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ON THE BESOV REGULARITY OF THE BIFRACTIONAL BROWNIAN MOTION

BY

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Abstract. Our aim in this paper is to improve Hölder continuity results for the bifractional Brownian motion (bBm) $(B^{\alpha,\beta}(t))_{t\in[0,1]}$ with $0<\alpha<1$ and $0<\beta\leqslant 1$. We prove that almost all paths of the bBm belong to (resp. do not belong to) the Besov spaces $\mathbf{Bes}(\alpha\beta,p)$ (resp. $\mathbf{bes}(\alpha\beta,p)$) for any $\frac{1}{\alpha\beta}<p<\infty$, where $\mathbf{bes}(\alpha\beta,p)$ is a separable subspace of $\mathbf{Bes}(\alpha\beta,p)$. We also show similar regularity results in the Besov–Orlicz space $\mathbf{Bes}(\alpha\beta,M_2)$, with $M_2(x)=e^{x^2}-1$. We conclude by proving Itô-Nisio theorem for the bBm with $\alpha\beta>\frac{1}{2}$ in the Hölder spaces $\mathcal{C}^{\gamma}$, with $\gamma<\alpha\beta$.

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1. INTRODUCTION

Let $(B^{\alpha,\beta}(t))_{t\geqslant 0}$ be a bifractional Brownian motion (bBm for short), i.e., a centred real-valued Gaussian process with covariance function

$$R^{\alpha,\beta}(s,t):=R(s,t)=\frac{1}{2^{\beta}}\left((t^{2\alpha}+s^{2\alpha})^{\beta}-|t-s|^{2\alpha\beta}\right), \tag{1.1}$$

where $\alpha \in (0,1)$ and $\beta \in (0,1]$. Observe that, when $\beta = 1$, $B^{\alpha,1}$ is a fractional Brownian motion with Hurst parameter $\alpha \in (0,1)$. However the increments of $B^{\alpha,\beta}$ are not stationary except for the case when $\beta = 1$. The bBm has the following general properties: it is self-similar with index $\alpha\beta$, that is, for every $a > 0$,

$$\{B^{\alpha,\beta}(at),\ t \geqslant 0\} \stackrel{d}{=} \{a^{\alpha\beta} B^{\alpha,\beta}(t),\ t \geqslant 0\}, \tag{1.2}$$

where $X \stackrel{d}{=} Y$ means that the two processes have the same finite-dimensional distributions. It is a quasi-helix (see [17] and [18] for various properties and applications of quasi-helices) since for every $s, t \in [0,T]$, we have

$$2^{-\beta}|t-s|^{2\alpha\beta} \leqslant \mathbb{E}\left(B^{\alpha,\beta}(t) - B^{\alpha,\beta}(s)\right)^2 \leqslant 2^{1-\beta}|t-s|^{2\alpha\beta}. \tag{1.3}$$

Based on the fractional Brownian motion structure, Houdré and Villa [14] have constructed the bifractional Brownian motion as a more general self-similar Gaussian process. Russo and Tudor [27] have shown that the bBm behaves, in terms of sample paths properties, like a fractional Brownian motion with Hurst parameter $\alpha\beta$ (one can see that clearly in Lei and Nualart decomposition). There is a rich literature investigating the properties of the bifractional Brownian motion, we refer for example to the following non-exhaustive list: Bojdecki et al. [4], El-Nouty [10], El-Nouty and Journé [11], Kruk et al. [20], Es-Sebaiy and Tudor [12], Tudor and Xiao [30], Bardina and Es-Sebaiy [3] and Lei and Nualart [22], just to mention a few. It was shown essentially in this last paper the following decomposition of the bBm

$$\{C_2 B^{\alpha\beta}(t),\ t \geqslant 0\} \stackrel{d}{=} \{C_1 X^{\alpha,\beta}(t) + B^{\alpha,\beta}(t),\ t \geqslant 0\},$$

where C_1, C_2 are two constants and $(B^{\alpha\beta}(t))_{t \geqslant 0}$ is a fractional Brownian motion (fBm) with parameter $\alpha\beta$ and $(X^{\alpha,\beta}(t))_{t \geqslant 0}$ is a Gaussian process with infinitely differentiable trajectories on $(0,+\infty)$ and absolutely continuous on $[0,+\infty)$. On the other hand, we know from Ciesielski et al. [9] that almost all paths of the fBm $(B^{\alpha\beta}(t))_{t \geqslant 0}$ belong (resp. do not belong) to the Besov spaces $\mathbf{Bes}(\alpha\beta, p)$

(resp. to the separable subspaces $\mathbf{bes}(\alpha\beta, p)$). In fact, a stronger regularity result was obtained in the Besov-Orlicz space $\mathbf{Bes}(\alpha\beta, M_2)$, where $M_2(x) = e^{x^2} - 1$, (definitions are given in Section 2). Needless to mention that, if we take $0 < a < b$ one can deduce directly by Lei and Nualart decomposition that the sample paths of $(B^{\alpha,\beta}(t))_{a \leqslant t \leqslant b}$ satisfy the same Besov regularity as those of fractional Brownian motion of parameter $\alpha\beta$. Otherwise, we are unable to get the Hölder regularity of $X^{\alpha,\beta}$ on intervals of type $[0, \varepsilon]$, for $\varepsilon > 0$, since the trajectories of this process are only absolutely continuous near 0. Hence we can not derive directly from Lei and Nualart decomposition the Besov regularity for the bBm on the interval $[0, \varepsilon]$. Our main purpose in this paper is to investigate the Besov regularity for sample paths of the bBm $(B^{\alpha,\beta}(t))$ for $t \in [0, 1]$.

Besov spaces $\mathbf{Bes}(\gamma, p)$ are a general framework to investigate the modulus of smoothness in L^p-norms for trajectories of continuous time stochastic processes. In our paper we are concerned by a particular class of Besov spaces of real functions $(f(t)\,,\ \ t \in [0, 1])$ (for a more general context we can see Triebel [29]). One can get an improvement of the regularity in Besov spaces $\mathbf{Bes}(\gamma, p)$ by using Besov–Orlicz spaces $\mathbf{Bes}(\gamma, M_2)$, because of the following injections: for all $p \geqslant 1$,

$$\mathbf{Bes}(\gamma, M_2) \hookrightarrow \mathbf{Bes}(\gamma, p).$$

We note by $\mathcal{C}^\gamma$ the space of functions satisfying a Hölder condition of order $\gamma > 0$ endowed with the usual norm. It's known that Besov spaces cover the Hölder spaces as particular cases, more precisely $\mathcal{C}^\gamma = \mathbf{Bes}(\gamma, \infty)$. In addition, for any $\varepsilon > 0$, and p large enough, we have the following continuous injections (see Section 2):

$$\mathcal{C}^{\gamma+\varepsilon}(I) \hookrightarrow \mathbf{bes}(\gamma, p)(I) \quad \text{and} \quad \mathcal{C}^{\gamma}(I) \hookrightarrow \mathbf{Bes}(\gamma, p)(I) \hookrightarrow \mathcal{C}^{\gamma - \frac{1}{p}}(I),$$

where $\mathbf{bes}(\gamma, p)$ is a separable subspace of the Besov space $\mathbf{Bes}(\gamma, p)$. It is well known that almost surely the sample paths of the bBm $(B^{\alpha,\beta}(t))_{t \geqslant 0}$ belong to the Hölder spaces $\mathcal{C}^\gamma$ for $\gamma < \alpha\beta$, and do not belong a.s. to $\mathcal{C}^{\alpha\beta}$. Our aim is to improve these classical results by showing that we can get smoothness of order $\alpha\beta$ in the Besov spaces $\mathbf{Bes}(\alpha\beta, p)$ for $\frac{1}{\alpha\beta} < p < \infty$, or even in the Besov–Orlicz space $\mathbf{Bes}(\alpha\beta, M_2)$. This is the best regularity one can get in the context of Besov spaces, because we also prove that almost surely the trajectories of the bBm do not belong to the separable spaces $\mathbf{bes}(\alpha\beta, p)$ for $\frac{1}{\alpha\beta} < p < \infty$. So the above injections explain clearly the sharpness of our results. Note that our paper leads to some previous Besov regularity results: When $\alpha = \frac{1}{2}$ and $\beta = 1$ we get the standard Brownian motion regarded in [7] and [8], for $\beta = 1$ we recover the fBm situation considered in [9], and the case $\alpha = \beta = \frac{1}{2}$ corresponds to the regularity of the mild solution for a linear stochastic heat equation driven by a white noise (see [6]).

