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Optimal margin requirement

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ABSTRACT

We investigate the optimal level of margin requirement in centralized or decentralized clearing and settlement systems. We prove in an analytical model that aggressively risk-sensitive margins are not optimal, and the actual position of the clients and the general macroeconomic conditions, like overall funding liquidity should also be taken into consideration. For example, in a crisis period characterized by high volatility, the clearing institution may be tempted to require larger margins to cover potential losses, however, clients just have difficulties to finance higher margins, hence, the probability of non-payment increases, and the expected loss of the clearing institution may also increase. We show that under realistic specifications, there exists a unique optimal margin minimizing the expected loss of the clearing institution. Characterizing this optimum, we provide a micro-level foundation for anti-cyclical risk management techniques (margins, collaterals, haircuts etc.) in general.

1. Introduction

Margin requirement serves as a collateral for risky transactions and is designed to cover the potential losses of the period between two settlements at a high significance level. The settlement of trading positions can be operated either by a central counterparty, or in a decentralized network through a distributed ledger technology (Csóka and Herings, 2017; Benos et al., 2017). In smart settlement systems, it is possible to shorten the trade-to-settlement time, hence to reduce the exposure to counterparty risk, however, in any case, margining remains a relevant risk management tool (Khapko and Zoican, 2018).

Originally, margin requirements were determined as a fixed percentage of the initial value of the transaction, but later, with the development of information technologies and in line with financial regulation promoting micro-level "prudency", margins became more and more risk-sensitive, i.e. in volatile periods margin requirements increased and vice versa. It turned out, however, that risk-sensitive margins aggravate the clients' situation in crisis periods, hence they have strong pro-cyclical effects (Danielsson et al., 2001, 2013). Now, in the aftermath of the crisis of 2007–2008, financial regulators propagate the introduction of anti-cyclical risk management techniques (capital buffer, collaterals, haircuts etc.) including anti-cyclical margin algorithms, as well (EMIR, 2012).

Several authors discussed optimal margin algorithms by empirical analysis or simulation techniques (Figlewski, 1984; Brennan, 1986; Fenn and Kupiec, 1993; Shanker and Balakrishnan, 2005; Koeppl et al., 2012; and Loon and Zhong, 2014). In this paper, we develop an analytical model to investigate the optimal margin level with respect to the confronting requirements of micro-level prudency and anti-cyclicality. The model can be considered as a combination and generalization of (Merton, 1974) and the Credit Portfolio View developed by McKinsey and described in (Crouhy et al., 2000) which is now adapted to the margin setting problem. In our model, the optimal margin minimizes the expected counterparty loss and takes into account the effects of margin requirements on

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the behavior of the counterparty, as well. We find that the optimal margin is not mechanically adjusted to the expected volatility as for example Value-at-Risk or Expected Shortfall based methods would suggest but considers also the client's actual account balance and the overall funding liquidity situation in the economy, because these factors determine the counterparty's willingness and ability-to-pay, respectively. Our findings constitute a strong foundation for anti-cyclical margin policies required by EMIR (2012).

In Section 2, after setting the assumptions, we derive formulas for the conditional and unconditional losses of the clearing system as a function of the margin. In Section 3, we show that there is a unique optimal choice of the margin to minimize the unconditional loss. We do not get an explicit formula for this optimal margin, but we establish all important characteristics of its behavior. In Section 4, we extend our investigation to some other specifications. Finally, in Section 5, we derive conclusions.

2. The model

We develop a general and theoretical model of margin requirements taking account of the major factors and trade-offs and derive the optimal margin level minimizing counterparty risk. The model is a combination and generalization of Merton (1974) and the Credit Portfolio View developed by McKinsey and described in Crouhy et al. (2000) as default is conditional on the value of a stochastic underlying asset and is triggered above or below a certain threshold like in Merton (1974); and it is a discrete-time model where default probabilities depend on macroeconomical factors related to credit cycles in the economy like in the McKinsey model (Crouhy et al., 2000).

Assumption 1. (Futures price)

The clearing and settlement system is operating in a futures market of a single asset where the settlement period is one day. The closing futures price F is given exogenously; hence, the trading activity has no effect on the price evolution (no price impact). The price change ΔF is a stochastic variable and has a normal distribution $N(0, \sigma)$ with a mean $\mu = 0$ and a standard deviation σ.

Assumption 2. *(Margin account)*

We consider one representative client who is supposed to have a long futures position that will expire the next day. Today evening, $t = 0$, the positions are settled at a closing price of F_0, and the client has a balance $A_0 \in \mathbb{R}$ on his account. Now, the system has to decide on the margin requirement valid for the next day, thus, M is the collateral the client has to deposit in his account to cover the counterparty risk arising from its position. The client is required to pay the difference of the margin and his actual account balance $M - A_0$ (if the difference is negative, the client can withdraw this amount). After the next day's settlement, at $t = 1$, the account balance of the client A_1 is a normally distributed stochastic variable and its expected value depends on whether the client fulfilled the margin requirement at $t = 0$ or not:

$$A_1 = \begin{cases} M + \Delta F & \text{if the requirement is fulfilled} \\ A_0 + \Delta F & \text{if the requirement is not fulfilled} \end{cases} \tag{1}$$

It follows from Assumptions 1 and 2 that A_1 is a stochastic variable, too, and has a normal distribution with a mean M *or* A_0 and a standard deviation σ.

