suitably rescaled versions of $\tilde{F}_t^{(i)}$ of $F_t^{(i)}$, we achieve the following convergence result.

Theorem 2.11 (see also Theorem 7.2)

Let the above conditions prevail. Then the matrix $C = (C_{ij})_{1 \leq i,j \leq m}$*, where* $C_{ij} = |W_i \cap W_j|$*, is the asymptotic covariance matrix of the* m*-dimensional random vector* $\tilde{F}_t$*.*
Moreover, assume that $t^2 \epsilon_t^d \to \infty$ *as* $t \to \infty$*. Let* $X \sim \mathcal{N}(0, C)$ *be a centred Gaussian with covariance matrix* C*. Then for* $t \geq 1$ *large enough,*

- *if* $\alpha_1, \ldots, \alpha_m > -\frac{d}{4}$*, there is a constant* $c_1 > 0$ *such that*

$$d_3(\tilde{F}_t, X) \leq c_1 \left(\epsilon_t + \left(t^{-1/2} \vee (t^2 \epsilon_t^d)^{-1/2}\right)\right). \tag{2.32}$$

- *if* $-\frac{d}{2} < \min\{\alpha_1, \ldots, \alpha_m\} \leq -\frac{d}{4}$*, then for any* $1 < p < -\frac{d}{2} \min\{\alpha_1, \ldots, \alpha_m\}^{-1}$*, there is a constant* $c_2 > 0$ *such that*

$$d_3(\tilde{F}_t, X) \leq c_2 \left(\epsilon_t + \left(t^{-1+1/p} \vee (t^2 \epsilon_t^d)^{-1+1/p}\right)\right). \tag{2.33}$$

If the matrix C *is positive definite, then the bounds* (7.12) *and* (7.13) *apply to* $d_2(\tilde{F}_t, X)$ *as well for different constants* $c_1, c_2 > 0$.

In Remark 7.3 we also discuss some criteria implying that the matrix C is positive-definite.

Roadmap

- The full discussion can be found in Chapter 7;
- proofs are presented in Section 9.7;
- additional discussions about the Gilbert Graph are located in Sections 2.2.2 and 6.2.

2.2.4 The k-Nearest Neighbour Graph

Our next object of study is the k-Nearest Neighbour Graph. This graph is constructed by connecting two points x and y if y is among the k points closest to x or vice-versa. Given a Poisson measure η on $\mathbb{R}^d$ with Lebesgue intensity, we construct the k-Nearest Neighbour Graph G_t inside the domain tH, where $H \subset \mathbb{R}^d$ is a convex body and $t \geq 1$ is a positive parameter.

The functional we study is again a sum of edge-functionals, but these functionals are more general than simple power-weighted edge-lengths. Let $\phi : (0, \infty) \to (0, \infty)$ be a decreasing function such that there is an $\epsilon > 0$ verifying

$$\int_0^1 \phi(s)^{2+\epsilon} s^{d-1} ds < \infty. \tag{2.34}$$

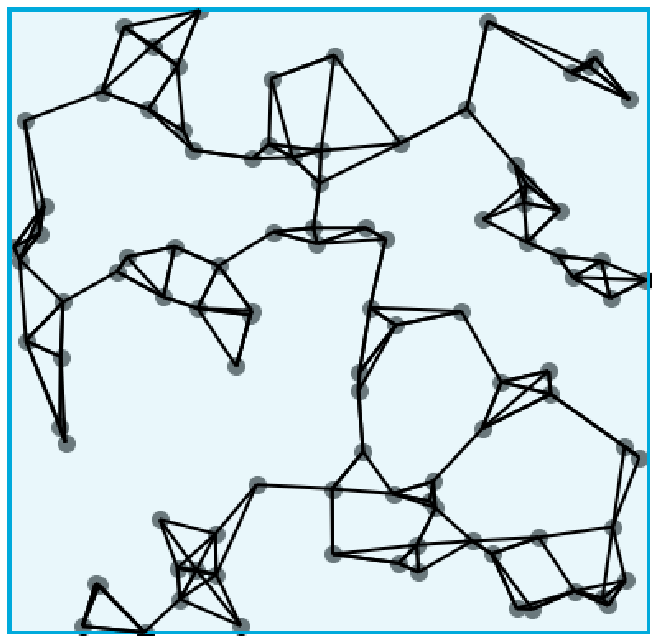


Figure 2.3: An illustration of a 3-Nearest Neighbour Graph

This condition ensures that $2+\epsilon$-moments of $\phi(e(x, \eta))$ are finite, where $e(x, \eta)$ is the length from the point x to its closest neighbour in η. The set of function ϕ verifying the above conditions includes the functions $x \mapsto x^{-\alpha}$, for $0 < \alpha < \frac{d}{2}$. Define the functional F_t by

$$F_t := \sum_{\text{edges } e \text{ in } G_t} \phi(|e|). \tag{2.35}$$

The use of k-Nearest Neighbour Graphs is wide-spread, from applications in computational geometry [EPY97, Smi99, PS12] to the use as a basis for the nearest-neighbour chain algorithm to find clusters [Mur02, Tuf11]. See also [Wad07] for a discussion of applications. Quantitative central limit theorems for edge-related quantities were shown in [BB83, AB93, PY01, BY05, PY05, Pen07, LPS16], where the first bound on the Kolmogorov distance for the total edge-length was given in [AB93] and it was of the order $\log(t)^{7/4}t^{-1/4}$. This speed was improved in [PY05] to $\log(t)^{3d}t^{-1/2}$. A presumably optimal speed was achieved in [LPS16], where the authors deduce a quantitative central limit theorem for the sum of power-weighted edge-lengths with powers $\alpha \geq 0$, at a speed of convergence of $t^{-d/2}$.

We complement the existing results by dealing with the case $\alpha \in \left(-\frac{d}{2}, 0\right)$. Note that in this case, the CLT is new even in its qualitative version. In the regime $\alpha \in \left(-\frac{d}{4}, 0\right)$, we also recover the same, presumably optimal, speed of convergence of $t^{-d/2}$, whereas in the case $\alpha \in \left(-\frac{d}{2}, -\frac{d}{4}\right]$, we find a speed of convergence that decreases as α approaches $-\frac{d}{2}$. It is natural to ask what happens when $\alpha \leq -\frac{d}{2}$. We consider this a separate issue and leave it open for further research.

Theorem 2.12 (see also Theorem 6.5)

For any $p \in (1, 2]$ *such that* $p < 1 + \frac{\epsilon}{2}$, *there is a constant* $c > 0$ *such that, for* $t \geq 1$,

$$\max \left\{ d_W \left(\frac{F_t - \mathbb{E}F_t}{\sqrt{\operatorname{Var}(F_t)}}, N \right), d_K \left(\frac{F_t - \mathbb{E}F_t}{\sqrt{\operatorname{Var}(F_t)}}, N \right) \right\} \leq c t^{d(1/p-1)}. \tag{2.36}$$

This inequality holds in particular for the function $\phi(x) = x^{-\alpha}$ *with* $0 < \alpha < \frac{d}{2}$, *for any* $p \in (1, 2]$ *such that* $p < \frac{d}{2\alpha}$.

Roadmap

- For the full presentation, seek out Section 6.3;
- for proofs, look up Section 9.8.

2.2.5 The Radial Spanning Tree

Our final application is the Radial Spanning Tree. Take a convex body $H \subset \mathbb{R}^d$ containing the point 0 in its interior and a Poisson measure η of Lebesgue intensity on $\mathbb{R}^d$. Construct the Radial Spanning Tree G_t by connecting each point x of η within tH to its closest neighbour among all points in $\eta_{|tH} \cup \{0\}$ that have norm smaller than x.

This defines an 'inward-looking tree', rooted at the point 0. It is a type of minimal spanning tree under conditions, and aside from an earlier reference in [Gil65] under the name of 'exodic graph', it was studied in [BB07] as a model related to the minimal directed spanning tree and to Poisson forests. The paper also discusses various applications, most notably in communication networks. Further work on the radial spanning tree has been done in [PW09, BCT13, ST17]. In [ST17], the authors give a quantitative central limit theorem for sums of power-weighted edge-lengths of the radial spanning tree for powers $\alpha \geq 0$. The framework is one where the intensity of the Poisson measure increases while the observation window stays constant. After rescaling to our framework of a constant intensity and a growing window, one obtains by [ST17, Theorem 1.2] a speed of convergence of $t^{-d/2}$. We add quantitative central limit theorems for $\alpha \in \left(-\frac{d}{2}, 0\right)$, recovering the same speed of $t^{-d/2}$ for $\alpha \in \left(-\frac{d}{4}, 0\right)$. Note that this CLT is new even in its qualitative version. As for the k-Nearest Neighbour Graph, the case $\alpha \leq -\frac{d}{2}$ will be the object of further research.

The functional we are interested in is akin to the one we studied for the k-Nearest Neighbour Graph. Let $\phi : (0, \infty) \to (0, \infty)$ be a decreasing function such that there is an $\epsilon > 0$ such that

$$\int_0^1 \phi(s)^{2+\epsilon} s^{d-1} ds < \infty. \tag{2.37}$$

Then define F_t by setting

$$F_t := \sum_{\text{edges } e \text{ in } G_t} \phi(|e|). \tag{2.38}$$

Using Theorem 2.1, we deduce the following CLT.

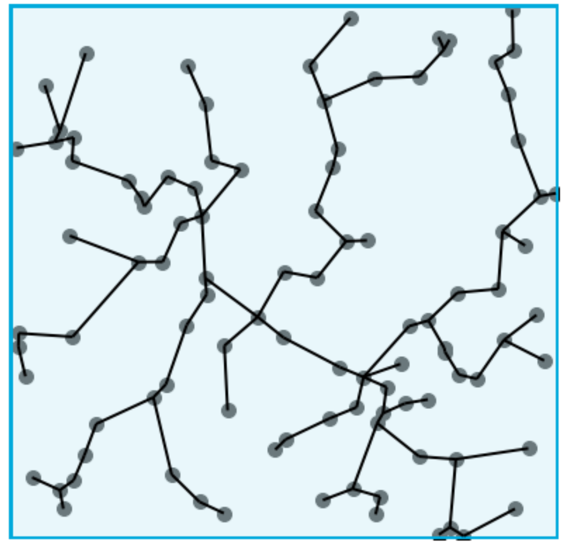


Figure 2.4: An illustration of a Radial Spanning Tree

Theorem 2.13 (see also Theorem 6.6)

For any $p \in (1, 2]$ *such that* $p < \frac{r}{2}$, *there is a constant* $c > 0$ *such that for* $t \geq 1$,

$$\max \left\{ d_W \left(\frac{F_t - \mathbb{E}F_t}{\sqrt{\operatorname{Var}(F_t)}}, N \right), d_K \left(\frac{F_t - \mathbb{E}F_t}{\sqrt{\operatorname{Var}(F_t)}}, N \right) \right\} \leq c t^{d(1/p-1)}. \tag{2.39}$$

This inequality holds in particular for the function $\phi(x) = x^{-\alpha}$ *with* $0 < \alpha < \frac{d}{2}$, *for any* $p \in (1, 2]$ *such that* $p < \frac{d}{2\alpha}$.

Roadmap

- A detailed presentation of the results given here is located in Section 6.4;
- proofs are to be found in Section 9.9;
- preliminary discussions about the Radial Spanning Tree have been given in Sections 1.6 and 1.7.

2.3 New Technical Contributions

An important part of this thesis consists of functional inequalities derived by various methods. These inequalities are essential in the development of improved second-order Poincaré inequalities, but we believe them to be of independent interest, too. Note that we provide for example moment bounds for Skorohod integrals, for moments between 1 and 2. To the best of our knowledge, this result is the first of its kind in the literature.

2.3.1 Novel Functional Inequalities

The proofs of Theorems 2.1, 2.3 and 2.5 rely on new functional inequalities for certain types of Poisson functionals. A classical result, used to prove the bounds provided in [LPS16], is the Poincaré inequality.

Poincaré Inequality [Las16, Thm. 10]

Let η be a $(\mathbb{X},\lambda)$-Poisson measure and F an integrable Poisson functional. Then

$$\mathbb{E}F^2-(\mathbb{E}F)^2\leq\int_{\mathbb{X}}\mathbb{E}(\mathrm{D}_x F)^2\lambda(dx). \tag{2.40}$$

This inequality states that the value of the variance is always bounded by the average of the second moment of the first add-one cost of the functional. It turns out that one can bound general p-moments of F, for $p\in[1,2]$, in the same way.

Proposition 2.14 p-Poincaré Inequality (see also (5.7))

Let η be a $(\mathbb{X},\lambda)$-Poisson measure and F an integrable Poisson functional. Then for $p\in[1,2]$,

$$\mathbb{E}|F|^p-|\mathbb{E}F|^p\leq\int_{\mathbb{X}}\mathbb{E}|\,\mathrm{D}_x F|^p\lambda(dx). \tag{2.41}$$

In the case $F\geq 0$ or $\mathbb{E}F=0$, this inequality can be deduced from the so-called ϕ-Sobolev inequalities shown in [Cha04, (5.10)] or from the Beckner type p inequalities discussed in [APS22, Section 4.6] (but with a worse constant). In our context, this inequality is derived from a new and much more general inequality involving Skorohod integrals. Skorohod integrals are the dual operators to the add-one cost D, but they also have a path-wise interpretation as a difference of integrals. We refer to the discussion on page 41. In its full generality, our statement reads as follows.

Theorem 2.15 (see also Theorem 5.2)

Let η be a Poisson measure on $\mathbb{X}\times[0,1]$ with intensity $\lambda\otimes ds$. Let $h=h(\eta,x,s)$ be a square-integrable function such that $\mathrm{D}\,h$ is square-integrable. Let $\phi:\mathbb{R}\to\mathbb{R}$ be a differentiable function with $(p-1)$-Hölder continuous derivative, for some $p\in(1,2]$ and assume that $\phi(0)=0$. Then

$$\begin{aligned}|\mathbb{E}\phi(\delta(h))|\leq{}&\frac{c_\phi}{p}\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\,|h(\eta,y,s)|^p\\&+c_\phi\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\,\left|\mathrm{D}_{(y,s)}h(\eta,x,t)\right|\cdot\left|\mathrm{D}_{(x,t)}h(\eta,y,s)\right|^{p-1}\\&+2c_\phi\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\,\left|\mathrm{D}_{(y,s)}h(\eta,x,t)\right|\cdot|h(\eta,y,s)|^{p-1},\end{aligned} \tag{2.42}$$

where c_ϕ is the Hölder constant of ϕ'. In particular, this inequality holds with $\phi(x)=|x|^p$ and $c_\phi=p2^{2-p}$, for $p\in[1,2]$.

This inequality is very versatile and to the best of our knowledge, it is the first bound of its kind on functionals of general Poisson-Skorohod integrals. Partial results are known in the particular case when h is predictable, see Remark 2.16 below. In particular, Theorem 2.15 contains the first general estimate in terms of add-one costs for p-moments of the Skorohod integral, where $p \in [1,2]$, the cases $p=1$ and $p=2$ being the only ones known. The time component can be removed, hence a similar inequality holds for functions $h = h(\chi, x)$, where χ is an $(\mathbb{X}, \lambda)$-Poisson measure.

A general Poisson functional F (with or without time component) can be represented in a non-unique way as a Skorohod integral, e.g. as

$$F - \mathbb{E}F = \delta\left(\mathbb{E}\left[\mathrm{D}_{(x,s)}\, F | \eta_{|\mathbb{X}\times[0,s)}\right]\right) = \delta(\mathrm{D}\,\mathrm{L}^{-1}\, F) \quad \text{a.s.} \tag{2.43}$$

where the first representation is of Clark-Ocône type and the second uses the Malliavin operator L^{-1} (see (8.21)). It follows that Theorem 2.15 implies inequalities for general Poisson functionals F, among them the p-Poincaré inequality announced in Proposition 2.14. We refer to Section 5.2 for a discussion of further important corollaries.

Remark 2.16 (Literature review). Inequality (2.41) (and likewise (5.4), (5.5) and (5.7) of Section 5.2) can be seen as part of a larger family of functional inequalities on the Poisson space, the most classical one being the Poincaré inequality, which we already mentioned in (2.40). We stress that our inequality (2.41) implies the classical one when $p=2$. Another well-known inequality is the modified log-Sobolev inequality shown in [Wu00] (see also [AL00]). It is extended in [Cha04, (5.10)] to the so-called Φ-Sobolev inequalities, which in the case $\Phi(x) = x^p$, imply (5.7) when $F \geq 0$ or $\mathbb{E}F = 0$. Similarly, the Beckner type p inequalities discussed in [APS22, Section 4.6] imply (5.7) when $F \geq 0$ or $\mathbb{E}F = 0$, albeit with a worse constant.

The paper [LMS22] discusses CLTs for Skorohod integrals, while the thesis [Zhu10] provides inequalities for p-norms of Skorohod integrals of *adapted*, *Banach-space valued* functions h in [Zhu10, Thm. 3.3.2]. Although we did not check the details, it is reasonable to assume that one can deduce the special case of Theorem 2.15 when $\phi(x) = |x|^p$ and h is predictable from such a result when applied to $\mathbb{R}^d$ (compare also with (5.4) in Section 5.2). Note that we need a general result for anticipative integrands in order to provide the bound on the Kolmogorov distance given in Theorem 2.1.

In the special case $\phi(x) = x^2$, Theorem 2.15 follows from an isometry relation reported in formula (8.4) of Chapter 8. The proof of the full Theorem 2.15 relies on a new version of Itô formula for Poisson processes, given in Theorem 2.18 (see also Theorem 5.1). It is then combined with the Clark-Ocône type formula mentioned in (2.43) (see also (3.19)) to derive Proposition 2.14 and other results presented in Section 5.2. Both [Wu00] and [Cha04] use a similar method of combining a Clark-Ocône result with (classical) Itô formulae to deduce their respective functional inequalities.

Remark 2.17 (Comparison with the Gaussian case and extensions when $p \geq 2$).

1. Let X be an isonormal Gaussian process defined on a real separable Hilbert space $\mathcal{H}$, let $F = F(X)$ be a real-valued functional of X and denote its Malliavin derivative by DF (see [NP12, Sections 2.1–2.3] for precise definitions of these concepts). The following inequality now holds for $q \geq 1$ (see [AMR22, Thm. 2.7]):

$$\mathbb{E}\left[|F - \mathbb{E}F|^q\right] \leq c_q \mathbb{E}\|DF\|_{\mathcal{H}}^q, \tag{2.44}$$

where $c_q > 0$ is a constant depending on q and $\|\cdot\|_{\mathcal{H}}$ is the norm in the space $\mathcal{H}$.

If $q = 2$, this inequality is the exact analogue of (2.41) when $p = 2$. When $p \in (1,2)$ however, the analogue of (2.41) does not hold in a Gaussian setting. This can be seen by letting $\mathcal{H} = L^2(\mathbb{R})$ and taking $F = X(\mathbb{1}_{[0,t]})$ to be a standard Brownian motion. Then $D_t F = \mathbb{1}_{[0,t]}$ and the LHS of (2.41) would be equal to

$$\mathbb{E} t^{p/2} |N|^p, \tag{2.45}$$

where N is a standard Gaussian, and the RHS of (2.41) would be equal to

$$\int_{\mathbb{R}} \mathbb{1}_{[0,t]}(s) ds = t. \tag{2.46}$$

If the inequality were to hold, we would now have, for $p \in (1,2)$ and some multiplying constant $c_p > 0$, that

$$t^{p/2} \leq c_p t, \tag{2.47}$$

which yields a contradiction as $t \to 0$.

2. Inequality (2.41) is false in general for $p > 2$. Indeed, consider an $(\mathbb{R}^d, \lambda)$-Poisson measure η (where λ is the Lebesgue measure) and $G = \eta(A) - \lambda(A)$ for some measurable $A \subset \mathbb{R}^d$. Then $\mathbb{E}G = 0$, $\mathbb{E}G^2 = \lambda(A)$ and $\mathrm{D}_x G = \mathbb{1}_A(x)$. On the LHS we have $\mathbb{E}|G|^p \geq (\mathbb{E}G^2)^{p/2} = \lambda(A)^{p/2}$ by Jensen's inequality and on the RHS

$$\int_{\mathbb{R}^d} \mathbb{E}|\, \mathrm{D}_x G|^p \lambda(dx) = \lambda(A). \tag{2.48}$$

However, since $p > 2$, we have $\lambda(A)^{p/2} \gg \lambda(A)$ for $\lambda(A)$ large enough. Hence the inequality fails for any multiplying constant.

3. Moment estimates for $p \geq 2$ are given in [GST21, Theorem 4.1] and [APS22, Proposition 4.20]. The RHSs of these inequalities involve related, but different quantities.

Roadmap

- For full statements of the results in this section, look into Section 5.2. Here you will also find further corollaries of Theorem 2.15, in particular the Corollary 5.5, which is crucial for giving bounds on the Kolmogorov distance;
- For the proofs, we refer to Section 9.2.

2.3.2 Anticipative Calculus for Poisson Point Processes

The proof of Theorem 5.2 (also seen in Theorem 2.15) relies on a new version of Itô formula, shown in Theorem 5.1. In contrast to the classical Itô formula for Poisson point processes as given in [IW81, Theorem II.5.1], our version does not assume the process to be a semi-martingale or the integrand to be predictable. In turn, we only use the term corresponding to the integral with respect to a compensated Poisson measure. We believe this result to be of independent interest, as to the best of our knowledge no such formula for anticipative integrands and general Poisson processes exists in the literature.

The exact statement is as follows.

Theorem 2.18 (see also Theorem 5.1)

Let η be a $(\mathbb{X}, \lambda)$-Poisson measure and $h = h(\eta, x, s)$ be a bounded, integrable function. Let $X_0 \in \mathbb{R}$. For $t \in [0,1]$, define

$$X_t(\eta) := X_0 + \int_{\mathbb{X}\times[0,t]} h(\eta - \delta_{(y,s)}, y, s)\eta(dy, ds) - \int_{\mathbb{X}} \int_0^t h(\eta, y, s)\lambda(dy)ds. \tag{2.49}$$

Then the process $(X_t)_{t\in[0,1]}$ is well-defined and $\mathbb{P}$-a.s. càdlàg. Let $\phi \in \mathcal{C}^1(\mathbb{R})$. Then, $\forall\, t \in [0,1]$,

$$\begin{aligned}\phi(X_t) = \phi(X_0) + \int_{\mathbb{X}\times[0,t]} \left(\phi\left(X_{s-} + h(\eta - \delta_{(y,s)}, y, s)\right) - \phi(X_{s-})\right)\eta(dy, ds) \\ - \int_{\mathbb{X}} \int_0^t \phi'(X_s)h(\eta, y, s)\lambda(dy)ds \qquad \mathbb{P}\text{-a.s.}\end{aligned} \tag{2.50}$$

and the quantities in (2.50) are well-defined.

The main difference between (2.50) and [IW81, Thm. II.5.1] consists in the fact that we do not assume the integrand h to be predictable. There exist Itô formulae for anticipative integrands in various settings, e.g. in the Wiener case in [AN98] and [NP88]) and for pure jump and general Lévy processes in [DNMBOP05] and [ALV08] respectively. To the best of our knowledge, our setting of a general Poisson point process is new.

Below, we give a detailed discussion of the similarities and differences between our result and the classical Itô formula mentioned in [IW81, Theorem II.5.1].

Remark 2.19 (Comparison with the classical Itô formula). We can only compare (2.50) to the special case of [IW81, Theorem II.5.1] where the semi-martingale in the statement of [IW81, Theorem II.5.1] has the following properties:

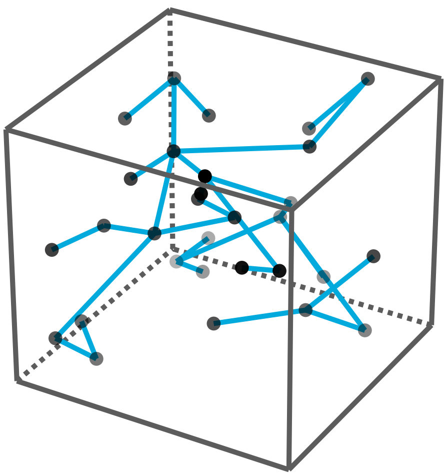
- the point process in question is a Poisson point process;
- the only non-zero part is the one with respect to the compensated Poisson measure (i.e. no Wiener part);
- the integrand h is both in L^2 and in L^1 (instead of $h \in L^{2,\text{loc}}$).

As mentioned before, the main difference between (2.50) and [IW81, Thm. II.5.1] consists then of the fact that we do not assume the integrand h to be predictable and thus achieve a formula for anticipative processes. This is achieved at the price of reducing the generality of processes our formula can be applied to. The proof of our result relies however on the same ideas as the proof of [IW81, Theorem II.5.1].

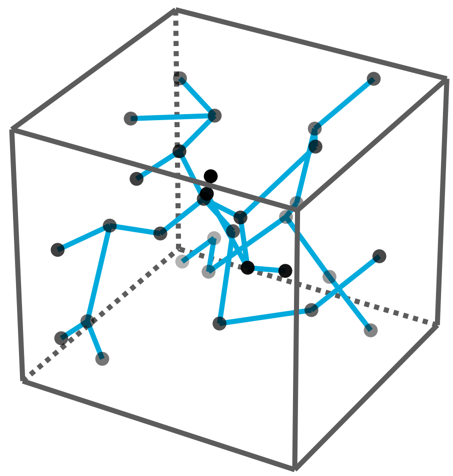
Note that when trying to extend (2.50) to general point processes, one encounters well-definedness issues. To the best of our knowledge, Skorohod integrals have not been defined for general point processes, and there are no isometry formulas available for anticipative integrands, nor Mecke-type formulas which would allow for a process to be well-defined.

Roadmap

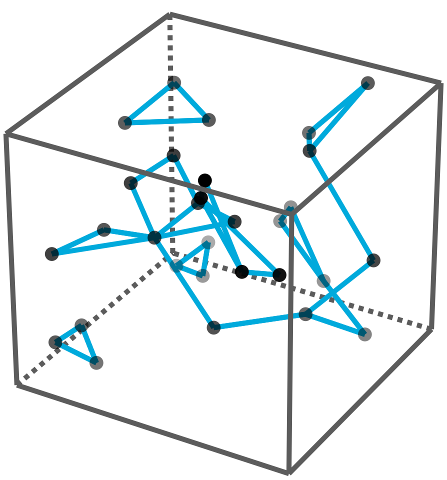
- For the full statement and surrounding discussions, see Section 5.1;
- if you are interested in the proofs, they are located in Section 9.1.



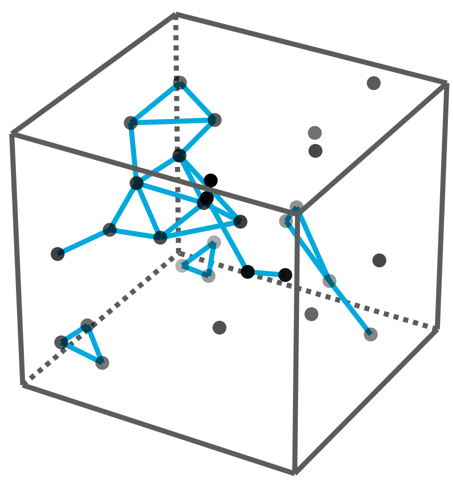
(a) The ONNG is a tree, and it is recognisable by its long edges, many of which connect to points with early arrival times.



(b) The Radial Spanning Tree is also a tree, connecting inwards and with generally shorter edges than in the ONNG.



(c) This is a 2-Nearest Neighbour Graph. It is not necessarily a tree, and it might have several connected components, but every node has at least 2 edges.



(d) In the Gilbert Graph, one usually sees clusters appearing for points which are grouped closely together. Isolated points are however possible.

Figure 2.5: A comparison of the different types of graphs in dimension 3.

Framework and Notations

We provide here an overview of the most relevant (in the context of this thesis) properties of Poisson point processes and elements of Poisson Malliavin calculus. Further definitions and properties that will be necessary for the proofs can be found in Chapter 8. We refer the reader to [Las16, LP18] for an exhaustive discussion of the material presented below.

Poisson random measure. Let $(\mathbb{W}, \mathcal{W}, \nu)$ be a σ-finite measure space and let $\mathbf{N}_{\mathbb{W}}$ be the set of $\mathbb{N}_0 \cup \{\infty\}$-valued measures on $(\mathbb{W}, \mathcal{W})$. Define the σ-algebra $\mathcal{N}_{\mathbb{W}}$ on $\mathbf{N}_{\mathbb{W}}$ as the smallest σ-algebra such that for all $W \in \mathcal{W}$, the map $\mathbf{N}_{\mathbb{W}} \ni \xi \mapsto \xi(W) \in \mathbb{N} \cup \{\infty\}$ is measurable. If it is clear from context which space we refer to, we will write $\mathbf{N}$ and $\mathcal{N}$ instead of $\mathbf{N}_{\mathbb{W}}$ and $\mathcal{N}_{\mathbb{W}}$.

A **Poisson random measure** with intensity ν is a $(\mathbf{N}, \mathcal{N})$-valued random element χ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- for all $W \in \mathcal{W}$ and all $k \in \mathbb{N}_0$, we have $\mathbb{P}(\chi(W) = k) = \exp(-\nu(W))\frac{\nu(W)^k}{k!}$ (with the convention that $\chi(W) = \infty$ (resp. $\chi(W) = 0$) $\mathbb{P}$-a.s. if $\nu(W) = \infty$ (resp. $\nu(W) = 0$));
- for $W_1, ..., W_n \in \mathcal{W}$ disjoint, the random variables $\chi(W_1), ..., \chi(W_n)$ are mutually independent.

Existence and uniqueness of such a measure is shown in [LP18, Chapter 3]. We denote by $\mathbb{P}_\chi$ the law of χ in $(\mathbf{N}, \mathcal{N})$ and we say that χ is a $(\mathbb{W}, \nu)$-**Poisson measure**.

In view of the σ-finiteness of $(\mathbb{W}, \nu)$ and using [LP18, Corollary 6.5] we can and will assume throughout the paper that the Poisson measure χ is **proper**, i.e. that there exist independent random elements $X_1, X_2, ... \in \mathbb{W}$ and an independent $\mathbb{N}_0 \cup \{\infty\}$-valued random variable κ such that $\mathbb{P}$-a.s.

$$\chi = \sum_{n=1}^{\kappa} \delta_{X_n}, \tag{3.1}$$

where δ_w is the Dirac mass at the point $w \in \mathbb{W}$. All our results only depend on the law of χ, hence this assumption has no impact on them. In this context, we will often identify χ with its support, i.e. with the random collection of points $\{X_1, X_2, ...\}$. Note that in the case where ν has atoms, these points might appear with higher multiplicities.

Poisson functionals. For $p \geq 0$, denote by $L^p(\mathbb{P}_\chi)$ the set of random variables F such that there is a measurable function $f : \mathbf{N} \to \mathbb{R}$ such that $F = f(\chi)$ $\mathbb{P}$-a.s. and, if $p > 0$, such that $\mathbb{E}|F|^p < \infty$.

We call F a **Poisson functional** and f a **representative** of F. All results that follow do not depend on the choice of the representative f and hence, throughout the thesis, we will use the symbol F indiscriminately to represent both f and F.

Add-one cost and Malliavin derivative. For a Poisson functional $F \in L^0(\mathbb{P}_\chi)$ and $w \in \mathbb{W}$, define the **add-one cost operator** of F as

$$\mathrm{D}_w F := F(\chi + \delta_w) - F(\chi), \tag{3.2}$$

and inductively set $\mathrm{D}^{(n)}_{w_1,\dots,w_n} F := \mathrm{D}_{w_n} \mathrm{D}^{(n-1)}_{w_1,\dots,w_{n-1}} F$ for $n \geq 1$ and $w_1, \dots, w_n \in \mathbb{W}$, where $\mathrm{D}^{(0)} F = F$ and $\mathrm{D}^{(1)} F = \mathrm{D}\, F$. It can be shown that, given a representative f, the function $(w_1, \dots, w_n, \chi) \mapsto \mathrm{D}^{(n)}_{w_1,\dots,w_n} f(\chi)$ is measurable and symmetric in $w_1, \dots, w_n$ (cf. [Las16, p. 5]). We denote by dom D the set of all $F \in L^2(\mathbb{P}_\chi)$ such that

$$\mathbb{E} \int_{\mathbb{W}} (\mathrm{D}_w F)^2 \nu(dx) < \infty. \tag{3.3}$$

The restriction of the operator D to dom D is called the **Malliavin derivative** of F (see [Las16, Theorem 3]). Note that for $F \in L^1(\mathbb{P}_\eta)$, the LHS of (3.3) is well-defined and (3.3) is sufficient for F to be in dom D (as follows from the $L^1(\mathbb{P}_\eta)$-Poincaré inequality as stated in [Las16, Cor. 1]). For $F, G \in L^0(p_\chi)$, we have the following formula for the add-one cost of a product:

$$\mathrm{D}(FG) = (\mathrm{D}\, F)G + F(\mathrm{D}\, G) + (\mathrm{D}\, F)(\mathrm{D}\, G). \tag{3.4}$$

Chaotic decomposition. For a function $g \in L^2(\mathbb{W}^n, \nu^{(n)})$, denote by $\mathrm{I}_n(g)$ the nth **Wiener-Itô integral** of g (cf. [Las16, Chapter 3]). Then for $F \in L^2(\mathbb{P}_\chi)$, we have the **Wiener-Itô chaos expansion**

$$F = \sum_{n=0}^{\infty} \mathrm{I}(f_n), \tag{3.5}$$

where $f_n(w_1, \dots, w_n) = \frac{1}{n!} \mathbb{E}\, \mathrm{D}^{(n)}_{w_1,\dots,w_n} F$ and the series converges in $L^2(\mathbb{P}_\chi)$ (cf. [Las16, Theorem 2]). If in addition $F \in$ dom D, then $\mathrm{D}\, F \in L^2(\mathbb{P}_\chi \otimes \nu)$ and it holds $\mathbb{P}_\chi-$a.s. and for $\nu-$a.e. $w \in \mathbb{W}$ that

$$\mathrm{D}_w F = \sum_{n=1}^{\infty} n\, \mathrm{I}_{n-1}(f_n(w, .)), \tag{3.6}$$

cf. [Las16, Theorem 3].

Mecke formula. Denote by $L^p(\mathbf{N} \times \mathbb{W})$ the quotient set of all measurable functions $h : \mathbf{N} \times \mathbb{W} \to \mathbb{R}$ such that, if $p > 0$, one has $\mathbb{E} \int_W |h(\chi, w)|^p \nu(dw) < \infty$.

Next, we introduce the so-called **Mecke formula** (cf. e.g. [Las16, formula (7)]), which holds for $h \in L^1(\mathbf{N} \times \mathbb{W})$ and for $h : \mathbf{N} \times \mathbb{W} \to [0, \infty)$ measurable:

$$\mathbb{E} \int_{\mathbb{W}} h(\chi, w)\chi(dw) = \mathbb{E} \int_{\mathbb{W}} h(\chi + \delta_w, w)\nu(dw). \tag{3.7}$$

In particular, combined with the fact that χ is assumed to be proper, this implies that for a function $h \in L^1(\mathbf{N} \times \mathbb{W})$, the integral

$$\int_{\mathbb{W}} h(\chi - \delta_w, w)\chi(dw) \tag{3.8}$$

is well-defined.

Skorohod integrals. If $h \in L^2(\mathbf{N} \times \mathbb{W})$, then for ν-a.e. $w \in \mathbb{W}$, we have $h(., w) \in L^2(\mathbb{P}_\chi)$ and thus we can write

$$h(\chi, w) = \sum_{n=0}^{\infty} \mathrm{I}_n(h_n(w, .)), \tag{3.9}$$

with $h_n(w, w_1, ..., w_n) = \frac{1}{n!}\mathbb{E}\, \mathrm{D}^{(n)}_{w_1,...,w_n} h(\chi, w)$ (cf. [Las16, formula (42)]). We say that $h \in \operatorname{dom} \delta$ if

$$\sum_{n=0}^{\infty} (n+1)! \int_{\mathbb{W}^{n+1}} \tilde{h}_n^2 d\nu^{n+1} < \infty, \tag{3.10}$$

where $\tilde{h}_n$ is the symmetrisation of h_n given by

$$\tilde{h}_n(w_1, ..., w_{n+1}) = \frac{1}{n+1} \sum_{k=1}^{n+1} h_n(w_k, w_1, ..., w_{k-1}, w_{k+1}, ..., w_{n+1}). \tag{3.11}$$

For $h \in \operatorname{dom} \delta$ we define the **Skorohod integral** of h by

$$\delta(h) := \sum_{n=0}^{\infty} \mathrm{I}_{n+1}(h_n), \tag{3.12}$$

which converges in $L^2(\mathbb{P}_\chi)$. Note that, by [Las16, Theorem 5], the following condition is sufficient for $h \in L^2(\mathbf{N} \times \mathbb{W})$ to be in $\operatorname{dom} \delta$:

$$\mathbb{E} \int_{\mathbb{W}} \int_{\mathbb{W}} (\mathrm{D}_x\, h(\chi, y))^2\, \nu(dx)\nu(dy) < \infty. \tag{3.13}$$

If $h \in L^1(\mathbf{N} \times \mathbb{W}) \cap \operatorname{dom} \delta$, then by [Las16, Theorem 6], we have $\mathbb{P}$-a.s.

$$\delta(h) = \int_{\mathbb{W}} h(\chi - \delta_w, w)\chi(dw) - \int_{\mathbb{W}} h(\chi, w)\nu(dw), \tag{3.14}$$

where the RHS is well-defined for any $h \in L^1(\mathbf{N} \times \mathbb{W})$ by (3.8).

Extension to a marked space. It will often be convenient to endow the space $\mathbb{W}$ with marks representing time. As we are only interested in the law of the Poisson functionals in question, we can always suppose that the $(\mathbb{W}, \nu)$-Poisson measure χ is the marginal of a $(\mathbb{W} \times [0,1], \nu \otimes ds)$-Poisson measure η. Indeed, $\eta(. \times [0,1])$ has the same law as χ when both are regarded as random elements in $\mathbf{N}_{\mathbb{W}}$. For a functional $F \in L^0(\mathbb{P}_\chi)$, define

$$G_F(\eta) := F(\eta(. \times [0,1])). \tag{3.15}$$

Then $G_F(\eta)$ has the same law under $\mathbb{P}_\eta$ as $F(\chi)$ under $\mathbb{P}_\chi$. Moreover, for any $(x, s) \in \mathbb{W} \times [0,1]$,

$$\mathrm{D}_{(x,s)}\, G_F(\eta) = \mathrm{D}_x\, F(\eta(. \times [0,1])), \tag{3.16}$$

which is equal in law to $\mathrm{D}_x\, F(\chi)$.

Predictability. We call a measurable function $h : \mathbf{N}_{\mathbb{W}\times[0,1]} \times \mathbb{W} \times [0,1] \to \mathbb{R}$ **predictable** if for all $(y,s) \in \mathbb{W} \times [0,1]$ and all $\nu \in \mathbf{N}_{\mathbb{W}\times[0,1]}$

$$h(\nu, y, s) = h(\nu_{|\mathbb{W}\times[0,s)}, y, s). \tag{3.17}$$

This definition of predictability appears e.g. in [LP11a, (2.5)], where it is argued that this version of predictability is comparable to predictability in the classical sense (as defined e.g. in [IW81, Definition I.5.2]). It is also shown in [LP11a, Proposition 2.4] that if $h \in L^2(\mathbf{N} \times \mathbb{W} \times [0,1])$ satisfies (3.17), then $h \in \operatorname{dom}\delta$.

Conditional expectations and Clark-Ocône formula. Let η be a $(\mathbb{W}\times[0,1], \nu\otimes ds)$-Poisson measure. Using that the measures $\eta_{|\mathbb{W}\times[0,s)}$ and $\eta_{|\mathbb{W}\times[s,1]}$ are independent, one can define a version of conditional expectation for any non-negative or integrable random variable $G \in L^0(\mathbb{P}_\eta)$ by

$$\mathbb{E}[G|\eta_{|\mathbb{W}\times[0,s)}] := \int G(\eta_{|\mathbb{W}\times[0,s)} + \xi)\Pi_s(d\xi), \tag{3.18}$$

where Π_s is the law of $\eta_{|\mathbb{W}\times[s,1]}$. If it is finite, the conditional expectation $\mathbb{E}[G|\eta_{|\mathbb{W}\times[0,s)}]$ is predictable (cf. [LP11b, formula (2.3)] and the discussion thereafter). In particular, for $F \in L^2(\mathbb{P}_\eta)$ the quantity $\mathbb{E}[\mathrm{D}_{(x,s)} F|\eta_{|\mathbb{W}\times[0,s)}]$ is well-defined, finite and predictable and the following Clark-Ocône type formula is shown in [LP11a, Theorem 2.1] (see also [Wu00, HP02]):

$$F = \mathbb{E}F + \delta(\mathbb{E}[\mathrm{D}_{(x,s)} F|\eta_{|\mathbb{W}\times[0,s)}]) \qquad \mathbb{P}_\chi - \text{a.s.} \tag{3.19}$$

This formula will be essential in the proof of Corollary 5.3.

Generic sets. Let $\mu \subset \mathbb{R}^d$ be a finite set. We say that μ is **generic** if all pairwise distances between points are distinct. We say that a set $\mu \subset \mathbb{R}^d$ is **generic with respect to points** $x, y \in \mathbb{R}^d$ if $x, y \notin \mu$ and $\mu \cup \{x, y\}$ is generic. Note that for compact sets $H \subset \mathbb{R}^d$, any (H, dx)-Poisson measure χ can a.s. be identified with its support and this support is a.s. generic. To simplify the presentation, we will at times adopt the notation

$$F(\mu) := F(\xi_\mu), \qquad \text{where } \xi_\mu = \sum_{x\in\mu} \delta_x \tag{3.20}$$

for a finite set $\mu \in \mathbb{R}^d$ and a measurable functional $F : \mathbf{N}_{\mathbb{R}^\mathbf{d}} \to \mathbb{R}$. Similar notation will be used for $\mathrm{D}\, F(\mu)$, $\mathrm{D}^{(2)}\, F(\mu)$ etc.

3.1 Notation

For $x \in \mathbb{R}^d$ and $r > 0$, we write $B^d(x,r)$ to indicate the (open) ball of centre x and radius r. In the context of this thesis, a convex body in $\mathbb{R}^d$ denotes a closed, compact, convex set of non-empty interior. For a measurable set $A \subset \mathbb{R}^d$, we denote by $|A|$ the Lebesgue measure of A, unless A is finite, in which case $|A|$ denotes the number of elements in A. We use $\overline{A}$ to denote the closure of A. Throughout this paper, $\kappa_d = |B^d(0,1)|$. We use the symbols $\wedge$ (resp. $\vee$) to denote a minimum (resp. maximum) of two elements. We shall use LHS and RHS to denote 'left hand side' and 'right hand side' and use both $|x|$ and $\|x\|$ to denote the Euclidean norm of $x \in \mathbb{R}^d$. For a square matrix A, we call $\|A\|_{op}$ the operator norm of A.

The supremum norm of a function $f : \mathbb{R}^m \to \mathbb{R}$ is denoted by $\|f\|_\infty$ and its Lipschitz norm by $\|f\|_{Lip}$. We use the notation $\|f'\|_\infty$ for $\sup_{x\in\mathbb{R}^m} \sup_{1\le i\le m} \left|\frac{\partial f}{\partial x_i}(x)\right|$ and adopt similar conventions with $\|f''\|_\infty$ and $\|f'''\|_\infty$ for the second and third derivatives respectively. By $\mathcal{C}^k(\mathbb{R}^m)$ we mean the set of functions $f : \mathbb{R}^m \to \mathbb{R}$ which are k times continuously differentiable. Denote the Hessian of f by Hess f.

By $\stackrel{d}{=}$ and $\stackrel{d}{\longrightarrow}$ we mean equality and convergence in distribution respectively. We use the symbol $\simeq$ (resp. $\lesssim$) if there is equality (resp. inequality) up to multiplication by a positive constant.

4 Second-order p-Poincaré Inequalities

In this section we state in detail our new bounds on the distance between the distribution of a Poisson functional and the normal law. Note that we have introduced these bounds already in the introductory Chapter 2. Here we present our results with all technical details, and we stress that every result presented in this section is novel.

We split this chapter into two parts, talking separately about the univariate and the multivariate case. Section 4.1 about the univariate case is based on the preprint [Tra22], while Section 4.2 is based on ongoing work.

4.1 The Univariate Case

Recall that for an integrable random variable F and a standard Gaussian N, the Wasserstein distance between the distributions of F and N is given by

$$d_W(F,N) = \sup_{h\in\mathcal{H}} |\mathbb{E}h(F) - \mathbb{E}h(N)| \tag{4.1}$$

where $\mathcal{H}$ is the set of Lipschitz-continuous functions $h : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant $\|h\|_{Lip} \leq 1$. On the other hand, the Kolmogorov distance between the distributions of F and N is defined as

$$d_K(F,N) = \sup_{z\in\mathbb{R}} |\mathbb{P}(F \leq z) - \mathbb{P}(N \leq z)|. \tag{4.2}$$

See e.g. [NP12, Appendix C], and the references therein, for a discussion of the basic properties of d_W and d_K.

Remark 4.1. For the rest of this section, we fix a σ-finite measure space $(\mathbb{X}, \mathcal{X}, \lambda)$. Before we state our main theorems, we introduce some simplified notation to improve the legibility of the following results. Write $\mathbb{Y} := \mathbb{X} \times [0,1]$ and $\bar{\lambda} := \lambda \otimes dt$. We introduce a partial order on $\mathbb{Y}$ by saying that $x < y$ if $x = (z,s)$, $y = (w,u)$ and $s < u$. In the following, η will be a $(\mathbb{Y}, \bar{\lambda})$-Poisson measure and χ will be a $(\mathbb{X}, \lambda)$-Poisson measure. We will write η_x for $\eta_{|\mathbb{X}\times[0,s)}$ when $x = (z,s)$. Integrals with respect to $\bar{\lambda}$ are taken over $\mathbb{Y}$ and integrals with respect to λ are taken over $\mathbb{X}$.

The next statement contains the general abstract bounds on which our analysis will rely.

Theorem 4.2. *Let $F \in L^2(\mathbb{P}_\eta) \cap \operatorname{dom} \mathrm{D}$ such that $\mathbb{E}F = 0$. Then for any $q \in [1,2]$,*

$$d_W(F,N) \le \sqrt{\frac{2}{\pi}}\,\mathbb{E}\left|1 - \int_{\mathbb{Y}} \mathrm{D}_y F\,\mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)\right| + 2\,\mathbb{E}\int_{\mathbb{Y}} |\mathbb{E}[\mathrm{D}_y F|\eta_y]| \cdot |\mathrm{D}_y F|^q \bar{\lambda}(dy) \tag{4.3}$$

and

$$\begin{aligned} d_K(F,N) \le \mathbb{E}\left|1 - \int_{\mathbb{Y}} \mathrm{D}_y F\,\mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)\right| \\ + \sup_{z\in\mathbb{R}} \mathbb{E}\int_{\mathbb{Y}} \left|\mathbb{E}\left[\mathrm{D}_y F\middle|\eta_y\right]\right| \mathrm{D}_y F \cdot \mathrm{D}_y(F f_z(F) + \mathbb{1}_{\{F>z\}})\bar{\lambda}(dy), \end{aligned} \tag{4.4}$$

where f_z is the canonical solution to Stein's equation in the Kolmogorov case, as defined in (9.76).

This result, as it is presented here, is new. It combines ideas from [PSTU10, Thm. 3.1] and [BOPT20, Thm. 3.1] for the Wasserstein distance, and builds on [LRPY22, Thm. 1.15] for the Kolmogorov distance, while systematically replacing the use of the operator L^{-1} by a conditional expectation. The proof can be found in Section 9.3.

As a next step, we derive the upper bounds we use in applications. Define the following quantities:

$$\begin{aligned}
\beta_1 &:= \frac{2^{2/p}\sqrt{2}}{\sqrt{\pi}}\sigma^{-2}\left(\int_{\mathbb{Y}}\left(\int_{\mathbb{Y}} \mathbb{E}\left[\mathbb{E}[|\mathrm{D}_y F|\big|\eta_y]^{2p}\right]^{\frac{1}{2p}} \cdot \mathbb{E}\left[\mathbb{E}[|\mathrm{D}^{(2)}_{x,y} F|\big|\eta_{x\vee y}]^{2p}\right]^{\frac{1}{2p}} \bar{\lambda}(dy)\right)^p \bar{\lambda}(dx)\right)^{1/p} \\
\beta_2 &:= \frac{2^{2/p}}{\sqrt{2\pi}}\sigma^{-2}\left(\int_{\mathbb{Y}}\left(\int_{\mathbb{Y}} \mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}[|\mathrm{D}^{(2)}_{x,y} F|\big|\eta_y]^{2p}\right]^{1/p} \bar{\lambda}(dy)\right)^p \bar{\lambda}(dx)\right)^{1/p} \\
\beta_3 &:= 2\sigma^{-(q+1)}\mathbb{E}\int_{\mathbb{Y}} \left|\mathbb{E}\left[\mathrm{D}_y F|\,\eta_y\right]\right|^{q+1}\bar{\lambda}(dy), \\
\beta_4 &:= 2^{3-q}\sigma^{-(q+1)}\int_{\mathbb{Y}}\int_{\mathbb{Y}} \mathbb{1}_{\{y\le x\}}\mathbb{E}\left[\mathbb{E}[\mathrm{D}_y F|\eta_y]^2\right]^{1/2} \cdot \mathbb{E}\left[\left|\mathbb{E}[\mathrm{D}^{(2)}_{x,y} F|\eta_x]\right|^{2q}\right]^{1/2} \bar{\lambda}(dx)\bar{\lambda}(dy)
\end{aligned}$$

The following statement is our first bound on Wasserstein distances, expressed in terms of moments of the first- and second-order add-one costs conditional on past behaviour. We refer also to the introduction, where we have presented a version of this result in Theorem 2.3.

Theorem 4.3. *Let η be a $(\mathbb{Y},\bar{\lambda})$-Poisson-measure and let $F \in L^2(\mathbb{P}_\eta) \cap \operatorname{dom} \mathrm{D}$. Define $\sigma := \sqrt{\operatorname{Var}(F)}$ and $\hat{F} := (F - \mathbb{E}F)\sigma^{-1}$. Let $p, q \in [1,2]$. Then*

$$d_W(\hat{F}, N) \le \beta_1 + \beta_2 + \beta_3 + \beta_4. \tag{4.5}$$

The proof can be found on page 84 in Section 9.3.

Now define

$$\begin{aligned}
\gamma_1 &:= \frac{2^{2/p}\sqrt{2}}{\sqrt{\pi}}\sigma^{-2}\left(\int_{\mathbb{X}}\left(\int_{\mathbb{X}} \mathbb{E}\left[|\mathrm{D}_y F|^{2p}\right]^{\frac{1}{2p}} \cdot \mathbb{E}\left[|\mathrm{D}^{(2)}_{x,y} F|^{2p}\right]^{\frac{1}{2p}} \lambda(dy)\right)^p \lambda(dx)\right)^{1/p} \\
\gamma_2 &:= \frac{2^{2/p}}{\sqrt{2\pi}}\sigma^{-2}\left(\int_{\mathbb{X}}\left(\int_{\mathbb{X}} \mathbb{E}\left[|\mathrm{D}^{(2)}_{x,y} F|^{2p}\right]^{1/p} \lambda(dy)\right)^p \lambda(dx)\right)^{1/p} \\
\gamma_3 &:= 2\sigma^{-(q+1)}\int_{\mathbb{X}} \mathbb{E}\,|\mathrm{D}_y F|^{q+1}\,\lambda(dy)
\end{aligned}$$

and

$$\gamma_4 := \sigma^{-2}\left(4\int_{\mathbb{X}} \mathbb{E}\left[|\operatorname{D}_y F|^{2p}\right]\lambda(dy)\right)^{1/p}$$
$$\gamma_5 := \sigma^{-2}\left(4p\int_{\mathbb{X}}\int_{\mathbb{X}} \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F|^{2p}\right]\lambda(dy)\lambda(dx)\right)^{1/p}$$
$$\gamma_6 := \sigma^{-2}\left(2^{2+p}p\int_{\mathbb{X}}\int_{\mathbb{X}} \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F|^{2p}\right]^{1/2}\cdot \mathbb{E}\left[|\operatorname{D}_x F|^{2p}\right]^{1/2}\lambda(dy)\lambda(dx)\right)^{1/p}$$
$$\gamma_7 := \sigma^{-2}\left(8p\int_{\mathbb{X}}\int_{\mathbb{X}} \left(\mathbb{E}|\operatorname{D}^{(2)}_{x,y} F|^{2p}\right)^{\frac{1}{2p}}\cdot \left(\mathbb{E}|\operatorname{D}_x F|^{2p}\right)^{\frac{1}{2p}}\cdot\left(\mathbb{E}\,|\operatorname{D}_y F|^{2p}\right)^{1-1/p}\lambda(dy)\lambda(dx)\right)^{1/p}.$$

Note that the quantities $\beta_1, \ldots, \beta_4$ and $\gamma_1, \ldots, \gamma_7$ only contain expressions related to $\operatorname{D} F$ and $\operatorname{D}^{(2)} F$.

