RESEARCH ARTICLE

On Besov regularity and local time of the solution to the stochastic heat equation

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Abstract
Sharp Besov regularities in time and space variables are investigated for $(u(t,x),\ t\in[0,T],\ x\in\mathbb{R})$, the mild solution to the stochastic heat equation driven by space-time white noise. Existence, Hölder continuity, and Besov regularity of local times are established for $u(t,x)$ viewed either as a process in the space variable or time variable. Hausdorff dimensions of their corresponding level sets are also obtained.

KEYWORDS
Stochastic heat equation; White noise; Besov-Orlicz spaces; Schauder functions; Haar basis; Local times; Hölder continuity; Hausdorff dimension

AMS CLASSIFICATION
60G15; 60G17; 60H05; 60H15

Introduction

Stochastic partial differential equations (SPDEs) model various random phenomena in physics, finance, fluid mechanics, among others, see, e.g. [12, 27, 35]. They have been widely investigated by different approaches in the last three decades. Let us mention the analytic method [36–38], the semigroup point of view [25], and the probabilistic setting using the theory of martingale measures [52]. The particular case of stochastic heat equations has been intensively studied from many perspectives: See, e.g., [2, 4, 24, 45, 46, 53] for regularity investigations and [13, 16, 40] for many other studies. The regularity in Besov spaces of SPDEs has received much attention in the past decade. Among other things, such studies are closely related to the theme of adaptive numerical wavelet methods. For more details on this subject, we refer to [17–19].

In this paper, we consider the following linear stochastic heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \frac{\partial^2}{\partial t\partial x}W(t,x), \quad t>0,\ x\in\mathbb{R}, \\ &u(0,x)=0, \qquad x\in\mathbb{R}, \end{aligned} \tag{1}$$

where W is a space-time white noise. In Section 2.1, following Walsh's random field

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approach [52], we will briefly give a rigorous formulation of the formal equation (1). The purpose of this paper is to investigate the Besov regularity for the process $(u(t,x),\ t \in [0,T],\ x \in [a,b])$ as well as its local time, respectively in the space variable x (for fixed t) and in the time variable t (for fixed x), with $T > 0$ and $a < b$ are arbitrary real values.

Besov spaces, involving a bounded or unbounded interval $I \subseteq \mathbb{R}$, usually noted in the literature by $\mathcal{B}^{\alpha}_{p,q}(I)$ with $0 < \alpha < 1$, $1 \leq p, q \leq +\infty$, are a set of functions of $L^p(I)$ having a smoothness of order α. They cover some classical function spaces as special cases. Namely, $\mathcal{B}^{\alpha}_{p,p}(\mathbb{R})$ coincides with the classical Sobolev space $W^{\alpha,p}(\mathbb{R})$ and $\mathcal{B}^{\alpha}_{\infty,\infty}(I)$ is the classical α-Hölder space $\mathcal{H}^{\alpha}(I)$. When the Orlicz norm is used in place of the L^p-norm, we obtain the well-known Besov-Orlicz spaces.
In this paper, we are essentially concerned by the class of Besov spaces $\mathcal{B}^{\alpha}_{p,\infty}([0,1])$ and also by Besov-Orlicz spaces $\mathcal{B}^{\alpha}_{\mathcal{N},\infty}([0,1])$, $\mathcal{N}$ is the Young function $\mathcal{N}(x) = e^{x^2} - 1$. We will note these spaces respectively by $\mathcal{B}^{\alpha}_{p}$ and $\mathcal{B}^{\alpha}_{\mathcal{N}}$. In this case, for $\alpha p > 1$ and for any $\varepsilon > 0$, we have the following continuous injections:

$$\mathcal{H}^{\alpha+\varepsilon} \hookrightarrow \mathcal{B}^{\alpha,0}_{p} \hookrightarrow \mathcal{B}^{\alpha}_{p} \hookrightarrow \mathcal{H}^{\alpha-1/p}, \tag{2}$$

where $\mathcal{B}^{\alpha,0}_{p}$ is a separable subspace of $\mathcal{B}^{\alpha}_{p}$. Furthermore, it is important to recall that the Besov-Orlicz space $\mathcal{B}^{\alpha}_{\mathcal{N}}$ is continuously embedded in $\mathcal{B}^{\alpha}_{p}$ for any $p \geq 1$. See section 1.3 for more details and [48] for an introduction to Besov spaces.
In [10], the authors have investigated the Besov regularity of the bifractional Brownian motion. However, we know that, for each fixed $x \in \mathbb{R}$, the Gaussian process $(u(t,x)\,;\ t \geq 0)$ is a bifractional Brownian motion (bBm for short) with parameters $H = K = \frac{1}{2}$, multiplied by a constant (see Remark 6). Therefore, $(u(t,x)\,;\ t \in [0,1])$ has the seem Besov regularity as the bBm $B^{\frac{1}{2},\frac{1}{2}}$ (see Subsection 2.2). Otherwise, for each fixed $t > 0$, the Gaussian process $(u(t,x)\,;\ x \in \mathbb{R})$ is equal to a Brownian motion plus a centered Gaussian process with C^{∞} sample paths. Hence, $(u(t,x)\,;\ x \in [0,1])$ has the seem Besov regularity as the Brownian motion (see Subsection 2.3). We have the following, for any $p \geq 4$,
i) For any fixed space variable x,

$$\mathbb{P}\left[u(\cdot,x) \in \mathcal{B}^{1/4}_{p}\right] = 1 \ \text{ and } \ \mathbb{P}\left[u(\cdot,x) \in \mathcal{B}^{1/4,0}_{p}\right] = 0.$$

ii) For any fixed time variable t,

$$\mathbb{P}\left[u(t,\cdot) \in \mathcal{B}^{1/2}_{p}\right] = 1 \ \text{ and } \ \mathbb{P}\left[u(t,\cdot) \in \mathcal{B}^{1/2,0}_{p}\right] = 0.$$

In fact, we even get a better result showing that i) and ii) are true respectively in the Besov-Orlicz space $\mathcal{B}^{1/4}_{\mathcal{N}}$ and $\mathcal{B}^{1/2}_{\mathcal{N}}$. We should point out here that, by a suitable affine change of variables, the interval $[0,1]$ may be replaced in i) (resp. ii)) by any arbitrary compact interval $[0,T]$ (resp. $[a,b]$).

On the other hand, injections (2) show that the obtained regularity results are the best one can get in the scale of Besov spaces. They improve the classical Hölder regularities of $u(t,x)$; namely, the process $u(t,x)$ satisfies a.s. α- Hölder condition with $\alpha < \frac{1}{4}$ (resp. $\alpha < \frac{1}{2}$) in the time (resp. space) variable, see e.g. [24], [46] and references therein.
As far as we know, only the spatial Besov regularity of $u(t,x)$ has been partially

investigated firstly in Deaconu's thesis [26], where the author has also claimed the temporal regularity as in i) but she failed to provide a proof for her finding. Our approach is certainly much more technical than that of [26], but it allowed us to obtain sharp results. Our proof is based on the characterization of Besov spaces in terms of sequences spaces (Theorem 1.4). We use essentially the same arguments, with some adjustments, like those used by Ciesielski, Kerkyacharian, and Roynette in [15], where the authors have investigated Besov regularity for a large class of Gaussian processes. Recently, a more direct method, which uses the usual modulus-of-continuity definition of the Besov norms, has been employed in [41] to prove the temporal regularity in the Besov-Orlicz space for solutions to parabolic stochastic differential equations. Many other works investigating different problems have been published using the sequential characterization of Besov topology, we refer e.g., to this non-exhaustive list [44], [15], [9], [7].

Our second aim is to investigate existence, Hölder continuity, and Besov regularity of local times of $u(t,x)$, viewed as a process respectively in time and space variables. We will use the notion of local nondeterminism (LND), initiated by Berman in [6] and extended later in [20] to the strong local nondeterminism concept (SLND). These two notions are the most important mathematical tools used by several authors to study sample path properties for various Gaussian processes and random fields, see, e.g. [3, 8, 54–58] and references therein.

We must note that, for fixed x, the process $(u(t,x), t \in [0,T])$ is identically distributed as the so-called bifractional Brownian motion (see [47]). So, many sample path properties of the process $(u(t,x), t \in [0,T])$, including the Hölder continuity of its local time, may be deduced from [50]. In this last paper, to prove that the bifractional Brownian motion satisfies the SLND property, the authors have used the Lamperti transformation to connect self-similar and stationary Gaussian processes. In our paper, to verify the LND condition for the processes $(u(t,x), t \in [0,T], x \in \mathbb{R})$ respectively in time and space variables, we use fine estimates on the Green kernel G and careful calculations. Furthermore, we will improve the knowledge of the sample paths properties for these two processes by proving regularity results of their local times in the modular Besov spaces $\mathcal{B}_p^\omega$, with respect to modulus $\omega(t) = t^\alpha (\log 1/t)^\lambda$, with $\alpha = 1/4$ (resp. $1/2$) and $p\lambda > 1$. In contrast with the Besov regularity obtained for the Brownian local time in [11], our results are not optimal, but they have the merit of being new and sharper than what is known.

Let us give an overview of the results existing in the literature related to the context of our paper. First, it was raised in [57] the relevance of studying the sample paths properties for the solutions to stochastic heat or wave equations through their local times and LND concept. Ouahhabi and Tudor [42] have studied, for fixed space variable, the Hölder regularity of the local time of the solution to the d-dimensional stochastic linear heat equation driven by a fractional-white noise, with Hurst parameter $H > \frac{1}{2}$. The investigation of the sample paths properties, both in time and space variables, of the solution to the stochastic heat equation driven by a fractional-colored noise is considered in [49] and [58].