The timing of the events is summarized in Fig. 1.

Fig. 1 shows that at the end of day 0, the clearing system calculates the account balance of the client (A_0) according to the exogeneously given futures settlement price (F_0) and determines the margin requirement for the next day (M). The client decides whether he pays or not, and a new day starts. The settlement price at the end of the next day (F_1) is again an exogenous random variable which determines the account balance (A_1) and if it is negative, the client may decide to pay or not, and in the latter case, a loss is realized by the clearing system.

Assumption 3. *(The loss generating process)*

The client has a limited liability, i.e. cannot be forced to pay more than he has on his account. The probability that the client does not fulfill the margin requirement at $t = 0$ is denoted by P_0. In the case the client pays, we are in the upward branch of the tree and the account balance at the beginning of day 1 is M. Otherwise, if the client does not pay the margin, we are on the downward branch of the tree, the account balance of the client at the beginning of day 1 is only A_0, and the client's position will be liquidated at the settlement price F_1 of the next day. The clearing system realizes a loss at $t = 1$ if the account balance A_1 is negative and the client

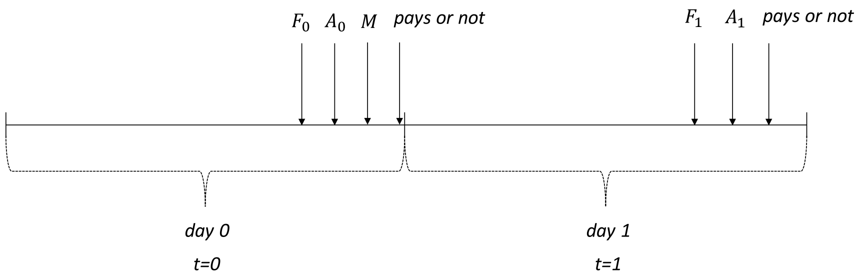

Fig. 1. Timing.

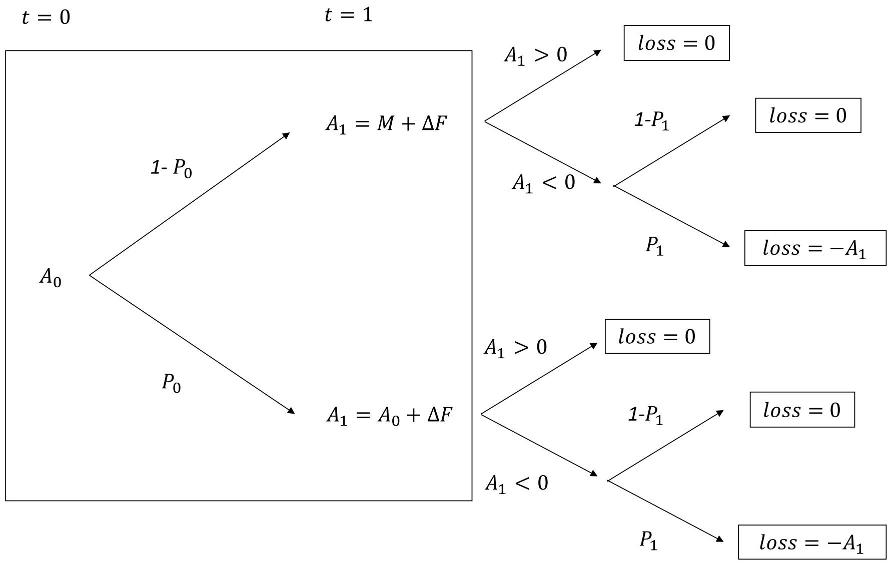

Fig. 2. The event tree of the loss generating process.

refuses to pay it, which can happen both in the downward and upward branches with a probability P_1. Fig. 2 summarizes all possible states and probabilities.

If the client does not pay the margin at $t = 0$, we are on the downward branch of the tree and the account balance of the client remains A_0. Otherwise, if the client pays the margin, we are in the upward branch of the tree and the account balance becomes M. Once the actual account balance is determined, the counterparty loss depends only on ΔF. Therefore, we investigate the conditional expected loss for a general value of A instead of M or A_0.

Definition 1. ***(Conditional expected counterparty loss)***

The conditional expected value of the counterparty loss denoted by L is the expected loss at $t = 1$ which is conditional on whether the client paid the margin in $t = 0$ or not, i.e. we are in the upper or lower branch of the tree. et A stand for a general value of the client's balance at the beginning of day 1 which is either A_0 or M. The conditional expected loss $L(A, \sigma)$ can be expressed as

$$L(A,\sigma) = L(A_1) \ = -\int_{-\infty}^{0} D(A_1)\cdot A_1\cdot P_1(A_1)dA_1. \tag{2}$$

where and $D(A_1) = D(A, \sigma)$ is the density function of A_1, and $P_1(A_1) = P_1(A,\sigma)$ is the probability of non-payment at $t = 1$.