The next statement contains our main estimates on Wasserstein and Kolmogorov distances, given in terms of moments of first- and second-order add-one costs (without conditioning). Compare it also with the version given in Theorem 2.1 in the introduction. The proof can be found on page 87 in Section 9.3.

Theorem 4.4. *Let χ be a $(\mathbb{X}, \lambda)$-Poisson measure and let $F \in L^2(\mathbb{P}_\chi) \cap \operatorname{dom} \operatorname{D}$. Define $\sigma := \sqrt{\operatorname{Var} F}$ and $\hat{F} := (F - \mathbb{E}F)\sigma^{-1}$. Then for any $p, q \in [1, 2]$,*

$$d_W(\hat{F}, N) \leq \gamma_1 + \gamma_2 + \gamma_3 \tag{4.6}$$

and

$$d_K(\hat{F}, N) \leq \sqrt{\tfrac{\pi}{2}}\gamma_1 + \sqrt{\tfrac{\pi}{2}}\gamma_2 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7. \tag{4.7}$$

Remark 4.5. Using Hölder's inequality, one can replace the term γ_7 by the slightly larger but simpler bound

$$\sigma^{-2}\left(8p\int_{\mathbb{X}}\int_{\mathbb{X}} \left(\mathbb{E}|\operatorname{D}^{(2)}_{x,y} F|^{2p}\right)^{\frac{1}{2p}}\cdot \left(\mathbb{E}|\operatorname{D}_x F|^{2p}\right)^{1-\frac{1}{2p}}\lambda(dx)\lambda(dy)\right)^{1/p}. \tag{4.8}$$

We will use this bound in the proof of Theorem 6.6 in the context of the Radial Spanning Tree.

We recall that Theorem 4.4 is an extension of the results given in [LPS16] to minimal moment assumptions. For a discussion of the related literature, we refer to Section 2.1 of the introduction. Here, we give a more detailed overview of the similarities and differences between our results and those of [LPS16].

The proofs of Theorems 4.2, 4.3 and 4.4 follow in spirit the ideas from [LPS16] and [LRPY22, Theorem 1.12] (for the Kolmogorov distance). However, we work on a space $\mathbb{X} \times [0, 1]$ extended by a time component and systematically replace the operator L^{-1} by the conditional expectation $\mathbb{E}[\operatorname{D}_{(x,t)} \cdot | \eta_{|\mathbb{X}\times[0,t)}]$ (see [PT13] for a similar approach for Poisson measures on the real line). Moreover, we apply the inequalities established in Chapter 5 to achieve the improvement in the exponent. For the Wasserstein distance, we also use an improvement due to [BOPT20] to obtain the terms $\beta_3, \beta_4, \gamma_3$. For the Kolmogorov distance, our bound in Theorem 4.4 makes use of an improvement implemented in [LRPY22], but we remove a strong condition on F. The resulting bound is close in spirit to the one given in [LPS16, Theorem 1.2], but with an improvement from 4th moments to $2p$th moments. Moreover, our bound does not need the term corresponding to [LPS16, term γ_3, p. 670] and replaces the term corresponding to [LPS16, term γ_4, p. 671] by a term depending only on the add-one cost operators of F instead of $\mathbb{E}F^4$.

4.2 The Multivariate Case

In this section, we present multivariate extensions to the bounds discussed in Section 4.1. Recall that we use $\|\phi'\|_\infty$ (resp. $\|\phi''\|_\infty$ resp. $\|\phi'''\|_\infty$) to denote the supremum of the absolute values of all first (resp. second resp. third) partial derivatives of the function ϕ.

We now introduce the distances to be used in this context. Define the following two sets of functions:

- Let $\mathcal{H}_m^{(2)}$ be the set of all $\mathcal{C}^2$ functions $h : \mathbb{R}^m \to \mathbb{R}$ such that $|h(x) - h(y)| \leq \|x - y\|$ for all $x, y \in \mathbb{R}^m$ and $\sup_{x\in\mathbb{R}^m} \|\text{Hess}\, h(x)\|_{op} \leq 1$;
- let $\mathcal{H}_m^{(3)}$ be the set of all $\mathcal{C}^3$ functions $h : \mathbb{R}^m \to \mathbb{R}$ such that $\|h''\|_\infty$ and $\|h'''\|_\infty$ are bounded by 1.

Let X, Y be two m-dimensional random vectors. Define the d_2 and d_3 distances between X and Y as follows:

$$d_2(X,Y) := \sup_{h\in\mathcal{H}_m^{(2)}} |\mathbb{E}h(X) - \mathbb{E}h(Y)|\,, \quad \text{if } \mathbb{E}\|X\|, \mathbb{E}\|Y\| < \infty \tag{4.9}$$

$$d_3(X,Y) := \sup_{h\in\mathcal{H}_m^{(3)}} |\mathbb{E}h(X) - \mathbb{E}h(Y)|\,, \quad \text{if } \mathbb{E}\|X\|^2, \mathbb{E}\|Y\|^2 < \infty. \tag{4.10}$$

With these definitions, we can state the multivariate analogue of Theorem 4.4 (see also Theorem 2.5 for a version of this result).

Define the terms $\zeta_1, ..., \zeta_4$ by

$$\zeta_1 := \sum_{i,j=1}^m |C_{ij} - \text{Cov}(F_i, F_j)|$$

$$\zeta_2 := 2^{2/p-1} \sum_{i,j=1}^m \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \mathbb{E}\left[|\, \mathrm{D}_{x,y}^{(2)} F_i|^{2p} \right]^{1/2p} \mathbb{E}\left[|\, \mathrm{D}_{x,y}^{(2)} F_j|^{2p} \right]^{1/2p} \lambda(dx) \right)^p \lambda(dy) \right)^{1/p}$$

$$\zeta_3 := 2^{2/p} \sum_{i,j=1}^m \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \mathbb{E}\left[|\, \mathrm{D}_x F_i|^{2p} \right]^{1/2p} \mathbb{E}\left[|\, \mathrm{D}_{x,y}^{(2)} F_j|^{2p} \right]^{1/2p} \lambda(dx) \right)^p \lambda(dy) \right)^{1/p}$$

$$\zeta_4 := m^{q-1} \sum_{i,j=1}^m \int_{\mathbb{X}} \mathbb{E}\left[|\, \mathrm{D}_x F_i|^{q+1} \right]^{1/(q+1)} \mathbb{E}\left[|\, \mathrm{D}_x F_j|^{q+1} \right]^{1-1/(q+1)} \lambda(dx).$$

Then the following statement holds.

Theorem 4.6. *Let χ be a $(\mathbb{X}, \lambda)$-Poisson measure. Let $m \geq 1$ and let $F = (F_1, ..., F_m)$ be an $\mathbb{R}^m$-valued random vector such that for $1 \leq i \leq m$, we have $F_i \in L^2(\mathbb{P}_\chi) \cap \text{dom}\, \mathrm{D}$ and $\mathbb{E}F_i = 0$. Let $C = (C_{ij})_{1\leq i,j\leq m}$ be a symmetric positive-semidefinite matrix and let $X \sim \mathcal{N}(0, C)$. Then for all $p, q \in [1, 2]$,*

$$d_3(F, X) \leq \tfrac{1}{2}(\zeta_1 + \zeta_2 + \zeta_3) + \zeta_4. \tag{4.11}$$

If moreover the matrix C is positive-definite, then for all $p, q \in [1, 2]$,

$$d_2(F, X) \leq \|C^{-1}\|_{op} \|C\|_{op}^{1/2} (\zeta_1 + \zeta_2 + \zeta_3) + \left(2\|C^{-1}\|_{op} \|C\|_{op}^{1/2} \vee \tfrac{\sqrt{2\pi}}{8} \|C^{-1}\|_{op}^{3/2} \|C\|_{op} \right) \zeta_4. \tag{4.12}$$

The proof of these bounds is located in Section 9.4. It relies on the work [PZ10], where the bound on the d_2 distance uses the Malliavin-Stein method, whereas the one for the d_3 distance uses an interpolation technique taken from the context of spin glasses (see [Tal03]). Building on the results of [PZ10], we adapt the univariate improvement of [BOPT20] to a multivariate setting and make use of the p-Poincaré inequality (5.6) as we did for the proof of Theorem 4.3. Theorem 4.6 is an extension of [SY19, Thm. 1.1] to minimal moment requirements. In [SY19, Thm. 1.2], the authors also develop a bound for the convex distance — extending such a bound to minimal moment assumptions is under investigation.

For a discussion of the surrounding literature we refer to Section 1.8.

Ancillary Results: New Estimates for Skorohod Integrals

5.1 A version of Itô formula

We start this section by giving a new version of Itô formula for Poisson integrals with anticipative integrands. This is a crucial ingredient for the proof of the new estimates given in Theorem 5.2. In the following, we will take η to be a $(\mathbb{X}\times[0,1], \mathcal{X}\otimes\mathcal{B}([0,1]), \lambda(dx)\otimes ds)$-Poisson measure, where $(\mathbb{X}, \mathcal{X}, \lambda)$ is a σ-finite measure space. We refer also to Theorem 2.18, where we have anticipated the following statement.

Theorem 5.1 Itô formula for non-adapted integrands**.** *Let $h \in L^1(\mathbf{N} \times \mathbb{X} \times [0,1])$ be bounded and let $X_0 \in \mathbb{R}$. For $t \in [0,1]$, define*

$$X_t(\eta) := X_0 + \int_{\mathbb{X}\times[0,t]} h(\eta - \delta_{(y,s)}, y, s)\eta(dy, ds) - \int_{\mathbb{X}}\int_0^t h(\eta, y, s)\lambda(dy)ds. \tag{5.1}$$

Then the process $(X_t)_{t\in[0,1]}$ is well-defined and $\mathbb{P}$-a.s. càdlàg. Let $\phi \in \mathcal{C}^1(\mathbb{R})$. Then, $\forall\, t \in [0,1]$,

$$\phi(X_t) = \phi(X_0) + \int_{\mathbb{X}\times[0,t]} \left(\phi\left(X_{s-} + h(\eta - \delta_{(y,s)}, y, s)\right) - \phi(X_{s-})\right)\eta(dy, ds) \\ - \int_{\mathbb{X}}\int_0^t \phi'(X_s)h(\eta, y, s)\lambda(dy)ds \qquad \mathbb{P}\text{-a.s.} \tag{5.2}$$

and the quantities in (5.2) are well-defined.

We recall that the main difference between (5.2) and the classical Itô formula given in [IW81, Theorem II.5.1] is that we allow for integrands to be anticipative, in a more restrictive framework of stochastic integrals w.r.t. Poisson measures without Wiener part. For a detailed comparison between the two results, we refer to Remark 2.19 in the introduction.

5.2 Moment Inequalities

In this section, we present a number of new functional inequalities that are of independent interest and also crucial to the improved bounds on Wasserstein and Kolmogorov distances presented in earlier sections. The proofs of all three results are located in Section 9.2.

We recall that, to the best of the author's knowledge, Theorem 5.2 is the first result giving moment bounds of general Skorohod integrals. Partial results when h is predictable are discussed in Remark 2.16 of the introduction, where one can also find a detailed literature review concerning related results.

In the special case $\phi(x) = x^2$, the theorem below follows immediately from the isometry relation reported in formula (8.4) of Chapter 8. Note that we stated this result in Theorem 2.15 as part of the introduction.

Theorem 5.2. *Let $h \in L^2(\mathbf{N}\times\mathbb{X}\times[0,1])$ satisfy (3.13). Let $\phi : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $(p-1)$-Hölder continuous derivative, for some $p \in (1,2]$ and assume that $\phi(0) = 0$. Then*

$$\begin{aligned} |\mathbb{E}\phi(\delta(h))| \leq{}& \frac{c_\phi}{p}\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\, |h(\eta,y,s)|^p \\ &+ c_\phi\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\, \left|\mathrm{D}_{(y,s)}\, h(\eta,x,t)\right|\cdot\left|\mathrm{D}_{(x,t)}\, h(\eta,y,s)\right|^{p-1} \\ &+ 2c_\phi\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\, \left|\mathrm{D}_{(y,s)}\, h(\eta,x,t)\right|\cdot|h(\eta,y,s)|^{p-1}, \end{aligned} \tag{5.3}$$

where c_ϕ is the Hölder constant of ϕ'. In particular, this inequality holds with $\phi(x) = |x|^p$ and $c_\phi = p2^{2-p}$, for $p \in [1,2]$.

Our first corollary is a version of the above inequality for predictable functions h and contains a generalisation of the classical Poincaré inequality.

Corollary 5.3. *Let $h \in L^2(\mathbf{N}\times\mathbb{X}\times[0,1])$ be predictable in the sense of (3.17). Let $\phi : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $(p-1)$-Hölder continuous derivative, for some $p \in (1,2]$. Assume $\phi(0) = 0$. Then*

$$|\mathbb{E}\phi(\delta(h))| \leq \frac{c_\phi}{p}\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds|h(\eta,y,s)|^p. \tag{5.4}$$

Moreover, for $F \in L^2(\mathbb{P}_\eta)$ and $p \in [1,2]$,

$$\mathbb{E}|F|^p - |\mathbb{E}F|^p \leq 2^{2-p}\mathbb{E}\int_{\mathbb{X}}\int_0^1 \left|\mathbb{E}[\mathrm{D}_{(x,t)}\, F|\eta_{\mathbb{X}\times[0,t)}]\right|^p\lambda(dx)dt. \tag{5.5}$$

Remark 5.4. 1. We can extend inequality (5.5) to $F \in L^1(\mathbb{P}_\eta)$ at the cost of introducing an additional absolute value on the RHS:

$$\mathbb{E}|F|^p - |\mathbb{E}F|^p \leq 2^{2-p}\mathbb{E}\int_{\mathbb{X}}\int_0^1 \mathbb{E}[|\,\mathrm{D}_{(x,t)}\, F||\eta_{\mathbb{X}\times[0,t)}]^p\lambda(dx)dt. \tag{5.6}$$

This can be seen easily by approximating F by $F_n := (F\wedge n)\vee(-n)$ and using monotone and dominated convergence.

2. When removing the conditional expectation in (5.5) using Jensen's inequality, the inequality can be extended to functionals $G \in L^1(\mathbb{P}_\chi)$, where χ is a $(\mathbb{X}, \lambda)$-Poisson measure without time component. Indeed, as discussed in Chapter 3, the marginal $\eta(. \times [0,1])$ has the same law as χ, which means one can see G as a functional on $\mathbf{N}_{\mathbb{X}\times[0,1]}$. We have then

$$\mathbb{E}|G|^p - |\mathbb{E}G|^p \leq 2^{2-p}\mathbb{E}\int_{\mathbb{X}} |\mathrm{D}_x G|^p \, \lambda(dx). \tag{5.7}$$

Compare also with Proposition 2.14, where we already stated (5.7).

The versatility of Theorem 5.2 can be appreciated when considering the following corollary, which will be crucial in finding a bound on the Kolmogorov distance.

Corollary 5.5. *Let* $h \in L^1(\mathbf{N} \times \mathbb{X} \times [0,1])$ *and* $G \in L^0(\mathbb{P}_\eta)$ *bounded by a constant* $c_G > 0$. *Then for any* $p \in [1,2]$,

$$\begin{aligned}
&\left|\mathbb{E}\int_{\mathbb{X}}\int_0^1 h(\eta,x,s)\,\mathrm{D}_{(x,s)}\,G\lambda(dx)ds\right| \\
&\leq c_G\Bigg(2^{2-p}\mathbb{E}\int_{\mathbb{X}}\int_0^1 |h(\eta,x,s)|^p\lambda(dx)ds \\
&+ p2^{2-p}\mathbb{E}\int_{\mathbb{X}}\int_0^1\int_{\mathbb{X}}\int_0^1 |\,\mathrm{D}_{(x,s)}\,h(\eta,y,u)|^p\lambda(dx)ds\lambda(dy)du \\
&+p2^{3-p}\int_{\mathbb{X}}\int_0^1\int_{\mathbb{X}}\int_0^1 \mathbb{1}_{\{s<u\}}\mathbb{E}[|\,\mathrm{D}_{(y,u)}\,h(\eta,x,s)|^p]^{1/p}\mathbb{E}[|h(\eta,y,u)|^p]^{1-1/p}\lambda(dx)ds\lambda(dy)du\Bigg)^{1/p}.
\end{aligned} \tag{5.8}$$

Remark 5.6. Provided that we upper bound the indicator in the third term on the RHS of (5.8) by 1, this inequality can be extended to a space $\mathbb{X}$ without time component.

6 Univariate Applications

In this chapter, we look at four types of graphs built on Poisson measures and assess the speeds of convergence to normality of sums of functions of edge-lengths, in particular sums of α-power-weighted edge-lengths such as (2.13). As was found in previous work [LPS16, ST17], we find for certain ranges of exponents α that the speed is given by $t^{-d/2}$, which corresponds to the order of the square root of the variance. This is the presumably optimal speed corresponding to the one in the classical Berry-Esseen theorem (see e.g. [Pet75, Theorem 4, p. 111]). Beyond a certain threshold, we find a slower speed of convergence that depends on α. Generally speaking, a $2p$th moment integrability of the first and second order add-one costs of the functionals leads to a speed of convergence of $t^{-d(1-1/p)}$. Whether this speed is optimal or not is an open question.

6.1 Online Nearest Neighbour Graph

We conducted a preliminary discussion of the ONNG in Section 2.2.1 of the introduction. In this section, we will give a more detailed definition and state our results in their full generality.

Let $\mu \subset \mathbb{R}^d \times [0,1]$ be a finite set such that the projection of μ onto $\mathbb{R}^d$ is generic and does not contain any multiplicities and the projections onto $[0,1]$ are distinct. The ONNG on μ is an (undirected) graph in $\mathbb{R}^d$ constructed as follows:

- Vertices are given by $\{x \in \mathbb{R}^d : (x,s) \in \mu\}$
- Let $(x,s) \in \mu$. If $\mu \cap \left(\mathbb{R}^d \times [0,s)\right)$ is non-empty, then the online nearest neighbour of (x,s) is given by the point $(z,u) \in \mu \cap \left(\mathbb{R}^d \times [0,s)\right)$ which min-

Figure 6.1: Realisation of the Online Nearest Neighbour graph

imises $|x - z|$. In this case there is an edge from x to z and we denote this event by $\{(x, s) \to (z, u) \text{ in } \mu\}$.

For a point $(x, s) \in \mu$, the coordinate s can be seen as the arrival time of the point $x \in \mathbb{R}^d$, or its mark. Any point $(x, s) \in \mu$ has exactly one online nearest neighbour, except for the point in μ whose mark is minimal, which has none. Even though the graph is undirected, we think of arrows going from a point to its nearest neighbour, as this simplifies the discussion.

To define our functional of interest, let

$$e(x, s, \mu) := \begin{cases} \inf\{|x - z| : (z, u) \in \mu \cap \big(\mathbb{R}^d \times [0, s)\big)\}, & \text{if } \mu \cap \big(\mathbb{R}^d \times [0, s)\big) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \tag{6.1}$$

for $(x, s) \in \mu$. This is the length of the edge from x to its online nearest neighbour if there is one, and zero otherwise. Note that the online nearest neighbour in μ is unique for any point $(x, s) \in \mathbb{R}^d \times [0, 1]$ such that the time coordinate s and the position x do not occur in μ. For convenience, we shall extend the above definitions to any such $(x, s) \in \mathbb{R}^d \times [0, 1]$ and tacitly adopt the corresponding notation.

We will be studying the sums of power-weighted edge-lengths defined as follows: for $\alpha > 0$, let

$$F^{(\alpha)}(\mu) := \sum_{(x,s)\in\mu} e(x, s, \mu)^\alpha. \tag{6.2}$$

Note that here we make use of the convention explained in Chapter 3 to identify a set of points μ with the point measure whose support is given by μ.

Let η be a Poisson measure on $\mathbb{R}^d \times [0, 1]$ with Lebesgue intensity. Let $H \subset \mathbb{R}^d$ be a convex body. For $t \geq 1$, define

$$F_t^{(\alpha)} := F^{(\alpha)}(\eta_{|tH\times[0,1]}). \tag{6.3}$$

We recall that a quantitative CLT was shown for this functional in [Pen05, Thm. 3.6] in the exponent range $0 \leq \alpha < \frac{d}{4}$, and a similar CLT was conjectured in [Wad09] for the exponent range $\frac{d}{4} \leq \alpha \leq \frac{d}{2}$ (see the discussion in Section 2.2.1 and Conjecture 2.7).

Our forthcoming Theorem 6.1 confirms Conjecture 2.7 by giving quantitative central limit theorems for $\alpha \in \big(0, \frac{d}{2}\big]$ and upper and lower bounds for the variances that match the conjectured orders. Upper bounds of the conjectured orders were already given in [Wad09, Theorem 2.1] for the variances involved. They are shown for an ONNG built on n uniformly distributed random variables and the corresponding result for the Poisson version follows by Poissonisation. For the sake of completeness, we will give purely Poissonian proofs of the upper bounds, following however a similar strategy as in [Wad09].

For a further discussion of the related literature, we refer to Section 2.2.1 of the introduction. We recall our convergence statement which we already alluded to in Theorem 2.8 of the introduction.

Theorem 6.1. *For $0 < \alpha < \frac{d}{2}$, and for every $1 < p < \frac{d}{2\alpha}$ such that $p \leq 2$, there is a constant $c_1 > 0$ such that for all $t \geq 1$ large enough*

$$\max\left\{ d_W\left(\frac{F_t^{(\alpha)} - \mathbb{E}F_t^{(\alpha)}}{\sqrt{\operatorname{Var}\big(F_t^{(\alpha)}\big)}}, N\right), d_K\left(\frac{F_t^{(\alpha)} - \mathbb{E}F_t^{(\alpha)}}{\sqrt{\operatorname{Var}\big(F_t^{(\alpha)}\big)}}, N\right)\right\} \leq c_1 t^{-d\left(1-\frac{1}{p}\right)}, \tag{6.4}$$

where N denotes a standard normal random variable. Moreover, there are constants $c_2, C_2 > 0$ such that for all $t \geq 1$ large enough

$$c_2 t^d < \mathrm{Var}(F_t^{(\alpha)}) < C_2 t^d. \tag{6.5}$$

For $\alpha = \frac{d}{2}$, there is a constant $c_3 > 0$ such that for all $t \geq 1$ large enough

$$d_W \left(\frac{F_t^{(d/2)} - \mathbb{E} F_t^{(d/2)}}{\sqrt{\mathrm{Var}\left(F_t^{(d/2)}\right)}}, N \right) \leq c_3 \log(t)^{-1}. \tag{6.6}$$

Moreover, there are constants $c_4, C_4 > 0$ such that for all $t \geq 1$ large enough

$$c_4 t^d \log(t^d) < \mathrm{Var}(F_t^{(d/2)}) < C_4 t^d \log(t^d). \tag{6.7}$$

The constants $c_1, c_2, C_2, c_3, c_4, C_4$ may depend on $H, \ \alpha, \ d$ and p.

We recall that, in the special case $0 < \alpha < \frac{d}{4}$, we find a speed of convergence of $t^{-d/2}$, which corresponds to the square root of the order of the variance. The proof of this theorem can be found in Section 9.5.

6.2 Gilbert Graph

For a finite set $\mu \subset \mathbb{R}^d$ and a real number $\epsilon > 0$, the Gilbert graph $G(\mu, \epsilon)$ has vertex set μ and an edge between $x, y \in \mu$, $x \neq y$ if and only if $|x - y| < \epsilon$. To construct our functional of interest, we consider

- $W \subset \mathbb{R}^d$ a convex body;
- for every $t > 0$, we take η^t an $(\mathbb{R}^d, t\,dx)$-Poisson measure;
- $(\epsilon_t)_{t>0}$ a sequence of positive real numbers s.t. $\epsilon_t \to 0$ as $t \to \infty$.

Then for $\alpha \in \mathbb{R}$, define

$$\begin{aligned} L_t^{(\alpha)}(W) &:= \sum_{e \in G(\eta^t_{|W}, \epsilon_t)} |e|^\alpha \\ &= \frac{1}{2} \sum_{x,y \in \eta^t_{|W}, x \neq y} \mathbb{1}_{\{|x-y| \leq \epsilon_t\}} |x-y|^\alpha, \end{aligned} \tag{6.8}$$

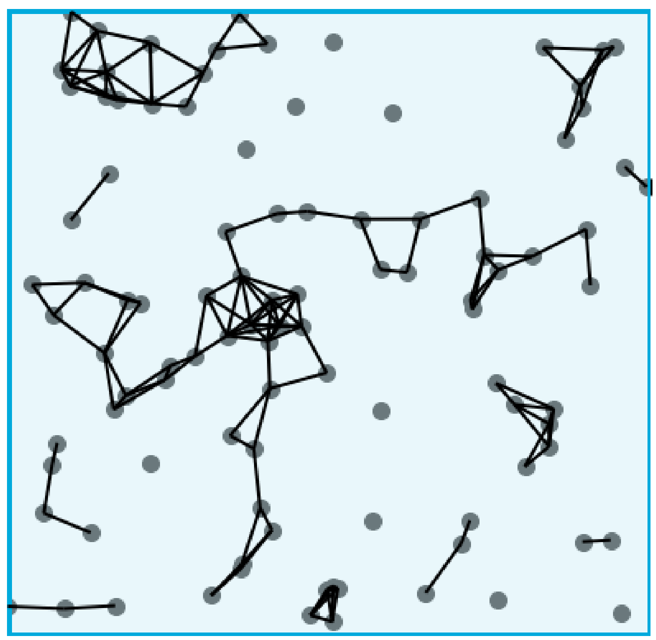


Figure 6.2: Realisation of the Gilbert graph

where e denote the edges of the graph and $|e|$ their length. When it is clear from context, and in particular in this section, we write $L_t^{(\alpha)}$ without specifying the set W.

Define

$$\hat{L}_t^{(\alpha)} := \frac{L_t^{(\alpha)} - \mathbb{E} L_t^{(\alpha)}}{\sqrt{\mathrm{Var}\left(L_t^{(\alpha)}\right)}}. \tag{6.9}$$

Remark 6.2. For the sake of continuity with the article [RST17], we use the convention that the intensity $t\,dx$ of η^t grows and the observation window W stays constant. In the other applications presented in this article, we keep the intensity constant and instead let the observation window grow as tW. Note that one can pass from one setting to the other by a simple rescaling. Indeed, consider $\tilde{\eta}$ a Poisson measure on $\mathbb{R}^d$ and Lebesgue intensity and for $s \geq 1$, construct a Gilbert graph on $\tilde{\eta}_{|sH}$ by connecting two points $x \neq y \in \tilde{\eta}_{|sH}$ if and only if $|x-y| < \tilde{\epsilon}_s$. For $\alpha \in \mathbb{R}$, let

$$F_s^{(\alpha)} := \frac{1}{2} \sum_{x,y \in \tilde{\eta}_{|sW}, x \neq y} \mathbb{1}_{\{|x-y|<\tilde{\epsilon}_s\}} |x-y|^{\alpha}. \tag{6.10}$$

Then $F_t^{(\alpha)}$ is equal in law to $s^{\alpha} L_{s^d}^{(\alpha)}$ with $\epsilon_{s^d} = s^{-1}\tilde{\epsilon}_s$. The central limit theorem for $F_t^{(\alpha)}$ can be deduced from the one for $L_t^{(\alpha)}$.

For a discussion of the surrounding literature and the history of the Gilbert graph, we refer to Section 2.2.2 of the introduction. Here we recall that a qualitative CLT for the functional $\hat{L}^{(\alpha)}$ for exponents $\alpha > -\frac{d}{2}$ was already given [RST17], along with a quantitative Kolmogorov bound when $\alpha > -\frac{d}{4}$. We recover the same speed of convergence in the case $\alpha > -\frac{d}{4}$ and extend to the regime where $-\frac{d}{2} < \alpha \leq -\frac{d}{4}$. Below, we repeat the statement of our convergence result (see also Theorem 2.9). The proof is presented in Section 9.6.

Theorem 6.3. *Let $\alpha > -\frac{d}{2}$ and assume that $t^2\epsilon_t^d \to \infty$ as $t \to \infty$. Then for $t \geq 1$ large enough*

- *if $\alpha > -\frac{d}{4}$, there is a constant $c_1 > 0$ such that*

$$\max\left\{d_W\left(\hat{L}_t^{(\alpha)}, N\right), d_K\left(\hat{L}_t^{(\alpha)}, N\right)\right\} \leq c_1\left(t^{-1/2} \vee (t^2\epsilon_t^d)^{-1/2}\right). \tag{6.11}$$

- *if $-\frac{d}{2} < \alpha \leq -\frac{d}{4}$, then for any $1 < p < -\frac{d}{2\alpha}$, there is a constant $c_2 > 0$ such that*

$$\max\left\{d_W\left(\hat{L}_t^{(\alpha)}, N\right), d_K\left(\hat{L}_t^{(\alpha)}, N\right)\right\} \leq c_2\left(t^{-1+1/p} \vee (t^2\epsilon_t^d)^{-1+1/p}\right). \tag{6.12}$$

The speed of convergence varies according to the asymptotic behaviour of $t\epsilon_t^d$. Indeed, one has

$$t^{-1+1/p} \vee (t^2\epsilon_t^d)^{-1+1/p} = \begin{cases} (t^2\epsilon_t^d)^{-1+1/p} & \text{if } t\epsilon_t^d \to 0 \text{ (sparse regime)} \\ t^{-1+1/p} & \text{if } t\epsilon_t^d \to \theta > 0 \text{ (thermodynamic regime)} \\ t^{-1+1/p} & \text{if } t\epsilon_t^d \to \infty \text{ (dense regime)} \end{cases} \tag{6.13}$$

Remark 6.4. A careful inspection of the bounds applied to γ_3 in the proof of Theorem 6.3 reveals that in the sparse regime ($t\epsilon_t^d \to 0$) when $-\frac{d}{2} < \alpha \leq -\frac{d}{4}$, a slightly improved rate can be found for the Wasserstein distance. Indeed, for any $1 < p < -\frac{d}{2\alpha}$ and any $0 < r < -\frac{d}{\alpha} - 2$, there is a constant $c > 0$ such that

$$d_W(\hat{L}_t^{(\alpha)}, N) \leq c\left(t^{-1+1/p} \vee (t^2\epsilon_t^d)^{-r/2}\right). \tag{6.14}$$

Since $\frac{r}{2} \in \left(0, -\frac{d}{2\alpha} - 1\right)$ and $1 - \frac{1}{p} \in \left(0, 1 + \frac{2\alpha}{d}\right)$ and $1 + \frac{2\alpha}{d} < -\frac{d}{2\alpha} - 1$, one can choose $\frac{r}{2} > 1 - \frac{1}{p}$. This then gives a slightly faster convergence rate. As an illustrating example, consider the case where $\epsilon_t^d = t^{-\theta}$ with $1 < \theta < 2$. Then $t\epsilon_t^d \to 0$ and $t^2\epsilon_t^d \to \infty$. Theorem 6.3 provides the rate of convergence $t^{(-1+\frac{1}{p})(2-\theta)}$, and by following this strategy it can be improved to $t^{-1+\frac{1}{p}} \vee t^{-\frac{r}{2}(2-\theta)}$ with $\frac{r}{2} > 1 - \frac{1}{p}$.

6.3 k-Nearest Neighbour Graphs

For a finite generic set $\mu \subset \mathbb{R}^d$ (see page 42 of Chapter 3 for the definition of **generic**) and a positive integer $k \in \mathbb{N}$, the k-**Nearest Neighbour graph** has vertex set μ and an edge between $x, y \in \mu$ if and only if y is one of the k nearest points to x or vice-versa.

For our functional of interest, consider the following framework:

- $H \subset \mathbb{R}^d$ is a convex body;
- η is an $(\mathbb{R}^d, dx)$-Poisson measure;
- $\phi : (0, \infty) \to (0, \infty)$ is a decreasing function such that there is an $r > 2$ verifying

$$\int_0^1 \phi(s)^r s^{d-1} ds < \infty. \tag{6.15}$$

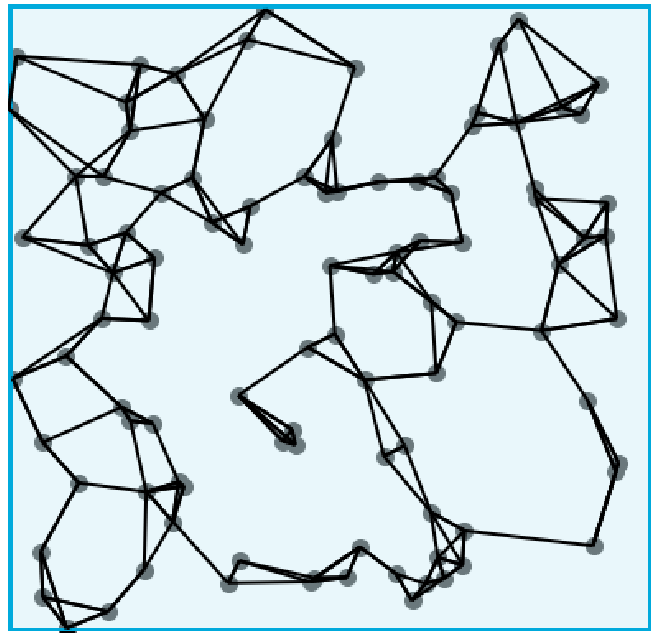


Figure 6.3: Realisation of a 3-Nearest Neighbour graph

For any finite generic set $\mu \subset \mathbb{R}^d$, define

$$F(\mu) := \frac{1}{2} \sum_{x,y \in \mu, x \neq y} \mathbb{1}_{\{x \in N(y,\mu) \text{ or } y \in N(x,\mu)\}} \phi(|x - y|), \tag{6.16}$$

where $N(x, \mu)$ is the set of k-nearest neighbours of x in μ. For $t \geq 1$, define $F_t := F(\eta_{|tH})$ and set $\hat{F}_t := (F_t - \mathbb{E}F_t)\,\mathrm{Var}(F_t)^{-1/2}$.

For a discussion of the literature, we refer to Section 2.2.4. We recall that a quantitative CLT for sums of power-weighted edge-lengths has been given in [LPS16] for the case $\alpha \geq 0$, at a speed of convergence of $t^{-d/2}$, which we complement by a result in the range $\alpha \in \left(-\frac{d}{2}, 0\right)$, recovering the same speed of convergence when $\alpha > -\frac{d}{4}$. We repeat our convergence result below, having anticipated the statement in Theorem 2.12. The proof can be found in Section 9.8.

Theorem 6.5. *Under the conditions stated above, for any $p \in (1, 2]$ such that $p < \frac{r}{2}$, there is a constant $c > 0$ such that, for $t \geq 1$,*

$$\max\left\{d_W\left(\hat{F}_t, N\right), d_K\left(\hat{F}_t, N\right)\right\} \leq ct^{d(1/p-1)}. \tag{6.17}$$

This inequality holds in particular for the function $\phi(x) = x^{-\alpha}$ with $0 < \alpha < \frac{d}{2}$, for any $p \in (1, 2]$ such that $p < \frac{d}{2\alpha}$.

6.4 Radial Spanning Tree

Let $\mu \subset \mathbb{R}^d \setminus \{0\}$ be a finite set, generic with respect to the point 0. The radial spanning tree on μ, in short $RST(\mu)$, is constructed as follows:

- The set of vertices is given by $\mu \cup \{0\}$;
- for every $x \in \mu$, we add exactly one edge to the point $z \in \left(\mu \cap B^d(0,|x|) \cup \{0\}\right)$ which minimises $|x-z|$. We call z the radial nearest neighbour of x and say 'x connects to z', denoted by '$x \to z$ in μ'. We denote the length $|x-z|$ by $g(x,\mu)$.

In order to define our functional of interest, consider the following setting:

- $H \subset \mathbb{R}^d$ a convex body such that $B^d(0,\epsilon) \subset H$ for some $\epsilon > 0$;
- η is an $(\mathbb{R}^d, dx)$-Poisson measure;
- $\phi : (0,\infty) \to (0,\infty)$ is a decreasing function such that there is an $r > 2$ satisfying

$$\int_0^1 \phi(s)^r s^{d-1} ds < \infty. \tag{6.18}$$

For any finite set $\mu \subset \mathbb{R}^d$ generic with respect to 0, define

$$F(\mu) := \sum_{x \in \mu} \phi(g(x,\mu)) \tag{6.19}$$

and for $t \geq 1$, define $F_t := F(\eta_{|tH})$. Set $\hat{F}_t := (F_t - \mathbb{E}F_t)\,\mathrm{Var}(F_t)^{-1/2}$.

For a discussion of literature, see Section 2.2.5. We recall that rescaling the result from [ST17, Theorem 1.2], one derives a CLT of speed $t^{-d/2}$ for α-power-weighted edge-lengths where $\alpha \geq 0$. We extend the known range of convergence to exponents $\alpha \in \left(-\frac{d}{2}, 0\right)$, recovering the same speed when $\alpha > -\frac{d}{4}$. This CLT is new, even as a qualitative result. We also recall our full convergence result below, after having mentioned it in Theorem 2.13 already. The proof of Theorem 6.6 is located in Section 9.9.

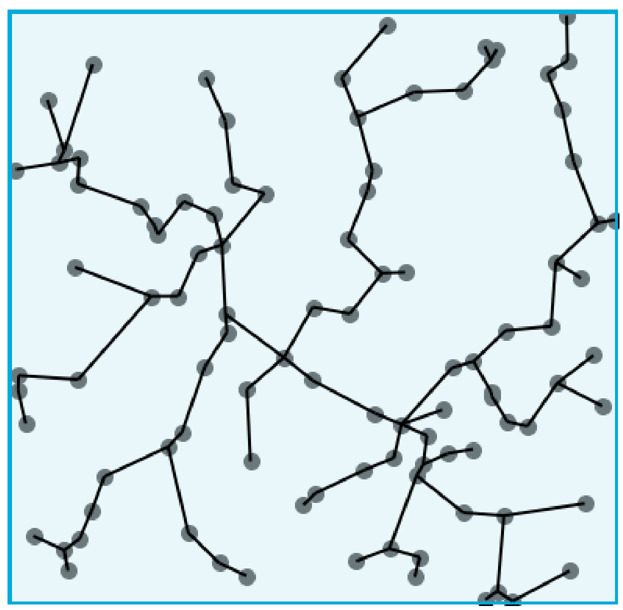
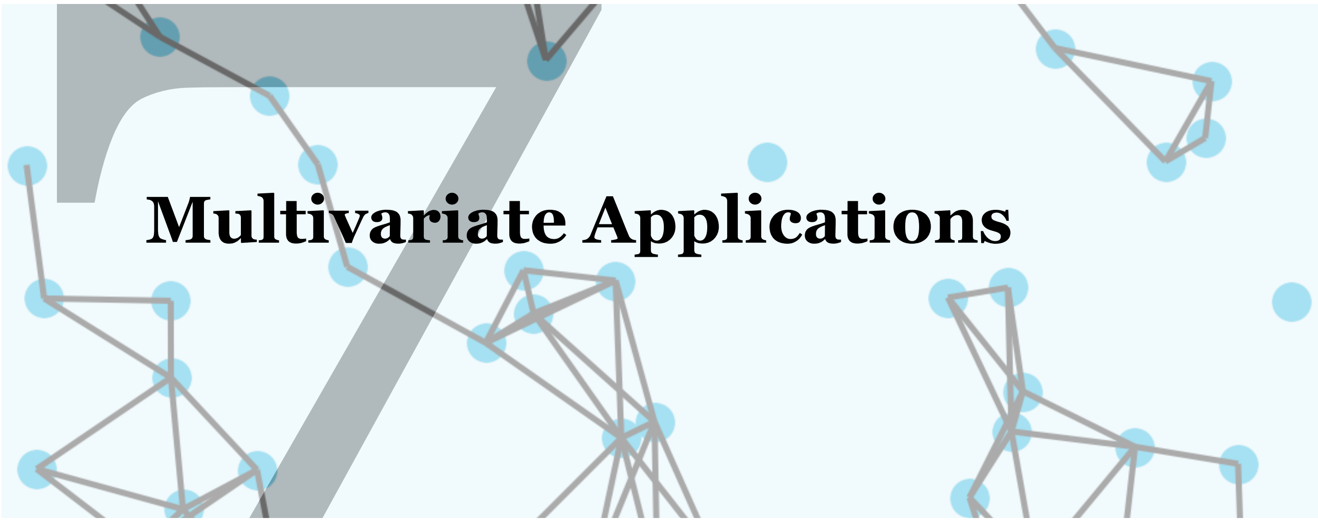


Figure 6.4: Realisation of a Radial Spanning Tree

Theorem 6.6. *Under the conditions stated above, for any $p \in (1,2]$ such that $p < \frac{r}{2}$, there is a constant $c > 0$ such that for $t \geq 1$,*

$$\max\left\{d_W\left(\hat{F}_t, N\right), d_K\left(\hat{F}_t, N\right)\right\} \leq ct^{d(1/p-1)}. \tag{6.20}$$

This inequality holds in particular for the function $\phi(x) = x^{-\alpha}$ with $0 < \alpha < \frac{d}{2}$, for any $p \in (1,2]$ such that $p < \frac{d}{2\alpha}$.



7 Multivariate Applications

In this section, we study two multivariate functionals of the Gilbert graph. Both of these functionals consist of sums of power-weighted edge-lengths — in the first functional, we vary the powers of the edges-lengths, and in the second functional, we restrict the graph to different domains. In both cases, we achieve quantitative central limit theorems with speeds akin to what we found in the univariate case.

Note that the results in this chapter are work in progress by the author and have not yet appeared in any preprint or publication.

7.1 Varying the Powers

We use the same setting as in Section 6.2. Let $W \subset \mathbb{R}^d$ be a convex body and consider real numbers $\alpha_1, \dots, \alpha_m$ such that $\alpha_i + \alpha_j > -d$ for all $i, j \in \{1, \dots, m\}$. For every $1 \leq i \leq m$, define $L_t^{(\alpha_i)}(W) = L_t^{(\alpha_i)}$ as in Section 6.2 and set

$$\tilde{L}_t^{(\alpha_i)} := \left(t\epsilon_t^{\alpha_i+d/2} \vee t^{3/2}\epsilon_t^{\alpha_i+d}\right)^{-1}\left(L_t^{(\alpha_i)} - \mathbb{E}L_t^{(\alpha_i)}\right). \tag{7.1}$$

Set furthermore

$$\sigma_{ij}^{(1)} := \frac{d\kappa_d}{2|\alpha_i + \alpha_j + d|} \quad \text{and} \quad \sigma_{ij}^{(2)} := \frac{d^2\kappa_d^2}{(\alpha_i + d)(\alpha_j + d)} \tag{7.2}$$

and define the matrix $C = (C_{ij})_{1 \leq i,j \leq m}$ by $C_{ij} = |W| c_{ij}$, where

$$c_{ij} := \begin{cases} \sigma_{ij}^{(1)} & \text{if } \lim_{t\to\infty} t\epsilon_t^d = 0 \\ \left(\sigma_{ij}^{(1)} + \theta\sigma_{ij}^{(2)}\right) & \text{if } \lim_{t\to\infty} t\epsilon_t^d = \theta \leq 1 \\ \left(\frac{1}{\theta}\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}\right) & \text{if } \lim_{t\to\infty} t\epsilon_t^d = \theta > 1 \\ \sigma_{ij}^{(2)} & \text{if } \lim_{t\to\infty} t\epsilon_t^d = \infty \end{cases} \tag{7.3}$$

Defining the vector $\tilde{L}_t$ as

$$\tilde{L}_t := \left(\tilde{L}_t^{(\alpha_1)}, \dots, \tilde{L}_t^{(\alpha_m)}\right), \tag{7.4}$$

we recall that a CLT for $\tilde{L}_t$ has been shown in [RST17, Thm. 5.2], as well as convergence of the covariance matrix to C in [RST17, Thm. 3.3]. The matrix C is positive-definite in the sparse and thermodynamic regime (i.e. if $t\epsilon_t^d \to 0$ or $t\epsilon_t^d \to c$), while it is singular in the dense regime (i.e. if $t\epsilon_t^d \to \infty$), see [RST17, Prop. 3.4]. Define also

$$\beta_{ij}^{(t)} := \frac{\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} t\epsilon_t^d}{1 \vee t\epsilon_t^d} \tag{7.5}$$

and note that $\beta_{ij}^{(t)} \to c_{ij}$ as $t \to \infty$.

Theorem 7.1. *Assume that $t^2\epsilon_t^d \to \infty$ as $t \to \infty$. Let $X \sim \mathcal{N}(0, C)$ be a centred Gaussian with covariance matrix C. Then for $t \geq 1$ large enough*

- *if $\alpha_1, ..., \alpha_m > -\frac{d}{4}$, there is a constant $c_1 > 0$ such that*

$$d_3(\tilde{L}_t, X) \leq c_1 \left(\epsilon_t + \max_{1 \leq i,j \leq m} \left| \beta_{ij}^{(t)} - c_{ij} \right| + \left(t^{-1/2} \vee (t^2\epsilon_t^d)^{-1/2} \right) \right). \tag{7.6}$$

- *if $-\frac{d}{2} < \min\{\alpha_1, ..., \alpha_m\} \leq -\frac{d}{4}$, then for any $1 < p < -\frac{d}{2}\min\{\alpha_1, ..., \alpha_m\}^{-1}$, there is a constant $c_2 > 0$ such that*

$$d_3(\tilde{L}_t, X) \leq c_2 \left(\epsilon_t + \max_{1 \leq i,j \leq m} \left| \beta_{ij}^{(t)} - c_{ij} \right| + \left(t^{-1+1/p} \vee (t^2\epsilon_t^d)^{-1+1/p} \right) \right). \tag{7.7}$$

  If $\lim_{t\to\infty} t\epsilon_t^d < \infty$, then the bounds (7.6) and (7.7) apply to $d_2(\tilde{L}_t, X)$ as well for different constants $c_1, c_2 > 0$.

The proof of this theorem can be found in Section 9.7. The bounds given in Theorem 7.1 vary according to the limit of $t\epsilon^d$. We give a precise discussion of the bounds (7.7) in Table 7.1. The bounds for (7.6) follow when setting $p = 2$.

	c_{ij}	speed of convergence in (7.7)	bound holds for
$t\epsilon_t^d \to 0$	$\sigma_{ij}^{(1)}$	$\epsilon_t + \left(\max_{1\leq i,j\leq m} \sigma_{ij}^{(2)} \right) t\epsilon_t^d + \left(t^2\epsilon_t^d \right)^{-1+1/p}$	d_2 and d_3 distances
$t\epsilon_t^d \to \theta \leq 1$	$\sigma_{ij}^{(1)} + \theta\sigma_{ij}^{(2)}$	$\epsilon_t + \left(\max_{1\leq i,j\leq m} \sigma_{ij}^{(2)} \right) \left\| \theta - t\epsilon_t^d \right\| + t^{-1+1/p}$	d_2 and d_3 distances
$t\epsilon_t^d \to \theta > 1$	$\frac{1}{\theta}\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}$	$\epsilon_t + \left(\max_{1\leq i,j\leq m} \sigma_{ij}^{(1)} \right) \left\| \frac{1}{\theta} - \frac{1}{t\epsilon_t^d} \right\| + t^{-1+1/p}$	d_2 and d_3 distances
$t\epsilon_t^d \to \infty$	$\sigma_{ij}^{(2)}$	$\epsilon_t + \left(\max_{1\leq i,j\leq m} \sigma_{ij}^{(1)} \right) \frac{1}{t\epsilon_t^d} + t^{-1+1/p}$	d_3 distance

Table 7.1: The different bounds depending on the limit of $t\epsilon_t^d$

7.2 Varying the Domains

We use the same setting as in Section 6.2. Let $\alpha > -\frac{d}{2}$ be a real number and let $W_1, ..., W_m \subset \mathbb{R}^d$ be convex bodies. For every $1 \leq i \leq m$, define $F_t^{(i)} := L_t^{(\alpha)}(W_i)$ using the notation in Section 6.2 and set

$$\tilde{F}_t^{(i)} := t^{-1}\epsilon_t^{-\alpha-d/2}\left(\tfrac{1}{2}\sigma_1 + \sigma_2 t\epsilon_t^d\right)^{-1/2}\left(F_t^{(i)} - \mathbb{E}F_t^{(i)}\right) \tag{7.8}$$

and

$$\tilde{F}_t := \left(\tilde{F}_t^{(1)}, ..., \tilde{F}_t^{(m)}\right), \tag{7.9}$$

where

$$\sigma_1 := \frac{d\kappa_d}{d+2\alpha} \quad \text{and} \quad \sigma_2 := \left(\frac{d\kappa_d}{\alpha+d}\right)^2. \tag{7.10}$$

Define the matrix $C = (C_{ij})_{1\leq i,j\leq m}$ by

$$C_{ij} := |W_i \cap W_j|. \tag{7.11}$$

Theorem 7.2. *Under the above conditions, the matrix C is the asymptotic covariance matrix of the m-dimensional random vector $\tilde{F}_t$.*

Moreover, assume that $t^2\epsilon_t^d \to \infty$ as $t \to \infty$. Let $X \sim \mathcal{N}(0, C)$ be a centred Gaussian with covariance matrix C. Then for $t \geq 1$ large enough,

- *if $\alpha_1, ..., \alpha_m > -\frac{d}{4}$, there is a constant $c_1 > 0$ such that*

$$d_3(\tilde{F}_t, X) \leq c_1\left(\epsilon_t + \left(t^{-1/2} \vee (t^2\epsilon_t^d)^{-1/2}\right)\right). \tag{7.12}$$

- *if $-\frac{d}{2} < \min\{\alpha_1, ..., \alpha_m\} \leq -\frac{d}{4}$, then for any $1 < p < -\frac{d}{2}\min\{\alpha_1, ..., \alpha_m\}^{-1}$, there is a constant $c_2 > 0$ such that*

$$d_3(\tilde{F}_t, X) \leq c_2\left(\epsilon_t + \left(t^{-1+1/p} \vee (t^2\epsilon_t^d)^{-1+1/p}\right)\right). \tag{7.13}$$

If the matrix C is positive definite, then the bounds (7.12) and (7.13) apply to $d_2(\tilde{F}_t, X)$ as well for different constants $c_1, c_2 > 0$.

The speed of convergence varies according to the asymptotic behaviour of $t\epsilon_t^d$. We refer to (6.13) for a precise discussion of the different cases. The proof of Theorem 7.2 can be found in Section 9.7.

Remark 7.3. The question whether C is positive definite is not entirely straightforward, but some things can be said. For a vector $x \in \mathbb{R}^m$, we have

$$x^T C x = \sum_{i,j=1}^m x_i x_j \int_{\mathbb{R}^d} \mathbb{1}_{W_i}(z)\mathbb{1}_{W_j}(z)dz = \int_{\mathbb{R}^d}\left(\sum_{i=1}^m x_i \mathbb{1}_{W_i}(z)\right)^2 dz. \tag{7.14}$$

Since $W_1, \ldots, W_m$ form a collection of convex bodies, it is clearly a necessary and sufficient condition that the family of indicators $\mathbb{1}_{W_1}(z), ..., \mathbb{1}_{W_m}(z)$ is linearly independent in $L^2(\mathbb{R}^d)$, which translates (to some extent) to none of the sets being obtainable from the other sets via certain combinations of unions, intersections and complements.

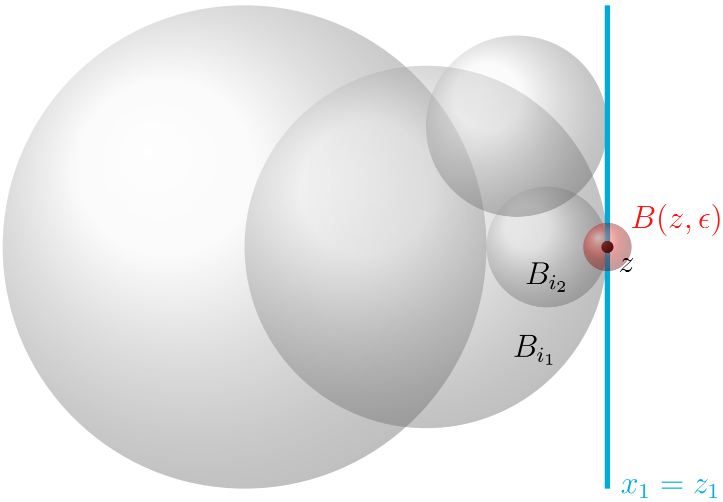


Figure 7.1: Constructing a sequence from a collection of distinct balls

A simple sufficient condition for positive definiteness is that for each $1 \le i \le m-1$, we have

$$W_i \setminus \bigcup_{j>i} W_j \neq \emptyset, \tag{7.15}$$

i.e. the sets $W_1, ..., W_m$ can be ordered in a sequence such that each set has a point not included in subsequent sets. This implies that the family of indicators $\mathbb{1}_{W_1}(z), ..., \mathbb{1}_{W_m}(z)$ is linearly independent, and it also entails positive-definiteness of C. Indeed, let $z_1 \in W_1 \setminus \bigcup_{j>1} W_j$. Since $\bigcup_{j>1} W_j$ is closed, there is an open set $U \subset \left(\bigcup_{j>1} W_j\right)^c$ such that $z_1 \in U$. As W_1 is a convex body, the intersection V of the interior of W_1 with U is open and non-empty. Assume that $x^T C x = 0$, then the function $f = x_1 \mathbb{1}_{W_1} + \ldots + x_m \mathbb{1}_{W_m}$ is zero almost everywhere. In particular, it is constant $f \equiv x_1$ on $V \subset W_1 \setminus \bigcup_{j>1} W_j$, hence we must have $x_1 = 0$. One can now iterate this argument to show that $x_2 = \ldots = x_m = 0$.

