Believe it or Not: The Role of Investor Beliefs for Private Equity Valuation*

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Abstract

I show that investors' beliefs can explain the undervaluation of private equity (PE) funds. The value of PE funds, in accordance with finance principles, is determined by the expected cash flows discounted for time and risk. Therefore, the undervaluation may stem from either an incorrect stochastic discount factor (SDF) or a discrepancy between investors' beliefs and the true distribution of cash flows. I propose an estimation method based on Empirical Likelihood (EL) to back out investors' beliefs from funds' cash flows. I validate estimated beliefs using investors' sentiment surveys. I find that investors' pessimism about PE cash flows and overoptimism about public market cash flows, rather than SDF misspecification, offer a potential explanation for the undervaluation of PE funds.

Keywords: Subjective Beliefs; Investors' Expectations; Private Equity Funds; Empirical Likelihood; Survey Data.

JEL Codes: G12, G23, G24, D84.

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1 Introduction

Private Equity (PE) refers to investments made in companies not publicly traded on a stock market. The PE industry has experienced robust growth over the years. As of 2021, its size was estimated at approximately \$8.2 trillion, which is over one-tenth of the \$54 trillion U.S. asset management industry[1]. While the broader scope of PE encompasses various investment strategies and structures, this paper narrows its focus to the valuation of PE funds from the viewpoint of Limited Partners (LPs).

In this paper, I address the performance measurement challenge for PE funds. Two reasons make this problem complex. First, conventional valuation methods, such as the Internal Rate of Return (IRR) and various multiples, fail to provide a risk adjustment in line with finance principles (LeRoy (1973)). Specifically, they do not account for the higher value of payoffs in bad states of the economy. Second, return time series data is not available for PE funds, mainly due to the non-periodic nature of cash flows and the potential manipulation of portfolio net asset values (NAVs) by PE managers[2].

To mitigate the problem of risk adjustment without readily available return data, Kaplan and Schoar (2005) proposed benchmarking PE investments against the public stock market, building on ideas in Long and Nickels (1996). This approach, known as the Public Market Equivalent (PME), uses a specific form of stochastic discount factor (SDF), as shown by Sorensen and Jagannathan (2015). Korteweg and Nagel (2016) augmented the PME metric by accommodating a broader class of SDFs in PE valuation, resulting in the generalized PME (GPME).

However, even the GPME method falls short in explaining PE funds valuation, especially for leveraged buyouts (BO). As Korteweg (2019) pointed out, average BO investments tend to outperform similarly risky stock market investments after risk adjustment. This phenomenon constitutes the valuation puzzle for PE funds. According to finance principles, understanding the PE valuation puzzle requires considering investors' risk preferences and beliefs about future cash flows. Therefore, I introduce a method to estimate investors' beliefs from cash flow data that allows to disentangle beliefs from potential SDF misspecification. The discrepancy between investors' beliefs about the distribution of future cash flows and the true probability distribution can provide insight into the undervaluation of PE funds.

I estimate investors' subjective beliefs from PE cash flow data using a two-stage estimation

[1] According to Preqin, a leading provider of data, analytics, and insights for PE funds, and the BCG Global Asset Management 2022 report (page 23).

[2] See Brown, Ghysels, and Gredil (2020), Brown, Gredil, and Kaplan (2019), Jenkinson, Landsman, Rountree, and Soonawalla (2020), and Phalippou and Gottschalg (2009).

procedure. In the first step, I utilize the generalized method of moments (GMM) to estimate SDF parameters. This procedure involves pricing benchmark funds which invest in both the public stock and bond markets, following the methodology suggested by Korteweg and Nagel (2016). To ensure robustness, I repeat the procedure for several commonly used SDFs, each of which has different assumptions about the structure of the underlying economy. I then apply the SDFs with estimated parameters to price the excess cash flows, which are constructed as the difference between the aggregate PE fund cash flows and the cash flows of a benchmark fund that invests in the stock market. In theory, the SDF should accurately price the excess cash flows in each period, eliminating any abnormal performance for PE funds. However, if there remains unexplained abnormal performance after accounting for time and risk discounting, this indicates a discrepancy between investors' beliefs about PE cash flows and the true distribution.

In the second step, I utilize an empirical likelihood (EL) estimation method to estimate investors' beliefs that reconcile with the condition of zero excess cash flow at each period. This process involves minimizing the discrepancy of beliefs using the Kullback-Leibler divergence as a measure of the difference between the estimated probability distribution - which I term 'subjective beliefs' - and the observed data, termed as the 'true probability distribution'. The estimated new distribution generally differs from the true one, thereby contradicting the Rational Expectation (RE) hypothesis, which presumes that investors have perfect knowledge of the data generating process.

Figure 1 visualises the discrepancy between true (observed) and belief-adjusted cash flows, illustrating the impact of investors' beliefs on the valuation of PE funds. It shows that the aggregate excess cash flows perceived by investors are smaller than what has been historically observed. The belief-adjusted aggregate excess cash flows exhibit a fatter negative tail and a more left-skewed distribution compared to the historical cash flows. Therefore, this finding constitutes the first result: on average, investors tend to be overly pessimistic about a PE fund's ability to generate cash flows. This pessimism prompts investors to demand compensation for the additional risk associated with uncertainty about future cash flows, a factor not captured by commonly used asset pricing models.

Furthermore, I validate estimated beliefs by demonstrating their correspondence with actual investor sentiment as reflected in survey data. The main focus is on sentiment among private equity investors. For them I analyze two key indices: the Silicon Valley VC Confidence Index (SVVCCI) and the Central Europe PE Confidence Index (CEPECI). The SVVCCI is a quarterly survey of venture capitalists in the San Francisco Bay Area, conducted by Cannice and Goldberg (2009), while the CEPECI is a semi-annual survey of private equity professionals in Central Europe,

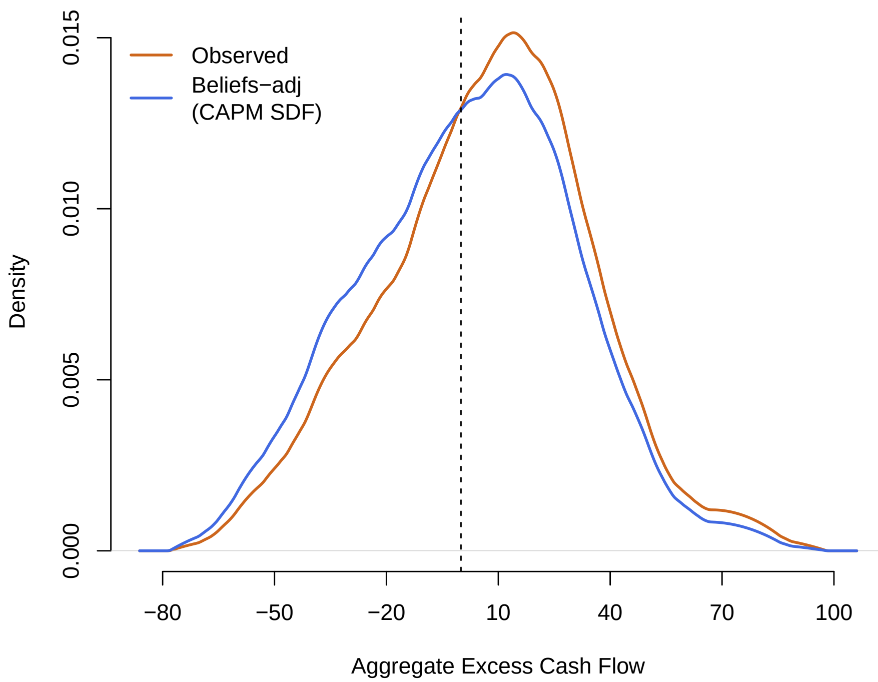

Figure 1: Discrepancy in PE Excess Cash Flows: True vs. Belief-Adjusted. This figure plots the observed and belief-adjusted aggregate excess cash flows for PE funds from Q1 1996 to Q4 2014. Adjustments are based on beliefs estimated under the CAPM model. Values are presented in millions of dollars, with density estimations derived using the Epanechnikov kernel method.

conducted by Deloitte. The results reveal a positive correlation between these PE sentiment indices and belief-adjusted aggregate excess cash flows, suggesting that estimated beliefs do indeed contain information about investors' expectations.

In addition to PE investor sentiment, I also consider the sentiments of individual and institutional stock market investors. For this purpose, I utilize data from the Gallup Investor Survey, the Economic Sentiment Indicator (ESI) from Eurostat, and Robert Shiller's Investor Survey. The analysis of these surveys validates that the stock market sentiment, like PE sentiment, correlates in the expected way with belief-adjusted excess cash flows via the public market component, which is constructed using a benchmark fund.

Despite the results of the correlation analysis, concerns may persist regarding the potential impact of SDF misspecification on the estimated beliefs. To further address this, I orthogonalize estimated beliefs against public market factors that are not included in the utilized SDF, but which could still influence investor risk preferences. Specifically, I consider potential risk factors identified in the literature: the equity risk premium (Haddad, Loualiche, and Plosser (2017)), Moody's BBB to AAA credit spread (Schmid, Huether, and Steri (2019)), the Chicago Board Options Exchange S&P 100 Volatility Index (Gompers, Kovner, Lerner, and Scharfstein (2008)), and the aggregate liquidity factor proposed by Pástor and Stambaugh (2003) (Franzoni, Nowak, and Phalippou (2012)). The results reveal that these orthogonalized beliefs continue to exhibit a strong correlation with survey data, underscoring their robustness. The orthogonalized beliefs show a particularly strong correlation with original beliefs when using SDFs that are most adept at pricing excess cash flows in terms of a pricing error.

Among the omitted factors in existing SDFs, market volatility notably emerges as a consistent and significant factor, highlighting its potential role in forming investor risk preferences. Credit market conditions and the expected market premium also appear as potentially influential factors. However, their significance varies depending on the specific SDF in use. In contrast, the liquidity factor consistently proves to be insignificant across all models tested. This observation reinforces the notion that PE investors are long-horizon investors who do not frequently trade and, therefore, require minimal compensation for illiquidity, aligning with the ideas proposed by Constantinides (1986).

The findings of this paper hold when beliefs are estimated under a range of asset pricing models, including the Consumption-CAPM (C-CAPM) by Rubinstein (1976), the External Habit model by Campbell and Cochrane (1999), and the Long-Run Risks model based on Epstein-Zin preferences by Bansal and Yaron (2004). Each of these models offers an alternative perspective on the structure of the economy and the risk preferences of potential investors in PE.

The remainder of the paper is organised as follows. In Section 2, I present the methodology and the moment condition that I use to estimate the investors' beliefs from the observed excess cash flows. Section 3 details the results of the model-implied subjective beliefs and explains the existence of the positive excess cash flow. Additionally, I explore various risk preferences assumptions and conduct a bootstrap exercise to compare the economic implications of estimated beliefs in Section 3. Section 4 compares estimated beliefs with survey data from individual, institutional, and PE investors. Finally, the paper concludes with suggestions for future research.

Related Literature. The role of investors' beliefs in shaping asset prices has been the subject of growing interest in the literature. The financial literature has long been dominated by the rational expectations paradigm, as proposed by Muth (1961), which equates the beliefs held by investors with the true (objective) data generating distribution. This approach effectively excludes any potential impact of investors' opinions on prices and eliminates the need to estimate them. However, Miller (1977) was one of the first to highlight the discrepancy between investors' subjective beliefs and the true probability distribution of returns, which results in trades between heterogeneous groups of investors based on differing perceptions of future uncertainty.

Stefan Nagel's discussion in Brunnermeier et al. (2021) acknowledges the significance of the rational expectations hypothesis (RE) as the first approximation of investors' beliefs, but also draws attention to its inherent limitations and fallacies. These limitations stem from several sources. The first challenge is associated with knowledge of the true DGP (data generating process) parameter values by agents within the model. As pointed out by Hansen (2007), the RE hypothesis implies that agents possess more precise information than econometricians, with the Law of Large Numbers often serving as a heuristic defense for the practical application of this hypothesis. The complexity of applying the RE approach to PE performance measurement is further amplified by the lack of frequent and reliable valuations of PE stakes to construct the time series of returns, the relatively recent history of the asset class dating back only to the 1980s, and the overall data opacity for outside investors.

The second challenge emerges from the disconnect between investors' expectations, as documented in survey data, and the predictions derived from the RE model. Greenwood and Shleifer (2014) is the first work that explicitly compares investors' survey expectations with expected returns predicted by the RE model. Their findings, which revealed a strong negative correlation between the two, cast doubt on the validity of the RE paradigm. Digging deeper into the potential mismatch, Adam, Matveev, and Nagel (2021) find no evidence that investors confound their beliefs and risk preferences. In particular, they refute the idea that investors report risk-neutral forecasts in surveys or provide only pessimistically-tilted forecasts. The literature broadly affirms that survey expectations data are informative about investor expectations, and they are not in line with the RE. While the survey data for PE is scarce, this paper attempts to utilize it to validate the beliefs of PE investors.

The third challenge involves investors' behavioral biases. Barberis and Thaler (2003) demonstrates that behavioral biases can lead to deviations from rationality. Manski (2004) liked the survey data approach to validate and potentially relax the RE assumption. Recent literature also suggests that lifetime experiences shape individuals' macroeconomic expectations. Notably, Nagel and Xu (2022) propose a fading memory mechanism within the asset pricing model, reconciling the discrepancy between subjective beliefs and the objective DGP. This approach builds on micro-evidence from household portfolio choice and survey expectations (Malmendier and Nagel (2011); Malmendier and Nagel (2016)). They argue that even if investors retain a perfect memory of the realized process, their choices might be strongly influenced by the most recent data, leading to a discrepancy between their beliefs and the objective DGP. This paper does not exploit the concept of behavioral biases when justifying the use of subjective beliefs, maintaining agnosticism about the belief-formation process.

In this paper, I acknowledge the limitations of applying the RE paradigm to PE funds valuation. Measuring PE performance is hard and cannot be fully explained using standard asset pricing models. I posit that the uncertainty associated with discrepancies in beliefs should be priced along with compensation for time and risk, an aspect potentially overlooked by existing models. To quantitatively evaluate the asset pricing implications of the belief discrepancy, I propose a general methodology for unconditionally estimating investors' beliefs using cash flow data.

This paper contributes to the rich empirical literature on the evaluation of PE funds' performance, following the works such as Kaplan and Schoar (2005), Korteweg and Nagel (2016), Gupta and Van Nieuwerburgh (2021), Gredil, Sorensen, and Waller (2019), Ang, Chen, Goetzmann, and Phalippou (2018), Harris, Jenkinson, and Kaplan (2014), Robinson and Sensoy (2016), among others. I demonstrate that incorporating subjective beliefs can significantly improve existing models. My approach closely aligns with Korteweg and Nagel (2016), wherein PE performance is evaluated based on a comparison with the public market. I construct the excess cash flow as the difference between the original PE fund cash flow and the mimicking cash flow of a benchmark fund investing in the public stock market. Using the principle of no arbitrage, I require the SDF to accurately value the excess cash flow over the public stock market. I show that incorporating subjective beliefs can help to eliminate the remaining valuation error after the SDF. Furthermore, I extend the SDF to a broader set linked to consumption-based asset pricing models, following Gredil, Sorensen, and Waller (2019).

Moreover, this paper contributes to the literature on belief recovery and the EL with its application in finance by proposing a novel method to extract investor beliefs from cash flow data and validate them using survey data. The theoretical development of the belief recovery literature started with Ross (2015) and Chen, Hansen, and Hansen (2020), followed by a growing body of empirical studies aimed at uncovering investors' beliefs. For example, Ghosh and Roussellet (2020) relied on Kitamura, Tripathi, and Ahn (2004) methodology to recover price-consistent conditional beliefs for a representative investor. Ghosh, Korteweg, and Xu (2020) also focused on recovering beliefs of heterogeneous investors, providing a concise analysis of recovered beliefs. For a broader overview of the literature on investors' beliefs, readers may refer to Adam and Nagel (2022). In this paper, I augment this literature by extracting the unconditional beliefs of investors in PE funds.

The paper also relates to the EL literature. The EL method was developed by Owen (1988) and extended to moment conditions models by Qin and Lawless (1994). Smith (1997) and Newey and Smith (2004) proposed the Generalised Empirical Likelihood (GEL) class of estimators, which includes the exponential tilting (ET) estimator. This paper utilises the ET, which goes back at least to Efron (1981), to estimate investors' beliefs. I focus on selecting the probability measure that has the least Kullback-Leibler discrepancy from the true probability distribution while satisfying the moment condition for excess cash flows of PE funds.

The use of EL-type methods has gained popularity in the financial literature. The methodology of this paper mainly builds upon Stutzer (1996) and Julliard and Ghosh (2012). Stutzer (1996) used EL-type methods to extract the risk-neutral probability distribution, while Julliard and Ghosh (2012) applied EL-type methods to evaluate the empirical validity of the rare disasters hypothesis within the equity premium puzzle context. Other notable applications of EL-type methods in asset pricing literature include Almeida and Garcia (2012, 2017), Ghosh, Julliard, and Taylor (2017), Almeida, Ardison, and Garcia (2020) and Almeida and Freire (2022).

2 Methodology

2.1 SDF Pricing

Traditional asset pricing theory determines the fair value of an asset (V_t) as the expected future cash flow (CF_{t+1}) plus residual value (V_{t+1}), discounted for time and risk using an SDF ($M_{t,t+1}$):

$$V_t = \mathbb{E}^{\mathbb{P}}[M_{t,t+1}(CF_{t+1} + V_{t+1})] \tag{1}$$

An asset's value is not solely determined by the risks associated with the investment but also by investors' beliefs about the distribution of future cash flows. The Euler equation (1) captures this essence by focusing on three pivotal components: (i) the stochastic future cash flows, (ii) the one-step-ahead SDF, and (iii) the true probability measure ($\mathbb{P}$). Valuation errors might arise not only from inaccuracies in reflecting risk preferences in the SDF but also from potential discrepancies in capturing the representative investor's understanding of the DGP. This latter challenge becomes particularly crucial given the RE assumption, which suggests that agents are aware of the true DGP.

This paper investigates the violation of the RE hypothesis as the main source of valuation errors. While the application of RE is widely acknowledged for analytical purposes, this paper introduces an alternative version of the Euler equation that incorporates investors' beliefs, represented by the measure $\mathbb{P}^*$, which generally might deviate from the true measure $\mathbb{P}$.

$$V_t = \mathbb{E}^{\mathbb{P}^*}[M_{t,t+1}(CF_{t+1} + V_{t+1})] \tag{2}$$

In this paper, I emphasize the discrepancy between the true market beliefs ($\mathbb{P}$) and investors' subjective beliefs ($\mathbb{P}^*$) and its significance in the valuation of PE funds. I demonstrate that the standard SDF approach cannot adequately explain the outperformance of PE funds relative to the public market. To assess the performance of PE funds, I utilize the benchmarking technique known as the Generalized Public Market Equivalent (GPME) metric introduced by Korteweg and Nagel (2016). The GPME approach computes the Net Present Value (NPV) of PE cash flows using the SDF as a discount factor. My findings indicate an abnormal performance for PE funds of 25 cents for every invested dollar on average[3].

A detailed discussion of the GPME framework and my adaptation is available in Appendix C. This paper primarily focuses on the following NPV formulation:

$$NPV_i^{PE} = \sum_{t=1}^{T} M_{1,t} CF_{i,t}^{PE} \tag{3}$$

This formulation ensures that all cash flows are consistently compared, as they are discounted to a single reference point for each fund. This consistency is crucial for my analysis since the beliefs estimation requires aggregation to form the underlying time series for the moment condition. Here, $t = 1, ..., T$ represents the timeline of observed PE cash flows, while $M_{1,t}$ denotes the multi-period SDF, which is derived by compounding the single-period discount factors:

$$M_{1,1} = 1; M_{1,2} = M_{1,2}; M_{1,3} = M_{1,2}M_{1,3}; M_{1,t} = M_{1,2}M_{1,3}...M_{1,t}$$

Under the null hypothesis, the NPV of a PE fund is expected to be zero, that is, $\mathbb{E}[NPV_i^{PE}] = 0$, given that the cash flows ($CF_{i,t}^{PE}$) include both the initial investments (capital calls) and the returned cash flows (capital distributions). If this condition is met for each individual fund, then the aggregated NPV across all funds should also be zero:

$$\mathbb{E}^{\mathbb{P}} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{PE} = 0 \tag{4}$$

Similarly, a benchmark fund, which simulates cash flows from investments in a market index like CRSP, should also possess an NPV of zero:

$$\mathbb{E}^{\mathbb{P}} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{mrkt} = 0 \tag{5}$$

Combining the NPV conditions for both the PE funds and the benchmark funds yields the condition for excess cash flows:

[3] For estimation details, see Table 3.

$$\mathbb{E}^{\mathbb{P}}\sum_{t=1}^{T}M_{1,t}\sum_{i=1}^{N}[CF_{i,t}^{PE}-CF_{i,t}^{mrkt}]=\mathbb{E}^{\mathbb{P}}\sum_{t=1}^{T}M_{1,t}\sum_{i=1}^{N}CF_{i,t}^{exc}=0 \tag{6}$$

Equation 6 implies that the NPV of the aggregate excess cash flows equals zero. If the GPME metric deviates from zero, the excess cash flows stand out as the sole source of abnormal performance. The occurrence of abnormal performance for PE funds implies that this equation is not satisfied, given the combination of risk preferences ($M_{1,t}$) and probability distribution ($\mathbb{P}$).

In my research approach, the initial step entails estimating the parameters of the SDF under the empirical probability distribution, $\hat{\mathbb{P}}$. This estimation relies on the moment condition depicted by Equation 5. I assume that the benchmark fund's investment in the public market is perfectly priced[4]. In the subsequent step, I apply an empirical likelihood-type methodology. The aim here is to pinpoint the subjective probability distribution, represented by $\hat{\mathbb{P}^*}$. This alternative distribution eliminates the pricing errors in combination with the SDF determined in the prior step.

2.2 Empirical Likelihood Estimation

To estimate investors' subjective beliefs, I seek an alternative probability distribution $\mathbb{P}^*$ that is closest to $\mathbb{P}$ in the Kullback-Leibler Information Criterion (KLIC) sense and satisfies the moment condition for excess average cash flows, as shown in Equation (6):

$$\begin{aligned}&\min_{\mathbb{P}^*} D(\mathbb{P}^*||\mathbb{P}) \equiv \min_{\mathbb{P}^*}\int \log\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}\right)d\mathbb{P}^*\\ &\text{subject to} \quad \mathbb{E}^{\mathbb{P}^*}\sum_{t=1}^{T}M_{1,t}\sum_{i=1}^{N}CF_{i,t}^{exc}=0\end{aligned} \tag{7}$$

Here, $D(\mathbb{P}^*||\mathbb{P})$ represents the closeness between the subjective probability distribution $\mathbb{P}^*$ and the true probability distribution $\mathbb{P}$. The KL divergence is non-negative and equals zero if and only if $\mathbb{P}^*=\mathbb{P}$, which holds under the RE hypothesis. The probability distribution closest to the true one represents the investors' subjective beliefs.

The Exponential Tilting (ET) estimator, introduced by Efron (1981), Kitamura and Stutzer (1997), and Imbens, Spady, and Johnson (1998), is employed to solve this problem. Anatolyev and Gospodinov (2011) demonstrates that this solution yields identical model-implied probabili-

[4] For specific SDFs requiring supplementary moment conditions, I introduce a benchmark fund: for the K-N CAPM SDF, it invests in the bond market, while for the CAPM+SMB SDF, the benchmark invests in small-cap firms.

ties to those formulated for the KL divergence minimisation problem. However, since $\mathbb{P}$ and $\mathbb{P}^*$ are unobservable, the estimated subjective beliefs are the solution to the following minimization problem:

$$\begin{aligned}
\min_{p_t} D(\hat{\mathbb{P}^*}||\hat{\mathbb{P}}) &= \sum_{t=1}^{T} p_t \log\left(\frac{p_t}{\frac{1}{T}}\right) \\
\text{subject to} \quad & \sum_{t=1}^{T} p_t M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0, \\
& \sum_{t=1}^{T} p_t = 1.
\end{aligned} \tag{8}$$

Here, $1/T$ denotes the empirical probability distribution, and $\{p_t\}_{t=1}^{T}$ is the discrete distribution of subjective beliefs that assigns probabilities to the same points in time as $CF_{i,t}^{exc},\ t = 1, \ldots, T$. Introducing Lagrange multipliers to accommodate constraints, the Lagrangian is formed as:

$$L(p_t, \lambda, \mu) = \sum_{t=1}^{T} p_t \log\left(\frac{p_t}{\frac{1}{T}}\right) + \lambda' \sum_{t=1}^{T} p_t M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} + \mu\left(\sum_{t=1}^{T} p_t - 1\right) \tag{9}$$

Here, λ and μ enforce the moment and normalization constraints, respectively. With subsequent first-order conditions and normalization requirements, the optimal probability measure assumes a specific form as per Csiszár (1975)'s I-projection solution, leading to:

$$p_t = \frac{e^{\lambda'(M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc})}}{\sum_{t=1}^{T} e^{\lambda'(M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc})}} \tag{10}$$

In this equation, p_t is the probability measure satisfying the moment condition while remaining as close as possible to the empirical distribution $\frac{1}{T}$ in the KLIC sense. It is important to note that investors' subjective beliefs and model-implied probabilities, while intricately connected, represent distinct statistical objects. Model-implied probabilities derive from the alternative probability measure, while subjective beliefs emerge from applying these implied probabilities to specific variables of interest, such as excess cash flows in this case.

2.3 Data

To estimate investors' subjective beliefs, I use cash flow data for PE funds employing a buyout strategy sourced from Preqin, spanning the period from 1980 to 2020. Furthermore, I filtered out funds incepted between 1980 and 2015 to exclude premature funds, characterized by substantial undistributed capital, from the analysis.

The cash flow data comprises three elements: capital calls, capital distributions, and valuations. Capital calls occur when the General Partner (GP) requests capital from the Limited Partner (LP) to facilitate investments. Conversely, capital distributions take place when the LP begins receiving returns from past investments. Valuations offer insights into the remaining capital awaiting distribution to investors. While valuations are crucial, particularly for premature funds, serving as estimates of the fund's remaining capital, they are often subject to potential manipulation by GPs. For a detailed discussion and other issues related to valuations, see Brown, Ghysels, and Gredil (2020), Brown, Gredil, and Kaplan (2019), Jenkinson, Landsman, Rountree, and Soonawalla (2020), and Phalippou and Gottschalg (2009).

To circumvent issues related to valuations and to employ more reliable cash flows, I restrict the sample to closed and liquidated funds. Liquidated funds refer to those that have terminated their operations and returned all capital to investors, while closed funds denote those that have concluded their fundraising campaigns but have not yet distributed all capital.

2.3.1 Excess Cash Flows. The key quantity for performing the ET estimation is excess cash flows. I use quarterly excess cash flows from 1996 Q3 to 2014 Q4, truncating and averaging the sample due to concerns about the stationarity of the underlying time series. This is imperative to consistently estimate investors' subjective distribution of cash flows in the ET estimation procedure and to make a valid statistical inference. Appendix A provides detailed insights into the construction and stationarity checks.