While writing this article, we discovered the papers [33, 41, 51] where a direct method was used to handle Besov norms. They gave alternative proofs of Besov regularities of Brownian motion and fractional Brownian motion, without the equivalent wavelet description used in [14] and [15]. By this new approach, we expect that we will establish sharp Besov regularities for the solution and its local time to a d-dimensional stochastic heat (or wave) equation driven by a fractional-colored noise. We intend to study these questions in a future paper.

This paper is organized as follows: In the first paragraph, we briefly recall some notions on local times, Besov and Besov-Orlicz spaces. The second paragraph is devoted to studying the Besov-Orlicz regularity for the sample paths of $t \mapsto u(t,x)$ and $x \mapsto u(t,x)$. In the third paragraph, we investigate the existence and Besov regularity of local times of the process $u(t,x)$ in time and space variables and conclude by considering Hausdorff dimensions of the level sets $\{t \in [0,T];\ u(t,x) = u(t_0,x)\}$ and $\{x \in [a,b];\ u(t,x) = u(t,x_0)\}$.

1. Preliminaries

In this paragraph, we give some basic notions on local times and Besov spaces.

1.1. The local times

We recall here the concept of the local time, in the Fourier analytic sense, as it has been initiated by S. Berman in [5].

Let $I \subset \mathbb{R}$ be a compact interval, and $\theta : s \in I \to \theta_s \in \mathbb{R}$, a deterministic Borel function. $\mathcal{B}(I)$ be the Borel σ-algebra on I, for any $B \in \mathcal{B}(I)$ we define the occupation measure μ_B by

$$\mu_B(A) = \lambda\{s \in B,\ \theta_s \in A\}, \quad A \in \mathcal{B}(\mathbb{R}).$$

If μ_B is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we say that θ has a local time on B, and we define its local time $L(.,B)$ as the Radon Nikodym derivative of μ_B with respect to the Lebesgue measure. If $B = [0,t]$ (resp. $B = I$) we write $L(\xi,t)$ (resp. $L(\xi)$) instead of $L(.,[0,t])$ (resp. $L(\xi,I)$). This definition can be extended to any measurable and bounded (or positive) function f to get the so-called occupation density formula:

$$\int_0^t f(\theta_s)ds = \int_{\mathbb{R}} f(\xi)L(\xi,t)d\xi.$$

The idea of Berman is to relate properties of $L(.,B)$ with the integrability of the Fourier transform of θ. Recall the following essential result:

Proposition 1.1. *Let $I \subset \mathbb{R}$ be a compact interval, $\theta : s \in I \to \theta_s \in \mathbb{R}$ a deterministic Borel function, and $B \subseteq I$ a Borel set. The function θ has a square integrable local time $L(\xi,B)$ iff*

$$\int_{\mathbb{R}} \left| \int_B \exp(iu\,\theta_s)ds \right|^2 du < \infty.$$

Moreover, we have the following representation of the local time, for almost every $\xi \in \mathbb{R}$,

$$L(\xi,B) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_B \exp(iu(\theta_s - \xi))dsdu. \tag{3}$$

The deterministic function θ can be chosen to be a sample path of a stochastic process $(X_t,\ t \in [0,T])$. In this regard, we say that the process X has a local time (resp. square integrable local time) if for almost all ω, the trajectory $t \to X_t(\omega)$ has a local time (resp. square integrable local time). To prove that X has a square integrable local time, $L(\xi, B)$, it is enough to establish that

$$\mathbb{E} \int_{\mathbb{R}} \left| \int_B \exp(iuX_s) ds \right|^2 du < \infty.$$

When $(X_t,\ t \in [0,T])$ is Gaussian, then we get the well-known Berman's criterion:

Proposition 1.2 ([5])**.** *If $(X_t,\ t \in [0,T])$ is a centered Gaussian process, and satisfies*

$$\int_0^T \int_0^T \left[\mathbb{E}(X_s - X_t)^2\right]^{-1/2} dsdt < \infty.$$

Then for almost all ω, for any $B \in \mathcal{B}([0,T])$, the trajectory $s \mapsto X_s(\omega)$ has a local time $L(\xi, B, \omega)$ which is square integrable with respect to ξ.

Remark 1. In the rest of this paper, we will note the local time of a process by $L(\xi, B)$ instead of $L(\xi, B, \omega)$ unless a confusion exists.

The following theorem will clarify Berman's principle, providing the link between the regularity of the local time and the irregularity of its underlying stochastic process:

Theorem 1.3 ([5])**.** *Let $(X_t,\ t \in [0,T])$ be a centered Gaussian process, such that for some $p \geq 0$,*

$$\int_0^T \int_0^T [\mathbb{E}(X_s - X_t)^2]^{-(p+1)/2} dsdt < \infty. \tag{4}$$

Then for almost all ω, the trajectory $s \mapsto X_s(\omega)$ has a local time $L(\xi)$, such that the first $[p/2]$ derivatives of $L(\xi)$ exist and are square integrable ($[v]$ = integer part of v). Moreover, when $[p]$ is even, the sample functions of X are nowhere $(2/(p+1))$-Hölder continuous.

Remark 2. A particular situation is obtained if,

$$\mathbb{E}(X_t - X_s)^2 \geq c|t-s|^{\beta},$$

where $0 < \beta < 2$ and c is a positive constant. We can easily deduce that a.s. the trajectories of the process X are nowhere β-Hölder continuous.

To derive from Kolmogorov continuity theorem, the joint continuity of the local time and the Hölder continuity in t of $L(\xi, t)$, our starting point is the following identities about the moments of the increments of the local time

$$\begin{aligned}&\mathbb{E}[L(\xi+k, t+h) - L(\xi, t+h) - L(\xi+k, t) + L(\xi, t)]^n \\ =&(2\pi)^{-n} \int_{[t,t+h]^n} \int_{\mathbb{R}^n} \prod_{j=1}^n (e^{-i(\xi+k)u_j} - e^{-i\xi u_j}) \mathbb{E}[\exp(i \sum_{j=1}^n u_j X_{t_j})] d\bar{u} d\bar{t},\end{aligned}$$

and

$$\mathbb{E}[L(\xi,t+h)-L(\xi,t)]^n=(2\pi)^{-n}\int_{[t,t+h]^n}\int_{\mathbb{R}^n}\exp(-i\xi\sum_{j=1}^n u_j)\mathbb{E}[\exp(i\sum_{j=1}^n u_jX_{t_j})]d\overline{u}d\overline{t},$$

where $\overline{u}=(u_1,...,u_n)$, $\overline{t}=(t_1,...,t_n)$, ξ, $\xi+k\in\mathbb{R}$ and t, $t+h\in[0,T]$. To find suitable upper bounds for these increments for a given centered Gaussian process X, the expression $\mathrm{Var}[\sum_{j=1}^n v_j(X_{t_j}-X_{t_{j-1}})]$ appearing by an appropriate change of variables in the characteristic function above, is lower bounded by $C_m\sum_{j=1}^n v_j^2\,\mathrm{Var}(X_{t_j}-X_{t_{j-1}})$ when X verifies the LND property (see, [6, Lemma 2.3.]). This last property reads as follows:

$$\lim_{c\to 0}\inf_{0\le t-r\le c,\ r<s<t}\frac{\mathrm{Var}(X_t-X_s|X_\tau,\ r\le\tau\le s)}{\mathrm{Var}(X_t-X_s)}>0. \tag{5}$$

To analyze the local time of the solution to the stochastic heat equation, the LND property will be verified in the next for $u(t,x)$ viewed either as a process in time variable for any fixed $x\in\mathbb{R}$ or a process in space variable for any fixed t.

1.2. Modular Besov spaces

In this paragraph, we recall some notions of modular Besov spaces and their characterizations in terms of some sequences spaces. Let $I\subset\mathbb{R}$ be a compact interval, $1\le p<\infty$ and $f\in L^p(I\,;\mathbb{R})$. We can measure the smoothness of f by its modulus of continuity computed in the L^p-norm. For this end, let us define for any $t>0$,

$$\Delta_p(f,I)(t)=\sup_{|s|\le t}\left\{\int_{I_s}|f(x+s)-f(x)|^p dx\right\}^{\frac{1}{p}},$$

where $I_s=\{x\in I;\ \ x+s\in I\}$.
We can find in the literature various ways to define Besov norms. They are generally defined with respect to some moduli, which are non-decreasing and continuous positive functions ω defined on $[0,+\infty[$, s.t. $\omega(0)=0$. The most typical example of moduli is:

$$\omega_{\alpha,\lambda}(t)=t^{\alpha}(\log(1/t))^{\lambda}, \tag{6}$$

with $0<\alpha<1$ and $\lambda\in\mathbb{R}$. We refer to [15] for a more details on this subject.

Remark 3. From now on, we will only be concerned with the moduli $\omega_{\alpha,\lambda}$ defined in (6) and write ω instead of $\omega_{\alpha,\lambda}$ unless a confusion exists.

Let ω be any modulus defined by (6), we consider the norm

$$||f||_{\omega,p}:=||f||_{L^p(I)}+\sup_{0<t\le 1}\frac{\Delta_p(f,I)(t)}{\omega(t)}.$$

The modular Besov space is given by

$$\mathcal{B}_p^{\omega}(I)=\{f\in L^p(I);\ \ ||f||_{\omega,p}<\infty\}.$$

The space $(\mathcal{B}_p^\omega(I), ||\,.\,||_{\omega,p})$ is a non separable Banach space. We also define

$$\mathcal{B}_p^{\omega,0}(I) = \{f \in L^p(I);\ \ \Delta_p(f,I)(t) = o(\omega(t)) \text{ as } t \to 0^+\}.$$

$\mathcal{B}_p^{\omega,0}(I)$ is a separable subspace of $\mathcal{B}_p^\omega(I)$. For $p=\infty$, the space $\mathcal{B}_\infty^\omega(I)$ is defined in the same way by using the usual L^∞-norm. In this case it coincides with $\mathcal{H}^\omega(I)$, the ω-Hölder space defined by

$$\mathcal{H}^\omega(I) := \left\{ f\ : I \longrightarrow \mathbb{R}\ \mid\ \operatorname*{ess\,sup}_{\substack{x,y\in I\\ x\neq y}} \frac{|f(x)-f(y)|}{\omega(|x-y|)} < \infty \right\}, \tag{7}$$

endowed with the norm $||f||_{\omega,\infty} = \operatorname*{ess\,sup}_{x\in I} |f(x)| + \operatorname*{ess\,sup}_{\substack{x,y\in I\\ x\neq y}} \dfrac{|f(x)-f(y)|}{\omega(|x-y|)}$.