Among Itô's accomplishments, there is the Itô-Nisio theorem (cf. [16]), in which the authors have established on one hand a general improvement of the Fourier series decomposition of the Brownian motion, and on the other hand a generalization of Wiener's construction of the Brownian motion. They have given the expansion as the convergence of normalized sums of independent random variables. Later, Kerkyacharian and Roynette [19] have proved the same result of the Itô-Nisio in Hölder spaces with a sample proof. In this paper we show the Itô-Nisio theorem for the bBm with $\alpha\beta > \frac{1}{2}$ in the Hölder spaces $\mathcal{C}^\gamma$, with $\gamma < \alpha\beta$. The case $\beta = 1$ corresponds to the fBm with Hurst parameter $\alpha > 1/2$.

This paper is organized as follows. In the second paragraph we give a brief introduction to Besov spaces. The third paragraph is devoted to study the Besov regularity for the sample paths of the bifractional Brownian motion. In the fourth paragraph we investigate the Itô-Nisio theorem for bBm with $\alpha\beta > \frac{1}{2}$. The proofs of our results use technical and very fine calculations based on dyadic coordinate expansions of the bifractional Brownian motion and descriptions of the Besov norms in terms of the corresponding expansion coefficients of a function.

2. PRELIMINARIES

2.1. Besov spaces. Let $I \subset \mathbb{R}$ be a compact interval , $1 \leqslant p < \infty$ and $f \in L^p(I\,;\mathbb{R})$. We define for any $t > 0$

$$\Delta_p(f,I)(t) = \sup_{|s|\leqslant t} \left\{ \int_{I_s} |f(x+s) - f(x)|^p dx \right\}^{\frac{1}{p}},$$

where $I_s = \{x \in I;\;\; x+s \in I\}$. $\Delta_p(f,I)(t)$ denotes the modulus of continuity of f in the L^p-norm. For $\gamma > 0$, we consider the norm

$$||f||_{\gamma,p} := ||f||_{L^p(I)} + \sup_{0<t\leqslant 1} \frac{\Delta_p(f,I)(t)}{t^\gamma}.$$

The Besov space is given by $\mathbf{Bes}(\gamma,p)(I) = \{f \in L^p(I);\;\; ||f||_{\gamma,p} < \infty\}$. The space $(\mathbf{Bes}(\gamma,p)(I), ||.||_{\gamma,p})$ is a non separable Banach space. We also define $\mathbf{bes}(\gamma,p)(I) = \{f \in L^p(I);\;\; \Delta_p(f,I)(t) = o(t^\gamma) \text{ as } t \to 0^+\}$ a separable subspace of $\mathbf{Bes}(\gamma,p)(I)$. For $p=\infty$, the space $\mathbf{Bes}(\gamma,\infty)(I)$ is defined in the same way by using the usual L^∞-norm.

In the case of unit interval $I = [0,1]$ Besov spaces are characterized in terms of sequences of the coefficients of the expansion of continuous functions with respect to the Schauder basis. The following isomorphism theorem has been established by Ciesielski [7] (see also Ciesielski *et al.* [9])

THEOREM 2.1. Let $1 < p < \infty$ and $\frac{1}{p} < \gamma < 1$, we have

1. $\mathbf{Bes}(\gamma,p)([0,1])$ is linearly isomorphic to a sequences space and we have the following equivalence of norms:

$$||f||_{\gamma,p} \sim \max\left\{ |f_0|, |f_1|, \sup_j 2^{-j\left(\frac{1}{2}-\gamma+\frac{1}{p}\right)} \left[\sum_{k=1}^{2^j} |f_{jk}|^p\right]^{\frac{1}{p}} \right\},$$

where the coefficients $\left\{f_0, f_1, f_{jk}\,, j \geqslant 0\,, 1 \leqslant k \leqslant 2^j\right\}$ are given by

$$f_0 = f(0), \qquad f_1 = f(1) - f(0),$$

$$f_{jk} = 2 \cdot 2^{j/2} \left\{ f\left(\frac{2k-1}{2^{j+1}}\right) - \frac{1}{2} f\left(\frac{2k}{2^{j+1}}\right) - \frac{1}{2} f\left(\frac{2k-2}{2^{j+1}}\right) \right\}.$$

2. f is in $\mathbf{bes}(\gamma, p)([0,1])$ if and only if

$$\lim_{j\to\infty} 2^{-j\left(\frac{1}{2}-\gamma+\frac{1}{p}\right)} \left[\sum_{k=1}^{2^j} |f_{jk}|^p\right]^{\frac{1}{p}} = 0.$$

2.2. Besov–Orlicz spaces. Let $I \subset \mathbb{R}$, be a compact interval, $\gamma \in (0,1)$ and M_2 is the Young function defined by $M_2(x) = e^{x^2} - 1$. The Orlicz space $L_{M_2}(I)$ is the space of measurable functions $f : I \mapsto \mathbb{R}$, such that

$$\|f\|^*_{M_2} := \inf_{\lambda>0} \frac{1}{\lambda} \left[1 + \int_I M_2(\lambda f(t))dt\right] < \infty.$$

It is more suitable to use an equivalent norm to $\|\cdot\|^*_{M_2}$ (see Fernique [13] or Ciesielski [8] for a proof):

$$\|f\|_{M_2} = \sup_{p\geqslant 1} \frac{\|f\|_{L^p(I)}}{\sqrt{p}}.$$

Let $\Delta_{M_2}(f, I)(t)$ be the modulus of continuity of f in the Orlicz space $L_{M_2}(I)$ defined as:

$$\Delta_{M_2}(f, I)(t) = \sup_{p\geqslant 1} \frac{\Delta_p(f, I)(t)}{\sqrt{p}}.$$

For $\gamma > 0$, we consider the following norm

$$\|f\|_{\gamma,M_2} = \|f\|_{M_2} + \sup_{0<t\leqslant 1} \frac{\Delta_{M_2}(f, I)(t)}{t^\gamma}.$$

The Besov-Orlicz space is defined by

$$\mathbf{Bes}(\gamma, M_2)(I) := \{f \in L_{M_2}(I);\ \|f\|_{\gamma,M_2} < \infty\}.$$

$\mathbf{Bes}(\gamma, M_2)(I)$ endowed with the norm $\|\cdot\|_{\gamma,M_2}$ is a non separable Banach space. We introduce $\mathbf{bes}(\gamma, M_2)(I) = \{f \in L_{M_2}(I);\ \ \Delta_{M_2}(f, I)(t) = o(t^\gamma) \text{ as } t \to 0^+\}$ a separable subspace of $\mathbf{Bes}(\gamma, M_2)(I)$.