Assumption 4. (*Probability of non-payment*)

First, we specify the probability of non-payment at $t = 0$ as

$$P_0 = P(M - A_0, \lambda) = \begin{cases} 1 - e^{-\lambda\cdot(M-A_0)} & \textit{for } M \geq A_0 \\ 0 & \textit{for } M < A_0 \end{cases} \tag{3}$$

and for the sake of simplicity, we assume

$$P_1 = 1 \tag{4}$$

In Eq. (3), λ is a positive parameter characterizing the overall funding (il)liquidity conditions in the economy where higher λ indicates less funding liquidity available. According to Assumption 4, the probability of non-payment at $t = 0$ is an increasing function of two symmetric factors, the overall funding illiquidity (λ) which determines the ability-to-pay and the payment burden ($M - A_0$) which determines the willingness-to-pay.

In Section 4, we examine other specifications for P_0 and P_1, as well.

Theorem 1. ***(Conditional expected counterparty loss)***

The conditional expected value of the loss $L(A)$ is

$$L(A) = L(A,\sigma) = \sigma\cdot\varphi\left(\frac{A}{\sigma}\right) - A\cdot\phi\left(-\frac{A}{\sigma}\right) \tag{5}$$

where ϕ is the density function, and φ is the distribution function of the standard normal distribution.

Proof.

$$L(A) = L(A, \sigma) = -\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-A)^2}{2\sigma^2}} x\,dx = \sigma \cdot \varphi\left(\frac{A}{\sigma}\right) - A \cdot \phi\left(-\frac{A}{\sigma}\right) \tag{6}$$

□

Corollary 1. The counterparty loss ***L*(*A*)** is a differentiable, strictly decreasing, and strictly convex function of the customer's account balance *A*, and

$$\lim_{A\to\infty} L(A) = \lim_{A\to\infty} L'(A) = 0. \tag{7}$$

Proof. Using equation $\phi'(x) = -x \cdot \phi(x)$, we obtain

$$L'(A) = -\phi\left(-\frac{A}{\sigma}\right) = \phi\left(\frac{A}{\sigma}\right) - 1. \tag{8}$$

Hence, $\lim_{A\to\infty} L'(A) = \lim_{y\to\infty} \phi(y) - 1 = 0.$

Also, $L'(A) < 0$ implies that $L(A)$ is strictly decreasing.

Further,

$$L''(A) = \frac{\varphi\left(\frac{A}{\sigma}\right)}{\sigma} > 0 \tag{9}$$

which proves strict convexity.

Finally,

$$\begin{aligned} \lim_{A\to\infty} L(A) &= \lim_{A\to\infty} \left(\sigma \cdot \varphi\left(\frac{A}{\sigma}\right) - A \cdot \phi\left(-\frac{A}{\sigma}\right)\right) \\ &= \lim_{A\to\infty} \frac{\phi\left(-\frac{A}{\sigma}\right)}{-\frac{1}{A}} = \lim_{A\to\infty} \frac{\varphi\left(-\frac{A}{\sigma}\right)}{\frac{1}{A^2}} = \lim_{A\to\infty} A^2 \cdot \varphi\left(\frac{A}{\sigma}\right) = 0 \end{aligned}$$

□

Definition 2. ***(Unconditional expected counterparty loss)***

The unconditional expected counterparty loss *UL* is calculated at $t = 0$ right after the daily settlement and before the margin setting and it is the expected value of potential losses in accordance with Fig. 2:

$$UL(A_0, M, \sigma, \lambda) = (1 - P_0) \cdot L(M, \sigma, \lambda) + P_0 \cdot L(A_0, \sigma, \lambda). \tag{10}$$

A_0 is exogenously given, and *M* is determined by the system; hence this latter is the control variable.

Definition 3. ***(Optimal margin)***

The optimal margin M^* minimizes the unconditional expected counterparty loss *UL* in (10).

In the next section, we characterize the optimal margin.

3. Characteristics of the optimum

For the sake of completeness and later reference, we deal first with a special "benchmark" case when there is no volatility at all, contrary to Assumption 1.

Proposition 1. (*Benchmark*)

We assume $\Delta F = 0$, i.e. the price does not change.

If $A_0 < 0$, then there is a unique minimum of *UL* at

$$M^* = \min\left\{A_0 + \frac{1}{\lambda}, 0\right\}$$

If $A_0 \geq 0$, then *UL* is 0 for any $M \geq 0$, and this is the minimal value of *UL*.

Remark: For $A_0 \geq 0$, it is worth choosing $M^* = A_0$ for practical reasons as it is better and simpler not to bother clients to pay more than they have on their accounts, nor let them withdraw the excess money.

Thus, we can summarize the optimal margin as

$$M^* = \begin{cases} A_0 + \frac{1}{\lambda} & if \quad A_0 \leq -\frac{1}{\lambda} \\ 0 & if \quad -\frac{1}{\lambda} \leq A_0 \leq 0 \\ A_0 & if \quad A_0 \geq 0 \end{cases} \tag{11}$$

Proof. Now $L(A) = \max\{-A, 0\}$ as we can see it from Fig. 1 with $\Delta F = 0$.