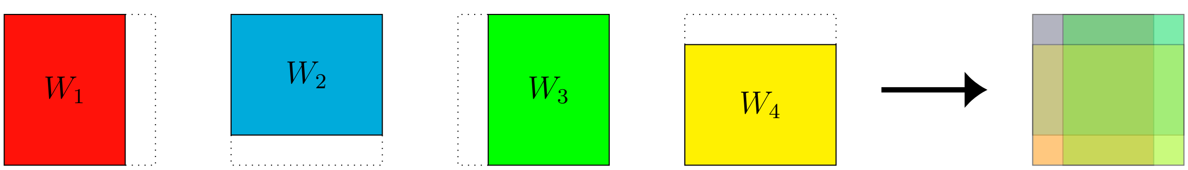


Figure 7.2: Four convex bodies

The condition (7.15) suffices to prove positive-definiteness for collections of distinct balls. Indeed, let $B_1, \ldots, B_m$ be a collection of closed distinct balls in $\mathbb{R}^d$. The union $\bigcup_{i\ge1} B_i$ is closed and bounded, and admits thus at least one point z whose first coordinate achieves the maximum over all first coordinates of points in $\bigcup_{i\ge1} B_i$. Let $B_{i_1}, \ldots, B_{i_n}$ be the balls tangent to the hyperplane $x_1 = z_1$, sorted by decreasing radius (i.e. B_{i_1} is the largest ball among $B_{i_1}, \ldots, B_{i_n}$). There is an open ball $B(z,\epsilon) \subset \left(\bigcup_{j\neq i_1,\ldots,i_n} B_j\right)^c$, and the intersection $B(z,\epsilon) \cap B_{i_1}$ contains a point not included in $B_{i_2}, \ldots, B_{i_n}$. The ball B_{i_1} can thus be made the first ball in the sequence. This construction can be iterated to satisfy condition (7.15). See Figure 7.1 for an illustration.

The condition (7.15) is however not necessary for the matrix C to be positive definite. Indeed, consider the sets $W_1, \ldots, W_4$ of Figure 7.2

One can show easily that if $x_1 \mathbb{1}_{W_1} + \ldots + x_4 \mathbb{1}_{W_4} = 0$ almost everywhere, then $x_1 = \ldots = x_4 = 0$. However, these sets do not fulfil condition (7.15).

8 Background on Malliavin Calculus

In this section, we present several useful notions related to Malliavin calculus. Unless otherwise indicated, these results are explained in [Las16]. We work in the setting of Chapter 3: in particular, χ indicates a $(\mathbb{W}, \nu)$-Poisson measure.

We start with three useful **isometry** relations. Let $f \in L^2(\mathbb{W}^n, \nu^{(n)})$ and $g \in L^2(\mathbb{W}^m, \nu^{(m)})$. Then

$$\mathbb{E}\, \mathrm{I}_n(f)\, \mathrm{I}_m(g) = \mathbb{1}_{\{m=n\}} n! \int_{\mathbb{W}^n} \tilde{f}(x)\tilde{g}(x)\, \nu^{(n)}(dx), \tag{8.1}$$

where $\tilde{f}$ and $\tilde{g}$ are the **symmetrisations** of f and g defined by

$$\tilde{f}(x_1, ..., x_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(x_{\sigma(1)}, ..., x_{\sigma(n)}), \tag{8.2}$$

the set Σ_n being the set of all permutations of $\{1, ..., n\}$. See [Las16, Lemma 4] and the remark thereafter on page 10 for a proof.

Relation (8.1) implies that for $F, G \in L^2(\mathbb{P}_\chi)$ having an expansion (3.5) with kernels f_n and g_n respectively,

$$\mathbb{E}FG = \mathbb{E}F\mathbb{E}G + \sum_{n=1}^{\infty} n! \int_{\mathbb{W}^n} f_n(x) g_n(x)\, \nu^{(n)}(dx). \tag{8.3}$$

By [Las16, Theorem 5], if $h \in L^2(\mathbf{N} \times \mathbb{W})$ satisfies (3.13), then

$$\mathbb{E}\delta(h)^2 = \mathbb{E} \int_{\mathbb{W}} h(\chi, w)^2 \nu(dw) + \mathbb{E} \int_{\mathbb{W}} \int_{\mathbb{W}} \mathrm{D}_z\, h(\chi, w)\, \mathrm{D}_w\, h(\chi, z)\, \nu(dz)\nu(dw). \tag{8.4}$$

A well-known relation in Malliavin calculus is the so-called **integration by parts** formula: for $F \in \operatorname{dom} \mathrm{D}$ and $h \in \operatorname{dom} \delta$, we have $\mathbb{E} \int_{\mathbb{W}} h(\chi, w)\, \mathrm{D}_w\, F \nu(dw) = \mathbb{E}[F\delta(h)]$ (cf. [Las16, Theorem 4]). The condition on F is however suboptimal in our context, which is why we need a version of integration by parts under slightly different assumptions.

Lemma 8.1. *Let* $h \in \operatorname{dom} \delta \cap L^1(\mathbf{N} \times \mathbb{W})$ *and* $F \in L^0(\mathbb{P}_\chi)$ *bounded. Then*

$$\mathbb{E} \int_{\mathbb{W}} h(\chi, w)\, \mathrm{D}_w\, F \nu(dw) = \mathbb{E}[F\delta(h)]. \tag{8.5}$$

Proof. Since $h \in \operatorname{dom}\delta \cap L^1(\mathbf{N} \times \mathbb{W})$ and F is bounded, it is easy to check that the expectations appearing in the statement are well-defined and finite. Note that

$$\begin{aligned}\mathbb{E}\int_{\mathbb{X}} \mathrm{D}_w F\, h(\chi, w)\nu(dw) &= \mathbb{E}\int_{\mathbb{X}} (F(\chi+\delta_w) - F(\chi))h(\chi, w)\nu(dw) \\ &= \mathbb{E}\int_{\mathbb{X}} F(\chi+\delta_w)h(\chi, w)\nu(dw) - \mathbb{E}\int_{\mathbb{X}} F(\chi)h(\chi, w)\nu(dw), \qquad (8.6)\end{aligned}$$

where the last line is justified by the fact that F is bounded and $h \in L^1(\mathbf{N} \times \mathbb{W})$, so both integrals are well-defined. We now apply Mecke formula (3.7) to deduce that (8.6) equals

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{X}} F(\chi)h(\chi-\delta_w, w)\chi(dw) - \mathbb{E}\int_{\mathbb{X}} F(\chi)h(\chi, w)\nu(dw) \\ &\qquad = \mathbb{E}F(\chi)\left(\int_{\mathbb{X}} h(\chi-\delta_w, w)\chi(dw) - \int_{\mathbb{X}} h(\chi, w)\nu(dw)\right). \qquad (8.7)\end{aligned}$$

Since $h \in \operatorname{dom}\delta \cap L^1(\mathbf{N} \times \mathbb{X})$,

$$\int_{\mathbb{X}} h(\chi-\delta_w, w)\chi(dw) - \int_{\mathbb{X}} h(\chi, w)\nu(dw) = \delta(h) \qquad \mathbb{P}\text{-a.s.} \qquad (8.8)$$

The result follows. ■

Next, we introduce the **Ornstein-Uhlenbeck** operators P_τ, with $\tau \in [0,1]$. For $F \in L^1(\mathbb{P}_\chi)$ and $\tau \in [0,1]$, we define

$$P_\tau F = \int \mathbb{E}\big[F(\chi^\tau + \xi)\big|\chi\big]\Pi_\tau(d\xi), \qquad (8.9)$$

where χ^τ is a τ-thinning of χ (see [Las16, p. 24] and the reference given therein) and Π_τ is the law of an independent Poisson measure with intensity measure $(1-\tau)\nu$. It follows by Jensen's inequality that for all $p \geq 1$, one has

$$\mathbb{E}|P_\tau F|^p \leq \mathbb{E}|F|^p. \qquad (8.10)$$

By [Las16, Lemma 6], for all $F \in L^2(\mathbb{P}_\chi)$ and all $\tau \in [0,1]$, for $\nu^{(n)}$-a.e. $w_1, ..., w_n \in \mathbb{W}$ it holds $\mathbb{P}$-a.s. that

$$\mathrm{D}^{(n)}_{w_1,\dots,w_n}(P_\tau F) = \tau^n P_\tau\, \mathrm{D}^{(n)}_{w_1,\dots,w_n} F. \qquad (8.11)$$

This implies that for $F \in L^2(\mathbb{P}_\chi)$, the following expansion holds (see also [Las16, formula (79)]):

$$P_\tau F = \mathbb{E}F + \sum_{n=1}^{\infty} \tau^n\, \mathrm{I}_n(f_n). \qquad (8.12)$$

As such, the Ornstein-Uhlenbeck operators we use are a reparametrised version of the usual Ornstein-Uhlenbeck semigroup. Given $F \in L^2(\mathbb{P}_\chi)$, one defines the standard Ornstein-Uhlenbeck semigroup as

$$T_s F := \mathbb{E}F + \sum_{n=1}^{\infty} e^{-ns}\, \mathrm{I}_n(f_n), \qquad (8.13)$$

with $s \geq 0$. We refer to the discussion in [Las16, p. 24].

The following lemma summarises some useful approximation properties of the Ornstein-Uhlenbeck operator.

Lemma 8.2. *Let* $h \in L^2(\mathbf{N}_{\mathbb{W}} \times \mathbb{W})$ *and let* $\tau \in (0,1)$. *Then* $P_\tau h$ *satisfies condition* (3.13) *and* $P_\tau h \to h$ *in* $L^2(\mathbf{N}_{\mathbb{W}} \times \mathbb{W})$ *as* $\tau \to 1$. *Moreover, for* $w, z \in \mathbb{W}$, *and all* $p \geq 1$,

$$\mathbb{E}|P_\tau h(\chi, w)|^p \leq \mathbb{E}|h(\chi, w)|^p \tag{8.14}$$

and

$$\mathbb{E}|\operatorname{D}_z P_\tau h(\chi, w)|^p \leq \mathbb{E}|\operatorname{D}_z h(\chi, w)|^p. \tag{8.15}$$

Under the additional assumption that $h \in \operatorname{dom}\delta$, *it holds that* $\delta(P_\tau h) \to \delta(h)$ *in* $L^2(\mathbb{P}_\chi)$ *as* $\tau \to 1$.

Proof. By the isometry property (8.3) and the expansions (3.6), (8.12) and (3.9), we infer that

$$\mathbb{E}\int_{\mathbb{W}}\int_{\mathbb{W}} (\operatorname{D}_w P_\tau h(\chi, z))^2 \,\nu(dw)\nu(dz) = \sum_{n=0}^{\infty} n \cdot n!\, \tau^{2n} \|h_n\|_{n+1}^2, \tag{8.16}$$

where $\|.\|_n$ is the norm in $L^2(\mathbb{W}^n, \nu^{(n)})$. Now note that $\sup_{n\geq 1} n\tau^{2n} < \infty$ and

$$\sum_{n=0}^{\infty} n!\, \|h_n\|_{n+1}^2 = \int_{\mathbb{W}} \mathbb{E}h(w)^2 \nu(dw) < \infty, \tag{8.17}$$

hence $P_\tau h$ satisfies (3.13). Similarly using the expansions, we deduce that

$$\mathbb{E}\int_{\mathbb{W}} (P_\tau h(\chi, w) - h(\chi, w))^2 \,\nu(dw) = \sum_{n=1}^{\infty} n!\, (\tau^n - 1)^2 \|h_n\|_{n+1}^2. \tag{8.18}$$

By dominated convergence, this expression tends to 0 as $\tau \to 1$. Properties (8.14) and (8.15) follow immediately from (8.10) and (8.11). For the last point, note that

$$\delta(P_\tau h) - \delta(h) = \sum_{n=0}^{\infty} \operatorname{I}_{n+1}((\tau^n - 1)h_n) \tag{8.19}$$

and

$$\mathbb{E}\,(\delta(P_\tau h) - \delta(h))^2 = \sum_{n=0}^{\infty} (n+1)!(1-\tau^n)^2 \|\tilde{h}_n\|_{n+1}^2, \tag{8.20}$$

which converges to 0 as $\tau \to 1$ by dominated convergence since $h \in \operatorname{dom}\delta$. ■

For the multivariate bound in Theorem 4.6, we will also make use of the **(pseudo) inverse of the Ornstein-Uhlenbeck generator** L^{-1} defined for all $F \in L^2(\mathbb{P}_\chi)$ having expansion (3.5) by

$$\operatorname{L}^{-1} F := -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{I}_n(f_n). \tag{8.21}$$

The Ornstein-Uhlenbeck generator L itself is defined as

$$\operatorname{L} F := -\sum_{n=1}^{\infty} n \operatorname{I}(f_n), \tag{8.22}$$

for all $F \in L^2(\mathbb{P}_\chi)$ satisfying

$$\sum_{n=1}^{\infty} n^2 n! \|f_n\|_n^2 < \infty. \tag{8.23}$$

For any F satisfying this condition, one sees that

$$\lim_{s\to 0} \frac{T_s F - F}{s} = \mathrm{L}\, F \tag{8.24}$$

in $L^2(\mathbb{P}_\chi)$. We refer to [Las16, p. 24] for further details.

The following properties of L^{-1} can all be found in [LPS16], see [LPS16, Lemma 3.4] and the beginning of the proof of [LPS16, Prop. 4.1].

Lemma 8.3 ([LPS16]). *For $F \in L^2(\mathbb{P}_\chi)$ and any $r \geq 1$, we have*

$$\mathbb{E}\left|\mathrm{D}_w \mathrm{L}^{-1} F\right|^r \leq \mathbb{E}\left|\mathrm{D}_w F\right|^r, \quad \textit{for } \nu - \textit{a.e. } w \in \mathbb{W}, \tag{8.25}$$

and

$$\mathbb{E}\left|\mathrm{D}^{(2)}_{w,z} \mathrm{L}^{-1} F\right|^r \leq \mathbb{E}\left|\mathrm{D}^{(2)}_{w,z} F\right|^r, \quad \textit{for } \nu^{(2)} - \textit{a.e. } (w,z) \in \mathbb{W}^2. \tag{8.26}$$

Moreover, let $F, G \in \operatorname{dom} \mathrm{D}$ with $\mathbb{E}F = \mathbb{E}G = 0$. Then

$$\operatorname{Cov}(F,G) = \mathbb{E}\int_{\mathbb{W}} (\mathrm{D}_w F) \cdot \left(-\mathrm{D}_w \mathrm{L}^{-1} G\right) \nu(dw). \tag{8.27}$$

Analogously, the covariance between two functionals can be written in terms of conditional expectations.

Lemma 8.4 ([LP11b, Theorem 1.5]). *Let η be a $(\mathbb{W} \times [0,1], \nu \otimes ds)$-Poisson measure and let $F, G \in L^2(\mathbb{P}_\eta)$. Then*

$$\mathbb{E}\int_{\mathbb{W}}\int_0^1 \mathbb{E}[\mathrm{D}_{(y,s)} F | \eta_{|\mathbb{W}\times[0,s)}]^2 \lambda(dy) ds < \infty, \tag{8.28}$$

and an analogous estimate holds for G. Moreover,

$$\operatorname{Cov}(F,G) = \mathbb{E}\int_{\mathbb{W}}\int_0^1 \mathbb{E}[\mathrm{D}_{(y,s)} F | \eta_{|\mathbb{W}\times[0,s)}] \mathbb{E}[\mathrm{D}_{(y,s)} G | \eta_{|\mathbb{W}\times[0,s)}] \lambda(dy) ds, \tag{8.29}$$

which expresses the covariance in terms of conditional expectations.

9 Proofs

9.1 Proof of Theorem 5.1

Each summand on the RHS of (5.1) is well defined by virtue of Mecke formula (3.7) and the discussion thereafter. By Mecke formula (3.7) it can be seen that $h(\eta - \delta_{(y,s)}, y, s)$ is almost surely integrable with respect to the measure $\eta(dy, ds)$. By assumption, h is also integrable with respect to $\lambda(dy)ds$. It now follows by dominated convergence that the process (X_t) is càdlàg.

As a next step, we show that the integrals on the RHS of (5.2) are well-defined. For this, note that $(X_t)_{t\in[0,1]}$ (and $(X_{t-})_{t\in[0,1]}$) are a.s. bounded on $[0, 1]$. Indeed,

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,1]} |X_t| &\leq \mathbb{E} \int_{\mathbb{X}\times[0,1]} |h(\eta - \delta_{(y,s)}, y, s)|\eta(dy, ds) + \mathbb{E} \int_{\mathbb{X}} \int_0^1 |h(\eta, y, s)|\lambda(dy)ds \\ &= 2\mathbb{E} \int_{\mathbb{X}} \int_0^1 |h(\eta, y, s)|\lambda(dy)ds < \infty, \end{aligned} \tag{9.1}$$

where the equality follows by Mecke formula (3.7). Now write

$$\begin{aligned}&\int_{\mathbb{X}\times[0,t]} \left|\phi(X_{s-} + h(\eta - \delta_{(y,s)}, y, s)) - \phi(X_{s-})\right| \eta(dy, ds) \\ &\qquad\leq \int_{\mathbb{X}\times[0,1]} \left| \int_0^1 \phi'\left(X_{s-} + uh(\eta - \delta_{(y,s)}, y, s)\right) h(\eta - \delta_{(y,s)}, y, s)du \right| \eta(dy, ds). \end{aligned} \tag{9.2}$$

By the boundedness of h, we infer from (9.1) that $X_{s-} + uh(\eta - \delta_{(y,s)}, y, s)$ almost surely takes values in a compact interval. Since the function ϕ' is continuous, this entails

$$\sup_{s,u\in[0,1],y\in\mathbb{X}} |\phi'(X_{s-} + uh(\eta - \delta_{(y,s)}, y, s))| < \infty \qquad \mathbb{P}\text{-a.s.} \tag{9.3}$$

Hence the RHS of (9.2) is bounded by

$$\left(\sup_{s,u\in[0,1],y\in\mathbb{X}} |\phi'(X_{s-} + uh(\eta - \delta_{(y,s)}, y, s))| \right) \int_{\mathbb{X}\times[0,1]} |h(\eta - \delta_{(y,s)}, y, s)|\eta(dy, ds) < \infty \qquad \mathbb{P}\text{-a.s.} \tag{9.4}$$

which implies that the first integral on the RHS of (5.2) is well-defined. Similarly,

$$\sup_{s\in[0,1]} |\phi'(X_s)| < \infty \qquad \mathbb{P}\text{-a.s.} \tag{9.5}$$

and hence

$$\int_{\mathbb{X}}\int_0^1 |\phi'(X_s)h(\eta,x,s)|\lambda(dx)ds < \infty \qquad \mathbb{P}\text{-a.s.} \tag{9.6}$$

This concludes the proof that all terms in (5.2) are well-defined.

To show (5.2), we start by showing it for an approximation of X_t. Let $(U_m)_{m\in\mathbb{N}} \subset \mathbb{X}$ s.t. $\bigcup_m U_m = \mathbb{X}$ and $\forall\, m \in \mathbb{N}$, $\lambda(U_m) < \infty$ and $U_m \subset U_{m+1}$. Define

$$X_t^{(m)}(\eta) := X_0 + \int_{U_m\times[0,t]} h(\eta - \delta_{(y,s)}, y, s)\eta(dy, ds) - \int_{U_m}\int_0^t h(\eta, y, s)\lambda(dy)ds. \tag{9.7}$$

Define the event $\Omega_0 := \{\eta(U_m \times [0,1]) < \infty, m \geq 1\}$. Then $\mathbb{P}(\Omega_0) = 1$ and, since η is proper, $\forall\omega \in \Omega_0$ and all $m \geq 1$ there exists a finite collection of points $(y_1, s_1), ..., (y_{n_m}, s_{n_m}) \in U_m \times [0,1]$ (all depending on ω) s.t.

$$\eta_{|U_m\times[0,1]} = \sum_{i=1}^{n_m} \delta_{(y_i,s_i)}. \tag{9.8}$$

W.l.o.g. we can assume that $0 < s_1 < s_2 < ... < s_{n_m} < 1$ and set $s_0 := 0$ and $s_{n_m+1} := 1$. Now the process $X^{(m)}$ can be written as

$$X_t^{(m)} = \sum_{i=1}^{n_m} \mathbb{1}_{\{s_i\leq t\}} h(\eta - \delta_{(y_i,s_i)}, y_i, s_i) - \int_{U_m}\int_0^t h(\eta, y, s)\lambda(dy)ds, \tag{9.9}$$

and one has the telescopic sums:

$$\begin{aligned}
\phi\left(X_t^{(m)}\right) - \phi(X_0) &= \sum_{i=1}^{n_m+1}\left(\phi\left(X_{s_i\wedge t}^{(m)}\right) - \phi\left(X_{s_{i-1}\wedge t}^{(m)}\right)\right) \\
&= \sum_{i=1}^{n_m}\left(\phi\left(X_{s_i\wedge t}^{(m)}\right) - \phi\left(X_{(s_i-)\wedge t}^{(m)}\right)\right) + \sum_{j=1}^{n_m+1}\left(\phi\left(X_{(s_j-)\wedge t}^{(m)}\right) - \phi\left(X_{s_{j-1}\wedge t}^{(m)}\right)\right) \\
&= I_1(t) + I_2(t),
\end{aligned} \tag{9.10}$$

where

$$X_{(s-)\wedge t}^{(m)} = \begin{cases} X_{s-}^{(m)} & \text{if } s \leq t \\ X_t^{(m)} & \text{if } s > t. \end{cases} \tag{9.11}$$

The sum $I_1(t)$ represents what is happening at jump times, whereas $I_2(t)$ shows what happens in between jump times. The indices $i = 0$ and $i = n_m + 1$ do not appear in the sum $I_1(t)$ because $\mathbb{P}$-a.s. there are no jumps at times $t = 0$ and $t = 1$.

We first study $I_1(t)$.

$$I_1(t) = \sum_{i=1}^{n_m} \left(\phi\left(X^{(m)}_{s_i\wedge t}\right) - \phi\left(X^{(m)}_{(s_i-)\wedge t}\right)\right)$$
$$= \sum_{i=1}^{n_m} \mathbb{1}_{\{s_i\le t\}} \left(\phi\left(X^{(m)}_{s_i-} + h(\eta - \delta_{(y_i,s_i)}, y_i, s_i)\right) - \phi\left(X^{(m)}_{s_i-}\right)\right)$$
$$= \int_{U_m\times[0,t]} \left(\phi\left(X^{(m)}_{s-} + h(\eta - \delta_{(y,s)}, y, s)\right) - \phi\left(X^{(m)}_{s-}\right)\right) \eta(dy, ds). \tag{9.12}$$

Now consider $I_2(t)$. For $s \in [s_{i-1}, s_i)$,

$$X^{(m)}_s = \sum_{j=1}^{i-1} h(\eta - \delta_{(y_j,s_j)}) - \int_{U_m}\int_0^s h(\eta, y, u)\lambda(dy)du \tag{9.13}$$

and so for $s \in (s_{i-1}, s_i)$

$$\frac{d}{ds}X^{(m)}_s = -\int_{U_m} h(\eta, y, s)\lambda(dy). \tag{9.14}$$

This implies that

$$\phi\left(X^{(m)}_{(s_i-)\wedge t}\right) - \phi\left(X^{(m)}_{s_{i-1}\wedge t}\right) = -\int_{s_{i-1}\wedge t}^{(s_i-)\wedge t} \phi'\left(X^{(m)}_s\right)\int_{U_m} h(\eta, y, s)\lambda(dy)ds. \tag{9.15}$$

We conclude that

$$I_2(t) = \sum_{i=1}^{n_m+1} \phi\left(X^{(m)}_{(s_i-)\wedge t}\right) - \phi\left(X^{(m)}_{s_{i-1}\wedge t}\right)$$
$$= -\sum_{i=1}^{n_m+1} \int_{s_{i-1}\wedge t}^{(s_i-)\wedge t}\int_{U_m} \phi'\left(X^{(m)}_s\right) h(\eta, y, s)\lambda(dy)ds$$
$$= -\int_0^t\int_{U_m} \phi'\left(X^{(m)}_s\right) h(\eta, y, s)\lambda(dy)ds. \tag{9.16}$$

We have shown until now that

$$\phi\left(X^{(m)}_t\right) = \phi(X_0) + \int_{U_m\times[0,t]} \left(\phi\left(X^{(m)}_{s-} + h(\eta - \delta_{(y,s)}, y, s)\right) - \phi\left(X^{(m)}_{s-}\right)\right)\eta(dy, ds)$$
$$- \int_{U_m}\int_0^t \phi'\left(X^{(m)}_s\right) h(\eta, x, s)\lambda(dx)ds \qquad \mathbb{P}\text{-a.s.} \tag{9.17}$$

Our aim is now to let $m \to \infty$. By dominated convergence, $X^{(m)}_t \to X_t$ a.s. for fixed $t \in [0, 1]$. We would like to use dominated convergence for both the second and third terms on the RHS of (9.17). Start by noting that $(X^{(m)}_t)_{t\in[0,1]}$ (as well as $(X^{(m)}_{t-})_{t\in[0,1]}$ and $(X_t)_{t\in[0,1]}$) are $\mathbb{P}$-a.s. uniformly bounded on $[0, 1]$ and in m. Indeed,

$$\sup_{\substack{m\in\mathbb{N}\\ t\in[0,1]}} \max\left\{\left|X^{(m)}_t\right|, \left|X^{(m)}_{t-}\right|, |X_t|\right\}$$
$$\le \int_{\mathbb{X}\times[0,1]} |h(\eta - \delta_{(y,s)}, y, s)|\eta(dy, ds) + \int_{\mathbb{X}}\int_0^1 |h(\eta, y, s)|\lambda(dy)ds < \infty \qquad \mathbb{P}\text{-a.s.} \tag{9.18}$$

We start with the second term on the RHS of (9.17) and write

$$\int_{U_m\times[0,t]}\left|\phi\left(X_{s-}^{(m)}+h(\eta-\delta_{(y,s)},y,s)\right)-\phi\left(X_{s-}^{(m)}\right)\right|\eta(dy,ds)$$
$$\leq\int_{\mathbb{X}\times[0,1]}\left|\int_0^1\phi'\left(X_{s-}^{(m)}+uh(\eta-\delta_{(y,s)},y,s)\right)h(\eta-\delta_{(y,s)},y,s)du\right|\eta(dy,ds). \quad (9.19)$$

By boundedness of h, we get that $X_{s-}^{(m)}+uh(\eta-\delta_{(y,s)},y,s)$ almost surely takes values in a compact interval independent of m. The function ϕ' being continuous, we deduce

$$\sup_{\substack{m\in\mathbb{N}\\ s,u\in[0,1]\\ y\in\mathbb{X}}}\left|\phi'\left(X_{s-}^{(m)}+uh(\eta-\delta_{(y,s)},y,s)\right)\right|<\infty \qquad \mathbb{P}\text{-a.s.} \quad (9.20)$$

Hence the RHS of (9.19) is bounded by

$$\left(\sup_{\substack{m\in\mathbb{N}\\ s,u\in[0,1]\\ y\in\mathbb{X}}}\left|\phi'\left(X_{s-}^{(m)}+uh(\eta-\delta_{(y,s)},y,s)\right)\right|\right)\int_{\mathbb{X}\times[0,1]}|h(\eta-\delta_{(y,s)},y,s)|\eta(dy,ds)<\infty \qquad \mathbb{P}\text{-a.s.} \quad (9.21)$$

We can thus apply dominated convergence and deduce that

$$\int_{U_m\times[0,t]}\left(\phi\left(X_{s-}^{(m)}+h(\eta-\delta_{(y,s)},y,s)\right)-\phi\left(X_{s-}^{(m)}\right)\right)\eta(dy,ds)$$
$$\overset{m\to\infty}{\longrightarrow}\int_{\mathbb{X}\times[0,t]}\left(\phi\left(X_{s-}+h(\eta-\delta_{(y,s)},y,s)\right)-\phi\left(X_{s-}\right)\right)\eta(dy,ds) \qquad \mathbb{P}\text{-a.s.} \quad (9.22)$$

Now, similarly,

$$\int_{U_m}\int_0^t\left|\phi'\left(X_s^{(m)}\right)h(\eta,x,s)\right|\lambda(dx)ds$$
$$\leq\left(\sup_{\substack{m\in\mathbb{N}\\ s\in[0,1]}}\left|\phi'\left(X_s^{(m)}\right)\right|\right)\int_{\mathbb{X}}\int_0^t|h(\eta,x,s)|\lambda(dx)ds<\infty \qquad \mathbb{P}\text{-a.s.} \quad (9.23)$$

and by dominated convergence

$$\int_{U_m}\int_0^t\phi'\left(X_s^{(m)}\right)h(\eta,x,s)\lambda(dx)ds\overset{m\to\infty}{\longrightarrow}\int_{\mathbb{X}}\int_0^t\phi'\left(X_s\right)h(\eta,x,s)\lambda(dx)ds \qquad \mathbb{P}\text{-a.s.} \quad (9.24)$$

The fact that $\phi\left(X_t^{(m)}\right)\to\phi\left(X_t\right)$ a.s. follows by continuity of ϕ. This concludes the proof. ■

9.2 Proofs of Theorem 5.2 and Corollaries 5.3 and 5.5

Lemma 9.1. *For $p \in (1,2]$, the function $\phi : \mathbb{R} \to \mathbb{R} : x \mapsto |x|^p$ is continuously differentiable and its derivative ϕ' is $(p-1)$-Hölder continuous with Hölder constant $c_\phi = p2^{2-p}$.*

Proof. We want to show that

$$c_\phi := \sup_{\substack{a\neq b\\ a,b\in\mathbb{R}}} \frac{|\phi'(a)-\phi'(b)|}{|a-b|^{p-1}} = p2^{2-p}. \tag{9.25}$$

First we observe that $\phi'(x) = p\,\mathrm{sgn}(x)|x|^{p-1}$, with the convention that $\mathrm{sgn}(0) = 1$, where sgn is the sign function. Let $a \neq b$ and assume without loss of generality that $|a| \geq |b| \geq 0$. Then $a \neq 0$ and

$$\frac{|\mathrm{sgn}(a)|a|^{p-1} - \mathrm{sgn}(b)|b|^{p-1}|}{|a-b|^{p-1}} = \frac{\left|1-\mathrm{sgn}\left(\frac{b}{a}\right)\left|\frac{b}{a}\right|^{p-1}\right|}{\left|1-\frac{b}{a}\right|^{p-1}}. \tag{9.26}$$

It follows that

$$c_\phi = p \sup_{x\in[-1,1)} \frac{1-\mathrm{sgn}(x)\,|x|^{p-1}}{(1-x)^{p-1}} =: p \sup_{x\in[-1,1)} f(x) \tag{9.27}$$

The function f is differentiable on $[-1,1)$ with derivative $f'(x) = (p-1)(1-x)^{-p}(1-|x|^{p-2}) \leq 0$, so f is decreasing and therefore

$$f(x) \leq f(-1) = 2^{2-p}. \tag{9.28}$$

We conclude that $c_\phi = p2^{2-p}$. ∎

Proof of Theorem 5.2. We start by showing the result for an approximation of h. Let $(U_m)_{m\in\mathbb{N}} \subset \mathbb{X}$ s.t. $\bigcup_m U_m = \mathbb{X}$ and $\forall\, m \in \mathbb{N}$, $\lambda(U_m) < \infty$ and $U_m \subset U_{m+1}$. Define for $\mu \in \mathbf{N}$, $(y,s) \in \mathbb{X}\times[0,1]$:

$$h_m(\mu,y,s) := [(h(\mu,y,s)\wedge m)\vee(-m)]\,\mathbb{1}_{\{y\in U_m\}}. \tag{9.29}$$

Now $h_m \in L^1(\mathbf{N}\times\mathbb{X}\times[0,1]) \cap L^2(\mathbf{N}\times\mathbb{X}\times[0,1])$ and $h_m \to h$ in L^2 as $m\to\infty$. Moreover, $|h_m| \leq |h|$ and $|\,\mathrm{D}_{(x,t)}\,h_m(\eta,y,s)| \leq |\,\mathrm{D}_{(x,t)}\,h(\eta,y,s)|$ for all $(x,t),(y,s) \in \mathbb{X}\times[0,1]$. This implies that h_m satisfies (3.13) and hence $h_m \in \mathrm{dom}\,\delta$.
Finally, h_m is also bounded. In particular, for any $\mu \in \mathbf{N}$, $(y,s) \in \mathbb{X}\times[0,1]$,

$$|h_m(\mu,y,s)| \leq m\mathbb{1}_{\{y\in U_m\}}. \tag{9.30}$$

We conclude that $\delta(h_m)$ is well-defined and has by (3.14) the pathwise expression

$$\delta(h_m) = \int_{\mathbb{X}\times[0,1]} h_m(\eta-\delta_{(y,s)},y,s)\eta(dy,ds) - \int_{\mathbb{X}}\int_0^1 h_m(\eta,y,s)\lambda(dy)ds. \tag{9.31}$$

Define X_t as in Theorem 5.1 with $h = h_m$ and $X_0 = 0$. Then $\delta(h_m) = X_1$ and we infer from (5.2) that

$$\begin{aligned}\phi(\delta(h_m)) - \phi(0) = \int_{\mathbb{X}\times[0,1]} &\left(\phi(X_{s-}+h_m(\eta-\delta_{(y,s)},y,s)) - \phi(X_{s-})\right)\eta(dy,ds)\\ &- \int_{\mathbb{X}}\int_0^1 \phi'(X_s)h_m(\eta,y,s)\lambda(dy)ds \qquad \mathbb{P}\text{-a.s.}\end{aligned} \tag{9.32}$$

We would now like to take expectations on both sides of (9.32), but in order to do so, we must first show that the terms in question are in $L^1(\mathbb{P}_\eta)$. Start with a simple estimate for ϕ that uses Hölder continuity of ϕ'. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}|\phi(a+b)-\phi(a)| &= \left|\int_0^1 \phi'(a+ub)b\,du\right| \\ &\le |\phi'(a)b| + \int_0^1 |b|\cdot|\phi'(a+ub)-\phi'(a)|\,du \\ &\le |\phi'(0)b| + |\phi'(a)-\phi'(0)|\cdot|b| + |b|\int_0^1 c_\phi |ub|^{p-1}du \\ &\le |\phi'(0)|\cdot|b| + c_\phi|a|^{p-1}\cdot|b| + \frac{c_\phi}{p}|b|^p. \end{aligned} \tag{9.33}$$

We are also going to need a bound on $\sup_{s\in[0,1]}\left\{\max\left\{|X_s|, |X_{s-}(\eta+\delta_{(y,s)})|\right\}\right\}$. Using (9.30), we find the following:

$$\begin{aligned}\max\left\{|X_s|, |X_{s-}(\eta+\delta_{(y,s)})|\right\} &\le \int_{\mathbb{X}\times[0,1]} m\mathbb{1}_{\{x\in U_m\}}\eta(dx,dt) + \int_{\mathbb{X}}\int_0^1 m\mathbb{1}_{\{x\in U_m\}}\lambda(dx)dt \\ &\le m\left(\eta(U_m\times[0,1]) + \lambda(U_m)\right). \end{aligned} \tag{9.34}$$

Using (9.33) with $a=0$ and $b=\delta(h_m)$, we get for the LHS in (9.32)

$$\mathbb{E}\,|\phi(\delta(h_m))| \le |\phi'(0)|\cdot\mathbb{E}|\delta(h_m)| + \frac{c_\phi}{p}\mathbb{E}|\delta(h_m)|^p < \infty \tag{9.35}$$

which is finite since $\delta(h_m)\in L^2(\mathbb{P}_\eta)$. To show that the first term on the RHS in (9.32) is integrable, we first apply Mecke formula (3.7) to get

$$\begin{aligned}&\mathbb{E}\left|\int_{\mathbb{X}\times[0,1]}\left(\phi(X_{s-}+h_m(\eta-\delta_{(y,s)},y,s))-\phi(X_{s-})\right)\eta(dy,ds)\right| \\ &\qquad\le \mathbb{E}\int_{\mathbb{X}}\int_0^1\left|\phi(X_{s-}(\eta+\delta_{(y,s)})+h_m(\eta,y,s))-\phi(X_{s-}(\eta+\delta_{(y,s)}))\right|\lambda(dy)ds. \end{aligned} \tag{9.36}$$

Now we combine (9.33), (9.34) and (9.30) to get that the RHS of (9.36) is bounded by

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{X}}\int_0^1\Big(|\phi'(0)|\cdot m\mathbb{1}_{\{y\in U_m\}} + c_\phi|m\left(\eta(U_m\times[0,1])+\lambda(U_m)\right)|^{p-1}\cdot m\mathbb{1}_{\{y\in U_m\}} \\ &\qquad\qquad + \frac{c_\phi}{p}m^p\mathbb{1}_{\{y\in U_m\}}\Big)\lambda(dy)ds \\ &\le m\lambda(U_m)\left(|\phi'(0)| + c_\phi m^{p-1}\mathbb{E}\left[(\eta(U_m\times[0,1])+\lambda(U_m))^{p-1}\right] + \frac{c_\phi m^{p-1}}{p}\right) < \infty \end{aligned} \tag{9.37}$$

which is finite since $\eta(U_m\times[0,1])$ is a Poisson random variable with parameter $\lambda(U_m)$ and thus all its moments are finite. This also shows that $\phi(X_{s-}(\eta+\delta_{(y,s)})+h_m(\eta,y,s))-\phi(X_{s-}(\eta+\delta_{(y,s)}))\in L^1(\mathbf{N}\times\mathbb{X}\times[0,1])$. The second term on the RHS can be treated by the same method.
We can now take expectations on both sides of (9.32) and apply Mecke formula (3.7) to get:

$$\begin{aligned}\mathbb{E}\phi(\delta(h_m)) &= \mathbb{E}\int_{\mathbb{X}}\int_0^1 \phi(X_{s-}(\eta+\delta_{(y,s)})+h_m(\eta,y,s))-\phi(X_{s-}(\eta+\delta_{(y,s)}))\lambda(dy)ds \\ &\qquad - \mathbb{E}\int_{\mathbb{X}}\int_0^1\phi'(X_s(\eta))h_m(\eta,y,s)\lambda(dy)ds \end{aligned} \tag{9.38}$$

Now add and subtract the integral of $\phi(X_{s-}(\eta)+h_m(\eta,y,s))-\phi(X_{s-}(\eta))$ (which can be shown to be in $L^1(\mathbf{N}\times\mathbb{X}\times[0,1])$ by the same methods as above). We obtain:

$$\begin{aligned}\mathbb{E}\phi(\delta(h_m)) = \mathbb{E}\int_{\mathbb{X}}\int_0^1 &\phi(X_{s-}(\eta+\delta_{(y,s)})+h_m(\eta,y,s))-\phi(X_{s-}(\eta+\delta_{(y,s)}))\\ &-\phi(X_{s-}(\eta)+h_m(\eta,y,s))+\phi(X_{s-}(\eta))\lambda(dy)ds\\ &+\mathbb{E}\int_{\mathbb{X}}\int_0^1 \phi(X_{s-}(\eta)+h_m(\eta,y,s))-\phi(X_{s-}(\eta))-\phi'(X_s(\eta))h_m(\eta,y,s)\lambda(dy)ds\\ &=: I_1+I_2. \end{aligned} \tag{9.39}$$

To deal with I_1, note that for $a,b,c\in\mathbb{R}$

$$|\phi(a+c)-\phi(a)-\phi(b+c)+\phi(b)| = \left|\int_a^b \phi'(u+c)-\phi'(u)du\right| \tag{9.40}$$

$$\leq \int_{a\wedge b}^{a\vee b} c_\phi |c|^{p-1}du \tag{9.41}$$

$$= c_\phi|a-b|\cdot|c|^{p-1}. \tag{9.42}$$

Hence

$$|I_1| \leq c_\phi \mathbb{E}\int_{\mathbb{X}}\int_0^1 \left|X_{s-}(\eta+\delta_{(y,s)})-X_{s-}(\eta)\right|\cdot|h_m(\eta,y,s)|^{p-1}\lambda(dy)ds. \tag{9.43}$$

Now we bound and rewrite part of the integrand on the RHS of (9.43) to find

$$\begin{aligned}&\left|X_{s-}(\eta+\delta_{(y,s)})-X_{s-}(\eta)\right|\\ &=\Big|\int_{\mathbb{X}\times[0,s)} h_m(\eta+\delta_{(y,s)}-\delta_{(x,t)},x,t)\eta(dx,dt)-\int_{\mathbb{X}}\int_0^s h_m(\eta+\delta_{(y,s)},x,t)\lambda(dx)dt\\ &\quad -\int_{\mathbb{X}\times[0,s)} h_m(\eta-\delta_{(x,t)},x,t)\eta(dx,dt)+\int_{\mathbb{X}}\int_0^s h_m(\eta,x,t)\lambda(dx)dt\Big|\\ &=\left|\int_{\mathbb{X}\times[0,s)} \mathrm{D}_{(y,s)}\,h_m(\eta-\delta_{(x,t)},x,t)\eta(dx,dt)-\int_{\mathbb{X}}\int_0^s \mathrm{D}_{(y,s)}\,h_m(\eta,x,t)\lambda(dx)dt\right|\\ &\leq \int_{\mathbb{X}\times[0,s)}\left|\mathrm{D}_{(y,s)}\,h_m(\eta-\delta_{(x,t)},x,t)\right|\eta(dx,dt)+\int_{\mathbb{X}}\int_0^s\left|\mathrm{D}_{(y,s)}\,h_m(\eta,x,t)\right|\lambda(dx)dt. \end{aligned} \tag{9.44}$$

Multiplying this by $|h_m(\eta,x,t)|^{p-1}$ and taking expectations, after an application of Mecke formula (3.7) we deduce that

$$\begin{aligned}|I_1| \leq c_\phi\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\,&\left|\mathrm{D}_{(y,s)}\,h_m(\eta,x,t)\right|\\ &\left(|h_m(\eta+\delta_{(x,t)},y,s)|^{p-1}+|h_m(\eta,y,s)|^{p-1}\right)\lambda(dx)dt\lambda(dy)ds. \end{aligned} \tag{9.45}$$

Since $|a-b|^{p-1}\leq|a|^{p-1}+|b|^{p-1}$ for all $a,b\in\mathbb{R}$, one has that

$$|h_m(\eta+\delta_{(x,t)},y,s)|^{p-1}\leq|\,\mathrm{D}_{(x,t)}\,h_m(\eta,y,s)|^{p-1}+|h_m(\eta,y,s)|^{p-1} \tag{9.46}$$

This implies that

$$|I_1| \leq c_\phi \mathbb{E} \int_{\mathbb{X}} \lambda(dy) \int_0^1 ds \int_{\mathbb{X}} \lambda(dx) \int_0^s dt \left| \mathrm{D}_{(y,s)} h_m(\eta,x,t) \right| \cdot | \mathrm{D}_{(x,t)} h_m(\eta,y,s)|^{p-1}$$
$$+ 2c_\phi \mathbb{E} \int_{\mathbb{X}} \lambda(dy) \int_0^1 ds \int_{\mathbb{X}} \lambda(dx) \int_0^s dt \left| \mathrm{D}_{(y,s)} h_m(\eta,x,t) \right| \cdot |h_m(\eta,y,s)|^{p-1}. \tag{9.47}$$

Now we consider $|I_2|$. For this, note that for $a, b \in \mathbb{R}$,

$$\begin{aligned} |\phi(a+b) - \phi(a) - \phi'(a)b| &= \left| \int_0^b \phi'(a+u) - \phi'(a) du \right| \\ &\leq c_\phi \int_0^{|b|} u^{p-1} du \\ &= \frac{c_\phi}{p} |b|^p. \end{aligned} \tag{9.48}$$

Applying this to $a = X_s(\eta)$ and $h = h_m(\eta, y, s)$ yields

$$|I_2| \leq \frac{c_\phi}{p} \mathbb{E} \int_{\mathbb{X}} \int_0^1 |h_m(\eta,y,s)|^p \lambda(dy) ds. \tag{9.49}$$

Observe that in the previous computation we implicitly used the fact that, $\mathbb{P}$-a.s., the set $\{s \in [0,1] : X_s(\eta) \neq X_{s-}(\eta)\}$ has zero Lebesgue measure. We have therefore shown the following inequality:

$$|\mathbb{E}\phi(\delta(h_m))| \leq \frac{c_\phi}{p} \mathbb{E} \int_{\mathbb{X}} \lambda(dy) \int_0^1 ds \, |h_m(\eta,y,s)|^p$$
$$+ c_\phi \mathbb{E} \int_{\mathbb{X}} \lambda(dy) \int_0^1 ds \int_{\mathbb{X}} \lambda(dx) \int_0^s dt \left| \mathrm{D}_{(y,s)} h_m(\eta,x,t) \right| \cdot | \mathrm{D}_{(x,t)} h_m(\eta,y,s)|^{p-1}$$
$$+ 2c_\phi \mathbb{E} \int_{\mathbb{X}} \lambda(dy) \int_0^1 ds \int_{\mathbb{X}} \lambda(dx) \int_0^s dt \left| \mathrm{D}_{(y,s)} h_m(\eta,x,t) \right| \cdot |h_m(\eta,y,s)|^{p-1}. \tag{9.50}$$

By the construction of h_m, the RHS of this inequality is upper bounded by

$$\frac{c_\phi}{p} \mathbb{E} \int_{\mathbb{X}} \lambda(dy) \int_0^1 ds \, |h(\eta,y,s)|^p$$
$$+ c_\phi \mathbb{E} \int_{\mathbb{X}} \lambda(dx) \int_0^1 ds \int_{\mathbb{X}} \lambda(dx) \int_0^s dt \left| \mathrm{D}_{(y,s)} h(\eta,x,t) \right| \cdot | \mathrm{D}_{(x,t)} h(\eta,y,s)|^{p-1}$$
$$+ 2c_\phi \mathbb{E} \int_{\mathbb{X}} \lambda(dy) \int_0^1 ds \int_{\mathbb{X}} \lambda(dx) \int_0^s dt \left| \mathrm{D}_{(y,s)} h(\eta,x,t) \right| \cdot |h(\eta,y,s)|^{p-1}. \tag{9.51}$$

In order to conclude the proof, it remains to show that $\mathbb{E}\phi(\delta(h_m)) \to \mathbb{E}\phi(\delta(h))$, as $m \to \infty$. For this, we use (9.33) with $a = \delta(h)$ and $b = \delta(h_m) - \delta(h)$ to get

$$\mathbb{E}|\phi(\delta(h_m)) - \phi(\delta(h))| \leq \mathbb{E}|\delta(h_m) - \delta(h)| \left(|\phi'(0)| + c_\phi |\delta(h)|^{p-1} + \frac{c_\phi}{p} |\delta(h_m) - \delta(h)|^{p-1} \right). \tag{9.52}$$

Using, in order, the Cauchy-Schwarz, Minkowski and Jensen inequalities, we obtain

$$\begin{aligned}
&\mathbb{E}|\phi(\delta(h_m)) - \phi(\delta(h))| \\
&\qquad \leq \mathbb{E}\left[(\delta(h_m)-\delta(h))^2\right]^{1/2} \mathbb{E}\left[\left(|\phi'(0)| + c_\phi|\delta(h)|^{p-1} + \frac{c_\phi}{p}|\delta(h_m)-\delta(h)|^{p-1}\right)^2\right]^{1/2} \\
&\qquad \leq \mathbb{E}\left[\delta(h_m-h)^2\right]^{1/2}\left(|\phi'(0)| + c_\phi\mathbb{E}\left[|\delta(h)|^2\right]^{(p-1)/2} + \frac{c_\phi}{p}\mathbb{E}\left[|\delta(h_m-h)|^2\right]^{(p-1)/2}\right).
\end{aligned} \tag{9.53}$$

By the isometry property (8.4) and the Cauchy-Schwarz inequality, we infer that

$$\mathbb{E}\left[|\delta(h)|^2\right] \leq \mathbb{E}\int_{\mathbb{X}}\int_0^1 h(\eta,y,s)^2\lambda(dy)ds + \mathbb{E}\int_{\mathbb{X}}\int_0^1\int_{\mathbb{X}}\int_0^1 \left(\mathrm{D}_{(x,t)}\,h(\eta,y,s)\right)^2\lambda(dy)ds\lambda(dx)ds \tag{9.54}$$

and

$$\begin{aligned}
\mathbb{E}\left[|\delta(h_m-h)|^2\right] \leq{}& \mathbb{E}\int_{\mathbb{X}}\int_0^1 (h_m(\eta,y,s)-h(\eta,y,s))^2\lambda(dy)ds \\
&+\mathbb{E}\int_{\mathbb{X}}\int_0^1\int_{\mathbb{X}}\int_0^1 \left(\mathrm{D}_{(x,t)}\,h_m(\eta,y,s) - \mathrm{D}_{(x,t)}\,h(\eta,y,s)\right)^2\lambda(dy)ds\lambda(dx)ds.
\end{aligned} \tag{9.55}$$

The RHS of (9.54) is finite because h satisfies (3.13), and the RHS of (9.55) tends to 0 as $m \to \infty$ by dominated convergence and the assumption (3.13). This implies convergence to 0 on the RHS of (9.53).
To show that the inequality holds in particular for $\phi(x) = |x|^p$ with $p \in (1,2]$, it suffices to note that by Lemma 9.1, this function satisfies the conditions of the theorem. For $\phi(x) = |x|$, we define h_m as in (9.29). Then $h_m \in L^1(\mathbf{N}\times\mathbb{X}\times[0,1]) \cap L^2(\mathbf{N}\times\mathbb{X}\times[0,1]) \cap \operatorname{dom}\delta$ and the pathwise representation (3.14) holds. By the triangle inequality, we have

$$\mathbb{E}|\delta(h_m)| \leq \mathbb{E}\int_{\mathbb{X}\times[0,1]} |h_m(\eta-\delta_{y,s},y,s)|\eta(dy,ds) + \mathbb{E}\int_{\mathbb{X}}\int_0^1 |h_m(\eta,y,s)|\lambda(dy)ds. \tag{9.56}$$

By Mecke formula (3.7) and the fact that $|h_m| \leq |h|$, inequality (5.3) follows with h_m on the LHS instead of h, and the second and third term on the RHS being zero. As we have shown before, $\delta(h_m) \to \delta(h)$ in $L^2(\mathbb{P}_\eta)$ as $m \to \infty$, hence the inequality follows for h. ■

Proof of Corollary 5.3. **Step 1.** We first prove the following slightly modified version of (5.3) which holds under the weaker assumption $h \in \operatorname{dom}\delta$:

$$\begin{aligned}
&|\mathbb{E}\phi(\delta(h))| \\
&\leq \frac{c_\phi}{p}\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds|h(\eta,y,s)|^p \\
&+ c_\phi\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\left(\mathbb{E}\left|\mathrm{D}_{(y,s)}\,h(\eta,x,t)\right|^p\right)^{1/p}\cdot\left(\mathbb{E}\left|\mathrm{D}_{(x,t)}\,h(\eta,y,s)\right|^p\right)^{1-1/p} \\
&+ 2c_\phi\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\left(\mathbb{E}\left|\mathrm{D}_{(y,s)}\,h(\eta,x,t)\right|^p\right)^{1/p}\cdot\left(\mathbb{E}\left|h(\eta,y,s)\right|^p\right)^{1-1/p}.
\end{aligned} \tag{9.57}$$

For h satisfying (3.13), this is an immediate consequence of Theorem 5.2 by applying Hölder inequality.
Now let $h \in L^2(\mathbf{N} \times \mathbb{X} \times [0,1])$ and for $\tau \in (0,1)$ define $P_\tau h$. By Lemma 8.2, we have that $P_\tau h$ satisfies (3.13), and so inequality (9.57) holds when h is replaced by $P_\tau h$. Using (8.14) and (8.15), we deduce

$$\begin{aligned} &|\mathbb{E}\phi(\delta(P_\tau h)) - \phi(0)| \qquad (9.58)\\ &\leq \frac{c_\phi}{p}\mathbb{E}\int_{\mathbb{X}} \lambda(dy)\int_0^1 ds|h(\eta,y,s)|^p \\ &+ c_\phi \int_{\mathbb{X}} \lambda(dy)\int_0^1 ds \int_{\mathbb{X}} \lambda(dx)\int_0^s dt \left(\mathbb{E}\left|\mathrm{D}_{(y,s)} h(\eta,x,t)\right|^p\right)^{1/p} \cdot \left(\mathbb{E}\left|\mathrm{D}_{(x,t)} h(\eta,y,s)\right|^p\right)^{1-1/p} \\ &+ 2c_\phi \int_{\mathbb{X}} \lambda(dy)\int_0^1 ds \int_{\mathbb{X}} \lambda(dx)\int_0^s dt \left(\mathbb{E}\left|\mathrm{D}_{(y,s)} h(\eta,x,t)\right|^p\right)^{1/p} \cdot \left(\mathbb{E}\left|h(\eta,y,s)\right|^p\right)^{1-1/p}. \qquad (9.59)\end{aligned}$$