Aggregate excess cash flows are constructed by subtracting the aggregate benchmark public market mimicking cash flows ($\sum_{i=1}^{N} CF_{i,t}^{\text{mrkt}}$) from the aggregate PE cash flows ($\sum_{i=1}^{N} CF_{i,t}^{PE}$). The market benchmark fund is formulated following the methodology described in Korteweg and Nagel (2016). To construct mimicking cash flows, I set the date of transactions and assume that an LP investor allocates identical amounts of capital to both funds, ensuring capital calls of the two funds are identical in timing and magnitude.

While capital distributions share timing, they differ in magnitude. The size of capital distributions from benchmark funds is contingent upon public market performance. Specifically, the

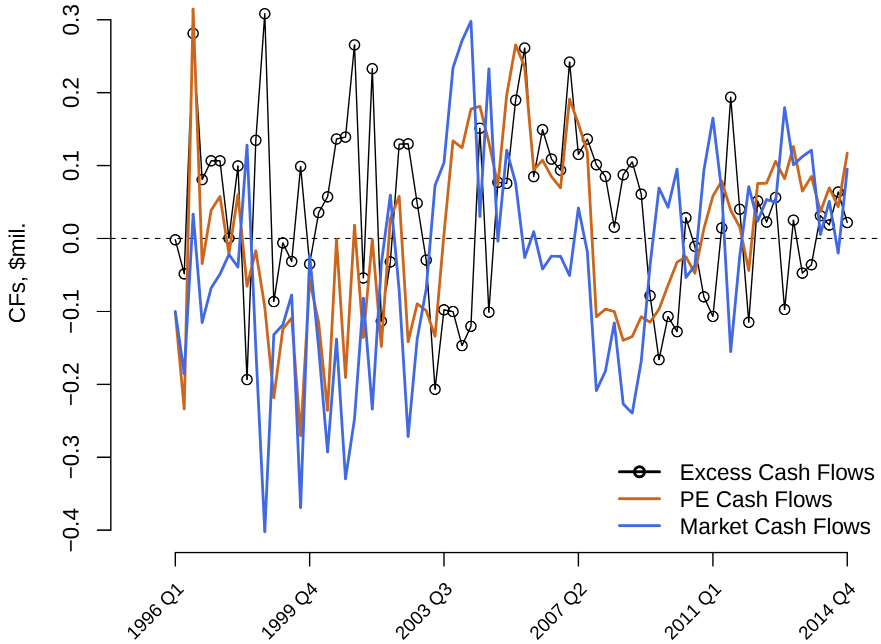

Figure 2: Average Excess Cash Flow and Component Analysis. This figure illustrates the value-weighted average excess cash flow of PE funds (black line) from Q1 1996 to Q4 2014. The average excess cash flow is derived from the difference between the average PE cash flows (brownish line) and the average market (CRSP) cash flows (blue line).

payout equals the accumulated returns between two consecutive dates, plus a fraction of the capital in a benchmark fund. This fraction is meticulously selected to align with the fund's lifespan.

Figure 2 presents the value-weighted average excess cash flow for PE funds spanning from 1996 Q1 to 2014 Q4. The excess cash flow fluctuates between a low of −\$0.21 million and a high of \$0.31 million, with a mean of \$0.04 million. It is essential to note that my analysis focuses on average excess cash flows due to their superior stationarity properties. Such properties are critical for ensuring robust beliefs estimation. All the cash flows considered are adjusted for inflation and are expressed in 1990 dollars. To better grasp the economic relevance of these excess cash flows, I compare them with the average payout, which is the average capital distributions across all funds for each period. The mean of this ratio is 6.6%, representing a substantial portion of the typical PE cash flow.

2.3.2 Summary Statistics. Table 1 presents the descriptive statistics for the sample of PE funds used in the analysis. The sample encompasses 976 funds orchestrated by 454 PE firms. On average, a PE firm in the sample has raised two funds, each with an average commitment of \$1.9 billion. The typical fund in the sample experiences 30 unique cash flows (inclusive of capital calls and distributions) and spans a period of 10 years between its initial and final observed cash flow.

The performance metrics are notably striking, both in absolute and relative terms. The median (average) fund posts an IRR of 10.78% (11.34%) and a total value-to-paid-in (TVPI) ratio of 1.52 (1.65). The GMPE metric is computed based on the GMM estimation of risk preferences in the SDF[5]. The abnormal performance for PE funds in the sample is affirmatively positive, equating to 25 cents and 15 cents for the GMPE and PME metrics, respectively. Notably, buyout funds have a reputation for outperforming the public market, in stark contrast to, for instance, VC funds, see Korteweg (2019). This divergence underscores the reason for centring the paper's focus on these funds: to avert interpretational ambiguity in the findings. Nevertheless, this focus is not predominantly derived from a methodological perspective.

Table 1: Summary Statistics: PE Fund Data. Data sourced from Preqin focuses on closed and liquidated funds incepted between 1980-2015, with a commitment threshold of at least $5 million (in 1990 dollars). Variables showcased include 'Fund size' (total commitment in millions), 'Fund effective years' (duration between the first and last observed cash flows), and performance measures such as IRR, TVPI and GPME. The GPME (Generalized Public Market Equivalent) captures discounted cash flows across funds using a specific SDF.

	Mean	Median	St.dev
# funds	976		
PE firms	454		
Funds/PE Firms	2.15	2.00	1.64
Fund size	1848	783	2812
Fund effective years	10.31	10.63	4.60
# of cash flows / fund	29.37	30.00	12.29
IRR, %	11.34	10.78	13.96
TVPI	1.65	1.52	0.76
PME, % – a/r per $1 com.	15.43	11.85	39.64
GPME, % – a/r per $1 com.	25.43	-0.71	125.38

3 Empirical Results

My methodology estimates beliefs using a two-step approach. First, I estimate the parameters of the SDF. In the second step, using the estimated SDF, I derive a subjective probability measure that aligns with observed PE valuations while minimizing deviation from the empirical distribution of cash flows. The organization of this section is as follows: 1) I elaborate on how I construct and estimate various mainstream SDFs in sub-section 3.1. 2) Next, utilizing the estimated SDFs, I delve into the economic implications of these beliefs, using the K-N CAPM SDF from the prior PE

[5] Results from the GMM estimation are detailed in Table 3.

literature, in sub-section 3.2. 3) I then generalize the economic implications across a broader class of SDFs in sub-section 3.3. 4) Finally, by conducting a counterfactual exercise using model-implied probabilities, I demonstrate the statistical significance of the subjective beliefs narrative in the PE context in sub-section 3.4.

3.1 Mainstream SDFs Construction and Estimation

My methodology facilitates the extraction of beliefs under diverse assumptions regarding risk preferences and the economic structure. Models can be constructed with parameters either estimated from cash flow data or directly sourced from existing literature ('off-the-shelf'). This adaptability is essential because LPs, including pension funds, endowments, and sovereign funds—whose beliefs I aim to quantify—may have varying investment objectives. For instance, some might be more willing to bear the additional liquidity costs associated with committing their capital for extended periods, often exceeding a decade. Others might anchor their investment decisions on the necessity to redistribute income across generations. Therefore, it is logical to consider a broad array of SDFs, each rooted in unique assumptions about the underlying economy.

In this sub-section, my objective is to construct and estimate these models to determine if they can explain PE valuations under the RE hypothesis. Furthermore, I explore how beliefs, when estimated under these various models, differ from one another. I consider a range of consumption-based asset pricing models, moving beyond the exponential CAPM (K-N CAPM) of Korteweg and Nagel (2016). This spectrum includes the C-CAPM (Consumption-CAPM) by Rubinstein (1976), the EHB (external habit formation) model by Campbell and Cochrane (1999), the LRR (long-run risks) model by Bansal and Yaron (2004), the exponential CAPM complemented with the size premium (CAPM+SMB), and Kaplan and Schoar (2005) version of the CAPM SDF (K-S CAPM).

Table 2 outlines the functional form of each SDF. For all the models, I estimate the relative risk aversion (RRA) parameter using a GMM procedure that incorporates model-specific unconditional moment restrictions. This estimation procedure is designed to select SDF parameters that can accurately price the benchmark fund that invests in the public equity market. In the CAPM and CAPM+SMB models, I introduce additional factors such as the Treasury Bill and the small growth portfolio benchmark. This approach facilitates the estimation of both the temporal component and the risk price tied to investing in a small-growth portfolio, one of the six portfolios highlighted by Fama and French (1993).

Constructing the LRR and EHB models involves significant complexity. I adhere to the method-

Table 2: SDFs Construction. Displayed in this table are the specific models and their functional forms used to derive Stochastic Discount Factors (SDFs), as shown in Panel A. Models require the estimation of parameters (like risk-preferences) distinctly from subjective beliefs, and if estimated, they are denoted with 'hats', e.g., $\hat{a}$, $\hat{b}$. For models like LRR and EHB, Panel B reveals how to build the one-period-ahead SDF, specifying the consumption-based time series and the required calibration parameters. Notably, lowercase variables represent their log values. The consumption data is sourced from the National Income and Product Accounts of the Bureau of Economic Analysis, spanning from 1949 Q4 to 2019 Q4.

Panel A: Functional form		
Model	One-period-ahead SDF (*log*)	Paper
K-N CAPM	$m_{t+1} = \hat{a} - \hat{b} r_{m,t+1}$	Korteweg & Nagel (2016)
K-S CAPM	$m_{t+1} = -r_{m,t+1}$	Kaplan & Schoar (2005)
CAPM+SMB	$m_{t+1} = \hat{a} - \hat{b} r_{m,t+1} - \hat{c} r_{smb,t+1}$	Korteweg & Nagel (2016)
C-CAPM	$m_{t+1} = log(\delta) - \hat{b}\Delta c_t$	Rubinstein (1976)
EHB	$m_{t+1} = log(\delta) - \hat{b}\Delta(c_t - h_t)$	Campbell & Cochrane (1999)
LRR	$m_{t+1} = log(\delta) - \frac{1}{\psi} x_t - \kappa_c \frac{1-\psi\hat{b}}{\psi(1-\rho_x\kappa_c)} e_{x,t+1} - \hat{b} e_{c,t+1}$	Bansal & Yaron (2004)

Panel B: Construction	
Model	Consumption-based time series
EHB	$H_t = C_t - C_t S_t$ $s_t = (1-\phi_{QQ})\log(\bar{S}) + \phi_{QQ} s_{t-1} + \lambda(s_{t-1}) u_t$ $\lambda(S_t) = 1/\bar{S}\sqrt{1-2(s_{t-1} - \log(\bar{S}))} - 1$, $s_t \leq s_{\max}$, and 0 otherwise $S_{\max} = \exp(\log(\bar{S}) + \frac{1}{2}(1-\bar{S}^2))$ $\bar{S} = \sigma(\Delta c_t)\sqrt{\frac{\gamma}{(1-\phi_{QQ})}}$ $\phi_{QQ} = \phi_{mm}^3 = 0.89^3$, $\gamma = 2$, $\delta = 0.998$
LRR	$x_t = \Delta\hat{c}_t$, $\Delta c_t = [\Delta c_{t-1}, \Delta cy_{t-1}, pd_{t-1}, rf_{t-1}, spread_{t-1}]$ $e_{x,t+1} = x_{t+1} - \rho_x x_t$, $e_{c,t+1} = \delta\hat{c}_{t+1} - x_t$ $sd(e_{c,t+1}) = \sigma_{QQ}$, $sd(e_{x,t+1}) = \phi_e \sigma_{QQ}$ $\rho_x = \rho_{mm}^3 = 0.987^3$, $\psi = 2$, $\kappa_c = 0.9649$, $\sigma_{QQ} = \sqrt{3}\cdot\sigma_{mm} = \sqrt{3}\cdot 0.0078$, $\phi_e = 0.1085$ and $\delta = 0.998$

ology outlined in the original paper for the EHB model and to the approach described by Colacito and Croce (2011) for the LRR model. Investors in the EHB model of Campbell and Cochrane (1999) aim to keep their consumption above the current habit in the economy. This behaviour aligns with the objective of the PE investors to smooth their future consumption. On the other hand, the LRR model, as introduced by Bansal and Yaron (2004), posits that consumption has a small but persistent component inducing uncertainty. Investors in the LRR model are concerned with long-term consumption uncertainty and aim to mitigate it. Although the central idea resonates with the habit model, the fundamental mechanisms are distinct, leading to different asset pricing implications.

Table 3 showcases the estimation outcomes of all models. The point estimate of the RRA coefficient for the K-N CAPM model stands at 3.63, which is considerably higher than the assumption dictated by the log-utility model proposed by Kaplan and Schoar (2005). The J-test results do not negate the null hypothesis of zero pricing errors. However, the GPME value, being economically significant at 25 cents of outperformance per committed dollar, indicates potential model misspecification.

Table 3: GPME Assessment of PE Funds Using Various SDFs. The table examines the abnormal performance of PE funds by utilizing the SDFs referenced from Table 2. The following models are estimated: *K-N CAPM* – Korteweg and Nagel (2016) SDF, *K-S CAPM* – Kaplan and Schoar (2005) SDF, *C-CAPM* – Rubinstein (1976) SDF, *EHB* – Campbell and Cochrane (1999) SDF, *LRR* – Bansal and Yaron (2004) SDF. These models are calibrated to price benchmark funds, resembling the inflow structure of PE funds, and investing in the CRSP value-weighted index and T-bills. Each SDF is mapped to its respective abnormal performance and parameters. Additionally, t-tests are applied to these parameters, with p-values provided in round brackets. For SDF parameters with absent p-values, values are sourced directly from existing literature. The table concludes with the J-test, analyzing if the abnormal performance is statistically equal to zero.

SDF	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	EHB	LRR
Abnorm.perf. (per \$1)	0.25	0.15	0.38	0.14	0.13	0.10
			SDF parameters			
a	0.161	0	0.276			
	(0.01)		(0.00)			
b	3.63	1	4.00	2.01	3.23	8.01
	(0.00)		(0.00)	(0.00)	(0.02)	(0.09)
J–stat.	2.46	12.17	22.18	13.30	2.89	3.14
$P(\chi_1^2 > J)$	0.11	0.00	0.00	0.00	0.09	0.07

The alternate models unanimously reject the null hypothesis of zero-pricing error at the 10% significance level. The findings for the K-S CAPM model reflect a GPME estimate of 15 cents. Similarly, the C-CAPM model by Rubinstein (1976) exhibits similar estimates, 14 cents. Both the LRR model and the EHB model show relatively lower pricing errors, with abnormal performance of 10 and 13 cents, respectively. These findings highlight the important role of the EHB and LRR models in understanding PE investor behaviour. All else being equal, the beliefs implied by these models should be more accurate for the pricing kernels since their pricing error is the smallest under the RE hypothesis.

To shed light on the dynamics of pricing errors, which underpins the observed outperformance of PE funds using the GMPE metric, I applied the estimated SDFs to the aggregate excess cash flows. Figure 3 graphically represents the pricing error in terms of millions of dollars over time, supplemented by the Mean Standard Errors (MSE) for each SDF. Both the EHB and LRR models consistently exhibit superior precision in their pricing of the excess cash flows. The differences among the pricing errors for the K-N CAPM, K-S CAPM, and C-CAPM SDFs are small. By minimizing these pricing errors, I estimate an alternative probability distribution for excess cash flows. Subsequent sections delve into the economic implications of extracted investor beliefs.

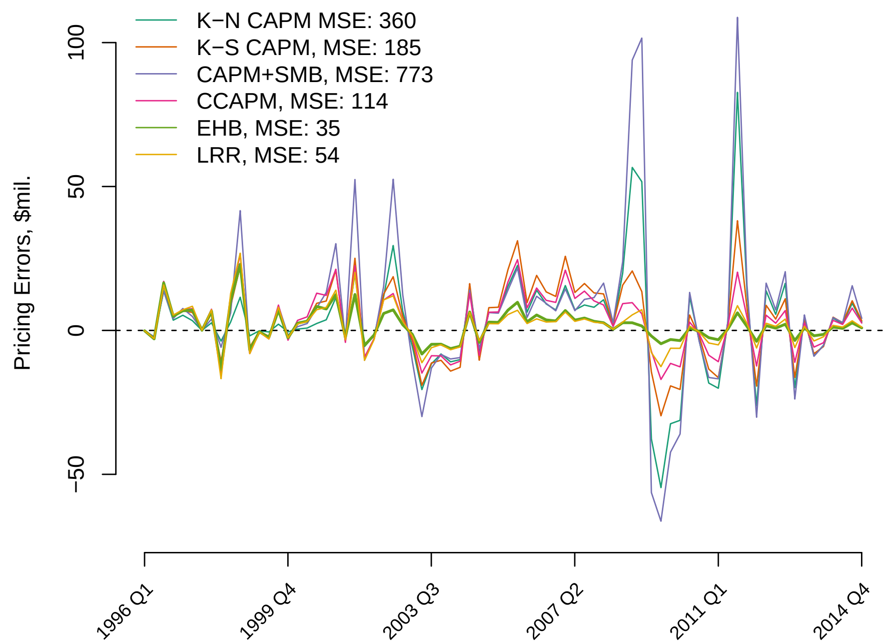

Figure 3: Analysis of Pricing Errors Across Time and SDFs. The figure plots the period-wise errors of average excess cash flow pricing using different SDFs for 1996 Q1 – 2014 Q4. The legend of the plot presents considered SDFs and their corresponding Mean Squared Errors (MSE). All numbers are in millions of dollars.

3.2 Subjective Beliefs of PE Investor

Having estimated the SDF parameters, I can identify subjective beliefs consistent with the observed excess cash flows, thereby eliminating the pricing error resulting from SDF misspecification. To accomplish this, I ascertain an alternative probability distribution using Equation (10), ensuring that $\sum_{t=1}^{T} p_t M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0$, where p_t denotes model-implied probabilities. Subsequently, I apply these probabilities to excess cash flows to obtain investors' subjective beliefs. In this subsection, my focus is centred on the beliefs inferred from the K-N CAPM SDF, aiming to unravel the primary economic narrative they provide. I later demonstrate that this narrative extends to other SDFs as well.

The results illustrating model-implied probabilities and belief-adjusted components of excess cash flows are depicted in Figure 4. The figure's upper panel showcases the profile of model-implied probabilities for the K-N CAPM SDF. Evidently, these probabilities exhibit heightened sensitivity during recessionary periods, particularly in the aftermath of the dot-com bubble and the global financial crisis—periods defined by shaded grey regions. The spikes observed in 2003 Q2 and 2009 Q3 indicate that investors took time to recalibrate their expectations, with the recalibration duration shortening between these two disastrous events.

The figure's dashed line corresponds to the empirical (or historical, represented by $1/T$) probabilities associated with the occurrence of excess cash flow. Deviations of implied probabilities from this line suggest discrepancies between investor expectations and market trends. The ET methodology strives to minimize these discrepancies, aligning them with the zero excess moment condition stipulated in Equation 8. For instance, in the aftermath of pronounced recessions, the inferred probabilities diverge significantly from those implied by the RE hypothesis, mirroring the pronounced pricing errors illustrated in Figure 3. Such an outcome highlights the mechanism underpinning belief extraction. It underscores the pivotal role of subjective beliefs during crises, especially when asset pricing models based on the RE hypothesis fail to reflect market dynamics.

In the middle and bottom panels of Figure 4, I examine the belief-adjusted components of excess cash flow – the PE cash flow and the public market cash flow. I aim to identify the underlying reasons for the positive abnormal performance of PE funds. The middle panel displays the difference between the belief-adjusted and observed aggregate PE cash flows, termed as the *beliefs discrepancy*. Remarkably, for around 63% of the observed period, the beliefs discrepancy was negative. This suggests that investors often underestimate the performance of PE funds, especially during crises. However, patterns vary. At the onset of the global financial crisis, investors considerably underestimated its effect on PE cash flows. This optimism faded by 2009 Q3 when the discrepancy between expected and actual PE cash flow reached about $25 million, the largest observed difference.

The bottom panel of Figure 4 presents an analysis of the belief-adjusted public market component. The findings contrast sharply with the PE cash flow results. Around 65% of the observed

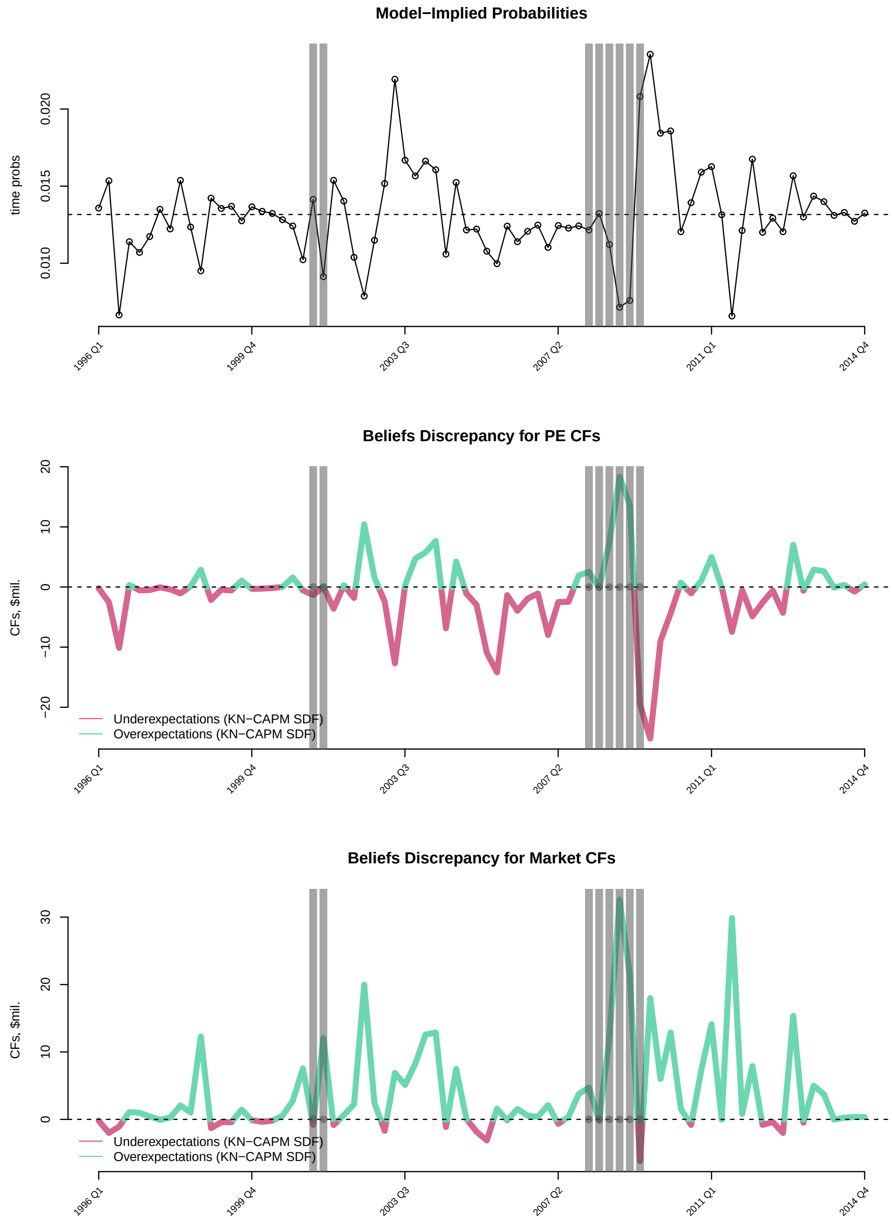

Figure 4: Model-Implied Probabilities and Subjective Beliefs. The figure is structured into three panels, capturing different dimensions of model-implied probabilities and belief-adjusted cash flows over the period 1996 Q1 to 2014 Q4. On the upper panel, the figure plots the model-implied probabilities estimated under K-N CAPM SDF. On the middle and bottom panel, the figure captures the *Beliefs Discrepancy*, which is the difference between belief-adjusted and observed aggregate cash flows, presented in $ million. The middle panel delves into the PE fund cash flows, while the bottom panel delves into the mimicking cash flow of the CRSP benchmark fund.

period showed a positive beliefs discrepancy, which was also greater in magnitude. Investors typically overestimate the public market cash flow, even in crisis times. A notable discrepancy was in

2009 Q2 when they expected the CRSP cash flow to be \$6 million less than the actual. However, in the crisis apex in 2008 Q4, PE investors held a strong belief in the public market's cash flow, resulting in a \$32 million overestimation of public market cash flow.

The results of my analysis indicate that investors are pessimistic about PE cash flows and, concurrently, overly optimistic about the public market. Such beliefs lead to excessive compensation for the risk of holding PE fund stakes, resulting in the observed positive abnormal performance of PE funds. I do not take a stance on the potential sources of observed biases in expectations. PE investors might still be entirely rational, use all available information, and update their expectations using the Bayes rule. The bias may arise from the opacity of the asset class. An alternative explanation might be behavioural. Overconfidence about current portfolio holdings is a well-documented phenomenon in behavioural finance, as noted by Glaser and Weber (2007). This type of behavioural bias is not confined to unsophisticated investors; even experts, as shown by Elan and Goodrich (2010), can overestimate their ability to forecast the future of their investments compared to the average person.

3.3 Subjective Beliefs for Various SDFs

In this sub-section, I extend the economic implications of subjective beliefs, as discussed in the previous sub-section, to a broader set of SDFs. Figures 5 and 6 display the *belief discrepancies* for the aggregate PE and market cash flows across various models. Notably, patterns of investor expectations for components of excess cash flows appear consistent when compared to beliefs estimated under the K-N CAPM SDF.

PE Cash Flows. Table 4 shows that the beliefs discrepancy is predominantly negative for PE cash flows across all evaluated models. The beliefs extracted from the EHB and LRR SDFs present the least discrepancy, being more pessimistic than the actual PE cash flow just 52% of the time. Nonetheless, the magnitude of negative expectations about the PE cash flows notably surpasses the positive ones. For example, the $min - max$ discrepancy range for the EHB model spans from -13.41 million dollars to 7.8 million dollars. The range for the LRR model is from -16.26 million dollars to 9.2 million dollars. Contrary to beliefs inferred from the K-N CAPM SDF, such pessimism is not specifically centred around major catastrophic events in all models. For instance, with the K-N CAPM, K-S CAPM, CAPM+SMB, and C-CAPM models, the beliefs are considerably more sensitive to the GFC occurrence rather than the dot-com bubble. However, for the EHB and LRR models, the influence of the dot-com bubble is more pronounced.

Table 4: Beliefs Discrepancy for PE Cash Flows. The table provides statistics related to the *beliefs discrepancy* for PE cash flows, as visualized in Figure 5.