With the same notations as in Theorem 2.1, we have the following characterization of Besov–Orlicz spaces (see Ciesielski [8] or Ciesielski *et al.* [9]):

THEOREM 2.2. We have

1. $\mathbf{Bes}(\gamma, M_2)([0,1])$ is linearly isomorphic to a sequences space and we have the following equivalence of norms:

$$||f||_{\gamma,M_2} \sim \max\left\{ |f_0|, |f_1|, \sup_{p,j} \frac{1}{\sqrt{p}} 2^{-j\left(\frac{1}{2}-\gamma+\frac{1}{p}\right)} \left[\sum_{k=1}^{2^j} |f_{jk}|^p\right]^{\frac{1}{p}} \right\},$$

2. f belongs to $\mathbf{bes}(\gamma, M_2)([0,1])$ if and only if

$$\lim_{j\to\infty} \sup_{p\geqslant 1} \frac{1}{\sqrt{p}} 2^{-j\left(\frac{1}{2}-\gamma+\frac{1}{p}\right)} \left[\sum_{k=1}^{2^j} |f_{jk}|^p\right]^{\frac{1}{p}} = 0.$$

Remark 2.3. 1. Let $1 \leqslant p < \infty$ and $0 < \gamma < \gamma' < 1$, then we have

$$\mathbf{Bes}(\gamma', p)(I) \hookrightarrow \mathbf{bes}(\gamma, p)(I).$$

2. We denote by $\mathcal{C}^\gamma(I)$ the Hölder space define by

$$\mathcal{C}^\gamma(I) := \left\{ f \in C(I), \quad \sup_{\substack{x,y\in I\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\gamma} < \infty \right\}, \tag{2.1}$$

endowed with the norm $||f||_\gamma = \sup_{x\in I}|f(x)| + \sup_{\substack{x,y\in I\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\gamma}$.

- $\mathbf{Bes}(\gamma,\infty)(I) = \mathcal{C}^\gamma(I)$.
- For $1 \leqslant p < \infty$, we have $\mathcal{C}^\gamma(I) \hookrightarrow \mathbf{Bes}(\gamma,p)(I)$.
- For any $1 \leqslant p < \infty$ and $\varepsilon > 0$, we get $\mathcal{C}^{\gamma+\varepsilon}(I) \hookrightarrow \mathbf{bes}(\gamma,p)(I)$.
- For all $1 < p < \infty$ and $\frac{1}{p} < \gamma < 1$, we obtain $\mathbf{Bes}(\gamma,p)(I) \hookrightarrow \mathcal{C}^{\gamma-\frac{1}{p}}(I)$.
- For each $p \in [1,\infty)$ and $0 < \gamma < 1$,

$$\mathbf{Bes}(\gamma, M_2)(I) \hookrightarrow \mathbf{Bes}(\gamma,p)(I) \quad \text{and} \quad \mathbf{bes}(\gamma, M_2)(I) \hookrightarrow \mathbf{bes}(\gamma,p)(I).$$

In the next, we will restrict ourselves to the interval $I = [0,1]$, so we will omit to precise the interval I in our notations, e.g. $\mathbf{Bes}(\gamma,p) := \mathbf{Bes}(\gamma,p)(I)$.

3. BESOV REGULARITY OF THE BIFRACTIONAL BROWNIAN MOTION

Our main result is the following theorem

THEOREM 3.1. For each $\alpha \in (0,1)$, $\beta \in (0,1]$ and $\frac{1}{\alpha\beta} < p < \infty$, we have

$$\mathbb{P}(B^{\alpha,\beta}(.) \in \mathbf{Bes}(\alpha\beta, p)) = 1 \text{ and } \mathbb{P}(B^{\alpha,\beta}(.) \in \mathbf{bes}(\alpha\beta, p)) = 0,$$

where $B^{\alpha,\beta}(.)$ are the sample paths $t \in [0,1] \to B^{\alpha,\beta}(t)$.

To show this theorem, we will adapt the techniques in [9]. Let us first give some preliminary results.

The below lemma is a useful tool to obtain precise estimations in the calculations of this paper. For the proof we refer to [9].

LEMMA 3.2. Let (X,Y) be a mean zero Gaussian vector such that $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = 1$ and $\rho = |\mathbb{E}XY|$. So for any measurable functions f and g such that $\mathbb{E}(f(X))^2 < \infty, \mathbb{E}(f(Y))^2 < \infty$ and $f(X)$, $f(Y)$ are centred, we have

$$|\mathbb{E}f(X)g(Y)| \leqslant \rho \left\{\mathbb{E}(f(X))^2\right\}^{1/2} \left\{\mathbb{E}(g(Y))^2\right\}^{1/2},$$

when f (or g) is even we can replace ρ by ρ^2 in the previous inequality.

We define

(3.1)
$$u_{jk} := 2 \cdot 2^{j/2} \left\{ B^{\alpha,\beta}\left(\frac{2k-1}{2^{j+1}}\right) - \frac{1}{2} B^{\alpha,\beta}\left(\frac{2k}{2^{j+1}}\right) - \frac{1}{2} B^{\alpha,\beta}\left(\frac{2k-2}{2^{j+1}}\right) \right\}.$$

We set

$$v_{jk} = \frac{u_{jk}}{\sigma_{jk}} \qquad \text{with} \qquad \sigma_{jk} = \left\{\mathbb{E}[|u_{jk}|^2]\right\}^{1/2}. \tag{3.2}$$

By using (1.1) and (3.1), we have for all $j \geqslant 1$ and $k, k' \in \{1, ..., 2^j\}$

$$\mathbb{E}[u_{jk} u_{jk'}] = \frac{2^{j(1-2\alpha\beta)}}{2^{\beta+2\alpha\beta}} (\Delta_y^2 \Delta_x^2 \Psi_{k,k'}(0,0) - \Delta^4 \Phi_{k,k'}(0)), \tag{3.3}$$

where Δ_y^2 (resp. Δ_x^2) is the one step progressive difference of order 2 in the y variable (resp. x variable), and Δ^4 is the one step progressive difference of order 4, i.e.

$$\Delta^4 f(0) = f(4) - 4f(3) + 6f(2) - 4f(1) + f(0),$$

and

$$\begin{aligned}\Delta_y^2\Delta_x^2 g(0,0) =& g(2,2)-2g(2,1)+g(2,0)-2g(1,2)+4g(1,1)\\&-2g(1,0)+g(0,2)-2g(0,1)+g(0,0).\end{aligned}$$

The functions in (3.3) are

$$\Psi_{k,k'}(x,y)=\left((2k-2+x)^{2\alpha}+(2k'-2+y)^{2\alpha}\right)^{\beta},$$

and

$$\Phi_{k,k'}(x)=\left|2(k-k')-2+x\right|^{2\alpha\beta}.$$

LEMMA 3.3. For all $\alpha\in(0,1)$, $\beta\in(0,1]$, $j\geqslant 1$ and $k,k'\in\{1,...,2^j\}$ with $k'<k$, there exist $C>0$, $\kappa_{k,k'}\in(0,2)$ and $c_{k,k'}\in(0,4)$ such that

$$\begin{aligned}(3.4)\qquad \left|\mathbb{E}[u_{jk}u_{jk'}]\right|\leqslant C2^{j(1-2\alpha\beta)}\Bigg\{&\frac{1}{(2k'-2+\kappa_{k,k'})^{4-2\alpha\beta}}\\&+\frac{1}{(2(k-k')-2+c_{k,k'})^{4-2\alpha\beta}}\Bigg\}.\end{aligned}$$

And there exist two constants $m_1,m_2>0$ such that, for all $j\geqslant 1$ and $k\in\{1,...,2^j\}$,

$$(3.5)\qquad m_1\,2^{j(1-2\alpha\beta)}\leqslant\mathbb{E}[|u_{jk}|^2]\leqslant m_2\,2^{j(1-2\alpha\beta)}.$$