Therefore, by (3) and (10),

$$UL(M) = \begin{cases} e^{-\lambda\cdot(M-A_0)} max\{-M, 0\} + (1 - e^{-\lambda\cdot(M-A_0)}) max\{-A_0, 0\} & if\ M \geq A_0 \\ max\{-M, 0\} & if\ M \leq A_0 \end{cases}$$

Clearly, $UL(M) \geq 0$.

Consider first $A_0 \geq 0$. Then $\max\{-A_0, 0\} = 0$.

If $M \leq A_0$, then $UL(M) = \max\{-M, 0\} = 0$ if $0 \leq M \leq A_0$, and is positive if $M < 0$.

If $M \geq A_0$, then $UL(M) = 0$.

This proves Proposition 1 for $A_0 \geq 0$.

Turning to the case $A_0 < 0$, now $\max\{-A_0, 0\} = -A_0$.

If $M \leq A_0$, then $UL(M) = -M \geq -A_0 > UL(0) = (1 - e^{\lambda A_0})(-A_0)$.

If $M \geq A_0$, then we have to distinguish two cases.

If $M \geq 0$, then $UL(M) = (1 - e^{\lambda A_0})(-A_0)$ which is strictly decreasing since $\frac{\partial UL}{\partial M} < 0$.

If $A_0 \leq M \leq 0$, then $UL(M) = e^{-\lambda\cdot(M-A_0)}(-M) + (1 - e^{-\lambda\cdot(M-A_0)})(-A_0)$, so

$$\frac{\partial UL}{\partial M} = e^{-\lambda\cdot(M-A_0)}(\lambda\cdot(M - A_0) - 1)$$

which is negative for $M < A_0 + \frac{1}{\lambda}$, is 0 for $M = A_0 + \frac{1}{\lambda}$, and positive for $M > A_0 + \frac{1}{\lambda}$. Therefore, UL assumes its minimum at $M^* = A_0 + \frac{1}{\lambda}$ if $A_0 \leq -\frac{1}{\lambda}$, and at the endpoint $M^* = 0$ of the interval if $-\frac{1}{\lambda} \leq A_0 \leq 0$. This completes the proof of Proposition 1. □

From now on, we return to Assumption 1 again.

Theorem 2. ***(Characterization of the optimal margin)***

Let

$$h(M) = \lambda\cdot(L(A_0) - L(M)) + L'(M).$$

Then, there is a unique optimal margin M^* satisfying

$$h(M^*) = \lambda\cdot(L(A_0) - L(M^*)) + L'(M^*) = 0. \tag{12}$$

Proof. Obviously, $M^* < A_0$ cannot be optimal; hence we consider when M^* is in the interval $[A_0, \infty)$. We have to minimize UL with respect to M. Its derivative in (A_0, ∞) and its right-derivative in A_0 is

$$\begin{aligned} \frac{\partial UL}{\partial M}(A_0, M) &= \lambda e^{-\lambda\cdot(M-A_0)}\cdot(L(A_0) - L(M)) + e^{-\lambda\cdot(M-A_0)}\cdot L'(M) \\ &= e^{-\lambda\cdot(M-A_0)}\cdot h(M) \end{aligned} \tag{13}$$

Here we have $h(A_0) < 0$ (hence UL cannot be minimal in $M = A_0$), and $\lim_{M\to\infty} h(M) = \lambda\cdot L(A_0) > 0$. (8) implies that $L'(M)$ is increasing and is negative, hence $L(M)$ is decreasing. Therefore $h(M)$ is (continuous and) strictly increasing, so there is exactly one M satisfying $h(M) = 0$, and here h changes sign from negative to positive. Multiplying this with $e^{-\lambda\cdot(M-A_0)} > 0$ does not change this behavior, so this is the unique optimum. □

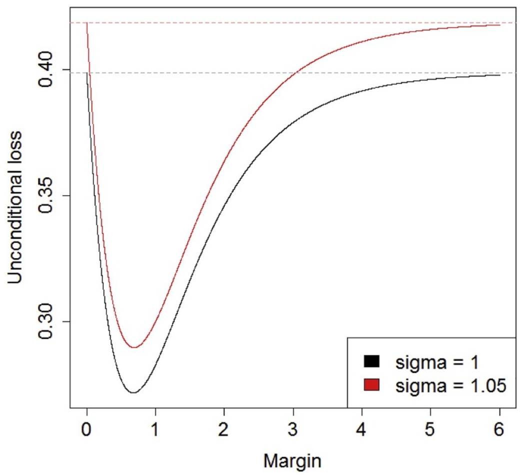

Fig. 3. Unconditional counterparty loss in function of the margin requirement.

Fig. 3 depicts the unconditional loss depending on the margin if $A_0 = 0$ and $\lambda = 1$.

We can see in Fig. 3 that the loss is not a monotone function of the margin and as Theorem 2 states, there is a unique optimum. In general, if M goes to infinity, then $\lim_{M\to\infty} UL(A_0, M) = L(A_0)$, as no matter how large margin the system requests if it is not paid, see (3). The result will be the same when no margin is requested.