It remains to show that $\phi(\delta(P_\tau h)) \to \phi(\delta(h))$ in $L^1(\eta)$ as $\tau \to 1$. Using arguments similar to the ones in the proof of Theorem 5.2, one shows that

$$\begin{aligned} &\mathbb{E}|\phi(\delta(P_\tau h)) - \phi(\delta(h))| \\ &\qquad \leq \|\delta(P_\tau h - h)\|_{L^2(\mathbb{P}_\eta)} \left(|\phi'(0)| + c_\phi\|\delta(h)\|^{p-1}_{L^2(\mathbb{P}_\eta)} + \frac{c_\phi}{p}\|\delta(P_\tau h - h)\|^{p-1}_{L^2(\mathbb{P}_\eta)}\right). \qquad (9.60)\end{aligned}$$

By the isometry property (8.1), the second moment $\mathbb{E}\delta(h)^2$ is finite since $h \in \operatorname{dom}\delta$ and Lemma 8.2 implies that $\|\delta(P_\tau h - h)\|_{L^2(\mathbb{P}_\eta)} \to 0$ as $\tau \to 1$.
Step 2. Let $h \in L^2(\mathbf{N}\times\mathbb{X}\times[0,1])$ be predictable. Then $h \in \operatorname{dom}\delta$ (see page 42) and (9.57) holds. If $t < s$, then by predictability of h one has $h(\eta+\delta_{(y,s)}, x, t) = h(\eta, x, t)$ and hence $\mathrm{D}_{(y,s)} h(\eta,x,t) = 0$. It follows that the second and third terms on the RHS of (9.57) are 0. Inequality (5.4) ensues.
Step 3. To prove (5.5) for $p \in (1,2]$, let $h(\eta,x,s) := \mathbb{E}[\mathrm{D}_{(x,s)} F|\eta_{\mathbb{X}\times[0,s)}]$. This is predictable in the sense of (3.17) (see Chapter 3) and by the Clark-Ocône representation (3.19), one has that

$$F = \mathbb{E}F + \delta(h) \qquad \mathbb{P}\text{-a.s.} \qquad (9.61)$$

Define $\phi(x) := |x+\mathbb{E}F|^p - |\mathbb{E}F|^p$. Then $\phi(0) = 0$ and the derivative ϕ' is $(p-1)$-Hölder continuous with Hölder constant 2^{2-p} by Lemma 9.1. Moreover, it holds that

$$\phi(\delta(h)) = |F|^p - |\mathbb{E}F|^p \qquad \mathbb{P}\text{-a.s.} \qquad (9.62)$$

We can now apply (5.4) for this choice of ϕ and h and inequality (5.5) follows.
For $p = 1$, assume the RHS of the inequality (5.5) to be finite (else there is nothing to prove). Then $h(\eta,x,s) = \mathbb{E}[\mathrm{D}_{(x,s)} F|\eta_{\mathbb{X}\times[0,s)}] \in \operatorname{dom}\delta \cap L^1(\mathbb{N}\times\mathbb{X}\times[0,1])$ and hence

$$\delta(h) = \int_{\mathbb{X}\times[0,1]} h(\eta - \delta_{(x,s)}, x, s)\eta(dx,ds) - \int_{\mathbb{X}}\int_0^1 h(\eta,x,s)\lambda(dx)ds. \qquad (9.63)$$

Using triangle inequality and Mecke formula, we deduce the chain of inequalities

$$\begin{aligned} \mathbb{E}|F| - |\mathbb{E}F| &\leq \mathbb{E}|F - \mathbb{E}F| \\ &= \mathbb{E}|\delta(h)| \qquad (9.64)\\ &\leq \mathbb{E}\int_{\mathbb{X}\times[0,1]} |h(\eta-\delta_{(x,t)}, x, t)|\eta(dx,dt) + \mathbb{E}\int_{\mathbb{X}}\int_0^1 |h(\eta,x,t)|\lambda(dx)dt \\ &\leq 2\mathbb{E}\int_{\mathbb{X}}\int_0^1 |h(\eta,x,t)|\lambda(dx)dt, \qquad (9.65)\end{aligned}$$

which concludes the proof. ■

Proof of Corollary 5.5. Let $(U_n)_{n\geq 1} \subset \mathbb{X}$ be an increasing sequence of subsets such that $\bigcup_{n\geq 1} U_n = \mathbb{X}$ and $\lambda(U_n) < \infty$. Define for $n \in \mathbb{N}$

$$h_n(\eta, y, s) := \mathbb{1}_{\{y\in U_n\}}[(h(\eta, y, s) \wedge n) \vee (-n)]. \tag{9.66}$$

Clearly $h_n \in L^1(\mathbf{N} \times \mathbb{X} \times [0,1]) \cap L^2(\mathbf{N} \times \mathbb{X} \times [0,1])$ and we can define $P_\tau h_n$ for $\tau \in (0,1)$. We will start by showing (5.8) for $P_\tau h_n$ instead of h. Using the definition (8.9), one easily checks that $P_\tau h_n \in L^1(\mathbf{N} \times \mathbb{X} \times [0,1])$. By Lemma 8.2, it also follows that $P_\tau h_n \in \operatorname{dom}\delta$ and hence we can apply Lemma 8.1 and deduce that

$$\mathbb{E}\int_{\mathbb{X}}\int_0^1 P_\tau h_n(\eta, x, s)\, \mathrm{D}_{(x,s)}\, G\lambda(dx)ds = \mathbb{E}\left[G\delta(P_\tau h_n)\right]. \tag{9.67}$$

This implies by Jensen's inequality that for any $p \in [1,2]$

$$\left|\mathbb{E}\int_{\mathbb{X}}\int_0^1 P_\tau h_n(\eta, x, s)\, \mathrm{D}_{(x,s)}\, G\lambda(dx)ds\right| \leq c_G \left(\mathbb{E}|\delta(P_\tau h_n)|^p\right)^{1/p}. \tag{9.68}$$

As by Lemma 8.2 the quantity $P_\tau h_n$ also satisfies (3.13), we can apply Theorem 5.2 to $\mathbb{E}|\delta(P_\tau h_n)|^p$ with $\phi(x) = |x|^p$ (which satisfies the required conditions by Lemma 9.1). After a further application of Hölder inequality, this yields for $p \in (1,2]$ that

$$\begin{aligned}
&\mathbb{E}|\delta(P_\tau h_n)|^p \\
&\leq 2^{2-p}\mathbb{E}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds|P_\tau h_n(\eta, y, s)|^p \\
&+ p2^{2-p}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\mathbb{E}\left[\left|\mathrm{D}_{(y,s)}\, P_\tau h_n(\eta, x, t)\right|^p\right]^{1/p}\cdot\mathbb{E}\left[\left|\mathrm{D}_{(x,t)}\, P_\tau h_n(\eta, y, s)\right|^p\right]^{1-1/p} \\
&+ p2^{3-p}\int_{\mathbb{X}}\lambda(dy)\int_0^1 ds\int_{\mathbb{X}}\lambda(dx)\int_0^s dt\mathbb{E}\left[\left|\mathrm{D}_{(y,s)}\, P_\tau h_n(\eta, x, t)\right|^p\right]^{1/p}\cdot\mathbb{E}\left[|P_\tau h_n(\eta, y, s)|^p\right]^{1-1/p}.
\end{aligned} \tag{9.69}$$

Using Lemma 8.2 and the definition of h_n, we find that

$$\mathbb{E}|\,\mathrm{D}_{(x,s)}\, P_\tau h_n(\eta, y, u)|^p \leq \mathbb{E}|\,\mathrm{D}_{(x,s)}\, h(\eta, y, u)|^p \tag{9.70}$$

and

$$\mathbb{E}|P_\tau h_n(\eta, y, u)|^p \leq \mathbb{E}|h(\eta, y, u)|^p \tag{9.71}$$

hence (9.69) is upper bounded by the RHS of (5.8). For $p = 1$, we reason as in the proof of Corollary 5.3 to deduce that

$$\mathbb{E}|\delta(P_\tau h_n)| \leq 2\mathbb{E}\int_{\mathbb{X}}\int_0^1 |P_\tau h_n(\eta, x, s)|\lambda(dx)ds, \tag{9.72}$$

which is again upper bounded by the RHS of (5.8).
It remains to take $\tau \to 1$ and $n \to \infty$ in the LHS of (9.68). By (8.9), one sees that $P_\tau h_n(\eta, x, s) = \mathbb{1}_{\{x\in U_n\}} P_\tau h_n(\eta, x, s)$, and hence by the Cauchy-Schwarz inequality

$$\mathbb{E}\int_{\mathbb{X}}\int_0^1 |P_\tau h_n(\eta, x, s) - h_n(\eta, x, s)|\cdot|\,\mathrm{D}_{(x,s)}\, G|\lambda(dx)ds \leq 2c_G\lambda(U_n)\|P_\tau h_n - h_n\|_{L^2(\mathbf{N}\times\mathbb{X}\times[0,1])}, \tag{9.73}$$

which converges to zero as $\tau \to 1$, by Lemma 8.2. Therefore inequality (5.8) holds with h_n instead of h on the LHS. By dominated convergence, $h_n\,\mathrm{D}\,G \to h\,\mathrm{D}\,G$ in $L^1(\mathbf{N} \times \mathbb{X} \times [0,1])$, thus inequality (5.8) holds for h. ■

9.3 Proofs of Theorems 4.2, 4.3 and 4.4

Throughout this section, we work with the simplified notation adopted in Remark 4.1; recall also the definitions of the distances d_W and d_K given in (4.1) and (4.2), and write $\mathcal{H}$ to denote the set of Lipschitz-continuous functions $h : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant $\|h\|_{Lip} \leq 1$. Let $N \sim \mathcal{N}(0,1)$.

Given $h \in \mathcal{H}$ and $\phi(z) = \mathbb{P}(N \leq z)$, Stein's equation

$$f'(x) = xf(x) + h(x) - \int_{\mathbb{R}} h(y)\phi'(y)dy, \qquad x \in \mathbb{R}, \tag{9.74}$$

admits a **canonical solution** f_h satisfying the two properties: (a) $f_h \in \mathcal{C}^1(\mathbb{R})$ and $\|f_h'\|_\infty \leq \sqrt{\frac{2}{\pi}}$ and (b) f_h' is Lipschitz-continuous with $\|f_h'\|_{Lip} \leq 2$, see e.g. [PR16, Theorem 3, Chapter 6] and the references therein. In particular, one has that

$$d_W(F,N) = \sup_{h\in\mathcal{H}} |\mathbb{E}f_h'(F) - \mathbb{E}Ff_h(F)|. \tag{9.75}$$

Similarly, for fixed $z \in \mathbb{R}$, the canonical solution f_z to Stein's equation

$$f_z'(x) = xf_z(x) + \mathbb{1}_{(-\infty,z]}(x) - \phi(z), \qquad x \in \mathbb{R}, \tag{9.76}$$

is differentiable everywhere except at z, where it is customary to define $f_z'(z) = zf_z(z) + 1 - \phi(z)$. One has that

- $\|f_z\|_\infty \leq \frac{\sqrt{2\pi}}{4}$ and $\|f_z'\|_\infty \leq 1$
- $|xf_z(x)| \leq 1$ for all $x \in \mathbb{R}$ and the function $x \mapsto xf_z(x)$ is non-decreasing for all $z \in \mathbb{R}$,

(see e.g. [CGS11, Lemma 2.3]). We consequently have that

$$d_K(F,N) = \sup_{z\in\mathbb{R}} |\mathbb{E}f_z'(F) - \mathbb{E}Ff_z(F)|. \tag{9.77}$$

Proof of Thm. 4.2. We will show the upper bounds in (4.3) and (4.4) by exploiting the representations (9.75) and (9.77). The proof is divided into three steps. First, we are going to derive the first terms on the RHSs of (4.3) and (4.4) for both Wasserstein and Kolmogorov distances at the same time. Second, we deduce the second term on the RHS of (4.3) and, as a last step, we find the second term on the RHS of (4.4). Throughout the proof, we fix $h \in \mathcal{H}$ and $z \in \mathbb{R}$ and consider the corresponding canonical solutions f_h and f_z.

Step 1. Write f for either f_h or f_z. Then, f is Lipschitz and there is a version of f' which is bounded. Since $F \in \operatorname{dom}\mathrm{D}$ and f' is bounded, the expression

$$\mathbb{E}f'(F)\int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy) \tag{9.78}$$

is well-defined. Add and subtract this term to $\mathbb{E}f'(F) - \mathbb{E}Ff(F)$ and bound the resulting first term as follows:

$$\begin{aligned}
&\left|\mathbb{E}f'(F) - \mathbb{E}f'(F)\int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)\right| \\
&\leq \mathbb{E}|f'(F)| \cdot \left|1 - \int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)\right| \\
&\leq \|f'\|_\infty\, \mathbb{E}\left|1 - \int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)\right|.
\end{aligned} \tag{9.79}$$

As $\|f_h'\|_\infty \le \sqrt{\frac{2}{\pi}}$ and $\|f_z'\|_\infty \le 1$, the bounds follow.
We are left to deal with

$$\left|\mathbb{E}Ff(F) - \mathbb{E}f'(F)\int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)\right|. \tag{9.80}$$

Since f is Lipschitz and $F \in L^2(\mathbb{P}_\eta)$, it follows that $f(F) \in L^2(\mathbb{P}_\eta)$. Hence by Lemma 8.4

$$\mathbb{E}Ff(F) = \mathrm{Cov}(F, f(F)) = \mathbb{E}\int_{\mathbb{Y}} \mathbb{E}[\mathrm{D}_y F|\eta_y]\, \mathbb{E}[\mathrm{D}_y f(F)|\eta_y]\bar{\lambda}(dy). \tag{9.81}$$

Again by Lipschitzianity of f, it follows that $|\mathrm{D}_y f(F)| \le |\mathrm{D}_y F|$ and hence an application of Cauchy-Schwarz inequality together with the fact that $F \in \operatorname{dom}\mathrm{D}$ justifies that

$$\mathbb{E}Ff(F) = \mathbb{E}\int_{\mathbb{Y}} \mathbb{E}[\mathrm{D}_y F|\eta_y]\, \mathrm{D}_y f(F)\bar{\lambda}(dy). \tag{9.82}$$

Therefore we are left to bound

$$\mathbb{E}\int_{\mathbb{Y}} |\mathbb{E}[\mathrm{D}_y F|\eta_y]| \cdot |f'(F)\, \mathrm{D}_y F - \mathrm{D}_y f(F)|\, \bar{\lambda}(dy). \tag{9.83}$$

Step 2. To bound (9.83) for the Wasserstein distance, we use an argument borrowed from the proof of [BOPT20, Theorem 3.1] to upper bound $|f_h(b) - f_h(a) - f_h'(a)(b-a)|$. Let $a, b \in \mathbb{R}$. Then by the properties stated above, f_h is Lipschitz and hence

$$|f_h(b) - f_h(a) - f_h'(a)(b-a)| \le |f_h(b) - f_h(a)| + |f_h'(a)(b-a)| \le 2\sqrt{\tfrac{2}{\pi}}|b-a|. \tag{9.84}$$

But at the same time by Lipschitzianity of f_h',

$$|f_h(b) - f_h(a) - f_h'(a)(b-a)| = \left|\int_a^b f_h'(x) - f_h'(a)dx\right| \le 2(b-a)^2. \tag{9.85}$$

We deduce that for any $q \in [1,2]$

$$|f_h(b) - f_h(a) - f_h'(a)(b-a)| \le 2\min\{|b-a|, (b-a)^2\} \le 2|b-a|^q. \tag{9.86}$$

It follows that

$$\begin{aligned} |f_h'(F)\, \mathrm{D}_y F - \mathrm{D}_y f_h(F)| &= |f_h(F(\eta+\delta_y)) - f_h(F(\eta)) - f_h'(F(\eta))(F(\eta+\delta_y) - F(\eta))| \\ &\le 2|\mathrm{D}_y F|^q \end{aligned} \tag{9.87}$$

and therefore (9.83) is bounded by

$$2\,\mathbb{E}\int_{\mathbb{Y}} |\mathbb{E}[\mathrm{D}_y F|\eta_y]|\, |\mathrm{D}_y F|^q \bar{\lambda}(dy). \tag{9.88}$$

The required bound in the Wasserstein distance follows suit.
Step 3. We reason as in the proof of [LRPY22, Theorem 1.12] to conclude

$$|f_z'(F)\, \mathrm{D}_y F - \mathrm{D}_y f_z(F)| \le \mathrm{D}_y F \cdot \mathrm{D}_y(Ff_z(F) + \mathbb{1}_{\{F>z\}}) \tag{9.89}$$

Thus (9.83) is upper bounded by

$$\mathbb{E}\int_{\mathbb{Y}} |\mathbb{E}[\mathrm{D}_y F|\eta_y]|\, \mathrm{D}_y F \cdot \mathrm{D}_y\left(Ff_z(F) + \mathbb{1}_{\{F>z\}}\right)\bar{\lambda}(dy). \tag{9.90}$$

The desired bound now follows by taking the supremum over all $z \in \mathbb{R}$. ■

Proof of Theorem 4.3. We apply Theorem 4.2 and bound the ensuing RHS of (4.3). The proof will be split into two steps.
Step 1. We start by showing that under the conditions of the theorem,

$$\mathbb{E}\left|1-\int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)\right| \leq \sqrt{\frac{\pi}{2}}\beta_1+\sqrt{\frac{\pi}{2}}\beta_2. \tag{9.91}$$

We can assume that $\mathbb{E}F=0$ and $\sigma=1$ (indeed, the result then follows since $\mathrm{D}\,\hat{F}=\sigma^{-1}\,\mathrm{D}\,F$). For ease of notation, define

$$G:=\int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)-1. \tag{9.92}$$

As a first step, we show that $\mathbb{E}G=0$. As $F\in\operatorname{dom}\mathrm{D}$, we can use Fubini's theorem and Lemma 8.4 to deduce

$$\mathbb{E}\int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)=\mathbb{E}\int_{\mathbb{Y}} \mathbb{E}[\mathrm{D}_y F|\eta_y]^2\bar{\lambda}(dy)=\mathbf{Var}(F)=1. \tag{9.93}$$

Now by the modification (5.6) of Corollary 5.3 given in Remark 5.4 and since $G\in L^1(\mathbb{P}_\eta)$, we have for $p\in[1,2]$,

$$\mathbb{E}|G|\leq(\mathbb{E}|G|^p)^{1/p}\leq\left(2^{2-p}\mathbb{E}\int_{\mathbb{Y}}\mathbb{E}[|\,\mathrm{D}_x G||\eta_x]^p\bar{\lambda}(dx)\right)^{1/p}. \tag{9.94}$$

Let us now study the term

$$\mathrm{D}_x G=\mathrm{D}_x\int_{\mathbb{Y}} \mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy) \tag{9.95}$$

and define

$$h(\eta,y):=\mathrm{D}_y F\, \mathbb{E}[\mathrm{D}_y F|\eta_y]. \tag{9.96}$$

We want to show that since $h\in L^1(\mathbf{N}\times\mathbb{Y})$,

$$\left|\mathrm{D}_x\int_{\mathbb{Y}} h(\eta,y)\bar{\lambda}(dy)\right|\leq\int_{\mathbb{Y}}|\mathrm{D}_x h(\eta,y)|\,\bar{\lambda}(dy) \tag{9.97}$$

and to do so we need an argument taken from the proof of [LPS16, Proposition 4.1]: Assume the RHS of (9.97) to be finite (if it is not, then the inequality (9.97) is trivially true). Then

$$\int_{\mathbb{Y}}|h(\eta+\delta_x,y)|\bar{\lambda}(dy)\leq\int_{\mathbb{Y}}|\mathrm{D}_x h(\eta,y)|+|h(\eta,y)|\bar{\lambda}(dy)<\infty \tag{9.98}$$

and hence

$$\mathrm{D}_x\int_{\mathbb{Y}} h(\eta,y)\bar{\lambda}(dy)=\int_{\mathbb{Y}}\mathrm{D}_x h(\eta,y)\bar{\lambda}(dy). \tag{9.99}$$

The inequality (9.97) follows. We have therefore shown

$$\begin{aligned}\mathbb{E}|G|&\leq\left(2^{2-p}\mathbb{E}\int_{\mathbb{Y}}\mathbb{E}\left[\int_{\mathbb{Y}}|\,\mathrm{D}_x h(\eta,y)|\bar{\lambda}(dy)\middle|\,\eta_x\right]^p\bar{\lambda}(dx)\right)^{1/p}\\&=\left(2^{2-p}\int_{\mathbb{Y}}\mathbb{E}\left(\int_{\mathbb{Y}}\mathbb{E}[|\,\mathrm{D}_x h(\eta,y)||\eta_x]\,\bar{\lambda}(dy)\right)^p\bar{\lambda}(dx)\right)^{1/p},\end{aligned} \tag{9.100}$$

where the second line follows from Tonelli's theorem. By Minkowski's integral inequality,

$$\mathbb{E}\left(\int_{\mathbb{Y}} \mathbb{E}\big[|\operatorname{D}_x h(\eta, y)|\big|\eta_x\big]\bar{\lambda}(dy)\right)^p \le \left(\int_{\mathbb{Y}} \mathbb{E}\big[\mathbb{E}\big[|\operatorname{D}_x h(\eta, y)|\big|\eta_x\big]^p\big]^{1/p}\bar{\lambda}(dy)\right)^p. \tag{9.101}$$

By the formula (3.4) for products,

$$\begin{aligned}\operatorname{D}_x h(\eta, y) &= \operatorname{D}_x\left(\operatorname{D}_y F\, \mathbb{E}[\operatorname{D}_y F|\eta_y]\right)\\ &= \operatorname{D}^{(2)}_{x,y} F \cdot \mathbb{E}[\operatorname{D}_y F|\eta_y] + \operatorname{D}_y F \cdot \operatorname{D}_x \mathbb{E}[\operatorname{D}_y F|\eta_y] + \operatorname{D}^{(2)}_{x,y} F \cdot \operatorname{D}_x \mathbb{E}[\operatorname{D}_y F|\eta_y].\end{aligned} \tag{9.102}$$

Since $\mathbb{E}[\operatorname{D}_y F|\eta_y]$ depends on η only through η_y, it follows that $\operatorname{D}_x \mathbb{E}[\operatorname{D}_y F|\eta_y] = 0$ whenever $x \ge y$. Moreover, since $F \in L^2(\mathbb{P}_\eta)$ implies that $\operatorname{D}_y F, \operatorname{D}^{(2)}_{x,y} F \in L^1(\mathbb{P}_\eta)$ by [LP11b, Theorem 1.1], if $x < y$, we can put the add-one cost operator inside the conditional expectation. Therefore

$$\operatorname{D}_x \mathbb{E}[\operatorname{D}_y F|\eta_y] = \mathbb{1}_{\{x<y\}}\mathbb{E}[\operatorname{D}^{(2)}_{x,y} F|\eta_y]. \tag{9.103}$$

By Minkowski's norm inequality, it follows that

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\big[|\operatorname{D}_x h(\eta, y)|\big|\eta_x\big]^p\right]^{1/p} \le{}& \mathbb{E}\left[\mathbb{E}\big[|\operatorname{D}^{(2)}_{x,y} F \cdot \mathbb{E}[\operatorname{D}_y F|\eta_y]|\big|\eta_x\big]^p\right]^{1/p}\\ &+ \mathbb{E}\left[\mathbb{E}\big[|\operatorname{D}_y F \cdot \mathbb{1}_{\{x<y\}}\mathbb{E}[\operatorname{D}^{(2)}_{x,y} F|\eta_y]|\big|\eta_x\big]^p\right]^{1/p}\\ &+ \mathbb{E}\left[\mathbb{E}\big[|\operatorname{D}^{(2)}_{x,y} F \cdot \mathbb{1}_{\{x<y\}}\mathbb{E}[\operatorname{D}^{(2)}_{x,y} F|\eta_y]|\big|\eta_x\big]^p\right]^{1/p}.\end{aligned} \tag{9.104}$$

By the properties of conditional expectations and splitting the first term into two parts, (9.104) is bounded by

$$\begin{aligned}&\mathbb{1}_{\{y\le x\}}\mathbb{E}\left[|\mathbb{E}[|\operatorname{D}^{(2)}_{x,y} F||\eta_x]^p \cdot \mathbb{E}[\operatorname{D}_y F|\eta_y]|^p\right]^{1/p}\\ &+ \mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}\big[\mathbb{E}[|\operatorname{D}^{(2)}_{x,y} F||\eta_y] \cdot |\mathbb{E}[\operatorname{D}_y F|\eta_y]|\big|\eta_x\big]^p\right]^{1/p}\\ &+ \mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}\big[\mathbb{E}[|\operatorname{D}_y F||\eta_y] \cdot |\mathbb{E}[\operatorname{D}^{(2)}_{x,y} F|\eta_y]|\big|\eta_x\big]^p\right]^{1/p}\\ &+ \mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}\big[\mathbb{E}[|\operatorname{D}^{(2)}_{x,y} F||\eta_y] \cdot |\mathbb{E}[\operatorname{D}^{(2)}_{x,y} F|\eta_y]|\big|\eta_x\big]^p\right]^{1/p}.\end{aligned} \tag{9.105}$$

By an application of Jensen's inequality (9.105) is now bounded by

$$\begin{aligned}&\mathbb{E}\left[|\mathbb{E}[\operatorname{D}_y F|\eta_y]|^p \cdot \mathbb{E}\big[|\operatorname{D}^{(2)}_{x,y} F|\big|\eta_{x\vee y}\big]^p\right]^{1/p}\\ &+ \mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}\big[|\operatorname{D}_y F|\big|\eta_y\big]^p \cdot \big|\mathbb{E}\big[\operatorname{D}^{(2)}_{x,y} F\big|\eta_y\big]\big|^p\right]^{1/p}\\ &+ \mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}\big[|\operatorname{D}^{(2)}_{x,y} F|\big|\eta_y\big]^{2p}\right]^{1/p}.\end{aligned} \tag{9.106}$$

Plugging the conclusion from (9.101) - (9.106) into (9.100) and applying Minkowski's norm inequality again yields

$$\begin{aligned}\mathbb{E}|G| \le{}& 2^{2/p-1}\left(\int_{\mathbb{Y}}\left(\int_{\mathbb{Y}}\mathbb{E}\left[\big|\mathbb{E}[\operatorname{D}_y F|\eta_y]\big|^p \cdot \mathbb{E}\big[|\operatorname{D}^{(2)}_{x,y} F|\big|\eta_{x\vee y}\big]^p\right]^{1/p}\bar{\lambda}(dy)\right)^p\bar{\lambda}(dx)\right)^{1/p}\\ &+ 2^{2/p-1}\left(\int_{\mathbb{Y}}\left(\int_{\mathbb{Y}}\mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}\big[|\operatorname{D}_y F|\big|\eta_y\big]^p \cdot \big|\mathbb{E}[\operatorname{D}^{(2)}_{x,y} F|\eta_y]\big|^p\right]^{1/p}\bar{\lambda}(dy)\right)^p\bar{\lambda}(dx)\right)^{1/p}\\ &+ 2^{2/p-1}\left(\int_{\mathbb{Y}}\left(\int_{\mathbb{Y}}\mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}\big[|\operatorname{D}^{(2)}_{x,y} F|\big|\eta_y\big]^{2p}\right]^{1/p}\bar{\lambda}(dy)\right)^p\bar{\lambda}(dx)\right)^{1/p}\end{aligned} \tag{9.107}$$

which is in turn bounded by

$$2^{2/p}\left(\int_{\mathbb{Y}}\left(\int_{\mathbb{Y}}\mathbb{E}\left[\mathbb{E}[|\operatorname{D}_y F||\eta_y]^p\cdot\mathbb{E}[|\operatorname{D}^{(2)}_{x,y} F||\eta_{x\vee y}]^p\right]^{1/p}\bar{\lambda}(dy)\right)^p\bar{\lambda}(dx)\right)^{1/p}$$
$$+\,2^{2/p-1}\left(\int_{\mathbb{Y}}\left(\int_{\mathbb{Y}}\mathbb{1}_{\{x<y\}}\mathbb{E}\left[\mathbb{E}[|\operatorname{D}^{(2)}_{x,y} F||\eta_y]^{2p}\right]^{1/p}\bar{\lambda}(dy)\right)^p\bar{\lambda}(dx)\right)^{1/p}. \qquad (9.108)$$

The result follows by an application of the Cauchy-Schwarz inequality.
Step 2. We want to show that

$$\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}[\operatorname{D}_y F|\eta_y]|\cdot|\operatorname{D}_y F|^q\bar{\lambda}(dy)\le\tfrac{1}{2}\beta_3+\tfrac{1}{2}\beta_4. \qquad (9.109)$$

Again it suffices to show (9.109) for F with $\mathbb{E}F=0$ and $\mathbb{E}F^2=1$. Assume the β_3,β_4 to be finite (otherwise there is nothing to prove). Then, in particular

$$\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}\left[\operatorname{D}_y F|\,\eta_y\right]|^{q+1}\bar{\lambda}(dy)<\infty \qquad (9.110)$$

and we can add and subtract this term to the LHS of (9.109), yielding $\frac{1}{2}\beta_3$ and the following rest term:

$$\left|\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}[\operatorname{D}_y F|\eta_y]|\cdot|\operatorname{D}_y F|^q\bar{\lambda}(dy)-\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}\left[\operatorname{D}_y F|\,\eta_y\right]|^{q+1}\bar{\lambda}(dy)\right|$$
$$=\left|\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}[\operatorname{D}_y F|\eta_y]|\cdot\mathbb{E}[|\operatorname{D}_y F|^q|\eta_y]\bar{\lambda}(dy)-\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}\left[\operatorname{D}_y F|\,\eta_y\right]|^{q+1}\bar{\lambda}(dy)\right|$$
$$\le\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}[\operatorname{D}_y F|\eta_y]|\cdot\left|\mathbb{E}[|\operatorname{D}_y F|^q|\eta_y]-|\mathbb{E}\left[\operatorname{D}_y F|\,\eta_y\right]|^q\right|\bar{\lambda}(dy), \qquad (9.111)$$

where the equality is justified by the fact that $\mathbb{E}[G|\eta_y]$ is defined for all $G\in L^0(\mathbb{P}_\eta)$ which are integrable or non-negative (see the definition (3.18)).
The term $\mathbb{E}[|\operatorname{D}_y F|^q|\eta_y]-|\mathbb{E}\left[\operatorname{D}_y F|\,\eta_y\right]|^q$ can be rewritten as

$$\tilde{\mathbb{E}}|g_y(\chi)|^q-|\tilde{\mathbb{E}}g_y(\chi)|^q, \qquad (9.112)$$

where $g_y(\chi)=\operatorname{D}_y F(\eta_y+\chi)$ and $\tilde{\mathbb{E}}$ is the expectation with respect to χ, a Poisson measure on the space $\{x\in\mathbb{Y}:x\ge y\}$ with intensity $\lambda\otimes ds$, independent of η. As $F\in\operatorname{dom}\operatorname{D}$, it can be shown that $g_y\in L^2(\mathbb{P}_\chi)$ for a.e. $y\in\mathbb{Y}$.
By (5.5) in Corollary 5.3,

$$\tilde{\mathbb{E}}|g_y(\chi)|^q-|\tilde{\mathbb{E}}g_y(\chi)|^q\le 2^{2-q}\tilde{\mathbb{E}}\int_{\mathbb{Y}}\mathbb{1}_{\{y\le x\}}\left|\tilde{\mathbb{E}}[\operatorname{D}_x g_y(\chi)|\chi_x]\right|^q\bar{\lambda}(dx). \qquad (9.113)$$

Plugging (9.113) into (9.111), we deduce that the RHS of (9.111) is upper bounded by

$$2^{2-q}\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}[\operatorname{D}_y F|\eta_y]|\int_{\mathbb{Y}}\mathbb{1}_{\{y\le x\}}\tilde{\mathbb{E}}\left|\tilde{\mathbb{E}}[\operatorname{D}_x g_y(\chi)|\chi_x]\right|^q\bar{\lambda}(dx)\bar{\lambda}(dy). \qquad (9.114)$$

Since χ and η are independent and have the same intensity measure,

$$\mathbb{1}_{\{y\le x\}}|\mathbb{E}[\operatorname{D}_y F|\eta_y]|\cdot\tilde{\mathbb{E}}\left|\tilde{\mathbb{E}}[\operatorname{D}_x g_y(\chi)|\chi_x]\right|^q \qquad (9.115)$$

has the same law as

$$\mathbb{1}_{\{y\leq x\}}|\mathbb{E}[\mathrm{D}_y F|\eta_y]| \cdot \mathbb{E}\big[|\mathbb{E}[\mathrm{D}^{(2)}_{x,y} F|\eta_x]|^q\big|\eta_y\big]. \tag{9.116}$$

This implies finally that the RHS of (9.111) is upper bounded by

$$2^{2-q}\mathbb{E}\int_{\mathbb{Y}}\int_{\mathbb{Y}} \mathbb{1}_{\{y\leq x\}}|\mathbb{E}[\mathrm{D}_y F|\eta_y]| \cdot |\mathbb{E}[\mathrm{D}^{(2)}_{x,y} F|\eta_x]|^q \bar{\lambda}(dx)\bar{\lambda}(dy) \tag{9.117}$$

and the result follows by the Cauchy-Schwarz inequality. ■

Proof of Theorem 4.4. It suffices to prove this result for F such that $\mathbb{E}F = 0$ and $\sigma = 1$. If Theorem 4.4 holds for all σ-finite spaces $\mathbb{X}$, it holds in particular for the space $\mathbb{X}\times[0,1]$. In fact, it suffices to prove Theorem 4.4 for functionals $F \in L^2(\mathbb{P}_\eta)\cap\mathrm{dom}\,\mathrm{D}$, where η is a $(\mathbb{X}\times[0,1], \lambda\otimes ds)$-Poisson measure. Indeed, as discussed in Chapter 3, we can regard any functional $F \in L^2(\mathbb{P}_\chi)$, with the $(\mathbb{X}, \lambda)$-Poisson measure χ, as a functional of η without changing the law of F or its add-one costs. If we replace $\mathbb{X}$ by $\mathbb{X}\times[0,1]$ and χ by η in the terms $\gamma_1, ..., \gamma_7$, the expressions do not change either since for any $F \in L^2(\mathbb{P}_\chi)\cap\mathrm{dom}\,\mathrm{D}$, the integrands do not depend on time. For the rest of this proof, we let thus $F \in L^2(\mathbb{P}_\eta)\cap\mathrm{dom}\,\mathrm{D}$ and recall the notation from Remark 4.1 where $\mathbb{Y} := \mathbb{X}\times[0,1]$ and $\bar{\lambda} := \lambda\otimes ds$ are explicitly defined.
We divide this proof into two steps: first, we discuss the bound on the Wasserstein distance, second, we show the bound on the Kolmogorov distance.
Step 1. By a combination of Theorem 4.2 and the proof of Theorem 4.3, we see that

$$\begin{aligned} d_W(F,N) &\leq \sqrt{\frac{2}{\pi}}\,\mathbb{E}\left|1-\int_{\mathbb{Y}}\mathrm{D}_y F\,\mathbb{E}[\mathrm{D}_y F|\eta_y]\bar{\lambda}(dy)\right| + 2\,\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}[\mathrm{D}_y F|\eta_y]|\cdot|\mathrm{D}_y F|^q\bar{\lambda}(dy) \\ &\leq \beta_1+\beta_2+2\,\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}[\mathrm{D}_y F|\eta_y]|\cdot|\mathrm{D}_y F|^q\bar{\lambda}(dy). \end{aligned} \tag{9.118}$$

To bound β_1 and β_2, we simply apply Jensen's inequality and bound $\mathbb{1}_{\{x<y\}}$ by 1. This gives

$$\beta_1+\beta_2 \leq \gamma_1+\gamma_2. \tag{9.119}$$

For the second term, apply Hölder's inequality to deduce

$$\mathbb{E}\int_{\mathbb{Y}}|\mathbb{E}[\mathrm{D}_y F|\eta_y]|\cdot|\mathrm{D}_y F|^q\bar{\lambda}(dy) \leq \int_{\mathbb{Y}}\left(\mathbb{E}\,|\mathbb{E}[\mathrm{D}_y F|\eta_y]|^{q+1}\right)^{1/(q+1)}\cdot\left(\mathbb{E}|\,\mathrm{D}_y F|^{q+1}\right)^{1-1/(q+1)}\bar{\lambda}(dy). \tag{9.120}$$

A further application of Jensen's inequality yields the result.
Step 2. We start from inequality (4.4) in Theorem 4.2. The first term was dealt with in Step 1 of this proof. Let us deal with the second term. Define

$$h(\eta,y) := \mathrm{D}_y F\cdot|\mathbb{E}[\mathrm{D}_y F|\eta_y]| \tag{9.121}$$

and for $z\in\mathbb{R}$,

$$Z_z := Ff_z(F)+\mathbb{1}_{\{F>z\}}. \tag{9.122}$$

The term we want to bound is thus given by

$$\sup_{z\in\mathbb{R}}\mathbb{E}\int_{\mathbb{Y}}h(\eta,y)\,\mathrm{D}_y Z_z\bar{\lambda}(dy). \tag{9.123}$$

Since $F \in \operatorname{dom} \mathrm{D}$, it follows by Cauchy-Schwarz inequality that $h \in L^1(\mathbf{N} \times \mathbb{Y})$. Moreover, by the properties of f_z, we have $|Z_z| \le 2$ for all $z \in \mathbb{R}$. Hence we can apply Corollary 5.5 and get for any $p \in [1, 2]$

$$\begin{aligned}
\mathbb{E} \int_{\mathbb{Y}} h(\eta, y) \, \mathrm{D}_y \, Z_z \bar{\lambda}(dy) &\le 2 \Bigg(2^{2-p} \mathbb{E} \int_{\mathbb{Y}} |h(\eta, y)|^p \bar{\lambda}(dy) \\
&\quad + p 2^{2-p} \mathbb{E} \int_{\mathbb{Y}} \int_{\mathbb{Y}} |\mathrm{D}_y \, h(\eta, x)|^p \, \bar{\lambda}(dy) \bar{\lambda}(dx) \\
&\quad + p 2^{3-p} \int_{\mathbb{Y}} \int_{\mathbb{Y}} \mathbb{1}_{\{x<y\}} \left(\mathbb{E} |\mathrm{D}_y \, h(\eta, x)|^p\right)^{1/p} \cdot \left(\mathbb{E} |h(\eta, y)|^p\right)^{1-1/p} \bar{\lambda}(dy) \bar{\lambda}(dx) \Bigg)^{1/p} \\
&= (I_1 + I_2 + I_3)^{1/p} \\
&\le (I_1)^{1/p} + (I_2)^{1/p} + (I_3)^{1/p}, \qquad (9.124)
\end{aligned}$$

since $|a + b|^{1/p} \le |a|^{1/p} + |b|^{1/p}$.
Let us first look at I_1. By applying the Cauchy-Schwarz and Jensen inequalities, it follows that

$$\begin{aligned}
I_1 &= 4 \, \mathbb{E} \int_{\mathbb{Y}} | \, \mathrm{D}_y \, F|^p \cdot |\mathbb{E}[\mathrm{D}_y \, F | \eta_y]|^p \bar{\lambda}(dy) \\
&\le 4 \int_{\mathbb{Y}} \mathbb{E} \left[| \, \mathrm{D}_y \, F|^{2p} \right] \bar{\lambda}(dy). \qquad (9.125)
\end{aligned}$$

Now we deal with I_2. By the formula (3.4),

$$\mathrm{D}_y \, h(\eta, x) = \mathrm{D}^{(2)}_{x,y} \, F \cdot \mathrm{D}_y \, |\mathbb{E}[\mathrm{D}_x \, F | \eta_x]| + \mathrm{D}^{(2)}_{x,y} \, F \cdot |\mathbb{E}[\mathrm{D}_x \, F | \eta_x]| + \mathrm{D}_x \, F \cdot \mathrm{D}_y \, |\mathbb{E}[\mathrm{D}_x \, F | \eta_x]|. \qquad (9.126)$$

By Minkowski's norm inequality,

$$\begin{aligned}
\left(\mathbb{E} \int_{\mathbb{Y}} \int_{\mathbb{Y}} |\mathrm{D}_y \, h(\eta, x)|^p \, \bar{\lambda}(dy) \bar{\lambda}(dx) \right)^{1/p} &\le \left(\mathbb{E} \int_{\mathbb{Y}} \int_{\mathbb{Y}} \left| \mathrm{D}^{(2)}_{x,y} \, F \cdot \mathrm{D}_y \, |\mathbb{E}[\mathrm{D}_x \, F | \eta_x]| \right|^p \bar{\lambda}(dy) \bar{\lambda}(dx) \right)^{1/p} \\
&\quad + \left(\mathbb{E} \int_{\mathbb{Y}} \int_{\mathbb{Y}} \left| \mathrm{D}^{(2)}_{x,y} \, F \cdot |\mathbb{E}[\mathrm{D}_x \, F | \eta_x]| \right|^p \bar{\lambda}(dy) \bar{\lambda}(dx) \right)^{1/p} \\
&\quad + \left(\mathbb{E} \int_{\mathbb{Y}} \int_{\mathbb{Y}} | \, \mathrm{D}_x \, F \cdot \mathrm{D}_y \, |\mathbb{E}[\mathrm{D}_x \, F | \eta_x]| |^p \bar{\lambda}(dy) \bar{\lambda}(dx) \right)^{1/p}. \qquad (9.127)
\end{aligned}$$

By the triangle inequality, and as in the proof of Theorem 4.3,

$$\left| \, \mathrm{D}_y \, |\mathbb{E}[\mathrm{D}_x \, F | \eta_x]| \right| \le \mathbb{1}_{\{y<x\}} | \, \mathrm{D}_y \, \mathbb{E}[\mathrm{D}_x \, F | \eta_x]| \le |\mathbb{E}[\mathrm{D}^{(2)}_{x,y} \, F | \eta_x]|. \qquad (9.128)$$

By the Cauchy-Schwarz and Jensen inequalities,

$$\begin{aligned}
(I_2)^{1/p} &\le (4p)^{1/p} \left(\int_{\mathbb{Y}} \int_{\mathbb{Y}} \mathbb{E} \left[| \, \mathrm{D}^{(2)}_{x,y} \, F|^{2p} \right] \bar{\lambda}(dy) \bar{\lambda}(dx) \right)^{1/p} \\
&\quad + 2^{2/p+1} p^{1/p} \left(\int_{\mathbb{Y}} \int_{\mathbb{Y}} \mathbb{E} \left[| \, \mathrm{D}^{(2)}_{x,y} \, F|^{2p} \right]^{1/2} \cdot \mathbb{E} \left[| \, \mathrm{D}_x \, F|^{2p} \right]^{1/2} \bar{\lambda}(dy) \bar{\lambda}(dx) \right)^{1/p}. \qquad (9.129)
\end{aligned}$$

Lastly, we look at the term I_3. Since $\mathbb{E}[\mathrm{D}_x F|\eta_x]$ depends only on η_x, we get

$$\mathbb{1}_{\{x<y\}} \mathrm{D}_y |\mathbb{E}[\mathrm{D}_x F|\eta_x]| = 0. \tag{9.130}$$

Hence

$$\mathbb{1}_{\{x<y\}} \left(\mathbb{E}\,|\mathrm{D}_y h(\eta,x)|^p\right)^{1/p} = \mathbb{1}_{\{x<y\}} \left(\mathbb{E}\left|\mathrm{D}^{(2)}_{x,y} F \cdot \mathbb{E}[\mathrm{D}_x F|\eta_x]\right|^p\right)^{1/p}. \tag{9.131}$$

Applying Cauchy-Schwarz and Jensen again yields

$$I_3 \le 8p \int_{\mathbb{Y}}\int_{\mathbb{Y}} \left(\mathbb{E}|\,\mathrm{D}^{(2)}_{x,y} F|^{2p}\right)^{\frac{1}{2p}} \cdot \left(\mathbb{E}|\,\mathrm{D}_x F|^{2p}\right)^{\frac{1}{2p}} \cdot \left(\mathbb{E}\,|\mathrm{D}_y F|^{2p}\right)^{1-1/p} \bar\lambda(dy)\bar\lambda(dx). \tag{9.132}$$

This inequality concludes the proof. ∎

9.4 Proof of Theorem 4.6

The proof of Theorem 4.6 uses both an interpolation technique and the multivariate Stein method. For a positive-definite symmetric matrix $C = (C_{ij})_{1\le i,j\le m}$, the multivariate Stein equation is given by

$$g(x) - \mathbb{E}g(X) = \langle x, \nabla f(x)\rangle_{\mathbb{R}^m} - \langle C, \mathrm{Hess}\, f(x)\rangle_{H.S.}, \tag{9.133}$$

for $x \in \mathbb{R}^m$ and $X \sim \mathcal{N}(0,C)$. The inner product $\langle .,.\rangle_{H.S.}$ is the Hilbert-Schmidt inner product defined as $\langle A,B\rangle_{H.S.} := Tr(AB^T)$ for real $m\times m$ matrices A and B and where $Tr(.)$ denotes the trace. If $g \in \mathcal{C}^2(\mathbb{R}^m)$ has bounded first- and second-order partial derivatives, then a solution to (9.133) is given by

$$f_g(x) := \int_0^1 \frac{1}{2t}\mathbb{E}\left[g(\sqrt{t}x + \sqrt{1-t}X) - g(X)\right] dt. \tag{9.134}$$

The solution f_g satisfies the following bounds:

$$\|f_g''\|_\infty \le \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\|g\|_{Lip} \tag{9.135}$$

and

$$\|f_g'''\|_\infty \le \frac{\sqrt{2\pi}}{4}\|C^{-1}\|_{op}^{3/2}\|C\|_{op} \sup_{x\in\mathbb{R}^m}\|\mathrm{Hess}\, g(x)\|_{op}. \tag{9.136}$$

These results can be found in [PZ10, Lemma 2.17].

Before we can give the proof of Theorem 4.6, we also need some technical estimates which improve the corresponding estimates given in [PZ10]. The first lemma is an extension of Lemma 3.1 in [PZ10]: contrary to what was done in [PZ10], we bound the rest term R_x in the development below by a q^{th} power of the add-one costs, with $q \in [1,2]$. Lemma 3.1 in [PZ10] corresponds to the choice of $q = 2$.

Lemma 9.2. *Let $F = (F_1, ..., F_m)$ for some $m \ge 1$, where $F_i \in L^2(\mathbb{P}_\chi) \cap \mathrm{dom}\,\mathrm{D}$ and $\mathbb{E}F_i = 0$ for $1 \le i \le m$. Then for all $\phi \in \mathcal{C}^2(\mathbb{R}^m)$ with $\|\phi'\|_\infty, \|\phi''\|_\infty < \infty$, it holds that for a.e. $x \in \mathbb{X}$ and all $q \in [1,2]$,*

$$\mathrm{D}_x \phi(F) = \sum_{i=1}^m \frac{\partial\phi}{\partial x_i}(F)\,\mathrm{D}_x F_i + R_x, \tag{9.137}$$

where

$$|R_x| \le \left(2\|\phi'\|_\infty \vee \tfrac{1}{2}\|\phi''\|_\infty\right)\left(\sum_{i=1}^m |\,\mathrm{D}_x F_i|\right)^q. \tag{9.138}$$

Proof. One has that

$$\begin{aligned} \mathrm{D}_x \phi(F) &= \phi(F + \mathrm{D}_x F) - \phi(F) \\ &= \int_0^1 \sum_{i=1}^m \frac{\partial \phi}{\partial x_i}(F + t\,\mathrm{D}_x F_i) \cdot \mathrm{D}_x F_i \, dt \\ &= \sum_{i=1}^m \frac{\partial \phi}{\partial x_i}(F) \cdot \mathrm{D}_x F_i + R_x, \end{aligned} \tag{9.139}$$

where

$$R_x = \sum_{i=1}^m \int_0^1 \left(\frac{\partial \phi}{\partial x_i}(F + t\,\mathrm{D}_x F) - \frac{\partial \phi}{\partial x_i}(F) \right) \cdot \mathrm{D}_x F_i \, dt. \tag{9.140}$$

Note that

$$\left| \frac{\partial \phi}{\partial x_i}(F + t\,\mathrm{D}_x F) - \frac{\partial \phi}{\partial x_i}(F) \right| \leq (2\|\phi'\|_\infty) \wedge (t\|\phi''\|_\infty \|\mathrm{D}_x F\|) \tag{9.141}$$

by the mean value theorem. Hence

$$\begin{aligned} |R_x| &\leq \left(2\|\phi'\|_\infty \sum_{i=1}^m |\,\mathrm{D}_x F_i| \right) \wedge \left(\tfrac{1}{2}\|\phi''\|_\infty \|\mathrm{D}_x F\| \sum_{i=1}^m |\,\mathrm{D}_x F_i| \right) \\ &\leq \left(2\|\phi'\|_\infty \vee \frac{1}{2}\|\phi''\|_\infty \right) \left[\sum_{i=1}^m |\,\mathrm{D}_x F_i| \wedge \left(\sum_{i=1}^m |\,\mathrm{D}_x F_i| \right)^2 \right] \\ &\leq \left(2\|\phi'\|_\infty \vee \frac{1}{2}\|\phi''\|_\infty \right) \left(\sum_{i=1}^m |\,\mathrm{D}_x F_i| \right)^q . \end{aligned} \tag{9.142}$$

■

The next lemma is an improvement of Lemma 4.1 in [PZ10], where the improvement comes from the fact that we use Lemma 9.2 in the final step.

Lemma 9.3. *Let* $m \geq 1$ *and for* $0 \leq i \leq m$*, let* $F_i \in L^2(\mathbb{P}_\chi) \cap \operatorname{dom} \mathrm{D}$ *and assume* $\mathbb{E}F_i = 0$*. Then for all* $g \in \mathcal{C}^2(\mathbb{R}^m)$ *such that* $\|g'\|_\infty, \|g''\|_\infty < \infty$*, we have*

$$\mathbb{E}g(F_1, ..., F_m)F_0 = \mathbb{E}\sum_{i=1}^m \frac{\partial g}{\partial x_i}(F_1, ..., F_m)\langle \mathrm{D}\,F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_0\rangle_{L^2(\lambda)} + \mathbb{E}\langle R, -\mathrm{D}\,\mathrm{L}^{-1} F_0\rangle_{L^2(\lambda)}, \tag{9.143}$$

where for all $q \in [1, 2]$*,*

$$\left|\mathbb{E}\langle R, -\mathrm{D}\,\mathrm{L}^{-1} F_0\rangle\right| \leq \left(2\|g'\|_\infty \vee \tfrac{1}{2}\|g''\|_\infty\right) \int_{\mathbb{X}} \mathbb{E}\left(\sum_{k=1}^m |\,\mathrm{D}_x F_k| \right)^q |\,\mathrm{D}_x \mathrm{L}^{-1} F_0| \lambda(dx). \tag{9.144}$$

Proof. As detailed in the proof of [PZ10, Lemma 4.1],

$$\mathbb{E}g(F_1, ..., F_m)F_0 = \mathbb{E}\langle \mathrm{D}\,g(F_1, ..., F_m), -\mathrm{D}\,\mathrm{L}^{-1} F_0\rangle \tag{9.145}$$

$$= \mathbb{E}\sum_{i=1}^m \frac{\partial g}{\partial x_i}(F_1, ..., F_m)\langle \mathrm{D}\,F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_0\rangle + \mathbb{E}\langle R, -\mathrm{D}\,\mathrm{L}^{-1} F_0\rangle, \tag{9.146}$$

where R is a rest term satisfying the bound in Lemma 9.2. The claimed result follows. ■

The next proposition is an extension of Theorems 3.3 and 4.2 in [PZ10] and exploits much of the arguments rehearsed in the proofs of these theorems, which are combined with the content of Lemmas 9.2 and 9.3.