As it is mentioned in the introduction, standard changes of variables permit us to restrict ourselves to the unit interval $I=[0,1]$, so we will omit to specify the interval I in our notations. The following theorem (Theorem 1.4) is a characterization of modular Besov spaces $\mathcal{B}_p^\omega$ in terms of progressive differences of a function in dyadic points. Its proof has been established for general moduli in [15].

Theorem 1.4. *Let ω be as in (6), $p>1$, $\frac{1}{p} < \alpha < 1$ and $\lambda \geq 0$, then we have*

(1) $\mathcal{B}_p^\omega$ is linearly isomorphic to a sequences space and we have the following equivalence of norms:

$$||f||_{\omega,p} \sim \max\left\{ |f_0|, |f_1|, \sup_j \frac{2^{-j\left(\frac{1}{2}+\frac{1}{p}\right)}}{\omega(2^{-j})} \left[\sum_{k=1}^{2^j} |f_{jk}|^p\right]^{\frac{1}{p}} \right\},$$

where the coefficients $\left\{f_0, f_1, f_{jk}\,, j\geq 0\,, 1\leq k\leq 2^j\right\}$ are given by

$$f_0 = f(0), \qquad f_1 = f(1) - f(0),$$

$$f_{jk} = 2\cdot 2^{j/2}\left\{ f\left(\frac{2k-1}{2^{j+1}}\right) - \frac{1}{2} f\left(\frac{2k}{2^{j+1}}\right) - \frac{1}{2} f\left(\frac{2k-2}{2^{j+1}}\right) \right\}.$$

(2) f is in $\mathcal{B}_p^{\omega,0}$ if and only if

$$\lim_{j\to 0} \frac{2^{-j\left(\frac{1}{2}+\frac{1}{p}\right)}}{\omega(2^{-j})} \left[\sum_{k=1}^{2^j} |f_{jk}|^p\right]^{\frac{1}{p}} = 0.$$

When $\omega(t) = t^\alpha$, we will use the notations:

$$||f||_{\alpha,p} := ||f||_{\omega,p}, \quad \mathcal{B}_p^\alpha := \mathcal{B}_p^\omega \quad \text{and} \quad \mathcal{B}_p^{\alpha,0} := \mathcal{B}_p^{\omega,0}.$$

Remark 4. One can easily show, by Theorem 1.4, the following useful continuous injections

- For any $\varepsilon > 0$, $1 \le p < \infty$ and $\frac{1}{p} < \alpha < 1$,

$$\mathcal{H}^{\alpha+\varepsilon} \hookrightarrow \mathcal{B}_p^{\alpha,0} \hookrightarrow \mathcal{B}_p^{\alpha} \hookrightarrow \mathcal{H}^{\alpha-\frac{1}{p}}.$$

- Let $1 \le p < \infty$ and $0 < \alpha < \beta < 1$, we have

$$\mathcal{B}_p^{\beta} \hookrightarrow \mathcal{B}_p^{\alpha,0}.$$

- Let $\omega(t) = t^{\alpha}(\log(1/t))^{\lambda}$, $0 < \alpha < 1$ and $\lambda \ge 0$, then

$$\mathcal{B}_p^{\omega} \hookrightarrow \mathcal{H}^{\gamma}, \tag{8}$$

  for any $\gamma < \alpha$ and p sufficiently large.

1.3. Besov–Orlicz spaces

Let $I \subset \mathbb{R}$, be a compact interval, and $\mathcal{N}$ is the Young function defined by $\mathcal{N}(x) = e^{x^2} - 1$. The Orlicz space $L_{\mathcal{N}}(I)$ is the space of measurable functions $f : I \mapsto \mathbb{R}$, such that

$$\|f\|_{\mathcal{N}}^{*} := \inf_{\lambda>0} \frac{1}{\lambda}\left[1 + \int_I \mathcal{N}(\lambda f(t))dt\right] < \infty.$$

It is more suitable to use an equivalent norm to $\|\cdot\|_{\mathcal{N}}^{*}$ (see e.g. Ciesielski [14]):

$$\|f\|_{\mathcal{N}} = \sup_{p\ge 1} \frac{\|f\|_{L^p(I)}}{\sqrt{p}}.$$

Let $\Delta_{\mathcal{N}}(f, I)(t)$ be the modulus of continuity of f in the Orlicz space $L_{\mathcal{N}}(I)$ defined as:

$$\Delta_{\mathcal{N}}(f, I)(t) = \sup_{p\ge 1} \frac{\Delta_p(f, I)(t)}{\sqrt{p}}.$$

For $0 < \alpha < 1$, we consider the following norm

$$\|f\|_{\alpha,\mathcal{N}} = \|f\|_{\mathcal{N}} + \sup_{0<t\le 1} \frac{\Delta_{\mathcal{N}}(f, I)(t)}{t^{\alpha}}.$$

The Besov-Orlicz space is defined by

$$\mathcal{B}_{\mathcal{N}}^{\alpha}(I) := \{f \in L_{\mathcal{N}}(I);\ \|f\|_{\alpha,\mathcal{N}} < \infty\}.$$

$\mathcal{B}_{\mathcal{N}}^{\alpha}(I)$ endowed with the norm $\|\cdot\|_{\alpha,\mathcal{N}}$ is a non separable Banach space. We introduce $\mathcal{B}_{\mathcal{N}}^{\alpha,0}(I) = \{f \in L_{\mathcal{N}}(I);\ \ \Delta_{\mathcal{N}}(f, I)(t) = o(t^{\alpha}) \text{ as } t \to 0^{+}\}$ a separable subspace of $\mathcal{B}_{\mathcal{N}}^{\alpha}(I)$.

We will restrict ourselves to the interval $I = [0, 1]$, so we will omit to precise the interval I in our notations, e.g. we will use $\mathcal{B}_{\mathcal{N}}^{\alpha}$ to denote the Besov-Orlicz space

$\mathcal{B}^{\alpha}_{\mathcal{N}}([0,1])$. With the same notations as in Theorem 1.4, we have the following isomorphism theorem (see Ciesielski [14] or Ciesielski *et al.* [15]):

Theorem 1.5. *We have*

(1) $\mathcal{B}^{\alpha}_{\mathcal{N}}$ *is linearly isomorphic to a sequences space and we have the following equivalence of norms:*

$$||f||_{\alpha,\mathcal{N}} \sim \max\left\{ |f_0|, |f_1|, \sup_{p,j} \frac{1}{\sqrt{p}} 2^{-j\left(\frac{1}{2}-\alpha+\frac{1}{p}\right)} \left[\sum_{k=1}^{2^j} |f_{jk}|^p\right]^{\frac{1}{p}} \right\},$$

(2) f *belongs to* $\mathcal{B}^{\alpha,0}_{\mathcal{N}}$ *if and only if*

$$\lim_{j\to\infty} \sup_{p\geq 1} \frac{1}{\sqrt{p}} 2^{-j\left(\frac{1}{2}-\alpha+\frac{1}{p}\right)} \left[\sum_{k=1}^{2^j} |f_{jk}|^p\right]^{\frac{1}{p}} = 0.$$

Remark 5. For any $p \in [1,\infty)$ and $0 < \alpha < 1$, the following injections are easy to verify

$$\mathcal{B}^{\alpha}_{\mathcal{N}} \hookrightarrow \mathcal{B}^{\alpha}_{p} \quad \text{and} \quad \mathcal{B}^{\alpha,0}_{\mathcal{N}} \hookrightarrow \mathcal{B}^{\alpha,0}_{p}.$$

2. Besov regularity of solution to stochastic heat equation

2.1. Linear stochastic heat equation

Consider the linear stochastic heat equation defined by (1), where $W = (W(t,A),\ t \geq 0,\ A \in \mathcal{B}_b(\mathbb{R}))$ is a space-time white noise, i.e., W is a centered Gaussian process with covariance function given by

$$\mathbb{E}[W(t,A)W(t',A')] = (t \wedge t')\,\lambda(A \cap A'), \quad A, A' \in \mathcal{B}_b(\mathbb{R}),\ t,t' \geq 0\,,$$

where λ is the Lebesgue measure, $\mathcal{B}_b(\mathbb{R})$ is the collection of bounded Borel sets. Set $W(t,x) := W(t,[0,x])$. It is well known that there exist a unique (mild) solution of this equation given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s,x-y)dW(s,y), \tag{9}$$

where the integral is the Wiener integral with respect to the Gaussian process W and G is the Green kernel of the heat equation given by

$$G(t,x) = \begin{cases} (2\pi t)^{-1/2}\exp(-\frac{|x|^2}{2t}) & \text{if } t>0,\ x\in\mathbb{R} \\ 0 & \text{if } t\leq 0,\ x\in\mathbb{R}. \end{cases} \tag{10}$$

The mild solution $(u(t,x)\,;\ t \geq 0\,,\ x \in \mathbb{R})$ is a centered two-parameter Gaussian process (also called a Gaussian random field). The covariance of the solution (9), when $x \in \mathbb{R}$ is fixed, satisfies

$$\mathbb{E}\left(u(t,x)u(s,x)\right) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{t+s} - \sqrt{|t-s|}\right), \qquad \text{for every } s,t \geq 0. \tag{11}$$

2.2. Besov regularity of $t \to u(t,x)$

Our main result in this paragraph, is the following theorem

Theorem 2.1. *For all $x \in \mathbb{R}$ and $4 < p < \infty$, we have*

$$\mathbb{P}(u(\cdot,x) \in \mathcal{B}_p^{1/4}) = 1 \qquad \text{and} \qquad \mathbb{P}(u(\cdot,x) \in \mathcal{B}_p^{1/4,0}) = 0, \tag{12}$$

$$\mathbb{P}(u(\cdot,x) \in \mathcal{B}_{\mathcal{N}}^{1/4}) = 1 \qquad \text{and} \qquad \mathbb{P}(u(\cdot,x) \in \mathcal{B}_{\mathcal{N}}^{1/4,0}) = 0, \tag{13}$$

where $u(\cdot,x)$ is the sample path $t \in [0,1] \longrightarrow u(t,x)$ and $\mathcal{N}(x) = e^{x^2} - 1$.