	Beliefs Discrepancy under:					
	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	EHB	LRR
Underexepctations, % time	63%	55%	58%	56%	52%	52%
Max Discrepancy, $ millions	18.32	15.44	22.98	13.8	7.8	9.2
Max Discrepancy - YQ	2008 Q4	2004 Q2	2008 Q4	2004 Q2	1999 Q3	1999 Q3
Min Discrepancy, $ millions	-25.21	-25.12	-26.92	-21.66	-13.41	-16.26
Min Discrepancy - YQ	2009 Q3	2005 Q4	2009 Q3	2005 Q4	1996 Q3	1996 Q3

Figure 5 illustrates the evolution of beliefs discrepancy over time. For the C-CAPM, LRR, and EHB models, scepticism regarding the cash flow-generating abilities of PE funds is strong at the start of the sample period. This could be attributed to the phenomenon of 'regret aversion' or 'loss aversion' among investors, as introduced by Tversky and Kahneman (1991). Their research indicates that investors tend to be more sensitive to potential losses than to gains. At the inception of the new asset class, PE might have been perceived as a riskier investment compared to the public market. For instance, BO funds often invest in portfolios of companies, understanding that only some might see significant success. Such an approach might seem riskier to LPs. Moreover, there is a pronounced variation in return-generating skills among fund managers, even among those specializing in BO strategies, as highlighted by Korteweg and Sorensen (2017). All these elements could contribute to the negative beliefs discrepancy at the sample's outset. Yet, for the K-N CAPM, K-S CAPM, and CAPM+SMB, the GFC led to higher volatility in the beliefs discrepancy.

Concurrently, the pre dot-com bubble era resulted in a positive beliefs discrepancy across all models. Especially, EHB and LRR investors demonstrated notable optimism during the late 1990s when the dot-com bubble was inflating. The peak was observed in 1999 Q3, several quarters before the bubble burst, leading to the subsequent recession in 2001 Q2-Q3.

These findings underscore the varying perceptions across different models regarding distinct recession periods. According to the findings of Ning, Wang, and Yu (2015), the impact of the 2008 financial crisis on the venture industry was less profound than that of the 2000 tech crash, potentially because venture firms were still rebounding from the steep decline post the 2000 peak. The subjective beliefs estimated for the EHB, LRR, and C-CAPM investors validate these findings for the PE sector.

In sum, the insights derived from the analysis of PE cash flows reveal a persistent pattern of pessimistic beliefs among investors, regardless of their underlying risk preferences. Episodes of pronounced optimism frequently antecede intervals of marked pessimism. While catastrophic events

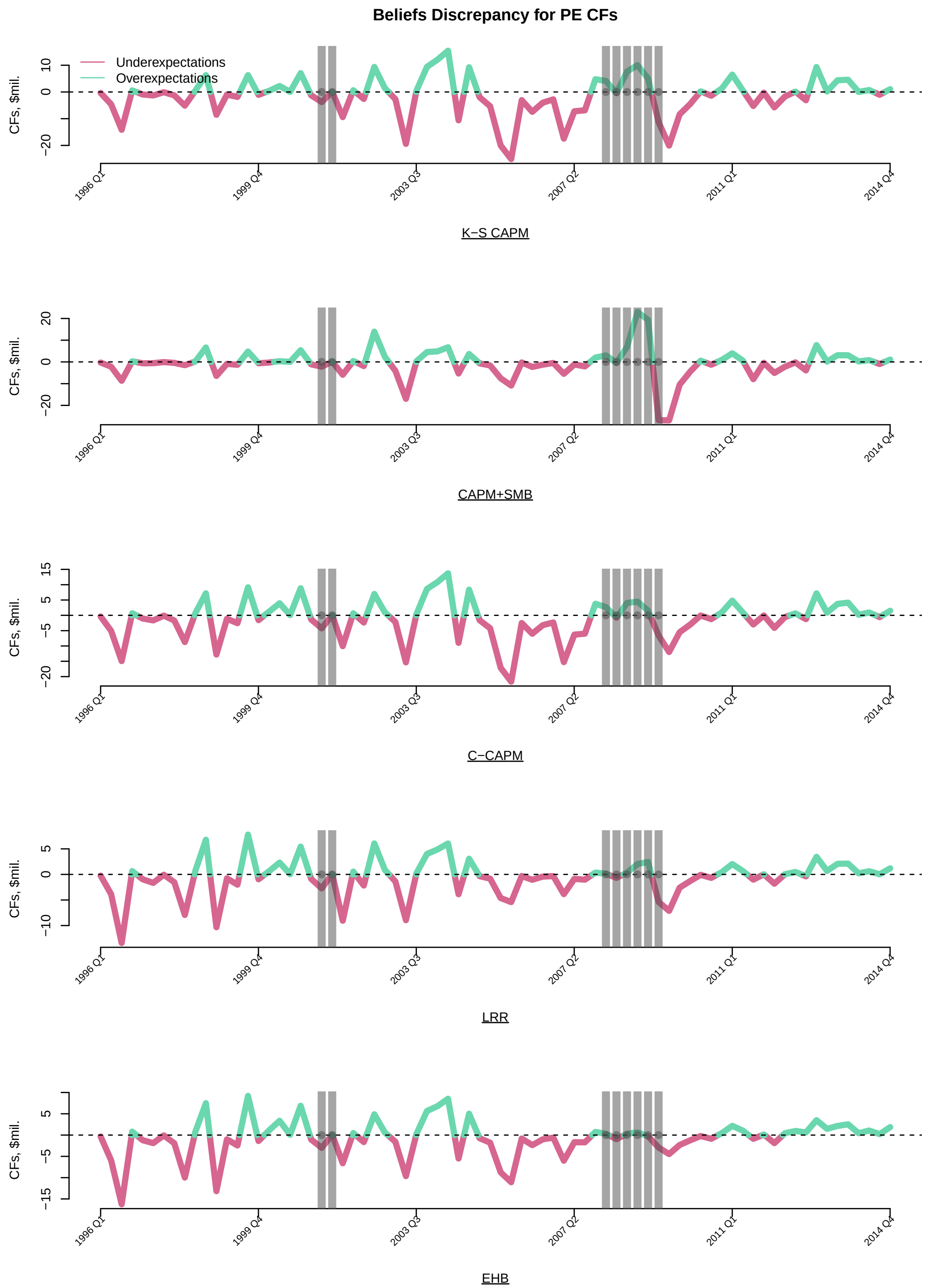

Figure 5: Aggregate PE Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate cash flows for PE funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived.

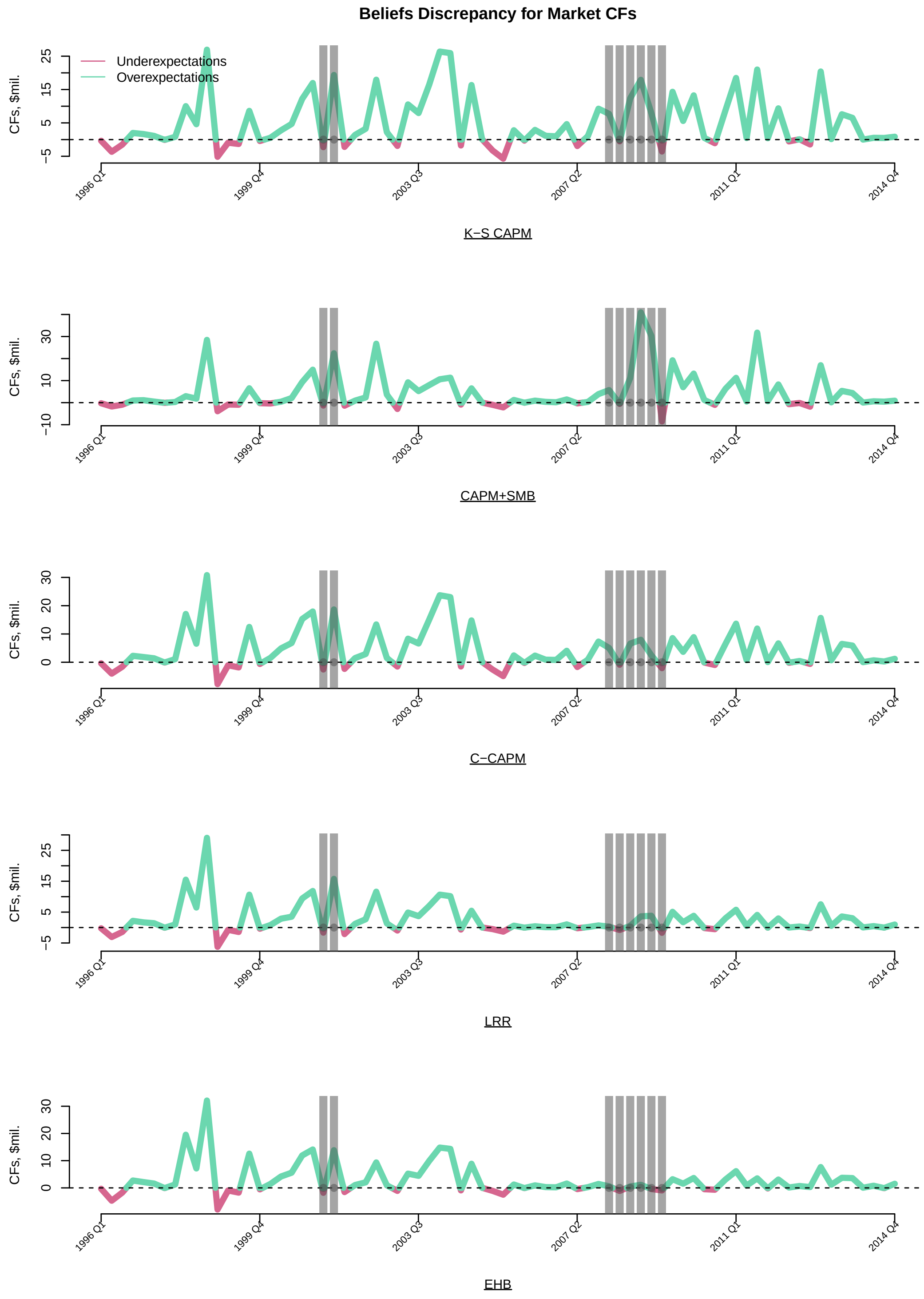

Figure 6: Aggregate Market Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate mimicking cash flows for CRSP-benchmark funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived.

undeniably influence the shaping of PE investors' expectations, the pattern is not consistently observed across all subjective beliefs. This underscores the complexity inherent in the investors' expectation formation process.

Market Cash Flows. Table 5 illustrates that the beliefs discrepancy for market cash flows is notably distinct from that of PE cash flows. Across all models, investors consistently hold an optimistic view of the CRSP cash flow, doing so 64%-72% of the time. Figure 6 suggests that investors' subjective beliefs consistently lean towards overestimating the public market's cash flow-generating capabilities. Considering the $min-max$ discrepancy ratio, this optimism is pronounced, with the narrowest range observed for the K-S CAPM from -5.72 million dollars to 26.94 million dollars.

Post-2005, the expectations of EHB and LRR investors regarding the CRPS index are consistently overoptimistic. Nevertheless, the magnitude of their beliefs discrepancy is significantly more subdued than those extracted under other SDFs. Moreover, belief discrepancies of EHB, LRR, and C-CAPM investors were not profoundly influenced by the 2008 financial crisis, in contrast to the beliefs underpinned by K-N CAPM, K-S CAPM, and CAPM+SMB SDFs.

Furthermore, Figure 6 depicts a prevailing trend of overoptimism preceding the dot-com bubble across all models. As highlighted in Table 5, the most pronounced surges of optimism, corresponding with the most significant beliefs discrepancies, are prominent for the EHB, LRR, and C-CAPM models around 1998 Q3.

Such trends can be attributed to the documented phenomenon of overoptimism during market booms, as pointed out by Kieren, Müller-Dethard, and Weber (2023). The unprecedented rise of the NASDAQ index, which began its ascent in 1995, might have galvanized investors to cultivate exceedingly optimistic beliefs about the public market.

Table 5: Beliefs Discrepancy for Market Cash Flows. The table provides statistics related to the *beliefs discrepancy* for market cash flows, as visualized in Figure 6.

	Beliefs Discrepancy under:					
	K-N CAPM	K-S CAPM	CAPM+SMB	CCAPM	EHB	LRR
Overexepctations, % time	64%	72%	70%	71%	69%	70%
Max Discrepancy, $ millions	32.62	26.94	40.94	30.80	29.05	32.09
Max Discrepancy - YQ	2008 Q4	1998 Q3	2008 Q4	1998 Q3	1998 Q3	1998 Q3
Min Discrepancy, $ millions	-6.19	-5.72	-8.51	-7.73	-6.23	-7.96
Min Discrepancy - YQ	2009 Q2	2005 Q3	2009 Q2	1998 Q4	1998 Q4	1998 Q4

In summary, PE investors exhibit greater optimism regarding public market cash flows and are more pessimistic about PE cash flows. This dichotomy culminates in a positive excess cash flow, which in turn translates into positive performance benchmarking metrics, such as GMPE. Consequently, I show that the primary source of positive abnormal performance for PE funds—unexplained by conventional asset pricing models—lies in the discrepancy between investors' beliefs about excess cash flows and those suggested by the RE hypothesis. The economic significance of the implications stemming from investor subjective beliefs is undeniable. The statistical significance of these beliefs discrepancy patterns will be formally tested in the subsequent sub-section, employing a counterfactual exercise based on model-implied probabilities.

3.4 Counterfactual Evidence

In this sub-section, I analyse the counterfactual cash flows implied by alternative probability distributions. A similar exercise was conducted by Julliard and Ghosh (2012) to evaluate the empirical validity of the rare disasters hypothesis in solving the equity premium puzzle. They used an alternative probability distribution to construct the counterfactual distribution of rare disasters (in terms of size and frequency). Similarly, I use alternative probability distributions to construct the counterfactual average risk-adjusted excess cash flow and its components based on the estimated beliefs. The results of the bootstrap analysis are presented in Table 6 in three distinct panels.

Panel A in Table 6 shows the results of the counterfactual analysis for excess cash flows. Here, the historical mean of discounted excess cash flows is juxtaposed against the bootstrapped mean. For all subjective beliefs, the bootstrapped mean of excess cash flows is virtually zero. The CAPM+SMB SDF emerges as the least precise pricing kernel in terms of pricing error (as also evident from Figure 3). The subjective beliefs inferred from this SDF appear to yield the least leptokurtic bootstrap distribution of excess cash flows, as depicted in Figure 7. Consequently, the 95% confidence interval for excess cash flows is the most wide for this distribution (refer to the third row of Panel A in Table 6). Nevertheless, the counterfactual mean for beliefs derived under the CAPM+SMB SDF remains below 1% of the Mean Average Payout (MAP). The MAP represents the average payout, where the aggregate payout is calculated as the cumulative capital distributions across all funds for each period.

Meanwhile, the SDF valuation under the RE hypothesis (first row, Panel A, Table 6) results in economically significant non-zero means for excess cash flows. The historical mean of risk-adjusted excess cash flows ranges from 5.53% for the LRR SDF to 12.27% for the CAPM+SMB. The significant difference between the historical and bootstrapped average highlights a conflict

Table 6: Counterfactual Excess, PE, and Market Cash Flows. This table presents the findings from a counterfactual analysis using 100,000 simulations, emphasizing subjective beliefs as importance weights for cash flows from 1996 Q1 to 2014 Q4. Each column represents an SDF, indicating where subjective beliefs originate. Panel A, B, and C each display results for Excess Cash Flows, PE Cash Flows, and Market Cash Flows, respectively. Statistics in each panel include the historical mean, bootstrap mean, 95% confidence interval for the bootstrap mean, and the probability of the counterfactual mean surpassing the historical mean. Values are portrayed as percentages of the Mean Average Payout (MAP).

	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	LRR	EHB
			Panel A: Excess Cash Flows (in % of *MAP*)			
$\overline{CF_T^{exc}}$	6.39	8.36	12.27	8.11	5.53	5.54
$\overline{CF_{boot}^{exc}}$	-0.02	0	0.88	0	-0.01	-0.01
$[\overline{CF_{boot2.5\%}^{exc}};\overline{CF_{boot97.5\%}^{exc}}]$	[-6.18;6.15]	[-5.73;5.74]	[-8.49;10.25]	[-5.45;5.46]	[-4.86;4.84]	[-3.99;3.97]
$Pr(CF_{boot}^{exc} > \overline{CF_T^{exc}})$	2.14%	0.24%	1.09%	0.2%	1.32%	0.33%
			Panel B: PE Cash Flows (in % of *MAP*)			
$\overline{CF_T^{PE}}$	-3.29	-1.12	-12.15	-2.53	-4.68	-3.07
$\overline{CF_{boot}^{PE}}$	-4.27	-3.76	-13.99	-5.23	-6.68	-5.41
$[\overline{CF_{boot2.5\%}^{PE}};\overline{CF_{boot97.5\%}^{PE}}]$	[-10.8;2.26]	[-9.88;2.37]	[-24.13;-3.85]	[-11.28;0.82]	[-11.72;-1.64]	[-9.99;-0.82]
$Pr(CF_{boot}^{PE} > \overline{CF_T^{PE}})$	39.05%	19.98%	37.47%	19.11%	21.77%	15.88%
			Panel C: Market Cash Flows (in % of *MAP*)			
$\overline{CF_T^{mrkt}}$	-9.68	-9.48	-24.43	-10.65	-10.21	-8.62
$\overline{CF_{boot}^{mrkt}}$	-4.26	-3.76	-14.87	-5.23	-6.67	-5.4
$[\overline{CF_{boot}^{mrkt}}_{2.5\%};\overline{CF_{boot}^{mrkt}}_{97.5\%}]$	[-12.19;3.68]	[-10.95;3.43]	[-29.4;-0.35]	[-12.13;1.67]	[-12.31;-1.03]	[-10.36;-0.44]
$Pr(CF_{boot}^{mrkt} > \overline{CF_T^{mrkt}})$	90.43%	93.89%	89.54%	93.63%	88.84%	89.53%

between beliefs implied by the RE hypothesis and the subjective beliefs estimated from PE cash flows.

In the third row of Panel A, the results of the counterfactual exercise are used to test the validity of using the RE hypothesis in PE valuation. The 95% confidence interval for the counterfactual excess cash flows across all models does not include the historically observed mean. This suggests that the RE hypothesis should be rejected. The discrepancy between subjective beliefs and those implied by the RE hypothesis is, therefore, the primary source of pricing error for PE funds. The SDF misspecification cannot solely offer a plausible explanation for the observed outperformance of PE funds. The valuation of PE funds based on subjective beliefs, as captured in the simulations, outputs the zero risk-adjusted excess cash flow that resolves the emerging valuation puzzle.

The fourth row of Panel A reveals that investors tend to value PE funds, assuming a very low

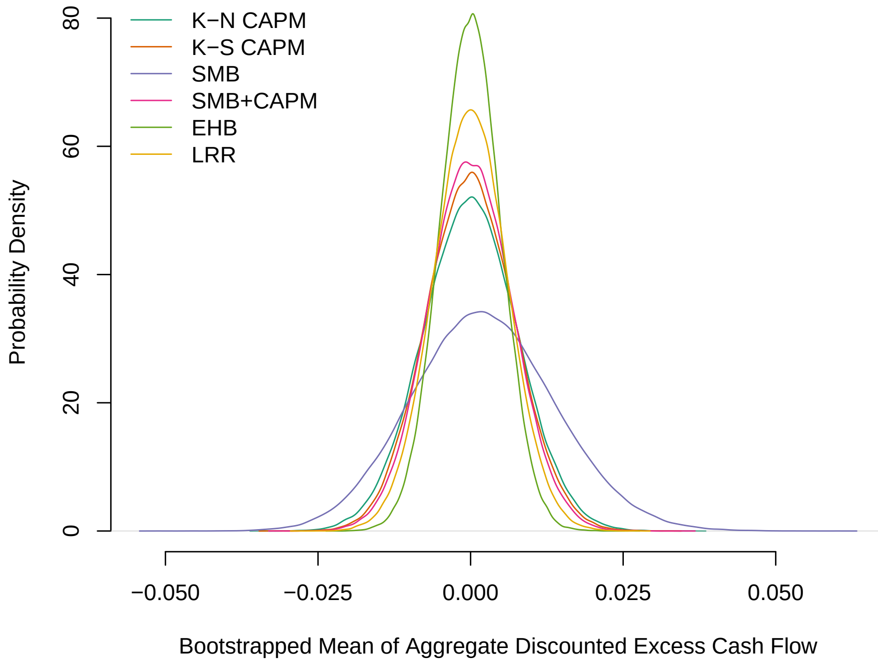

Figure 7: Bootstrap Density of Aggregate Risk-Adjusted Excess Cash Flows. The figure plots the bootstrap density of counterfactual excess cash flows implied by six distinct SDFs indicated in the legend of this graph.

probability of large excess cash flows. On average, the likelihood of observing a counterfactual excess cash flow that is at least as large as the historical one is less than 3% across all models. In the following panels, I will delve deeper into the implications of these findings for the components of the excess cash flow.

Panels B and C of Table 6 present the counterfactual distributional properties of the components of the excess cash flow. The third row of Panel B indicates that the historical PE cash flow is within the 95% confidence interval. Similarly, the average market cash flow that was observed empirically is also plausible across all models. Thus, the counterfactual experiment reveals that both components are plausible separately, but the likelihood of their combination leads to the unrealistically high historical excess cash flow.

The results from the fourth row of each panel in Table 6 provide insight into why the RE hypothesis is rejected for excess cash flows. With a probability greater than 88%, investors expect market cash flows to be higher than what was actually observed. On the other hand, with a probability not higher than 40%, investors expect PE cash flows to be higher than observed. As shown in Figures 5 and 6, investors tend to be more optimistic about market cash flows compared to PE cash flows. Thus, the bootstrap exercise supports the idea that the PE valuation puzzle is due to a combination of pessimism regarding PE cash flows and optimism regarding stock market cash flows.

In summary, the results of the bootstrap exercise show that subjective beliefs play a crucial role in explaining the abnormal excess cash flows of PE funds. The RE hypothesis can be rejected with substantial statistical confidence, indicating that investors tend to overestimate public market performance while underestimating PE funds' performance. Such discrepancy in beliefs leads investors to demand additional compensation for the uncertainty associated with future cash flows, resulting in the observed undervaluation of funds. These findings are consistent across all mainstream asset pricing models.

However, there are several unaddressed concerns. The first concern is how to ensure that estimated beliefs represent the investors' sentiment. The following section addresses this issue. The second concern is investors' (LPs) ability to translate their beliefs into a price discount. The potential heterogeneity in fee schemes and its influence on the economic implications of estimated beliefs is elaborated upon in Appendix B. The third concern is about alternative approaches for beliefs estimation. Appendix D provides insights on this, with Appendix D.1 examining the SDF estimation based on the same Euler equation employed for beliefs estimation and its subsequent influence on beliefs implications. Appendix D.2 investigates the feasibility of estimating subjective beliefs through the moment condition for aggregate cash flows despite their suboptimal stationarity properties compared to average cash flows. Lastly, Appendix D.3 contemplates the prospect of estimating investors' subjective beliefs on a fund-by-fund basis, thereby harnessing the available cross-sectional heterogeneity.

4 Analysis of Beliefs

In this section, I will address the question of whether extracted subjective beliefs accurately reflect the opinions of investors. To do this, I will compare estimated beliefs with the results of publicly available surveys, as described in sub-section 4.2. In sub-section 4.1, I will provide an overview of the survey data. Additionally, I will address potential concerns regarding the sensitivity of estimated beliefs to misspecification of the SDF, as well as the adequacy of PE sentiment indices in reflecting opinions of investors, in sub-sections 4.3 and 4.4, respectively.

4.1 Survey Data

I make use of survey data for three types of investors: individual, institutional, and Private Equity investors. The opinions of individual investors have received significant attention in the paper by Greenwood and Shleifer (2014). I utilize two surveys from their paper: 1) the monthly Gallup survey (*Gallup*) conducted between 1996 and 2012 with occasional gaps and 2) Robert Shiller's survey of wealthy individual investors (*Shiller (Ind)*) available sporadically between 1999 and July 2001 and conducted monthly afterwards. The opinions of institutional investors are represented by Robert Shiller's institutional investor survey (*Shiller (Inst)*) conducted at six-month intervals from July 1989 to July 2001 and then monthly afterwards.

Table 7: Sentiment Indices Correlation. The table reports the correlations between Institutional (*Shiller (inst)*), Individual (*Shiller (ind), Gallup, ESI*) and PE (*CEPECI, SVVCCI*) sentiment indices.

	Shiller (inst)	Shiller (ind)	Gallup	ESI	CEPECI	SVVCCI
Shiller (inst)	—	0.51***	-0.52***	-0.55***	0.09	-0.05
Shiller (ind)	0.51***	—	-0.10	-0.17	0.57***	0.43***
Gallup	-0.52***	-0.10	—	0.78***	0.90***	0.83***
ESI	-0.55***	-0.17	0.78***	—	0.86***	0.76***
CEPECI	0.09	0.57***	0.90***	0.86***	—	0.82***
SVVCCI	-0.05	0.43***	0.83***	0.76***	0.82***	—

Note: *p<0.1; **p<0.05; ***p<0.01

The Silicon Valley Venture Capital Confidence Index (*SVVCCI*) and Central Europe Private Equity Confidence Index (*CEPECI*) capture the beliefs of PE investors. As there is limited data specifically for investors focused on buyout (BO) strategies, these indices serve as a proxy for the opinions of PE investors in general. Figure 8 provides a visual representation of the time series for the PE indices and highlights that aggregated sentiments are similar. Table 7 shows a high correlation of 0.82 between the moods of PE and VC investors in the San Francisco area and Central Europe, suggesting that the sentiment of PE investors is closely tied to that of VC investors.

The SVVCCI is based on a recurring quarterly survey of San Francisco Bay Area venture capitalists, which has been conducted since Q1 2004. This index measures and reports the opinions of professional venture capitalists regarding their estimates of the high-growth entrepreneurial environment for the next 6 to 18 months. For more information about the quarterly survey, please see Appendix 2 of Cannice and Goldberg (2009).