P r o o f. Denote by $\Phi^{(4)}_{k,k'}$ the derivative of order 4 of $\Phi_{k,k'}$. So by the mean value theorem and (3.3), there exist three constants $c_{1,k,k'},c_{2,k,k'}\in(0,2)$ and $c_{3,k,k'}\in(0,4)$ such that

$$(3.6)\qquad \mathbb{E}[u_{jk}u_{jk'}]=\frac{2^{j(1-2\alpha\beta)}}{2^{\beta+2\alpha\beta}}(\partial_y^2\partial_x^2\Psi_{k,k'}(c_{1,k,k'},c_{2,k,k'})-\Phi^{(4)}_{k,k'}(c_{3,k,k'})),$$

where $\partial_y=\frac{\partial}{\partial y}$ and $\partial_x=\frac{\partial}{\partial x}$. On the other hand, we get for all $j\geqslant 1$ and $k,k'\in$

$\{1,...,2^j\}$,

(3.7)
$$
\begin{aligned}
&\partial_y^2\partial_x^2\Psi_{k,k'}(c_{1,k,k'},c_{2,k,k'})\\
&= 4\alpha^2(2\alpha-1)^2\beta(\beta-1)(2k-2+c_{1,k,k'})^{2\alpha-2}(2k'-2+c_{2,k,k'})^{2\alpha-2}\\
&\qquad\times\big((2k-2+c_{1,k,k'})^{2\alpha}+(2k'-2+c_{2,k,k'})^{2\alpha}\big)^{\beta-2}\\
&+8\alpha^3(2\alpha-1)\beta(\beta-1)(\beta-2)(2k-2+c_{1,k,k'})^{2\alpha-2}(2k'-2+c_{2,k,k'})^{4\alpha-2}\\
&\qquad\times\big((2k-2+c_{1,k,k'})^{2\alpha}+(2k'-2+c_{2,k,k'})^{2\alpha}\big)^{\beta-3}\\
&+8\alpha^3(2\alpha-1)\beta(\beta-1)(\beta-2)(2k-2+c_{1,k,k'})^{4\alpha-2}(2k'-2+c_{2,k,k'})^{2\alpha-2}\\
&\qquad\times\big((2k-2+c_{1,k,k'})^{2\alpha}+(2k'-2+c_{2,k,k'})^{2\alpha}\big)^{\beta-3}\\
&+16\alpha^4\beta(\beta-1)(\beta-2)(\beta-3)(2k-2+c_{1,k,k'})^{4\alpha-2}(2k'-2+c_{2,k,k'})^{4\alpha-2}\\
&\qquad\times\big((2k-2+c_{1,k,k'})^{2\alpha}+(2k'-2+c_{2,k,k'})^{2\alpha}\big)^{\beta-4}\\
&= I_1+I_2+I_3+I_4.
\end{aligned}
$$

And for all $j \geqslant 1$ and $k,k' \in \{1,...,2^j\}$ such that $k'<k$, we have

$$\Phi^{(4)}_{k,k'}(c_{3,k,k'}) = \prod_{l=0}^{3}(2\alpha\beta-l)(2(k-k')-2+c_{3,k,k'})^{2\alpha\beta-4}. \tag{3.8}$$

Let us first investigate the inequality (3.4). Combining (3.6), (3.7) and (3.8), we get for all $j \geqslant 1$ and $k,k' \in \{1,...,2^j\}$ such that $k'<k$

$$\big|\mathbb{E}[u_{jk}u_{jk'}]\big| \leqslant \frac{2^{j(1-2\alpha\beta)}}{2^{\beta+2\alpha\beta}}\left\{\sum_{l=1}^{4}|I_l| + \frac{\prod_{l=0}^{3}|2\alpha\beta-l|}{(2(k-k')-2+c_{3,k,k'})^{4-2\alpha\beta}}\right\}. \tag{3.9}$$

· For $\alpha \leqslant \frac{1}{2}$. Let $j \geqslant 1$ and $k,k' \in \{1,...,2^j\}$ such that $k'<k$, we have by (3.7)

$$\sum_{l=1}^{4}|I_l| \leqslant \frac{\tilde{C}(\alpha,\beta)}{(2k'-2+c_{1,k,k'}\wedge c_{2,k,k'})^{4-2\alpha\beta}}, \tag{3.10}$$

where

$$
\begin{aligned}
\tilde{C}(\alpha,\beta) = 2^\beta\alpha^2|\beta-1|\{&\beta(2\alpha-1)^2+2\alpha|2\alpha-1|\beta|\beta-2|\\
&+\alpha^2\beta|\beta-2||\beta-3|\}.
\end{aligned} \tag{3.11}
$$

Combining (3.9) and (3.10), we get

$$
\begin{aligned}
\big|\mathbb{E}[u_{jk}u_{jk'}]\big| \leqslant \frac{2^{j(1-2\alpha\beta)}}{2^{\beta+2\alpha\beta}}\Bigg\{&\frac{\tilde{C}(\alpha,\beta)}{(2k'-2+c_{1,k,k'}\wedge c_{2,k,k'})^{4-2\alpha\beta}}\\
&+\frac{\prod_{l=0}^{3}|2\alpha\beta-l|}{(2(k-k')-2+c_{3,k,k'})^{4-2\alpha\beta}}\Bigg\}.
\end{aligned} \tag{3.12}
$$

· For $\alpha > \frac{1}{2}$. Let $j \geqslant 1$ and $k, k' \in \{1, ..., 2^j\}$ such that $k' < k$, we remark that

$$I_2 + I_3 = \frac{8\alpha^3(2\alpha-1)\beta(\beta-1)(\beta-2)}{(2k-2+c_{1,k,k'})^{2-2\alpha}(2k'-2+c_{2,k,k'})^{2-2\alpha}} \times \frac{1}{\left((2k-2+c_{1,k,k'})^{2\alpha}+(2k'-2+c_{2,k,k'})^{2\alpha}\right)^{2-\beta}}. \tag{3.13}$$

And by the inequality $\frac{ab}{a^2+b^2} \leqslant \frac{1}{2}$, we have

$$|I_4| \leqslant \frac{4\alpha^4\beta|\beta-1||\beta-2||\beta-3|}{(2k-2+c_{1,k,k'})^{2-2\alpha}(2k'-2+c_{2,k,k'})^{2-2\alpha}} \times \frac{1}{\left((2k-2+c_{1,k,k'})^{2\alpha}+(2k'-2+c_{2,k,k'})^{2\alpha}\right)^{2-\beta}}. \tag{3.14}$$

Combining (3.7), (3.13) and (3.14), we get

$$\sum_{l=1}^{4}|I_l| \leqslant \frac{\tilde{C}(\alpha,\beta)}{(2k'-2+c_{1,k,k'}\wedge c_{2,k,k'})^{4-2\alpha\beta}}, \tag{3.15}$$

where $\tilde{C}(\alpha,\beta)$ is given by (3.11). According to (3.9) and (3.15), we obtain

$$\left|\mathbb{E}[u_{jk}u_{jk'}]\right| \leqslant \frac{2^{j(1-2\alpha\beta)}}{2^{\beta+2\alpha\beta}}\left\{\frac{\tilde{C}(\alpha,\beta)}{(2k'-2+c_{1,k,k'}\wedge c_{2,k,k'})^{4-2\alpha\beta}} + \frac{\prod_{l=0}^{3}|2\alpha\beta-l|}{(2(k-k')-2+c_{3,k,k'})^{4-2\alpha\beta}}\right\}. \tag{3.16}$$

We put

$$C = \frac{\tilde{C}(\alpha,\beta)\vee\prod_{l=0}^{3}|2\alpha\beta-l|}{2^{\beta+2\alpha\beta}}.$$

So for all $\alpha \in (0,1)$, $\beta \in (0,1]$, $j \geqslant 1$ and $k, k' \in \{1, ..., 2^j\}$ such that $k' < k$, we have

$$\left|\mathbb{E}[u_{jk}u_{jk'}]\right| \leqslant C2^{j(1-2\alpha\beta)}\left\{\frac{1}{(2k'-2+\kappa_{k,k'})^{4-2\alpha\beta}} + \frac{1}{(2(k-k')-2+c_{k,k'})^{4-2\alpha\beta}}\right\}, \tag{3.17}$$

where $\kappa_{k,k'} = c_{1,k,k'} \wedge c_{2,k,k'}$ and $c_{k,k'} = c_{3,k,k'}$. This finishes the proof of (3.4).