Corollary 2. (***Upper and lower bounds of the optimal margin***)

The following upper and lower bounds hold for the optimal margin:

$$\forall\ \lambda, \sigma: \quad A_0 < M^* < A_0 + \frac{1}{\lambda} \tag{14}$$

Proof. We have already proved $A_0 < M^*$. For the upper bound, we divide (12) by λ and by $A_0 - M^*$

$$\frac{L(A_0) - L(M^*)}{A_0 - M^*} = \frac{L'(M^*)}{\lambda(M^* - A_0)}$$

By Lagrange's mean value theorem there exists $A_0 < \hat{M} < M^*$ such that $\frac{L(A_0) - L(M^*)}{A_0 - M^*} = L'(\hat{M})$. This means:

$$\frac{-L'(\hat{M})}{-L'(M^*)} = \frac{1}{\lambda(M^* - A_0)} \tag{15}$$

By Corollary 1, L is strictly convex, i.e. $L'' > 0$, thus L' is increasing, so the left hand side is greater than 1. Thus, $\lambda(M^* - A_0) < 1 \Leftrightarrow M^* < A_0 + \frac{1}{\lambda}$. □

Corollary 3. (***Limits in*** σ)

If $\sigma \to 0$, we get back the benchmark case (see Proposition 1).

If $\sigma \to \infty$, then $M^* \to A_0 + \frac{1}{\lambda}$ for all fixed A_0 and λ.

Proof. First let $\sigma \to 0$, then

$$L(A) = L(A, \sigma) = \sigma \cdot \varphi\left(\frac{A}{\sigma}\right) - A \cdot \phi\left(-\frac{A}{\sigma}\right) \to \max\{-A, 0\}.$$

Here, $\max\{-A, 0\}$ is the loss function of the benchmark case (11) in Proposition 1, see also Fig. 1 now with $\Delta F = 0$.

If $\sigma \to \infty$, then

$$\lambda \cdot (L(A_0) - L(M^*)) + L'(M^*) \to \lambda \cdot \frac{1}{2} \cdot (-A_0 - (-M^*)) - \frac{1}{2}$$

uniformly on every compact subset of $\mathbb{R}$, so the only zero of the left hand side converges to the zero of the right hand side, which is $M^* = A_0 + \frac{1}{\lambda}$. □

Corollary 4. (***Limit in*** A_0)

$$\lim_{A_0\to-\infty} M^* - A_0 = \frac{1}{\lambda}$$

Proof. We rely on Eq. (15) in Corollary 2:

$$\frac{-L'(\hat{M})}{-L'(M^*)} = \frac{1}{\lambda(M^* - A_0)}$$

If $A_0 \to -\infty$, then $\hat{M}$, $M^* \to -\infty$, as well, and from Eq. (8) it is clear that $\lim_{A_0\to-\infty} L' = -1$, hence the left hand side of (15) goes to 1. So does the right hand side, hence $M^* - A_0 \to \frac{1}{\lambda}$. □

Fig. 4 illustrates Corollaries 2, 3, and 4 together with the benchmark case.

Definition 4. *(Risk-sensitive margin)*

The margin setting mechanism is *weakly* (*strongly*) risk-sensitive if the margin is a non-decreasing (increasing and linear) function of the expected volatility σ.

We note that margin systems directly linked to Value-at-Risk or Expected Shortfall are strongly risk-sensitive (for normal distribution).

Theorem 3. *(Sensitivity of the optimum)*

The optimal margin M^*

(i) is a strictly increasing function of A_0 and a strictly decreasing function of λ, and
(ii) is not risk-sensitive even in a *weak* sense as its relation to σ depends on the parameters.

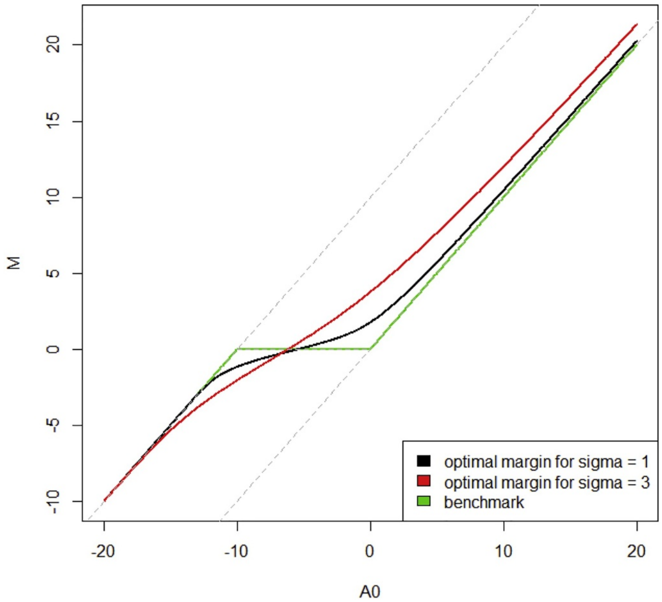

Fig. 4. Unconditional counterparty loss in function of the margin requirement.

Proof. Using the law of implicit derivation in formula (12), we compute $\frac{\partial M^*}{\partial A_0}$, $\frac{\partial M^*}{\partial \lambda}$, and $\frac{\partial M^*}{\partial \sigma}$.