The CEPECI is a semi-annual survey conducted by Deloitte of private equity professionals focused on Central Europe, see Deloitte's website. The index is based on the responses to the first

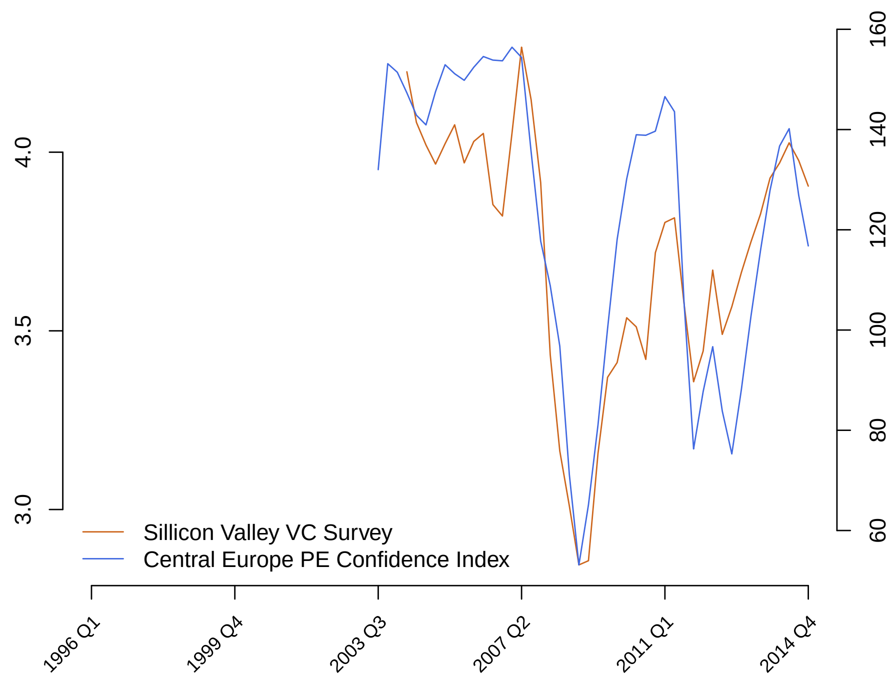

Figure 8: PE Investors Sentiments. The figure plots investors' expectations about the private equity market in Central Europe (blue line) and the San Francisco Bay area (brownish line). For Europe, the confidence index is represented by a survey of PE professionals conducted by Deloitte, while investors' moods of Silicon Valley VC capitalists are represented by a survey conducted by the University of San Francisco.

seven questions of the survey, which focus on topics such as economic climate, debt availability, investors' focus, size of transactions, market activity, investment return, and investors' activities. The positive answer ratio is calculated for each period by dividing the number of positive answers by the sum of positive and negative answers. This ratio is then compared to the base period of spring 2003 to obtain the final index value.

Further, I investigate how market expectations are related across different types of investors. Figure 9 depicts the evolution of three indices: Shiller Individual, Shiller Institutional, and CEPECI. The sentiment about PE and the public market is often misaligned. Even public institutional and individual investors do not always agree on the movement of the aggregate stock market, as shown by the correlation in the first column of Table 7. This finding is consistent with previous literature, as shown in the analysis of beliefs of heterogeneous public market investors by Ghosh, Korteweg, and Xu (2020).

Additionally, it is important to note that there can be disagreements within the same type of investor group. For example, the correlation between the Gallup survey and Shiller Individual survey is low and negative (-0.1), but it is not statistically significant. This discrepancy could be due to the fact that the two groups of investors are not as homogeneous as they may seem. As noted by Adam and Nagel (2022), the Shiller index surveys wealthy individuals, while the Gallup index surveys the general population.

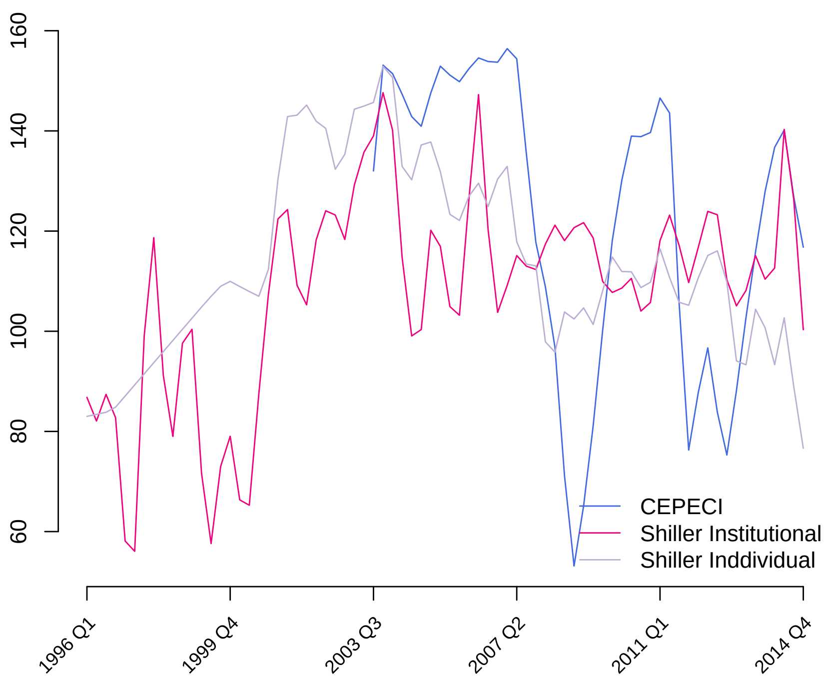

Figure 9: Sentiments for PE, Institutional, and Individual Investors. The figure plots investors' sentiments towards the stock market, as captured by the U.S. Institutional One-Year Confidence Index. It contrasts Shiller's Institutional Investors (pink line) and Individual Investors (violet line) using the left scale. Simultaneously, sentiments towards the private equity market in Central Europe are visualized by a blue line, interpreted using the right scale.

4.2 Subjective Beliefs and Investors' Sentiment

The central question of this sub-section is whether beliefs extracted from models align with survey data. We must first understand how expectations influence excess cash flows to answer this question. One important point to consider is that investors are asked in questionnaires to forecast cash flows over some distant future. To reflect this aspect of sentiment indices, I include them in Table 8 with a one-year lag. Also, it is crucial to understand that subjective beliefs estimated using my approach are related to the excess market cash flow. At the same time, investors usually make predictions about other quantities. Therefore, survey data may affect excess cash flows indirectly through PE and stock market components.

First, I inspect the PE survey data, represented by CEPECI and SVVCCI indices. For example,

Table 8: Correlations between Surveys and Excess Cash Flow. The table reports the correlations between various sentiment indices, presented row-wise, and the two key metrics: 1) belief-adjusted excess cash flow (shown in Panel A), and 2) a placebo test that omits the nuanced term structure of investors' expectations (shown in Panel B). Across the columns, the subjective beliefs are estimated using distinct SDFs.

	Panel A: Average Excess Cash Flow adjusted for Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.25**	-0.18	-0.18	-0.12	-0.08	-0.11
Shiller (ind)	-0.02	-0.02	-0.02	0.03	0.08	0.08
Gallup	0.45***	0.36***	0.36***	0.29**	0.26**	0.29**
ESI	0.39***	0.32***	0.32***	0.26**	0.24**	0.26**
CEPECI	0.60***	0.57***	0.57***	0.57***	0.57***	0.58***
SVVCCI	0.73***	0.70***	0.70***	0.69***	0.68***	0.68***
	Panel B: Placebo Test for Contemporaneous Correlations					
Shiller (inst)	-0.16	-0.14	-0.14	-0.10	-0.05	-0.07
Shiller (ind)	-0.13	-0.17	-0.17	-0.13	-0.07	-0.08
Gallup	-0.03	-0.11	-0.11	-0.16	-0.19	-0.16
ESI	0.33***	0.23**	0.23**	0.16	0.13	0.17
CEPECI	0.14	0.06	0.06	0.05	0.07	0.10
SVVCCI	0.29*	0.21	0.21	0.19	0.21	0.24

Note: *p<0.1; **p<0.05; ***p<0.01

San Francisco VC investors forecast the 'future high-growth venture entrepreneurial environment' on a scale from one to five. I interpret the level of optimism (pessimism) as their subjective probability of a positive (negative) PE cash flow, which in turn would result in a positive (negative) excess cash flow. As a result, it is expected that PE survey indices would positively correlate with the belief-adjusted average excess and PE cash flows with a one-year lag. Panel A of Table 8 shows that the survey data aligns well with subjective beliefs, as the correlation is high and statistically significant. These results suggest that the opinions of PE investors play a significant role in predicting the excess cash flows in the PE market.

Further, I inspect the individual and institutional sentiment indices. The results show a spuriously puzzling finding: individual investors correctly predict the belief-adjusted cash flows, while institutional investors do not. The correlation between the belief-adjusted cash flows and the institutional sentiment index, represented by *Shiller (inst)*, is negative and mostly insignificant, except for the K-N CAPM model. This suggests that institutional sentiments still may impact PE excess cash flows, but to a lesser extent than those of PE and individual investors.

The explanation for this finding is provided by Greenwood and Shleifer (2014), who discovered that individual investors' expectations in survey data are inconsistent with the rational expectations model. They find that individual investors' positive expectations generally lead to lower average stock returns. Thus, the optimism of individual investors leads to lower public market cash flows (misprediction), which in turn results in higher excess cash flows. This is why a positive correlation is observed. On the other hand, Ghosh, Korteweg, and Xu (2020) find that institutional investors, who are countercyclical in their expectations, accurately predict the aggregate stock market. Their positive expectations result in higher public market cash flows, reducing excess cash flows. This explains the negative correlation sign.

Panel B of Table 8 reports the results of a placebo test. In this test, I examine the correlation between contemporaneous levels of sentiments and belief-adjusted excess cash flows, without considering the potential lag between the time when the answers were given and the time when they might affect cash flows. The results reveal no correlation for the estimated beliefs, except for four pairs: CEPECI and K-N CAPM, ESI and K-N CAPM, ESI and K-S CAPM, and ESI and SMB+CAPM. It is worth noting that the subjective beliefs estimated under these models were the least accurate, as discussed in sub-section 3.3. Therefore, SDFs less contaminated by misspecification lead to a more precise estimation of subjective beliefs.

In summary, I show that estimated beliefs are in line with survey data. The opinions of PE investors have the strongest impact on predicting excess cash flows from investments in PE funds. The beliefs of individual and institutional investors affect excess cash flows through their impact on the public market component. The results of the placebo test highlight the potential influence of SDF misspecification on the estimated beliefs. The next sub-section addresses this issue in more detail.

4.3 Beliefs Adjusted for SDF Misspecification

If risk preferences are not adequately reflected in SDF, it may result in significant pricing errors. The errors associated with model misspecification might distort beliefs. In this sub-section, I test the potential impact of misspecification on estimated beliefs and investigate the potential sources of risks not captured by SDFs used in my analysis.

Public Market Conditions. Mainly, I investigate market fundamentals as a potential source of omitted factors. In Table 9, I test for expected market premia (*expected premia*), Moody's BBB to AAA credit spread (*credit spread*), the Chicago Board Options Exchange S&P 100 Volatility Index (*vxo*), and the aggregate liquidity factor of Pástor and Stambaugh (2003) (*liquidity*). This exercise aims to 'purify' the model-implied probabilities estimated in the previous section by adjusting for the potential SDF misspecification. The residuals from all six regressions will be used to construct the 'purified' beliefs. Furthermore, Table 9 provides insights into the most plausible public market factors that might have been missed in the SDFs considered in this paper.

Table 9: How Model-Implied Probabilities Align with Market Fundamentals. The table reports the regression analysis of model-implied probabilities against market fundamentals. The market fundamentals are presented by expected market premia, credit spread, volatility index, aggregate liquidity factor and their contemporaneous changes (Δ_t).

	Model-Implied Probabilities under:					
	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	EHB	LRR
	(1)	(2)	(3)	(4)	(5)	(6)
expected premia	0.0001	0.0003**	0.0003**	0.0003	0.0001	0.0001
	(0.0001)	(0.0002)	(0.0002)	(0.0002)	(0.0002)	(0.0001)
Δ_t(credit spread)	−0.008**	−0.001	−0.001	0.004	0.007	0.006
	(0.003)	(0.005)	(0.005)	(0.005)	(0.005)	(0.004)
Δ_t(vxo)	−0.011***	−0.015***	−0.015***	−0.014***	−0.011**	−0.010**
	(0.003)	(0.004)	(0.004)	(0.005)	(0.005)	(0.004)
Δ_t(liq.agg)	0.004	0.012	0.012	0.012	0.009	0.010
	(0.006)	(0.009)	(0.009)	(0.010)	(0.010)	(0.008)
Constant	0.013***	0.011***	0.011***	0.012***	0.013***	0.013***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
Observations	75	75	75	75	75	75
R^2	0.412	0.313	0.313	0.219	0.132	0.159
Adjusted R^2	0.378	0.273	0.273	0.174	0.083	0.111
F Statistic	12.248***	7.956***	7.956***	4.909***	2.667**	3.313**
Note:				*p<0.1; **p<0.05; ***p<0.01		

The results from Table 9 indicate that changes in market volatility have a statistically significant effect on all model-implied probabilities. This is in line with the research by Gompers, Kovner, Lerner, and Scharfstein (2008), who find that VC firms increase their investments during periods of high volatility in the public market, with no effect on their performance. Limited Partners appear to respond similarly by allocating more capital into private equity, as they seek to escape the volatility of the public stock market[6]. As market volatility increases, it signals a contraction in public market cash flows, which leads to an increase in excess cash flows, all else being equal. As a result, changes in volatility are expected to positively correlate with average excess cash flows and negatively with model-implied probabilities[7].

[6] June 1, 2022, Altexchange.com; May 26, 2022,Wall Street Journal

[7] The distinction between subjective beliefs and model-implied probabilities lies in their calculation. Model-implied

The credit spread is another important factor to consider. For example, Schmid, Huether, and Steri (2019) explore the potential to value PE funds using information about their loan portfolio. They develop the credit market equivalent metric and show the significance of credit market movements on PE valuation. An increase in the credit spread signifies a decrease in public market cash flow and, therefore, a corresponding increase in the excess cash flow. Only beliefs estimated for the K-N CAPM model show the expected negative relationship and statistical significance for this variable.

The expected market premia can play a crucial role in the determination of PE excess cash flows. The prior research by Haddad, Loualiche, and Plosser (2017) show that changes in the aggregate risk premium can impact the PE industry. They find that a low risk premium increases the present value of performance gains and reduces the cost of holding an illiquid investment. As the expected market premia increases, it signifies a higher public market cash flow and, as a result, a lower excess cash flow. The K-S, K-N CAPM and CAP+SMB models show a positive and statistically significant relationship between expected market premia and model-implied beliefs. Overall, the credit market conditions and expected market premia are potentially important omitted factors, but their significance is lower compared to the market volatility factor.

Another factor to consider is the aggregate market liquidity, as suggested by Pástor and Stambaugh (2003). Despite its potential importance in PE, see, for example, Franzoni, Nowak, and Phalippou (2012), the regression results in Table 9 indicate that the aggregate market liquidity does not have a significant impact on model-implied probabilities. It is possible that the effect of aggregate market liquidity is overshadowed by other public market conditions considered here.

In conclusion, it appears that changes in market volatility, expected market premia, and credit conditions may be factors omitted in existing asset pricing models applied to PE cash flows. The question remains: how much of an impact does this potential misspecification have on subjective beliefs? To address this question, I use the residuals from the presented regressions as ('purified') orthogonalized model-implied probabilities to construct belief-adjusted cash flows and repeat the validation process as outlined in sub-section 4.2.

Orthogonalized Beliefs. To further examine the reliability of the results, I assess the robustness of the correlations observed in Table 8 when using 'purified' beliefs. To do this, I extract orthogonalized probabilities in the form of residuals from the regressions in Table 9. They are orthogonalized to the market conditions potentially omitted in the risk preferences. Then, I scale

probabilities are the alternative probability measure ($\mathbb{P}^*$), which eliminates the pricing errors, while subjective beliefs are obtained by applying a new probability measure to cash flows.

Table 10: A Correlation Between Survey Data and Orthogonalized Beliefs. The table reports the correlations between distinct sentiment indices and excess cash flow that has been adjusted for belief. These subjective beliefs, estimated using different models, are further refined to be orthogonal to prevailing market conditions, as detailed in Table 9.

	Average Excess Cash Flow adjusted for Purified Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.24**	-0.18	-0.18	-0.11	-0.08	-0.10
Shiller (ind)	-0.01	-0.01	-0.01	0.03	0.08	0.08
Gallup	0.42***	0.36***	0.36***	0.30**	0.28**	0.30**
ESI	0.38***	0.33***	0.33***	0.28**	0.26**	0.28**
CEPECI	0.60***	0.58***	0.58***	0.58***	0.58***	0.58***
SVVCCI	0.71***	0.69***	0.69***	0.67***	0.66***	0.67***

the time series of the residuals to ensure that they satisfy the probability conditions of summing to one and being non-negative.

As shown in Figure 10, the 'purified' probabilities strongly correlate with the original model-implied probabilities, indicating that subjective beliefs extracted using my methodology capture investors' opinions rather than misspecification. The correlation for the probabilities estimated under different pricing kernels is as follows: K-S CAPM – 0.83; CAPM+SMB – 0.83; CCAPM – 0.89; LRR – 0.92; EHB – 0.93. A higher correlation is observed for beliefs estimated under the SDFs that produce a lower pricing error for excess cash flows, which suggests that a more calibrated SDF outputs more informative beliefs.

I repeat the correlation analysis for subjective beliefs obtained using orthogonalized model-implied probabilities and survey indices. The results of Table 10 are similar to those found previously. The correlation between subjective beliefs and PE survey indices remains strong, with little to no change in statistical significance. The correlation between subjective beliefs and both individual and institutional investors also remains unchanged. Therefore, I conclude that my methodology robustly captures investors' subjective beliefs, even in the presence of potential misspecification of SDFs. The misspecification does not significantly impact model-implied probabilities and associated subjective beliefs.

4.4 PE Market Conditions and Investors' Sentiment

Beyond the concerns associated with estimated beliefs, I also examine the relationship between sentiment indices and the PE market. My main objective is to assess if there is a correlation between PE market conditions and investors' opinions and if this relationship has been supported

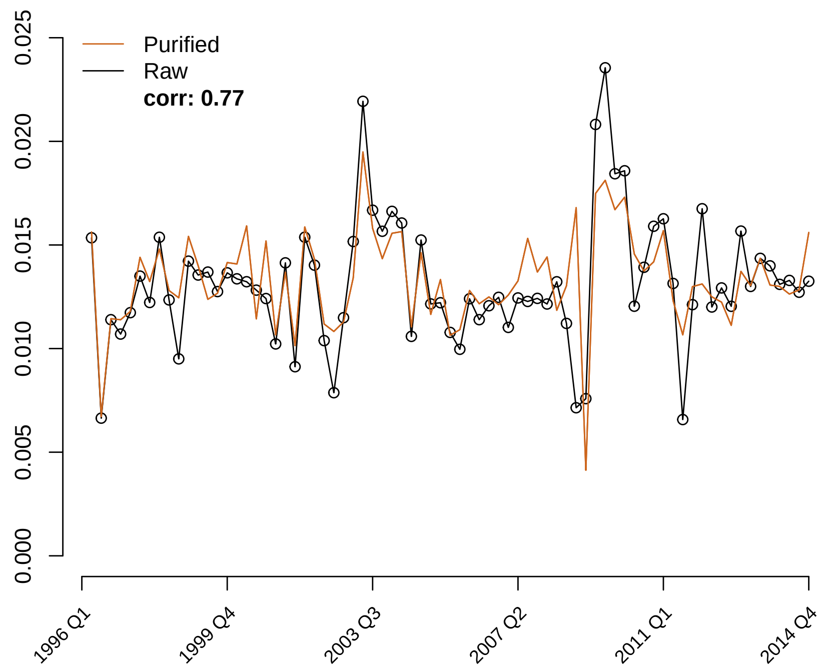

Figure 10: Original Versus Orthogonalized Model-Implied Probabilities. The figure plots the model-implied original probabilities and their orthogonalized counterparts. These orthogonalized probabilities are obtained post-projection to public market conditions as detailed in Table 9 for K-N CAPM SDF for 1996 Q1 – 2014 Q4.

by previous research. This examination sheds light on potential factors influencing the belief formation process. I will take into account factors like entry and exit conditions and transactions in the PE secondary market. Table 11 presents the results of the correlation analysis between these factors and one-step-ahead subjective beliefs.

The correlation analysis for PE sentiment indices reveals that PE market entry conditions, represented by fundraising activities, exert a pronounced influence on investor expectations. High levels of fundraising correlate with pessimistic expectations among PE investors. This observation aligns with Brown et al. (2021) findings, which indicate that periods of increased fundraising precede periods marked by subpar performance. This implies that fundraising activities adversely affect investors' outlook on future cash flows.

An active secondary market tends to have a negative influence on PE investors' sentiment. High levels of transactions in the secondary market correspond with more pessimistic expectations for future cash flows. This sentiment might be rooted in the notion that observing a surge in investors eager to exit the market dampens confidence in PE funds' future cash flow generation capabilities. This perspective gains credence from Nadauld, Sensoy, Vorkink, and Weisbach (2019), who demonstrate that buyers in the secondary market outperform sellers by offering liquidity to those eager to exit. Consequently, a spike in secondary market transactions could be perceived

Table 11: PE Market Dynamics and Surveyed Sentiments. This table reports the correlations between PE market dynamics and sentiments reflected in surveys targeting PE investors. Key PE market conditions encompass factors like the volume of capital raised by PE funds (*fundraise*), the number of secondary transactions (*scnd. transactions*), the number of BO IPO deals (*#BO IPO deals*), and the mean value of exit deals (*avg. BO exit size*). Moreover, contemporaneous changes in these factors (Δ_t) are considered. IPO data is provided by Ritter's website.

	CEPECI	SVVCCI
fundraise	-0.55***	-0.34**
Δ_t(fundraise)	0.17	0.14
scnd. transactions	-0.35**	-0.14
Δ_t(scnd. transactions)	0.03	0.03
#BO IPO deals	0.17	0.37**
Δ_t(#BO IPO deals)	0.24	0.23
avg. BO exit size	-0.19	0.04
Δ_t(avg. BO exit size)	0.02	0.11

as indicative of a less favourable PE market environment, leading to a negative correlation with investor sentiment. The data in Table 11 supports this perspective.

Traditionally, activity in the IPO market serves as a barometer of investor sentiment. Favorable exit conditions typically trigger positive investor expectations. As evidenced by Nanda, Samila, and Sorenson (2020), each additional IPO among a VC firm's first ten investments forecasts an elevated IPO rate—up to 8% higher—for subsequent investments. As illustrated in Table 11, there exists a positive correlation between the sentiment of VC investors, as captured by the SVVCCI index, and a dynamic IPO market. It is noteworthy that the volume of IPOs exerts a greater influence on shaping investor sentiment than the magnitude of exit transactions.

In summary, I show that the PE survey data, employed to validate estimated subjective beliefs, aligns sensibly with the prevailing conditions of the PE market. The analysis shows that robust fundraising activity and a bustling secondary market tend to dampen the optimism of PE investors. In contrast, a buoyant IPO market is linked with heightened investor optimism. These observations are consistent with prior research.

5 Conclusion

The valuation of PE funds is challenging. The existing asset pricing models fail to explain the PE valuation from the perspective of Limited Partners. In this paper, I argue that the conventional approach of discounting for time and risk using SDF is insufficient for explaining the observed abnormal performance of PE funds. The overlooked component in existing models, it appears, is the uncertainty tied to future cash flows. Investors in PE may demand additional compensation for this uncertainty beyond just the time and risk factors.

I find that investors tend to be pessimistic about the cash flows of PE funds but overly optimistic about public market cash flows. This results in general pessimism concerning excess cash flows, which is the sole source of abnormal performance when using benchmarking methods like GPME. To estimate investors' subjective beliefs from cash flow data, I introduce a novel methodology based on an Empirical Likelihood-type estimator. This approach estimates investors' beliefs from the time series of excess cash flows unconditionally, requiring only that the underlying time series is stationary.

The method operates in two stages: In the first stage, I determine the parameters of existing SDFs. In the second stage, I eliminate any remaining pricing errors from the time series of excess cash flow after discounting with the estimated SDF parameters. This allows me to back out an alternative probability distribution for excess cash flows. When paired with the SDFs used, this distribution can rationalize the observed PE fund valuations. The resulting model-implied probabilities, or alternative distribution, represent investors' subjective beliefs about excess cash flows after the relevant variable has been reweighted. Moreover, the estimated subjective beliefs are the closest in the Kullback-Leibler sense to the historical excess cash flows, implied by the Rational Expectation (RE) hypothesis.

However, I further show that there is a significant discrepancy, both economically and statistically, between investors' subjective beliefs and the cash flow distribution under the RE hypothesis. To validate the practical significance of these estimated beliefs, I have conducted a correlation analysis using survey data from PE investors. I also consider sentiment indices from public market investors, as these could indirectly influence the observed excess cash flow through the public market component. I show that estimated beliefs have the largest statistical significance and economic magnitude for the core PE investors. Furthermore, the public market sentiment shows substantial alignment with the observed public market component of excess cash flow. Overall, these findings underscore the practical significance of my proposed methodology.

Lastly, I have executed a battery of robustness tests to determine how resilient these beliefs are against potential contamination due to the model misspecification. I highlight that SDF misspecification can not critically distort the estimated beliefs. Moreover, this methodology proves robust even when alternative moment conditions that underlie the EL-type estimations are considered. It is worth noting that this method could be extended to other asset classes similar to PE, such as Venture Capital or Real Estate funds. It could also be used to discern investors' beliefs for various public market applications.

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Appendices

A Stationary Excess Cash Flows

The ET estimator relies on the stationarity of the underlying time series of excess cash flows for estimating unconditional subjective beliefs. However, challenges with the aggregate cash flows spanning the entire sample are apparent, as depicted in Figure A.1, top panel. One issue emerging at the sample's outset is the limited availability of cash flows, attributed to the mere existence of a handful of funds. Predominantly, these cash flows arise from capital calls, aligning directly with the benchmark market fund's cash flows and leading to limited variability. Another challenge emerges at the sample's end: cash flows from funds incepted post-2015 (the last inception year in my dataset) display cash flows heavily influenced by valuations, leading to a surge in excess cash flows.

Truncating the time series to a shorter period partially addresses the non-stationarity inherent in the aggregate excess cash flows. The enhancements in stationarity properties become evident in Figure A.1, middle panel. However, it is important to note that the excess cash flows before 2000 are less volatile than those post-2000. The rising number of funds in the industry has increased both the magnitude and fluctuations of the aggregate excess cash flows. These patterns cast doubt on their use in belief extraction due to stationarity concerns.

A potential resolution lies in averaging the time series, shifting from the moment condition of $\mathbb{E}^{P^*} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0$ to $\mathbb{E}^{P^*} \sum_{t=1}^{T} M_{1,t} \frac{1}{N_t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0$. This strategy aims to 'normalize' the aggregate cash flows by accounting for the active fund count within each time interval.

Assessing the impact of averaging on the stationarity characteristics of the aggregate excess cash flows series involves dividing the original series by the count of funds exhibiting non-zero cash flows, defined N_t. The bottom panel of Figure A.1 suggests that the average excess cash flows display marginally enhanced stationarity compared to aggregate excess cash flows. The autocorrelation function (ACF) in Figure A.2 supports this observation, showcasing a swift descent to zero after one lag for the average excess cash flows. Using this averaged series appears to attenuate the data's persistent nature, proving beneficial for subjective beliefs estimation.