Recall that a stochastic process $(B^{H,K}(t))_{t\in[0,1]}$ is called a bifractional Brownian motion (bBm), if $B^{H,K}$ is a mean-zero Gaussian process with covariance function

$$\mathbb{E}\left[B^{H,K}(t)B^{H,K}(s)\right] = \frac{1}{2^K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right), \tag{14}$$

where $H \in (0,1)$ and $K \in (0,1]$. In particular, when $K = 1$ then $B^{H,1}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$, and this process is denoted by B^H. And, if $K = 1$ and $H = \frac{1}{2}$ then $B^{\frac{1}{2},1}$ is the ordinary Brownian motion, and we denote this process by B.

Remark 6. According to (11), we remark that, when $x \in \mathbb{R}$ is fixed, the Gaussian process $(u(t,x)\,,\ t \geq 0)$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$, multiplied by a constant.

By using Theorem 1.4 and 1.5, the authors have widely investigated in [10] the Besov regularity of bBm, we mention the following theorem.

Theorem 2.2 (Theorem 3.1 and 3.7 in [10])**.** *Let $(B^{H,K}(t))_{t\in[0,1]}$ be a bBm with parameters $H \in (0,1)$ and $K \in (0,1]$. For all $\frac{1}{HK} < p < \infty$, we have*

$$\mathbb{P}(B^{H,K}(\cdot) \in \mathcal{B}_p^{HK}) = 1 \qquad \text{and} \qquad \mathbb{P}(B^{H,K}(\cdot) \in \mathcal{B}_p^{HK,0}) = 0, \tag{15}$$

$$\mathbb{P}(B^{H,K}(\cdot) \in \mathcal{B}_{\mathcal{N}}^{HK}) = 1 \qquad \text{and} \qquad \mathbb{P}(B^{H,K}(\cdot) \in \mathcal{B}_{\mathcal{N}}^{HK,0}) = 0, \tag{16}$$

where $B^{H,K}(\cdot)$ is the sample path $t \in [0,1] \longrightarrow B^{H,K}(t)$ and $\mathcal{N}(x) = e^{x^2} - 1$.

Proof of Theorem 2.1. The proof is a straightforward result of Remark 6 and Theorem 2.2. □

Remark 7. In Lei and Nualart [39], it was shown essentially the following decomposition of the bBm

$$\{C_2 B^{HK}(t),\ t \geq 0\} \overset{d}{=} \{C_1 X^{H,K}(t) + B^{H,K}(t),\ t \geq 0\},$$

where C_1, C_2 are two constants and $(B^{HK}(t))_{t\geq 0}$ is a fBm with parameter HK and $(X^{H,K}(t))_{t\geq 0}$ is a Gaussian process with infinitely differentiable trajectories on $(0,+\infty)$ and absolutely continuous on $[0,+\infty)$. On the other hand, we know from Ciesielski et al. [15] (see also [10]) that almost all paths of the fBm $(B^{HK}(t))_{t\geq 0}$ belong (resp. do not belong) to the Besov spaces $\mathcal{B}_p^{HK}$ (resp. to the separable subspaces $\mathcal{B}_p^{HK,0}$). In fact, a stronger regularity result was obtained in the Besov-Orlicz space $\mathcal{B}_{\mathcal{N}}^{HK}$, where $\mathcal{N}(x)=e^{x^2}-1$. However, according to Remark 6 and Lei and Nualart decomposition, if we take $0<a<b$, one can deduce directly that, for any fixed $x\in\mathbb{R}$, the sample paths of $(u(t,x)\,,\ t\in[a,b])$ satisfy the same Besov regularity as those of fBm of parameter $\frac{1}{4}$. Otherwise, we are unable to get the Hölder regularity of $X^{H,K}$ on intervals of type $[0,\varepsilon]$, for $\varepsilon>0$, since the trajectories of this process are only absolutely continuous near 0. Hence, we can not derive directly from Lei and Nualart decomposition the Besov regularity, when $x\in\mathbb{R}$ is fixed, of $(u(t,x)\,,\ t\in[0,\varepsilon])$ or more generally the Besov regularity of the bBm on the interval $[0,\varepsilon]$. Therefore, To fill this gap, the authors have investigated in [10] the Besov regularity for sample paths of the bBm $(B^{H,K}(t))$ for $t\in[0,1]$.

2.3. Besov regularity of $x\to u(t,x)$

In this paragraph we will study the Besov regularity of $(u(t,x),\ x\in[0,1])$ for any fixed $t>0$.

Theorem 2.3. *For all* $t\in(0,\infty)$ *and* $2<p<\infty$*, we have*

$$\mathbb{P}(u(t,\cdot)\in\mathcal{B}_p^{1/2})=1 \qquad \text{and} \qquad \mathbb{P}(u(t,\cdot)\in\mathcal{B}_p^{1/2,0})=0, \tag{17}$$

$$\mathbb{P}(u(t,\cdot)\in\mathcal{B}_{\mathcal{N}}^{1/2})=1 \qquad \text{and} \qquad \mathbb{P}(u(t,\cdot)\in\mathcal{B}_{\mathcal{N}}^{1/2,0})=0, \tag{18}$$

where $u(t,\cdot)$ *is the sample path* $x\in[0,1]\longrightarrow u(t,x)$ *and* $\mathcal{N}(x)=e^{x^2}-1$.

To prove this theorem, we need first to state the following key result.

Proposition 2.4 (Theorem 3.3 in [34])**.** *Let* $t\in(0,\infty)$ *be fixed. Hence, the solution* $u(t,x)$ *to the linear stochastic heat equation* (1)*, at time* t*, can be decomposed as*

$$u(t,x)=CB(x)+S(x) \qquad \forall x\in\mathbb{R}, \tag{19}$$

where C *is a positive and finite constant,* $(B(x))_{x\in\mathbb{R}}$ *is a 2-sided Brownian motion, and* $(S(x))_{x\in\mathbb{R}}$ *is a centered Gaussian process with* C^∞ *sample functions.*

Proof of Theorem 2.3. The proof is a simple consequence of Proposition 2.4 and Theorem 2.2. □

Remark 8. Theorem 2.3 can be generalized to a more general setting. Let us consider the following stochastic partial differential equation

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,x)=-(-\Delta)^{\alpha/2}v(t,x)+\frac{\partial^2}{\partial t\partial x}W(t,x), \quad t>0,\ x\in\mathbb{R},\\ &v(0,x)=0, \qquad x\in\mathbb{R}, \end{aligned} \tag{20}$$

where $\alpha\in(1,2]$ is a fixed parameter, $-(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian to the power $\alpha/2$, and $\frac{\partial^2}{\partial t \partial x} W(t,x)$ is a space-time white noise. The mild solution of this equation is given by

$$v(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) dW(s,y), \tag{21}$$

where $\{p_t(x)\}_{t>0, x\in\mathbb{R}}$ is the heat kernel of the fractional Laplacian. According to [30, Proposition 3.1], for any fixed $t > 0$, we have the following decomposition of the solution $v(t,x)$ to the linear stochastic heat equation (20)

$$v(t,x) = C_\alpha\, B^{(\alpha-1)/2}(x) + S(x) \qquad \forall x \in \mathbb{R}, \tag{22}$$

where C_α is a positive and finite constant, $(B^{(\alpha-1)/2}(x))_{x\in\mathbb{R}}$ is the fBm with Hurst index $(\alpha-1)/2$, and $(S(x))_{x\in\mathbb{R}}$ is a centered Gaussian process with C^∞ sample functions. By this and Theorem 2.2, we conclude that for all $t \in (0,\infty)$ and $\frac{2}{\alpha-1} < p < \infty$,

$$\mathbb{P}(v(t,\cdot) \in \mathcal{B}_p^{(\alpha-1)/2}) = 1 \qquad \text{and} \qquad \mathbb{P}(v(t,\cdot) \in \mathcal{B}_p^{(\alpha-1)/2,0}) = 0, \tag{23}$$

$$\mathbb{P}(v(t,\cdot) \in \mathcal{B}_{\mathcal{N}}^{(\alpha-1)/2}) = 1 \qquad \text{and} \qquad \mathbb{P}(v(t,\cdot) \in \mathcal{B}_{\mathcal{N}}^{(\alpha-1)/2,0}) = 0, \tag{24}$$

where $v(t,\cdot)$ is the sample path $x \in [0,1] \longrightarrow v(t,x)$ and $\mathcal{N}(x) = e^{x^2} - 1$.

3. Existence and regularity of local times

3.1. Local time of the process $t \to u(t,x)$

In this section we will consider the process $(u(t,x),\ t \in [0,T])$, for some fixed $T > 0$ and $x \in \mathbb{R}$.