Now we will prove (3.5). For this end, let us start with proving the upper bound, for all $j \geqslant 1$ and $k \in \{1, ..., 2^j\}$, we have by (3.3) and the mean value theorem, there exist $c_{1,k}, c_{2,k} \in (0,2)$ such that

$$\begin{aligned}\mathbb{E}[|u_{jk}|^2] &= \frac{2^{j(1-2\alpha\beta)}}{2^{\beta+2\alpha\beta}}(\partial_y^2\partial_x^2\Psi_{k,k}(c_{1,k}, c_{2,k}) - \Delta^4\Phi_{k,k}(0)) \\ &= \frac{2^{j(1-2\alpha\beta)}}{2^{\beta+2\alpha\beta}}(\partial_y^2\partial_x^2\Psi_{k,k}(c_{1,k}, c_{2,k}) + 8 - 2^{2\alpha\beta+1}).\end{aligned} \tag{3.18}$$

· For $\alpha \leqslant \frac{1}{2}$. By (3.7) we remark that $\partial_y^2\partial_x^2\Psi_{k,k}(c_{1,k}, c_{2,k}) \leqslant 0$, so by using (3.18) we have

$$\mathbb{E}[|u_{jk}|^2] \leqslant \frac{8 - 2^{2\alpha\beta+1}}{2^{\beta+2\alpha\beta}}2^{j(1-2\alpha\beta)}. \tag{3.19}$$

· For $\alpha > \frac{1}{2}$. By (3.7) we remark that $I_1, I_4 \leqslant 0$, and by (3.13) we get, for all $j \geqslant 1$ and $k \in \{2, ..., 2^j\}$,

$$I_2 + I_3 \leqslant \frac{\alpha^3(2\alpha-1)\beta(\beta-1)(\beta-2)}{2^{3-2\alpha\beta-\beta}}. \tag{3.20}$$

So (3.18) and (3.20) entail

$$\begin{aligned}&\mathbb{E}[|u_{jk}|^2] \\ &\leqslant 2^{j(1-2\alpha\beta)}\left(\frac{\alpha^3(2\alpha-1)\beta(\beta-1)(\beta-2)}{8} + \frac{8 - 2^{2\alpha\beta+1}}{2^{\beta+2\alpha\beta}}\right).\end{aligned} \tag{3.21}$$

Now for $j \geqslant 1$ and $k = 1$, we have by (3.1) and (1.2),

$$\mathbb{E}[|u_{j1}|^2] = 2^{j(1-2\alpha\beta)}\frac{4\mathbb{E}\left[B^{\alpha,\beta}(1) - \frac{1}{2}B^{\alpha,\beta}(2)\right]^2}{2^{2\alpha\beta}}. \tag{3.22}$$

Put

$$\begin{aligned}m_2 = \max\Bigg\{&\left(\frac{\alpha^3(2\alpha-1)\beta(\beta-1)(\beta-2)}{8} + \frac{8 - 2^{2\alpha\beta+1}}{2^{\beta+2\alpha\beta}}\right); \\ &\frac{4\mathbb{E}\left[B^{\alpha,\beta}(1) - \frac{1}{2}B^{\alpha,\beta}(2)\right]^2}{2^{2\alpha\beta}}\Bigg\}.\end{aligned} \tag{3.23}$$

Combining (3.21), (3.22), (3.23) and (3.19), we get for all $\alpha \in (0,1)$, $\beta \in (0,1]$, $j \geqslant 1$ and $k \in \{1, ..., 2^j\}$,

$$\mathbb{E}[|u_{jk}|^2] \leqslant m_2 2^{j(1-2\alpha\beta)}. \tag{3.24}$$

This finishes the proof of the upper bound in (3.5). Let us now investigate the lower bound of (3.5). According to (3.1) and (1.2), it follows that for all $j \geqslant 1$ and $k \in \{1, ..., 2^j\}$,

$$\mathbb{E}[|u_{jk}|^2] = 2^{j(1-2\alpha\beta)} \frac{4C(k)}{2^{2\alpha\beta}}, \tag{3.25}$$

where

$$C(k) = \mathbb{E}\left[B^{\alpha,\beta}(2k-1) - \frac{1}{2}B^{\alpha,\beta}(2k) - \frac{1}{2}B^{\alpha,\beta}(2k-2)\right]^2.$$

We know by [1, Lemma 3.3] (see also [24]) that the process $(B^{\alpha,\beta}(t))_{t\geqslant 0}$ is locally non-deterministic i.e. for all $0 = t_0 < t_1 < ... < t_m < 1$ with $t_m - t_1 < \delta$ and $(u_1, ..., u_m) \in \mathbb{R}^m$,

(3.26)

$$\operatorname{Var}\left(\sum_{j=1}^{m} u_j[B^{\alpha,\beta}(t_j) - B^{\alpha,\beta}(t_{j-1})]\right) \geqslant C_m \sum_{j=1}^{m} u_j^2 \operatorname{Var}\left(B^{\alpha,\beta}(t_j) - B^{\alpha,\beta}(t_{j-1})\right).$$

On the other hand by (1.2), we have for some $\varepsilon < \frac{\delta}{2} \wedge \frac{1}{2k}$,

$$\begin{aligned} C(k) = \varepsilon^{-2\alpha\beta}\mathbb{E}\Bigg[\frac{1}{2}\left[B^{\alpha,\beta}(\varepsilon(2k-1)) - B^{\alpha,\beta}(\varepsilon(2k-2))\right] \\ -\frac{1}{2}\left[B^{\alpha,\beta}(\varepsilon(2k)) - B^{\alpha,\beta}(\varepsilon(2k-1))\right]\Bigg]^2. \end{aligned} \tag{3.27}$$

Combining (3.26), (3.27) and (1.3), we get $C(k) \geqslant \frac{C_3}{2^{1+\beta}}$. Hence

$$\mathbb{E}[|u_{jk}|^2] \geqslant m_1 2^{j(1-2\alpha\beta)} \qquad \text{with} \qquad m_1 = \frac{4C_3}{2^{2\alpha\beta+\beta+1}}.$$

This finishes the proof of Lemma 3.3. ∎

Remark 3.4. We remark that when $\beta = 1$, $\partial_y^2\partial_x^2\Psi_{k,k'}(c_{1,k,k'}, c_{2,k,k'}) = 0$ and hence $\Delta_y^2\Delta_x^2\Psi_{k,k'}(0,0) = 0$, so equation (3.3) becomes

$$\mathbb{E}[u_{jk}u_{jk'}] = -\frac{2^{j(1-2\alpha)}}{2^{1+2\alpha}}\Delta^4\Phi_{k,k'}(0), \tag{3.28}$$

and this is the same equation as the (IV.9) in [9], for the fractional Brownian motion.

LEMMA 3.5. There exists a constant $M > 0$ such that, for all $j \geqslant 1$ and $k, k' \in \{1, ..., 2^j\}$, we have

$$\sum_{k,k'=1}^{2^j} |\mathbb{E}v_{jk}v_{jk'}|^2 \leqslant M2^j, \tag{3.29}$$

where the v_{jk} are given by (3.2).