Subsequent formalized testing, detailed in Table A.1, employ the Augmented Dickey-Fuller (ADF), Phillips-Perron (PP), and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) tests. Each test adopts its unique approach to stationarity, featuring distinct null hypotheses:

- **ADF test**: The null hypothesis asserts the presence of a unit root in the series, indicating

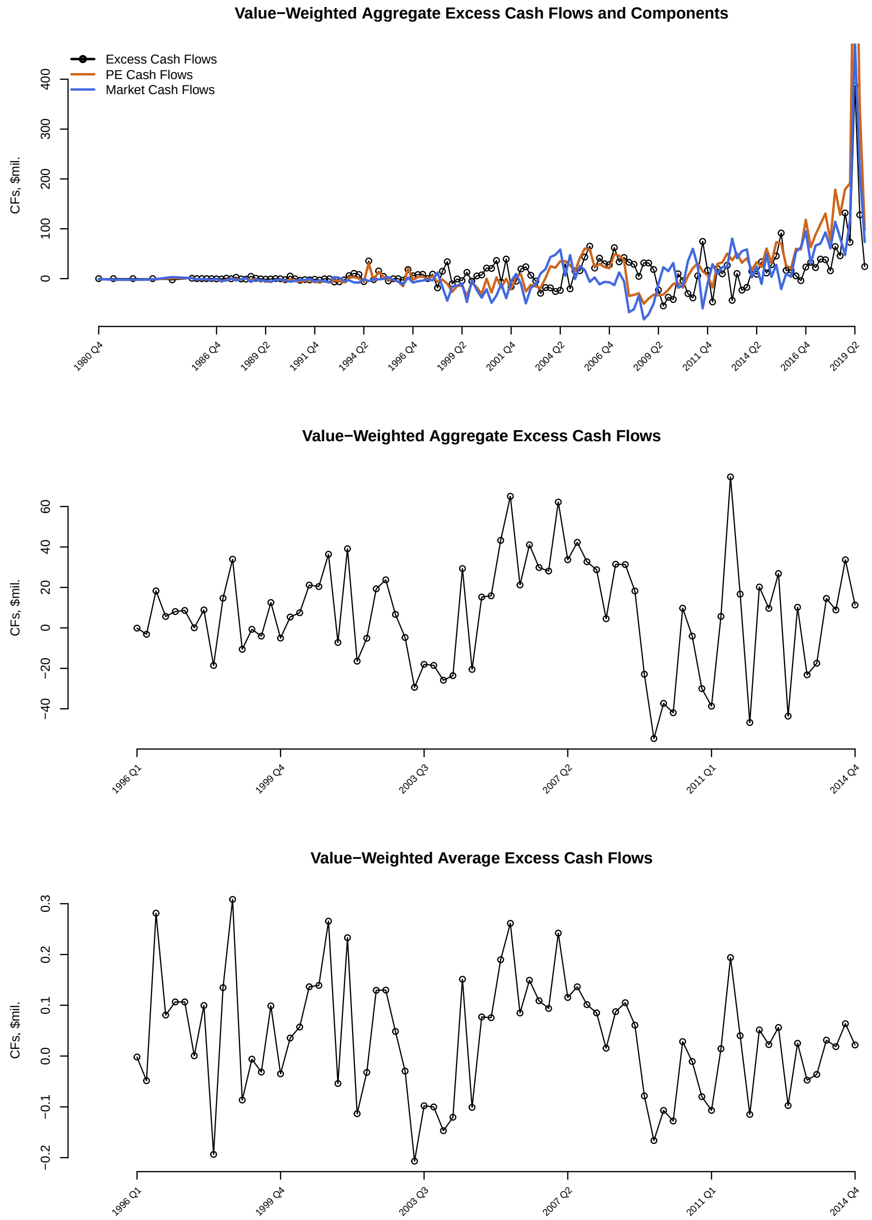

Figure A.1: Excess Cash Flow and Its Components. This figure plots the excess cash flow dynamics for PE funds. The top panel contrasts aggregate PE cash flow with aggregate CRSP cash flows from 1980 Q4 to 2019 Q4. The middle and bottom panels offer a detailed view of the aggregate and average excess cash flows for PE funds between 1996 Q1 and 2014 Q4, respectively.

non-stationarity. A significant p-value implies this hypothesis's rejection

- **PP test**: Aligning with the ADF test, the PP test's null hypothesis posits that the series contains a unit root. However, this test adjusts for any serial correlation within error terms,

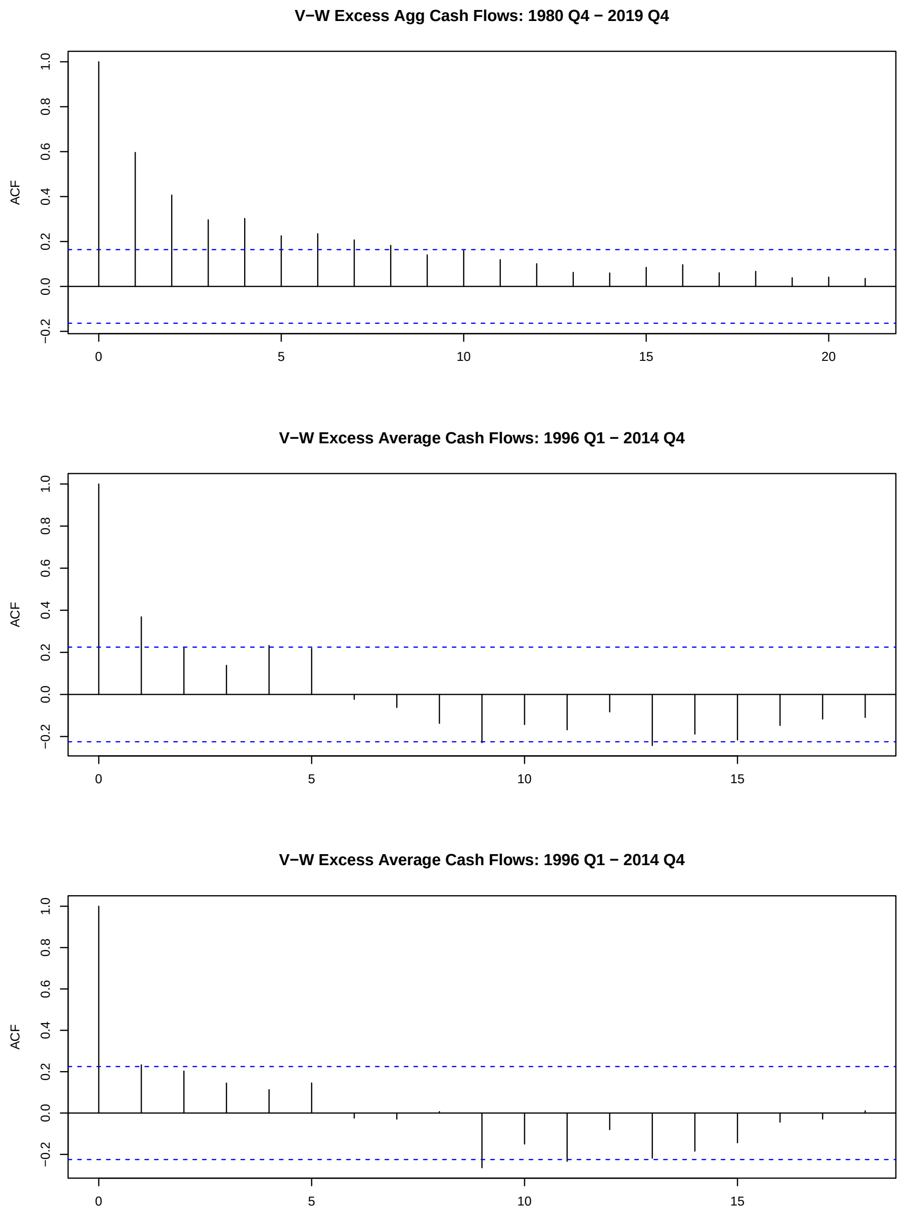

Figure A.2: Autocorrelation Function for Excess Cash Flows. The figure plots the autocorrelation function (ACF) for aggregate and average excess cash flows for PE funds. The x-axis at each plot shows the number of lags ACF is computed for. Dashed blue lines represent a zero bound.

ensuring a robust analysis.

- **KPSS test**: Its null hypothesis posits the series is stationary around a deterministic trend. A significant p-value negates this hypothesis.

Both time series produce conflicting conclusions between the ADF and PP tests. While ADF

Table A.1: Stationarity Test Results for Aggregate and Average Excess Cash Flows. The table presents the results of stationarity tests applied to both aggregate and average excess cash flows. Columns provide the test statistic and corresponding p-values for each applied test. Rows denote specific stationarity tests: Augmented Dickey-Fuller (ADF), Phillips-Perron (PP), and Kwiatkowski-Phillips-Schmidt-Shin (KPSS).

	Aggregate Excess CFs		Average Excess CFs	
Test	Test Statistic	p-value	Test Statistic	p-value
ADF	-2.30	0.45	-2.68	0.30
PP	-47.88	0.01	-63.48	0.01
KPSS	0.099	0.10	0.183	0.10

results insinuate non-stationarity, the PP test leans towards a stationary stance. The inferences drawn from the KPSS test align with those of the PP, further hinting at stationarity.

Switching to the averaged series seems to reduce the persistence and the longer memory in the data, as evidenced by the diminished and faster-decaying autocorrelations. The averaged series appears closer to stationarity, although formal statistical tests do not unequivocally favor one over the other. Hence, focusing on the average excess cash flows from 1996 Q1 to 2014 Q4 appears suitable for the EL-type methodology used to extract investors' beliefs.

However, it is important to acknowledge that, given the foundation laid by the GMPE-linked moment condition derivation for aggregate cash flows, extracting beliefs using this approach remains rational. A detailed examination of this approach is conducted in Appendix D.2.

B 'Billion Dollar Club'

In this appendix, I investigate the capability of Limited Partners (LPs) to translate their beliefs into price discounts during negotiations for investments in PE funds. While the main analysis has demonstrated the significant economic role of beliefs in driving excess cash flows, it remains uncertain whether the average investor possesses sufficient bargaining power to negotiate more favorable contract terms based on these beliefs. If this holds true, the premium tied to future cash flow uncertainties might be at least partly priced through the fee agreement. To probe this further, I focus on a subset of funds representing investments made by members of the so-called 'Billion Dollar Club', investors committing $1bn or more to PE.

According to Preqin, in 2018, there were 359 investors that met the criteria, allocating a total of $1.54tn to private equity. The majority of these investors (54%) are based in North America, while 28% are based in Europe, and 9% are based in Asia and the rest of the world. As pointed out by Christopher Elvin[8], this elite group of investors exert significant influence over the PE industry due to their ability to shape standards, negotiate fees, and gain access to oversubscribed vehicles[9]. This advantage could feasibly enable them to obtain price discounts via negotiating fee structures or accessing the best deals offered by the fund. The concentration of such bargaining power might lead to additional compensation relative to time and risk, potentially aligning subjective beliefs more closely with the RE hypothesis.

The academic literature indicates that high-performing General Partners (GPs) do not increase the size of their subsequent funds in line with the rising demand for their stakes. Consequently, these funds often become oversubscribed. In this situation, large investors such as those in the 'Billion Dollar Club' have a distinct advantage in investing in top-tier GPs, a trend underlined by the research of Keating (2006). As highlighted by Lerner and Leamon (2011), certain practitioners, like David Swensen[10], have strategically reinvested in these leading GPs to ensure greater accessibility to their future funds. It is also important to note that only the best GPs typically are able to raise the follow-on funds. This is supported by studies from Sensoy, Wang, and Weisbach (2014), Chung, Sensoy, Stern, and Weisbach (2012), and Kaplan and Schoar (2005), underscoring that the prospects of launching a new fund hinge heavily on the historical performance of their previous ones.

To construct the sample of investments made by the 'Billion Dollar Club', I rely on data from

[8] a former head of PE products at Preqin
[9] https://docs.preqin.com/press/PE-1bn-Club-Jun-18.pdf
[10] a former head of Yale University endowment

Pitchbook[11] detailing LPs' capital allocation to PE funds. This dataset encompasses details about the funds LPs invest in, their investment sizes, and the associated dates. An LP qualifies as a club member if they have committed at least 1 billion dollars specifically to buyout funds—this is notably stricter than the typical definition which pertains to the broader category of PE. The timing of investments also plays a role in the classification. For example, if an LP achieves club membership in 2008, only the funds they committed to post-2008 are taken into account. CalPERS (California Public Employees' Retirement System) was the pioneering LP to meet these standards in 1995, and the club's roster expanded to 58 members by 2014.

Table B.2: Descriptive Statistics for the 'Billion Dollar Club': LPs Investment Data. The table reports characteristics of the LPs in the "Billion Dollar Club" for the years 2005 and 2014 from the Pitchbook dataset. Detailed statistics include the count of funds invested, the cumulative capital commitment (presented in $ millions), and the mean fund size. The LPs are divided by their membership status in the club (members and non-members) and further differentiated by their institutional types.

Type of LP	# LPs		Total Commitment		Average Fund Size	
	club	non-club	club	non-club	club	non-club
Fund of Funds	2	52	11820.5	3497.2	636.4	539.8
Government Agency	1	3	4068.2	466.0	513.8	522.9
Insurance Company	2	71	2307.6	4941.8	498.3	475.0
Public Pension Fund	21	61	58302.2	9654.3	526.9	537.9
Wealth Management Firm	1	5	1059.5	451.0	440.6	608.8
Total (2005):	27	570	77558.0	39404.2	539.7	521.1
Corporate Pension	4	79	6894.0	9837.9	564.3	538.2
Endowment	3	10	7762.0	974.8	499.0	551.5
Foundation	1	41	1024.0	3126.6	627.6	504.7
Fund of Funds	4	31	12570.7	6356.7	669.4	544.5
Government Agency	1	1	7947.8	13.4	530.9	230.5
Insurance Company	4	69	10982.2	12585.6	543.2	500.5
Public Pension Fund	39	80	233139.9	18734.2	549.1	534.7
Sovereign Wealth Fund	1	1	1550.5	301.3	623.2	710.7
Wealth Management Firm	1	4	1066.6	210.0	430.1	642.1
Total (2014):	58	470	282937.7	58282.9	567.3	541.7

Table B.2 presents the characteristics of LPs, segregating between members (*club*) and non-members (*non-club*) of the 'Billion Dollar Club' as of the midpoint year 2005 and the concluding year 2014. The table indicates that Public Pension Funds are the primary investors in buyout funds, constituting over 50% of club members. Additionally, the collective commitment from the

[11] a comprehensive provider of M&A, private equity, and venture capital data

club has seen a consistent uptrend, amplifying its investments threefold to culminate at $283 billion by 2014. In contrast, the non-members, comprising 470 LPs, had aggregated investments of merely $58 billion in 2014. Generally speaking, club members tend to commit to larger funds compared to their non-member counterparts across nearly all LP classifications. Still, this discrepancy is not vast. For context, back in 2005, the mean fund size chosen by Public Pension Funds within the club was marginally less than that picked by non-members. However, this trend pivoted in the opposite direction by 2014.

Table B.3 showcases the characteristics of funds from the Preqin sample invested by both members (*club*) and non-members (*non-club*) of the 'Billion Dollar Club'. The median fund size is almost double for club members; thus, the difference is significantly higher than in the Pitchbook sample. However, the performance of funds invested by club members is not significantly better, as the median IRR (with a 1 bp advantage for club members) and TVPI are nearly identical. It should be noted that this performance parity could stem from the fact that many of the same funds attract investments from LPs across both categories. And, given that performance metrics are tracked at the fund level and not at the individual LP level, such nuances might not surface. Further insights from Table B.3 suggest that club LPs have a propensity towards more liquid funds. This is inferred from the fact that the funds they lean towards tend to turn over capital more swiftly, as indicated by *Fund effective years* column. Overall, there are no significant differences between member and non-member funds.

Table B.3: Fund Performance Metrics for the 'Billion Dollar Club' Investors. This table reports the performance metrics of the funds in which members of the "Billion Dollar Club" and their non-member counterparts have invested. The metrics assessed include the median fund size, represented as the overall commitment (in $ millions), the IRR (%), TVPI, and the fund's effective years, which indicate the span between the first and last observed cash flows of a particular fund.

	Fund Size	IRR,%	TVPI	Fund effective years
club	912	11	1.54	10.25
non-club	456	10	1.50	11.25

Beliefs. Figure B.3 shows belief-adjusted average excess cash flows for the subset of funds where investments were made by the 'Billion Dollar Club'. Using the K-N CAPM SDF for estimating beliefs, the belief-adjusted distribution closely mirrors the distribution implied by the RE hypothesis, especially when juxtaposed against Figure 1. Nonetheless, this distribution reveals a marginally fatter negative tail and exhibits a slight left skew. On the other hand, the EHB SDF provides a very accurate estimation for the sub-sample of 'billionaires' funds by producing very low pricing errors for excess cash flows. In fact, the belief-adjusted distribution, estimated through the EHB SDF, nearly aligns seamlessly with the one suggested by the RE hypothesis. Such observations bolster the premise that club members can translate their subjective beliefs and the associated uncertainties about cash flows into price discounts through limited partner agreements (LPAs).

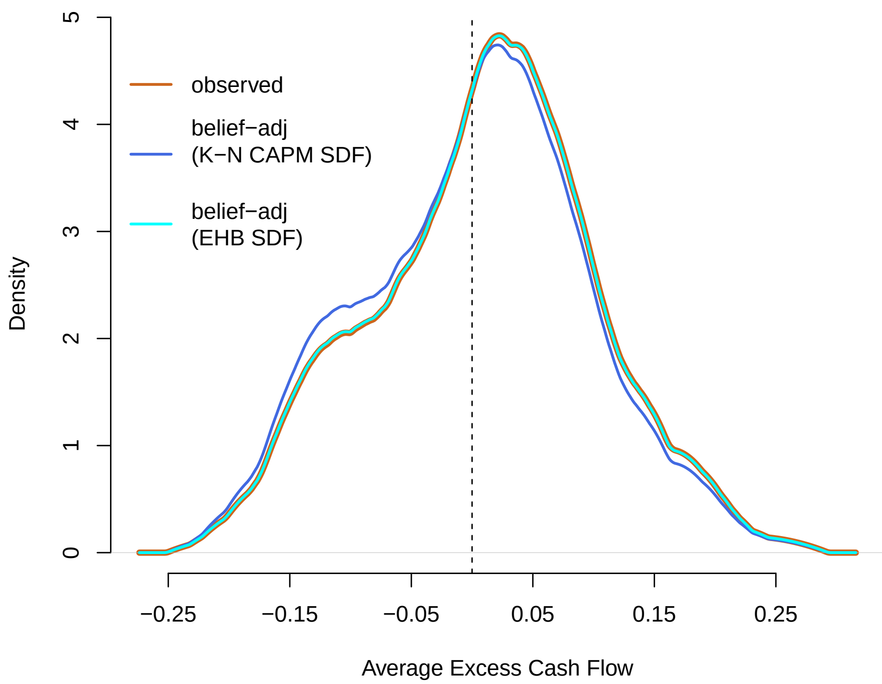

Figure B.3: Discrepancy in PE Excess Cash Flows: True vs. Belief-Adjusted. This figure plots the observed and belief-adjusted average excess cash flows for 'Billion Dollar Club' PE funds from Q1 1996 to Q4 2014. Adjustments are based on beliefs estimated under the K-N CAPM and EHB models. Values are presented in millions of dollars, with density estimations derived using the Epanechnikov kernel method.

In Figure B.4, the top panel displays the impact of beliefs estimated under the K-N CAPM SDF on the components of excess cash flow for the sub-sample of "billionaire" funds. The figure underscores that the discrepancy between subjective beliefs and those suggested by the RE hypothesis persists. However, the magnitude of this discrepancy is reduced in comparison to the full sample of funds. Despite this reduction, the magnitude remains considerable, indicating that even in the 'billionaire' funds sub-sample, the belief discrepancy carries economic significance when assessed using the K-N CAPM SDF. Thus, the narrative of subjective beliefs remains relevant, even for the 'Billion Dollar Club' sub-sample.

In Figure B.4, the bottom panel illustrates the influence of beliefs calculated under the EHB SDF on the components of excess cash flow for the sub-sample of "billionaire" funds. The panel reveals a diminished presence of the 'pessimism - optimism' pattern, and the magnitude of the errors has substantially decreased, spanning several orders of magnitude. This suggests that the combination of well-calibrated risk preferences (as in the EHB SDF) and their application to funds backed by investors with significant bargaining clout can notably mitigate pricing errors. This further underscores a potential linkage between the alignment of subjective beliefs and the terms set forth in LPAs.

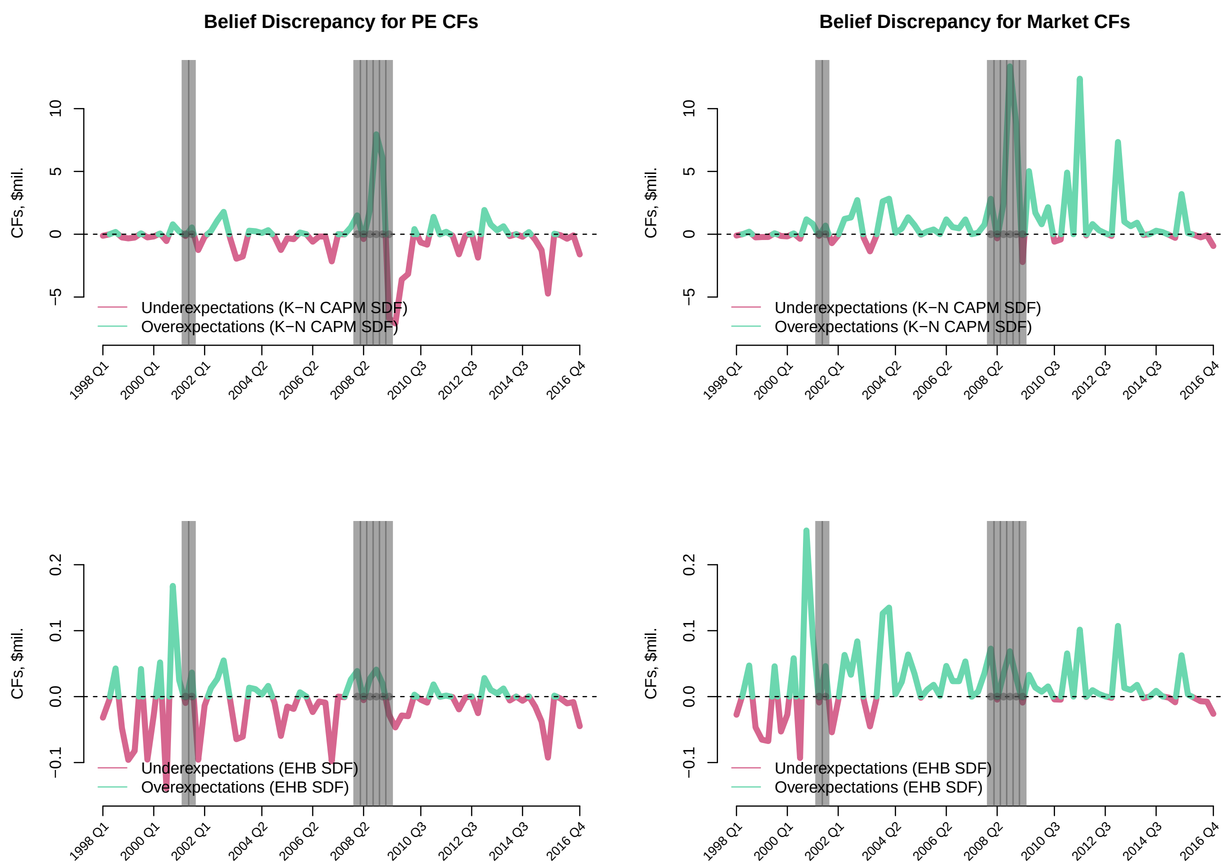

Figure B.4: Comparison of Belief-Adjusted and Observed Aggregate Cash Flows: PE vs. Market. This figure illustrates the discrepancy between belief-adjusted and observed aggregate cash flows for PE funds and market benchmark funds spanning from 1996 Q1 to 2014 Q4. Organized in two columns, the left graph details the aggregate cash flows of PE funds, while the right graph showcases those of CRSP-mimicking funds. The rows differentiate the subjective beliefs: the top row adheres to the K-N CAPM SDF, and the bottom row adheres to the EHB SDF. NBER-designated recession periods are highlighted with shaded areas.

Furthermore, in Table B.4, I validate the estimated beliefs against survey data. The results emphasize a positive and substantial correlation between subjective beliefs and survey indices. Nonetheless, the correlation's magnitude is notably lower. This could suggest that sentiment indices like CEPECI and SVVCCI encompass a broader spectrum of investors, not just those from the 'Billion Dollar Club', which is a logical assumption.

Table B.4: Correlations between Surveys and Excess Cash Flow. The table reports the correlations between various sentiment indices, presented row-wise, and belief-adjusted average excess cash flows for 'Billion Dollar Club' PE funds. Across the columns, the subjective beliefs are estimated using K-N CAPM SDF and EHB SDF.

	Average Excess Cash Flow adjusted for Purified Beliefs estimated for:	
	K-N CAPM	EHB
Shiller (inst)	-0.15	-0.13
Shiller (ind)	0.10	0.12
Gallup	0.38***	0.33**
ESI	0.26**	0.23*
CEPECI	0.52***	0.50***
SVVCCI	0.62***	0.56***

In this appendix, I examine the subjective beliefs of LPs from the 'Billion Dollar Club'. The findings suggest that these LPs remain over-optimistic about the stock market cash flows while harboring pessimistic views on the cash flows of PE funds. Yet, it appears that members of the 'Billion Club' can use their bargaining power to secure price discounts, which, in turn, narrows the discrepancy in beliefs, especially when well-calibrated SDFs are in play. These insights bolster the main analysis, underscoring that subjective beliefs persistently influence the investment-making process, particularly when bargaining power among investors is not apparent and they 'vote with their feet', shifting the demand for PE fund stakes.

C Different NPV Methodologies

The GPME framework, as described by Korteweg and Nagel (2016), employs an NPV methodology that discounts each fund's cash flows back to its individual inception date. While theoretically robust, this approach encounters challenges when it comes to the beliefs estimation process, which requires aggregation across funds. Since each fund has a distinct inception point, it complicates the aggregation process. To elaborate, the original GPME framework computes the NPV as:

$$NPV_i^{PE,KN16} = \sum_{t=1}^{T} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{PE}$$

Here, $M_{\emptyset(i)}$ represents the multi-period SDF at the inception of fund i. $t = 1, ..., T$ denotes the range of all dates for which PE cash flows are observed. Furthermore, $M_{1,t}$ is the multi-period SDF, derived by successively compounding the single-period discount factors.

This approach captures the economic conditions at the start of the fund and aligns with the GMM estimation of SDF parameters tailored for PE investors. The GPME methodology also accounts for irregularly spaced PE cash flows by referring them back to each fund's inception. Consequently, despite potential time inconsistencies among data points, the GMM seeks to minimize pricing errors by optimally selecting the parameters of the asset pricing model.

However, in the context of the belief estimation, which aims to minimize the KLIC divergence, the temporal aggregation of consistent economic objects becomes crucial, particularly given the overlapping structure of PE funds. Directly aggregating cash flows with varied discounting points poses challenges—akin to combining 'apples and oranges'. One potential avenue is to consider segmenting funds based on inception dates or similar criteria, aggregating within cohorts, and analyzing these aggregated values. However, such segmentation is impractical here due to overlapping investment horizons and difficulty delineating consistent fund cohorts.