3.1.1. Existence of the local time

We will need the following estimates, which can be easily shown by standard arguments as change of variables and Parseval's equality:

Lemma 3.1. *For any $x \in \mathbb{R}$ and $s < t$ with $s,t \in [0,T]$,*

(1) $\int_0^s \int_{\mathbb{R}} |G(t-r,x-y) - G(s-r,x-y)|^2 dydr \le C|t-s|^{1/2}$
(2) $\int_s^t \int_{\mathbb{R}} G^2(t-r,x-y) dydr = C_1|t-s|^{1/2}$.

Proposition 3.2. *For any $x \in \mathbb{R}$ and $0 \le p < 3$, we have*

$$\int_0^T \int_0^T [\mathbb{E}(u(t,x) - u(s,x))^2]^{-(p+1)/2} dsdt < \infty\,.$$

Proof. According to Remark 6, we know that, for any fixed $x \in \mathbb{R}$, the Gaussian process $(u(t,x)\,,\ t \ge 0)$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$, multiplied by a constant. Therefore, by [32, Proposition 3.1] we conclude the proof of Proposition 3.2. □

Theorem 3.3. *Let $x \in \mathbb{R}$ be fixed, then*

(1) There exists a square integrable version of the local time of $(u(t,x),\ t \in [0,T])$*.*
(2) The process $(u(t,x),\ t \in [0,T])$ *satisfies the LND property i.e., formula* (5)*.*

Proof. The existence of a square integrable version of the local time of $u(t,x)$ is a consequence of Berman's theory (cf. Theorem 1.3) and Proposition 3.2. We will denote this version by $(L(\xi,t),\ t \geq 0,\ \xi \in \mathbb{R})$.

Let us now prove that $(u(t,x),\ t \in [0,T])$ satisfies the LND condition. So we have to prove,

$$\lim_{c \to 0} \inf_{0 \leq t-r \leq c,\, r<s<t} \frac{\mathrm{Var}(u(t,x)-u(s,x)|u(\tau,x),\ r \leq \tau \leq s)}{\mathrm{Var}(u(t,x)-u(s,x))} > 0. \tag{25}$$

First, remark that we have the following inclusion of σ-algebras

$$\sigma(u(\tau,x),\ r \leq \tau \leq s) \subset \mathcal{F}_s^W,$$

where we have noted $\mathcal{F}_s^W = \sigma((W(r,A),\ 0 \leq r \leq s,\ A \in \mathcal{B}_b(\mathbb{R})) \vee \mathcal{N})$. However, recall that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are σ-algebras such that $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then for every Gaussian random variable $Y \in L^2(\mathbb{P})$,

$$\mathrm{Var}(Y|\mathcal{G}_1) \geq \mathrm{Var}(Y|\mathcal{G}_2).$$

We then get

$$\frac{\mathrm{Var}(u(t,x)-u(s,x)|u(\tau,x),\ r \leq \tau \leq s)}{\mathrm{Var}(u(t,x)-u(s,x))} \geq \frac{\mathrm{Var}(u(t,x)-u(s,x)|\mathcal{F}_s^W)}{\mathrm{Var}(u(t,x)-u(s,x))}. \tag{26}$$

Now

$$\begin{aligned} &\mathrm{Var}(u(t,x)-u(s,x)|W(\tau,A),\ 0 \leq \tau \leq s,\ A \in \mathcal{B}_b(\mathbb{R})) \\ &= \mathrm{Var}(\int_0^s \int_{\mathbb{R}} (G(t-\tau,x-y)-G(s-\tau,x-y))dW(\tau,y) \\ &\qquad + \int_s^t \int_{\mathbb{R}} G(t-\tau,x-y)dW(\tau,y)|\mathcal{F}_s^W). \end{aligned} \tag{27}$$

But since $\int_0^s \int_{\mathbb{R}} (G(t-\tau,x-y)-G(s-\tau,x-y))dW(\tau,y)$ is $\mathcal{F}_s^W-$measurable and $\int_s^t \int_{\mathbb{R}} G(t-\tau,x-y)dW(\tau,y)$ is independent of $\mathcal{F}_s^W$, we have

$$\mathrm{Var}(u(t,x)-u(s,x)|W(\tau,A),\ 0 \leq \tau \leq s,\ A \in \mathcal{B}_b(\mathbb{R})) = \int_s^t \int_{\mathbb{R}} G^2(t-\tau,x-y)dyd\tau. \tag{28}$$

On the other hand

$$\begin{aligned} \mathrm{Var}(u(t,x)-u(s,x)) &= \int_0^s \int_{\mathbb{R}} (G(t-\tau,x-y)-G(s-\tau,x-y))^2 dyd\tau \\ &\qquad + \int_s^t \int_{\mathbb{R}} G^2(t-\tau,x-y)dyd\tau \\ &:= A(s,t). \end{aligned} \tag{29}$$

Combining (26), (27), (28) and (29) we obtain

$$\begin{aligned}
&\lim_{c\to 0}\inf_{0\le t-r\le c,\ r<s<t} \frac{\mathrm{Var}(u(t,x)-u(s,x)|u(\tau,x),\ r\le\tau\le s)}{\mathrm{Var}(u(t,x)-u(s,x))}\\
&\ge \lim_{c\to 0}\inf_{0\le t-r\le c,\ r<s<t} \frac{\int_s^t\int_{\mathbb{R}} G^2(t-\tau,x-y)dyd\tau}{A(s,t)}.
\end{aligned}$$

Remark that

$$\begin{aligned}
&\lim_{c\to 0}\inf_{0\le t-r\le c,\ r<s<t} \frac{\int_s^t\int_{\mathbb{R}} G^2(t-\tau,x-y)dyd\tau}{A(s,t)} > 0\\
\Leftrightarrow &\lim_{c\to 0}\inf_{0\le t-r\le c,\ r<s<t} \frac{\int_s^t\int_{\mathbb{R}} G^2(t-\tau,x-y)dyd\tau}{\int_0^s\int_{\mathbb{R}}(G(t-\tau,x-y)-G(s-\tau,x-y))^2dyd\tau} > 0.
\end{aligned}$$

The last property of G is assured by Lemma 3.1, which ends the proof of Theorem 3.3. □

Proposition 3.4. *For all $x,y\in\mathbb{R}$, $t,t+h\in(0,T]$ and for any even positive integer n, there exists $C_n>0$ such that*

$$\mathbb{E}[L(y,t+h)-L(x,t+h)-L(y,t)+L(x,t)]^n \le C_n|x-y|^{\zeta n}|h|^{n(3/4-\zeta/4)}, \qquad (30)$$

$$\mathbb{E}[L(x,t+h)-L(x,t)]^n \le C_n|h|^{3n/4}, \qquad (31)$$

where $0<\zeta<1$.

Proof. We prove just the first inequality; the second one follows the same lines. For simplicity of notations we use X_t to denote the process $(u(t,x)\,,\ t\in[0,T])$. We consider only $h>0$ such that $t+h\in[0,T]$ the other case follows the same way. Following [31] or [28] we have

$$\begin{aligned}
&\mathbb{E}[L(y,t+h)-L(x,t+h)-L(y,t)+L(x,t)]^n\\
&= (2\pi)^{-n}\int_{[t,t+h]^n}\int_{\mathbb{R}^n}\prod_{j=1}^n[e^{-iyu_j}-e^{-ixu_j}]\mathbb{E}[e^{i\sum_{j=1}^n u_jX_{t_j}}]d\overline{u}d\overline{t}.
\end{aligned}$$

The elementary inequality $|1-e^{i\theta}|\le 2^{1-\zeta}|\theta|^\zeta$ for any $0<\zeta<1$ and $\theta\in\mathbb{R}$, leads to

$$\mathbb{E}[L(y,t+h)-L(x,t+h)-L(y,t)+L(x,t)]^n \le 2^{-n\zeta}\pi^{-n}|y-x|^{n\zeta}\,T(n,\zeta), \qquad (32)$$

where

$$T(n,\zeta)=\int_{[t,t+h]^n}\int_{\mathbb{R}^n}\prod_{j=1}^n|u_j|^\zeta\mathbb{E}[e^{i\sum_{j=1}^n u_jX_{t_j}}]d\overline{u}d\overline{t}.$$

In order to apply the LND property for the Gaussian process X_t, we do two transformations:

- We replace the integration over the domain $[t,t+h]^n$ by the integration over the subset $t<t_1<t_2...<t_n<t+h$.