Proposition 9.4. *Let $m \geq 1$ and let $F = (F_1, ..., F_m)$ be an $\mathbb{R}^m$-valued random vector such that for $1 \leq i \leq m$, we have $F_i \in L^2(\mathbb{P}_\chi) \cap \operatorname{dom} \mathrm{D}$ and $\mathbb{E}F_i = 0$. Let $C = (C_{ij})_{1\leq i,j\leq m}$ be a symmetric positive-semidefinite matrix and let $X \sim \mathcal{N}(0, C)$. Then for all $q \in [1, 2]$,*

$$d_3(F, X) \leq \tfrac{1}{2} \sum_{i,j=1}^{m} \mathbb{E}\left[|C_{ij} - \langle \mathrm{D}\, F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_j\rangle_{L^2(\lambda)}|\right] + \sum_{i=1}^{m} \int_{\mathbb{X}} \mathbb{E}\left[\left(\sum_{j=1}^{m} |\,\mathrm{D}_x F_j|\right)^q |\,\mathrm{D}_x \mathrm{L}^{-1} F_i|\right] \lambda(dx). \quad (9.147)$$

If moreover the matrix C is positive-definite, then for all $q \in [1, 2]$,

$$d_2(F, X) \leq \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sum_{i,j=1}^{m} \mathbb{E}\left[|C_{ij} - \langle \mathrm{D}\, F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_j\rangle_{L^2(\lambda)}|\right] + \left(2\|C^{-1}\|_{op}\|C\|_{op}^{1/2} \vee \frac{\sqrt{2\pi}}{8}\|C^{-1}\|_{op}^{3/2}\|C\|_{op}\right) \sum_{i=1}^{m} \int_{\mathbb{X}} \mathbb{E}\left[\left(\sum_{j=1}^{m} |\,\mathrm{D}_x F_j|\right)^q |\,\mathrm{D}_x \mathrm{L}^{-1} F_i|\right] \lambda(dx). \quad (9.148)$$

Proof. To show the bound on the d_3 distance, we proceed as in the proof of Theorem 4.2 in [PZ10], but we replace the use of Lemma 4.1 therein with our Lemma 9.3. Indeed, we only need to show that

$$|\mathbb{E}[\phi(F)] - \mathbb{E}[\phi(X)]| \leq \tfrac{1}{2}\|\phi''\|_\infty \sum_{i,j=1}^{m} \mathbb{E}\left[|C_{ij} - \langle \mathrm{D}\, F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_j\rangle_{L^2(\lambda)}\right] + \tfrac{1}{2}\left(2\|\phi''\|_\infty \vee \tfrac{1}{2}\|\phi'''\|_\infty\right) \sum_{i=1}^{m} \int_{\mathbb{X}} \mathbb{E}\left[\left(\sum_{j=1}^{m} |\,\mathrm{D}_x F_j|\right)^p |\,\mathrm{D}_x \mathrm{L}^{-1} F_i|\right] \lambda(dx) \quad (9.149)$$

for any $\phi \in \mathcal{C}^3(\mathbb{R}^m)$ with bounded second and third partial derivatives. Defining

$$\psi(t) := \mathbb{E}\phi(\sqrt{1-t}F + \sqrt{t}X), \quad (9.150)$$

it is clear that

$$|\mathbb{E}\phi(F) - \mathbb{E}\phi(X)| \leq \sup_{t\in(0,1)} |\psi'(t)|. \quad (9.151)$$

Defining moreover

$$\phi_i^{t,b}(x) := \frac{\partial \phi}{\partial x_i}(\sqrt{1-t}x + \sqrt{t}b) \quad (9.152)$$

for any vector $b \in \mathbb{R}^m$, it is shown in the proof of [PZ10, Thm. 4.2] that $\psi'(t)$ can be written as

$$\psi'(t) = \frac{1}{2\sqrt{t}}\mathcal{A} - \frac{1}{2\sqrt{1-t}}\mathcal{B}, \quad (9.153)$$

where

$$\mathcal{A} = \sqrt{t} \sum_{i,j=1}^{m} C_{ij} \mathbb{E} \frac{\partial^2 \phi}{\partial x_i \partial x_j} (\sqrt{1-t}F + \sqrt{t}X) \tag{9.154}$$

and

$$\mathcal{B} = \sum_{i=1}^{m} \mathbb{E} \frac{\partial \phi}{\partial x_i} (\sqrt{1-t}F + \sqrt{t}X) F_i. \tag{9.155}$$

Conditioning on X in $\mathcal{B}$, one can apply Lemma 9.3 and deduce that

$$\mathcal{B} = \sqrt{1-t} \sum_{i,j=1}^{m} \mathbb{E} \frac{\partial^2 \phi}{\partial x_i \partial x_j} (\sqrt{1-t}F + \sqrt{t}X) \langle \mathrm{D}\, F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_j \rangle \\ + \sum_{i=1}^{m} \mathbb{E}\left[\mathbb{E}[\langle R_b^i, -\mathrm{D}\,\mathrm{L}^{-1} F_i \rangle | X = b] \right], \tag{9.156}$$

where R_b^i satisfies

$$\mathbb{E}\left| \langle R_b^i, -\mathrm{D}\,\mathrm{L}^{-1} F_i \rangle \right| \\ \leq \left(\tfrac{1}{2} \|(\phi_i^{t,b})''\|_\infty \vee 2\|(\phi_i^{t,b})'\|_\infty \right) \int_{\mathbb{X}} \mathbb{E} \left(\sum_{j=1}^{m} |\,\mathrm{D}_x F_j| \right)^q |\,\mathrm{D}_x \mathrm{L}^{-1} F_i| \lambda(dx). \tag{9.157}$$

It suffices now to observe that

$$\left| \frac{\partial^2 \phi}{\partial x_i \partial x_j} (\sqrt{1-t}F + \sqrt{t}X) \right| \leq \|\phi''\|_\infty \tag{9.158}$$

and

$$\left| \frac{\partial \phi_i^{t,b}}{\partial x_i}(x) \right| = \sqrt{1-t} \left| \frac{\partial^2 \phi}{\partial x_i \partial x_j} (\sqrt{1-t}F + \sqrt{t}x) \right| \leq \sqrt{1-t} \|\phi''\|_\infty \tag{9.159}$$

and

$$\left| \frac{\partial^2 \phi_i^{t,b}}{\partial x_i \partial x_j}(x) \right| = (1-t) \left| \frac{\partial^3 \phi}{\partial x_i \partial x_j \partial x_k} (\sqrt{1-t}F + \sqrt{t}x) \right| \leq (1-t) \|\phi'''\|_\infty. \tag{9.160}$$

We deduce that

$$\sup_{t \in (0,1)} |\psi'(t)| \leq \tfrac{1}{2} \|\phi''\|_\infty \sum_{i,j=1}^{m} \mathbb{E}\left[\left| C_{ij} - \langle \mathrm{D}\, F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_j \rangle_{L^2(\lambda)} \right| \right] \\ + \sup_{t \in (0,1)} \frac{1}{2\sqrt{1-t}} \left(2\sqrt{1-t} \|\phi''\|_\infty \vee \tfrac{1}{2}(1-t) \|\phi'''\|_\infty \right) \\ \sum_{i=1}^{m} \int_{\mathbb{X}} \mathbb{E}\left[\left(\sum_{j=1}^{m} |\,\mathrm{D}_x F_j| \right)^q |\,\mathrm{D}_x \mathrm{L}^{-1} F_i| \right] \lambda(dx). \tag{9.161}$$

The bound for the d_3 distance now follows.

For the d_2 distance, as argued in the proof of Theorem 3.3 in [PZ10], it is enough to show that

$$\begin{aligned} |\mathbb{E}[g(F)-g(X)]| \leq & A\|C^{-1}\|_{op}\|C\|_{op}^{1/2}\sum_{i,j=1}^m \mathbb{E}\left|C_{ij}-\langle \mathrm{D}\, F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_j\rangle_{L^2(\lambda)}\right| \\ & + \left(2A\|C^{-1}\|_{op}\|C\|_{op}^{1/2} \vee \frac{\sqrt{2\pi}}{8}B\|C^{-1}\|_{op}^{3/2}\|C\|_{op}\right) \\ & \sum_{i=1}^m \int_{\mathbb{X}} \mathbb{E}\left[\left(\sum_{j=1}^m |\,\mathrm{D}_x F_j|\right)^q |\,\mathrm{D}_x \mathrm{L}^{-1} F_i|\right]\lambda(dx), \end{aligned} \tag{9.162}$$

for smooth functions $g \in \mathcal{C}^\infty(\mathbb{R}^m)$ whose first- and second-order derivatives are bounded in such a way that $\|g\|_{Lip} \leq A$ and $\sup_{x\in\mathbb{R}^m}\|\mathrm{Hess}\, g(x)\|_{op} \leq B$. We now proceed as in the proof of Theorem 3.3 in [PZ10] to deduce that

$$\mathbb{E}g(F) - \mathbb{E}g(X) = \sum_{i,j=1}^m \mathbb{E}\left[C_{ij}\frac{\partial^2 f_g}{\partial x_i \partial x_j}(F)\right] - \sum_{k=1}^m \mathbb{E}\left[\langle \mathrm{D}\left(\frac{\partial f_g}{\partial x_k}(F)\right), -\mathrm{D}\,\mathrm{L}^{-1} F_k\rangle_{L^2(\lambda)}\right], \tag{9.163}$$

where f_g is the canonical solution (9.134) to the multivariate Stein equation (9.133). Define $\phi_k(x) := \frac{\partial f_g}{\partial x_k}(x)$. By Lemma 9.2, we have that

$$\mathrm{D}_x\, \phi_k(F) = \sum_{i=1}^m \frac{\partial \phi_k}{\partial x_i}(F)\cdot \mathrm{D}_x F_i + R_{x,k}, \tag{9.164}$$

where

$$|R_{x,k}| \leq \left(2\|\phi_k'\|_\infty \vee \tfrac{1}{2}\|\phi_k''\|_\infty\right)\left(\sum_{i=1}^m |\,\mathrm{D}_x F_i|\right)^q. \tag{9.165}$$

It follows that

$$\begin{aligned} |\mathbb{E}g(F) - \mathbb{E}g(X)| \leq & \|f_g''\|_\infty \sum_{i,j=1}^m \mathbb{E}\left|C_{ij}-\langle \mathrm{D}\, F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_j\rangle_{L^2(\lambda)}\right| \\ & + \sup_{1\leq k\leq m}\left(2\|\phi_k'\|_\infty \vee \tfrac{1}{2}\|\phi_k''\|_\infty\right)\sum_{i=1}^m \int_{\mathbb{X}} \mathbb{E}\left[\left(\sum_{j=1}^m |\,\mathrm{D}_x F_j|\right)^q |\,\mathrm{D}_x \mathrm{L}^{-1} F_i|\right]\lambda(dx). \end{aligned} \tag{9.166}$$

To see that (9.162) holds, it suffices now to see that by (9.135) and (9.136), we have

$$\|f_g''\|_\infty \leq \|C^{-1}\|_{op}\|C\|_{op}^{1/2}\|g\|_{Lip} \leq A\|C^{-1}\|_{op}\|C\|_{op}^{1/2} \tag{9.167}$$

$$\|\phi_k'\|_\infty \leq \|f_g''\|_\infty \leq A\|C^{-1}\|_{op}\|C\|_{op}^{1/2} \tag{9.168}$$

$$\|\phi_k''\|_\infty \leq \|f_g'''\|_\infty \leq \frac{\sqrt{2\pi}}{4}\|C^{-1}\|_{op}^{3/2}\|C\|_{op}\sup_{x\in\mathbb{R}^m}\|\mathrm{Hess}\, g(x)\|_{op} \leq \frac{\sqrt{2\pi}}{4}B\|C^{-1}\|_{op}^{3/2}\|C\|_{op}. \tag{9.169}$$

This concludes the proof. ■

Proof of Theorem 4.6. Using Proposition 9.4, it suffices to show that

$$\sum_{i,j=1}^m \mathbb{E}\left[\left|C_{ij}-\langle \mathrm{D}\, F_i, -\mathrm{D}\,\mathrm{L}^{-1} F_j\rangle_{L^2(\lambda)}\right|\right] \leq \zeta_1+\zeta_2+\zeta_3 \tag{9.170}$$

and

$$\sum_{i=1}^{m} \int_{\mathbb{X}} \mathbb{E}\left[\left(\sum_{j=1}^{m} |\operatorname{D}_x F_j|\right)^q |\operatorname{D}_x \operatorname{L}^{-1} F_i|\right] \lambda(dx) \leq \zeta_4. \tag{9.171}$$

Fix $i, j \in \{1, ..., m\}$. By the triangle inequality, we have that

$$\begin{aligned}\mathbb{E}&\left[|C_{ij} - \langle \operatorname{D} F_i, -\operatorname{D}\operatorname{L}^{-1} F_j\rangle_{L^2(\lambda)}|\right] \\ &\leq \mathbb{E}\left[|C_{ij} - \operatorname{Cov}(F_i, F_j)|\right] + \mathbb{E}\left[|\operatorname{Cov}(F_i, F_j) - \langle \operatorname{D} F_i, -\operatorname{D}\operatorname{L}^{-1} F_j\rangle_{L^2(\lambda)}|\right]. \end{aligned} \tag{9.172}$$

Define $G_{ij} := \operatorname{Cov}(F_i, F_j) - \langle \operatorname{D} F_i, -\operatorname{D}\operatorname{L}^{-1} F_j\rangle_{L^2(\lambda)}$. Then since $F_i, F_j \in \operatorname{dom}\operatorname{D}$, we have $G_{ij} \in L^1(\mathbb{P}_\chi)$ and by Lemma 8.3, one has $\mathbb{E}G_{ij} = 0$.
Using the p-Poincaré inequality (5.7) given in Remark 5.4, we deduce that for any $p \in [1, 2]$,

$$\mathbb{E}|G_{ij}| \leq \mathbb{E}\left[|G_{ij}|^p\right]^{1/p} \leq 2^{2/p-1}\mathbb{E}\left[\int_{\mathbb{X}} |\operatorname{D}_x G_{ij}|^p \lambda(dx)\right]^{1/p}. \tag{9.173}$$

Now note that

$$\operatorname{D}_x G_{ij} = \operatorname{D}_x \int_{\mathbb{X}} (\operatorname{D}_y F_i) \cdot (-\operatorname{D}_y \operatorname{L}^{-1} F_j)\, \lambda(dy), \tag{9.174}$$

and by the argument in the proof of [LPS16, Prop. 4.1, p. 689], we have that

$$\left|\operatorname{D}_x \int_{\mathbb{X}} (\operatorname{D}_y F_i) \cdot (-\operatorname{D}_y \operatorname{L}^{-1} F_j)\, \lambda(dy)\right| \leq \int_{\mathbb{X}} \left|\operatorname{D}_x \left((\operatorname{D}_y F_i) \cdot (-\operatorname{D}_y \operatorname{L}^{-1} F_j)\right)\right| \lambda(dy). \tag{9.175}$$

Using Minkowski's integral inequality, it follows that

$$\begin{aligned}\mathbb{E}&\left[|\operatorname{Cov}(F_i, F_j) - \langle \operatorname{D} F_i, -\operatorname{D}\operatorname{L}^{-1} F_j\rangle_{L^2(\lambda)}|\right] \\ &\leq \left(\int_{\mathbb{X}}\left(\int_{\mathbb{X}} \mathbb{E}\left[\left|\operatorname{D}_x \left((\operatorname{D}_y F_i) \cdot (-\operatorname{D}_y \operatorname{L}^{-1} F_j)\right)\right|^p\right]^{1/p} \lambda(dy)\right)^p \lambda(dx)\right)^{1/p}. \end{aligned} \tag{9.176}$$

By (3.4), we have that

$$\begin{aligned}\operatorname{D}_x &\left((\operatorname{D}_y F_i) \cdot (-\operatorname{D}_y \operatorname{L}^{-1} F_j)\right) \\ &= \operatorname{D}^{(2)}_{x,y} F_i \cdot (-\operatorname{D}_y \operatorname{L}^{-1} F_j) + \operatorname{D}_y F_i \cdot \left(-\operatorname{D}^{(2)}_{x,y} \operatorname{L}^{-1} F_j\right) + \operatorname{D}^{(2)}_{x,y} F_i \cdot \left(-\operatorname{D}^{(2)}_{x,y} \operatorname{L}^{-1} F_j\right). \end{aligned} \tag{9.177}$$

Using Minkowski's norm inequality, the Cauchy-Schwarz inequality and Lemma 8.3, one sees that

$$\begin{aligned}\mathbb{E}\left[\left|\operatorname{D}_x \left((\operatorname{D}_y F_i) \cdot (-\operatorname{D}_y \operatorname{L}^{-1} F_j)\right)\right|^p\right]^{1/p} \leq& \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F_i|^{2p}\right]^{1/2p} \mathbb{E}\left[|\operatorname{D}_y F_j|^{2p}\right]^{1/2p} \\ &+ \mathbb{E}\left[|\operatorname{D}_y F_i|^{2p}\right]^{1/2p} \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F_j|^{2p}\right]^{1/2p} \\ &+ \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F_i|^{2p}\right]^{1/2p} \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F_j|^{2p}\right]^{1/2p}. \end{aligned} \tag{9.178}$$

Combining (9.172), (9.176) and (9.178) yields (9.170).
To show (9.171), note that

$$\left(\sum_{j=1}^{m} |\operatorname{D}_x F_j|\right)^q \leq m^{q-1} \sum_{j=1}^{m} |\operatorname{D}_x F_j|^q. \tag{9.179}$$

The bound (9.171) now follows by Hölder inequality and Lemma 8.3. ■

9.5 Proofs for the Online Nearest Neighbour Graph

The technical proofs in this section sometimes use ideas and techniques close to [Pen05, Section 3.4] and [Wad09] (and previous papers). We will discuss the connections with this part of the literature as the proofs unfold.

Throughout this section, we work under the setting of Section 6.1. We will adopt the following notation: for a measurable subset $A \subset \mathbb{R}^d$, we will write η_A for $\eta_{|A\times[0,1]}$. Moreover, we need to adapt the definition of 'generic' to sets in the marked space $\mathbb{R}^d \times [0,1]$. In this section only, we call a finite set $\mu \subset \mathbb{R}^d \times [0,1]$ **generic** if

- all projections of μ onto $\mathbb{R}^d$ are distinct;
- all pairwise distances between projections of μ onto $\mathbb{R}^d$ are distinct;
- all projections of μ onto $[0,1]$ are distinct.

Note that for any convex body $D_0 \subset \mathbb{R}^d$, the restriction of an $(\mathbb{R}^d\times[0,1], dx\otimes ds)$-Poisson measure η to $D_0 \times [0,1]$ has generic support. We call μ **generic with respect to** a point (or points) $(x,s),(y,u) \in \mathbb{R}^d \times [0,1]$ if $\mu \cup \{(x,s),(y,u)\}$ is generic.

We start with a short discussion of the properties of the space H defined in Section 6.1. Since H has non-empty interior, there exist $\delta > 0$ and $y_0 \in H$ such that $B^d(y_0,\delta) \subset H$. Fix these δ and y_0 throughout this section. For $\epsilon > 0$ define

$$H_\epsilon := \{x \in H : \text{dist}(x, \partial H) > \epsilon\}, \tag{9.180}$$

where dist denotes the Euclidean distance and ∂H the boundary of H. For small ϵ, this set is non-empty. By [HLS16, (3.19)], one has

$$|H \setminus H_\epsilon| \le |H + B^d(0,\epsilon)| - |H|, \tag{9.181}$$

where the sum is the Minkowski sum of sets. By Steiner's formula (cf. [SW08, (14.5)]), it follows that there is a constant $\beta_H > 0$ such that

$$|H \setminus H_\epsilon| \le \beta_H\, \epsilon. \tag{9.182}$$

9.5.1 Moment estimates of conditional expectations of add-one costs

The goal of this subsection is to prove the following proposition:

Proposition 9.5. *Assume the conditions in the statement of Theorem 6.1. Let $\alpha > 0$ and $r \ge 1$. Then for all $t \ge 1$ and all $(x,s),(y,u) \in tH \times (0,1]$ with $s < u$, it holds that*

$$\mathbb{E}\left[\mathbb{E}\big[|\,\mathrm{D}_{(x,s)}\, F_t^{(\alpha)}|\big|\eta_{|tH\times[0,s)}\big]^r\right]^{1/r} \le c_1(s^{-\alpha/d} \wedge t^\alpha) \tag{9.183}$$

$$\mathbb{E}\left[\mathbb{E}\big[|\,\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F_t^{(\alpha)}|\big|\eta_{|tH\times[0,u)}\big]^r\right]^{1/r} \le c_1(u^{-\alpha/d} \wedge t^\alpha). \tag{9.184}$$

$$\mathbb{P}\left(\mathbb{E}\big[|\,\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F_t^{(\alpha)}|\big|\eta_{|tH\times[0,u)}\big] \ne 0\right) \le C_2 \exp(-c_2 u|x-y|^d). \tag{9.185}$$

where $c_1 > 0$ is a constant depending on α, r and ϵ and $c_2, C_2 > 0$ depend only on d and H.

Remark 9.6. The bound (9.183) can be compared to Lemma 4.2 in [Wad09], with the difference that we work in a Poisson setting and consider general moments $r \geq 1$, whereas [Wad09] considers uniform random variables and $r = 2$.

We start with two technical lemmas.

Lemma 9.7. *Let $\epsilon, \ell > 0$. There exists $0 < q < 1$ depending only on ℓ, ϵ and d such that for every convex body $D_0 \subset \mathbb{R}^d$ satisfying $\operatorname{diam}(D_0) \leq \ell$ and $B^d(y_0, \epsilon) \subset D_0$ for some $y_0 \in D_0$, for every $w \in D_0$ and $0 < s \leq \ell$, there is a point $w' \in D_0$ with*

$$B^d(w', qs) \subset D_0 \cap B^d(w, s). \tag{9.186}$$

In particular, this implies that

$$\inf_{\substack{w \in D_0 \\ 0 < s \leq \ell}} \frac{|D_0 \cap B^d(w, s)|}{s^d} \geq c_1(\epsilon, \ell, d) \tag{9.187}$$

with $c_1(\epsilon, \ell, d) > 0$ a constant depending on ϵ, ℓ and d.

In words, this lemma means that there is a constant rate q such that for any not-too-big ball intersecting D_0, we can find a smaller ball shrunken by the factor q within the intersection. The ratio q at which the ball is shrunk depends only on a lower bound for the in-radius of D_0 and an upper bound for the diameter of D_0. This will become important in the proof of Proposition 9.5, where D_0 is a probabilistic object, but has deterministic upper and lower bounds.

Remark 9.8. While geometric lemmas in the spirit of Lemma 9.7 are already shown in [Pen05, Wad09] and appear in some form in many papers on stochastic geometry, we need a version that allows for fine control over the constants, in order to be able to state our results in the full generality of convex sets.

Proof. Since the ball $B^d(y_0, \epsilon)$ is a subset of D_0 and D_0 is closed and convex, for any $w \in D_0$, the convex hull of $w \cup \overline{B}^d(y_0, \epsilon)$ is inside D_0. If $d \geq 2$ and for $w \notin \overline{B}^d(y_0, \epsilon)$, this is a cone-like structure with angular radius $\omega_w = \arcsin(\epsilon / |w - y_0|)$. Since $\operatorname{diam}(D_0) \leq \ell$, it follows that

$$\frac{\pi}{4} \wedge \inf_{w \in D_0} \omega_w \geq \frac{\pi}{4} \wedge \arcsin\left(\frac{\epsilon}{\ell}\right) =: \theta. \tag{9.188}$$

For every $w \in D_0$, let $C(w)$ be the closed cone centred at w, of angular radius θ and central axis the half-line from w through y_0 (if $d = 1$, take $C(w)$ to be the closed half-line starting at w and containing y_0). The set $A(w, \epsilon) := C(w) \cap \overline{B}^d(w, \epsilon)$ is now included in D_0. This is clear if $d = 1$. If $d \geq 2$, then if $w \notin \overline{B}^d(y_0, \epsilon)$, it follows that $|y_0 - w| > \epsilon$ and since $\theta \leq \omega_w \wedge \frac{\pi}{4}$, one deduces that $A(w, \epsilon)$ is inside the convex hull of $w \cup \overline{B}^d(y_0, \epsilon)$ (here we need $\theta \leq \frac{\pi}{4}$). If $w \in \overline{B}^d(y_0, \epsilon)$, then by construction $A(w, \epsilon)$ is also inside $\overline{B}^d(y_0, \epsilon)$ (here we need $\theta \leq \frac{\pi}{3}$).
Since $s, \epsilon \leq \ell$, it follows that

$$A\left(w, \frac{\epsilon s}{\ell}\right) \subset A(w, \epsilon) \cap \overline{B}^d(w, s) \subset D_0 \cap \overline{B}^d(w, s). \tag{9.189}$$

The set $A(0, 1) = \overline{B}^d(0, 1) \cap C(0)$ is convex and of non-zero volume, therefore it contains an open ball of some radius $r > 0$, centred at some point $z \in A(0, 1)$. By translation and scaling,

$$B^d\left(\frac{\epsilon s}{\ell} \cdot z + w, \frac{\epsilon s r}{\ell}\right) \subset A\left(w, \frac{\epsilon s}{\ell}\right). \tag{9.190}$$

Thus with $q = \frac{\epsilon r}{\ell}$ and $w' = \frac{\epsilon s}{\ell} \cdot z + w$, we achieve

$$B^d(w', qs) \subset A\left(w, \frac{\epsilon s}{\ell}\right) \subset D_0 \cap \overline{B}^d(w, s). \tag{9.191}$$

Since $B^d(w', qs)$ is open, we have $B^d(w', qs) \subset B^d(w, s)$. The statement (9.187) follows easily by taking $c_1(\epsilon, \ell, d) := \kappa_d q^d$. ■

For the next part, we need some additional technical definitions. Let $C_1(0), \ldots, C_K(0)$ be a collection of closed cones centred at 0 and of angular radius $\frac{\pi}{12}$ such that $\bigcup_{i=1}^K C_i(0) = \mathbb{R}^d$ (for $d = 1$, take $C_1(0) := (-\infty, 0]$ and $C_2(0) := [0, \infty)$). Define $C_i(x) := C_i(0) + x$ for $x \in \mathbb{R}^d$ and let $C_1^+(x), \ldots, C_K^+(x)$ be the cones centred at x of angular radius $\frac{\pi}{6}$ such that $C_i^+(x)$ has the same central half-axis as $C_i(x)$ for each i (for $d = 1$, take $C_i^+(0) = C_i(0)$).
For $A \subset \mathbb{R}^d$ a convex body, $x \in A$ and $\mu \subset \mathbb{R}^d \times [0, 1]$ a finite set which is generic with respect to x, define

- $s_i(x, A) :=$ diam$(C_i(x) \cap A)$;
- $R_{i,\theta}(x, A, \mu) :=$ inf$\{|x - y| : (y, u) \in \big(A \cap C_i^+(x) \times [0, \theta)\big) \cap \mu\} \wedge s_i(x, A)$ for $\theta \in [0, 1]$ and where $\inf \emptyset = \infty$;
- $R_\theta(x, A, \mu) :=$ max$_{i=1,\ldots,K} R_{i,\theta}(x, A, \mu)$.

In practice, when it is clear what A and μ are, we will omit them from the notation and write $s_i(x), R_{i,\theta}(x), R_\theta(x)$. In words, $s_i(x, A)$ is the maximal distance within A from x to any point in the cone $C_i(x)$. The quantity $R_{i,\theta}(x, A, \mu)$ is the distance to the closest point of μ of mark smaller than θ within A and the larger cone $C_i^+(x)$. The quantity $R_\theta(x, A, \mu)$ is such that the ball $B^d(x, R_\theta(x, A, \mu))$ either contains a point of mark less than θ, or if it doesn't, then it contains all of A.

Remark 9.9. The use of cones and the construction of the radii R_θ follows the ideas given in [Pen05, Wad09], again with the difference that we work in a Poisson setting without passing through uniform random vectors.

Lemma 9.10. *Let $\epsilon, \ell > 0$ and $D_0 \subset \mathbb{R}^d$ be as in Lemma 9.7. Take $\lambda > 0$ and $x \in \lambda D_0$, and let $0 < r \le s_i(x, \lambda D_0)$ for some $i \in \{1, \ldots, K\}$. Then there is a constant $c_2(\epsilon, \ell, d) > 0$ such that*

$$|\lambda D_0 \cap C_i^+(x) \cap B^d(x, r)| \ge c_2(\epsilon, \ell, d) r^d. \tag{9.192}$$

Remark 9.11. A similar result was shown in the proof of [Wad09, Lemma 3.2], with a constant depending on D_0.

Proof. First, we show that there is a point $z \in \lambda D_0$ such that $B^d(z, \frac{r}{8}) \subset C_i^+(x) \cap B^d(x, r)$. Indeed, as $r \le s_i(x, \lambda D_0)$, there is by convexity a point $z \in \lambda D_0 \cap C_i(x)$ such that $|x - z| = \frac{r}{2}$. Now consider any point $y \in B^d(z, \frac{r}{8})$. For $d = 1$, one has that $y \in C_i^+(x) \cap B^1(x, r)$. For $d \ge 2$, the angle $\angle zxy$ will be largest when the line (xy) is tangent to $B^d(z, \frac{r}{8})$, in which case

$$\angle zxy \le \arcsin\left(\frac{r/8}{r/2}\right) = \arcsin\left(\frac{1}{4}\right) < \frac{\pi}{12}. \tag{9.193}$$

Since $z \in C_i(x)$, this implies that $y \in C_i^+(x)$ and we clearly also have $|x - y| \le |x - z| + |z - y| \le \frac{5}{8} r < r$. We infer that for all $d \ge 1$,

$$|\lambda D_0 \cap C_i^+(x) \cap B^d(x, r)| \ge |\lambda D_0 \cap B^d(z, \tfrac{r}{8})|. \tag{9.194}$$

Now $\frac{1}{\lambda}z \in D_0$ and $\frac{1}{\lambda}\frac{r}{8} \leq \text{diam}(D_0) \leq \ell$, therefore Lemma 9.7 implies

$$\left|\lambda D_0 \cap B^d\left(z, \tfrac{r}{8}\right)\right| = \lambda^d \left|D_0 \cap B^d\left(\tfrac{1}{\lambda}z, \tfrac{1}{\lambda}\tfrac{r}{8}\right)\right| \geq \lambda^d c_1(\epsilon, \ell, d)\left(\tfrac{r}{8\lambda}\right)^d = 8^{-d}c_1(\epsilon, \ell, d)r^d, \tag{9.195}$$

which yields the desired bound. ∎

Lemma 9.12. *Let $\epsilon, \ell > 0$ and $D_0 \subset \mathbb{R}^d$ be as in Lemma 9.7. Let $\lambda > 0$, $x \in \lambda D_0$ and $\theta \in (0,1)$. Then for all $\beta > 0$, there is a constant $c_3(\epsilon, d, \ell, \beta) > 0$ such that*

$$\mathbb{E}\left[R_\theta(x, \lambda D_0, \eta_{\lambda D_0})^\beta\right] \leq c_3(\epsilon, d, \ell, \beta)\left(\theta^{-\beta/d} \wedge \lambda^\beta\right). \tag{9.196}$$

Remark 9.13. This result is comparable to [Wad09, Lemma 3.2] and an argument in the proof of [Pen05, Lemma 3.3], both of which worked with uniform random vectors.

Proof. It is clear that by construction $R_\theta(x, \lambda D_0, \eta_{\lambda D_0}) \leq \text{diam}(\lambda D_0) \leq \ell\lambda$, thus we need only show the bound by $\theta^{-\beta/d}$. To simplify the notation, write $R_\theta(x)$ for $R_\theta(x, \lambda D_0, \eta_{\lambda D_0})$. We are going to study the probability $\mathbb{P}(\mathbb{R}_\theta(x) > r)$. If for some $r > 0$, we have $R_\theta(x) > r$, then $\max_{i=1,\dots,K} R_{i,\theta}(x) > r$, implying that there is an $i \in \{1, \dots, K\}$ such that $R_{i,\theta}(x) > r$. Since $R_{i,\theta}(x) \leq s_i(x)$, this implies in turn that $r < s_i(x)$ and that

$$\eta\left(\left(\lambda D_0 \cap B^d(x,r) \cap C_i^+(x)\right) \times [0,\theta)\right) = 0. \tag{9.197}$$

It follows that

$$\begin{aligned}
\mathbb{P}(\mathbb{R}_\theta(x) > r) &\leq \mathbb{P}\left(\bigcup_{i=1}^K \left\{\eta\left((\lambda D_0 \cap B^d(x,r) \cap C_i^+(x)) \times [0,\theta)\right) = 0,\ r < s_i(x)\right\}\right) \\
&\leq \sum_{i=1}^K \mathbb{1}_{\{r<s_i(x)\}}\mathbb{P}\left(\eta\left((\lambda D_0 \cap B^d(x,r) \cap C_i^+(x)) \times [0,\theta)\right) = 0\right) \\
&= \sum_{i=1}^K \mathbb{1}_{\{r<s_i(x)\}} \exp\left(-\theta|\lambda D_0 \cap B^d(x,r) \cap C_i^+(x)|\right).
\end{aligned} \tag{9.198}$$

By Lemma 9.10, there is a constant $C := c_2(\epsilon, \ell, d) > 0$ such that

$$|\lambda D_0 \cap B^d(x,r) \cap C_i^+(x)| \geq Cr^d. \tag{9.199}$$

Hence

$$\exp\left(-\theta|\lambda D_0 \cap B^d(x,r) \cap C_i^+(x)|\right) \leq \exp(-\theta C r^d) \tag{9.200}$$

and

$$\mathbb{P}(\mathbb{R}_\theta(x) > r) \leq K\exp(-\theta C r^d). \tag{9.201}$$

We can now estimate for all $\beta > 0$:

$$\begin{aligned}
\mathbb{E}[R_\theta(x)^\beta] &= \int_0^\infty \mathbb{P}(R_\theta(x) > r^{1/\beta})dr \\
&\leq \int_0^\infty K\exp(-\theta C r^{d/\beta})dr \\
&= K\frac{\beta}{d}\theta^{-\beta/d}C^{-\beta/d}\int_0^\infty u^{\beta/d-1}\exp(-u)du \\
&= KC^{-\beta/d}\Gamma\left(\frac{\beta}{d}+1\right)\theta^{-\beta/d},
\end{aligned} \tag{9.202}$$

by the properties of the Gamma function Γ (see e.g. [AS72, 6.1.1]). ∎

Before we pass to the next lemma, we note some useful properties of the add-one cost and introduce the quantity $\mathrm{L}_{(x,s)}^{-1} F^{(\alpha)}(\mu)$. Let $(x, s) \in \mathbb{R}^d \times [0, 1]$ and $\mu \subset \mathbb{R}^d \times [0, 1]$ be a generic set with respect to (x, s). Note that

$$\mathrm{D}_{(x,s)} F^{(\alpha)}(\mu) = e(x, s, \mu)^\alpha + \sum_{\substack{(y,u)\in \\ \mu\cap(\mathbb{R}^d\times(s,1])}} \left(e(y, u, \mu + \delta_{(x,s)})^\alpha - e(y, u, \mu)^\alpha\right). \tag{9.203}$$

Define the quantity

$$\mathrm{L}_{(x,s)}^{-1} F^{(\alpha)}(\mu) := \sum_{\substack{(y,u)\in \\ \mu\cap(\mathbb{R}^d\times(s,1])}} e(y, u, \mu)^\alpha \mathbb{1}_{\{(y,u)\to(x,s) \text{ in } \mu+\delta_{(x,s)}\}}, \tag{9.204}$$

where we use (as before) the notation $(y, u) \to (x, s)$ in $\mu + \delta_{(x,s)}$ in order to indicate that the point (y, u) connects to (x, s) in the collection of points $\mu + \delta_{(x,s)}$, as was explained in Section 6.1.

We claim the following: For any convex body $A \subset \mathbb{R}^d$ such that the projection onto $\mathbb{R}^d$ of μ is included in the interior of A, it holds that

$$\left|\mathrm{D}_{(x,s)} F^{(\alpha)}(\mu)\right| \le R_s(x, A, \mu)^\alpha + \mathrm{L}_{(x,s)}^{-1} F^{(\alpha)}(\mu). \tag{9.205}$$

To show the claim, we start by noting the following: if a point $(y, u) \in \mu \cap (\mathbb{R}^d \times (s, 1])$ has an online nearest neighbour in μ, then $e(y, u, \mu) \neq 0$. There are now two possibilities:

1. The point (y, u) connects to (x, s) in $\mu+\delta_{(x,s)}$. This implies that $e(y, u, \mu+\delta_{(x,s)}) < e(y, u, \mu)$.
2. The point (y, u) does not connect to (x, s) in $\mu + \delta_{(x,s)}$. Then $e(y, u, \mu + \delta_{(x,s)}) = e(y, u, \mu)$.

In both cases, it holds that

$$\left|e(y, u, \mu + \delta_{(x,s)})^\alpha - e(y, u, \mu)^\alpha\right| \le e(y, u, \mu)^\alpha \mathbb{1}_{\{(y,u)\to(x,s) \text{ in } \mu+\delta_{(x,s)}\}}. \tag{9.206}$$

As a next step, consider three scenarios:

1. If $\mu \cap \mathbb{R}^d \times [0, s) \neq \emptyset$, then any point $(y, u) \in \mu$ with $u > s$ has an online nearest neighbour in μ and (9.206) holds. Moreover, the point (x, s) has an online nearest neighbour in μ and by construction, $e(x, s, \mu) \le R_s(x, A, \mu)$. Combining this with (9.203) and (9.206) implies that (9.205) holds.

2. If $\mu \cap \mathbb{R}^d \times [0, s) = \emptyset$ but $\mu \cap \mathbb{R}^d \times (s, 1] \neq \emptyset$, then the point (x, s) does not have an online nearest neighbour in μ and $e(x, s, \mu) = 0$. However, there is now a point $(y_0, u_0) \in \mu \cap \mathbb{R}^d \times (s, 1]$ which is the point of lowest mark in μ and it does not have an online nearest neighbour in μ. Hence $e(y_0, u_0, \mu) = 0$ and $e(y_0, u_0, \mu + \delta_{(x,s)}) = |x - y_0|$ since this point will connect to (x, s) in $\mu + \delta_{(x,s)}$. Since $\mu \cap \mathbb{R}^d \times [0, s) = \emptyset$, we have that $A \subset \overline{B}^d(x, R_s(x, A, \mu))$ and hence $|x - y_0| \le R_s(x, A, \mu)$. Any point $(y, u) \in \mu$ different from (y_0, u_0) must have an online nearest neighbour in μ, since (y_0, u_0) is a potential neighbour. Thus for such (y, u), the inequality (9.206) holds and we deduce

$$\begin{aligned}\left|\mathrm{D}_{(x,s)} F^{(\alpha)}(\mu)\right| &= \left| |x - y_0|^\alpha + \sum_{\substack{(y_0,u_0)\neq(y,u)\in \\ \mu\cap(\mathbb{R}^d\times(s,1])}} \left(e(y, u, \mu + \delta_{(x,s)})^\alpha - e(y, u, \mu)^\alpha\right) \right| &(9.207)\\ &\le R_s(x, A, \mu)^\alpha + \mathrm{L}_{(x,s)}^{-1} F^{(\alpha)}(\mu), &(9.208)\end{aligned}$$

thus inequality (9.205) holds.

3. If $\mu = \emptyset$, then $\mathrm{D}_{(x,s)}\, F^{(\alpha)}(\mu) = 0$ and inequality (9.205) trivially holds.

This concludes the proof of the claim. In the next lemma, we now provide a bound for the quantity $\mathrm{L}^{-1}_{(x,s)}\, F^{(\alpha)}(\eta_{\lambda D_0})$.

Lemma 9.14. *Let $\epsilon, \ell > 0$ and $D_0 \subset \mathbb{R}^d$ be as in Lemma 9.7. Let $\lambda > 0$ and $(x, s) \in \lambda D_0 \times [0, 1]$. Then there is a constant $c_3(\epsilon, \ell, d, \alpha) > 0$ such that*

$$\mathbb{E}[\mathrm{L}^{-1}_{(x,s)}\, F^{(\alpha)}(\eta_{\lambda D_0})] \leq c_4(\epsilon, \ell, d, \alpha)(s^{-\alpha/d} \wedge \lambda^{\alpha}). \tag{9.209}$$

Proof. To prove (9.209), let $(y, u) \in \eta_{\lambda D_0}$ be a point connecting to (x, s) in $\eta_{\lambda D_0} + \delta_{(x,s)}$. We will proceed in three steps.
Step 1. We claim that $y \in \overline{B}^d(x, \mathbb{R}_\theta(x, \lambda D_0, \eta_{\lambda D_0}))$ for any $\theta < u$.
As the $C_i(x)$, $i = 1, ..., K$, cover $\mathbb{R}^d$, there is an i such that $y \in C_i(x)$. Assume that $|x - y| > R_\theta(x)$. Since $|x - y| \leq s_i(x, \lambda D_0)$, we must have

$$R_{i,\theta}(x) \leq R_\theta(x) < s_i(x, \lambda D_0) \tag{9.210}$$

and hence

$$\eta \cap ((\lambda D_0 \cap C_i^+(x) \cap \overline{B}^d(x, R_\theta(x))) \times [0, \theta)) \neq \emptyset. \tag{9.211}$$

So there is a point (z, v) within this set and in particular $v < \theta < u$. We now have:

- $|x - y| > R_\theta(x) \geq |x - z|$
- $z \in C_i^+(x)$
- $y \in C_i(x)$.

By the geometric properties of the cones $C_i(x)$ and $C_i^+(x)$, shown in [Pen05, Lemma 3.3], this implies that $|z - y| \leq |x - y|$. Since $|z - y| \neq |x - y|$ a.s., we deduce that the point y will connect to z rather than to x, which is a contradiction. We must thus have $|x - y| \leq R_\theta(x)$.
Step 2. We will now derive the first part of the bound in (9.209).
For $0 \leq \theta_1 < \theta_2 \leq 1$, define

$$H(\theta_1, \theta_2) := \sum_{\substack{(y,u)\in \\ \eta\cap(\lambda D_0\times(\theta_1,\theta_2])}} e(y, u, \eta_{\lambda D_0})^{\alpha} \mathbb{1}_{\{(y,u)\to(x,s) \text{ in } \eta_{\lambda D_0}+\delta_{(x,s)}\}}. \tag{9.212}$$

This is the contribution to the sum $\mathrm{L}^{-1}_{(x,s)}\, F^{(\alpha)}(\eta_{\lambda D_0})$ of points with marks in the interval $(\theta_1, \theta_2]$, so that

$$\mathrm{L}^{-1}_{(x,s)}\, F^{(\alpha)}(\eta_{\lambda D_0}) = H(0, 1) = H(s, 1) = \sum_{i=1}^{n} H(\theta_{i-1}, \theta_i), \tag{9.213}$$

for any partition $s = \theta_0 < \theta_1 < ... < \theta_n = 1$.
Let $(y, u) \in \eta \cap (\lambda D_0 \times (\theta_{i-1}, \theta_i])$ be a point connecting to x. By Step 1, we have $y \in \overline{B}(x, R_{\theta_{i-1}}(x))$. By construction, either $\lambda D_0 \cap B^d(x, R_{\theta_{i-1}}(x))$ contains a point (w, v) of η with mark less than θ_{i-1}, in which case $e(y, u, \eta_{\lambda D_0}) \leq |y - w| \leq |x - y| + |x - w|$, or $\lambda D_0 \subset B^d(x, R_{\theta_{i-1}}(x))$. In both cases,

$$e(y, u, \eta_{\lambda D_0}) \leq |x - y| + R_{\theta_{i-1}}(x) \leq 2R_{\theta_{i-1}}(x). \tag{9.214}$$

Moreover, the number of points in $\lambda D_0 \times (\theta_{i-1}, \theta_i]$ connecting to x is upper bounded by the total number of points in $\eta \cap \big((\lambda D_0 \cap \overline{B}^d(x, R_{\theta_{i-1}}(x))) \times (\theta_{i-1}, \theta_i]\big)$. From this we conclude that

$$H(\theta_{i-1}, \theta_i) \leq 2^\alpha R_{\theta_{i-1}}(x)^\alpha \eta \left((\lambda D_0 \cap \overline{B}^d(x, R_{\theta_{i-1}}(x))) \times (\theta_{i-1}, \theta_i]\right). \tag{9.215}$$

To upper bound the expectation of $H(\theta_{i-1}, \theta_i)$, we are going to use the fact that $\eta_{|\lambda D_0 \times [0,\theta_{i-1}]}$ and $\eta_{|\lambda D_0 \times (\theta_{i-1},\theta_i]}$ are independent and $R_{\theta_{i-1}}(x)$ is measurable with respect to $\eta_{|\lambda D_0 \times [0,\theta_{i-1}]}$. We can thus calculate

$$\begin{aligned}
\mathbb{E}\left[H(\theta_{i-1}, \theta_i)\right] &\leq 2^\alpha \mathbb{E}\left[R_{\theta_{i-1}}(x)^\alpha \eta \left((\lambda D_0 \cap \overline{B}^d(x, R_{\theta_{i-1}}(x))) \times (\theta_{i-1}, \theta_i]\right)\right] \\
&= 2^\alpha \mathbb{E}\left[R_{\theta_{i-1}}(x)^\alpha \mathbb{E}\left[\eta \left((\lambda D_0 \cap \overline{B}^d(x, R_{\theta_{i-1}}(x))) \times (\theta_{i-1}, \theta_i]\right) \Big| \, \eta_{|\lambda D_0 \times [0,\theta_{i-1}]}\right]\right] \\
&= 2^\alpha \mathbb{E}\left[R_{\theta_{i-1}}(x)^\alpha \left|\lambda D_0 \cap \overline{B}^d(x, R_{\theta_{i-1}}(x))\right| (\theta_i - \theta_{i-1})\right].
\end{aligned} \tag{9.216}$$

Upper bounding the volume of the intersection by $\kappa_d R_{\theta_{i-1}}(x)^d$ and applying Lemma 9.12 yields that (9.216) is bounded by

$$2^\alpha \kappa_d (\theta_i - \theta_{i-1}) \mathbb{E}\left[R_{\theta_{i-1}}(x)^{\alpha+d}\right] \leq 2^\alpha \kappa_d (\theta_i - \theta_{i-1}) c \, \theta_{i-1}^{-\alpha/d-1}, \tag{9.217}$$

with $c = c_3(\epsilon, d, \ell, \alpha + d)$. Combining this with (9.213), we infer that

$$\mathbb{E}\left[\mathrm{L}^{-1}_{(x,s)} F^{(\alpha)}(\eta_{\lambda D_0})\right] \leq 2^\alpha \kappa_d c \sum_{i=1}^{n} \theta_{i-1}^{-\alpha/d-1} (\theta_i - \theta_{i-1}) \tag{9.218}$$

for any partition $s = \theta_0 < \theta_1 < \ldots < \theta_n = 1$. Letting $n \to \infty$ and the mesh of the partition tend to 0, we get

$$\begin{aligned}
\mathbb{E}\left[\mathrm{L}^{-1}_{(x,s)} F^{(\alpha)}(\eta_{\lambda D_0})\right] &\leq 2^\alpha \kappa_d c \int_s^1 \theta^{-\alpha/d-1} d\theta \\
&= 2^\alpha \kappa_d c \frac{d}{\alpha} \left(s^{-\alpha/d} - 1\right) \\
&\leq c' s^{-\alpha/d},
\end{aligned} \tag{9.219}$$

with $c' := 2^\alpha \kappa_d c_3(\epsilon, d, \ell, \alpha + d) \frac{d}{\alpha}$, a constant independent of x, s and λ.

Step 3. To show the second part of (9.209), note that

$$\mathrm{L}^{-1}_{(x,s)} F^{(\alpha)}(\eta_{\lambda D_0}) \leq \mathrm{L}^{-1}_{(x,0)} F^{(\alpha)}(\eta_{\lambda D_0}). \tag{9.220}$$

Indeed, $e(y, u, \eta_{\lambda D_0})$ is independent of (x, s) and hence remains unchanged for any $(y, u) \in \eta_{\lambda D_0}$ if we reduce the arrival time of (x, s) to zero. Any point connecting to (x, s) will also connect to $(x, 0)$, hence no terms are deleted from the sum. Some positive terms might be added, since there may be points connecting to $(x, 0)$ but not to (x, s).