To circumvent this obstacle, I introduce a more straightforward strategy for beliefs extraction, which discounts all cash flows to the start of the dataset using the corresponding SDFs. This revision ensures the comparability of all aggregated cash flows—maintaining an 'apples to apples' comparison. Under this approach, the NPV for an individual PE fund is calculated as:

$$NPV_i^{PE} = \sum_{t=1}^{T} M_{1,t} CF_{i,t}^{PE}$$

This formulation guarantees uniformity in the cash flows. Notably, for the GMM estimation of SDF parameters, cash flows are normalized to one invested dollar. In contrast, for belief extraction, I resort to the original cash flows. Thus, I maintain the economic importance of each fund proportionate to its genuine size. Merging original (size-sensitive) cash flows with a unified reference point for discounting, without leaning on $M_{\varnothing(i)}$ normalization, offers a logical and economically meaningful solution.

D Alternative Approaches to Belief Estimation

D.1 Estimating Beliefs with 'Off-the-Shelf' SDFs. In the main analysis, I used the GMM with a moment condition for benchmark funds that invest in the public stock market to estimate the parameters of the SDF. To assess the robustness of the findings, it is suitable to examine the economic implications of subjective beliefs using an SDF estimated with the identical Euler equation employed for belief extraction. The alternative moment condition for an individual fund's GMM estimation is given by:

$$\mathbb{E}\sum_{t=1}^{T} M_{1,t}(\theta)\ CF_{i,t}^{exc} = 0,$$

Where θ denotes the relative risk aversion (RRA) coefficient. This methodology is consistently applied across mainstream asset pricing models. The results are summarized in Table D.5. One consistent observation is the necessity for high RRA coefficient values to explain the observed average excess cash flow:

Table D.5: GMM Estimation Results. The table reports the relative risk aversion (RRA) estimates for different SDFs using the GMM approach. P-values indicate the statistical significance of the RRA estimates. The last row, $P(\chi_1^2 > J)$, shows the over-identification test p-values for each SDF model.

SDF	K-N CAPM	C-CAPM	EHB	LRR
RRA	14.5	917.3	1611.9	39.9
p-value	0.2	0.1	0.0	1.0
$P(\chi_1^2 > J)$	< 0.01	< 0.01	< 0.01	0.02

High RRA coefficients, especially those exceeding the widely accepted threshold of 10, indicate an extreme risk aversion. In many economic contexts, these high values are often viewed as implausible, emphasizing potential limitations in the traditional Rational Expectations (RE) paradigm. These insights resonate with studies employing the consumption Euler Equation for public markets, such as Julliard and Ghosh (2012), who used the C-CAPM SDF to explain the equity premium puzzle across countries. They ascertain that data necessitates high RRA levels to rationalize the stock market risk premium.

A similar challenge emerges for results in the PE market: adhering strictly to the RE framework results in improbable SDF parameter estimations. This accentuates a potential model misspecification inherent in the RE models when determining RRA. This observation is not merely a critique of existing methodologies but rather emphasizes the importance of subjective beliefs in the PE context.

For the present analysis, I am inspired by the rationale of Julliard and Ghosh (2012). Given implausible SDF parameter estimates using the moment condition for average excess cash flow, I intend to re-estimate subjective beliefs using established 'off-the-shelf' SDF parameters. This analysis assesses how significantly this methodology tweak could influence the characteristics of subjective beliefs and their consequent asset pricing implications.

I focus on the four models presented in Table D.5 to construct 'off-the-shelf' SDFs. For the K-N CAPM, my reference is Korteweg and Nagel (2016), particularly to their GPME estimates for VC funds. Specifically, I set $a = 0.012$, on a quarterly basis, and RRA = 2.65. For the C-CAPM SDF, I use the upper bound for RRA as 10, as suggested by Julliard and Ghosh (2012). For the SDF determined by the LRR model, I rely on Bansal and Yaron (2004). The specifics of the LRR SDF construction are detailed in Table 2. In particular, I set $\sigma_{QQ} = 0.0135$, $\phi_e = 0.1085$, $\rho_x = \rho_{mm}^3 = 0.987^3$, $\psi = 2$, $\kappa_c = 0.9649$, and the RRA parameter to 10. For the EHB model, I follow Campbell and Cochrane (1999), and I set $\phi_{QQ} = \phi_{mm}^3 = 0.89^3$, $\gamma = 2$, $\delta = 0.998$, and the RRA parameter to 2. The $\sigma(\Delta c_t)$ is adjusted to the sample standard deviation of log consumption growth, following Ghosh, Julliard, and Taylor (2017).

With these calibrated SDFs, denoted as $M_{1,t}(\hat{\theta})$, I extract investors' subjective beliefs using Equation 8. The objective is three-fold. First, I aim to understand if the economic mechanism underscored in Figures 5 and 6 remains consistent for beliefs estimated under 'off-the-shelf' SDFs. Second, I evaluate if the statistical significance of this mechanism documented in Table 6 holds. Third, I test if beliefs retain their validity using survey data, as demonstrated in Table 8.

As evident from Figures D.5 and D.6, the dynamics of beliefs discrepancy for PE cash flows consistently exhibit the prevailing pessimism of PE investors across the majority of the observed periods. The amplitude of these fluctuations remains largely invariant. An exception to note is the C-CAPM model with upper-bound RRA parameters. This model demonstrates remarkable accuracy in pricing the PE cash flow time series, particularly for the post-2000 era, outperforming even the more sophisticated LRR and EHB models in this exercise.

Intriguingly, the C-CAPM model also excels in pricing market cash flows for the same post-2000 period. This accentuates the merit of relying on foundational, albeit simplistic models like C-CAPM. However, it is important to note that the magnitude of discrepancy for market cash flows remains largely unaltered, underscoring consistent performance across the two analyses.

Moreover, I examine whether the statistical significance of subjective beliefs persists upon the application of 'off-the-shelf' SDFs. Table D.6 reinforces the statistical robustness of the employed methodology. Beliefs determined under 'off-the-shelf' SDFs consistently eliminate the pricing

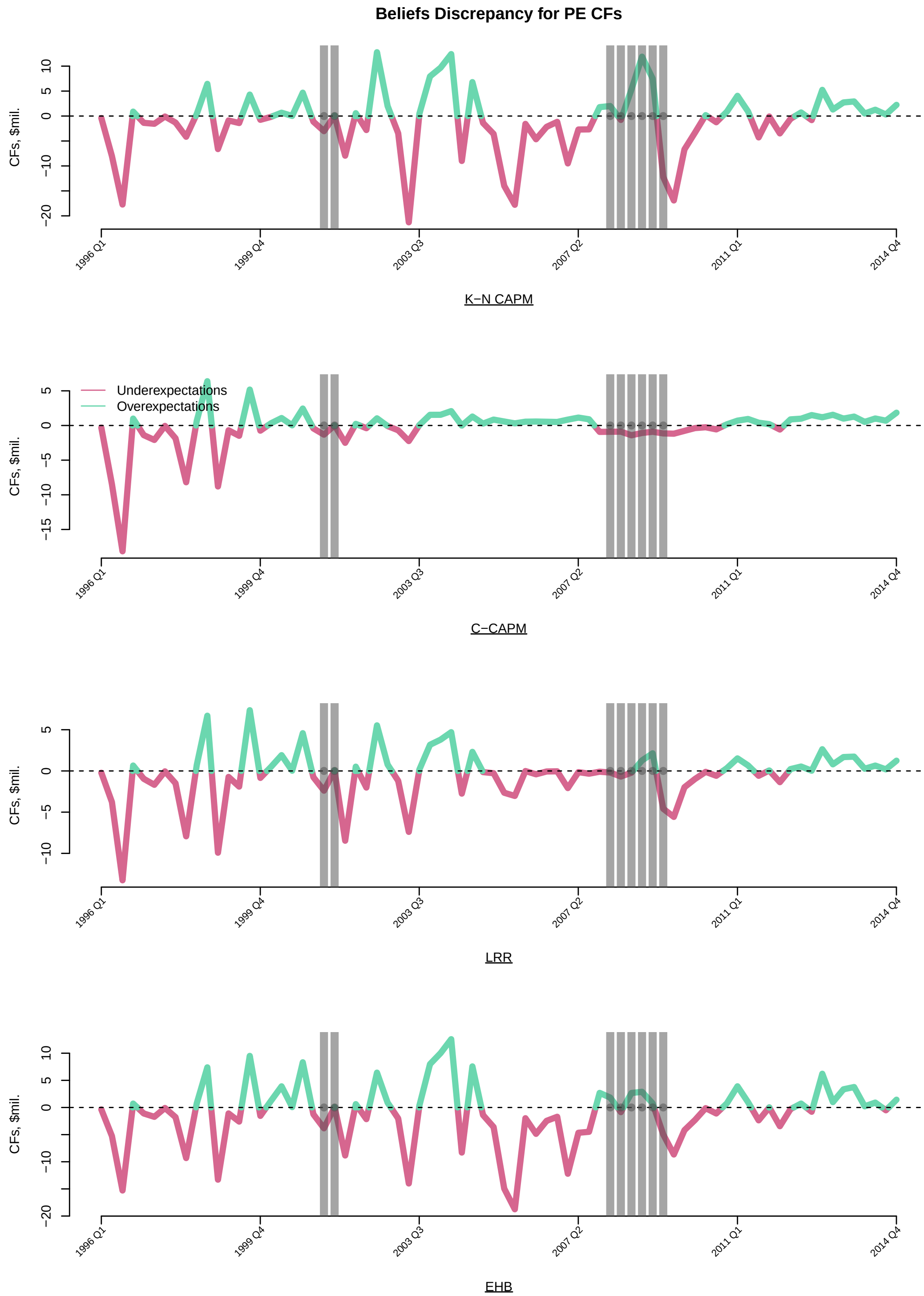

Figure D.5: Aggregate PE Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate cash flows for PE funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived. All mentioned SDFs are constructed using 'off-the-shelf' parameters.

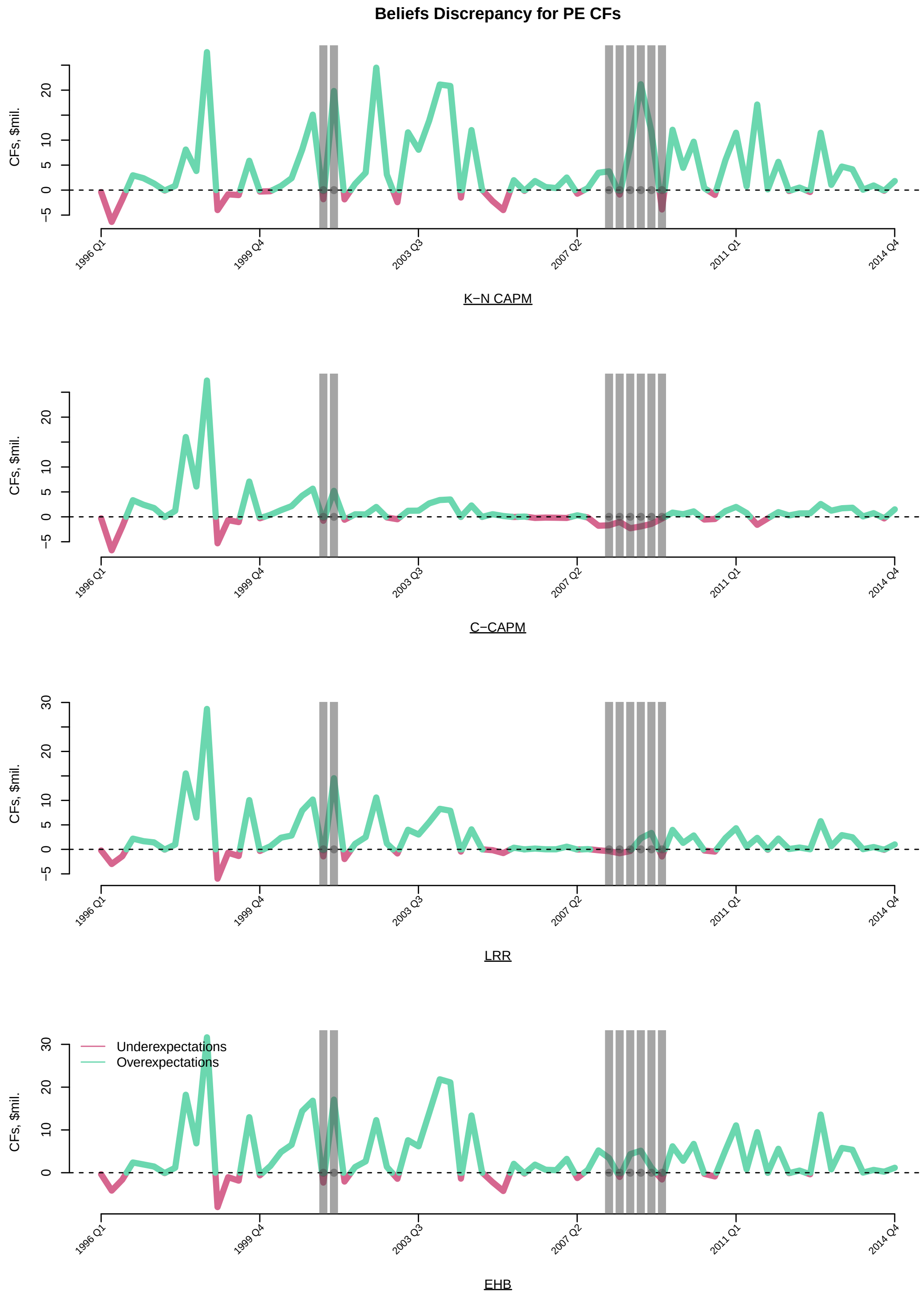

Figure D.6: Aggregate Market Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate mimicking cash flows for CRSP-benchmark funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived. All mentioned SDFs are constructed using 'off-the-shelf' parameters.

Table D.6: Counterfactual Excess, PE, and Market Cash Flows. This table presents the findings from a counterfactual analysis using 100,000 simulations, emphasizing subjective beliefs as importance weights for cash flows from 1996 Q1 to 2014 Q4. Each column represents an SDF, indicating where subjective beliefs originate. All presented SDFs are constructed using 'off-the-shelf' parameters. Panel A, B, and C each display results for Excess Cash Flows, PE Cash Flows, and Market Cash Flows, respectively. Statistics in each panel include the historical mean, bootstrap mean, 95% confidence interval for the bootstrap mean, and the probability of the counterfactual mean surpassing the historical mean. Values are portrayed as percentages of the Mean Average Payout (MAP).

	K-N CAPM	C-CAPM	LRR	EHB
		Panel A: Excess Cash Flows (in % of MAP)		
$\overline{CF_T^{exc}}$	4.26	2.55	5.02	7.41
$\overline{CF_{boot}^{exc}}$	-0.01	0	0	0
$[\overline{CF_{boot2.5\%}^{exc}};\overline{CF_{boot97.5\%}^{exc}}]$	[-2.7;2.69]	[-1.86;1.86]	[-4.61;4.61]	[-5.1;5.11]
$Pr(CF_{boot}^{exc} > \overline{CF_T^{exc}})$	0.14%	0.6%	1.69%	0.24%
		Panel B: PE Cash Flows (in % of MAP)		
$\overline{CF_T^{PE}}$	-1.59	-1.52	-4.72	-2.76
$\overline{CF_{boot}^{PE}}$	-3.54	-3.43	-6.62	-5.31
$[\overline{CF_{boot2.5\%}^{PE}};\overline{CF_{boot97.5\%}^{PE}}]$	[-7.02;-0.07]	[-6.23;-0.62]	[-11.31;-1.93]	[-11.01;0.39]
$Pr(CF_{boot}^{PE} > \overline{CF_T^{PE}})$	13.2%	7.89%	21.33%	19.03%
		Panel C: Market Cash Flows (in % of MAP)		
$\overline{CF_T^{mrkt}}$	-5.86	-4.07	-9.73	-10.17
$\overline{CF_{boot}^{mrkt}}$	-3.54	-3.42	-6.63	-5.31
$[\overline{CF_{boot}^{mrkt}}_{2.5\%};\overline{CF_{boot}^{mrkt}}_{97.5\%}]$	[-7.23;0.15]	[-6;-0.85]	[-11.85;-1.41]	[-11.76;1.14]
$Pr(CF_{boot}^{mrkt} > \overline{CF_T^{mrkt}})$	88.81%	70.08%	87.77%	92.75%

error remaining after SDF discounting. For all 'off-the-shelf' SDFs, the historically observed misvaluation falls outside the 95% bootstrap confidence interval, which is constructed based on the extracted beliefs. The C-CAPM model displays the narrowest confidence interval for excess cash flows, echoing the high precision of its 'off-the-shelf' SDF. Yet, despite the small discounted mean of excess cash flows yielded by the C-CAPM SDF, the formal test still rejects the validity of the RE hypothesis. This evidence supports that investors' subjective beliefs likely diverge from the empirical distribution of excess cash flows.

Furthermore, I check if extracted beliefs for 'off-the-shelf' SDFs match with survey data. As presented in Table D.7, even though the magnitude of these subjective beliefs appears slightly attenuated compared to those in Table 8, the statistical significance for core investors, as represented by the CEPECI and SVVCCI indices, remains pronounced. This accentuates that the economic significance of subjective beliefs can not be rejected by the survey evidence.

Moreover, for non-core individual and institutional investors in the public market, the correlation signs align with expectations, underscoring the economic coherence across diverse investor categories. However, a marginal decline in statistical significance is noted for individual investors, as exemplified by the Gallup and ESI surveys, while correlations stemming from Shiller's survey remain inconclusive.

In summary, based on the examination of beliefs estimated for 'off-the-shelf' SDFs, it is evident that the adopted methodology consistently extracts beliefs that are both statistically robust and economically compelling. This further corroborates the hypothesis that PE funds' observed positive abnormal performance can be attributed to the pronounced pessimism of LPs regarding the cash flow-generating capabilities of PE funds. Concurrently, there is a marked overestimation of the potential cash flows from their holdings in the public market.

Table D.7: Correlations between Surveys and Average Excess Cash Flow. The table reports the correlations between various sentiment indices, presented row-wise, and belief-adjusted average excess cash flows for PE funds. Across the columns, the subjective beliefs are estimated using 'off-the-shelf' SDFs.

	Average Excess Cash Flow adjusted for Beliefs estimated for:			
	K-N CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.18	-0.09	-0.10	-0.10
Shiller (find)	-0.01	0.12	0.04	0.09
Gallup	0.37***	0.28**	0.28**	0.29**
ESI	0.32***	0.27**	0.25**	0.26**
CEPECI	0.59***	0.58***	0.57***	0.58***
SVVCCI	0.71***	0.66***	0.69***	0.68***
Note:				*p<0.1; **p<0.05; ***p<0.01

D.2 Estimating Beliefs Using the Moment Condition for Aggregate Excess Cash Flows. In this appendix, I explore the consequences of using the moment condition from Equation 8. This comes with a particular focus on potential stationarity issues tied to aggregate excess cash flows. As mentioned in Appendix A, these cash flows do not have the best stationarity features, especially when compared to average cash flows.

The ET methodology I employ to derive beliefs is especially sensitive to any lack of stationarity in the series under consideration. Because of this, this section zooms in on how such shortcomings might influence the economic implications of beliefs estimated for aggregate cash flow. Ultimately, the aim is to see if these beliefs coincide with those pinpointed in the main analysis.

The inclination towards using aggregate excess cash flows arises from the Euler equation that is directly derived from the GPME moment condition. Opting for averaged excess cash flows may slightly deviate from the economic consistency of the original aggregate series, but it offers improved stationarity characteristics. The main goal of the current analysis is to comprehend the nuances of this trade-off.

I begin by examining the dynamics of belief discrepancies for the components of PE cash flows. As evident from Figures D.7 and D.8, the economic story appears more congruent than the analysis presented for the 'off-the-shelf' SDFs in Appendix D.1. The interpretation of the pessimism-optimism patterns remains consistent, with the only variation being a slight amplification in the magnitude of belief discrepancies for both PE and market cash flows.

Next, I turn to the statistical significance of the subjective beliefs narrative. The updated counterfactual exercise for aggregate cash flows for bootstrap based on model-implied probabilities for the aggregate excess cash flow moment condition is laid out in Table D.8. The findings are noteworthy for a couple of reasons. First, as previously, the ET methodology for extracting subjective beliefs effectively eliminates the observed pricing error, as evidenced in the first and second rows of Panel A. Second, the results from the formal test of the RE hypothesis are not as unanimous as before. Except for the K-N CAPM model, the constructed 95% confidence interval challenges the validity of the RE hypothesis.

The case of the K-N CAPM model deserves special attention. As mentioned, the occurrence of historically observed excess cash flow cannot be rejected at a 5% significance level. Moreover, the probability of the mean discounted excess cash flow being greater than its historical mean is almost 7%, which is considerably higher than for other models where this probability does not surpass 2%.

From Panel B of Table D.8, it is evident that the observed mean of PE cash flow nearly reaches the lower counterfactual bound, suggesting that LPs no longer exhibit excessive pessimism

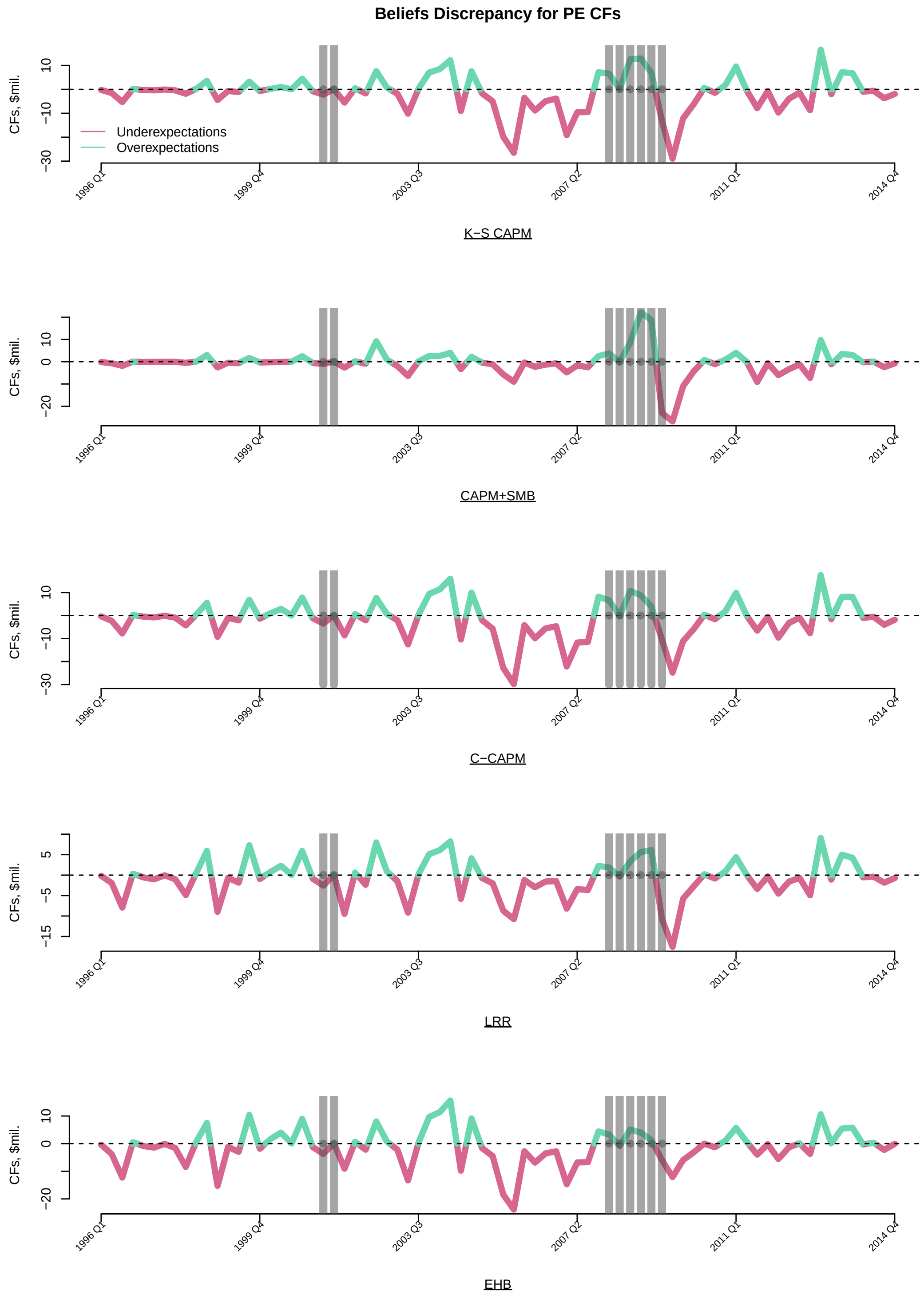

Figure D.7: Aggregate PE Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate cash flows for PE funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived. Subjective beliefs are estimated using the moment condition for aggregate cash flows.

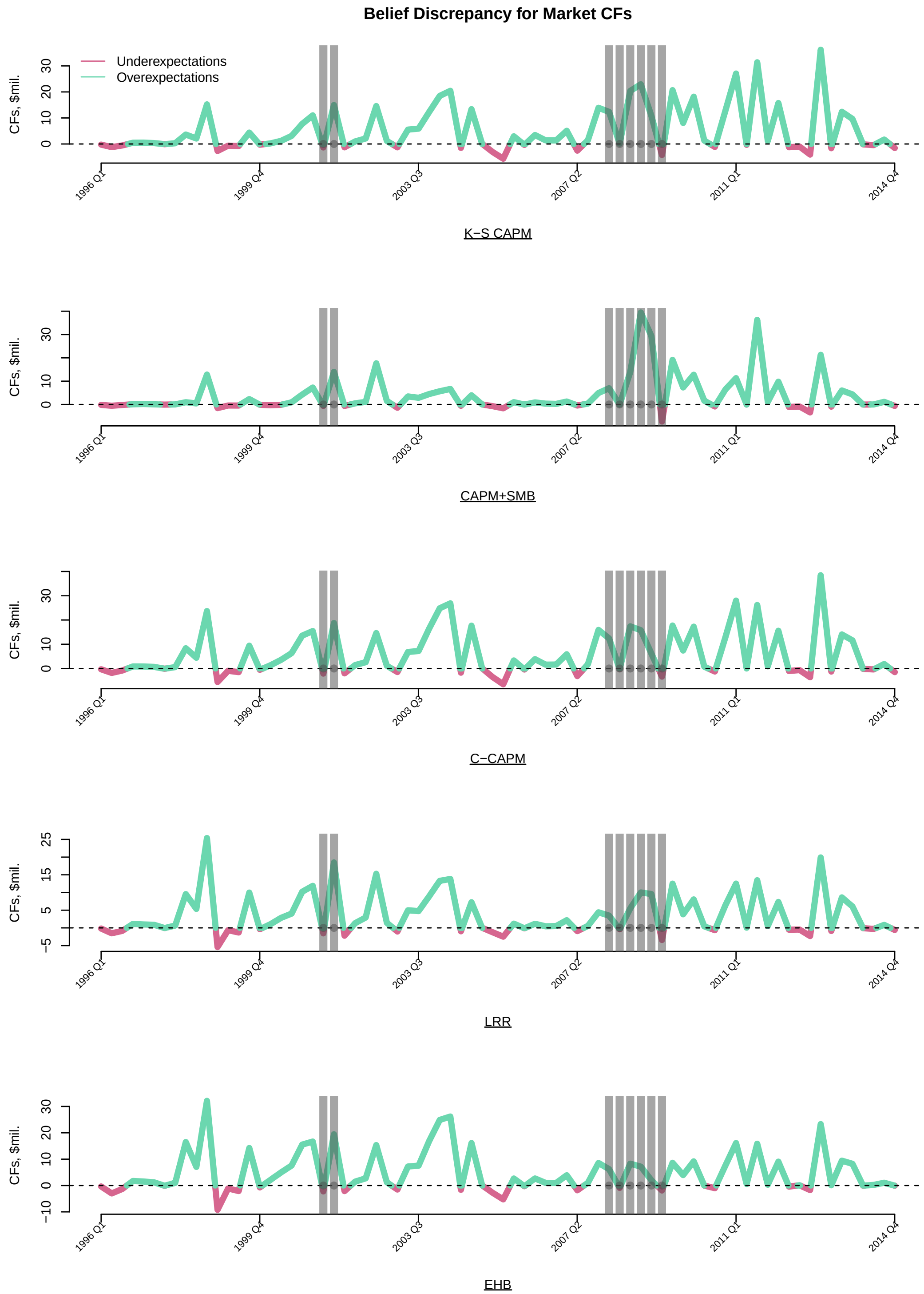

Figure D.8: Aggregate Market Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate mimicking cash flows for CRSP-benchmark funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived. Subjective beliefs are estimated using the moment condition for aggregate cash flows.

regarding the PE funds' cash flow generating abilities. When this result is compared with other models, it becomes clear that none yields a lower bound for PE cash flows proximate to the historical mean of the observed PE cash flow. As a consequence, the probability of observing the PE cash flow in a counterfactual world is almost 50%, reinforcing the lack of systematic pessimism toward PE funds.