For this, we need to compute first $\frac{\partial h}{\partial M^*}$, $\frac{\partial h}{\partial A_0}$, $\frac{\partial h}{\partial \lambda}$, and $\frac{\partial h}{\partial \sigma}$.

$$\frac{\partial h}{\partial M^*} = \lambda\left(1 - \Phi\left(\frac{M^*}{\sigma}\right)\right) + \varphi\left(\frac{M^*}{\sigma}\right)\frac{1}{\sigma} > 0 \tag{16}$$

$$\frac{\partial h}{\partial A_0} = \lambda\left(\Phi\left(\frac{A_0}{\sigma}\right) - 1\right) < 0 \tag{17}$$

$$\frac{\partial h}{\partial \lambda} = L(A_0) - L(M^*) > 0 \tag{18}$$

$$\frac{\partial h}{\partial \sigma} = \lambda\left(\varphi\left(\frac{A_0}{\sigma}\right) - \varphi\left(\frac{M^*}{\sigma}\right)\right) - \varphi\left(\frac{M^*}{\sigma}\right)\frac{M^*}{\sigma^2} \tag{19}$$

By (16), (17), (18), and (19), we obtain

$$\frac{\partial M^*}{\partial A_0} = -\frac{\frac{\partial h}{\partial A_0}}{\frac{\partial h}{\partial M^*}} > 0 \tag{20}$$

$$\frac{\partial M^*}{\partial \lambda} = -\frac{\frac{\partial h}{\partial \lambda}}{\frac{\partial h}{\partial M^*}} < 0 \tag{21}$$

$$\frac{\partial M^*}{\partial \sigma} = -\frac{\frac{\partial h}{\partial \sigma}}{\frac{\partial h}{\partial M^*}} \tag{22}$$

It is clear from (20) and (21) that M^* is a strictly increasing function of A_0 and a strictly decreasing function of λ. M^* is strictly increasing in σ iff $\frac{\partial h}{\partial \sigma} < 0$ by (16), (19), and (22). □

We derived an intuitive result that supports the idea of anti-cyclical margin from the point of view of micro-level financial stability. The optimal margin is a function of three exogenous parameters: the actual account balance of the client A_0, the overall funding illiquidity λ, and the volatility σ. The principle of *weak* risk-sensitivity would require *M* to be a non-decreasing function of σ. However, the optimal *M* does not necessarily satisfy it since other factors interact, as well. We can see in Fig. 4 that the optimal margin is a decreasing function of σ if A_0 is strongly negative. For example, in the case of a crisis, the client's actual account balance can be very low, even negative due to a recent shock (low A_0), and the client may have difficulties to get financing (high λ). In this situation, a less aggressive margin is optimal which is far below the risk-sensitive Value-at-Risk or Expected Shortfall level which is directly linked to the expected volatility σ. Formula (12) can be interpreted so that in the optimum the clearing system does not

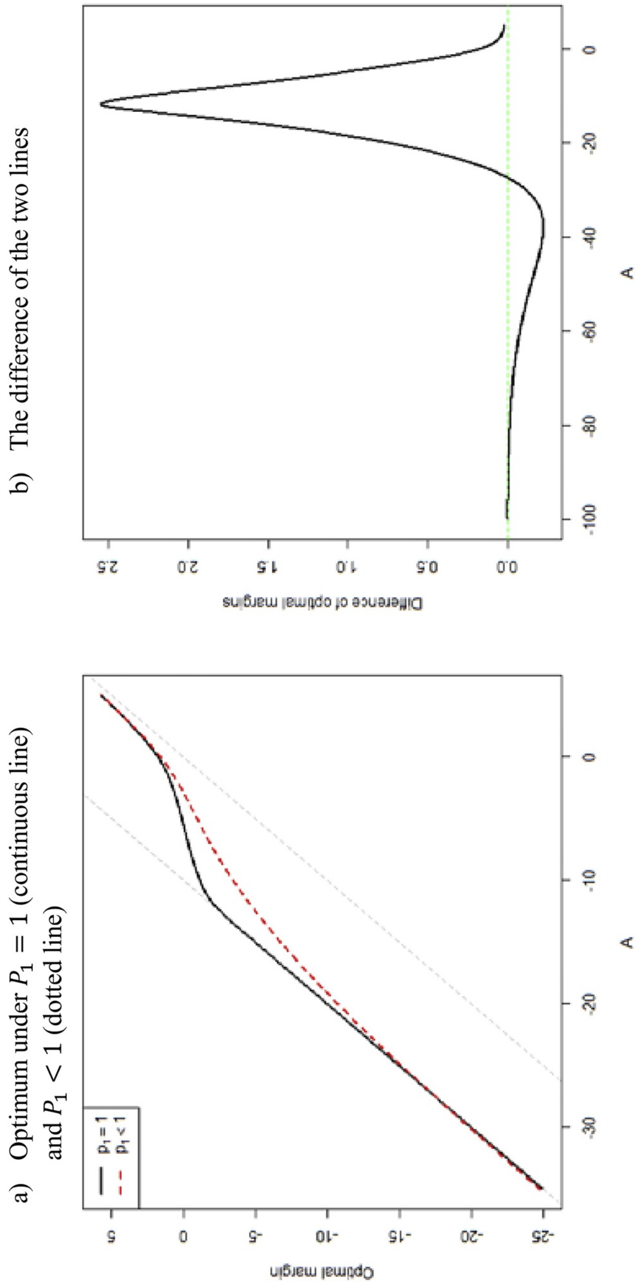

Fig. 5. Optimal margin as a function of the account balance.

mechanically adjust the margin to the volatility but also considers the payment burden $M - A_0$ and the funding illiquidity λ which determine the client's willingness-to-pay and ability-to-pay, respectively. It is remarkable that an anti-cyclical margin strategy should be operated merely to avoid larger losses without considering macro-level consequences (price impact, systemic risk, etc.) emphasized by the regulator.