For any $0 < \theta_0 < 1$,

$$\mathrm{L}^{-1}_{(x,0)} F^{(\alpha)}(\eta_{\lambda D_0}) = H(0, \theta_0) + \mathrm{L}^{-1}_{(x,\theta_0)} F^{(\alpha)}(\eta_{\lambda D_0}). \tag{9.221}$$

A crude upper bound for the first term on the RHS is as follows:

$$H(0, \theta_0) \leq \mathrm{diam}(\lambda D_0)^\alpha \eta(B^d(x, \lambda\, \mathrm{diam}(D_0)) \times [0, \theta_0)). \tag{9.222}$$

Taking expectation, we get

$$\mathbb{E} H(0, \theta_0) \leq \kappa_d \, \mathrm{diam}(D_0)^{\alpha+d} \lambda^{\alpha+d} \theta_0. \tag{9.223}$$

On the other hand, by Step 2 we have

$$\mathbb{E}\left[\mathrm{L}^{-1}_{(x,\theta_0)}\, F^{(\alpha)}(\eta_{\lambda D_0})\right] \le c'\theta_0^{-\alpha/d}. \tag{9.224}$$

Combining (9.220)-(9.224) yields for any $0 < \theta_0 < 1$ that

$$\mathbb{E}\left[\mathrm{L}^{-1}_{(x,s)}\, F^{(\alpha)}(\eta_{\lambda D_0})\right] \le \kappa_d \operatorname{diam}(D_0)^{\alpha+d}\lambda^{\alpha+d}\theta_0 + c'\theta_0^{-\alpha/d}. \tag{9.225}$$

Choosing $\theta_0 := \lambda^{-d}$, we deduce

$$\mathbb{E}\, \mathrm{L}^{-1}_{(x,s)}\, F^{(\alpha)}(\eta_{\lambda D_0}) \le \left(\kappa_d \operatorname{diam}(D_0)^{\alpha+d} + c'\right)\lambda^{\alpha}. \tag{9.226}$$

The bound in (9.209) follows with $c_4(\epsilon, \ell, d, \alpha) := \kappa_d \operatorname{diam}(D_0)^{\alpha+d} + c'$. ■

Proof of Prop. 9.5. **Proof of (9.183).** We will show that there is a constant $C_1 > 0$ such that for all $(x, s) \in tH \times [0, 1]$ and $t \ge 1$,

$$\mathbb{E}\big[|\, \mathrm{D}_{(x,s)}\, F_t^{(\alpha)}|\big|\eta_{|tH\times[0,s)}\big] \le C_1 R_s(x, tH, \eta_{tH})^{\alpha}, \tag{9.227}$$

where C_1 does not depend on $x,\ s$ or t. The claim (9.183) then follows by Lemma 9.12.
From here on, write $R_s(x)$ for $R_s(x, tH, \eta_{tH})$.
By (9.205) and since $e(x, s, \eta_{tH}) \le R_s(x)$, which is measurable with respect to $\eta_{|tH\times[0,s)}$, it is enough to show that

$$\mathbb{E}\big[|\, \mathrm{L}^{-1}_{(x,s)}\, F_t^{(\alpha)}|\big|\eta_{|tH\times[0,s)}\big] \le C_2 R_s(x)^{\alpha}, \tag{9.228}$$

for some constant $C_2 > 0$.
As established in the proof of Lemma 9.14, all points connecting to (x, s) are points of η inside the set $A(t, x, s) := (tH \cap \overline{B}^d(x, R_s(x))) \times (s, 1]$. Since this is included in tH, a set of finite measure, the total number of points in $\eta \cap A(t, x, s)$ is almost surely finite. Denote the points connecting to (x, s) by $(y_1, s_1), \ldots, (y_m, s_m)$, ordered by increasing mark, i.e. $s_1 < s_2 < \ldots < s_m$.
With this notation, we have

$$\mathrm{L}^{-1}_{(x,s)}\, F_t^{(\alpha)} = \sum_{i=1}^{m} e(y_i, s_i, \eta_{tH})^{\alpha}. \tag{9.229}$$

When removing all points of η outside of $A(t, x, s)$, i.e. all points further than $R_s(x)$ from x and all points of mark less than s, then for each of the points $y_2, \ldots, y_m$, the edge-length to its online nearest neighbour in η_{tH} will either stay constant or increase. It will never be zero because y_1 remains as a potential nearest neighbour. In formulas:

$$e(y_i, s_i, \eta_{tH}) \le e(y_i, s_i, \eta_{|A(t,x,s)}), \qquad \text{for } i = 2, \ldots, m. \tag{9.230}$$

For $i = 1$, we know as in the proof of Lemma 9.14 that

$$e(y_1, s_i, \eta_{tH}) \le 2R_s(x). \tag{9.231}$$

Combining (9.229) with the above discussion, we get

$$\mathrm{L}^{-1}_{(x,s)}\, F_t^{(\alpha)} \le 2^{\alpha} R_s(x)^{\alpha} + \sum_{i=2}^{m} e(y_i, s_i, \eta_{|A(t,x,s)}). \tag{9.232}$$

Using that edge-lengths are non-negative, we have

$$\mathrm{L}_{(x,s)}^{-1} F_t^{(\alpha)} \leq 2^\alpha R_s(x)^\alpha + \sum_{\substack{(y,u)\in \\ \eta\cap A(t,x,s)}} e(y,u,\eta_{|A(t,x,s)})^\alpha \mathbb{1}_{\{(y,u)\to(x,s) \text{ in } \eta_{|A(t,x,s)}+\delta_{(x,s)}\}}. \tag{9.233}$$

Here we have added the edge contributions from points that would not connect to (x,s) in $\eta_{tH} + \delta_{(x,s)}$, but that do connect to (x,s) in the smaller set $A(t,x,s) + \delta_{(x,s)}$. The sum on the RHS on (9.233) still includes the points (y_i, s_i) for $i \geq 2$.
Observe that $R_s(x) > 0$ a.s. and define

$$D_0 := R_s(x)^{-1} \left(tH \cap \overline{B}^d(x, R_s(x))\right). \tag{9.234}$$

Now we have $A(t,x,s) = R_s(x)D_0 \times (s,1]$ and we can write the second term in (9.233) as

$$\sum_{\substack{(y,u)\in \\ \eta\cap(R_s(x)D_0\times(s,1])}} e(y,u,\eta_{|R_s(x)D_0\times(s,1]})^\alpha \mathbb{1}_{\{(y,u)\to(x,s) \text{ in } \eta_{|R_s(x)D_0\times(s,1]}+\delta_{(x,s)}\}}. \tag{9.235}$$

By the properties of Poisson measures, $\eta_{|tH\times[0,s)}$ (and hence also $R_s(x)$) is independent of $\eta_{|tH\times(s,1]}$. Conditioning on $\eta_{|tH\times[0,s)}$, the sum (9.235) is equal in law to $\mathrm{L}_{(x,0)}^{-1} F^{(\alpha)}(\eta'_{|\lambda D_0\times[0,1-s)})$, where $\lambda = R_s(x)$ and η' is an independent copy of η.
We clearly have that

$$\mathrm{L}_{(x,0)}^{-1} F^{(\alpha)}(\eta'_{|\lambda D_0\times[0,1-s)}) \leq \mathrm{L}_{(x,0)}^{-1} F^{(\alpha)}(\eta'_{|\lambda D_0\times[0,1]}) = \mathrm{L}_{(x,0)}^{-1} F^{(\alpha)}(\eta'_{\lambda D_0}) \tag{9.236}$$

since adding points of higher mark only adds more edges to the graph.
By inequality (9.209) of Lemma 9.14,

$$\mathbb{E}_{\eta'}\left[\mathrm{L}_{(x,0)}^{-1} F^{(\alpha)}(\eta'_{\lambda D_0})\right] \leq c_4(\epsilon,\ell,d,\alpha)\lambda^\alpha = c_4(\epsilon,\ell,d,\alpha)R_s(x)^\alpha, \tag{9.237}$$

where ℓ is an upper bound for $\mathrm{diam}(D_0)$ and ϵ is such that there is a point $x_0 \in D_0$ with $B^d(x_0,\epsilon) \subset D_0$. Our claim (9.228) is proven if we can find such deterministic ℓ and ϵ that do not depend on $x,\ s$ or t.
We have that

$$D_0 = R_s(x)^{-1} tH \cap \overline{B}^d(R_s(x)^{-1}x, 1), \tag{9.238}$$

therefore $\mathrm{diam}(D_0) \leq 2 =: \ell$. To find an ϵ, let $w := t^{-1}x \in H$ and $r := t^{-1}R_s(x)$. Since $R_s(x) \leq t\,\mathrm{diam}(H)$, we have $r \leq \mathrm{diam}(H)$ and by Lemma 9.7, there exists a $0 < q < 1$ depending on H and its properties such that there is a point $w' \in H$ with

$$B^d(w', qr) \subset H \cap B^d(w,r). \tag{9.239}$$

Multiplying by $tR_s(x)^{-1}$ yields

$$B^d(tR_s(x)^{-1}w', q) \subset R_s(x)^{-1}\left(tH \cap B^d(x, R_s(x))\right) = D_0. \tag{9.240}$$

We can thus pick $\epsilon := q$ and the result is shown.
Proof of (9.184). By Lemma 9.12, it suffices to show that

$$\mathbb{E}\left[|\,\mathrm{D}_{(x,s),(y,u)}^{(2)} F_t^{(\alpha)}|\,|\eta_{|tH\times[0,u)}\right] \leq cR_u(y)^\alpha, \tag{9.241}$$

for some constant $c > 0$.
By the triangle inequality,

$$|\,\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F^{(\alpha)}_t| \leq |\,\mathrm{D}_{(y,u)}\, F^{(\alpha)}(\eta_{tH})| + |\,\mathrm{D}_{(y,u)}\, F^{(\alpha)}(\eta_{tH} + \delta_{(x,s)})|. \tag{9.242}$$

The inequality (9.227) deals with the first term on the RHS. For the second term, we use (9.205) to say

$$|\,\mathrm{D}_{(y,u)}\, F^{(\alpha)}(\eta_{tH} + \delta_{(x,s)})| \leq R_u(y, tH, \eta_{tH} + \delta_{(x,s)})^\alpha + \mathrm{L}^{-1}_{(y,u)}\, F^{(\alpha)}(\eta_{tH} + \delta_{(x,s)}). \tag{9.243}$$

Reusing arguments from the proof of Lemma 9.14, one sees that

$$R_u(y, tH, \eta_{tH} + \delta_{(x,s)}) \leq R_u(y, tH, \eta_{tH}), \tag{9.244}$$

which yields the required bound for this term. On the other hand, if we remove a point of mark lower than the one of y, more points might connect to y, and no points already connected to y will change neighbour. Therefore

$$\mathrm{L}^{-1}_{(y,u)}\, F^{(\alpha)}(\eta_{tH} + \delta_{(x,s)}) \leq \mathrm{L}^{-1}_{(y,u)}\, F^{(\alpha)}(\eta_{tH}). \tag{9.245}$$

The result now follows using (9.228).
Proof of (9.185). Using ideas explained in [LPS16, p. 673], we will show that for any convex body $A \subset \mathbb{R}^d$, for all $(x,s),(y,u) \in A \times [0,1]$ with $s < u$ and any finite set $\mu \subset A \times [0,1]$ generic with respect to (x,s) and (y,u), the condition

$$|x-y| > 3R_u(y, A, \mu), \tag{9.246}$$

implies that $\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F^{(\alpha)}(\mu) = 0$. This property is related to the theory of stabilisation, see [LPS16, p. 673] for a discussion. Recall that

$$\begin{aligned} &\mathrm{D}_{(y,u)}\, F^{(\alpha)}(\mu) \\ &\quad = e(y,u,\mu)^\alpha + \sum_{\substack{(z,v)\in \\ \mu\cap(\mathbb{R}^d\times(u,1])}} \left(e(z,v,\mu+\delta_{(y,u)})^\alpha - e(z,v,\mu)^\alpha\right) \mathbb{1}_{\{(z,v)\to(y,u) \text{ in } \mu+\delta_{(y,u)}\}} \end{aligned} \tag{9.247}$$

and

$$\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F^{(\alpha)}(\mu) = \mathrm{D}_{(y,u)}\, F^{(\alpha)}(\mu + \delta_{(x,s)}) - \mathrm{D}_{(y,u)}\, F^{(\alpha)}(\mu). \tag{9.248}$$

Condition (9.246) implies that $A \not\subset \overline{B}^d(y, R_u(y,A,\mu))$ and hence that (y,u) has an online nearest neighbour in μ and $e(y,u,\mu) \neq 0$. Thus $e(y,u,\mu+\delta_{(x,s)}) = |x-y| \wedge e(y,u,\mu)$, but at the same time

$$e(y,u,\mu) \leq R_u(y,A,\mu) < |x-y|. \tag{9.249}$$

It follows that $e(y,u,\mu+\delta_{(x,s)}) = e(y,u,\mu)$. Let (z,v) be a point that connects to (y,u) in $\mu+\delta_{(y,u)}$. Then as shown previously,

$$|z-y| \leq R_u(y,A,\mu) \quad \text{and} \quad e(z,v,\mu) \leq 2R_u(y,A,\mu). \tag{9.250}$$

By conditions (9.246) and (9.250),

$$|x-z| \geq |x-y| - |z-y| > 2R_u(y,A,\mu) \geq \max\{|z-y|, e(z,v,\mu)\} \tag{9.251}$$

and as before $e(z, v, \mu) \neq 0$, since (y, u) has an online nearest neighbour in μ. Put together, this implies that (z, v) will not connect to (x, s), neither in $\mu + \delta_{(x,s)}$, nor in $\mu + \delta_{(y,u)} + \delta_{(x,s)}$. It follows that

$$e(z, v, \mu + \delta_{(y,u)}) = e(z, v, \mu + \delta_{(y,u)} + \delta_{(x,s)}) \quad \text{and} \quad e(z, v, \mu) = e(z, v, \mu + \delta_{(x,s)}). \tag{9.252}$$

This means that the addition of (x, s) does not induce any changes to $\mathrm{D}_{(y,u)} F^{(\alpha)}(\mu)$ and we deduce that

$$\mathrm{D}_{(y,u)} F^{(\alpha)}(\mu + \delta_{(x,s)}) = \mathrm{D}_{(y,u)} F^{(\alpha)}(\mu) \tag{9.253}$$

and thus $\mathrm{D}^{(2)}_{(x,s),(y,u)} F^{(\alpha)}(\mu) = 0$.
Note that $R_u(y, A, \mu)$ and the reasoning above only depend on $\mu_{|A\times[0,u)}$ and thus for any finite $\chi \subset A \times (u, 1]$, by the same reasoning one concludes that $R_u(y, A, \mu_{|A\times[0,u)} \cup \chi) = R_u(y, A, \mu_{|A\times[0,u)}) = R_u(y, A, \mu)$ and

$$\mathrm{D}^{(2)}_{(x,s),(y,u)} F^{(\alpha)}(\mu_{|A\times[0,u)} \cup \chi) = 0 \qquad \text{if } |x - y| > 3R_u(y, A, \mu). \tag{9.254}$$

In particular, this implies that for $(x, s), (y, u) \in tH\times[0, 1]$ with $s < u$ and $|x-y| > 3R_u(y, tH, \eta_{tH})$,

$$\mathbb{E}\big[\,\mathrm{D}^{(2)}_{(x,s),(y,u)} F^{(\alpha)}_t \big| \eta_{|tH\times[0,u)}\big] = \int \mathrm{D}^{(2)}_{(x,s),(y,u)} F^{(\alpha)}_t(\eta_{|tH\times[0,u)} + \chi)\Pi_u(d\chi) = 0, \tag{9.255}$$

where Π_u is the law of $\eta_{|tH\times(u,1]}$.
We conclude that

$$\mathbb{P}\left(\mathbb{E}\big[\,\mathrm{D}^{(2)}_{(x,s),(y,u)} F^{(\alpha)}_t \big|\eta_{|tH\times[0,u)}\big] \neq 0\right) \leq \mathbb{P}\left(|x - y| < 3R_u(y, tH, \eta_{tH})\right). \tag{9.256}$$

As in the proof of Lemma 9.12, there is a constant $c > 0$ such that

$$\mathbb{P}\left(\tfrac{1}{3}|x - y| < R_u(y, tH, \eta_{tH})\right) \leq K \exp(-uc|x - y|^d), \tag{9.257}$$

which concludes the proof. ∎

Remark 9.15. The proofs of Lemma 9.14 and the inequality (9.183) given in Proposition 9.5 build on and extend ideas used in [Wad09]. In particular, the proof of (9.183) adapts a rescaling argument from the proof of [Wad09, Lemma 3.2] and extends it to the Poisson setting and to arbitrary convex bodies. As already discussed, in our proof we need the fine control over constants introduced in Lemma 9.7.

9.5.2 Moment estimates of add-one costs

To find the speed of convergence in the Kolmogorov distance, we need bounds on quantities of the type $\mathbb{E}|\,\mathrm{D}_{(x,s)} F^{(\alpha)}_t|^r$, with $r \geq 1$.

Proposition 9.16. *We work under the conditions of Theorem 6.1. Let $\alpha > 0$ and $r \geq 1$. For every $\epsilon > 0$ and for all $t \geq 1$ and all $(x, s), (y, u) \in tH \times (0, 1]$ with $s < u$,*

$$\mathbb{E}\left[|\,\mathrm{D}_{(x,s)} F^{(\alpha)}_t|^r\right]^{1/r} \leq c_1 s^{-\alpha/d-\epsilon} \tag{9.258}$$

$$\mathbb{E}\left[|\,\mathrm{D}^{(2)}_{(x,s),(y,u)} F^{(\alpha)}_t|^r\right]^{1/r} \leq c_1 u^{-\alpha/d-\epsilon} \tag{9.259}$$

$$\mathbb{P}\left(\mathrm{D}^{(2)}_{(x,s),(y,u)} F^{(\alpha)}_t \neq 0\right) \leq C_2 \exp(-c_2 u|x - y|^d), \tag{9.260}$$

where $c_1 > 0$ is a constant depending on α, r and ϵ and $c_2, C_2 > 0$ are absolute.

Remark 9.17. The bound (9.258) is an extension to arbitrary exponents $r \geq 1$ of what was shown in [Pen05, Lemma 3.4]. The proof below builds on the same arguments, but without using uniform random vectors.

Proof of (9.258). By (9.205) and (9.214), it can be seen that

$$|\,\mathrm{D}_{(x,s)}\, F_t^{(\alpha)}|^r \leq R_s(x)^{r\alpha} 2^{r-1} \left(1 + 2^{r\alpha} |A(x,s)|^r\right). \tag{9.261}$$

where $A(x,s) := \{(y,u) \in \eta_{|tH\times(s,1]} : \ (y,u) \to (x,s) \text{ in } \eta_{tH} + \delta_{(x,s)}\}$ is the set of points in η_{tH} that will connect to (x,s) upon addition of this point. Recall that any point connecting to (x,s) must be inside $\overline{B}^d(x, R_s(x))$. For each $i = 1, ..., K$, define

$$\begin{aligned} A_i(x,s) := \{&(y,u) \in \eta_{tH} \cap (C_i(x) \cap \overline{B}^d(x, R_s(x)) \times (s,1]) : \\ &|x-y| < |x-z| \ \forall (z,v) \in \eta_{tH} \cap (C_i(x) \cap \overline{B}^d(x, R_s(x)) \times (s,u))\}. \end{aligned} \tag{9.262}$$

This is the set of points (y,u) in η_{tH} inside the intersection of the cone $C_i(x)$ with $\overline{B}^d(x, R_s(x))$ that are closer to x than any other point of mark between s and u within this cone. Any point (y,u) connecting to (x,s) must be within such a set for some i, else there is a point closer to y than x and of lower mark, i.e. a potential neighbour. Hence

$$|A(x,s)|^r \leq \left(\sum_{i=1}^K |A_i(x,s)|\right)^r \leq K^{r-1} \left(\sum_{i=1}^K |A_i(x,s)|^r\right). \tag{9.263}$$

Define $m := \lceil r \rceil$ and fix $i \in \{1, ..., K\}$. Since $|A_i(x,s)|$ is a non-negative integer and $\frac{a}{m} \leq \frac{a-k}{m-k}$ for $0 \leq k \leq m-1$ and $a \geq m$, we have

$$|A_i(x,s)|^r \leq |A_i(x,s)|^m \leq m^m \binom{|A_i(x,s)|}{m} + (m-1)^m. \tag{9.264}$$

Our goal is now to estimate $\binom{|A_i(x,s)|}{m}$. First, let

$$G_i := \eta_{tH} \cap \left(C_i(x) \cap B^d(x, R_s(x)) \times (s,1]\right) \tag{9.265}$$

be the set of points in η_{tH} that are closer than $R_s(x)$ to x, within the cone $C_i(x)$ and of mark higher than s. Any point connecting to x must be within this set. Let $N_i := |G_i|$ be the random number of points inside G_i and note that N_i is almost surely finite. Given N_i, the points in G_i can be denoted by the random coordinates $\{(y_1, s_1), ..., (y_{N_i}, s_{N_i})\}$, where $y_1, ..., y_{N_i}$ are the spatial coordinates of the points and $s_1, ..., s_{N_i}$ are the marks of the points.
As N_i is almost surely finite, we can assume w.l.o.g. that the points $y_1, ..., y_{N_i}$ are ordered by increasing distance to x. We now condition on the event $N_i = n \neq 0$ and, given this conditioning, we also take conditional expectation with respect to the σ-algebra generated by the random coordinates y_j. Since $A_i(x,s) \subset G_i$, we have

$$\begin{aligned} \mathbb{E}\left[\binom{|A_i(x,s)|}{m} \,\middle|\, N_i = n, y_1, ..., y_n\right] &= \sum_{\substack{\{z_1,...,z_m\} \subset \{y_1,...,y_n\} \\ \text{distinct}}} \mathbb{E}\left[\mathbb{1}_{\{z_1,...,z_m \in A_i(x,s)\}} \,\middle|\, N_i = n, y_1, ..., y_n\right] \\ &= \sum_{1 \leq j_1 < j_2 < ... < j_m \leq n} \mathbb{P}\left(\{y_{j_1}, ..., y_{j_m}\} \subset A_i(x,s) | N_i = n, y_1, ..., y_n\right) \end{aligned} \tag{9.266}$$

Note that this expression is zero if $n < m$.
Conditional on the event $N_i = n$, the marks $s_1, ..., s_n$ are independent of the spatial coordinates $y_1, ..., y_n$ and i.i.d. uniformly distributed in $(s, 1]$. Hence any ordering of the marks is equally likely. Conditional on the spatial coordinates $y_1, ..., y_n$, the event $\{y_{j_1}, ..., y_{j_m}\} \subset A_i(x, s)$ happens if and only if y_{j_1} has the smallest mark among the points $y_1, y_2, ..., y_{j_1}$, and y_{j_2} has the smallest mark among the points $y_1, y_2, ..., y_{j_2}$ etc. The probability that this happens is exactly given by $(j_1 j_2 ... j_m)^{-1}$. Thus

$$\mathbb{E}\left[\binom{|A_i(x,s)|}{m}\middle| N_i = n, y_1, ..., y_n\right] = \sum_{1\le j_1<j_2<...<j_m\le n} (j_1 j_2 ... j_m)^{-1} \le \frac{1}{m!}\left(\sum_{j=1}^{n}\frac{1}{j}\right)^m. \tag{9.267}$$

It holds that

$$\sum_{j=1}^{n}\frac{1}{j} \le \int_1^n \frac{1}{x}dx + 1 = \log(n) + 1. \tag{9.268}$$

The bounds developed in (9.264)-(9.268) yield for $n \neq 0$:

$$\mathbb{E}\left[|A_i(x,s)|^r | N_i = n\right] \le m^m \frac{1}{m!}(\log(n) + 1)^m + (m-1)^m. \tag{9.269}$$

For any $\epsilon' > 0$, the function $g : [1, +\infty) \to \mathbb{R} : x \mapsto \left(m^m \frac{1}{m!}(\log(x) + 1)^m + (m-1)^m\right) x^{-\epsilon'}$ is bounded by a constant $c > 0$ (dependent on r and ϵ'), implying that

$$\mathbb{E}\left[|A_i(x,s)|^r | N_i = n\right] \le c n^{\epsilon'}. \tag{9.270}$$

If $n = 0$, then $|A_i(x, s)| = 0$, hence the bound (9.270) continues to hold. Therefore

$$\mathbb{E}\left[|A_i(x,s)|^r | N_i\right] \le c N_i^{\epsilon'}. \tag{9.271}$$

Conditional on $\eta_{|tH\times[0,s)}$, the random variable N_i is equal in law to $\eta'(tH \cap C_i(x) \cap \overline{B}^d(x, R_s(x)) \times (s, 1])$, where η' is an independent copy of η. This quantity in turn is upper bounded by $\eta'(\overline{B}^d(x, R_s(x)) \times [0, 1])$. Hence

$$\begin{aligned}\mathbb{E}\left[|A_i(x,s)|^r \middle| \eta_{|tH\times[0,s)}\right] &\le c\mathbb{E}\left[\eta'(\overline{B}^d(x, R_s(x)) \times [0,1])^{\epsilon'} \middle| \eta_{|tH\times[0,s)}\right] \\ &\le c(\kappa_d R_s(x)^d)^{\epsilon'},\end{aligned} \tag{9.272}$$

where we applied Jensen's inequality to pass to the second inequality. Plugging this bound into (9.263), we deduce

$$\mathbb{E}\left[|A(x,s)|^r | \eta_{|tH\times[0,s)}\right] \le c K^r \kappa_d^{\epsilon'} R_s(x)^{\epsilon' d}. \tag{9.273}$$

Combining (9.273) with (9.261) leads to

$$\mathbb{E}\left[|\,\mathrm{D}_{(x,s)}\, F_t^{(\alpha)}|^r\right] \le c_0 \mathbb{E} R_s(x)^{r\alpha}(1 + R_s(x)^{\epsilon' d}). \tag{9.274}$$

for some constant $c_0 > 0$. By Lemma 9.12, the RHS is bounded by $c_1 s^{-r\alpha/d-\epsilon'}$ for some constant $c_1 > 0$. For $\epsilon' = \epsilon r$, we get

$$\mathbb{E}\left[|\,\mathrm{D}_{(x,s)}\, F_t^{(\alpha)}|^r\right]^{1/r} \le c_1^{1/r} s^{-\alpha/d-\epsilon}. \tag{9.275}$$

Proof of (9.259). As in the proof of (9.184), we have

$$|\,\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F_t^{(\alpha)}| \le |\,\mathrm{D}_{(y,u)}\, F^{(\alpha)}(\eta_{tH})| + R_u(y)^\alpha + \mathrm{L}^{-1}_{(y,u)}\, F^{(\alpha)}(\eta_{tH}) \tag{9.276}$$

and the result follows by the proof of (9.258).
Proof of (9.260). As in the proof of (9.185), one see that if $|x-y| > 3R_u(y)$, then

$$\mathrm{D}_{(x,s),(y,u)}\, F_t^{(\alpha)} = 0. \tag{9.277}$$

Hence for some constants $c_1, c_2 > 0$,

$$\mathbb{P}(\mathrm{D}_{(x,s),(y,u)}\, F_t^{(\alpha)} \neq 0) \leq \mathbb{P}(|x-y| > 3R_u(y)) \leq c_1 \exp(-c_2 u|x-y|^d), \tag{9.278}$$

as seen in Lemma 9.12. ■

9.5.3 The orders of the variances

The goal of this subsection is to show the orders of the variances given in (6.5) and (6.7). For the sake of legibility, we will split the proof into several propositions.

Proposition 9.18. *There are constants $c_1, c_2 > 0$ such that for $0 < \alpha < \frac{d}{2}$ and all $t \geq 1$,*

$$\mathrm{Var}(F_t^{(\alpha)}) \leq c_1 t^d \tag{9.279}$$

and

$$\mathrm{Var}(F_t^{(d/2)}) \leq c_2 t^d \log t. \tag{9.280}$$

Remark 9.19. The bounds in this proposition were already shown in [Wad09, Theorem 2.1] via proving the result for uniform vectors and subsequent Poissonisation. We include a proof for the sake of completeness. It is similar in spirit to the one given in [Wad09], but works in a purely Poisson setting.

Proof. Let $0 < \alpha \leq \frac{d}{2}$. Since $F_t^{(\alpha)} \leq (\mathrm{diam}(H)t)^\alpha \eta(tH \times [0,1])$, it is clear that $F_t^{(\alpha)} \in L^2(\mathbb{P}_\eta)$. Hence by Lemma 8.4, we have

$$\mathrm{Var}\left(F_t^{(\alpha)}\right) = \mathbb{E}\int_{tH}\int_0^1 \mathbb{E}\big[\,\mathrm{D}_{(y,u)}\, F_t^{(\alpha)} \big|\eta_{|tH\times[0,u)}\big]^2\, dy du. \tag{9.281}$$

Applying (9.227) to the RHS of (9.281) yields

$$\mathrm{Var}\left(F_t^{(\alpha)}\right) \leq c\,\mathbb{E}\int_{tH}\int_0^1 R_u(y, tH, \eta_t H)^{2\alpha}\, dy du \tag{9.282}$$

for some constant $c > 0$. Now applying Lemma 9.12 to the RHS of (9.282) gives

$$\mathrm{Var}\left(F_t^{(\alpha)}\right) \leq c' \int_{tH}\int_0^1 \left(t^{2\alpha} \wedge u^{-2\alpha/d}\right) dy du \tag{9.283}$$

for some other constant $c' > 0$. Integrating now yields the result. ■

Proposition 9.20. *For $\alpha > 0$, there is a constant $c > 0$ such that*

$$ct^d \leq \mathrm{Var}\left(F_t^{(\alpha)}\right). \tag{9.284}$$

The proof will make use of [LPS16, Theorem 5.2].

Proof of Prop. 9.20. Recall that there are $y_0 \in H$ and $\delta > 0$ such that $B^d(y_0, \delta) \subset H$, as defined at the beginning of this Section 9.5. Also recall the definition of the cones $C_1^+(0), ..., C_K^+(0)$ from Section 9.5.1. For $i \in \{1, ..., K\}$, define

$$V_i := C_i^+(0) \cap B^d(0,1) \setminus B^d\big(0, \tfrac{1}{2}\big). \tag{9.285}$$

Now for $0 < \tau < \frac{\delta}{2}$, define

$$U_\tau := \Big\{(x, s, x + \tau z_1, s_1, ..., x + \tau z_K, s_K) : (x, s) \in tB^d\left(y_0, \tfrac{\delta}{2}\right) \times \left[\tfrac{1}{2}, 1\right], \\ \text{and } (z_i, s_i) \in V_i \times \left[0, \tfrac{1}{2}\right),\ i = 1, ..., K\Big\}. \tag{9.286}$$

Note that $U_\tau \subset (tH \times [0,1])^{K+1}$. For [LPS16, Theorem 5.2] to yield a lower bound as in (9.284), we need to show that for a suitably chosen τ to be defined later,

1. There is a constant $c_1 > 0$ such that for all $(x, s, \tilde{z}_1, s_1, ..., \tilde{z}_K, s_K) \in U_\tau$,

$$\left|\mathbb{E}F^{(\alpha)}\left(\eta_{tH} + \delta_{(x,s)} + \sum_{i=1}^{K} \delta_{(\tilde{z}_i, s_i)}\right) - F^{(\alpha)}\left(\eta_{tH} + \sum_{i=1}^{K} \delta_{(\tilde{z}_i, s_i)}\right)\right| \geq c_1. \tag{9.287}$$

2. There is a constant $c_2 > 0$ such that

$$\min_{\emptyset \neq J \subset \{1,...,K+1\}} \inf_{\substack{V \subset U_\tau \\ \lambda^{(K+1)}(V) \geq \lambda^{(K+1)}(U_\tau)/2^{K+2}}} \lambda^{(|J|)}\left(\Pi_J(V)\right) \geq c_2 t^d, \tag{9.288}$$

   where $\Pi_J(V)$ denotes the projection of V onto the coordinates indexed by J.

Proof of 1. We use a construction similar to the one in [LPS16, Lemma 7.1] for the k-Nearest Neighbour graph. Let $(x, s) \in tB^d\left(y_0, \frac{\delta}{2}\right) \times \left[\frac{1}{2}, 1\right]$ and $(z_i, s_i) \in V_i \times \left[0, \frac{1}{2}\right)$ for all $i = 1, ..., K$. Define

$$\mathcal{A}_\tau := \eta_{tH} + \sum_{i=1}^{K} \delta_{(x + \tau z_i, s_i)}. \tag{9.289}$$

By the choice of the points $z_1, ..., z_K$, we infer that

$$R_s(x, tH, \mathcal{A}_\tau) \leq \tau \tag{9.290}$$

and no point outside $B^d(x, \tau)$ will connect to x, since there is always a point z_i which is closer (by Lemma 3.3 in [Pen05]).
If $\eta(\overline{B}^d(x,\tau) \times [0,1]) = 0$, then no point at all will connect to x and the only change upon addition of x is the addition of the edge from x to its online nearest neighbour. Since there is no point of η in $\overline{B}^d(x, \tau)$, the online nearest neighbour of x must be one of the points $z_1, ..., z_K$. But $|x - z_i| \geq \frac{\tau}{2}$ for all $i = 1, ..., K$ and we deduce

$$\mathrm{D}_{(x,s)} F^{(\alpha)}(\mathcal{A}_\tau) \geq 2^{-\alpha}\tau^\alpha. \tag{9.291}$$

If $\eta(\overline{B}^d(x,\tau) \times [0,1]) \neq 0$, we use that by (9.205) and the proof of Lemma 9.14,

$$\begin{aligned} |\,\mathrm{D}_{(x,s)} F^{(\alpha)}(\mathcal{A}_\tau)| &\leq e(x, s, \mathcal{A}_\tau)^\alpha + \mathrm{L}_{(x,s)}^{-1} F^{(\alpha)}(\mathcal{A}_\tau) \\ &\leq R_s(x, tH, \mathcal{A}_\tau)^\alpha + (2R_s(x, tH, \mathcal{A}_\tau))^\alpha \eta(\overline{B}^d(x,\tau) \times [0,1]) \\ &\leq \tau^\alpha + (2\tau)^\alpha \eta(\overline{B}^d(x,\tau) \times [0,1]) \end{aligned} \tag{9.292}$$

to find the following bound:

$$\begin{aligned}\left|\mathbb{E}\mathbb{1}_{\{\eta(\overline{B}^d(x,\tau)\times[0,1])\neq 0\}}\,\mathrm{D}_{(x,s)}\,F^{(\alpha)}(\mathcal{A}_\tau)\right| &\leq \mathbb{P}(\eta(\overline{B}^d(x,\tau)\times[0,1])\neq 0)\tau^\alpha + (2\tau)^\alpha\mathbb{E}\eta(\overline{B}^d(x,\tau)\times[0,1]) \\ &= \tau^\alpha(1-\exp(-\kappa_d\tau^d)) + (2\tau)^\alpha\kappa_d\tau^d. \end{aligned} \tag{9.293}$$

Combining (9.291) and (9.293), we find

$$\begin{aligned}&\left|\mathbb{E}\,\mathrm{D}_{(x,s)}\,F^{(\alpha)}(\mathcal{A}_\tau)\right| \\ &\quad\geq \mathbb{E}\mathbb{1}_{\{\eta(B^d(x,\tau)\times[0,1])=0\}}\,\mathrm{D}_{(x,s)}\,F^{(\alpha)}(\mathcal{A}_\tau) - \left|\mathbb{E}\mathbb{1}_{\{\eta(B^d(x,\tau)\times[0,1])\neq 0\}}\,\mathrm{D}_{(x,s)}\,F^{(\alpha)}(\mathcal{A}_\tau)\right| \\ &\quad\geq 2^{-\alpha}\tau^\alpha\exp(-\kappa_d\tau^d) - \tau^\alpha(1-\exp(-\kappa_d\tau^d)) - (2\tau)^\alpha\kappa_d\tau^d \\ &\quad= \tau^\alpha\left(\exp(-\kappa_d\tau^d)2^{-\alpha} - (1-\exp(-\kappa_d\tau^d)) - 2^\alpha\kappa_d\tau^d\right) =: c_\tau. \end{aligned} \tag{9.294}$$

But we have

$$\lim_{\tau\to 0}\exp(-\kappa_d\tau^d)2^{-\alpha} - (1-\exp(-\kappa_d\tau^d)) - 2^\alpha\kappa_d\tau^d = 2^{-\alpha} > 0, \tag{9.295}$$

which means that we can choose $\tau > 0$ small enough such that $c_\tau > 0$. This choice of τ depends only on α and d. We fix this τ for the rest of the proof.

Proof of 2. We follow the same type of reasoning as was used in the proof of [LPS16, Theorem 5.3]. First, note that

$$|U_\tau| = \kappa_d 2^{-d-K-1}\delta^d|V_1|^K t^d. \tag{9.296}$$

Let $\emptyset \neq J = \{i_1,\dots,i_{|J|}\} \subset \{1,\dots,K+1\}$. For any $(y,u) = (y_1,u_1,\dots,y_{K+1},u_{K+1}) \in \left(\mathbb{R}^d\times[0,1]\right)^{K+1}$, write $(y,u)_J = (y_{i_1},u_{i_1},\dots,y_{i_{|J|}},u_{i_{|J|}})$. If $(y,u)\in U$, then for any $i,j\in\{1,\dots,K+1\}$, we have that $y_j\in B^d(y_i,2\tau)$ and $u_j\in\left[0,\frac{1}{2}\right)$. This implies that for any $(y,u)_J\in\left(\mathbb{R}^d\times[0,1]\right)^{|J|}$,

$$\lambda^{(K+1-|J|)}\left((y,u)_{J^c}\in\left(\mathbb{R}^d\times[0,1]\right)^{K+1-|J|} : ((y,u)_J,(y,u)_{J^c})\in U\right) \leq \left(2^{d-1}\tau^d\kappa_d\right)^{K+1-|J|}, \tag{9.297}$$

where we use λ for Lebesgue measure. Thus for any $V\subset U$,

$$\begin{aligned}\lambda^{(K+1)}(V) &\leq \int_{\left(\mathbb{R}^d\times[0,1]\right)^{K+1}}\mathbb{1}_{\{(y,u)_J\in\Pi_J(V)\}}\mathbb{1}_{\{((y,u)_J,(y,u)_{J^c})\in U\}}\lambda^{(K+1)}(d(y,u)) \\ &\leq \left(2^{d-1}\tau^d\kappa_d\right)^{K+1-|J|}\lambda^{(|J|)}\left(\Pi_J(V)\right). \end{aligned} \tag{9.298}$$

Hence any $V\subset U$ with $\lambda^{(K+1)}(V)\geq\lambda^{(K+1)}(U)2^{-(K+2)}$ satisfies

$$\lambda^{(|J|)}\left(\Pi_J(V)\right) \geq \left(2^{d-1}\tau^d\kappa_d\right)^{-(K+1-|J|)}2^{-(K+2)}\kappa_d 2^{-d-K-1}\delta^d|V_1|^K t^d. \tag{9.299}$$

This in turn is lower bounded by $c_2 t^d$, where

$$c_2 := \min\left\{1,\left(2^{d-1}\tau^d\kappa_d\right)^{-K}\right\}\kappa_d 2^{-d-2K-3}\delta^d|V_1|^K. \tag{9.300}$$

This concludes the proof. ∎

Proposition 9.21. *There are constants $c>0$ and $T_0\geq 1$ such that for all $t\geq T_0$,*

$$ct^d\log(t) \leq \mathrm{Var}\left(F_t^{(d/2)}\right). \tag{9.301}$$

Remark 9.22. By inspection of the arguments in Lemmas 9.23–9.29, one sees that

$$d\left(1-\tfrac{\pi}{2}+c(d)\right) \leq \liminf_{t\to\infty} \frac{\mathrm{Var}\left(F_t^{(d/2)}\right)}{t^d \log(t)}, \tag{9.302}$$

where $c(d)$ is a positive constant such that $c(d) > \frac{\pi}{2} - 1$ for $d \geq 1$. We believe this bound to correspond to the exact asymptotic order of $\mathrm{Var}\left(F_t^{(d/2)}\right)$. We can however only provide a closed form of $c(d)$ in dimension $d=1$. For dimension $d=2$, we numerically estimate $c(2)$ and for dimensions $d \geq 3$, we use a lower bound which is smaller than the LHS of (9.302):

$$d\left(1-\frac{\pi}{4}-\tilde{c}(d)\right) \leq \liminf_{t\to\infty} \frac{\mathrm{Var}\left(F_t^{(d/2)}\right)}{t^d \log(t)}, \tag{9.303}$$

where $0 < \tilde{c}(d) < 1 - \frac{\pi}{4}$ for $d \geq 3$ and $\tilde{c}(d) \to 0$ as $d \to \infty$.

Lemma 9.23. *Sief* $\alpha > 0$ *an* $t \geq 1$. *Sief* $\ell := \mathrm{diam}(H)$. *Dann as*

$$\mathrm{Var}(F_t^{(\alpha)}) = I_1(t) + I_2(t) - I_3(t) + I_4(t) - I_5(t), \tag{9.304}$$

wou d'Termen definéiert sinn als:

$$\begin{aligned}
I_1(t) &= \int_{tH} dy \int_0^1 dv \int_0^{(t\ell)^{2\alpha}} ds\, \left(\exp(-v|tH \cap B^d(y, s^{1/(2\alpha)})|) - \exp(-v|tH|)\right)\\
I_2(t) &= 2\int_{tH} dx \int_{tH} dy \int_0^1 dv \int_0^1 du \int_0^{(t\ell)^\alpha} ds \int_0^{(t\ell)^\alpha} dr\; \mathbb{1}_{\{u<v\}} \mathbb{1}_{\{s<|x-y|^\alpha\}}\\
&\qquad \exp\left(-u|tH \cap B^d(x, r^{1/\alpha})|\right) \exp\left(-v|tH \cap B^d(y, s^{1/\alpha})|\right)\\
&\qquad\qquad \left[\exp\left(u|tH \cap B^d(x, r^{1/\alpha}) \cap B^d(y, s^{1/\alpha})|\right) - 1\right]\\
I_3(t) &= 2\int_{tH} dx \int_{tH} dy \int_0^1 dv \int_0^1 du \int_0^{(t\ell)^\alpha} ds \int_0^{(t\ell)^\alpha} dr\; \mathbb{1}_{\{u<v\}} \mathbb{1}_{\{s>|x-y|^\alpha\}}\\
&\qquad \exp\left(-u|tH \cap B^d(x, r^{1/\alpha})|\right) \exp\left(-v|tH \cap B^d(y, s^{1/\alpha})|\right)\\
I_4(t) &= 2\int_{tH} dx \int_{tH} dy \int_0^1 dv \int_0^1 du \int_0^{(t\ell)^\alpha} ds \int_0^{(t\ell)^\alpha} dr\; \mathbb{1}_{\{u+v\geq 1\}}\\
&\qquad \exp\left(-u|tH \cap B^d(x, r^{1/\alpha})|\right) \exp\left(-v|tH|\right)\\
I_5(t) &= \int_{tH} dx \int_{tH} dy \int_0^1 dv \int_0^1 du \int_0^{(t\ell)^\alpha} ds \int_0^{(t\ell)^\alpha} dr \exp\left(-u|tH|\right) \exp\left(-v|tH|\right)
\end{aligned}$$

Proof. Mir fänken un mat e puer Identitéiten. D'Funktional $F_t^{(\alpha)}$ kann ee schreiwen als

$$F_t^{(\alpha)} = \int_{tH\times[0,1]} e(y, v, \eta)^\alpha \eta(dy, dv). \tag{9.305}$$

En plus, fir $(y, v), (x, u) \in tH \times [0, 1]$ mat $u < v$, hu mer

$$e(y, v, \eta)^\alpha = \mathbb{1}_{\{\eta(tH\times[0,v))\neq 0\}} \int_0^{(t\ell)^\alpha} ds\; \mathbb{1}_{\{\eta(tH\cap B^d(y,s^{1/\alpha})\times[0,v))=0\}} \tag{9.306}$$

an

$$e\left(y,v,\eta+\delta_{(x,u)}\right)^{\alpha}=|x-y|^{\alpha}\mathbb{1}_{\{\eta(tH\times[0,v))=0\}}\\ +\mathbb{1}_{\{\eta(tH\times[0,v))\neq 0\}}\int_0^{|x-y|^{\alpha}}ds\;\mathbb{1}_{\{\eta(tH\cap B^d(y,s^{1/\alpha})\times[0,v))=0\}}. \quad (9.307)$$

Mir stellen och fest dass $e(y,v,\eta)=e(y,v,\eta_{|tH\times[0,v)})$, e Fakt dee mer nach eng Rei Kéiere wäerte benotzen.
Kombinéiert een (9.305) an (9.306) mat der Mecke Equatioun (3.7), da kann een $\mathbb{E}F_t^{(\alpha)}$ ausrechnen:

$$\begin{aligned}\mathbb{E}F_t^{(\alpha)}&=\int_{tH}dy\int_0^1dv\int_0^{(t\ell)^{\alpha}}dr\;\mathbb{E}\mathbb{1}_{\{\eta(tH\times[0,v))\neq 0\}}\mathbb{1}_{\{\eta(tH\cap B^d(y,r^{1/\alpha})\times[0,v))=0\}}\\&=\int_{tH}dy\int_0^1dv\int_0^{(t\ell)^{\alpha}}dr\;\big(\exp\big(-v|tH\cap B^d(y,r^{1/\alpha})|\big)-\exp(-v|tH|)\big). \end{aligned}\quad (9.308)$$

Fir $\mathbb{E}\big[(F_t^{(\alpha)})^2\big]$ auszerechnen, stelle mer fir d'éischt fest dass

$$\mathbb{E}\left(\int_{tH\times[0,1]}e(y,v,\eta)^{\alpha}\eta(dy,dv)\right)^2\\ =\int_{tH}dy\int_0^1dv\;\mathbb{E}\,e(y,v,\eta)^{2\alpha}+\int_{tH}dx\int_{tH}dy\int_0^1du\int_0^1dv\;\mathbb{E}\,e(y,v,\eta+\delta_{(x,u)})^{\alpha}e(x,u,\eta+\delta_{(y,v)})^{\alpha}, \quad (9.309)$$

wat ee z.B. ka gesinn andeems een d'Mecke Equatioun zweemol uwent. Deen éischten Term op der rietser Säit vun (9.309) as gläich $\mathbb{E}\big[F_t^{(2\alpha)}\big]$, wat gläich $I_1(t)$ as duerch (9.308).
Deen zweeten Term op der rietser Säit vun (9.309) kann duerch Symmetrie geschriwwe ginn als

$$2\int_{tH}dx\int_{tH}dy\int_0^1du\int_0^1dv\;\mathbb{1}_{\{u<v\}}\mathbb{E}\,e(y,v,\eta+\delta_{(x,u)})^{\alpha}e(x,u,\eta)^{\alpha}. \quad (9.310)$$

Stieche mer (9.306) an (9.307) an déi Expressioun, a benotze mer datt de Produkt $\mathbb{1}_{\{\eta(tH\times[0,u))\neq 0\}}\mathbb{1}_{\{\eta(tH\times[0,v))=0\}}$ null as, da kréie mer

$$2\mathbb{E}\int_{tH}dx\int_{tH}dy\int_0^1du\int_0^1dv\int_0^{(t\ell)^{\alpha}}dr\int_0^{(t\ell)^{\alpha}}ds\;\mathbb{1}_{\{u<v\}}\mathbb{1}_{\{s<|x-y|^{\alpha}\}}\\ \mathbb{1}_{\{\eta(tH\times[0,u))\neq 0\}}\mathbb{1}_{\{\eta(tH\cap B^d(x,r^{1/\alpha})\times[0,u))=0\}}\mathbb{1}_{\{\eta(tH\cap B^d(y,s^{1/\alpha})\times[0,v))=0\}}. \quad (9.311)$$

Schreiwe mer dann

$$\mathbb{1}_{\{\eta(tH\cap B^d(y,s^{1/\alpha})\times[0,v))=0\}}=\mathbb{1}_{\{\eta(tH\cap B^d(y,s^{1/\alpha})\times[0,u))=0\}}\mathbb{1}_{\{\eta(tH\cap B^d(y,s^{1/\alpha})\times[u,v))=0\}} \quad (9.312)$$

an huele mer d'Espérance, woubäi mer och benotzen datt $\eta_{|tH\times[0,u)}$ an $\eta_{|tH\times[u,v)}$ onofhängesch vunenee sinn, da kënne mer weisen datt (9.311) gläich as mat

$$2\int_{tH}dx\int_{tH}dy\int_0^1du\int_0^1dv\int_0^{(t\ell)^{\alpha}}dr\int_0^{(t\ell)^{\alpha}}ds\;\mathbb{1}_{\{u<v\}}\mathbb{1}_{\{s<|x-y|^{\alpha}\}}\exp\big(-(v-u)|tH\cap B^d(y,s^{1/\alpha})|\big)\\ \big(\exp\big(-u|tH\cap(B^d(x,r^{1/\alpha})\cup B^d(y,s^{1/\alpha}))|\big)-\exp(-u|tH|)\big). \quad (9.313)$$

Kombinéiere mer elo (9.313) mat (9.308), dann hu mer

$$\mathrm{Var}\left(F_t^{(\alpha)}\right) = I_1(t) + A(t) + B(t) + C(t) \tag{9.314}$$

mat

$$\begin{aligned} A(t) &= 2\int_{tH} dx \int_{tH} dy \int_0^1 du \int_0^1 dv \int_0^{(t\ell)^\alpha} dr \int_0^{(t\ell)^\alpha} ds\; \mathbb{1}_{\{u<v\}} \mathbb{1}_{\{s<|x-y|^\alpha\}} \\ &\quad \exp\left(-(v-u)|tH \cap B^d(y, s^{1/\alpha})|\right) \exp\left(-u|tH \cap (B^d(x, r^{1/\alpha}) \cup B^d(y, s^{1/\alpha}))|\right) \\ B(t) &= -2\int_{tH} dx \int_{tH} dy \int_0^1 du \int_0^1 dv \int_0^{(t\ell)^\alpha} dr \int_0^{(t\ell)^\alpha} ds\; \mathbb{1}_{\{u<v\}} \mathbb{1}_{\{s<|x-y|^\alpha\}} \\ &\quad \exp\left(-(v-u)|tH \cap B^d(y, s^{1/\alpha})|\right) \exp(-u|tH|) \\ C(t) &= -\left(\int_{tH} dy \int_0^1 dv \int_0^{(t\ell)^\alpha} ds\; \left(\exp\left(-v|tH \cap B^d(y, s^{1/\alpha})|\right) - \exp(-v|tH|)\right)\right)^2. \end{aligned}$$

Et as einfach ze gesinn dass

$$C(t) = C_1(t) + C_2(t) - I_3(t) - I_5(t) \tag{9.315}$$

mat

$$\begin{aligned} C_1(t) &= -2\int_{tH} dx \int_{tH} dy \int_0^1 du \int_0^1 dv \int_0^{(t\ell)^\alpha} dr \int_0^{(t\ell)^\alpha} ds\; \mathbb{1}_{\{u<v\}} \mathbb{1}_{\{s<|x-y|^\alpha\}} \\ &\quad \exp(-v|tH \cap B^d(y, s^{1/\alpha})|) \exp(-u|tH \cap B^d(x, r^{1/\alpha})|) \\ C_2(t) &= 2\int_{tH} dx \int_{tH} dy \int_0^1 du \int_0^1 dv \int_0^{(t\ell)^\alpha} dr \int_0^{(t\ell)^\alpha} ds\; \exp(-v|tH \cap B^d(y, s^{1/\alpha})|) \exp(-u|tH|). \end{aligned}$$

Et as elo just nach ze weisen datt $I_2(t) + I_4(t) = A(t) + B(t) + C_1(t) + C_2(t)$. Fir d'éischt weise mer dass $A(t) + C_1(t) = I_2(t)$. En effet, schreiwt een $R := B^d(x, r^{1/\alpha})$ an $S := B^d(y, s^{1/\alpha})$, dann as

$$\begin{aligned} &\exp(-(v-u)|tH \cap S|) \exp(-u|tH \cap (S \cup R)|) \\ &\qquad = \exp(-v|tH \cap S|) \exp(-u|tH \cap R|) \exp(u|tH \cap S \cap R|) \end{aligned} \tag{9.316}$$

an dowéinst as

$$\begin{aligned} A(t) + C_1(t) &= 2\int_{tH} dx \int_{tH} dy \int_0^1 du \int_0^1 dv \int_0^{(t\ell)^\alpha} dr \int_0^{(t\ell)^\alpha} ds\; \mathbb{1}_{\{u<v\}} \mathbb{1}_{\{s<|x-y|^\alpha\}} \\ &\quad \exp(-v|tH \cap S|) \exp(-u|tH \cap R|) \left(\exp(u|tH \cap S \cap R|) - 1\right) = I_2(t). \end{aligned} \tag{9.317}$$

Fir ze gesinn dass $B(t) + C_2(t) = I_4(t)$, geet et duer en Changement de Variable am $B(t)$ ze maachen, andeems een $\tilde{v} := v - u$ an $d\tilde{v} = dv$ setzt. Domadder as de Beweis komplett. (Apologies for the easter egg and well done to the reader for making it this far into the thesis! This lemma and its proof have been written in Luxembourgish. A version in English is available in the preprint [Tra22] under Lemma E.19.) ■

In the following we are going to deal with the different terms in Lemma 9.23 in the case $\alpha = \frac{d}{2}$.

Lemma 9.24. *In the notation of Lemma 9.23 and with $\alpha = \frac{d}{2}$,*

$$I_1(t) \geq |H|\kappa_d^{-1} t^d \log(t^d) + O(t^d). \tag{9.318}$$

Proof. First, we show that the second part of the expression $I_1(t)$ is $O(t^d)$. Indeed,

$$\int_{tH} dy \int_0^1 dv \int_0^{(t\ell)^d} ds \exp(-v|tH|) = t^d \ell^d (1 - \exp(-t^d|H|)) = O(t^d). \tag{9.319}$$

Note that since $|tH \cap B^d(y, s^{1/d})| \leq \kappa_d s$, the first part of the expression $I_1(t)$ is lower bounded by

$$\int_{tH} dy \int_{t^{-d}}^1 dv \int_0^{(t\ell)^d} ds \ \exp(-v\kappa_d s), \tag{9.320}$$

which after integrating in y and s and the change of variables $u = t^d v$, is equal to

$$t^d |H| \kappa_d^{-1} \left(\log(t^d) - \int_1^{t^d} du\, u^{-1} \exp(-u\kappa_d \ell^d) \right). \tag{9.321}$$

Since $\int_1^\infty du\, u^{-1} \exp(-u\kappa_d \ell^d) < \infty$, this term is equal to $t^d |H| \kappa_d^{-1} \log(t^d) + O(t^d)$, concluding the proof. ■

Lemma 9.25. *In the notation of Lemma 9.23 and with $\alpha = \frac{d}{2}$, there is a constant $T_0 \geq 1$ such that for all $t \geq T_0$, we have*

$$I_3(t) \leq |H|\kappa_d^{-1} \frac{\pi}{2} t^d \log(t^d) + O(t^d). \tag{9.322}$$

See the proof of [Wad09, Lemma 3.6] for a similar computation to the one done below.

Proof. Start with a change of variables:

$$\begin{cases} \tilde{x} = t^{-1}x, \ dx = t^d d\tilde{x} \\ \tilde{y} = t^{-1}y, \ dy = t^d d\tilde{y} \\ \tilde{s} = t^{-d/2}s, \ ds = t^{d/2} d\tilde{s} \\ \tilde{r} = t^{-d/2}r, \ dr = t^{d/2} d\tilde{r} \\ \tilde{u} = t^d u, \ du = t^{-d} d\tilde{u} \\ \tilde{v} = t^d v, \ dv = t^{-d} d\tilde{v}, \end{cases} \tag{9.323}$$

which leads to

$$I_3(t) = 2t^d \int_H dx \int_H dy \int_0^{t^d} dv \int_0^v du \int_0^{\ell^{d/2}} ds \int_0^{\ell^{d/2}} dr \ \mathbb{1}_{\{s > |x-y|^{d/2}\}}$$
$$\exp\left(-u|H \cap B^d(x, r^{2/d})|\right) \exp\left(-v|H \cap B^d(y, s^{2/d})|\right). \tag{9.324}$$

We can reduce the integration interval of the variable v to $[\tau_0, t^d]$, for some large constant τ_0 and large enough t, since the rest term is clearly $O(t^d)$. For any $v \geq \tau_0$, we then have $|H \setminus H_{2v^{-1/(2d)}}| \leq \beta_H 2v^{-1/(2d)}$ by the discussion at the start of Section 6.1.
We can now split the above expression as follows:

$$
\begin{aligned}
I_3(t) = 2t^d \int_0^{\tau_0} dv \int_0^v du \int_H dx \int_H dy \int_0^{\ell^{d/2}} ds \int_0^{\ell^{d/2}} dr \; \mathbb{1}_{\{s>|x-y|^{d/2}\}} \\
\exp\left(-u|H \cap B^d(x, r^{2/d})|\right) \exp\left(-v|H \cap B^d(y, s^{2/d})|\right) \\
+ 2t^d \int_{\tau_0}^{t^d} dv \int_0^v du \int_{H \setminus H_{2v^{-1/(2d)}}} dx \int_H dy \int_0^{\ell^{d/2}} ds \int_0^{\ell^{d/2}} dr \; \mathbb{1}_{\{s>|x-y|^{d/2}\}} \\
\exp\left(-u|H \cap B^d(x, r^{2/d})|\right) \exp\left(-v|H \cap B^d(y, s^{2/d})|\right) \\
+ 2t^d \int_{\tau_0}^{t^d} dv \int_0^v du \int_{H_{2v^{-1/(2d)}}} dx \int_H dy \int_{v^{-1/4}}^{\ell^{d/2}} ds \int_0^{\ell^{d/2}} dr \; \mathbb{1}_{\{s>|x-y|^{d/2}\}} \\
\exp\left(-u|H \cap B^d(x, r^{2/d})|\right) \exp\left(-v|H \cap B^d(y, s^{2/d})|\right) \\
+ 2t^d \int_{\tau_0}^{t^d} dv \int_0^v du \int_{H_{2v^{-1/(2d)}}} dx \int_H dy \int_0^{v^{-1/4}} ds \int_{v^{-1/4}}^{\ell^{d/2}} dr \; \mathbb{1}_{\{s>|x-y|^{d/2}\}} \\
\exp\left(-u|H \cap B^d(x, r^{2/d})|\right) \exp\left(-v|H \cap B^d(y, s^{2/d})|\right) \\
+ 2t^d \int_{\tau_0}^{t^d} dv \int_0^v du \int_{H_{2v^{-1/(2d)}}} dx \int_H dy \int_0^{v^{-1/4}} ds \int_0^{v^{-1/4}} dr \; \mathbb{1}_{\{s>|x-y|^{d/2}\}} \\
\exp\left(-u|H \cap B^d(x, r^{2/d})|\right) \exp\left(-v|H \cap B^d(y, s^{2/d})|\right) \\
= O(t^d) + R_1(t) + R_2(t) + R_3(t) + I_3'(t).
\end{aligned}
\tag{9.325}
$$

Note that in $I_3'(t)$, due to the choice of sets we are integrating over, we have $B^d(x, r^{2/d}) \subset H$ and $B^d(y, s^{2/d}) \subset H$. Hence $I_3'(t)$ is upper bounded by

$$
2t^d \int_1^{t^d} dv \int_0^v du \int_0^\infty ds \int_0^\infty dr \int_H dx \int_{B^d(x,s^{2/d})} dy \; \exp\left(-u\kappa_d r^2\right) \exp\left(-v\kappa_d r^2\right) \tag{9.326}
$$

which is equal to $|H|\kappa_d^{-1}\frac{\pi}{2}t^d \log(t^d)$. Hence

$$
I_3'(t) \leq |H|\kappa_d^{-1}\tfrac{\pi}{2}t^d \log(t^d). \tag{9.327}
$$

It remains to show that the three rest terms $R_1(t), R_2(t), R_3(t)$ are $O(t^d)$. To this end, recall from Lemma 9.7 that there is a constant $c_H > 0$ such that $|H \cap B^d(w, s)| \geq c_H s^d$ for any $w \in H$ and $s \leq \ell$. This implies that

$$
R_1(t) \leq 2t^d \int_{\tau_0}^\infty dv \int_0^v du \int_0^\infty ds \int_0^\infty dr \int_{H \setminus H_{2v^{-1/(2d)}}} dx \int_{B^d(x,s^{2/d})} dy \; \exp\left(-uc_H r^2\right) \exp\left(-vc_H s^2\right) \tag{9.328}
$$

which in turn is upper bounded by

$$
t^d \pi \kappa_d c_H^{-2} \beta_H \int_{\tau_0}^\infty v^{-1-1/(2d)} = O(t^d). \tag{9.329}
$$

By the same reasoning, we get

$$R_2(t) \leq 2|H|\kappa_d t^d \int_{\tau_0}^{\infty} dv \int_0^v du \int_{v^{-1/4}}^{\infty} ds \int_0^{\infty} dr\ s^2 \exp\left(-uc_H r^2\right) \exp\left(-vc_H s^2\right) \tag{9.330}$$

and

$$R_3(t) \leq 2|H|\kappa_d t^d \int_{\tau_0}^{\infty} dv \int_0^v du \int_0^{\infty} ds \int_{v^{-1/4}}^{\infty} dr\ s^2 \exp\left(-uc_H r^2\right) \exp\left(-vc_H s^2\right). \tag{9.331}$$

both of which can be shown to be $O(t^d)$. ■

Lemma 9.26. *In the notation of Lemma 9.23 and with* $\alpha = \frac{d}{2}$, *one has* $I_4(t) = O(t^d)$ *and* $I_5(t) = O(t^d)$.