- In the integral over the u's, we change the variable of integration by the following transformation

$$u_n = v_n, \quad u_j = v_j - v_{j+1}, \quad j = 1, ..., n-1.$$

We obtain

$$\begin{aligned} T(n,\zeta) &= n! \int_{t<t_1<t_2...<t_n<t+h} \int_{\mathbb{R}^n} \prod_{j=1}^{n-1} |v_j - v_{j+1}|^\zeta |v_n|^\zeta \mathbb{E}[e^{i\sum_{j=1}^n v_j(X_{t_j} - X_{t_{j-1}})}] d\overline{v} d\overline{t} \\ &= n! \int_{t<t_1<t_2...<t_n<t+h} \int_{\mathbb{R}^n} \prod_{j=1}^{n-1} |v_j - v_{j+1}|^\zeta |v_n|^\zeta \, e^{-\frac{1}{2} var(\sum_{j=1}^n v_j(X_{t_j} - X_{t_{j-1}}))} \, d\overline{v} d\overline{t}, \end{aligned} \tag{33}$$

where $t_0 = 0$. Now, since $|a-b|^\zeta \leq |a|^\zeta + |b|^\zeta$ for all $0 < \zeta < 1$, it follows that

$$\prod_{j=1}^{n-1} |v_j - v_{j+1}|^\zeta |v_n|^\zeta \leq \prod_{j=1}^{n-1} (|v_j|^\zeta + |v_{j+1}|^\zeta)|v_n|^\zeta. \tag{34}$$

Note that the last term in the right is at most equal to a finite sum of terms each of the form $\prod_{j=1}^n |v_j|^{\epsilon_j \zeta}$, where $\epsilon_j = 0, 1,$ or 2 and $\sum_{j=1}^n \epsilon_j = n$. Let us write for simplicity $\sigma^2(j) = \mathbb{E}(X_{t_j} - X_{t_{j-1}})^2$. Using (34) and the LND property of X_t, i.e. the second point in Theorem 3.3, the term $T(n,\zeta)$ in (33) is dominated by the sum over all possible choice of $(\epsilon_1, ..., \epsilon_n) \in \{0,1,2\}^n$ of the following terms

$$\int_{t<t_1<t_2...<t_m<t+h} \int_{\mathbb{R}^n} \prod_{j=1}^{n} |v_j|^{\epsilon_j \zeta} \exp\left(-\frac{C_n}{2} \sum_{j=1}^n v_j^2 \sigma^2(j)\right) d\overline{v} d\overline{t}, \tag{35}$$

where C_n is a positive constant and h is small enough such that $0 < h < \delta_n$, (δ_n and C_n are given by the LND property). Now, by the change of variable $x_j = \sigma(j) v_j$, the term (35) becomes

$$\int_{t<t_1<t_2...<t_m<t+h} \prod_{j=1}^{n} \sigma(j)^{-1-\zeta\epsilon_j} \int_{\mathbb{R}^n} \prod_{j=1}^{n} |x_j|^{\epsilon_j \zeta} \exp\left(-\frac{C_n}{2} \sum_{j=1}^n x_j^2\right) d\overline{x} d\overline{t}. \tag{36}$$

Using the second point in Lemma 3.1, we have $\sigma^2(j) = \mathbb{E}(X_{t_j} - X_{t_{j-1}})^2 \geq C(t_j - t_{j-1})^{1/2}$, where C is a positive constant. This implies that the integral in (36) is dominated by

$$\begin{aligned} &\int_{t<t_1<t_2...<t_m<t+h} \prod_{j=1}^{n} |t_j - t_{j-1}|^{\frac{-1-\zeta\epsilon_j}{4}} \int_{\mathbb{R}^n} \prod_{j=1}^{n} |x_j|^{\epsilon_j \zeta} \exp\left(-\frac{C_n}{2} \sum_{j=1}^n x_j^2\right) d\overline{x} d\overline{t} \\ &\qquad = C(n,\zeta) \int_{t<t_1<t_2...<t_m<t+h} \prod_{j=1}^{n} |t_j - t_{j-1}|^{\frac{-1-\zeta\epsilon_j}{4}} d\overline{t}. \end{aligned} \tag{37}$$

Now, return to Equation (32). Combining (33), (35) and (37) we obtain

$$\mathbb{E}[L(y,t+h)-L(x,t+h)-L(y,t)+L(x,t)]^n \\ \leq C(n,\zeta)|y-x|^{n\zeta}\int_{t<t_1<t_2\ldots<t_m<t+h}\prod_{j=1}^{n}|t_j-t_{j-1}|^{\frac{-1-\zeta\epsilon_j}{4}}d\bar{t}. \tag{38}$$

Remark that the integral in the right hand side of (38) is finite. Moreover, by using an elementary calculations, we have for any $n \geq 1$, $h > 0$ and $b_j < 1$

$$\int_{t<t_1<t_2\ldots<t_m<t+h}\prod_{j=1}^{n}|t_j-t_{j-1}|^{-b_j}d\bar{t} = h^{n-\sum_{j=1}^{n}b_j}\frac{\prod_{j=1}^{n}\Gamma(1-b_j)}{\Gamma(1+n-\sum_{j=1}^{n}b_j)}.$$

Finally, taking $b_j = \frac{1+\zeta\epsilon_j}{4}$, we get

$$\mathbb{E}[L(y,t+h)-L(x,t+h)-L(y,t)+L(x,t)]^n \leq C(n,\zeta)|y-x|^{n\zeta}h^{n(\frac{3}{4}-\frac{\zeta}{4})}. \tag{39}$$

□

Remark 9. If we take $d = 1$ and $H = \frac{1}{2}$ in [42, Eq. (12) and Eq. (13)], we remark that these last equations coincide with respectively Eq. (1) and Eq. (9). However, by comparing [42, Theorem 4] with Proposition 3.4, we observe that [42, Eq. (28)] is not optimal, because of the fact that

$$0 < \xi < \frac{1-H-\frac{d}{4}}{2H-\frac{d}{2}}$$

Hence, by taking $d = 1$ and $H = \frac{1}{2}$, we get that $0 < \xi < \frac{1}{2}$. However, in Eq. (30), we have $0 < \zeta < 1$.

We can deduce by classical arguments (cf. D. Geman-J. Horowitz [31, Theorem 26.1] or Berman [6, Theorem 8.1.]) the following regularity result for the local time of the solution $(u(t,x),\ t \in [0,T])$

Theorem 3.5. *For any $x \in \mathbb{R}$, the solution $(u(t,x),\ t \in [0,T])$ has almost surely a jointly continuous local time $(L(\xi,t),\ t \in [0,T],\ \xi \in \mathbb{R})$ which satisfies for all $\alpha < 3/4$,*

$$\sup_{\xi}|L(\xi,t+h)-L(\xi,t)| \leq \eta' h^{\alpha}, \tag{40}$$

for any $t, t+h \in [0,T]$ such that $|h| < \eta$, where η and η' are random variables a.s. positive and finite.

We also obtain by Proposition 3.4 together with a version of Kolmogorov's continuity theorem in Besov norms (see Boufoussi et al. [9, Lemma 2.1.]), that

Theorem 3.6. *For all $\lambda > 0$, $p > \frac{1}{\lambda}$ and $\xi \in \mathbb{R}$,*

$$\mathbb{P}\left(L(\xi,.) \in \mathcal{B}_p^{\omega_\lambda}\right) = 1,$$

where $\omega_\lambda(t) = t^{3/4}(\log(1/t))^\lambda$ *and* $L(\xi, .)$ *is the sample paths* $t \to L(\xi, t)$*,* $t \in [0, 1]$ *.*

Remark 10. (1) Taking λ small enough, Theorem 3.6 ensures a more accurate regularity result. Particularly, we deduce by injection (8) that $L(\xi, .)$ satisfies a.s. a $\beta-$Hölder condition for any $\beta < \frac{3}{4}$.

(2) The process $u(., x)$ satisfies (4) of Theorem 1.3 with $0 \le p < 3$. Then there is a version of the local time $L(\xi, t)$, which is differentiable with respect to the space variable, and a.s. $L^{(1)}(\xi, t) = \partial_\xi L(\xi, t) \in L^2(\mathbb{R}, d\xi)$.
It is easy to verify that $L^{(1)}$ satisfies (32) with $T(n, \zeta + 1)$ instead of $T(n, \zeta)$. Following the same arguments as in Proposition 3.4, the finiteness of the integral in (38) (with $\zeta + 1$ in place of ζ) requires that $\zeta < 1/2$. Furthermore, we obtain that for all $x, y \in \mathbb{R}$, $t, t + h \in (0, T]$ and for any positive integer n, there exists $C_n > 0$ such that

$$\mathbb{E}[L^{(1)}(y, t+h) - L^{(1)}(x, t+h) - L^{(1)}(y, t) + L^{(1)}(x, t)]^n \le C_n |x - y|^{\zeta n} |h|^{n(1/2 - \zeta/4)},$$

where $0 < \zeta < 1/2$.

Consequently, we have the following regularity result

Theorem 3.7. *There is a jointly continuous version of* $(L^{(1)}(\xi, t), t \in [0, T], \xi \in \mathbb{R})$ *satisfying: For all compact* $U \subset \mathbb{R}$ *and for any* $\alpha < 1/2$

$$\sup_{x,y \in U, x \neq y} \frac{|L^{(1)}(x, t) - L^{(1)}(y, t)|}{|x - y|^\alpha} < \infty, \quad a.s.$$

3.1.2. Hausdorff dimension of level sets

Let $x \in \mathbb{R}$ be fixed. We define, for any $\xi \in \mathbb{R}$, the ξ-level set of $(u(t, x)\,,\ t \in [0, T])$ by

$$M^x(\xi) = \{t \in [0, T] \ : \ u(t, x) = \xi\}.$$

Our goal is to determine the Hausdorff dimension $\dim_H(M^x(u(t_0, x)))$ of $M^x(u(t_0, x))$. We can refer to [29, p. 27] for an introduction to Hausdorff measure and dimension. One of the crucial applications of the joint continuity of the local time is to extend $L(\xi, .)$ as a finite measure supported on the level set $M^x(\xi)$ see [1, Theorem 8.6.1]. To find a lower bound of the Hausdorff dimension of the level sets we need first the following Frostman's Lemma cf. [22, Lemma 6.10.]