4. Extentions

In this section, we briefly overview two extensions to the above model keeping Assumptions 1, 2, and 3 unchanged, but allowing that payment is possible also at $t = 1$, hence the probability of non-payment is $P_1 < 1$.

Extension 1. If $P_1 < 1$ is defined similarly to (3), the model gets complicated as $L(A_1)$ is not always convex, therefore, all the nice mathematical properties of the model get lost (even the uniqueness of the optimal margin), and we cannot express the result in an analytical form anymore. However, according to numerical simulations, there are no dramatic changes in the results, see Fig. 5.

We can see that as the account balance goes to minus infinity, the dotted line exceeds the continuous one (here, the difference is negative). When the account balance is infinitely large, the optima under $P_1 = 1$ and $P_1 < 1$ seem to be equal.

Extension 2. Now, we investigate other probabilities of non-payment:

$$P_0 = \begin{cases} 1 - \frac{1}{\lambda(M - A_0)} & for\ M - A_0 \geq \frac{1}{\lambda} \\ 0 & otherwise \end{cases} \tag{23}$$

$$P_1 = \begin{cases} 1 + \frac{1}{\lambda A_1} & for\ A_1 \leq -\frac{1}{\lambda} \\ 0 & otherwise \end{cases} \tag{24}$$

Note that (24) is the same as (23) but M is replaced by 0 and A_0 is replaced by A_1. Similarly to (3), the probability of non-payment is again an increasing function of two symmetric factors, the overall funding illiquidity (λ) and the payment burden which is $(M - A_0)$ in $t = 0$ and $-A_1$ in $t = 1$. Fig. 6 presents the probabilities of non-payment P_0 in the function of the margin M under the original (3) and the new (23) specifications for $A_0 = 0$ and $\lambda = 1$.

As we can observe also from Fig. 6, probabilities (23) and (24) express the willingness of clients to pay as long as the burden is not too big.

For this new specification, we combine the analogues of Theorem 1, Corollary 1, and Theorem 2 into

Theorem 4.

(i) The conditional expected value of the loss $L(A)$ is

$$L(A) = L(A, \sigma, \lambda) = \sigma \cdot \varphi\left(-\frac{A + \frac{1}{\lambda}}{\sigma}\right) - \left(A + \frac{1}{\lambda}\right) \cdot \phi\left(-\frac{A + \frac{1}{\lambda}}{\sigma}\right) \tag{25}$$

(ii) $L(A)$ is differentiable, strictly decreasing and strictly convex function of the customer's account balance A, and

$$\lim_{A \to \infty} L(A) = \lim_{A \to \infty} L'(A) = 0. \tag{26}$$

as in the basic model.

(i) The optimal margin can be expressed explicitly by

$$M^* = A_0 + \frac{1}{\lambda} \tag{27}$$

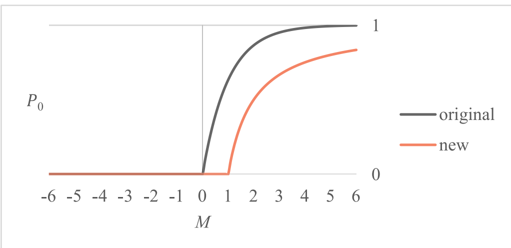

Fig. 6. Probability of non-payment P_0 in the function of the margin M under different specifications.

Remark. : (27) is a surprising result, because the optimal margin depends only on the account balance and the funding illiquidity, while volatility does not matter at all. Note that this "degenerated" optimum is just the upper bound of the optimum in the basic model, see Corollary 2 and Fig. 4.

Proof. (i)

$$\begin{aligned} L(A) = L(A,\sigma) &= -\int_{-\infty}^{0} D(A,\sigma)\cdot A_1\cdot P_1(A,\sigma)dA_1 \\ &= -\int_{-\infty}^{-\frac{1}{\lambda}} \frac{1}{\sqrt{2\pi}\sigma}\cdot e^{\frac{-(x-A)^2}{2\sigma^2}}\cdot x\cdot\left(1+\frac{1}{\lambda x}\right)\cdot dx \\ &= \sigma\cdot\varphi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right) - \left(A+\frac{1}{\lambda}\right)\cdot\phi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right) \end{aligned} \tag{28}$$

(ii)

Using equation $\phi'(x) = -x \cdot \phi(x)$, we obtain

$$L'(A) = -\phi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right) = \phi\left(\frac{A+\frac{1}{\lambda}}{\sigma}\right) - 1. \tag{29}$$

Hence, $\lim_{A\to\infty} L'(A) = \lim_{y\to\infty}\phi(y) - 1 = 0$.