P r o o f. Equality (3.2) and Hölder's inequality give

$$\begin{aligned}
\sum_{k,k'=1}^{2^j} |\mathbb{E}v_{jk}v_{jk'}|^2 &= 2\sum_{\substack{k'<k,\,k'\geqslant 2\\ k-k'\geqslant 2}}^{2^j} |\mathbb{E}v_{jk}v_{jk'}|^2 + 2\sum_{\substack{k'<k,\,k'\geqslant 2\\ k-k'=1}}^{2^j} |\mathbb{E}v_{jk}v_{jk'}|^2 + 2\sum_{k=2}^{2^j} |\mathbb{E}v_{jk}v_{j1}|^2 \\
&\quad + \sum_{k=1}^{2^j} \{\mathbb{E}[|v_{jk}|^2]\}^2 \\
&\leqslant 2\sum_{\substack{k'<k,\,k'\geqslant 2\\ k-k'\geqslant 2}}^{2^j} \left|\frac{\mathbb{E}u_{jk}u_{jk'}}{\sigma_{jk}\sigma_{jk'}}\right|^2 + 2(2^j-2) + 2(2^j-1) + 2^j \\
&= 2J + 5\,2^j - 6.
\end{aligned}$$

We will estimate J. For this end, let us note $A = \frac{2C^2}{m_2^2}$, so we obtain by (3.4) and (3.5),

$$\begin{aligned}
J &\leqslant A \sum_{\substack{k'<k,\,k'\geqslant 2\\ k-k'\geqslant 2}}^{2^j} \left\{\frac{1}{(2k'-2+\kappa_{k,k'})^{8-4\alpha\beta}} + \frac{1}{(2(k-k')-2+c_{k,k'})^{8-4\alpha\beta}}\right\} \\
&= A\sum_{k=4}^{2^j}\sum_{k'=2}^{k-2} \left\{\frac{1}{(2k'-2+\kappa_{k,k'})^{8-4\alpha\beta}} + \frac{1}{(2(k-k')-2+c_{k,k'})^{8-4\alpha\beta}}\right\} \\
&\leqslant A\sum_{k=4}^{2^j}\sum_{k'=2}^{k-2} \left\{\frac{1}{(2k'-2)^{8-4\alpha\beta}} + \frac{1}{(2(k-k')-2)^{8-4\alpha\beta}}\right\} \\
&\leqslant A\sum_{k=4}^{2^j}\sum_{k'=2}^{k-2} \left\{\int_{2k'-3}^{2k'-2} \frac{1}{x^{8-4\alpha\beta}}dx + \int_{2(k-k')-3}^{2(k-k')-2} \frac{1}{x^{8-4\alpha\beta}}dx\right\} \\
&= 2A\sum_{k=4}^{2^j}\sum_{k'=2}^{k-2}\int_{2k'-3}^{2k'-2} \frac{1}{x^{8-4\alpha\beta}}dx \\
&\leqslant 2A\sum_{k=4}^{2^j}\int_{1}^{2k-6} \frac{1}{x^{8-4\alpha\beta}}dx \\
&\leqslant \frac{2A}{7-4\alpha\beta}(2^j-3).
\end{aligned}$$

And this proves Lemma 3.5. ∎

LEMMA 3.6. For all $j \geqslant 1$ and $k \in \{1, ..., 2^j\}$, we have

$$\mathbb{E}\left[\sum_{k=1}^{2^j}(|v_{jk}|^p - c_p)\right]^2 \leqslant (c_{2p} - c_p^2)M2^j, \tag{3.30}$$

where $c_p = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |x|^p e^{-\frac{x^2}{2}}\,dx$.

P r o o f. First we get

$$\mathbb{E}\left[\sum_{k=1}^{2^j}(|v_{jk}|^p - c_p)\right]^2 = \sum_{k,k'=1}^{2^j}\mathbb{E}\left[(|v_{jk}|^p - c_p)(|v_{jk'}|^p - c_p)\right].$$

And by applying Lemma 3.2, with $f(x) = g(x) = |x|^p - c_p$, we obtain

$$\mathbb{E}\left[\sum_{k=1}^{2^j}(|v_{jk}|^p - c_p)\right]^2 \leqslant (c_{2p} - c_p^2)\sum_{k,k'=1}^{2^j}|\mathbb{E}[v_{jk}v_{jk'}]|^2.$$

Inequality (3.29) of Lemma 3.5 ends the proof of Lemma 3.6. ∎

Now, we are ready to show Theorem 3.1

P r o o f. [of Theorem 3.1] We are going to prove that, almost surely

$$2^{-j}\sum_{k=1}^{2^j}|v_{jk}|^p \underset{j\to\infty}{\longrightarrow} c_p. \tag{3.31}$$

For this end we will show that for all $\varepsilon > 0$ we have

$$\sum_{j\geqslant 1}\mathbb{P}\left\{2^{-j}\sum_{k=1}^{2^j}|v_{jk}|^p \notin [c_p - \varepsilon, c_p + \varepsilon]\right\} < \infty. \tag{3.32}$$

Markov's inequality gives

$$\mathbb{P}\left\{2^{-j}\sum_{k=1}^{2^j}|v_{jk}|^p \notin [c_p - \varepsilon, c_p + \varepsilon]\right\} \leqslant \frac{1}{\varepsilon^2 2^{2j}}\mathbb{E}\left[\sum_{k=1}^{2^j}(|v_{jk}|^p - c_p)\right]^2. \tag{3.33}$$

Combining the inequality (3.33) and Lemma 3.6, we get that (3.32) holds and (3.31) is then a consequence of Borel-Cantelli Lemma. Finally our main result, Theorem 3.1, is a simple consequence of Theorem 2.1. ∎

Below is a stronger regularity result than Theorem 3.1

THEOREM 3.7. For each $\alpha \in (0,1)$ and $\beta \in (0,1]$, we have

$$\mathbb{P}(B^{\alpha,\beta}(.) \in \mathbf{Bes}(\alpha\beta, M_2)) = 1 \text{ and } \mathbb{P}(B^{\alpha,\beta}(.) \in \mathbf{bes}(\alpha\beta, M_2)) = 0,$$

where $B^{\alpha,\beta}(.)$ are the sample paths $t \in [0,1] \to B^{\alpha,\beta}(t)$.

Proof. The proof is similar to that one of [9, Theorem II.5]. Indeed, taking into account Theorem 2.2, Lemma 3.6, and the fact that for positive integer p, we have

$$c_{2p} = (2p)!/(p!2^p) \underset{p\to\infty}{\sim} e^{-p}2^{p+1/2}p^p.$$

Therefore, there exists a constant c with $c > 1$, such that $c_{2p} \leqslant ce^{-p}(2p)^p$. ∎

Remark 3.8. 1. Let $(Y_t^\alpha)_{t\geqslant 0}$ be a sub-fractional Brownian motion i.e. a mean zero Gaussian process with covariance function

$$\mathbb{E}[Y_t^\alpha Y_s^\alpha] = s^{2\alpha} + t^{2\alpha} - \frac{1}{2}\left[(s+t)^{2\alpha} + |t-s|^{2\alpha}\right],$$

where $\alpha \in (0,1)$. We believe that by the same calculations as in the above one can get that almost all paths of the sub-fractional Brownian motion belong to (resp. do not belong to) the Besov spaces $\mathbf{Bes}(\alpha, p)$ and Besov–Orlicz space $\mathbf{Bes}(\alpha, M_2)$ (resp. $\mathbf{bes}(\alpha, p)$ and $\mathbf{bes}(\alpha, M_2)$).