Lemma 3.8. *Let* E *be a Borel set of* $\mathbb{R}$*.* $\mathcal{H}^s(E) > 0$ *if and only if there exists a finite Borel measure* μ *supported on* E *such that* $\mu(E) > 0$ *and a positive constant* c *such that*

$$\mu((y - r, y + r)) \le c r^s,$$

for all $y \in \mathbb{R}$ *and* $r > 0$*.*

Lemma 3.9. *For all* $x \in \mathbb{R}$*, we have almost surely and for almost every* $t_0 \in [0, T]$

$$\dim_H(M^x(u(t_0, x))) \ge \frac{3}{4}.$$

Proof. Let $x \in \mathbb{R}$ be fixed, we have by [5, Lemma 1.1.] that for almost every t_0

$$L(u(t_0, x), T) > 0.$$

We know that $L(u(t_0, x), .)$ is a measure supported on $M^x(u(t_0, x))$, and (40) entails that $L(u(t_0, x), .)$ satisfies a.s. a Hölder condition of any order smaller than $\frac{3}{4}$. So by Lemma 3.8 we have almost surely and for almost every t_0

$$\dim_H(M^x(u(t_0, x))) \geq \frac{3}{4}.$$

□

Lemma 3.10. *For all $x \in \mathbb{R}$, we have almost surely and for all $t_0 \in [0, T]$*

$$\dim_H(M^x(u(t_0, x))) \leq \frac{3}{4}.$$

Proof. We know that $u(., x)$ satisfies a.s. a Hölder condition of any order smaller than $\frac{1}{4}$. By Theorem 3.5 its local time is jointly continuous. The result then follows by [1, Theorem 8.7.3.]. □

Combining Lemma 3.9 and Lemma 3.10 we obtain

Corollary 3.11. *For all $x \in \mathbb{R}$, we have almost surely and for almost every t_0*

$$\dim_H(M^x(u(t_0, x))) = \frac{3}{4}.$$

3.2. Local time of the process $x \to u(t, x)$

3.2.1. Existence of the local time

Let $[a, b] \subset \mathbb{R}$, we will prove the existence of the local time of the process $(u(t, x)\,,\ x \in [a, b])$ where $t > 0$ is fixed. First we need the following result

Lemma 3.12. *For fixed $t > 0$, and for any $x, y \in [a, b]$, there exists a constant $c_t > 0$ such that*

$$c_t|x - y| \leq \mathbb{E}(u(t, x) - u(t, y))^2 \leq \frac{|x - y|}{2\pi}.$$

Remark 11. The above lemma is well known; cf. Lemmas 4.2 of [23]. For the sake of completeness, we give another proof.

Proof. Let $x, y \in [a, b]$ such that $x > y$, the change of variable $r = t - s$ together with

Parseval's identity give

$$\begin{aligned}\mathbb{E}(u(t,x)-u(t,y))^2 &= \int_0^t \int_{\mathbb{R}} (G(r,x-z)-G(r,y-z))^2 dz dr \\ &= \frac{1}{2\pi}\int_0^t \int_{\mathbb{R}} \left| e^{ixu}\exp(-\frac{ru^2}{2}) - e^{iyu}\exp(-\frac{ru^2}{2})\right|^2 du dr \\ &= \frac{1}{2\pi}\int_0^t \int_{\mathbb{R}} \exp(-ru^2)\left|e^{i(x-y)u}-1\right|^2 du dr.\end{aligned}$$

Again by the transformations $v = u(x-y)$ and $\tau = \frac{r}{(x-y)^2}$, we get

$$\mathbb{E}(u(t,x)-u(t,y))^2 = \frac{x-y}{2\pi}\int_0^{\frac{t}{(x-y)^2}} \int_{\mathbb{R}} \exp(-\tau v^2)\left|e^{iv}-1\right|^2 dv d\tau.$$

Using Fubini, we obtain

$$\begin{aligned}\mathbb{E}(u(t,x)-u(t,y))^2 &= \frac{x-y}{2\pi}\int_{\mathbb{R}}\int_0^{\frac{t}{(x-y)^2}} \exp(-\tau v^2)\left|e^{iv}-1\right|^2 d\tau dv \\ &= \frac{x-y}{2\pi}\int_{\mathbb{R}} (1-\exp(-\frac{t}{(x-y)^2}v^2))\frac{|e^{iv}-1|^2}{v^2}dv \\ &= \frac{x-y}{2\pi}\left\{\int_{\mathbb{R}} \frac{|e^{iv}-1|^2}{v^2}dv - \int_{\mathbb{R}} \exp(-\frac{t}{(x-y)^2}v^2)\frac{|e^{iv}-1|^2}{v^2}dv\right\} \\ &= \frac{x-y}{2\pi}\left\{1 - \int_{\mathbb{R}} \exp(-\frac{t}{(x-y)^2}v^2)\frac{|e^{iv}-1|^2}{v^2}dv\right\},\end{aligned}$$

where in the last line, Parseval's identity gives $\int_{\mathbb{R}} \frac{|e^{iv}-1|^2}{v^2}dv = \int_{\mathbb{R}} \chi_{[0,1]}(v)dv = 1$. So, on one hand, it is clear that

$$\mathbb{E}(u(t,x)-u(t,y))^2 \leq \frac{|x-y|}{2\pi}.$$

On the other hand, to find a lower bound for $\mathbb{E}(u(t,x)-u(t,y))^2$, we need to get a constant $0 \leq C_t < 1$ such that

$$C_t \geq \int_{\mathbb{R}} \exp(-\lambda v^2)\frac{|e^{iv}-1|^2}{v^2}dv := A\,,$$

where we have used the notation $\lambda = \frac{t}{(x-y)^2}$.
Now, denote by $f(v) = \frac{1}{\sqrt{2\pi\lambda}}e^{-v^2/2\lambda}$ and $g(v) = \chi_{[0,1]}(v)$. It follows by Parseval's identity,

$$\begin{aligned}A = \int_{\mathbb{R}} \left|\exp(-\frac{\lambda v^2}{2})\frac{e^{iv}-1}{iv}\right|^2 dv &= \int_{\mathbb{R}} |\widehat{f*g}(v)|^2 dv \\ &= \int_{\mathbb{R}} |f*g(v)|^2 dv.\end{aligned}$$

Then

$$A = \int_{\mathbb{R}} \left\{ \int_{[0,1]^2} \frac{1}{2\pi\lambda} e^{-(v-z_1)^2/2\lambda} e^{-(v-z_2)^2/2\lambda} dz_1 dz_2 \right\} dv.$$

By Fubini we have

$$\begin{aligned}
A &= \int_{[0,1]^2} \frac{1}{\sqrt{2\pi\lambda}} \left\{ \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\lambda}} e^{-(v-z_1)^2/2\lambda} e^{-(v-z_2)^2/2\lambda} dv \right\} dz_1 dz_2 \\
&= \int_{[0,1]^2} \frac{e^{-(z_1-z_2)^2/4\lambda}}{\sqrt{2\pi\lambda}} \left\{ \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\lambda}} \exp(-\frac{1}{\lambda}(v - \tfrac{z_1+z_2}{2})^2) dv \right\} dz_1 dz_2 \\
&= \int_{[0,1]^2} \frac{1}{\sqrt{4\pi\lambda}} \exp(-\frac{(z_1-z_2)^2}{4\lambda}) dz_1 dz_2 \\
&= \int_{[0,1]} \left\{ \int_{[0,1]} \frac{1}{\sqrt{2\pi(2\lambda)}} \exp(-\frac{(z_1-z_2)^2}{2(2\lambda)}) dz_2 \right\} dz_1 \\
&= \int_{[0,1]} \mathbb{P}[0 \le \sqrt{2\lambda} N + z_1 \le 1] dz_1,
\end{aligned}$$

where N is a standard Normal random variable. Then

$$\begin{aligned}
A &= \mathbb{E}\left[\int_{[0,1]} \chi_{[-\sqrt{2\lambda}N, 1-\sqrt{2\lambda}N]}(z_1) dz_1 \right] \\
&= \mathbb{E}\left[(1-\sqrt{2\lambda}N)\chi_{[0,1]}(\sqrt{2\lambda}N) + (1+\sqrt{2\lambda}N)\chi_{[-1,0]}(\sqrt{2\lambda}N) \right] \\
&= 2\mathbb{E}\left[(1-\sqrt{2\lambda}N)\chi_{[0,1]}(\sqrt{2\lambda}N) \right].
\end{aligned}$$

The last equality follows by the symmetry of the distribution of N. Now replace λ by its value, since $x, y \in [a,b]$ we obtain

$$\begin{aligned}
A = 2\mathbb{E}\left[(1 - \frac{\sqrt{2t}}{x-y} N)\chi_{[0, \frac{x-y}{\sqrt{2t}}]}(N) \right] &\le 2\mathbb{E}\left[(1 - \frac{\sqrt{2t}}{b-a} N)\chi_{[0, \frac{b-a}{\sqrt{2t}}]}(N) \right] \\
&\le 2\mathbb{P}[0 \le N \le \frac{\sqrt{2t}}{b-a}] < 1.
\end{aligned}$$

We then get $0 \le A < 1$, and this finishes the proof of the lemma 3.12. □

Consequently, we have

Proposition 3.13. *For all $t > 0$ and $0 \le p < 1$, we have*

$$\int_a^b \int_a^b [\mathbb{E}(u(t,x) - u(t,y))^2]^{-(p+1)/2} dx dy < \infty.$$

Proposition 3.14. *For all $t > 0$, there exists a square integrable version of the local time of $(u(t,x),\ x \in [a,b])$. We denote this version by $(L(\xi, y),\ y \in [a,b],\ \xi \in \mathbb{R})$, where $L(\xi, y) := L(\xi, [a,y])$.*

Proof. It is a consequence of Proposition 3.13, together with Theorem 1.3. □

3.2.2. Regularity of the local time

In order to study the regularity of the local time, we need to recall the fundamental tool for that, the strong local nondeterminism concept (SLND). This notion was introduced by Cuzick and DuPreez in [20] (see also [55]), and used by many authors to investigate the law of iterated logarithm, Chung's law of the iterated logarithm, modulus of continuity for various Gaussian processes.