Remark 9.27. Since $I_4(t) \geq 0$, it is not necessary to calculate the order of this term to find a lower bound of $\mathrm{Var}(F_t^{(d/2)})$. However, we include the proof for completeness.

Proof. Perform the same change of variables and upper bound as in the proof of Lemma 9.25 to find

$$I_4(t) \leq 2t^d \int_H dx \int_H dy \int_0^{t^d} dv \int_0^{t^d} du \int_0^{\ell^{d/2}} ds \int_0^{\ell^{d/2}} dr\ \mathbb{1}_{\{u+v\geq t^d\}} \exp\left(-uc_H r^2\right) \exp\left(-v|H|\right). \tag{9.332}$$

Integrating over x, y, s and v yields the upper bound

$$I_4(t) \leq 2|H|\ell^{d/2} t^d \int_0^{t^d} du \int_0^{\ell^{d/2}} dr\ \exp(-uc_H r^2)\left(\exp(-(t^d-u)|H|) - \exp(-t^d|H|)\right). \tag{9.333}$$

The integrand is bounded by $\exp(-(t^d-u)|H|)$. Introducing the change of variable $\tilde{u} = t^d - u$ yields the upper bound

$$I_4(t) \leq 2|H|\ell^{d/2} t^d \int_0^{t^d} du \int_0^{\ell^{d/2}} dr\ \exp(-u|H|) \leq 2\ell^d t^d = O(t^d). \tag{9.334}$$

As for $I_5(t)$, integrating over all variables yields that

$$I_5(t) = t^d \ell^d (1 - \exp(-t|H|))^2 = O(t^d). \tag{9.335}$$

■

Lemma 9.28 (Joint work with Pierre Perruchaud)**.** *In the notation of Lemma 9.23 and with* $\alpha = \frac{d}{2}$, *for every* $\epsilon > 0$, *there is a constant* $T_0 \geq 1$ *such that for all* $t \geq T_0$,

$$I_2(t) \geq (|H|\kappa_d^{-1} c(d) - \epsilon) t^d \log(t^d) + O(t^d), \tag{9.336}$$

where

$$c(d) = \int_{\mathbb{R}^d} dz \int_0^{\infty} dr\ \mathbb{1}_{\{|z|\geq 1\}} \left(\left(|B^d(0, r^{2/d}) \cup B^d(z,1)|\right)^{-1} - \kappa_d^{-1}(r^2+1)^{-1}\right). \tag{9.337}$$

Proof. Fix $\tilde{\epsilon} > 0$ such that $(|H| - \tilde{\epsilon})(c(d) - \tilde{\epsilon}) \geq |H|c(d) - \epsilon$ and let $\tilde{\delta} > 0$ be such that $\beta_H \tilde{\delta} < \tilde{\epsilon}$ and $|H_{\tilde{\delta}}| \geq |H| - \tilde{\epsilon}$ (which is possible by property (9.182)).
Assume that $x \in tH_{\tilde{\delta}}$, $y \in B^d(x, \frac{t\tilde{\delta}}{2})$, $r \leq (t\tilde{\delta})^{d/2}$ and $s \leq |x-y|^{d/2}$. Then $B^d(x, t\tilde{\delta}) \subset tH$, $B^d(x, r^{2/d}) \subset tH$ and $B^d(y, s^{2/d}) \subset tH$. In the integrals making up $I_2(t)$, we can reduce the intervals of integration to the ones stated here and hence $I_2(t)$ is lower bounded by

$$2\int_{tH_{\tilde{\delta}}} dx \int_{B^d(x,\frac{t\tilde{\delta}}{2})} dy \int_0^{(t\tilde{\delta})^{d/2}} dr \int_0^{|x-y|^{d/2}} ds \int_0^1 dv \int_0^v du$$
$$\exp(-u\kappa_d r^2)\exp(-v\kappa_d s^2)\left(\exp(u|B^d(0,r^{2/d}) \cap B^d(y-x, s^{2/d})|) - 1\right). \quad (9.338)$$

Now carry out the following changes of variables:

$$\begin{cases} \tilde{u} = v^{-1}u, \ d\tilde{u} = v^{-1}du \\ z = v^{1/d}(y-x), \ dz = vdy \\ \tilde{s} = v^{1/2}s, \ d\tilde{s} = v^{1/2}ds \\ \tilde{r} = v^{1/2}r, \ d\tilde{r} = v^{1/2}dr \end{cases} \quad (9.339)$$

and deduce that the above expression is equal to

$$2\int_0^1 dv \int_0^1 d\tilde{u} \int_{tH_{\tilde{\delta}}} dx \int_{B^d(0,tv^{1/d}\tilde{\delta}/2)} dz \int_0^{(tv^{1/d}\tilde{\delta})^{d/2}} d\tilde{r} \int_0^{|z|^{d/2}} d\tilde{s}$$
$$v^{-1}\exp(-\tilde{u}\kappa_d\tilde{r}^2)\exp(-\kappa_d\tilde{s}^2)\left(\exp(\tilde{u}|B^d(0,\tilde{r}^{2/d}) \cap B^d(z,\tilde{s}^{2/d})|) - 1\right). \quad (9.340)$$

Assume now that $t \geq \tilde{\delta}^{-1}\tau_0 =: T_0$ for some $\tau_0 > 0$ and lower bound this expression by reducing to the integration interval where $v^{1/d} \geq t^{-1}\tilde{\delta}^{-1}\tau_0$. Integrating additionally over x, we get the lower bound

$$t^d|H_{\tilde{\delta}}|\left(\int_{(t^{-1}\tilde{\delta}^{-1}\tau_0)^d}^1 dv\, v^{-1}\right) 2\int_0^1 du \int_{B^d(0,\tau_0/2)} dz \int_0^{\tau_0^{d/2}} dr \int_0^{|z|^{d/2}} ds$$
$$\exp(-u\kappa_d r^2)\exp(-\kappa_d s^2)\left(\exp(u|B^d(0,r^{2/d}) \cap B^d(z,s^{2/d})|) - 1\right). \quad (9.341)$$

For τ_0 large enough, this is lower bounded by

$$t^d(|H| - \tilde{\epsilon})(\log(t^d) - d\log(\tilde{\delta}^{-1}\tau_0))(c_0(d) - \tilde{\epsilon}) \geq (|H|c_0(d) - \epsilon)t^d\log(t^d) + O(t^d) \quad (9.342)$$

with

$$c_0(d) := 2\int_0^1 du \int_{\mathbb{R}^d} dz \int_0^\infty dr \int_0^{|z|^{d/2}} ds \exp(-u\kappa_d r^2)\exp(-\kappa_d s^2)$$
$$\left(\exp(u|B^d(0,r^{2/d}) \cap B^d(z,s^{2/d})|) - 1\right). \quad (9.343)$$

The last thing that remains to do is to show that $c_0(d) = \kappa_d^{-1}c(d)$. Perform the successive change of variables $\tilde{r} = u^{1/2}r$, $\tilde{z} = u^{1/d}z$, $\tilde{s} = u^{1/2}s$ and $\tilde{u} = u^{-1}$, then integrate over $\tilde{u}$ to get

$$c_0(d) = 2\int_{\mathbb{R}^d} dz \int_0^\infty dr \int_0^{|z|^{d/2}} ds\, \kappa_d^{-1}s^{-2}\exp(-\kappa_d s^2)\exp(-\kappa_d r^2)$$
$$\left(\exp(|B^d(0,r^{2/d}) \cap B^d(z,s^{2/d})|) - 1\right). \quad (9.344)$$

Then change variables $\tilde{z} = s^{-2/d}z$ and $\tilde{r} = s^{-1}r$ and integrate over s to find $c_0(d) = \kappa_d^{-1}c(d)$. ■

Combining Lemmas 9.24, 9.25, 9.26 and 9.28, we have shown that for every $\epsilon > 0$, there is a $T_0 \geq 1$ such that for all $t \geq T_0$,

$$\text{Var}\left(F_t^{(d/2)}\right) \geq t^d \log(t^d) \left[|H|\kappa_d^{-1}\left(1 + c(d) - \frac{\pi}{2}\right) - \epsilon\right] + O(t^d). \tag{9.345}$$

It remains thus to show that for $d \in \mathbb{N}$,

$$c(d) > \frac{\pi}{2} - 1 \approx 0.57, \tag{9.346}$$

which will be shown in the following lemma.

Lemma 9.29 (Joint work with Pierre Perruchaud)**.** *In the notation of Lemma 9.28, for $d \in \mathbb{N}$,*

$$c(d) > \frac{\pi}{2} - 1 \approx 0.57. \tag{9.347}$$

Proof. We start by showing a lower bound on $c(d)$. Indeed, for all $x \in (0, 1)$,

$$\frac{1}{1-x} - 1 = \frac{x}{1-x} \geq x. \tag{9.348}$$

Hence

$$\begin{aligned}
&\frac{1}{|B^d(0, r^{2/d}) \cup B^d(z, 1)|} - \frac{1}{\kappa_d(r^2 + 1)} \\
&= \kappa_d^{-1}(r^2+1)^{-1}\left(\frac{1}{1 - \kappa_d^{-1}(r^2+1)^{-1}|B^d(0, r^{2/d}) \cap B^d(z, 1)|} - 1\right) \\
&\geq \kappa_d^{-2}(r^2+1)^{-2}|B^d(0, r^{2/d}) \cap B^d(z, 1)|.
\end{aligned} \tag{9.349}$$

A lower bound for $c(d)$ is thus given by

$$\int_0^\infty dr\ \kappa_d^{-2}(r^2+1)^{-2} \int_{\mathbb{R}^d} dz\ \mathbb{1}_{\{|z| \geq 1\}} |B^d(0, r^{2/d}) \cap B^d(z, 1)|. \tag{9.350}$$

Looking only at the inner integral over z, note that we can rewrite it as follows:

$$\begin{aligned}
\int_{\mathbb{R}^d} dz\ \mathbb{1}_{\{|z|\geq 1\}} |B^d(0, r^{2/d}) \cap B^d(z, 1)| &= \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dx\ \mathbb{1}_{\{|z|\geq 1\}} \mathbb{1}_{\{|x| \leq r^{2/d}\}} \mathbb{1}_{\{|x-z| \leq 1\}} \\
&= \int_{\mathbb{R}^d} dx\ \mathbb{1}_{\{|x| \leq r^{2/d}\}} \int_{\mathbb{R}^d} dz\ \left(\mathbb{1}_{\{|x-z|\leq 1\}} - \mathbb{1}_{\{|x-z|\leq 1\}}\mathbb{1}_{\{|z| \leq 1\}}\right) \\
&= \int_{B^d(0, r^{2/d})} dx\ \left(\kappa_d - |B^d(0,1) \cap B^d(x, 1)|\right).
\end{aligned} \tag{9.351}$$

Plugging this into the lower bound (9.350), we obtain

$$\int_0^\infty dr\ \kappa_d^{-2}(r^2+1)^{-2} \int_{B^d(0, r^{2/d})} dx\ \left(\kappa_d - |B^d(0,1) \cap B^d(x,1)|\right), \tag{9.352}$$

which can be written as

$$\int_0^\infty dr\ (r^2+1)^{-2} r^2 - \tilde{c}(d) = \frac{\pi}{4} - \tilde{c}(d) \tag{9.353}$$

with

$$\tilde{c}(d) := \int_0^\infty dr\ \kappa_d^{-2}(r^2+1)^{-2} \int_{B^d(0,r^{2/d})} dx\ |B^d(0,1)\cap B^d(x,1)|. \tag{9.354}$$

Hence $c(d) \geq \frac{\pi}{4} - \tilde{c}(d)$ and to show that $c(d) \geq \frac{\pi}{2} - 1$, it suffices to show that $\tilde{c}(d) \leq 1 - \frac{\pi}{4} \approx 0.215$. First, note that

$$|B^d(0,1)\cap B^d(x,1)| = 2\kappa_{d-1}\mathbb{1}_{\{|x|\leq 2\}} \int_{\frac{|x|}{2}}^1 dy\ (1-y^2)^{\frac{d-1}{2}}. \tag{9.355}$$

which can be seen by integrating over the $d-1$-dimensional hyperspheres making up the spherical caps of the intersection, or by following the development in [Li11]. Hence $\tilde{c}(d)$ can be written

$$\begin{aligned} \tilde{c}(d) &= 2\kappa_{d-1}\kappa_d^{-2} \int_0^1 dy \int_0^\infty dr \int_{\mathbb{R}^d} dx\ \mathbb{1}_{\{|x|\leq r^{2/d}\wedge 2y\}}(r^2+1)^{-2}(1-y^2)^{\frac{d-1}{2}} \\ &= 2\kappa_{d-1}\kappa_d^{-1} \int_0^1 dy \int_0^\infty dr\ \left(r^2\wedge(2y)^d\right)(r^2+1)^{-2}(1-y^2)^{\frac{d-1}{2}}, \end{aligned} \tag{9.356}$$

where we integrated over x in the second line. For any $u \geq 0$, define

$$g(u) := 2\int_0^\infty dr\ (r^2\wedge u^2)(r^2+1)^{-2} = \frac{\pi}{2}u^2 + (1-u^2)\,\text{arctan}(u) - u. \tag{9.357}$$

Then it can be verified by standard methods that $g(0) = 0$, as well as $\lim_{u\to\infty} g(u) = \frac{\pi}{2}$ and the function is strictly increasing. Moreover, $g'(u) \leq \pi u$, therefore $g(x) \leq \frac{\pi}{2}x^2$.
We now have

$$\tilde{c}(d) = \kappa_{d-1}\kappa_d^{-1} \int_0^1 dy\ (1-y^2)^{\frac{d-1}{2}} g\left((2y)^{d/2}\right). \tag{9.358}$$

Change variables $u = \frac{1-y}{2}$ to get

$$\tilde{c}(d) = 2^d\kappa_{d-1}\kappa_d^{-1} \int_0^{1/2} du\ u^{\frac{d-1}{2}}(1-u)^{\frac{d-1}{2}} g\left(2^{d/2}(1-2u)^{d/2}\right) \tag{9.359}$$

and note that

$$2^d\kappa_{d-1}\kappa_d^{-1} \int_0^x du\ u^{\frac{d-1}{2}}(1-u)^{\frac{d-1}{2}} = I_x\left(\frac{d+1}{2}, \frac{d+1}{2}\right), \tag{9.360}$$

where $I_x(a,b)$ is the regularized incomplete beta function (see [Par22] for details).
Let us now deal with dimensions $d \in \{3, ..., 9\}$. Split the integration interval $[0, \frac{1}{2}]$ into intervals $[\frac{i-1}{40}, \frac{i}{40}]$ for $i \in \{1, 2, ..., 20\}$ and upper bound $g\left(2^{d/2}(1-2u)^{d/2}\right)$ by $g\left(2^{d/2}(1-\frac{i-1}{10})^{d/2}\right)$ for $u \in (\frac{i-1}{40}, \frac{i}{40}]$. We deduce the following bound:

$$\tilde{c}(d) \leq \sum_{i=1}^{20} g\left(\left(2 - \frac{i-1}{10}\right)^{d/2}\right)\left(I_{\frac{i}{40}}\left(\frac{d+1}{2}, \frac{d+1}{2}\right) - I_{\frac{i-1}{40}}\left(\frac{d+1}{2}, \frac{d+1}{2}\right)\right) =: \beta_1(d) \tag{9.361}$$

This results in the values in Table 9.1, all of which are smaller than $1 - \frac{\pi}{4} \approx 0.215$.

d	$\beta_1(d)$
3	0.203
4	0.175
5	0.150
6	0.128
7	0.110
8	0.094
9	0.081

Table 9.1: Values of $\beta_1(d)$

For $d \geq 10$, take $\theta = 0.32$ and split $\tilde{c}(d)$ as follows:

$$\begin{aligned}\tilde{c}(d) = 2^d \kappa_{d-1}\kappa_d^{-1}\Bigg(&\int_0^\theta du\, u^{\frac{d-1}{2}}(1-u)^{\frac{d-1}{2}} \underbrace{g\left(2^{d/2}(1-2u)^{d/2}\right)}_{\leq \frac{\pi}{2}} \\ &+ \int_\theta^{1/2} du\, u^{\frac{d-1}{2}}(1-u)^{\frac{d-1}{2}} \underbrace{g\left(2^{d/2}(1-2u)^{d/2}\right)}_{\leq \pi 2^{d-1}(1-2\theta)^d}\Bigg) \\ \leq \frac{\pi}{2} I_\theta\left(\frac{d+1}{2}, \frac{d+1}{2}\right) &+ \pi 2^{d-1}(1-2\theta)^d I_{1/2}\left(\frac{d+1}{2}, \frac{d+1}{2}\right), \end{aligned} \tag{9.362}$$

where the second line follows by (9.360). Since $I_{1/2}\left(\frac{d+1}{2}, \frac{d+1}{2}\right) = \frac{1}{2}$, it follows that $\tilde{c}(d)$ is upper bounded by

$$\beta_2(d) := \frac{\pi}{2}\left(I_\theta\left(\frac{d+1}{2}, \frac{d+1}{2}\right) + 2^{d-1}(1-2\theta)^d\right). \tag{9.363}$$

Now $\beta_2(10) \approx 0.208 < 1 - \frac{\pi}{4}$. By Proposition 4 in [BC21], for any $x \in (0, \frac{1}{2})$, the function $\alpha \mapsto I_x(\alpha, \alpha)$ is decreasing. Hence $\beta_2(d)$ is decreasing in d and thus for all $d \geq 10$, one has $\beta_2(d) \leq \beta_2(10)$.

For dimension $d = 1$, it is possible to show that $c(1) = \left(\frac{5}{2} - \sqrt{2}\right)\pi - 2\sqrt{2} \approx 0.583 > \frac{\pi}{2} - 1$. For dimension $d = 2$, one can numerically estimate that $c(2) \approx 0.606 > \frac{\pi}{2} - 1$.

The following python code was used to carry out this estimation with an approximate error of 1.5×10^{-8}. It can be used to estimate $c(d)$ at other small values of d.

```
import math
from math import gamma
import scipy as sc
import scipy.special
import scipy.integrate
import numpy as np

# volume(spherical cap)/volume(unit ball)
# r: radius, a: base distance to cap, d: dimension
def capvol(r,a,d):
    if a>=0:
        return 1/2*(r**d)*sc.special.betainc((d+1)/2,1/2,1-a**2/r
            **2) # smaller cap
```

```
        else:
            return r**d-capvol(r,-a,d) #larger cap

    # volume(intersection of balls)/volume(unit ball)
    # x: distance between centres of balls, r1,r2: radii, d:
       dimension
    def vol(x,r1,r2,d):
        if x >= r1+r2: #no intersection
            return 0
        elif x <= abs(r1-r2): #one ball within the other
            return min(r1,r2)**d
        else:
            c1 = (x**2+r1**2-r2**2)/(2*x) #distances to bases of caps
            c2 = (x**2-r1**2+r2**2)/(2*x)
            return capvol(r1,c1,d)+capvol(r2,c2,d) #sum of both
               spherical caps

    # the integrand
    # r,a: variables, d: dimension
    def integr(r,a,d):
        q1 = 1+a**d-vol(r,1,a,d)
        q2 = 1 + a**d
        return 2*d**2/4*r**(d-1)/a**(d/2+1)*(1/q1-1/q2)

    # the constant c(d)
    def cst(d):
        options={'limit':200}
        res = sc.integrate.nquad(lambda a,r:integr(r,a,d),\
            [lambda r:[0,r],[0,np.inf]],opts=[options,options])
        return res #returns the result and the maximal error made
```

■

Proof of Proposition 9.21. Combine Lemmas 9.23 - 9.29. ■

9.5.4 Proof of Theorem 6.1

The inequalities (6.5) and (6.7) are shown in Propositions 9.18, 9.20 and 9.21. In the following, we will write c to indicate the presence of a constant. The value of c might change from line to line, or indeed, within one line.
We have for $0 < \alpha \leq \frac{d}{2}$:

$$|F_t^{(\alpha)}| \leq (\text{diam}(H)t)^\alpha \eta(tH \times [0,1]) \tag{9.364}$$

and using (9.205),

$$|\,\mathrm{D}_{(x,s)}\, F_t^{(\alpha)}| \leq (\text{diam}(H)t)^\alpha (\eta(tH \times [0,1]) + 1), \tag{9.365}$$

hence $F_t^{(\alpha)} \in L^2(\mathbb{P}_\eta) \cap \operatorname{dom} \mathrm{D}$.

Part I: Wasserstein distance when $\alpha = \frac{d}{2}$

We use the bound given in Theorem 4.3 with $(\mathbb{Y}, \bar{\lambda}) = (tH \times [0,1], dx \otimes ds)$ and $p = q = 2$. Recall that, combining Propositions 9.5 and 9.21, we have for $r \geq 1$ the following bounds:

$$\mathbb{E}\left[\mathbb{E}[|\,\mathrm{D}_{(x,s)}\, F_t^{(d/2)}||\eta_{|tH\times[0,s)}]^r\right]^{1/r} \lesssim s^{-1/2} \wedge t^{d/2} \tag{9.366}$$

$$\mathbb{E}\left[\mathbb{E}[|\,\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F_t^{(d/2)}||\eta_{|tH\times[0,s\vee u)}]^4\right]^{1/4} \lesssim (s\vee u)^{-1/2}\exp(-c(s\vee u)|x-y|^d), \tag{9.367}$$

$$\mathrm{Var}\left(F_t^{(d/2)}\right) \gtrsim t^d \log(t), \tag{9.368}$$

where for the second line we used the Cauchy-Schwarz inequality to get

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}[|\,\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F_t^{(d/2)}||\eta_{|tH\times[0,s\vee u)}]^4\right]^{1/4}\\ &\leq \mathbb{E}\left[\mathbb{E}[|\,\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F_t^{(d/2)}||\eta_{|tH\times[0,s\vee u)}]^8\right]^{1/8} \mathbb{P}(\mathbb{E}[|\,\mathrm{D}^{(2)}_{(x,s),(y,u)}\, F_t^{(d/2)}||\eta_{|tH\times[0,s\vee u)}] \neq 0)^{1/2}.\end{aligned} \tag{9.369}$$

We start by plugging the bounds (9.366), (9.367), (9.368) into β_1 from Theorem 4.3. Then

$$\begin{aligned}\beta_1 &\lesssim \left(t^d\log(t)\right)^{-1}\\ &\left(\int_{tH} dx \int_0^1 ds \left(\int_{tH} dy \int_0^1 du\, u^{-1/2}(u\vee s)^{-1/2}\exp\left(-c(u\vee s)|x-y|^d\right)\right)^2\right)^{1/2}.\end{aligned} \tag{9.370}$$

We now change variables as follows:

$$\begin{cases}\tilde{x} = t^{-1}x,\ dx = t^d d\tilde{x}\\ \tilde{y} = t^{-1}y,\ dy = t^d d\tilde{y}\\ \tilde{s} = t^d s,\ ds = t^{-d} d\tilde{s}\\ \tilde{u} = t^d u,\ du = t^{-d} d\tilde{u}\end{cases} \tag{9.371}$$

and deduce that β_1 is bounded by

$$c\log(t)^{-1}\left(\int_H dx \int_0^\infty ds \left(\int_H dy \int_0^\infty du\, u^{-1/2}(u\vee s)^{-1/2}\exp(-c(u\vee s)|x-y|^d)\right)^2\right)^{1/2}. \tag{9.372}$$

This is $O(\log(t)^{-1})$ since the integral is finite. Indeed, changing variables in y and integrating over x, the integral is bounded by

$$|H|\int_0^\infty ds \left(\int_{B^d(0,2\,\mathrm{diam}(H))} dz \int_0^\infty du\, u^{-1/2}(u\vee s)^{-1/2}\exp(-c(u\vee s)|z|^d)\right)^2. \tag{9.373}$$

Integrating over z, this is equal to

$$c\int_0^\infty ds \left(\int_0^\infty du\, u^{-1/2}(u\vee s)^{-3/2}(1-\exp(-c(u\vee s)))\right)^2. \tag{9.374}$$

Now we use that $1 - \exp(-c(u \vee s)) \lesssim 1 \wedge (u \vee s)$ to bound (9.374) by

$$c \int_0^\infty ds \left(\int_0^\infty du\, u^{-1/2} (u \vee s)^{-3/2} (1 \wedge (u \vee s)) \right)^2 . \tag{9.375}$$

Splitting the integral over s and integrating over u, yields that (9.375) is equal to

$$c \int_0^1 ds\, (3 - \log(s))^2 + c \int_1^\infty ds\, s^{-2} < \infty. \tag{9.376}$$

For β_2, we have

$$\beta_2 \lesssim (t^d \log(t))^{-1} \left(\int_{tH} dx \int_0^1 ds \left(\int_{tH} dy \int_s^1 du\, u^{-1} \exp(-cu|x-y|^d) \right)^2 \right)^{1/2} . \tag{9.377}$$

This term can be dealt with in the same way as with β_1 and it is $O(\log(t)^{-1})$.
As for β_3, we get

$$\beta_3 \lesssim (t^d \log(t))^{-3/2} \int_{tH} dy \int_0^1 du\, (u^{-1/2} \wedge t^{d/2})^3. \tag{9.378}$$

Integrating over u and y gives that this is $O(\log(t)^{-3/2})$.
The last term is given by

$$\beta_4 \lesssim (t^d \log(t))^{-3/2} \int_{tH} dx \int_0^1 ds \int_{tH} dy \int_0^s du\, u^{-1/2} s^{-1} \exp(-cs|x-y|^d). \tag{9.379}$$

Proceeding in the same way we dealt with β_1 yields that $\beta_4 = O(\log(t)^{-3/2})$.
Combining our estimates above, we conclude that $\beta_1 + \beta_2 + \beta_3 + \beta_4 = O(\log(t)^{-1})$. ■

Part II: Wasserstein and Kolmogorov distances when $\alpha < \frac{d}{2}$

We use Theorem 4.4 with $p, q \in (1, 2]$ and $\epsilon > 0$ such that $2p(\alpha + \epsilon d) < d$ and $(q+1)(\alpha + \epsilon d) < d$. In the following we will show that all terms $\gamma_1, \ldots, \gamma_7$ are $O(t^{d(1/p-1)})$. By Propositions 9.20, 9.16 and Cauchy-Schwarz inequality, we have for all $r \geq 1$ the following bounds:

$$\mathbb{E}[|\, \mathrm{D}_{(x,s)}\, F_t^{(\alpha)}|^r]^{1/r} \lesssim s^{-\alpha/d-\epsilon} \tag{9.380}$$

$$\mathbb{E}[|\, \mathrm{D}^{(2)}_{(x,s),(y,u)}\, F_t^{(\alpha)}|^r]^{1/r} \lesssim (u \vee s)^{-\alpha/d-\epsilon} \exp\left(-c(u \vee s)|x-y|^d\right) \tag{9.381}$$

$$\mathrm{Var}(F_t^{(\alpha)}) \gtrsim t^d. \tag{9.382}$$

Introducing these bounds into γ_1, we get

$$\gamma_1 \lesssim t^{-d} \Bigg(\int_{tH} dx \int_0^1 ds \Bigg(\int_{tH} dy \int_0^1 du\, u^{-\alpha/d-\epsilon} (u \vee s)^{-\alpha/d-\epsilon} \exp\left(-c(u \vee s)|x-y|^d\right) \Bigg)^p \Bigg)^{1/p} . \tag{9.383}$$

Using a change of variables, we can bound the integral over y as follows:

$$\int_{tH} dy\, \exp(-c(u \vee s)|x-y|^d) \leq \int_{\mathbb{R}^d} dz\, \exp(-c(u \vee s)|z|^d) \lesssim (u \vee s)^{-1}. \tag{9.384}$$

Introducing (9.384) into (9.383) and integrating over x gives us

$$\gamma_1 \lesssim t^{d(1/p-1)} \left(\int_0^1 ds \left(\int_0^1 du\, u^{-\alpha/d-\epsilon} (u \vee s)^{-\alpha/d-\epsilon-1} \right)^p \right)^{1/p} . \tag{9.385}$$

Integrating over u, one sees that these integrals are bounded by

$$c \int_0^1 ds\, s^{-2p(\alpha/d+\epsilon)} < \infty. \tag{9.386}$$

Hence $\gamma_1 = O(t^{d(1/p-1)})$. The term γ_2 is bounded by

$$\gamma_2 \lesssim t^{-d} \left(\int_{tH} dx \int_0^1 ds \left(\int_{tH} dy \int_0^1 du\, (u \vee s)^{-2\alpha/d-2\epsilon} \exp\left(-c(u \vee s)|x-y|^d\right) \right)^p \right)^{1/p} . \tag{9.387}$$

This can be treated analogously to the bound on γ_1 and yields $\gamma_2 = O(t^{d(1/p-1)})$.
As for γ_3, it is bounded by

$$\gamma_3 \lesssim t^{-d(q+1)/2} \int_{tH} dy \int_0^1 du\, u^{-(q+1)(\alpha/d+\epsilon)}. \tag{9.388}$$

This is $O(t^{d(1-q)/2})$. Choose $q = 3 - \frac{2}{p}$, then $(q+1)(\alpha + \epsilon d) \leq 2p(\alpha + \epsilon d) < d$ and $q \in (1, 2]$, hence the conditions are satisfied. Moreover, $\frac{1-q}{2} = -1 + \frac{1}{p}$, thus we find the same rate of convergence. The term γ_4 is bounded by

$$\gamma_4 \lesssim t^{-d} \left(\int_{tH} dx \int_0^1 ds\, s^{-2p(\alpha/d-\epsilon)} \right)^{1/p} , \tag{9.389}$$

which is clearly $O(t^{d(1/p-1)})$. For the term γ_5, we deduce

$$\gamma_5 \lesssim t^{-d} \left(\int_{tH} dx \int_0^1 ds \int_{tH} dy \int_0^1 du\, (u \vee s)^{-2p(\alpha/d+\epsilon)} \exp(-c(u \vee s)|x-y|^d) \right)^{1/p} . \tag{9.390}$$

Integrating over x and y as before, we infer

$$\gamma_5 \lesssim t^{d(1/p-1)} \left(\int_0^1 ds \int_0^1 du\, (u \vee s)^{-2p(\alpha/d+\epsilon)-1} \right)^{1/p} , \tag{9.391}$$

which is $O(t^{d(1/p-1)})$. The terms γ_6 and γ_7 work similarly:

$$\gamma_6 \lesssim t^{-d} \Bigg(\int_{tH} dx \int_0^1 ds \int_{tH} dy \int_0^1 du\, (u \vee s)^{-p(\alpha/d+\epsilon)} s^{-p(\alpha/d+\epsilon)} \exp(-c(u \vee s)|x-y|^d) \Bigg)^{1/p} , \tag{9.392}$$

and

$$\gamma_7 \lesssim t^{-d} \Bigg(\int_{tH} dx \int_0^1 ds \int_{tH} dy \int_0^1 du\, (u \vee s)^{-(\alpha/d+\epsilon)} s^{-(\alpha/d+\epsilon)} u^{-2(\alpha/d+\epsilon)(p-1)} \exp(-c(u \vee s)|x-y|^d) \Bigg)^{1/p} \tag{9.393}$$

which can be shown to be $O(t^{d(1/p-1)})$ by the same method. This concludes the proof of Theorem 6.1. ■

9.6 Proofs for the Gilbert graph (univariate case)

Throughout this section, we work in the framework of Section 6.2 and Theorem 6.3. We start with a technical lemma.

Lemma 9.30. *Let Z be a Poisson random variable with intensity $\lambda > 0$ and let $r \geq 1$. Then there is a constant $c_r > 0$ such that*

$$\mathbb{E}[Z^r]^{1/r} \leq c_r(\lambda \vee \lambda^{1/r}). \tag{9.394}$$

Proof. Let $m \in \mathbb{N}$. As can be found in any standard reference (see e.g. [PT11, Proposition 3.3.2]),

$$\mathbb{E}Z^m = \sum_{i=1}^{m} \left\{ {m \atop i} \right\} \lambda^i, \tag{9.395}$$

where $\left\{ {m \atop i} \right\}$ are the Stirling numbers of second kind. This is bounded by $(\lambda \vee \lambda^m)B_m$, where $B_m = \sum_{i=1}^{m} \left\{ {m \atop i} \right\}$ is the mth Bell number. Hence

$$\mathbb{E}[Z^m]^{1/m} \leq B_m^{1/m}(\lambda \vee \lambda^{1/m}). \tag{9.396}$$

Now let $r > 1$, $r \notin \mathbb{N}$ and define $p_0 := \lfloor r \rfloor$ and $p_1 := \lceil r \rceil$. Then take $\theta := \frac{p_1}{r}\frac{r-p_0}{p_1-p_0} \in (0,1)$ such that we have $\frac{1}{r} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. By log-convexity of L^p norms (see e.g. [Tao22] or [Bre11, Remark 2 p. 93]) and the first part of the proof, we have

$$\mathbb{E}[Z^r]^{1/r} \leq \|Z\|_{p_0}^{1-\theta}\|Z\|_{p_1}^{\theta} \leq B_{p_0}^{(1-\theta)/p_0} B_{p_1}^{\theta/p_1}(\lambda \vee \lambda^{1/r}), \tag{9.397}$$

which provides the desired bound with $c_r := B_{p_0}^{(1-\theta)/p_0} B_{p_1}^{\theta/p_1}$. ■

As a next step, we prove bounds on the first and second order add-one costs of $L_t^{(\alpha)}$.

Proposition 9.31. *Let $\alpha > -\frac{d}{2}$ and $r \geq 1$ such that $d + r\alpha > 0$. Then $L_t^{(\alpha)} \in \operatorname{dom} \mathrm{D}$ and there is a constant $c > 0$ such that for all $x, y \in W$ and $t > 0$,*

$$\mathbb{E}\left[\left(\mathrm{D}_x L_t^{(\alpha)}\right)^r\right]^{1/r} \leq c\epsilon_t^{\alpha}(t\epsilon_t^d)^{1/r}\left(1 \vee t\epsilon_t^d\right)^{1-1/r} \tag{9.398}$$

$$\mathbb{P}(\mathrm{D}_x L_t^{(\alpha)} \neq 0) \leq 1 \wedge \kappa_d t \epsilon_t^d \tag{9.399}$$

$$\mathrm{D}_{x,y}^{(2)} L_t^{(\alpha)} = \mathbb{1}_{\{|x-y|<\epsilon_t\}}|x-y|^{\alpha}. \tag{9.400}$$

Proof. As explained in Chapter 3, to show that $L_t^{(\alpha)} \in \operatorname{dom} \mathrm{D}$ it suffices to argue that $L_t^{(\alpha)} \in L^1(\mathbb{P}_{\eta^t})$ and that (3.3) holds. By [RST17, Theorem 3.1], it is true that $L_t^{(\alpha)} \in L^1(\mathbb{P}_{\eta_t})$ and the fact that $\mathrm{D}_x L_t^{(\alpha)}$ is square-integrable follows from (9.398) with $r = 2$ and from the fact that W is bounded. Hence $L_t^{(\alpha)} \in \operatorname{dom} \mathrm{D}$ follows once we have shown (9.398).
Since $\mathrm{D}_x L_t^{(\alpha)} = L_t^{(\alpha)}(\eta^t + \delta_x) - L_t^{(\alpha)}(\eta^t)$, it is easy to see that

$$\mathrm{D}_x L_t^{(\alpha)} = \sum_{y \in \eta^t_{|W}} \mathbb{1}_{\{|x-y|<\epsilon_t\}}|x-y|^{\alpha}. \tag{9.401}$$

It now follows that

$$\mathrm{D}^{(2)}_{x,y}\, L^{(\alpha)}_t = \mathrm{D}_x\, L^{(\alpha)}_t(\eta^t + \delta_y) - \mathrm{D}_x\, L^{(\alpha)}_t(\eta^t) = \mathbb{1}_{\{|x-y|<\epsilon_t\}}|x-y|^\alpha, \tag{9.402}$$

which gives (9.400). Since all terms in the sum (9.401) are non-negative, we have $\mathrm{D}_x\, L^{(\alpha)}_t \neq 0$ if and only if $\eta^t(W \cap B^d(x,\epsilon_t)) \neq 0$. Therefore

$$\mathbb{P}(\mathrm{D}_x\, L^{(\alpha)}_t \neq 0) = \mathbb{P}(\eta^t(W \cap B^d(x,\epsilon_t)) \neq 0) = 1 - \exp(-t|W \cap B^d(x,\epsilon_t)|) \tag{9.403}$$

which is bounded by $1 \wedge t|W \cap B^d(x,\epsilon_t)| \leq 1 \wedge \kappa_d t \epsilon_t^d$. This gives (9.399).
To prove (9.398), note first that $\mathrm{D}_x\, L^{(\alpha)}_t$ can be written

$$\mathrm{D}_x\, L^{(\alpha)}_t = \int_{W \cap B^d(x,\epsilon_t)} |x-y|^\alpha \eta^t(dy). \tag{9.404}$$

This is smaller than

$$\int_{B^d(x,\epsilon_t)} |x-y|^\alpha \eta^t(dy), \tag{9.405}$$

By translation invariance of the law of η^t, this is equal in law to

$$\int_{B^d(0,\epsilon_t)} |y|^\alpha \eta^t(dy), \tag{9.406}$$

which in turn is equal in law to

$$\sum_{i=1}^{M_t} |U_i|^\alpha, \tag{9.407}$$

where M_t is a Poisson random variable of intensity $\kappa_d t \epsilon_t^d$ and $U_1, U_2, \ldots$ are i.i.d. uniform random variables in $B^d(0,\epsilon_t)$ independent of M_t.
By Jensen's inequality,

$$\mathbb{E}\left(\sum_{i=1}^{M_t} |U_i|^\alpha\right)^r \leq \mathbb{E} M_t^{r-1} \sum_{i=1}^{M_t} |U_i|^{r\alpha}. \tag{9.408}$$

By independence, this is equal to

$$\mathbb{E}\left[M_t^{r-1} \sum_{i=1}^{M_t} \mathbb{E}|U_i|^{r\alpha}\right] = \frac{d}{d+r\alpha} \epsilon_t^{r\alpha} \mathbb{E} M_t^r. \tag{9.409}$$

Using Lemma 9.30, we deduce

$$\mathbb{E}\left[M_t^r\right]^{1/r} \leq c_r \left(\kappa_d t \epsilon_t^d \vee (\kappa_d t \epsilon_t^d)^{1/r}\right). \tag{9.410}$$

Combining the above bounds leads to

$$\mathbb{E}\left[\left(\mathrm{D}_x\, L^{(\alpha)}_t\right)^r\right]^{1/r} \leq c_r(1 \vee \kappa_d) \left(\frac{d}{d+r\alpha}\right)^{1/r} \epsilon_t^\alpha \left(t\epsilon_t^d \vee (t\epsilon_t^d)^{1/r}\right), \tag{9.411}$$

which shows (9.398). ∎

Theorem 3.3 in [RST17] gives us the following variance asymptotics: for $\alpha > -\frac{d}{2}$,

$$\text{Var}\left(L_t^{(\alpha)}\right) = \left(\sigma_\alpha^{(1)} t^2 \epsilon_t^{2\alpha+d} + \sigma_\alpha^{(2)} t^3 \epsilon_t^{2\alpha+2d}\right) |W|(1 + O(\epsilon_t)), \tag{9.412}$$

where $\sigma_\alpha^{(1)} = \frac{d\kappa_d}{2(d+2\alpha)}$ and $\sigma_\alpha^{(2)} = \frac{d^2\kappa_d^2}{(\alpha+d)^2}$. Hence for large enough t, there is a constant $c > 0$ such that

$$\text{Var}\left(L_t^{(\alpha)}\right) \geq ct^2 \epsilon_t^{2\alpha+d} \left(1 \vee t\epsilon_t^d\right). \tag{9.413}$$

We are now in a position to prove Theorem 6.3.

Proof of Theorem 6.3. We plug the bounds from Proposition 9.31 and (9.413) into the terms $\gamma_1, ..., \gamma_7$ given in Theorem 4.4. Let $p \in (1, 2]$ be such that $2p\alpha + d > 0$.
For the choice of $r = 2p$ in (9.398), the first term yields

$$\gamma_1 \lesssim \left(t^2 \epsilon_t^{2\alpha+d} \left(1 \vee t\epsilon_t^d\right)\right)^{-1} \left(\int_W \left(\int_W \epsilon_t^\alpha (t\epsilon_t^d)^{1/(2p)} \left(1 \vee t\epsilon_t^d\right)^{1-1/(2p)} \mathbb{1}_{\{|x-y|<\epsilon_t\}} |x-y|^\alpha t dy\right)^p t dx\right)^{1/p}. \tag{9.414}$$

Note that

$$\int_W \mathbb{1}_{\{|x-y|<\epsilon_t\}} |x-y|^\alpha dy \leq \int_{B^d(x,\epsilon_t)} |x-y|^\alpha dy = \frac{d\kappa_d}{d+\alpha} \epsilon_t^{d+\alpha}. \tag{9.415}$$

Hence γ_1 is (up to multiplication by a positive constant) bounded by

$$t^{1/p-1} (t\epsilon_t^d)^{1/(2p)} (1 \vee t\epsilon_t^d)^{-1/(2p)} = t^{1/p-1} \left(1 \wedge (t\epsilon_t^d)^{1/(2p)}\right). \tag{9.416}$$

As for γ_2, we have

$$\gamma_2 \lesssim \left(t^2 \epsilon_t^{2\alpha+d} \left(1 \vee t\epsilon_t^d\right)\right)^{-1} \left(\int_W \left(\int_W \mathbb{1}_{\{|x-y|<\epsilon_t\}} |x-y|^{2\alpha} t dy\right)^p t dx\right)^{1/p}. \tag{9.417}$$

By (9.415), we deduce

$$\gamma_2 \lesssim t^{1/p-1} \left(1 \vee t\epsilon_t^d\right)^{-1}. \tag{9.418}$$

For γ_3, take $q = 3 - \frac{2}{p}$. Then $q \in (1, 2]$ and $(q+1)\alpha + d > 2p\alpha + d > 0$. Let $r > \frac{q+1}{2}$ such that $2r\alpha + d > 0$. Then, using

$$\mathbb{E}\left(\mathrm{D}_x L_t^{(\alpha)}\right)^{q+1} \leq \mathbb{E}\left[\left(\mathrm{D}_x L_t^{(\alpha)}\right)^{2r}\right]^{\frac{q+1}{2r}} \mathbb{P}(\mathrm{D}_x L_t^{(\alpha)} \neq 0)^{1-\frac{q+1}{2r}} \tag{9.419}$$

we deduce

$$\gamma_3 \lesssim \left(t^2 \epsilon_t^{2\alpha+d} \left(1 \vee t\epsilon_t^d\right)\right)^{1/p-2} \int_W \left(\epsilon_t^\alpha (t\epsilon_t^d)^{1/(2r)} \left(1 \vee t\epsilon_t^d\right)^{1-1/(2r)}\right)^{4-2/p} \left(1 \wedge t\epsilon_t^d\right)^{1-\frac{4-2/p}{2r}} t dx. \tag{9.420}$$

Simplifying, we infer

$$\gamma_3 \lesssim t^{1/p-1} (1 \vee (t\epsilon_t^d)^{1/p-1}). \tag{9.421}$$

With the same method, one can establish the upper bounds

$$\gamma_4 \lesssim t^{1/p-1} \left(1 \vee (t\epsilon_t^d)^{1/p-1}\right) \tag{9.422}$$

$$\gamma_5 \lesssim t^{1/p-1} (t\epsilon_t^d)^{1/p-1} \left(1 \vee t\epsilon_t^d\right)^{-1} \tag{9.423}$$

$$\gamma_6 \lesssim t^{1/p-1} (t\epsilon_t^d)^{1/p-1} \left(1 \wedge (t\epsilon_t^d)^{1/(2p)}\right) \tag{9.424}$$

$$\gamma_7 \lesssim t^{1/p-1} (t\epsilon_t^d)^{2/p-1-1/(2p^2)} \left(1 \vee (t\epsilon_t^d)\right)^{-2/p+1+1/(2p^2)} \tag{9.425}$$

All of these bounds are upper bounded by $t^{1/p-1}\left(1 \vee (t\epsilon_t^d)^{1/p-1}\right)$. If $\alpha > -\frac{d}{4}$, we can choose $p = 2$ and recover (6.11). To show (6.12), one chooses $1 < p < -\frac{d}{2\alpha}$, thus concluding the proof. ■

9.7 Proofs for the Gilbert graph (multivariate case)

Throughout this section, we work in the framework of Section 6.2 and Chapter 7. By c we denote a positive absolute constant whose value can change from line to line.