However, Panel C of Table D.8 indicates that the deviation in investors' subjective beliefs is notably greater for the public market component of excess cash flow. The bootstrap mean for counterfactual market cash flow is -1.97% of the Mean Aggregate Payout, compared to -7.15% for the historically observed market mean. Consequently, 86% of the time, LPs anticipate a higher cash flow from their public market portfolio.

Table D.8: Counterfactual Excess, PE, and Market Cash Flows. This table presents the findings from a counterfactual analysis using 100,000 simulations, emphasizing subjective beliefs as importance weights for cash flows from 1996 Q1 to 2014 Q4. Each column represents an SDF, indicating where subjective beliefs originate. Subjective beliefs are estimated using the moment condition for aggregate cash flows. Panel A, B, and C each display results for Excess Cash Flows, PE Cash Flows, and Market Cash Flows, respectively. Statistics in each panel include the historical mean, bootstrap mean, 95% confidence interval for the bootstrap mean, and the probability of the counterfactual mean surpassing the historical mean. Values are portrayed as percentages of the Mean Average Payout (MAP).

	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	LRR	EHB
			Panel A: Excess Cash Flows (in % of MAP)			
$\overline{CF_T^{exc}}$	5.31	6.19	10.06	5.48	2.78	3.03
$\overline{CF_{boot}^{exc}}$	-0.02	0.01	-0.27	-0.01	-0.01	0
$[\overline{CF_{boot2.5\%}^{exc}}; \overline{CF_{boot97.5\%}^{exc}}]$	[-7.07;7.03]	[-5.15;5.16]	[-10.53;9.99]	[-3.89;3.88]	[-2.68;2.66]	[-2.07;2.07]
$Pr(CF_{boot}^{exc} > \overline{CF_T^{exc}})$	6.88%	0.92%	2.71%	0.29%	2.08%	0.22%
			Panel B: PE Cash Flows (in % of MAP)			
$\overline{CF_T^{PE}}$	-1.83	2	-9.14	1	-1.77	-0.72
$\overline{CF_{boot}^{PE}}$	-2.99	0.67	-9.96	-0.21	-2.31	-1.42
$[\overline{CF_{boot2.5\%}^{PE}}; \overline{CF_{boot97.5\%}^{PE}}]$	[-1.99;0.67]	[-9.96;-0.21]	[-21.16;1.25]	[-4.28;3.85]	[-5.1;0.48]	[-3.73;0.9]
$Pr(CF_{boot}^{PE} > \overline{CF_T^{PE}})$	49.89%	30.26%	46.66%	28.15%	35.54%	28.09%
			Panel C: Market Cash Flows (in % of MAP)			
$\overline{CF_T^{mrkt}}$	-7.15	-4.19	-19.2	-4.49	-4.55	-3.75
$\overline{CF_{boot}^{mrkt}}$	-1.97	0.67	-9.69	-0.2	-2.3	-1.42
$[\overline{CF_{boot}^{mrkt}}_{2.5\%}; \overline{CF_{boot}^{mrkt}}_{97.5\%}]$	[-11.31;7.38]	[-5.68;7.01]	[-25.89;6.51]	[-5.25;4.85]	[-5.69;1.09]	[-4.16;1.33]
$Pr(CF_{boot}^{mrkt} > \overline{CF_T^{mrkt}})$	85.94%	93.2%	87.14%	95.15%	90.12%	94.94%

Figure D.9 offers insight into the economic implications of subjective beliefs estimated under the K-N CAPM SDF. It is evident that the K-N CAPM SDF successfully prices both PE and market cash flows relative to other models. Before the GFC, discrepancies in beliefs regarding both components of excess cash flows were negligible. However, the GFC had a marked impact, causing a substantial divergence between investors' subjective beliefs and those implied by the market.

The broader economic story portrayed by the K-N CAPM model indicates that LPs see the excess cash flow as having a tilt to the left with a noticeable negative tail. However, this viewpoint has two significant divergences for beliefs estimated for aggregate excess cash flow from our baseline understanding.

First, the shift in subjective beliefs about excess cash flow primarily stems from over-optimism regarding market cash flow rather than a mix of pessimism about PE cash flows combined with the said optimism. Second, the gap between these subjective beliefs and the RE hypothesis no longer holds strong statistically.

This drop in the importance of subjective beliefs can be logically connected to the results related to the GPME metric presented in Table 3. Specifically, the abnormal performance of the K-N CAPM does not stand out statistically at the 10% level, even though there is a large abnormal performance. While this could suggest that the model is not perfectly specified, with the current analysis, it is clear that the main influence comes from active funds around the time of the Global Financial Crisis (GFC).

Further, I assess whether the extracted beliefs for aggregate excess cash flow align with survey data. As per Table D.9, the validity of the estimated beliefs holds, even for the K-N CAPM model. While the magnitude of correlations with PE sentiment indices is slightly reduced, they remain statistically significant. Additionally, the correlation analysis for non-core individual and institutional investors in the public market remains largely consistent with the baseline analysis, even after introducing the new moment condition.

To sum it up, subjective beliefs that match observed PE valuations demonstrate a consistent trend: a pessimistic view towards aggregate excess cash flows. Yet, what drives this conclusion can change based on the specific conditions used to estimate these beliefs. For instance, in the case of the K-N CAPM SDF and the moment condition for aggregate cash flow, the main factors influencing these beliefs seem to alter. This shift challenges the prevailing notion of investor pessimism about the cash flow-generating capabilities of PE funds. Instead, it suggests a tilt in importance towards over-optimism about stock market cash flows.

More calibrated models, such as the EHB and LRR, yield smaller abnormal performances.

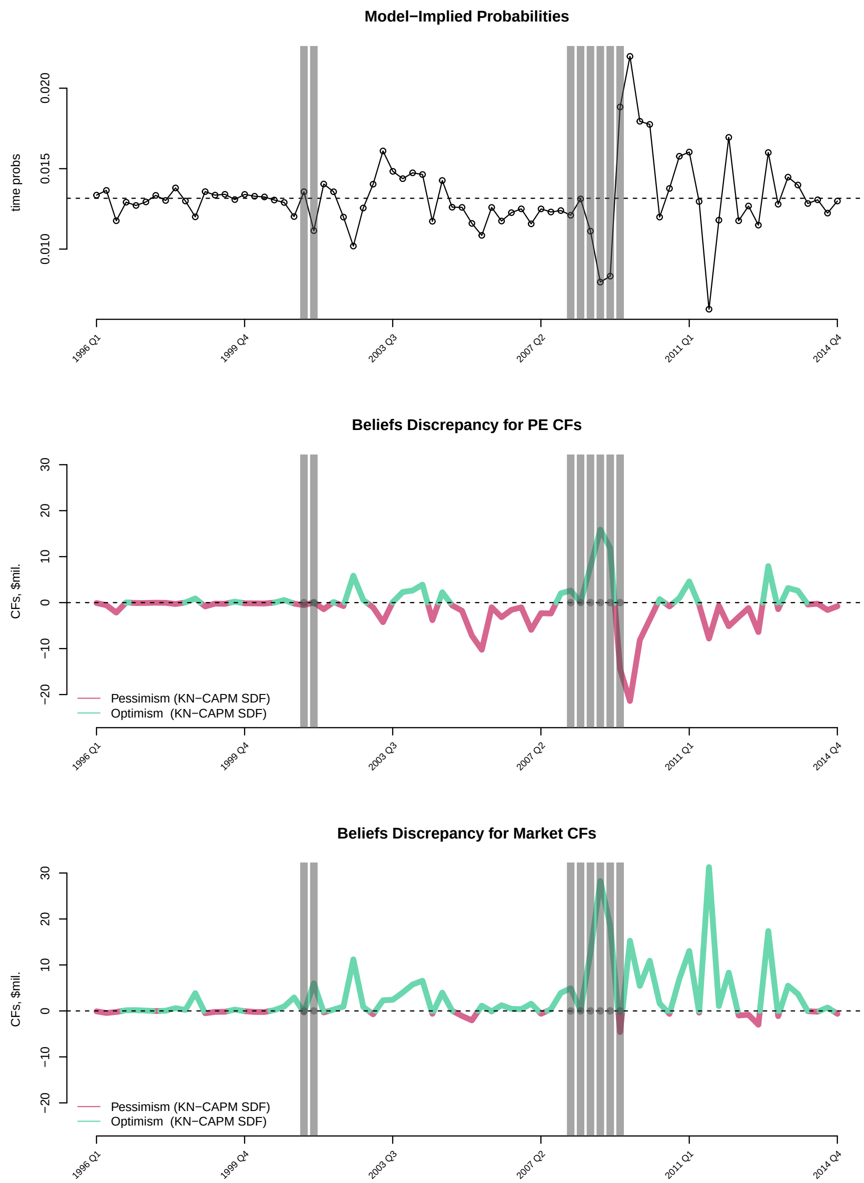

Figure D.9: Model-Implied Probabilities and Subjective Beliefs. The figure is structured into three panels, capturing different dimensions of model-implied probabilities and belief-adjusted cash flows over the period 1996 Q1 to 2014 Q4. Subjective beliefs are estimated using the moment condition for aggregate cash flows. On the upper panel, the figure plots the model-implied probabilities estimated under K-N CAPM SDF. On the middle and bottom panel, the figure captures the *Beliefs Discrepancy*, which is the difference between belief-adjusted and observed aggregate cash flows, presented in $ million. The middle panel delves into the PE fund cash flows, while the bottom panel delves into the mimicking cash flow of the CRSP benchmark fund.

Also, subjective beliefs estimated for them are statistically robust and match with survey data. Therefore, potential belief distortions for the K-N CAPM could arise from non-stationarity, severe model misspecification, or a mix of both. I argue that it is prudent to avoid at least one potential source of distraction in belief estimations. This rationale underpins the choice of average excess cash flows for the baseline analysis, as detailed in Appendix A.

Table D.9: Correlations between Surveys and Excess Cash Flow. The table reports the correlations between various sentiment indices, presented row-wise, and belief-adjusted excess cash flow. Across the columns, the subjective beliefs are estimated using distinct SDFs using the moment condition for aggregate cash flows.

	Panel A: Average Excess Cash Flow adjusted for Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.17	-0.19	-0.19	-0.17	-0.12	-0.13
Shiller (ind)	0.10	0.05	0.05	0.05	0.07	0.10
Gallup	0.45***	0.45***	0.45***	0.42***	0.36***	0.38***
ESI	0.38***	0.39***	0.39***	0.37***	0.32***	0.33***
CEPECI	0.58***	0.55***	0.55***	0.53***	0.54***	0.56***
SVVCCI	0.68***	0.65***	0.65***	0.64***	0.63***	0.65***
Note:						*p<0.1; **p<0.05; ***p<0.01

D.3 Estimating Beliefs Fund by Fund. Estimating investors' subjective beliefs on a fund-by-fund basis is attractive because it leverages the cross-sectional information from funds' cash flows. From a methodological perspective, this approach can be linked to the null hypothesis underlying the GPME calculation. Specifically, the individual PE fund cash flows, including the initial investment when discounted with the SDF, should be zero. As described in Appendix C, this null hypothesis can be formulated for the individual PE funds in NPV terms as:

$$NPV_i^{PE,KN16} = \sum_{t=1}^{T} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{PE}$$

Here, $M_{1,t}$ is the multi-period SDF, derived by successively compounding the single-period discount factors. $M_{\emptyset(i)}$ represents the multi-period SDF at the inception of fund i. $t = 1, ..., T$ denotes the range of all dates for which PE cash flows are observed.

The moment condition introduced in the original SDF estimation methodology by Korteweg and Nagel (2016) employs the mimicking cash flows of the benchmark fund investing in the public stock market. A similar NPV formulation for the market benchmark fund produces:

$$NPV_i^{mrkt,KN16} = \sum_{t=1}^{T} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{mrkt}$$

The difference between the two NPV moment conditions yields the excess moment condition for the individual fund, analogous to Equation 8. This moment constraint for excess cash flows is used to estimate beliefs using the ET methodology:

$$NPV_i^{exc,KN16} = \sum_{t=1}^{T} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{exc}$$

By leveraging ideas similar to those in the main analysis, I aim to estimate investors' subjective beliefs for individual funds. This process eliminates the pricing errors at the fund level instead of the aggregate level. The optimization problem at the fund level is:

$$
\begin{aligned}
\min_{p_{i,t}} D(\hat{\mathbb{P}}_i^* || \hat{\mathbb{P}}_i) &= \sum_{t=1}^{T_i} p_{i,t} \log\left(\frac{p_{i,t}}{\frac{1}{T_i}}\right) \\
\text{subject to} \quad \mathbb{E}^{\hat{\mathbb{P}}_i^*} &\sum_{t=1}^{T_i} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{exc} = 0, \\
&\sum_{t=1}^{T} p_{i,t} = 1.
\end{aligned} \tag{11}
$$

Solving Problem 11 provides the set of individual fund model-implied probabilities: $\hat{\mathbb{P}}_1^*, \hat{\mathbb{P}}_2^*, ..., \hat{\mathbb{P}}_N^*$. Subsequently, applying these probabilities to fund-level cash flows generates the alternative series of a fund's cash flows, referred to as individual fund subjective cash flows. I aggregate these cash flows across all funds, period by period, to examine whether this methodology yields distributional results similar to those at the aggregate level. All the results presented in this section are based on beliefs estimated using the K-N CAPM SDF.

The outcomes of the fund-specific estimation are showcased in Figure D.10. The top panel illustrates both the fund's excess cash flow, $CF_{i,t}^{exc}$, and its discounted version, $\frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{exc}$. The middle panel displays the specific probability measure for the fund, $\hat{\mathbb{P}}_i^*$, that minimizes the KLIC divergence from the historical distribution (delineated by the dashed line) while eliminating the pricing error at the individual fund level. The bottom panel demonstrates the outcome of applying the model-implied probabilities (from the middle panel) to discounted excess cash flows. When adjusted for beliefs, these discounted excess cash flows ensure the NPV identity holds true for every fund by balancing the sum of discounted investments (capital calls) with that of discounted capital payouts (capital distributions).

Figure D.11 visually portrays the discrepancy between the actual (observed) and belief-adjusted cash flows. The figure illustrates the influence of investors' beliefs during the fundraising phase for each fund within the Preqin sample from the primary analysis, affecting the valuation of PE funds. For a consistent comparison, I have limited the time series to the identical sample period (from Q1 1996 to Q4 2014) employed in the primary analysis. The patterns closely mirror those in Figure 1. Once investors' subjective beliefs are aggregated across the funds, the resulting time series reveals a fatter negative tail and a more left-skewed distribution than the historical cash flows. This new methodological approach reinforces the inference that investors must factor in a premium for uncertainties tied to future cash flows to rationalise the observed PE fund valuations.

In Figure D.12, I investigate the potential sources of investors' pessimism concerning excess cash

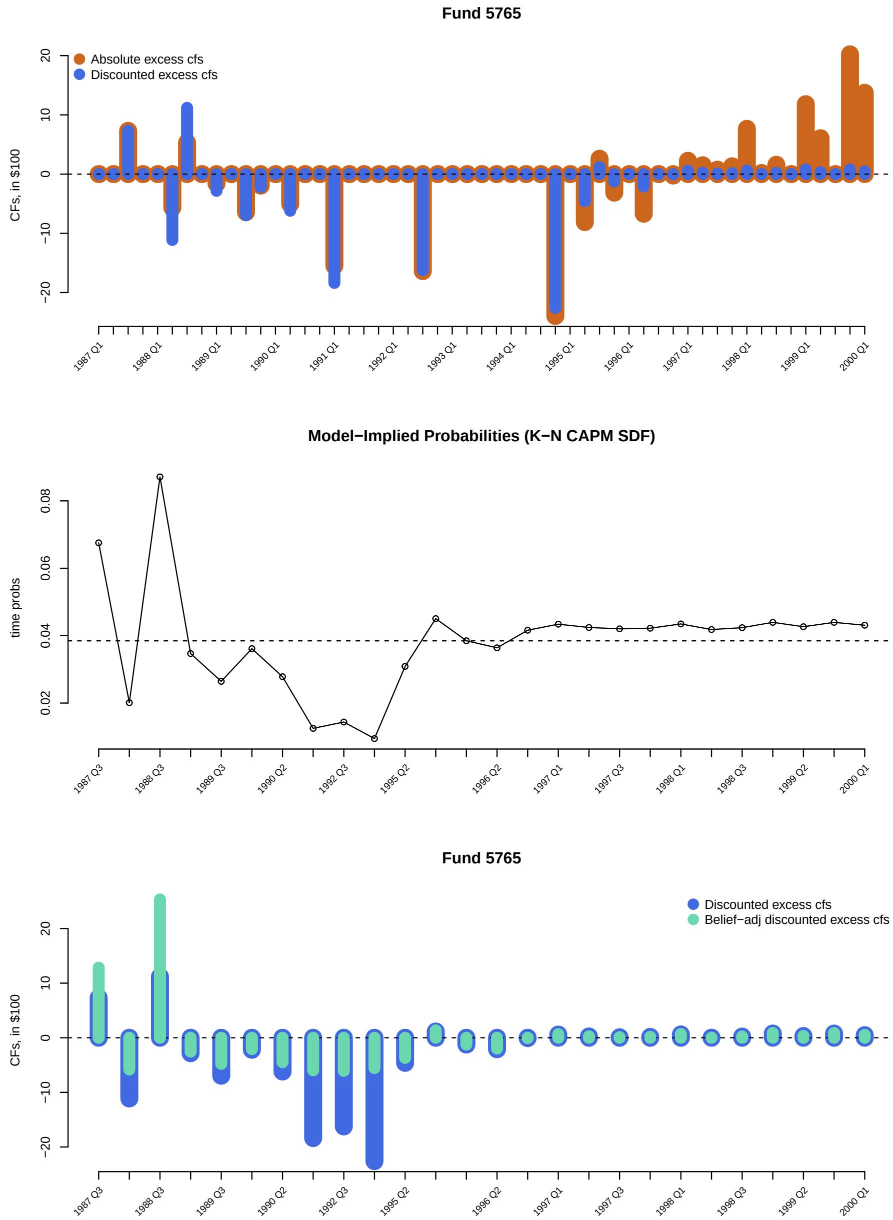

Figure D.10: Model-Implied Probabilities and Subjective Beliefs for One Fund. The figure plots the original and discounted excess cash flows for an individual fund on the upper panel. On the middle panel, the figure plots the model-implied probabilities obtained using an Empirical Likelihood–type estimator on the 'fund-by-fund' basis. On the bottom panel, the figure contrasts the risk-adjusted and belief-adjusted excess cash flows.

flows. As depicted in the top panel, their skepticism towards the cash flows of PE funds is more noticeable than in the primary analysis shown in Figure 4. In about 84% of the examined period,

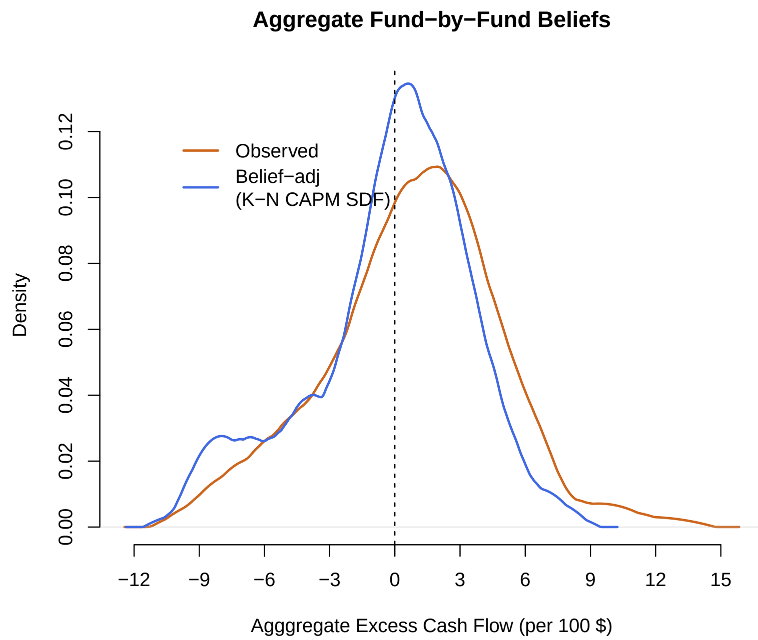

Figure D.11: Discrepancy in PE Excess Cash Flows: True vs. Belief-Adjusted. This figure plots the observed and belief-adjusted aggregate excess cash flows for PE funds from Q1 1996 to Q4 2014. Adjustments are based on beliefs estimated under the K-N CAPM SDF on a 'fund-by-fund' basis. Values are presented in 100 dollars, with density estimations derived using the Epanechnikov kernel method.

the discrepancy in beliefs is negative for fund-specific subjective beliefs. In contrast, this figure was only 63% when beliefs were estimated using average cash flows. A more detailed analysis of this discrepancy is available in Table D.10. The intensity of negative sentiments about PE cash flows significantly outweighs the positive sentiments. The $max - min$ range of discrepancy for the K-N CAPM model exceeds 5-to-1. Notably, this pessimism is not primarily clustered around significant market events such as the dot-com bubble or the Global Financial Crisis (GFC), corroborating evidence derived from subjective beliefs across different SDFs in the main analysis.

The bottom panel of Figure D.12 delivers an analysis of the belief-adjusted component linked to the public market. As discerned from the figure and further verified by Table D.10, investor sentiments towards public market cash flows are more restrained compared to the main analysis. Only about 51% of the time do they exhibit a positive discrepancy in beliefs, significantly less than the 60-70% observed in the primary analysis, and more in line with the historical distribution of cash flows, suggesting a 50% ratio. The discrepancy's magnitude also markedly diminished, plunging from the minimum 4.7-to-1 $max - min$ ratio (as per the K-S CAPM model in Table 5) to 1.2-to-1 for the K-N CAPM model in this analysis. Intriguingly, the impact of the GFC remains evident in fund-specific belief estimates; in Q4 2009, investors had an overly optimistic outlook for

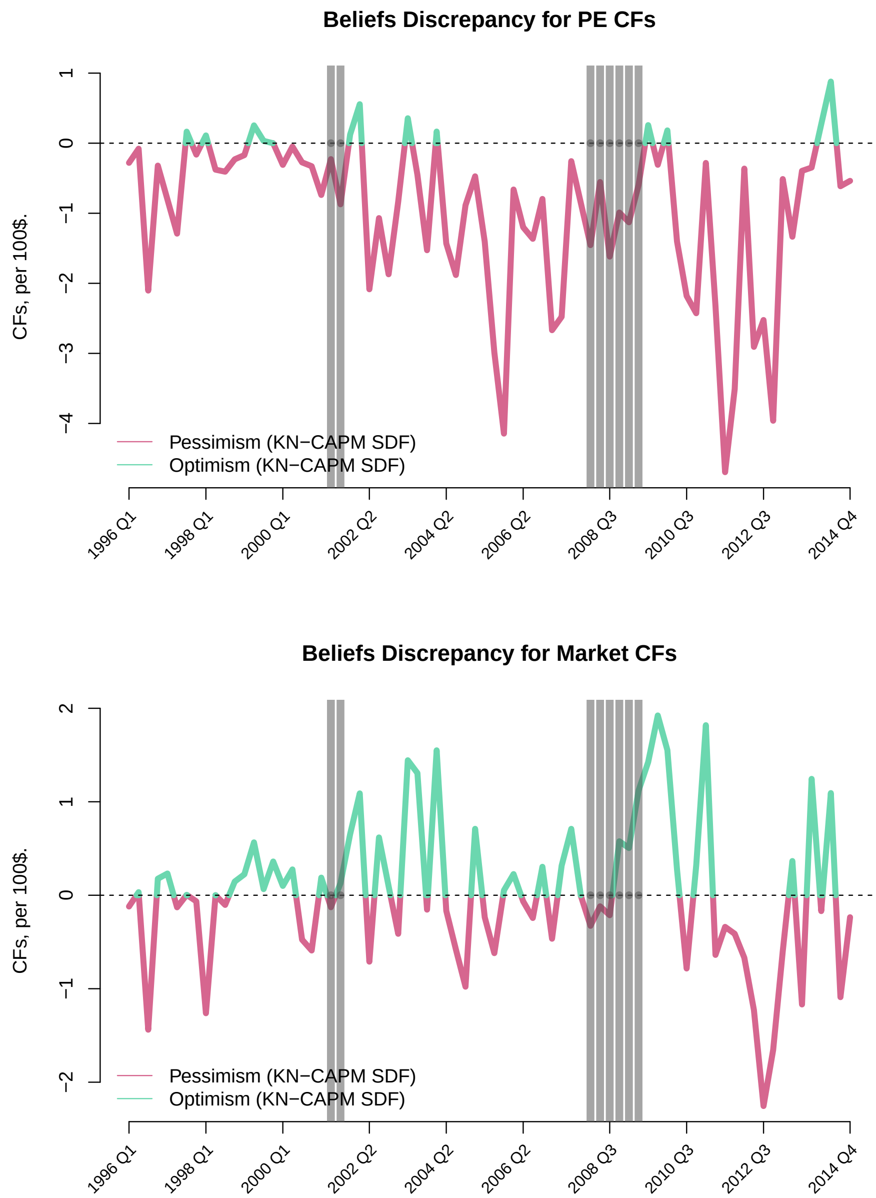

Figure D.12: Comparison of Belief-Adjusted and Observed Aggregate Cash Flows: PE vs. Market. This figure illustrates the discrepancy between belief-adjusted and observed aggregate cash flows for PE funds and market benchmark funds from 1996 Q1 to 2014 Q4. The top panel details the aggregate cash flows of PE funds, while the bottom panel showcases those of CRSP-mimicking funds. The discrepancy is calculated for beliefs estimated on a 'fund-by-fund' basis for the K-N CAPM SDF. NBER-designated recession periods are highlighted with shaded areas.

the market cash flow, expecting the CRSP cash flow to exceed the actual by $1.92 for every $100 invested. Predictably, this excessive optimism eventually gave way to heightened pessimism by Q3

2012, with a deviation of \$2.26 per \$100 invested, as detailed in Table D.10.