2. Let $r(t)$ be a real valued function, such that the kernel $K_r(t,s)$ defined by

$$K_r(t,s) := r(t) - r(s) - r(t-s),$$

is positive on the real line, and let φ be as follows

$$\varphi(t) = \int_0^\infty (1 - e^{-u|t|})dm(u),$$

where m is a positive measure on $[0,\infty)$ such that $\int_1^\infty dm(u) < \infty$. Therefore by [2, Theorem 5.1.], we get that

$$K(s,t) := \varphi(r(t) + r(s)) - \varphi(r(t-s)),$$

is a positive Kernel on the real line. Let $(X_t)_{t\geqslant 0}$ be a centred Gaussian process with covariance function

$$\mathbb{E}[X_tX_s] = K(s,t).$$

If in addition we assume that φ and r are in $\mathcal{C}^4((0,\infty))$, and that for all $a > 0$, we have $r(at) = a^\alpha r(t)$ and $\varphi(ax) = a^\beta\varphi(x)$ (i.e. X is a self-similar Gaussian process with index $\frac{\alpha\beta}{2}$). We expect that almost all paths of the process X belong to (resp. do not belong to) the Besov spaces $\mathbf{Bes}(\alpha\beta/2, p)$ (resp. $\mathbf{bes}(\alpha\beta/2, p)$). We intend to provide, in a future paper, a proof of this result using a new method dealing directly with Besov norms without using Ciesielski's isomorphism theorem. For this new approach, we refer to [15], [31], [23].

3. The bifractional Brownian motion $B^{\alpha,\beta}$, with $0 < \alpha < 1$, $1 < \beta < 2$ such that $\alpha\beta < 1$ has been introduced and studied in [3]. The case $\alpha > 1$ such that $2\alpha\beta \leqslant 1$ has been considered in [28]. One can show that the regularity results in Theorem 3.1 and 3.7 remain valid, since the derivatives $\partial_y^2\partial_x^2\Psi_{k,k'}(c_{1,k,k'}, c_{2,k,k'})$ and $\Phi^{(4)}_{k,k'}(c_{3,k,k'})$ do not change. So in the proof of Lemma 3.3, we need just to take into account the other cases (i.e. $0 < \alpha < 1$, $1 < \beta < 2$ s.t. $\alpha\beta < 1$ and $\alpha > 1$ with $2\alpha\beta \leqslant 1$).

4. AN ITÔ-NISIO THEOREM FOR THE BIFRACTIONAL BROWNIAN MOTION.

Let $\mathcal{E}$ be the linear space generated by the indicator functions $\mathbb{1}_{[0,t]}$ endowed with the inner product

$$< \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} >_{\mathcal{H}} = R(s,t)\,, \quad s,t \in [0,1]. \tag{4.1}$$

Define $\mathcal{H}$ as the completion of $\mathcal{E}$ w.r.t. the inner product $< .,. >_{\mathcal{H}}$. The application $\varphi \in \mathcal{H} \to B^{\alpha,\beta}(\varphi)$ is an isometry from $\mathcal{H}$ to the Gaussian space generated by $B^{\alpha,\beta}$. $B^{\alpha,\beta}(\varphi)$ is the Wiener integral of φ w.r.t. $B^{\alpha,\beta}$.

Our main result in this paragraph is the following Itô-Nisio theorem for the bifractional Brownian motion

THEOREM 4.1. Let $\alpha\beta > \frac{1}{2}$ and $(\varphi_n)_{n\geqslant 1}$ be an orthonormal basis of $\mathcal{H}$. Then we have almost surely

$$\sum_{n=1}^{N} < \varphi_n, \mathbb{1}_{[0,t]} >_{\mathcal{H}} B^{\alpha,\beta}(\varphi_n) \underset{N\to+\infty}{\longrightarrow} B^{\alpha,\beta}(t) \quad \text{in the Besov space } \mathbf{Bes}(\alpha\beta-\varepsilon, p),$$

where $\varepsilon > 0$ and $p \geqslant 1$ are such that $\frac{1}{2} < \alpha\beta - \varepsilon - \frac{1}{p}$.

By classical continuous injections we can deduce

COROLLARY 4.2. Suppose that $\alpha\beta > \frac{1}{2}$ and $(\varphi_n)_{n\geqslant 1}$ be an orthonormal basis of $\mathcal{H}$. Then we have almost surely

$$\sum_{n=1}^{N} < \varphi_n, \mathbb{1}_{[0,t]} >_{\mathcal{H}} B^{\alpha,\beta}(\varphi_n) \underset{N\to+\infty}{\longrightarrow} B^{\alpha,\beta}(t) \quad \text{in the Hölder space } \mathcal{C}^{\gamma},$$

for any $\gamma < \alpha\beta$.

Proof. [of Theorem 4.1] Put $\quad X_N(t) = \displaystyle\sum_{n=1}^{N} < \varphi_n, \mathbb{1}_{[0,t]} >_{\mathcal{H}} B^{\alpha,\beta}(\varphi_n).$
And define

$$z_{jk} := 2 \cdot 2^{j/2} \left\{ X_N\left(\frac{2k-1}{2^{j+1}}\right) - \frac{1}{2} X_N\left(\frac{2k}{2^{j+1}}\right) - \frac{1}{2} X_N\left(\frac{2k-2}{2^{j+1}}\right) \right\}.$$

Let $\{h_{jk},\ j \geqslant 0,\ k = 1, ..., 2^j\}$ be the Haar functions defined as follows

$$h_{jk} = \sqrt{2^j}\, \mathbb{1}_{[\frac{2k-2}{2^{j+1}}, \frac{2k-1}{2^{j+1}}[} - \sqrt{2^j}\, \mathbb{1}_{[\frac{2k-1}{2^{j+1}}, \frac{2k}{2^{j+1}}[} \qquad \text{and} \qquad h_1 = \mathbb{1}_{[0,1]}.$$

Remark that

$$\begin{aligned} u_{jk} - z_{jk} &= B^{\alpha,\beta}(h_{jk}) - X_N(h_{jk}) \\ &= B^{\alpha,\beta}(h_{jk}) - \sum_{n=1}^{N} < \varphi_n, h_{jk} >_{\mathcal{H}} B^{\alpha,\beta}(\varphi_n). \end{aligned}$$

Set

$$\omega_{jk}^N = \frac{u_{jk} - z_{jk}}{\varrho_{jk}^N} \qquad \text{with} \qquad \varrho_{jk}^N = \{\mathbb{E}[|u_{jk} - z_{jk}|^2]\}^{1/2}. \tag{4.2}$$

First, by Borel-Cantelli lemma, we can easily show that almost surely

$$2^{-j(1+\varepsilon p)} \sum_{k=1}^{2^j} |\omega_{jk}^N|^p \underset{j\to\infty}{\longrightarrow} 0. \tag{4.3}$$

Let $\phi, \psi \in \mathcal{H}$ and $\sum_{i=1}^{L_n} \lambda_i^n \mathbb{1}_{[0,t_i^n]}$, $\sum_{j=1}^{M_m} \mu_j^m \mathbb{1}_{[0,s_j^m]}$ be two sequences in $\mathcal{E}$ such that

$$\phi = \lim_{n\to\infty} \sum_{i=1}^{L_n} \lambda_i^n \mathbb{1}_{[0,t_i^n]} \qquad \text{and} \qquad \psi = \lim_{m\to\infty} \sum_{j=1}^{M_m} \mu_j^m \mathbb{1}_{[0,s_j^m]} \qquad \text{in } \mathcal{H}.$$