Proof of Theorem 7.1. It suffices to give bounds to the terms $\zeta_1, ..., \zeta_4$ of Theorem 4.6. We start by giving a bound to ζ_1. Note that is has been shown in [RST17, Thm. 3.3] that

$$|W| - \beta_W \epsilon_t < \frac{\mathrm{Cov}\left(L_t^{(\alpha_i)}, L_t^{(\alpha_j)}\right)}{\sigma_{ij}^{(1)} t^2 \epsilon_t^{\alpha_i+\alpha_j+d} + \sigma_{ij}^{(2)} t^3 \epsilon_t^{\alpha_i+\alpha_j+2d}} \leq |W|, \tag{9.426}$$

where β_W is a constant depending on W such that (9.182) holds. This implies that

$$\beta_{ij}^{(t)} \left(|W| - \beta_W \epsilon_t\right) < \mathrm{Cov}\left(\tilde{L}_t^{(\alpha_i)}, \tilde{L}_t^{(\alpha_j)}\right) \leq \beta_{ij}^{(t)} |W|, \tag{9.427}$$

where

$$\beta_{ij}^{(t)} := \frac{\sigma_{ij}^{(1)} t^2 \epsilon_t^{\alpha_i+\alpha_j+d} + \sigma_{ij}^{(2)} t^3 \epsilon_t^{\alpha_i+\alpha_j+2d}}{\left(t\epsilon_t^{\alpha_i+d/2} \vee t^{3/2}\epsilon_t^{\alpha_i+d}\right)\left(t\epsilon_t^{\alpha_j+d/2} \vee t^{3/2}\epsilon_t^{\alpha_j+d}\right)} = \frac{\sigma_{ij}^{(1)} + \sigma_{ij}^{(1)} t\epsilon_t^d}{1 \vee t\epsilon_t^d}. \tag{9.428}$$

Hence we have

$$\left|\mathrm{Cov}\left(\tilde{L}_t^{(\alpha_i)}, \tilde{L}_t^{(\alpha_j)}\right) - C_{ij}\right| \leq \beta_{ij}^{(t)} \beta_W \epsilon_t + \left||W|\beta_{ij}^{(t)} - C_{ij}\right| \tag{9.429}$$

and thus

$$\zeta_1 \leq \sum_{i,j=1}^{m} \left(\beta_{ij}^{(t)} \beta_W \epsilon_t + \left||W|\beta_{ij}^{(t)} - C_{ij}\right|\right) \leq c\left(\epsilon_t + \max_{1\leq i,j\leq m}\left|\beta_{ij}^{(t)} - c_{ij}\right|\right) \tag{9.430}$$

for some constant $c > 0$.
Next, we bound ζ_2. For this we use the bounds established in Proposition 9.31. We deduce that

$$\zeta_2 = 2^{2/p-1} \sum_{i,j=1}^{m} \left(t\epsilon_t^{\alpha_i+d/2} \vee t^{3/2}\epsilon_t^{\alpha_i+d}\right)^{-1} \left(t\epsilon_t^{\alpha_j+d/2} \vee t^{3/2}\epsilon_t^{\alpha_j+d}\right)^{-1} \left(\int_W \left(\int_W \mathbb{1}_{\{|x-y|<\epsilon_t\}} |x-y|^{\alpha_i+\alpha_j} t dx\right)^p t dy\right)^{1/p}. \tag{9.431}$$

Note that the inner integral is upper bounded by

$$\frac{d\kappa_d}{d+\alpha_i+\alpha_j} \epsilon_t^{d+\alpha_i+\alpha_j}, \tag{9.432}$$

and hence we deduce, after simplification,

$$\zeta_2 \leq 2^{2/p-1} |W|^{1/p} \left(\sum_{i,j=1}^{m} \frac{d\kappa_d}{d+\alpha_i+\alpha_j}\right) t^{-1+1/p} \left(1 \vee t\epsilon_t^d\right)^{-1}. \tag{9.433}$$

For ζ_3, after plugging in the bounds from Proposition 9.31, we get

$$\zeta_3 \leq 2^{2/p} c \sum_{i,j=1}^{m} \left(t\epsilon_t^{\alpha_i + d/2} \vee t^{3/2}\epsilon_t^{\alpha_i+d} \right)^{-1} \left(t\epsilon_t^{\alpha_j + d/2} \vee t^{3/2}\epsilon_t^{\alpha_j+d} \right)^{-1}$$
$$\left(\int_W \left(\int_W \epsilon_t^{\alpha_i} (t\epsilon_t^d)^{1/(2p)} (1 \vee t\epsilon_t^d)^{1-1/(2p)} \mathbb{1}_{\{|x-y|<\epsilon_t\}} |x-y|^{\alpha_j} t dx \right)^p t dy \right)^{1/p}. \quad (9.434)$$

After simplification, this bound yields

$$\zeta_3 \leq 2^{2/p} c |W|^{1/p} m \left(\sum_{i=1}^{m} \frac{d\kappa_d}{d+\alpha_i} \right) t^{-1+1/p} \left(1 \wedge t\epsilon_t^d\right)^{1/(2p)}. \quad (9.435)$$

For ζ_4, we plug in the bound (9.398) and deduce

$$\zeta_4 \leq m^{q-1} c^2 \sum_{i,j=1}^{m} \left(t\epsilon_t^{\alpha_i + d/2} \vee t^{3/2}\epsilon_t^{\alpha_i+d} \right)^{-1} \left(t\epsilon_t^{\alpha_j + d/2} \vee t^{3/2}\epsilon_t^{\alpha_j+d} \right)^{-q}$$
$$\int_W \epsilon_t^{\alpha_i} (t\epsilon_t^d)^{1/(q+1)} (1 \vee t\epsilon_t^d)^{1-1/(q+1)} \epsilon_t^{q\alpha_j} (t\epsilon_t^d)^{q/(q+1)} (1 \vee t\epsilon_t^d)^{q-q/(q+1)} t dx, \quad (9.436)$$

which, after simplification, yields

$$\zeta_4 \leq m^q c^2 |W| t^{(1-q)/2} (1 \wedge t\epsilon_t^d)^{(1-q)/2}. \quad (9.437)$$

If we take, as in the proof of Theorem 6.3, $q = 3 - \frac{2}{p}$, we get

$$\zeta_2 + \zeta_3 + \zeta_4 \leq c \left(t^{-1+1/p} \vee (t^2\epsilon_t^d)^{-1+1/p} \right), \quad (9.438)$$

which concludes the proof. ■

Proof of Theorem 7.2. As a first step, we compute the asymptotic covariance matrix of the vector $\tilde{F}_t$. Define the functions

$$h_i(x,y) := \mathbb{1}_{\{|x-y|<\epsilon_t\}} \mathbb{1}_{\{x \in W_i\}} \mathbb{1}_{\{y \in W_i\}} |x-y|^{\alpha}, \quad i = 1, \dots, m. \quad (9.439)$$

Then it holds that

$$F_t^{(i)} = \frac{1}{2} \iint_{(\mathbb{R}^d)^2} h_i(x,y) (\eta^t)^{(2)}(dx, dy) \quad (9.440)$$

and hence, for $i, j \in \{1, \dots, m\}$,

$$\mathrm{Cov}\left(F_t^{(i)}, F_t^{(j)}\right) = \frac{1}{4}\mathbb{E}\left[\iint_{(\mathbb{R}^d)^2} h_i(x,y)(\eta^t)^{(2)}(dx,dy) \iint_{(\mathbb{R}^d)^2} h_j(z,w)(\eta^t)^{(2)}(dz,dw) \right]$$
$$- \frac{1}{4}\mathbb{E}\left[\iint_{(\mathbb{R}^d)^2} h_i(x,y)(\eta^t)^{(2)}(dx,dy) \right] \mathbb{E}\left[\iint_{(\mathbb{R}^d)^2} h_j(z,w)(\eta^t)^{(2)}(dz,dw) \right]. \quad (9.441)$$

In the first term on the RHS of (9.441), it is possible to have $x = z$ or $x = w$ or similar equalities, which constitute the diagonals of the sets we are summing over. Using Mecke formula (3.7) and isolating these diagonals, one sees that

$$\begin{aligned}\text{Cov}\left(F_t^{(i)}, F_t^{(j)}\right) &= \frac{1}{4}\iint_{(\mathbb{R}^d)^2} h_i(x,y)t^2\,dxdy \cdot \iint_{(\mathbb{R}^d)^2} h_j(z,w)t^2\,dzdw \\ &+ \iiint_{(\mathbb{R}^d)^3} h_i(x,y)h_j(y,w)t^3\,dxdydw \\ &+ \frac{1}{2}\iint_{(\mathbb{R}^d)^2} h_i(x,y)h_j(x,y)t^2\,dxdy \\ &- \frac{1}{4}\iint_{(\mathbb{R}^d)^2} h_i(x,y)t^2\,dxdy \cdot \iint_{(\mathbb{R}^d)^2} h_j(z,w)t^2\,dzdw. \end{aligned} \tag{9.442}$$

The first and the last term cancel, thus we are left with

$$\text{Cov}\left(F_t^{(i)}, F_t^{(j)}\right) = \iiint_{(\mathbb{R}^d)^3} h_i(x,y)h_j(y,w)t^3\,dxdydw + \frac{1}{2}\iint_{(\mathbb{R}^d)^2} h_i(x,y)h_j(x,y)t^2\,dxdy. \tag{9.443}$$

We start by computing the first term on the RHS of (9.443). We have

$$\begin{aligned}&\iiint_{(\mathbb{R}^d)^3} h_i(x,y)h_j(y,w)t^3\,dxdydw \\ &\qquad = t^3\int_{W_i\cap W_j} dy \int_{W_i} dx \int_{W_j} dw \mathbb{1}_{\{x\in B(y,\epsilon_t)\}}\mathbb{1}_{\{w\in B(y,\epsilon_t)\}}|x-y|^\alpha|w-y|^\alpha. \end{aligned}\tag{9.444}$$

Recall from (9.180) the definition of the (possibly empty) inner parallel set

$$W_\epsilon := \{z \in W : \text{dist}(x, \partial W) > \epsilon\}, \tag{9.445}$$

for $\epsilon > 0$. Recall also from (9.182) that there is a constant $\beta_W > 0$ such that

$$|W \setminus W_\epsilon| \le \beta_W \epsilon. \tag{9.446}$$

We can now rewrite (9.444) as

$$t^3 \iiint_{(\mathbb{R}^d)^3} h_i(x,y)h_j(y,w)\,dxdydw = t^3\int_{(W_i\cap W_j)_{\epsilon_t}} dy \left(\int_{B(y,\epsilon_t)} dx\,|x-y|^\alpha\right)^2 + R_t, \tag{9.447}$$

where R_t is given by

$$R_t := t^3\int_{(W_i\cap W_j)\setminus(W_i\cap W_j)_{\epsilon_t}} dy \int_{W_i} dx \int_{W_j} dw \mathbb{1}_{\{x\in B(y,\epsilon_t)\}}\mathbb{1}_{\{w\in B(y,\epsilon_t)\}}|x-y|^\alpha|w-y|^\alpha. \tag{9.448}$$

The first term on the RHS of (9.447) is given by

$$t^3\int_{(W_i\cap W_j)_{\epsilon_t}} dy \left(\int_{B(y,\epsilon_t)} dx\,|x-y|^\alpha\right)^2 = |(W_i\cap W_j)_{\epsilon_t}| \left(\frac{d\kappa_d}{d+\alpha}\right)^2 t^3 \epsilon_t^{2d+2\alpha}. \tag{9.449}$$

Using (9.446), one sees that

$$\begin{aligned}0 \le R_t &\le t^3\,|(W_i\cap W_j)\setminus(W_i\cap W_j)_{\epsilon_t}| \left(\int_{B(y,\epsilon_t)} dx\,|x-y|^\alpha\right)^2 \\ &\le \beta_{W_i\cap W_j} t^3 \epsilon_t^{1+2d+2\alpha}\left(\frac{d\kappa_d}{d+\alpha}\right)^2. \end{aligned}\tag{9.450}$$

Combining (9.447), (9.449) and (9.450) with (9.446), one sees that

$$\left|t^3 \iiint_{(\mathbb{R}^d)^3} h_i(x,y)h_j(y,w)\,dxdydw - |W_i \cap W_j| \left(\frac{d\kappa_d}{d+\alpha}\right)^2 t^3 \epsilon_t^{2d+2\alpha}\right| \\ \leq \beta_{W_i\cap W_j} t^3 \epsilon_t^{1+2d+2\alpha} \left(\frac{d\kappa_d}{d+\alpha}\right)^2. \quad (9.451)$$

For the second term in (9.443), we proceed similarly. We have

$$\tfrac{1}{2}t^2 \iint_{(\mathbb{R}^d)^2} h_i(x,y)h_j(x,y)\,dxdy = \tfrac{1}{2}t^2 \iint_{W_i\cap W_j} \mathbb{1}_{\{|x-y|<\epsilon_t\}}|x-y|^{2\alpha}dxdy \quad (9.452)$$

$$= \tfrac{1}{2}t^2 \int_{(W_i\cap W_j)_{\epsilon_t}} dy \int_{B(y,\epsilon_t)} dx|x-y|^{2\alpha} + R_t', \quad (9.453)$$

where

$$0 \leq R_t' \leq \tfrac{1}{2}t^2 \beta_{W_i\cap W_j} \frac{d\kappa_d}{d+2\alpha}\epsilon_t^{d+2\alpha+1}. \quad (9.454)$$

Hence we get

$$\left|\tfrac{1}{2}t^2 \iint_{(\mathbb{R}^d)^2} h_i(x,y)h_j(x,y)\,dxdy - \tfrac{1}{2}|W_i\cap W_j| \frac{d\kappa_d}{d+2\alpha} t^2\epsilon^{d+2\alpha}\right| \\ \leq \tfrac{1}{2}\beta_{W_i\cap W_j}\frac{d\kappa_d}{d+2\alpha}t^2\epsilon^{d+2\alpha+1}. \quad (9.455)$$

From (9.443), (9.451) and (9.455), we deduce that

$$\left|\frac{\operatorname{Cov}\left(F_t^{(i)}, F_t^{(j)}\right)}{\frac{1}{2}\frac{d\kappa_d}{d+2\alpha}t^2\epsilon_t^{d+2\alpha} + \left(\frac{d\kappa_d}{d+\alpha}\right)^2 t^3\epsilon_t^{2d+2\alpha}} - |W_i\cap W_j|\right| \leq \beta_{W_i\cap W_j}\epsilon_t. \quad (9.456)$$

Now we use Theorem 4.6 and provide bounds on the terms $\zeta_1, ..., \zeta_4$. For the term ζ_1, we have by (9.456) that

$$\zeta_1 \leq \beta_{W_i\cap W_j}\epsilon_t. \quad (9.457)$$

Plugging in the bounds from Proposition 9.31, we get for ζ_2 that

$$\zeta_2 \leq 2^{2/p-1}ct^{-2}\epsilon_t^{-2\alpha-d}\left(1\vee t\epsilon_t^d\right)^{-1} \\ \sum_{i,j=1}^m \left(\int_{W_i\cap W_j}\left(\int_{W_i\cap W_j}\mathbb{1}_{\{|x-y|<\epsilon_t\}}|x-y|^{2\alpha}tdx\right)^p tdy\right)^{1/p}. \quad (9.458)$$

Simplifying, we deduce

$$\zeta_2 \leq ct^{-1+1/p}(1\vee t\epsilon_t^d)^{-1}\sigma_1 \sum_{i,j=1}^m |W_i\cap W_j|^{1/p}. \quad (9.459)$$

We proceed in the same way for ζ_3 and ζ_4, deducing

$$\zeta_3 \leq 2^{2/p}cmt^{-1+1/p}(1\wedge t\epsilon_t^d)^{1/(2p)}\frac{d\kappa_d}{d+\alpha}\sum_{i=1}^m |W_i|^{1/p} \quad (9.460)$$

and

$$\zeta_4 \leq m^{q-1} c t^{(1-q)/2} (1 \wedge t\epsilon_t^d)^{(1-q)/2} \sum_{i,j=1}^{m} |W_i \cap W_j|. \tag{9.461}$$

If we take, as in the proof of Theorem 6.3, $q = 3 - \frac{2}{p}$, we get

$$\zeta_2 + \zeta_3 + \zeta_4 \leq c \left(t^{-1+1/p} \vee (t^2 \epsilon_t^d)^{-1+1/p} \right), \tag{9.462}$$

which concludes the proof. ∎

9.8 Proofs for the k-Nearest Neighbour Graphs

In this section, we work in the setting of Theorem 6.5. We start with a technical lemma.

Lemma 9.32. *Let $\phi : (0, \infty) \to (0, \infty)$ be a non-increasing function satisfying (6.15) for some $r > 2$. Then for any $0 \leq q \leq r$, the following two integrals are finite:*

$$\int_0^1 \phi(s)^q s^{d-1} ds < \infty \tag{9.463}$$

and for any constant $c > 0$,

$$\int_{\mathbb{R}^d} \phi(|x|)^q \exp(-c|x|^d) dx < \infty. \tag{9.464}$$

Moreover, for $x \in \mathbb{R}^d$ and $\mu \subset \mathbb{R}^d$ a finite generic set with respect to x, define the following quantity:

$$e(x, \mu) := \begin{cases} 0 & \text{if } \mu = \emptyset \\ \min\{|x - z| : z \in \mu, z \neq x\} & \text{if } \mu \neq \emptyset, \end{cases} \tag{9.465}$$

(that is, $e(x, \mu)$ is the length from x to the point of μ nearest to it, or zero if μ is empty). Extend the definition of ϕ to $[0, \infty)$ by setting $\phi(0) = 0$. Then for any $0 < q \leq r$, there is a constant $c > 0$ such that for all $x \in tH$, $t \geq 1$,

$$\mathbb{E} \left[\phi(e(x, \eta_{|tH}))^q \right]^{1/q} < c. \tag{9.466}$$

Proof. We start by noting that $ds^{d-1} \mathbb{1}_{\{0<s<1\}}$ is a probability density. Hence, by Jensen's inequality,

$$\left(d \int_0^1 \phi(s)^q s^{d-1} ds \right)^{1/q} \leq \left(d \int_0^1 \phi(s)^r s^{d-1} ds \right)^{1/r}. \tag{9.467}$$

This is finite by virtue of condition (6.15), thus yielding (9.463). Using polar coordinates, the integral in (9.464) is equal to

$$d\kappa_d \int_0^\infty \phi(s)^q \exp(-cs^d) s^{d-1} ds. \tag{9.468}$$

Now use that ϕ is non-increasing to infer that, if $s \geq 1$, then $\phi(s) \leq \phi(1)$. Hence (9.468) is bounded by

$$d\kappa_d \left(\int_0^1 \phi(s)^q s^{d-1} ds + \int_1^\infty \phi(1)^q s^{d-1} \exp(-cs^d) ds \right). \tag{9.469}$$

The second integral is clearly finite and the first is finite by (9.463), thus implying (9.464). For the bound (9.466), note that by Jensen's inequality

$$\mathbb{E}\left[\phi(e(x,\eta_{|tH}))^q\right]^{1/q} \le \mathbb{E}\left[\phi(e(x,\eta_{|tH}))^r\right]^{1/r}, \tag{9.470}$$

therefore it suffices to show the bound for $q = r$.
Take $x \in tH$ and let us study the distribution of $e(x, \eta_{|tH})$. For any $a \ge 0$, we have

$$G(a) := \mathbb{P}(e(x,\eta_{|tH}) \le a) = \mathbb{P}(e(x,\eta_{|tH}) = 0) + \mathbb{P}(0 < e(x,\eta_{|tH}) \le a). \tag{9.471}$$

The event $e(x,\eta_{|tH}) = 0$ happens if and only if $\eta(tH) = 0$, whereas $0 < e(x,\eta_{|tH}) \le a$ is equivalent to the event that $\eta(tH \cap \overline{B}^d(x,a)) \ne 0$. Hence

$$G(a) = \mathbb{P}(\eta(tH) = 0) + \mathbb{P}(\eta(tH \cap \overline{B}^d(x,a)) \ne 0) = \exp(-|tH|) + 1 - \exp(-|tH \cap B^d(x,a)|). \tag{9.472}$$

We can compute the derivative of $1 - \exp(-|tH \cap B^d(x,a)|)$ as follows:

$$\frac{d}{da}(1 - \exp(-|tH \cap B^d(x,a)|)) = \exp(-|tH \cap B^d(x,a)|)\frac{d}{da}|tH \cap B^d(x,a)|. \tag{9.473}$$

Note that, by changing variables and moving to polar coordinates, we can rewrite the volume as

$$\begin{aligned} |tH \cap B^d(x,a)| &= \int_{\mathbb{R}^d} \mathbb{1}_{\{y \in tH\}} \mathbb{1}_{\{|y-x|<a\}} dy \\ &= \int_{\mathbb{R}^d} \mathbb{1}_{\{x+z \in tH\}} \mathbb{1}_{\{|z|<a\}} dz \\ &= \int_0^a du \int_{S^{d-1}} d\omega\, u^{d-1} \mathbb{1}_{\{x+u\omega \in tH\}}, \end{aligned} \tag{9.474}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$. Define thus

$$g(a) := \exp(-|tH \cap B^d(x,a)|) \int_{S^{d-1}} a^{d-1} \mathbb{1}_{\{x+a\omega \in tH\}} d\omega, \tag{9.475}$$

then

$$G(a) = \mathbb{P}(e(x,\eta_{|tH}) = 0) + \int_0^a g(u)du. \tag{9.476}$$

Going back to the bound we want to prove, we now have

$$\begin{aligned} \mathbb{E}\left[\phi(e(x,\eta_{|tH}))^r\right]^{1/r} &= \left(\phi(0)\mathbb{P}(e(x,\eta_{|tH}) = 0) + \int_0^\infty \phi(u)^r g(u)du\right)^{1/r} \\ &= \left(\int_0^\infty \phi(u)^r g(u)du\right)^{1/r}. \end{aligned} \tag{9.477}$$

We need to show that this integral is bounded by a constant independent of t. Since H has non-empty interior, there is a ball $B^d(x_0,\delta) \subset H$, with $x_0 \in H$ and $\delta > 0$. By Lemma 9.7 and rescaling, there is a constant $c_H > 0$ depending on H and d such that $|tH \cap B^d(x,a)| \ge c_H a^d$. Thus, upper bounding the indicator by 1, we find

$$g(a) \le d\kappa_d a^{d-1} \exp(-c_H a^d). \tag{9.478}$$

As a consequence

$$\left(\int_0^\infty \phi(u)^r g(u) du\right)^{1/r} \leq \left(d\kappa_d \int_0^\infty \phi(u)^r u^{d-1} \exp\left(-c_H u^d\right) du\right)^{1/r}, \tag{9.479}$$

and the RHS of this inequality is finite by (9.464). ■

Proposition 9.33. *Under the conditions of Theorem 6.5, there are absolute constants $C_2, c_2 > 0$ such that for any $t \geq 1$, any $x, y \in tH$, the following bound holds:*

$$\mathbb{P}\left(\mathrm{D}^{(2)}_{x,y} F_t \neq 0\right) < C_2 \exp\left(-c_2|x-y|^d\right). \tag{9.480}$$

For any $1 \leq p \leq \frac{r}{2}$ there is a constant $c_1 > 0$ such that for any $t \geq 1$, any $x, y \in tH$, the following two bounds hold:

$$\mathbb{E}\left[|\,\mathrm{D}_x F_t|^{2p}\right]^{1/(2p)} < c_1 \tag{9.481}$$

$$\mathbb{E}\left[|\,\mathrm{D}^{(2)}_{x,y} F_t|^{2p}\right]^{1/(2p)} < c_1\left(\phi(|x-y|)+1\right) \tag{9.482}$$

Moreover, $F_t \in \operatorname{dom} \mathrm{D}$.

The proof of (9.480) reuses arguments from [LPS16, Theorem 7.1].

Proof. **Step 1.** We start by showing that for any $x \in \mathbb{R}^d$ and any finite set $\mu \in \mathbb{R}^d$ generic with respect to x, we have $\mathrm{D}_x F(\mu) \geq 0$.
First, note that

$$F(\mu) = \sum_{e \in k\text{-NN}(\mu)} \phi(|e|), \tag{9.483}$$

where we take the sum over all edges e in the k-NN graph. Define a 'direction' on the graph in the following way:

- If a point w is a nearest neighbour of the point z, direct the edge from z to w and write $z \to w$;
- if z and w are reciprocal nearest neighbours, direct the edge both ways and write $z \leftrightarrow w$.

Upon addition of the point x, any of the following scenarios can happen:

1. An edge $z \to w$ is replaced by $z \to x$ (or $z \leftrightarrow x$), in which case $|z-w| > |z-x|$;
2. an edge $z \leftrightarrow w$ is replaced by $z \to x$ (or $z \leftrightarrow x$) and $w \to x$ (or $w \leftrightarrow x$), in which case $|z-w| > |z-x| \vee |w-x|$;
3. an edge $z \leftrightarrow w$ becomes $w \to z$ and the edge $z \to x$ (or $z \leftrightarrow x$) is added;
4. edges $x \to z$ are added.

Let E_a and E_r be the sets of added and removed edges respectively. Every removed edge is replaced by at least one added edge with shorter length. All other added edges increase $\mathrm{D}_x F(\mu)$. Since ϕ is decreasing, we have thus

$$\mathrm{D}_x F(\mu) = \sum_{e \in E_a} \phi(|e|) - \sum_{e \in E_r} \phi(|e|) \geq 0. \tag{9.484}$$

Step 2. We now prove (9.481). Fix $x \in \mathbb{R}^d$ and $\mu \subset \mathbb{R}^d$, a finite set generic with respect to x, and consider the k-NN built on μ. Define $e(x, \mu)$ as in Lemma 9.32.
Suppose that $\mu \neq \emptyset$ and consider $\mathrm{D}_x F(\mu)$. There is a constant $n_{d,k}$ such that for any k-NN graph in $\mathbb{R}^d$, no vertex has degree more than $n_{d,k}$. This fact was used in the proof of [LPS16, Lemma 7.2] and we refer to the references given therein for more details. When adding x to the graph, all added edges are incident to x, and hence we have $|E_a| \leq n_{d,k}$. Every added edge between points x and y must verify $|x - y| \geq e(x, \mu)$, since $e(x, \mu)$ is the minimally possible edge-length for any edge incident to x. Since ϕ is decreasing, we conclude

$$|\mathrm{D}_x F(\mu)| = \mathrm{D}_x F(\mu) \leq |E_a| \phi(e(x, \mu)) \leq n_{d,k} \phi(e(x, \mu)). \tag{9.485}$$

Note that this is well-defined since $e(x, \mu) > 0$ if $\mu \neq \emptyset$.
If $\mu = \emptyset$, then $\mathrm{D}_x F(\mu) = 0$. Extend the definition of ϕ to $[0, \infty)$ by setting $\phi(0) = 0$. Then in all cases

$$|\mathrm{D}_x F(\mu)| \leq n_{d,k} \phi(e(x, \mu)). \tag{9.486}$$

We have thus the following bound:

$$\mathbb{E}\left[|\mathrm{D}_x F_t|^{2p}\right]^{1/(2p)} \leq n_{d,k} \mathbb{E}\left[\phi(e(x, \eta_{|tH}))^{2p}\right]^{1/(2p)}, \tag{9.487}$$

which is bounded by a constant by Lemma 9.32.
Step 3. We now show (9.482). By Step 2, we have

$$\begin{aligned} |\mathrm{D}^{(2)}_{x,y} F_t| &\leq |\mathrm{D}_x F(\eta_{|tH} + \delta_x)| + |\mathrm{D}_x F(\eta_{|tH})| \\ &\leq n_{d,k} \phi(e(x, \eta_{|tH})) + n_{d,k} \phi(e(x, \eta_{|tH} + \delta_y)). \end{aligned} \tag{9.488}$$

For ease of notation, define the event $A_t := \{\eta(tH) \neq 0\}$. If $\eta(tH) = 0$, then $e(x, \eta_{|tH} + \delta_y) = |x - y|$, else $e(x, \eta_{|tH} + \delta_y) = e(x, \eta_{|tH}) \wedge |x - y|$. Hence if we split over both events, we get

$$\begin{aligned} \phi(e(x, \eta_{|tH} + \delta_y)) &= \mathbb{1}_{A_t^c} \phi(|x - y|) + \mathbb{1}_{A_t} \phi(e(x, \eta_{|tH}) \wedge |x - y|) \\ &\leq \mathbb{1}_{A_t^c} \phi(|x - y|) + \mathbb{1}_{A_t} \phi(|x - y|) + \mathbb{1}_{A_t} \phi(e(x, \eta_{|tH})) \\ &= \phi(|x - y|) + \mathbb{1}_{A_t} \phi(e(x, \eta_{|tH})). \end{aligned} \tag{9.489}$$

Hence we have

$$|\mathrm{D}^{(2)}_{x,y} F_t| \leq 2 n_{d,k} \phi(e(x, \eta_{|tH})) + n_{d,k} \phi(|x - y|). \tag{9.490}$$

Lemma 9.32 now yields the result.
Step 4. The inequality (9.480) follows immediately from the argument used in the proof of [LPS16, Theorem 7.1], which relies solely on the structure of the graph, and not on the function applied to the edge-lengths.
Step 5. Lastly, we show that $F_t \in \mathrm{dom}\, \mathrm{D}$. As was explained in Chapter 3, it suffices to show that F_t is integrable and $\mathrm{D}_x F_t$ square integrable. The second fact immediately follows from (6.15). By Mecke equation we also have

$$\mathbb{E} F_t = \int_{tH} \int_{tH} \phi(|x - y|) \mathbb{P}(x \in N(y, \eta_{|tH}) \text{ or } y \in N(x, \eta_{|tH})) dx dy, \tag{9.491}$$

where we recall that $N(y, \eta_{|tH})$ is the set of the k nearest neighbours of y in $\eta_{|tH}$. Upper bounding the probability in the integrand by 1 and changing variables, this is upper bounded by

$$t^d|H| \int_{B^d(0,t\,\mathrm{diam}(H))} \phi(|z|)dz. \tag{9.492}$$

Changing to polar coordinates, this is equal to

$$t^d|H|d\kappa_d \int_0^{t\,\mathrm{diam}(H)} s^{d-1}\phi(s)ds, \tag{9.493}$$

which is finite by Lemma 9.32. ■

Proposition 9.34. *Under the conditions of Theorem 6.5, there are constants* $c_1, c_2 > 0$ *such that*

$$c_1 t^d \leq \mathrm{Var}(F_t) \leq c_2 t^d. \tag{9.494}$$

Proof. The upper bound immediately follows from Poincaré inequality ((5.7) with $p = 2$) and (9.481) with $p = 1$. For the lower bound, we use [LPS16, Theorem 5.2] and a reasoning similar to what was done in the proof of [LPS16, Lemma 7.2].
Assume there are points $w_1, ..., w_m \in \mathbb{R}^d$ with $\frac{1}{2} < |w_i| < 1$ for $i \in \{1, ..., m\}$ such that for all $y \in \mathbb{R}^d$ with $|y| \geq \frac{1}{2}$,

$$|\{i \in \{1, ..., m\} : |w_i - y| < |y|\}| \geq k + 1. \tag{9.495}$$

This means that for any point y outside $B^d\left(0, \frac{1}{2}\right)$, there are at least $k + 1$ points among the $\{w_1, ..., w_m\}$ which are closer to y than the origin.
Now assume that there is $\tau > 0$ and $x \in tH$ such that $B^d(x, \tau) \in tH$. Define $\tilde{w}_i := x + \tau w_i$. Consider the collection of points

$$\mathcal{U} := \eta_{|tH} + \sum_{i=1}^{m} \delta_{\tilde{w}_i} \tag{9.496}$$

and let us evaluate $\mathrm{D}_x F(\mathcal{U})$.
First, note that by (9.495), all points outside $B^d(x, \tau/2)$ have at least k points closer than x among the $\tilde{w}_1, ..., \tilde{w}_m$, including these points themselves. Therefore any points in $\mathcal{U}$ connecting to x must be within $B^d(x, \tau/2)$.
Assume $\eta(B^d(x, \tau)) = 0$. Then the only edges added when adding x are those from x to its k nearest neighbours among $\tilde{w}_1, ..., \tilde{w}_m$. Since $|x - \tilde{w}_i| \leq \tau$ for any i, we have:

$$\mathbb{1}_{\{\eta(B^d(x,\tau))=0\}} \mathrm{D}_x F(\mathcal{U}) \geq \mathbb{1}_{\{\eta(B^d(x,\tau))=0\}} k\phi(\tau). \tag{9.497}$$

As shown in Step 1 of the proof of Proposition 9.33, we have $\mathrm{D}_x F(\mathcal{U}) \geq 0$ and

$$|\mathbb{E}\,\mathrm{D}_x F(\mathcal{U})| \geq \mathbb{E}\left[\mathbb{1}_{\{\eta(B^d(x,\tau))=0\}} k\phi(\tau)\right] = k\phi(\tau)\exp(-\kappa_d \tau^d). \tag{9.498}$$

We now find a set of $(x, \tilde{w}_1, ..., \tilde{w}_m)$ for which this bound is true.
Let $\tau > 0$ be such that there is a ball $B^d(x_0, 2\tau) \subset H$. For any $x \in B^d(tx_0, t\tau)$ we have $B^d(x, \tau) \subset tH$.
The closure of the set $B^d(0,1) \setminus B^d\left(0, \frac{1}{2}\right)$ is compact and can be covered by balls of radius $\frac{1}{4}$. Setting $k + 1$ points into the interior of each intersection of one such ball with the annulus $B^d(0,1) \setminus B^d\left(0, \frac{1}{2}\right)$ gives a collection of points $\{z_1, ..., z_m\}$. We claim that this collection satisfies (9.495). Indeed, any point inside $B^d(0,1) \setminus B^d\left(0, \frac{1}{2}\right)$ will be in one ball of the covering and

thus there are at least $k+1$ points from $\{z_1, ..., z_m\}$ at a distance of at most $\frac{1}{2}$. For any point y outside $B^d(0,1)$, there is a point $z \in \partial B^d(0,1)$ such that $|y-z| = |y|-1$ and this point has $k+1$ points z_i among $\{z_1, ..., z_m\}$ that are less than $\frac{1}{2}$ away. For any such z_i, one has $|y-z_i| \leq |y|-1+\frac{1}{2}$. Given a choice of $z_1, ..., z_m$, property (9.495) still holds if we slightly perturb the z_i: there is an $\epsilon > 0$ such that the collection of points $\{z_1+y_1, ..., z_m+y_m\}$ satisfies (9.495) for any $y_1, ..., y_m \in B^d(0,\epsilon)$.
The bound (9.498) is true for any $(x, \tilde{w}_1, ..., \tilde{w}_m) \in U$, where

$$U := \left\{(z, z+\tau(z_1+y_1), ..., z+\tau(z_m+y_m)) : z \in B^d(tx_0, t\tau), y_1, ..., y_m \in B^d(0,\epsilon)\right\}. \tag{9.499}$$

By [LPS16, Theorem 5.2] and a development analogous to the one in the proof of [LPS16, Theorem 5.3], we find that for some constant $c > 0$,

$$\operatorname{Var}(F_t) \geq c|U| = c\kappa_d \tau^d (\kappa_d \epsilon^d)^m t^d, \tag{9.500}$$

which yields the desired bound. ■

Proof of Theorem 6.5. For the rest of the proof, all constants will be referred to as c, to simplify notation. Take $p \in (1,2]$ such that $p < \frac{r}{2}$, where r is given by the condition (6.15).
Let us start with a bound on γ_1. We use that

$$\mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F_t|^{2p}\right]^{1/(2p)} \leq \mathbb{P}\left(\operatorname{D}^{(2)}_{x,y} F_t \neq 0\right)^{1/(2p)-1/r} \mathbb{E}\left[|\operatorname{D}^{(2)}_{x,y} F_t|^{r}\right]^{1/r} \tag{9.501}$$

and the bounds in Propositions 9.33 and 9.34 to conclude that

$$\gamma_1 \lesssim t^{-d}\left(\int_{tH}\left(\int_{tH} \exp(-c|x-y|^d)(\phi(|x-y|)+1)dy\right)^p dx\right)^{1/p}. \tag{9.502}$$

After changing variables and extending the domain of integration to $\mathbb{R}^d$, the inner integral is upper bounded by

$$\int_{\mathbb{R}^d} \exp(-c|y|^d)(\phi(|y|)+1)dy, \tag{9.503}$$

which is finite by Lemma 9.32. We deduce that

$$\gamma_1 \lesssim t^{d(1/p-1)}. \tag{9.504}$$

The terms $\gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ can be shown to be $O(t^{d(1/p-1)})$ by applying the same strategy.
For γ_3, take $q = 3 - \frac{2}{p}$. Then $q+1 < 2p < r$ and $\mathbb{E}|\operatorname{D}_y F_t|^{q+1} \leq \mathbb{E}\left[|\operatorname{D}_y F_t|^r\right]^{(q+1)/r}$ and we have

$$\gamma_3 \lesssim t^{-d(2-1/p)} \int_{tH} \mathbb{E}\left[|\operatorname{D}_y F_t|^r\right]^{(q+1)/r} dx \lesssim t^{d(1/p-1)}. \tag{9.505}$$

To show that in particular this bound is true for the function $\phi(x) = x^{-\alpha}$ with $0 < \alpha < \frac{d}{2}$, it suffices to check condition (6.15). Indeed, let $2 < r < \frac{d}{\alpha}$. Then

$$\int_0^1 s^{-\alpha r} s^{d-1} ds < \infty \tag{9.506}$$

and the bound on the speed of convergence holds for any $p < \frac{r}{2} < \frac{d}{2\alpha}$, with $p \in (1,2]$. ■

9.9 Proofs for the Radial Spanning Tree

In this section, we work in the setting of Theorem 6.6 and we call finite sets $\mu \subset \mathbb{R}^d$ generic only if μ is generic with respect to 0.

Proposition 9.35. *There are absolute constants $c_1, C_1 > 0$ (independent of ϕ) such that for any $t \geq 1$ and any $x, y \in tH$, the following bound holds:*

$$\mathbb{P}\left(\mathrm{D}^{(2)}_{x,y} F_t \neq 0\right) \leq C_1 \exp(-c_1|x-y|^d). \tag{9.507}$$

Moreover, for any $1 \leq p \leq \frac{r}{2}$, there are constants $c_2, C_2 > 0$ such that for any $t \geq 1$ and any $x, y \in tH$, the following bounds hold:

$$\mathbb{E}\left[|\,\mathrm{D}_x F_t|^{2p}\right]^{1/(2p)} \leq C_2\left(1 + \phi(|x|)\exp(-c_2|x|^d)\right) \tag{9.508}$$

$$\mathbb{E}\left[|\,\mathrm{D}^{(2)}_{x,y} F_t|^{2p}\right]^{1/(2p)} \leq C_2\left(\phi(|x-y|)\exp(-c_2|x-y|^d) + 1\right) \tag{9.509}$$

We also have that $F_t \in \operatorname{dom}\mathrm{D}$.

The proof relies on and reuses some arguments from the proofs of Lemmas 4.1 and 4.2 in [ST17].

Proof. The bound (9.507) follows immediately from the proof of Lemma 4.2 in [ST17]. Indeed, the proof given in [ST17] does not depend on the chosen functional, but only on the structure of the graph.
Next, we establish that for any $x \in \mathbb{R}^d$ and $\mu \subset \mathbb{R}^d$ finite and generic with respect to x, we have $\mathrm{D}_x F(\mu) \geq 0$. Indeed, it is true that

$$\mathrm{D}_x F(\mu) = \phi(g(x,\mu)) + \sum_{y\in\mu}\left(\phi(g(y,\mu+\delta_x)) - \phi(g(y,\mu))\right). \tag{9.510}$$

Since $0 < g(y, \mu+\delta_x) \leq g(y,\mu)$ and ϕ is non-increasing, the above expression is non-negative.
For the next part, define $e(x,\mu)$ as we did in Lemma 9.32 and extend ϕ by setting $\phi(0) = 0$. Then note that the summand on the RHS of (9.510) is zero, unless y connects to x, in which case $g(y,\mu+\delta_x) = |x-y|$. It follows that

$$0 \leq \mathrm{D}_x F(\mu) \leq \phi(g(x,\mu)) + \sum_{y\in\mu} \mathbb{1}_{\{y\to x \text{ in } \mu+\delta_x\}}\phi(|x-y|). \tag{9.511}$$

If μ is non-empty, then $|x-y| \geq e(x,\mu)$, hence the second term on the RHS of (9.510) is upper bounded by

$$\phi(e(x,\mu))\sum_{y\in\mu}\mathbb{1}_{\{y\to x \text{ in } \mu+\delta_x\}}, \tag{9.512}$$

a bound that continues to hold if $\mu = \emptyset$.
As for the first term on the RHS of (9.510), if $\mu \cap B^d(0,|x|) \cap B^d(x,|x|)$ is empty, then the term is equal to $\phi(|x|)$. If not, then $g(x,\mu) \geq e(x,\mu)$ and it is upper bounded by $\phi(e(x,\mu))$. Hence we deduce the bound

$$\mathrm{D}_x F(\mu) \leq \phi(|x|)\mathbb{1}_{\{\mu\cap B^d(0,|x|)\cap B^d(x,|x|)\neq\emptyset\}} + \phi(e(x,\mu))\left(1 + \sum_{y\in\mu}\mathbb{1}_{\{y\to x \text{ in } \mu+\delta_x\}}\right). \tag{9.513}$$

It follows by Hölder's and Minkowski's inequalities that

$$\mathbb{E}\left[(\mathrm{D}_x F(\mu))^{2p}\right]^{1/(2p)} \leq \phi(|x|)\mathbb{P}(\eta(tH \cap B^d(0,|x|) \cap B^d(x,|x|)) = 0)^{1/(2p)} \\ + \mathbb{E}\left[\phi(e(x,\eta_{|tH}))^r\right]^{1/r} \mathbb{E}\left[\left(1 + \sum_{y\in\mu} \mathbb{1}_{\{y\to x \text{ in } \mu+\delta_x\}}\right)^m\right]^{1/m}, \quad (9.514)$$

where $m = \lceil \frac{2pr}{r-2p} \rceil$.
We start with the first term on the RHS of (9.514). We know that

$$\mathbb{P}(\eta(tH \cap B^d(0,|x|) \cap B^d(x,|x|)) = 0) = \exp(-|tH \cap B^d(0,|x|) \cap B^d(x,|x|)|). \quad (9.515)$$

Note that $B^d(\frac{x}{2}, \frac{|x|}{2}) \subset B^d(0,|x|) \cap B^d(x,|x|)$, therefore

$$\left|tH \cap B^d(0,|x|) \cap B^d(x,|x|)\right| \geq \left|tH \cap B^d\left(\frac{x}{2}, \frac{|x|}{2}\right)\right|. \quad (9.516)$$

Since H is non-empty, there is a ball $B^d(x_0,\delta) \subset H$ with $x_0 \in H$ and $\delta > 0$. By scaling and use of Lemma 9.7, we conclude that there is a constant $c_H > 0$ depending on H and d such that

$$|tH \cap B^d\left(\tfrac{x}{2}, \tfrac{|x|}{2}\right)| \geq c_H |x|^d. \quad (9.517)$$

Hence

$$\phi(|x|)\mathbb{P}\left(\eta(tH \cap B^d(0,|x|) \cap B^d(x,|x|)) = 0\right)^{1/(2p)} \leq \phi(|x|) \exp\left(-\tfrac{c_H}{2p}|x|^d\right). \quad (9.518)$$

For the second term on the RHS of (9.514), we use that $\mathbb{E}\left[\phi(e(x,\eta_{|tH}))^r\right]^{1/r}$ is bounded by a constant independent of t and x, by Lemma 9.32. For the other part of the second term in (9.514), one can easily adapt the argument in the proof of [ST17, Lemma 4.1] to show that for any $m \in \mathbb{N}$,

$$\mathbb{E}\left[\left(1 + \sum_{y\in\mu} \mathbb{1}_{\{y\to x \text{ in } \mu+\delta_x\}}\right)^m\right]^{1/m} \quad (9.519)$$

is uniformly bounded by a constant. This concludes the proof of (9.508).
Now we prove (9.509). For $x, y \in \mathbb{R}^d$ and $\mu \subset \mathbb{R}^d$ finite generic with respect to x, y, we have

$$\mathrm{D}^{(2)}_{x,y} F(\mu) = \phi(g(x,\mu+\delta_y)) + \sum_{z\in\mu} (\phi(|x-z|) - \phi(g(z,\mu+\delta_y))) \mathbb{1}_{\{z\to x \text{ in } \mu+\delta_y+\delta_x\}} \\ + (\phi(|x-y|) - \phi(g(y,\mu))) \mathbb{1}_{\{y\to x \text{ in } \mu+\delta_y+\delta_x\}} \\ - \phi(g(x,\mu)) - \sum_{z\in\mu} (\phi(|x-z|) - \phi(g(z,\mu))) \mathbb{1}_{\{z\to x \text{ in } \mu+\delta_x\}}. \quad (9.520)$$

This expression is in fact symmetric in x and y since the operator $\mathrm{D}^{(2)}_{x,y}$ is symmetric. Assume without loss of generality that $|x| \leq |y|$. Then x cannot connect to y and hence $\phi(g(x,\mu+\delta_y)) = \phi(g(x,\mu))$. The point y will connect to x if and only if $|x-y| < |y|$ and $\mu \cap B^d(0,|y|) \cap B^d(y,|x-y|) = \emptyset$, thus

$$(\phi(|x-y|) - \phi(g(y,\mu))) \mathbb{1}_{\{y\to x \text{ in } \mu+\delta_y+\delta_x\}} \\ = \phi(|x-y|)\mathbb{1}_{\{|x-y|<|y|\}} \mathbb{1}_{\{\mu\cap B^d(0,|y|)\cap B^d(y,|x-y|)=\emptyset\}}. \quad (9.521)$$

Moreover, a point z that connects to x in $\mu+\delta_y+\delta_x$ also connects to x in $\mu+\delta_x$ and in this case $\phi(|x-z|)\geq\phi(g(z,\mu+\delta_y))$. We deduce that

$$\sum_{z\in\mu}(\phi(|x-z|)-\phi(g(z,\mu+\delta_y)))\,\mathbb{1}_{\{z\to x\text{ in }\mu+\delta_y+\delta_x\}}\leq\sum_{z\in\mu}\phi(|x-z|)\mathbb{1}_{\{z\to x\text{ in }\mu+\delta_x\}}. \tag{9.522}$$

Lastly, if z connects to x in $\mu+\delta_x$, then $\phi(|x-z|)\geq\phi(g(z,\mu))$, thus

$$\sum_{z\in\mu}(\phi(|x-z|)-\phi(g(z,\mu)))\,\mathbb{1}_{\{z\to x\text{ in }\mu+\delta_x\}}\leq\sum_{z\in\mu}\phi(|x-z|)\mathbb{1}_{\{z\to x\text{ in }\mu+\delta_x\}}. \tag{9.523}$$

Combining (9.521), (9.522) and (9.523), we deduce from (9.520) that

$$\left|\mathrm{D}^{(2)}_{x,y}F_t\right|\leq\phi(|x-y|)\mathbb{1}_{\{|x-y|<|y|\}}\mathbb{1}_{\{\eta(tH\cap B^d(0,|y|)\cap B^d(y,|x-y|))=0\}}+2\sum_{z\in\eta_{|tH}}\phi(|x-z|)\mathbb{1}_{\{z\to x\text{ in }\eta\}}. \tag{9.524}$$

The proof of (9.508) shows that the $2p$th moment of the second term is uniformly bounded. When $|x-y|<|y|$, then $B^d\left(y(1-\frac{|x-y|}{2|y|}),\frac{|x-y|}{2}\right)$ is included in the intersection $B^d(0,|y|)\cap B^d(y,|x-y|)$. By Lemma 9.7, and rescaling, we have thus for some constant $c_H>0$ that

$$\begin{aligned}&\mathbb{P}(\eta(tH\cap B^d(0,|y|)\cap B^d(y,|x-y|))=0)\\&\qquad=\exp(-|tH\cap B^d(0,|y|)\cap B^d(y,|x-y|)|)\leq\exp(-c_H|x-y|^d).\end{aligned} \tag{9.525}$$

The bound on the $2p$th moment of the first term in (9.524) follows.
The fact that $\mathrm{D}_x F_t$ is square-integrable follows from the bound (9.508) and Lemma 9.32. To show that $F_t\in\operatorname{dom}\mathrm{D}$, it suffices to show that $F_t\in L^1(\mathbb{P}_\eta)$, as was explained in Chapter 3. We apply Mecke equation and the fact that $g(x,\eta_{|tH}+\delta_x)=g(x,\eta_{|tH})$ to deduce that

$$\mathbb{E}F_t=\int_{tH}\mathbb{E}\phi(g(x,\eta_{|tH}))dx. \tag{9.526}$$

As shown in the discussion leading to (9.513), this is bounded by

$$\int_{tH}\left(\phi(|x|)\mathbb{P}\left(\eta(tH\cap B^d(0,|x|)\cap B^d(x,|x|))=0\right)+\phi(e(x,\mu))\right)dx. \tag{9.527}$$

By (9.518) and Lemma 9.32, this integral is finite. ■

Proposition 9.36. *Under the conditions above, there are constants $c_1,c_2>0$ such that for all $t\geq 1$ large enough*

$$c_1t^d\leq\operatorname{Var}F_t\leq c_2t^d. \tag{9.528}$$

Proof. The upper bound follows from (9.508) with $p=1$ and similar integration arguments as in the proof of Proposition 9.35. For the lower bound, we use Theorems 5.2 and 5.3 in [LPS16] again.
Recall that $B^d(0,\epsilon)\subset H$ and let $\delta<\frac{\epsilon}{4}$. Fix $t\geq 1$ and define the set

$$U:=\{(x,z):x\in tH\setminus B^d(0,4\delta),z\in B^d((1-\tfrac{\delta}{|x|})x,\delta)\}. \tag{9.529}$$

Note that $B^d\left(\left(1-\frac{\delta}{|x|}\right)x,\delta\right) \subset B^d(0,|x|)\cap B^d(x,|x|)$, therefore any $z\in B^d((1-\frac{\delta}{|x|})x,\delta)$ verifies $|x-z|<|x|$. Moreover, $|x-z|\le \left|x-z-\frac{\delta}{|x|}x\right|+\left|\frac{\delta}{|x|}x\right|\le 2\delta$.
Now for $(x,z)\in U$, consider $\mathrm{D}_x\, F(\eta_{|tH}+\delta_z)$. As can be seen from (9.510), we have

$$\mathrm{D}_x\, F(\eta_{|tH}+\delta_z)\ge \phi(g(x,\eta_{|tH}+\delta_z)). \tag{9.530}$$

By our choice of z, we have $g(x,\eta_{|tH}+\delta_z)\le |x-z|\le 2\delta$. As ϕ is decreasing, it follows that $\phi(g(x,\eta_{|tH}+\delta_z))\ge\phi(2\delta)$. By [LPS16, Theorem 5.2],

$$\mathrm{Var}(F_t)\ge \frac{\phi(2\delta)^2}{4^3\cdot 2}\min_{\emptyset\ne J\subset\{1,2\}}\inf_{\substack{V\subset U\\ \lambda(V)\ge\lambda(U)/8}}\lambda^{(|J|)}(\Pi_J(V)), \tag{9.531}$$

where λ is the Lebesgue measure.
By the reasoning in the proof of [LPS16, Theorem 5.3], this quantity is lower bounded by $\lambda(U)$ up to multiplication by a constant. We have

$$\lambda(U)=\kappa_d\delta^d|tH\setminus B^d(0,4\delta)|=\kappa_d\delta^d(t^d|H|-\kappa_d(4\delta)^d), \tag{9.532}$$

which is of order t^d for t large enough. ■

In the development below, most constants will be called c for convenience. These constants are by no means the same and can (and will) change from line to line or indeed, within lines.

Proof of Theorem 6.6. We apply the bounds found in Propositions 9.35 and 9.36 to the terms γ_1,...,γ_7 in Theorem 4.4. Let $p\in(1,2]$ such that $2p<r$ and choose $s>0$ such that $2p<s<r$. Then by Hölder's inequality and (9.509) and (9.507),

$$\mathbb{E}\left[|\mathrm{D}^{(2)}_{x,y}\, F_t|^{2p}\right]^{1/(2p)}\lesssim \exp(-c|x-y|^d)\mathbb{E}\big[|\mathrm{D}^{(2)}_{x,y}\, F_t|^s\big]^{1/s}\lesssim \exp(-c|x-y|^d)(1+\phi(|x-y|)). \tag{9.533}$$

With this bound, we find that γ_1 is bounded by

$$\gamma_1\lesssim t^{-d}\left(\int_{tH}\left(\int_{tH}\left(1+\phi(|x|)\exp(-c|x|^d)\right)\exp(-c|x-y|^d)\left(1+\phi(|x-y|)\right)dy\right)^p dx\right)^{1/p}. \tag{9.534}$$

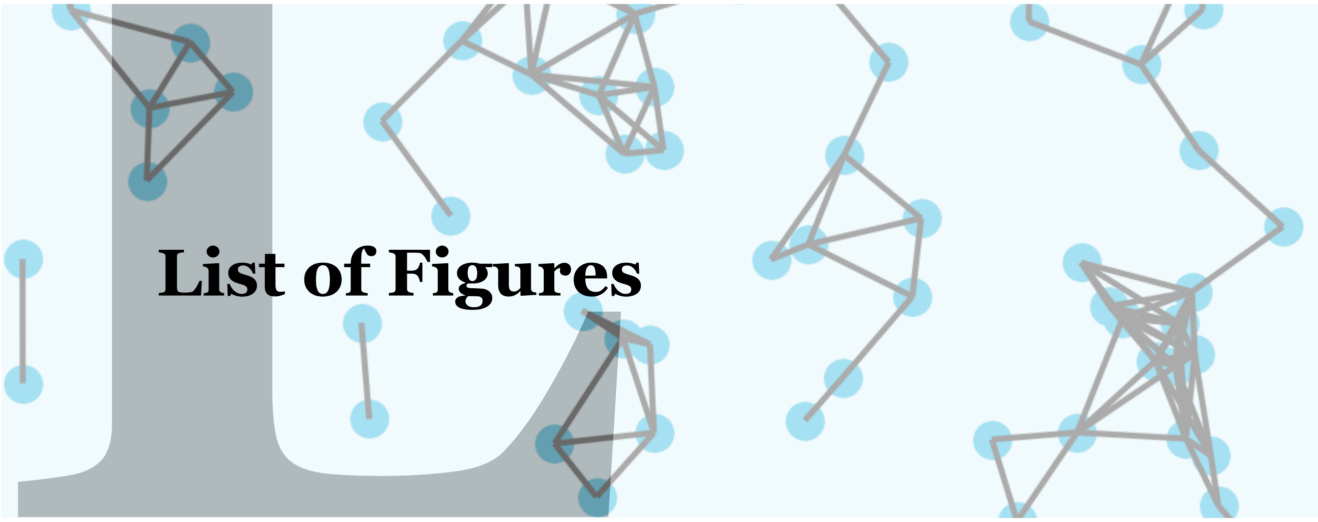
Using Lemma 9.32, we find $\gamma_1=O(t^{d(1/p-1)})$. The same way, we have $\gamma_2=O(t^{d(1/p-1)})$. For γ_3, take $q=3-\frac{2}{p}$, then $q+1<2p<s$ and $\mathbb{E}|\mathrm{D}_y\, F_t|^{q+1}\le\mathbb{E}\left[|\mathrm{D}_y\, F_t|^s\right]^{(q+1)/s}$. It follows that

$$\gamma_3\lesssim t^{-d(2-1/p)}\int_{tH}(1+\phi(|x|)\exp(-c|x|^d))^{q+1}dx, \tag{9.535}$$

which is of order $O(t^{d(1/p-1)})$. The terms γ_4,γ_5 and γ_6 work much the same. For γ_7 we use the simplified version proposed in Remark 4.5 and get

$$\gamma_7\lesssim t^{-d}\left(\int_{tH}\int_{tH}\exp(-c|x-y|^d)(\phi(|x-y|)+1)(1+\phi(|x|)\exp(-c|x|^d))^{2p-1}dxdy\right)^{1/p}. \tag{9.536}$$

By the same methods as applied before, this is $O(t^{d(1/p-1)})$.
The claim that the bound (6.20) is true in particular for $\phi(s)=s^{-\alpha}$ with $0<\alpha<\frac{d}{2}$ follows from the discussion at the end of the proof of Theorem 6.5. ■



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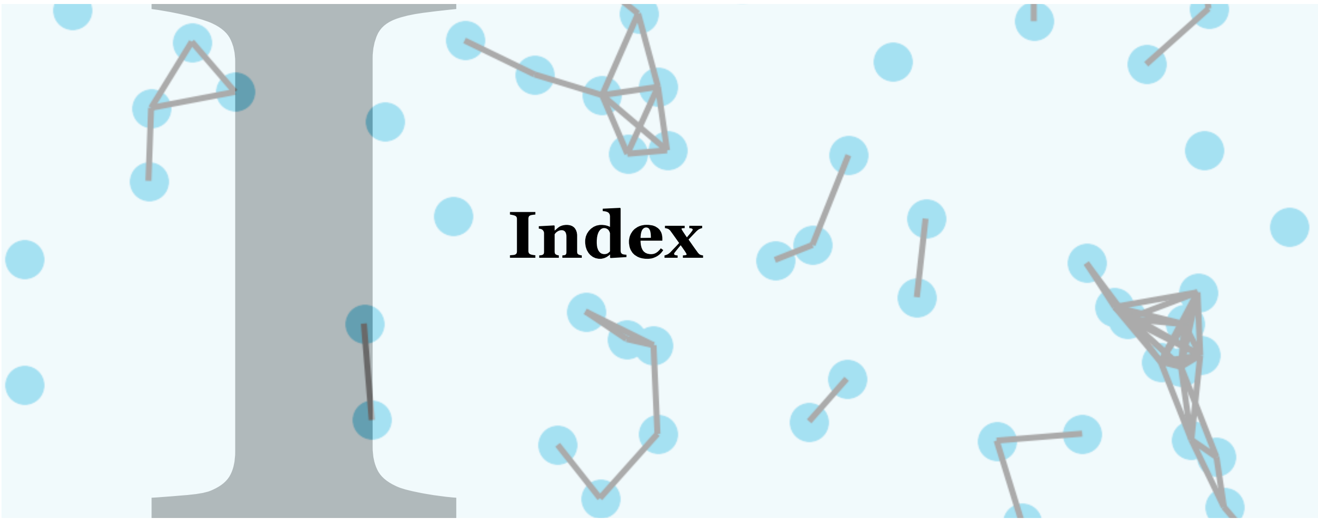

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