In summary, the fund-specific analysis of subjective beliefs suggests that investors tend to perceive excess cash flows with a negative skew. This is predominantly driven by deep-seated pessimism about PE cash flows. Interestingly, the secondary influential factor—public market cash flows—does not carry the same importance weight. This investigation implies that the observed undervaluation of PE funds primarily arises from investor skepticism regarding PE funds' cash flow-generating capacities, rather than any excessive optimism about their public market portfolios.

Table D.10: Discrepancy in Beliefs for PE and Market Cash Flows. The table provides statistics related to the *beliefs discrepancy* for PE and market cash flows, as visualized in Figure D.12.

	Beliefs Discrepancy for:	
	PE Cash Flows	Market Cash Flows
Underexpectations, % time	84.21%	48.68%
Overexpectations, % time	15.79%	51.32%
Max Discrepancy, \$	0.88	1.92
Max Discrepancy - YQ	2014 Q2	2009 Q4
Min Discrepancy, \$	-4.7	-2.26
Min Discrepancy - YQ	2011 Q3	2012 Q3
Avg Discrepancy, \$	-0.97	0.04

The validity assessment of these fund-specific subjective beliefs, as detailed in Table D.11, confirms that the estimated beliefs align with survey data. Although the magnitude of correlations with PE sentiment indices has somewhat diminished, it remains statistically significant when juxtaposed with the main analysis results in Table 8. It is worth noting that the correlation insights for non-core individual and institutional investors in the public market remain largely congruent with the baseline analysis. For both the Gallup and ESI surveys, the PE market beliefs narrative aligns with the public market, indicating positive correlation trends. However, the Shiller survey presents a potential anomaly: its statistically significant negative correlation may hint at the estimator's potential noisiness, which is somewhat echoed by the market cash flow findings earlier.

Table D.11: Correlations between Surveys and Excess Cash Flow. The table reports the correlations between various sentiment indices, presented column-wise, and belief-adjusted average excess cash flows. Subjective beliefs are estimated on a 'fund-by-fund' basis for the K-N CAPM SDF.

	Shiller (inst)	Shiller (ind)	Gallup	ESI	CEPECI	SVVCCI
K-N CAPM	0.02	-0.24**	0.50***	0.46***	0.57***	0.66***
Note:						*p<0.1; **p<0.05; ***p<0.01

While both methodologies (fund-by-fund versus average excess cash flow) produce similar findings, it is crucial to underscore some theoretical limitations connected with estimating subjective beliefs on an individual fund basis. The granularity afforded by the fund-by-fund method reveals cross-sectional nuances of investors' subjective beliefs. However, I spotlight three aspects that could critically distort the estimated beliefs and their economic implications when employing this novel approach: 1) the constraint of limited observations, 2) potential stationarity issues, and 3) the risk of overfitting.

First, the extraction of beliefs at the individual fund level intrinsically means that the analyses are often based on a constrained set of observations. This scarcity of data raises prominent concerns regarding the statistical reliability and robustness of the derived estimates. More succinctly, the limited data points might inhibit the estimator's ability to converge accurately to the true underlying measure.

Second, the ET methodology in use requires the stationarity of underlying processes. If the data exhibits non-stationary characteristics – such as trends, cycles, or other temporal patterns – the resultant estimates could be spurious, potentially leading to misleading inferences about the model-implied probabilities and corresponding subjective beliefs.

Third, engaging with a limited dataset invariably raises the risk of overfitting. In cases where the model excessively adapts to the specificities and noise within the data, like the fund-level cash flows, it might capture anomalous patterns rather than the true underlying dynamics. Consequently, a new approach trades the model's richer economic implications for the noisiness of beliefs' estimator.

Bearing these caveats in mind – nuances which I admit are not exhaustively explored or addressed in this paper – I proceed to examine the cross-sectional economic implications for PE funds using subjective beliefs estimated on a fund-by-fund basis.

Cross-Sectional Economic Implications.

The results of cross-sectional analysis for PE funds using subjective-beliefs primarily hinge on contrasting two time-series: the original cash-flows and their belief-adjusted counterparts. This comparison utilizes the concept of partial ordering between these two random variables via the notion of first-order stochastic dominance. In this context, both the original and belief-adjusted cash flows act as distinct random variables. The goal is to classify all funds into one of three categories based on this comparison: optimistic, pessimistic, and neutral. Such a classification provides insight into the LPs' sentiment at the time of capital commitment to the respective funds.

To fix ideas, I introduce the following notation. Let $CF^{\hat{P}}$ represent the original PE fund's cash flows as observed under the empirical probability measure, and $CF^{\hat{P}^*}$ symbolize the counterfactual cash-flows observed under model-implied probabilities, which reflect subjective beliefs about the fund's cash flows. I then define the *First-Order Stochastic Dominance* (FOSD) in terms of these cash flows:

Definition 1 (FOSD for Cash-Flows)**.** For any given cash-flows $F \equiv CF^{\hat{P}}$ and $G \equiv CF^{\hat{P}^*}$, F *first-order stochastically dominates* G if and only if

$$F_{CF}(cf) \leq G_{CF}(cf)$$

for all $cf \in CF$ with a strict inequality over some intervals. Here, $F_{CF}(cf)$ and $G_{CF}(cf)$ are the cumulative distribution functions (CDFs) of $CF^{\hat{P}}$ and $CF^{\hat{P}^*}$ respectively.

Using the above dominance criteria, I can categorize all funds into three mutually exclusive groups:

- **Optimistic Funds**: Here, $G_{CF}(cf)$ first-order stochastically dominates $F_{CF}(cf)$. All quantiles of the beliefs-adjusted cash-flows exceed those of the original series. This suggests that LPs anticipate that larger cash flows are more likely based on their subjective beliefs, indicating an optimistic outlook on future fund cash flows.

- **Pessimistic Funds**: In this case, $F_{CF}(cf)$ first-order stochastically dominates $G_{CF}(cf)$. All quantiles of the beliefs-adjusted cash-flows are less than those of the original series. This indicates that LPs, based on their beliefs, expect smaller cash flows, pointing to a pessimistic view of future fund cash flows.

- **Neutral Funds**: The *FOSD for Cash-Flows* criterion does not yield a conclusive result. Not all quantiles of the beliefs-adjusted cash-flows are consistently greater or smaller than the original series. This suggests that the LPs maintain a neutral outlook on the future cash flows of the fund.

To gain a deeper understanding of how subjective beliefs operate, consider the following simplified example. This illustration will shed light on the interaction between model-implied probabilities and subjective cash flows, as well as the mechanics of the FOSD criterion.

Assume a fund has three distinct cash flows, and we know the estimated model-implied probabilities:

1. **Original Cash Flows** (CF^P)

$$
\begin{aligned}
CF_i = CF_i^{\hat{P}_i} &= \{5, 10, -5\} \\
\hat{P}_i = \{p_1, p_2, p_3\} &= \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}
\end{aligned}
$$

2. **Belief-Adjusted Cash Flows** (CF^{P^*})

$$
\begin{aligned}
\hat{P}^* &= \{p_1^*, p_2^*, p_3^*\} = \left\{\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right\} \quad \text{(estimated using ET methodology for fund } i\text{)} \\
T &= \text{length of the cash flow series} \\
CF^{\hat{P}^*} &= \{p_1^* \cdot CF_1 \cdot T, p_2^* \cdot CF_2 \cdot T, p_3^* \cdot CF_3 \cdot T\} \\
&= \left\{\frac{2}{3} \cdot 5 \cdot 3, \frac{1}{6} \cdot 10 \cdot 3, \frac{1}{6} \cdot (-5) \cdot 3\right\} \\
&= \{10, 5, -2.5\}
\end{aligned}
$$

3. **FOSD**

$$
\begin{aligned}
F_{CF}(cf) = \{(Value; Prob)\} &= \{(-5; \frac{1}{3}), (5; \frac{2}{3}), (10; 1)\} \\
G_{CF}(cf) = \{(Value; Prob)\} &= \{(-2.5; \frac{2}{3}), (5; \frac{5}{6}), (10; 1)\}
\end{aligned}
$$

The *FOSD for Cash-Flows* criterion does not yield a conclusive result. Therefore, Fund i is classified as **neutral**.

As demonstrated by this example, the *FOSD for Cash-Flows* is a particularly conservative measure of LPs' sentiments. This design choice was deliberate, aiming to mitigate the potentially noisy nature of the fund-by-fund estimator for investors' subjective beliefs.

I start the cross-sectional analysis by examining the distribution of fund types over time. This will offer insights into how the PE industry has evolved through the years. As depicted in Figure D.13, during the early stages of the PE industry, LPs were predominantly neutral about fund cash flows. However, for funds with vintage years in the 1990s, LPs exhibited a markedly optimistic outlook on the PE industry. This trend shifted in the 2000s, coinciding with the aftermath of the first dot-com bubble burst. Specifically, for the 2000 vintage year, pessimistic LPs outnumbered their optimistic counterparts, registering at 17% versus 11%, respectively. Post-bubble, sentiments rebounded, and leading up to the GFC, LPs were primarily optimistic about PE funds. In contrast, post-GFC for funds launched in 2010, I observe a stark pessimism with 45% of LPs showing pessimistic sentiments as opposed to a mere 21% being optimistic. This shift indicates that following the GFC, PE investors were overwhelmingly pessimistic.

Another interesting trend is the consistent decline in neutral investors over time, suggesting a growing polarization in opinions among PE investors. This observation aligns well with the emergence of the secondary market around 2007–2009, as documented by Nadauld, Sensoy, Vorkink, and Weisbach (2019). A divergence in opinions is a critical precursor for trading—a concept that traces its roots back to Miller (1977).

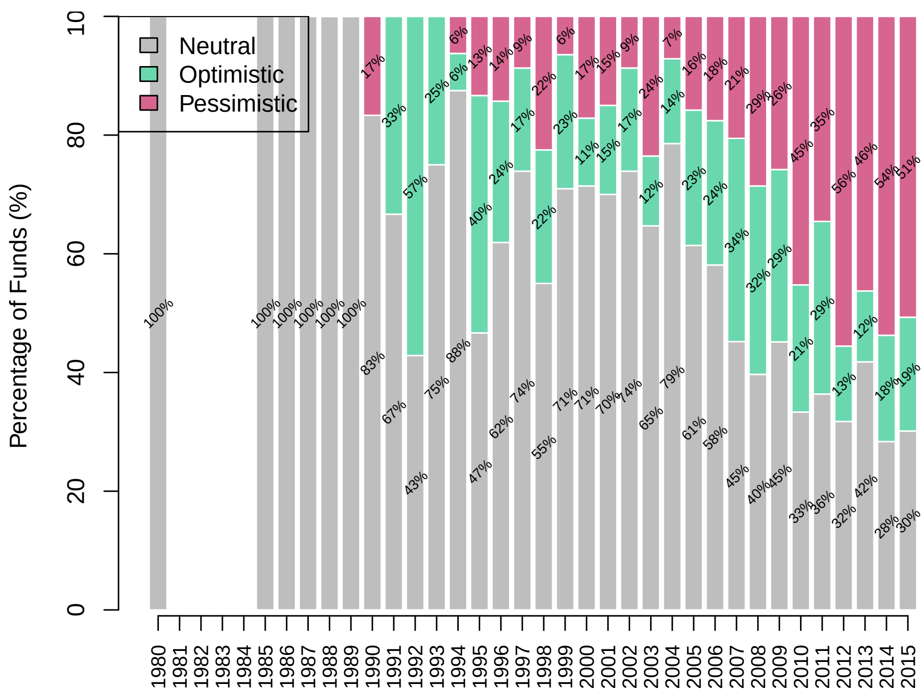

Figure D.13: Sentiment Distribution of Funds Across Vintage Years. The figure plots the distribution of LPs' sentiments (Optimism, Pessimism, or Neutral) regarding fund types, as deduced from quantile comparisons of belief-adjusted and original values. Each bar represents the distribution of sentiments for a specific vintage year, offering insights into how perceptions of different fund types have evolved over time.

I conduct a similar analysis in the context of geography. I focus on two key characteristics of funds: their primary geographical focus and the location of General Partners (GPs). For this analysis, I aggregate all geographies into four regions: Asia, Europe, North America, and the rest of the world.

The analysis concerning geographical focus is presented in Figure D.14. As anticipated, North America dominates in terms of the number of funds, substantially outnumbering the other regions. Funds originating in this region are mostly neutral according to my *FOSD for Cash-Flows* criterion. This neutrality is observed across all geographical categories. Asia showcases the most uniform distribution, with the highest proportion of optimistic LPs: 15 neutral funds, 15 optimistic funds, and 14 pessimistic funds. Conversely, funds focused on the rest of the world—likely less developed nations—reveal a substantial proportion of pessimistic investors, accounting for 34% over all time periods. This observation aligns with common sense, as investments in less-developed countries often entail heightened legal risks, a factor of paramount significance in PE investments.

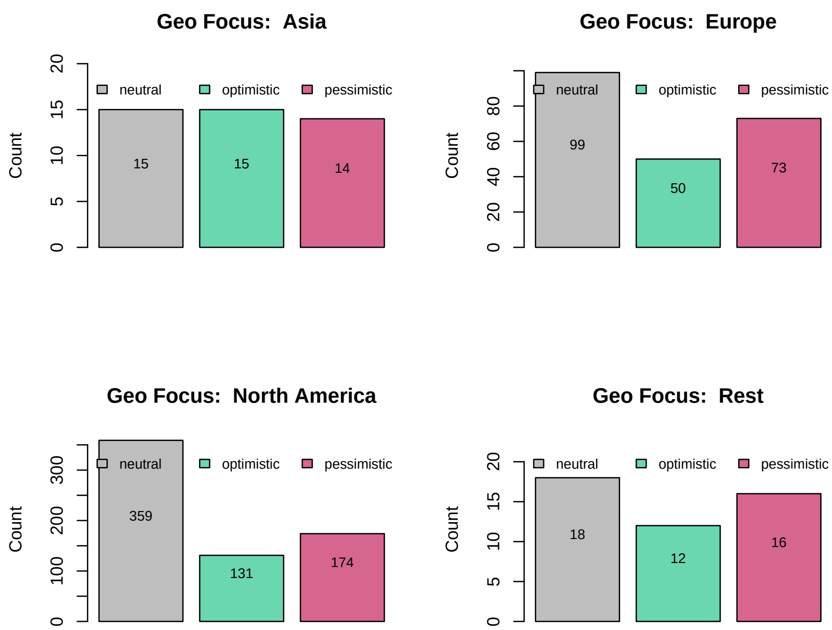

Figure D.14: Sentiment Distribution by Geographic Focus. The figure plots a breakdown of sentiments (Optimism, Pessimism, and Neutral) related to fund types, categorized by their primary geographic focus. Each subplot represents one of the enlarged regions - North America, Europe, Asia, and the Rest.

The distribution based on the locations of GPs is depicted in Figure D.15. North American GPs are highly sought after by investors. Mirroring the trend observed with geographical focus, investors maintain a predominantly neutral stance toward North American GPs. Intriguingly, both Europe and the rest of the world register the highest levels of pessimism concerning GPs' cash-flow generating capabilities. This skepticism may be rooted in the fee structures proposed by GPs in these regions. In contrast, investors display the most optimistic views toward GPs located in Asia, with the proportion standing at 36%.

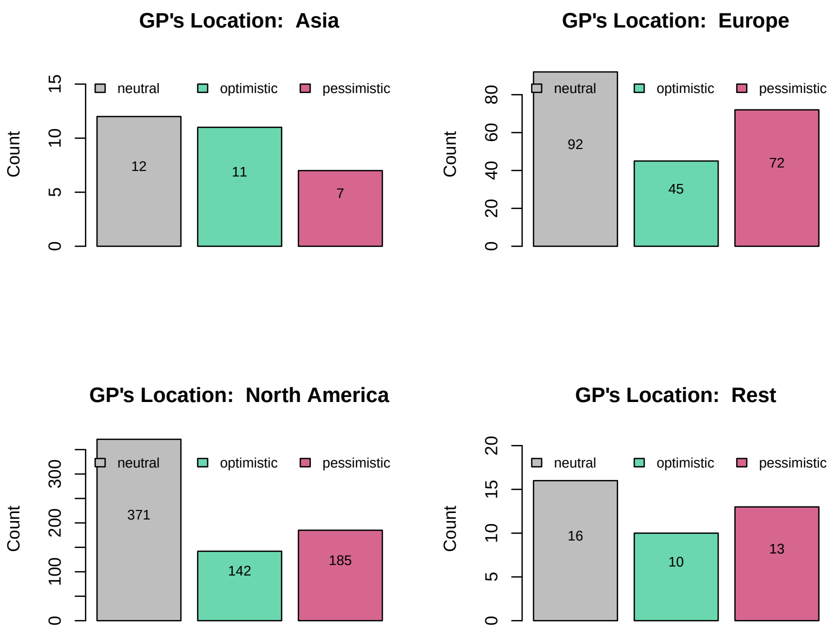

Figure D.15: Sentiment Distribution by GP's Location. The figure plots a segmentation of sentiments (Neutral, Optimistic, and Pessimistic) corresponding to fund types, arranged according to the location of their GP. Each subplot is dedicated to one of the aggregated regions - North America, Europe, Asia, and the Rest.

Finally, I analyze the industrial context of investing in PE. Before the analysis, I categorized funds into broader industrial groups. A detailed description of these categories can be found in Table D.12. The table presents the original Preqin industry classification in the second column, outlines which sub-industries each industry includes, and provides the rationale for consolidating these industries in the fourth column. As a result of this exercise, I identified six distinct consolidated industrial groups, which are listed in the first column: Tech and Media, Health and Services, Finance and Real Estate, Energy and Resources, Consumer and Industrial, and Diversified.

The sentiments of PE investors across these industries are depicted in Figure D.16. Most of the funds examined lean towards a diversified industrial focus, accounting for more than half of all funds. Investors are predominantly neutral about diversified funds. The Energy and Resources sector is viewed most pessimistically by LPs. About 41% of 17 initiated funds had investors who expressed a pessimistic sentiment. In contrast, the Finance and Real Estate industry, despite

Table D.12: Industry Groupings and Rationale. This table categorizes various industries into consolidated groups based on shared characteristics and functions. The rationale behind each grouping provides insight into the commonalities and dynamics that shape each category.

Consolidated Groups	Original Industries	Sub-Industries	Rationale
Tech & Media	Information Technology, Telecoms & Media	Electronics, Internet, IT Infrastructure, Software, Media, Telecoms	Both IT and Telecoms & Media companies are focused on the transmission, storage, and processing of data. Whether it's via hardware, software, or signals, these firms are central to the digital age, rendering their consolidation logical.
Health & Services	Healthcare, Business Services	Biotech, Healthcare IT, Pharmaceuticals, Business Support Services, Marketing/Advertising	While healthcare provides intangible services related to patient care, business services offer expert-driven intangible products/services. Both primarily deliver non-physical value, making them compatible.
Finance & Real Estate	Financial & Insurance Services, Real Estate	Financial Services, Insurance, Commercial Property, Real Estate Development	Both these sectors are pillars of the economy, dealing with assets, capital, and investments. Real estate, often financed through financial services, is intertwined with economic cycles, credit availability, and investor sentiment.
Energy & Resources	Energy & Utilities, Raw Materials & Natural Resources	Oil & Gas, Renewable Energy, Agribusiness, Chemicals, Mining	This consolidation is underpinned by the extraction, refinement, production, and distribution of natural resources, whether for energy generation or other purposes. These sectors form the backbone of primary industry operations, making their grouping cohesive.
Consumer & Industrial	Consumer Discretionary, Industrials	Consumer Products, Food, Retail, Aerospace, Defense, Industrial Machinery	Both sectors produce tangible goods, whether they're non-essential consumer items or goods used in construction and manufacturing. Their tangible output and close ties to consumer behavior and industrial activities underpin this grouping.
Diversified	Diversified	Multiple industries span this category.	As a standalone category, "Diversified" likely includes entities that span multiple industries, making it difficult to classify them under a specific sector.

its limited count of only 15 funds, had 40% of its funds characterized as optimistic. This is the only industry where optimistic funds outnumber both neutral and pessimistic funds. For other industries, the sentiment analysis shows that investors are primarily neutral, with a slight leaning towards pessimism.

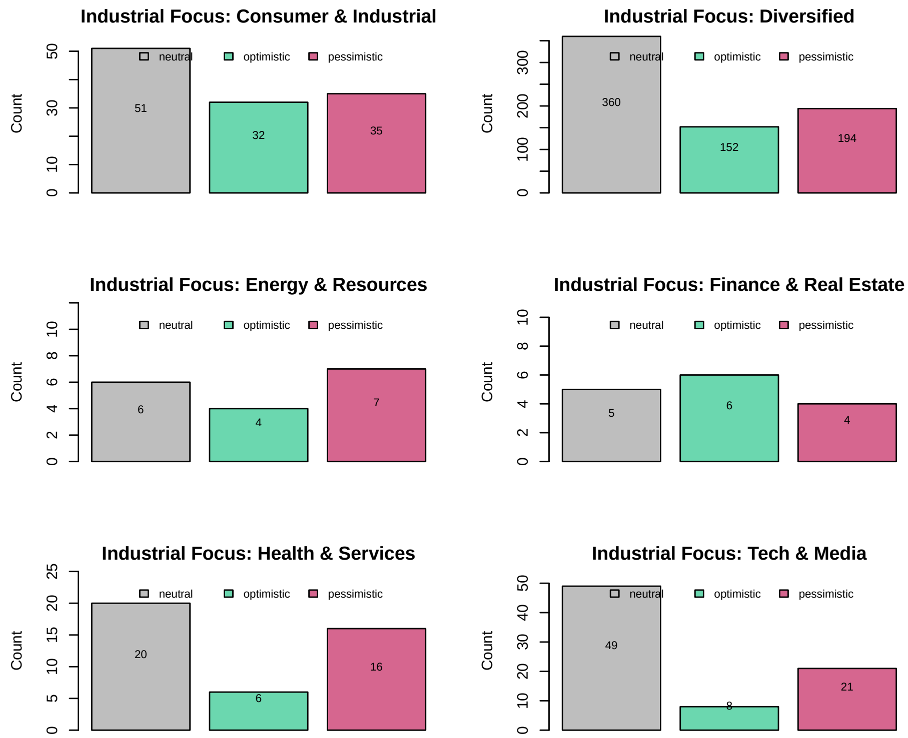

Figure D.16: Sentiment Distribution by Industrial Focus. The figure plots sentiment distributions (Neutral, Optimistic, and Pessimistic) across various enlarged industrial categories.

Further investigation into the Finance and Real Estate industry required an exploration of the temporal dynamics of pessimism-optimism patterns. These patterns are detailed in Figure D.17. The trends are quite telling. All years before the GFC were characterized by the inception of optimistic and neutral funds. However, 2008 saw a stark shift, and in the aftermath of the financial crisis, investors grew markedly pessimistic about the financial industry — a reaction that aligns with general sentiment. Notably, no funds initiated in the Finance and Real Estate industry after 2008 were identified as optimistic.

In conclusion, estimating investors' subjective beliefs on a fund-by-fund basis provides a fresh perspective distinct from the primary analysis. This approach permits a more detailed temporal exploration, as well as nuanced geographical and industrial analyses of the beliefs. These insights afford a deeper comprehension of the PE belief formation process and its implications for PE funds' cash flows. However, this novel subjective belief estimator may introduce a greater degree of noisiness compared to the estimator employed in the primary analysis. Nonetheless, this chapter presents robust evidence that, at the aggregate level, the implications of beliefs do not critically diverge. Therefore, cross-sectional results can be seen as informative to a certain degree.

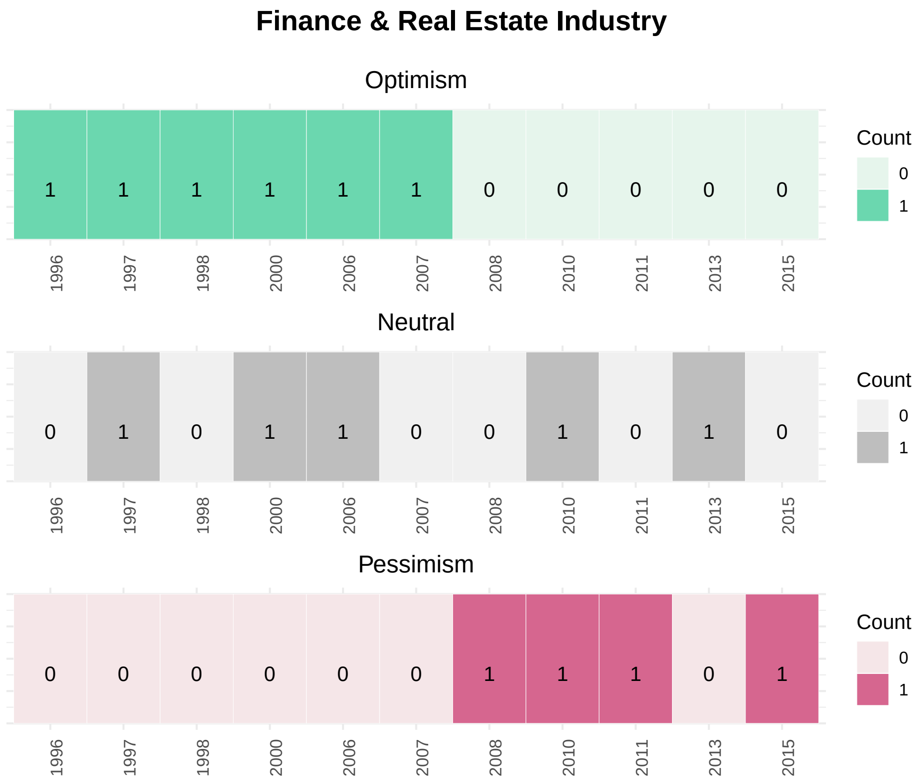

Figure D.17: Sentiment Distribution for 'Finance & Real Estate' Industry Across Vintage Years. The figure plots the sentiment distributions (Optimism, Neutral, and Pessimism) within the 'Finance & Real Estate' industry on a yearly basis. The distinct tiles for each year represent the count of funds, color-coded by their respective sentiment.