Also, $L'(A) < 0$ implies that $L(A)$ is strictly decreasing.

Further,

$$L''(A) = \frac{\varphi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right)}{\sigma} > 0 \tag{30}$$

which proves strict convexity.

Finally,

$$\begin{aligned} \lim_{A\to\infty} L(A) &= \lim_{A\to\infty}\left(\sigma\cdot\varphi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right) - \left(A+\frac{1}{\lambda}\right)\cdot\phi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right)\right) \\ &= \lim_{A\to\infty}\frac{\phi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right)}{-\frac{1}{A+\frac{1}{\lambda}}} = \lim_{A\to\infty}\frac{-\frac{1}{\sigma}\cdot\varphi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right)}{\frac{1}{\left(A+\frac{1}{\lambda}\right)^2}} = -\frac{1}{\sigma}\cdot\lim_{A\to\infty}\left(A+\frac{1}{\lambda}\right)^2\cdot\varphi\left(-\frac{A+\frac{1}{\lambda}}{\sigma}\right) = 0 \end{aligned}$$

(iii)

By (10) and (23),

$$UL(A_0, M, \sigma, \lambda) = \begin{cases} \frac{1}{\lambda(M-A_0)}\cdot L(M) + \left(1 - \frac{1}{\lambda(M-A_0)}\right)\cdot L(A_0) & \textit{if } M - A_0 \geq \frac{1}{\lambda} \\ L(M) & \textit{if } M - A_0 \leq \frac{1}{\lambda} \end{cases} \tag{31}$$

We have to show that $UL(M)$ assumes its minimum at $M = A_0 + \frac{1}{\lambda}$.

If $M \leq A_0 + \frac{1}{\lambda}$, then $UL(M) = L(M)$ is strictly decreasing by (ii). Therefore, $UL(M) \geq UL(A_0 + \frac{1}{\lambda})$. If $M \geq A_0 + \frac{1}{\lambda}$, then we prove that $UL(M)$ is strictly increasing, i.e. $\frac{\partial UL}{\partial M} > 0$.

$$\begin{gathered} \frac{\partial UL}{\partial M} = \frac{1}{\lambda}\left(\frac{L'(M)}{M-A_0} - \frac{L(M)}{(M-A_0)^2} + \frac{L(A_0)}{(M-A_0)^2}\right) > 0 \\ \Updownarrow \\ L'(M)(M-A_0) - L(M) + L(A_0) > 0 \\ \Updownarrow \\ L'(M) > \frac{L(M)-L(A_0)}{M-A_0} = L'(\hat{M}) \end{gathered} \tag{32}$$

for some $A_0 < \hat{M} < M$ again by Lagrange's mean value theorem. Indeed, (32) holds since

$$L'(A) = \phi\left(\frac{A+\frac{1}{\lambda}}{\sigma}\right) - 1$$

is monotone increasing. □

5. Conclusions

In the framework of an analytical model, we examine the fundamental trade-off between the confronting requirements of risk-sensitivity and anti-cyclicality. We prove that a well-chosen margin level can create significant value. The optimal strategy is non-trivial; in most cases, it lies inside the set of feasible strategies and represents a delicate compromise between different forces. The optimal margin is higher if the actual balance A_0 of the client is higher, and also if clients are supposed to get financing easily (i.e. funding illiquidity λ is low), but its relation to the expected volatility σ can be positive or negative depending on the other factors.

We derived the upper ($A_0 + \frac{1}{\lambda}$) and the lower (A_0) bounds of the optimal margin. The optimal margin lies within these boundaries, and its exact position depends on the volatility. The difference between the upper and the lower bounds equals the reciprocal of the funding illiquidity ($\frac{1}{\lambda}$), therefore, in a deep crisis when funding illiquidity is very high, boundaries are very close to each other, hence, volatility does not really matter, and the optimal margin is determined only by the account balance and the funding illiquidity. Moreover, a plausible specification of the probability of non-payment examined in Extension 2 leads to an optimal margin which equals just the previously seen upper bound ($A_0 + \frac{1}{\lambda}$) being completely independent of the volatility.

It follows that a risk-sensitive strategy is not always optimal; hence, the need for anti-cyclical margin setting can be justified not only at a macro-level (by the regulator) but also at a micro-level (by the interests of the clearing system).

Of course, margin setting is a much more complicated task in the real world than in the above model and its extensions. Firstly, margins may have a feedback effect on funding illiquidity (Brunnermeier and Pedersen, 2008) and also on the prices (Danielsson et al., 2001, 2013). Secondly, there are more clients, more assets, more maturities, and more periods. Margins of today influence the accounts of tomorrow, hence the margins of tomorrow and so on. These spillover effects make the margin setting a complex dynamic and stochastic optimization problem. Moreover, we only modeled the long side of a transaction, while in practice the long and the short sides should be managed simultaneously. And last but not least, there can be other risk factors influencing the probability and the magnitude of the losses. These issues remain the subject of further research.

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