Define for all $a > 0$,

$$\phi^a = \lim_{n\to\infty} \sum_{i=1}^{L_n} \lambda_i^n \mathbb{1}_{[0,at_i^n]} \qquad \text{and} \qquad \psi^a = \lim_{m\to\infty} \sum_{j=1}^{M_m} \mu_j^m \mathbb{1}_{[0,as_j^m]} \qquad \text{in } \mathcal{H}.$$

One can see easily by (1.2) and a density argument that

$$\mathbb{E}[B^{\alpha,\beta}(\phi^a)B^{\alpha,\beta}(\psi^a)] = a^{2\alpha\beta}\mathbb{E}[B^{\alpha,\beta}(\phi)B^{\alpha,\beta}(\psi)]. \tag{4.4}$$

Set $\theta^n_{j,k} =< \varphi_n, h_{jk} >_{\mathcal{H}}$, we get

$$\begin{aligned}|\varrho^N_{jk}|^2 &= \mathbb{E}\left[B^{\alpha,\beta}(h_{jk}) - \sum_{n=1}^{N} < \varphi_n, h_{jk} >_{\mathcal{H}} B^{\alpha,\beta}(\varphi_n)\right]^2 \\ &= ||h_{jk}||^2_{\mathcal{H}} + \sum_{n=1}^{N} |\theta^n_{j,k}|^2 - 2\sum_{n=1}^{N} |\theta^n_{j,k}|^2 = \sum_{n=N+1}^{\infty} |\theta^n_{j,k}|^2. \end{aligned}\tag{4.5}$$

Put

$$\tilde{h}_{jk} = \mathbb{1}_{[\frac{2k-2}{2^{j+1}},\frac{2k-1}{2^{j+1}}[} - \mathbb{1}_{[\frac{2k-1}{2^{j+1}},\frac{2k}{2^{j+1}}[}, \qquad \tilde{g}_{jk} = \mathbb{1}_{[\frac{2k-2}{2^{(j+1)/2}},\frac{2k-1}{2^{(j+1)/2}}[} - \mathbb{1}_{[\frac{2k-1}{2^{(j+1)/2}},\frac{2k}{2^{(j+1)/2}}[},$$

$$g_k = \mathbb{1}_{[2k-2,2k-1[} - \mathbb{1}_{[2k-1,2k[}. \tag{4.6}$$

Let $a_j = 2^{-(j+1)/2}$, we have by (4.4)

$$\begin{aligned}\theta^n_{j,k} &= \mathbb{E}[B^{\alpha,\beta}(h_{jk})B^{\alpha,\beta}(\varphi_n)] \\ &= 2^{j/2}\mathbb{E}[B^{\alpha,\beta}(\tilde{h}_{jk})B^{\alpha,\beta}(\varphi_n)] \\ &= 2^{j/2}\mathbb{E}[B^{\alpha,\beta}(\tilde{g}^{a_j}_{jk})B^{\alpha,\beta}((\varphi_n^{a_j^{-1}})^{a_j})] \\ &= 2^{-\alpha\beta}2^{j(1/2-\alpha\beta)}\mathbb{E}[B^{\alpha,\beta}(\tilde{g}_{jk})B^{\alpha,\beta}(\varphi_n^{a_j^{-1}})]. \end{aligned}\tag{4.7}$$

Hence combining (4.5) and (4.7), we get

$$\begin{aligned}|\varrho^N_{jk}|^2 &= 2^{-2\alpha\beta}2^{j(1-2\alpha\beta)}\sum_{n=N+1}^{\infty}\left\{\mathbb{E}[B^{\alpha,\beta}(\tilde{g}_{jk})B^{\alpha,\beta}(\varphi_n^{a_j^{-1}})]\right\}^2 \\ &\leqslant 2^{-2\alpha\beta}2^{j(1-2\alpha\beta)}\sup_{j,k}\sum_{n=N+1}^{\infty}\left\{\mathbb{E}[B^{\alpha,\beta}(\tilde{g}_{jk})B^{\alpha,\beta}(\varphi_n^{a_j^{-1}})]\right\}^2. \end{aligned}\tag{4.8}$$

The supremum in the last term of the inequality is finite. In fact we remark that $(a_j^{\alpha\beta}\varphi_n^{a_j^{-1}})_{n\geqslant 1}$ is an orthonormal basis of $\mathcal{H}$, therefore

$$\begin{aligned}\sum_{n=N+1}^{\infty}\left\{\mathbb{E}[B^{\alpha,\beta}(\tilde{g}_{jk})B^{\alpha,\beta}(\varphi_n^{a_j^{-1}})]\right\}^2 &= a_j^{-2\alpha\beta}\sum_{n=N+1}^{\infty} < \tilde{g}_{jk}, a_j^{\alpha\beta}\varphi_n^{a_j^{-1}} >^2_{\mathcal{H}} \\ &\leqslant a_j^{-2\alpha\beta}\sum_{n=1}^{\infty} < \tilde{g}_{jk}, a_j^{\alpha\beta}\varphi_n^{a_j^{-1}} >^2_{\mathcal{H}} \\ &= a_j^{-2\alpha\beta}||\tilde{g}_{jk}||^2_{\mathcal{H}} = a_j^{-2\alpha\beta}||g_k^{a_j}||^2_{\mathcal{H}} = ||g_k||^2_{\mathcal{H}}. \end{aligned}\tag{4.9}$$

On the other hand, we have by (3.24) that $\mathbb{E}[|u_{jk}|^2] \leqslant m_2 2^{j(1-2\alpha\beta)}$. Therefore by (3.1), (1.2), (4.6) and (4.1) we get for all $k \geqslant 1$,

$$||g_k||^2_{\mathcal{H}} = \mathbb{E}[2B^{\alpha,\beta}(2k-1) - B^{\alpha,\beta}(2k) - B^{\alpha,\beta}(2k-2)]^2 \leqslant m_2. \tag{4.10}$$

According to (4.9) we obtain for all $N \geqslant 1$,

$$A_N := \sup_{j,k} \sum_{n=N+1}^{\infty} \left\{ \mathbb{E}[B^{\alpha,\beta}(\tilde{g}_{jk}) B^{\alpha,\beta}(\varphi_n^{a_j^{-1}})] \right\}^2 < \infty. \tag{4.11}$$

$(A_N)_{N\geqslant 1}$ is a non-increasing real valued sequence such that

$$\lim_N A_N = 0 \qquad \text{a.s..} \tag{4.12}$$

We derive from (4.2), (4.8) and (4.11) the following inequality

$$2^{-j(1+\varepsilon p)} \sum_{k=1}^{2^j} |\omega^N_{jk}|^p \geqslant 2^{p\alpha\beta} 2^{-jp(\frac{1}{2}-(\alpha\beta-\varepsilon)+\frac{1}{p})} \sum_{k=1}^{2^j} \frac{|u_{jk} - z_{jk}|^p}{A_N^{p/2}}. \tag{4.13}$$

We remark that the sequence $2^{-jp(\frac{1}{2}-(\alpha\beta-\varepsilon)+\frac{1}{p})} \sum_{k=1}^{2^j} |u_{jk} - z_{jk}|^p$ is increasing in j for ε small enough, p large enough and $\alpha\beta > \frac{1}{2}$. Therefore by (4.3) and (4.13), we get that almost surely,

$$\sup_{j\geqslant 0} 2^{-jp(\frac{1}{2}-(\alpha\beta-\varepsilon)+\frac{1}{p})} \sum_{k=1}^{2^j} |u_{jk} - z_{jk}|^p \leqslant 2^{-p\alpha\beta} A_N^{p/2}.$$

Which finishes the proof of Theorem 4.1 by applying (4.12) and Theorem 2.1 . ■

Remark 4.3. When $\beta = 1$ we get the Itô-Nisio theorem for the fractional Brownian motion with $\alpha > \frac{1}{2}$.

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