Definition 3.15. Let $\{X_t,\ t \in I\}$ be a gaussian stochastic process with $0 < \mathbb{E}(X_t^2) < \infty$ for any $t \in J$ where J is a subinterval of I. Let ϕ be a function such that $\phi(0) = 0$ and $\phi(r) > 0$ for all $r > 0$. Then X is SLND on J if there exist constants $K > 0$ and $r_0 > 0$ such that for all $t \in J$ and all $0 < r \le \min\{|t|, r_0\}$,

$$\mathrm{Var}(X_t | X_s \ :\ s \in J,\ r \le |s-t| \le r_0) \ge K\phi(r).$$

Theorem 3.16. *For all $t > 0$, there exists a positive constant $K = K(t, a, b)$, such that for all $0 < r \le |b - a|$, we have*

$$\mathrm{Var}\left(u(t,y)|u(t,x) : x \in [a,b],\ r \le |y-x| \le |b-a|\right) \ge Kr. \tag{41}$$

Proof. It is enough to show that there exists a constant $K > 0$ such that,

$$\mathbb{E}\left(u(t,y) - \sum_{k=1}^{n} a_k u(t,x_k)\right)^2 \ge Kr, \tag{42}$$

for all integers $n \ge 1$, $(a_k)_1^n \in \mathbb{R}$ and $(x_k)_1^n \in [a,b] : r \le |y - x_k| \le |b-a|$, $\forall k \le n$.
Parseval's identity implies

$$\begin{aligned}
&\mathbb{E}\left(u(t,y) - \sum_{k=1}^{n} a_k u(t,x_k)\right)^2 \\
&= \int_0^t \int_{\mathbb{R}} \left(G(t-s, y-z) - \sum_{k=1}^{n} a_k G(t-s, x_k - z)\right)^2 dzds \\
&= \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}} \left|\exp(iyu) - \sum_{k=1}^{n} a_k \exp(ix_k u)\right|^2 \exp(-su^2) duds \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \left|\exp(iyu) - \sum_{k=1}^{n} a_k \exp(ix_k u)\right|^2 \frac{1-\exp(-tu^2)}{u^2} du := Q(r).
\end{aligned} \tag{43}$$

So we just need to prove that $Q(r) \ge Kr$.
Let $\varphi : \mathbb{R} \to [0,1]$ be a function in $C^\infty(\mathbb{R})$ such that $\varphi(0) = 1$ and $supp(\varphi) \subset]0,1[$. Denote by $\hat{\varphi}$ the Fourier transform of φ. Then $\hat{\varphi} \in C^\infty(\mathbb{R})$ and $\hat{\varphi}(u)$ decays rapidly as $|u| \to \infty$. Set

$$\varphi_r(\theta) = r^{-1}\varphi(r^{-1}\theta).$$

By the inversion theorem we have

$$\varphi_r(\theta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iu\theta} \widehat{\varphi}(ru) du. \tag{44}$$

Since $r \leq |y - x_i|$ and $supp(\varphi) \subset]0, 1[$, we have $\varphi_r(y - x_i) = 0$ for any $k = 1, ..., n$. This and (44) imply that

$$\begin{aligned} B &:= \int_{\mathbb{R}} (\exp(iyu) - \sum_{k=1}^{n} a_k \exp(ix_k u)) \exp(-iyu) \widehat{\varphi}(ru) du \\ &= 2\pi(\varphi_r(0) - \sum_{k=1}^{n} a_k \varphi_r(y - x_k)) = 2\pi r^{-1}. \end{aligned} \tag{45}$$

On the other hand, by (43) and Hölder inequality, we obtain

$$\begin{aligned} B^2 &\leq \int_{\mathbb{R}} \left| \exp(iyu) - \sum_{k=1}^{n} a_k \exp(ix_k u) \right|^2 \frac{1 - \exp(-tu^2)}{u^2} du \\ &\quad \times \int_{\mathbb{R}} \frac{u^2}{1 - \exp(-tu^2)} |\widehat{\varphi}(ru)|^2 du \\ &= \mathbb{E} \left(u(t, y) - \sum_{k=1}^{n} a_k u(t, x_k) \right)^2 \times \int_{\mathbb{R}} \frac{u^2}{1 - \exp(-tu^2)} |\widehat{\varphi}(ru)|^2 du \\ &\leq \mathbb{E} \left(u(t, y) - \sum_{k=1}^{n} a_k u(t, x_k) \right)^2 \frac{1}{r^3} \int_{\mathbb{R}} \frac{v^2}{1 - \exp(-\frac{tv^2}{|b-a|^2})} |\widehat{\varphi}(v)|^2 dv, \end{aligned}$$

where last inequality is justified by the change of variable $v = ru$ and $0 < r \leq |b - a|$. So by (45) we get

$$4\pi^2 \frac{1}{r^2} \leq \mathbb{E} \left(u(t, y) - \sum_{k=1}^{n} a_k u(t, x_k) \right)^2 \frac{1}{r^3} K,$$

where

$$K = \int_{\mathbb{R}} \frac{v^2}{1 - \exp(-\frac{tv^2}{|b-a|^2})} |\widehat{\varphi}(v)|^2 dv.$$

Finally, (42) holds. This finishes the proof of Theorem 3.16. □

Lemma 3.17. *Let $y, y + h \in [a, b]$. For any even positive integer n, we have*

$$\mathbb{E}[L(\xi, y + h) - L(\xi, y)]^n \leq C_n |h|^{n/2}, \tag{46}$$

where C_n is a positive constant.

Proof. For simplicity we will deal with $h > 0$ such that $y + h \in [a, b]$. The other case uses the same calculation. Let $I = [y, y + h]$, then following [31] or [28], we have

$$\begin{aligned}\mathbb{E}[L(\xi, I)^n] &= (2\pi)^{-n} \int_{I^n} \int_{\mathbb{R}^n} e^{-i<\overline{u},\overline{\xi}>} \mathbb{E}\left[e^{i\sum_{k=1}^n u_k u(t,x_k)}\right] d\overline{u} d\overline{x} \\ &= (2\pi)^{-n} \int_{I^n} \int_{\mathbb{R}^n} e^{-i<\overline{u},\overline{\xi}>} e^{-\frac{1}{2}\operatorname{Var}(\sum_{k=1}^n u_k u(t,x_k))} d\overline{u} d\overline{x},\end{aligned}$$

where $\overline{\xi} = (\xi, \cdots, \xi)$ and $\overline{u} = (u_1, \cdots, u_n)$, hence

$$\mathbb{E}[L(\xi, I)^n] \le (2\pi)^{-n} \int_{I^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\operatorname{Var}(\sum_{k=1}^n u_k u(t,x_k))} d\overline{u} d\overline{x}. \tag{47}$$

On the other hand, for distinct $x_1, x_2, \cdots, x_n$, the matrix $\operatorname{Cov}(u(t, x_1), u(t, x_2), \cdots, u(t, x_n))$ is invertible. Then the following function is a gaussian density

$$\frac{[\det \operatorname{Cov}(u(t, x_1), u(t, x_2), \cdots, u(t, x_n))]^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2}\overline{u} \operatorname{Cov}(u(t,x_1),u(t,x_2),\cdots,u(t,x_n))\overline{u}'}, \tag{48}$$

where $\overline{u}'$ denotes the transpose of $\overline{u}$. Therefore

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\operatorname{Var}(\sum_{k=1}^n u_k u(t,x_k))} d\overline{u} = \frac{(2\pi)^{n/2}}{[\det \operatorname{Cov}(u(t, x_1), u(t, x_2), \cdots, u(t, x_n))]^{1/2}}. \tag{49}$$

Combining (47) and (49), we get

$$\mathbb{E}[L(\xi, I)^n] \le (2\pi)^{-n/2} \int_{I^n} \frac{1}{[\det \operatorname{Cov}(u(t, x_1), u(t, x_2), \cdots, u(t, x_n))]^{1/2}} d\overline{x}. \tag{50}$$

It follows from (2.8) in [6] that

$$\begin{aligned}&\det \operatorname{Cov}(u(t, x_1), u(t, x_2), \cdots, u(t, x_n)) \\ &\quad = \operatorname{Var}(u(t, x_1)) \prod_{j=2}^n \operatorname{Var}(u(t, x_j) | u(t, x_1), \cdots, u(t, x_{j-1})).\end{aligned} \tag{51}$$

(51) together with (41) imply

$$\det \operatorname{Cov}(u(t, x_1), u(t, x_2), \cdots, u(t, x_n)) \ge K^n |x_1 - a| \prod_{j=2}^n \min_{1 \le i < j} |x_j - x_i|. \tag{52}$$

By using (52) in (50) we get

$$\begin{aligned}\mathbb{E}[L(\xi, I)^n] &\le C^n \int_{I^n} \frac{1}{|x_1 - a|^{1/2}} \prod_{j=2}^n \frac{1}{\min_{1 \le i < j} |x_j - x_i|^{1/2}} d\overline{x} \\ &\le C_n h^{n/2},\end{aligned} \tag{53}$$

where the last inequality is obtained by integrating in the order $dx_n, dx_{n-1}, \cdots, dx_1$ and with the help of some elementary arguments. This finishes the proof of the lemma 3.17. $\square$

Lemma 3.18. *For all $\xi, \xi+k \in \mathbb{R}$, $y, y+h \in [a,b]$ and for all even positive integer n, there exists a constant $C_n > 0$ such that*

$$\mathbb{E}[L(\xi+k, y+h) - L(\xi, y+h) - L(\xi+k, y) + L(\xi, y)]^n \leq C_n |k|^{n\delta} |h|^{n(1/2-\delta/2)},$$

where $0 < \delta < \frac{1}{2}$.

Proof. The proof uses the same techniques as those of Proposition 3.4. $\square$

We can deduce by classical arguments (cf. Berman [6, Theorem 8.1.] or Geman-J. Horowitz [31, Theorem 26.1]) the following regularity result on the local time of the process $(u(t,x),\ x \in [a,b])$

Theorem 3.19. *For any fixed $t > 0$, the process $(u(t,x),\ x \in [a,b])$ has almost surely, a jointly continuous local time $(L(\xi, y),\ \xi \in \mathbb{R}, y \in [a,b])$. It satisfies a.s. a γ-Hölder condition in y, uniformly in ξ, for every $\gamma < \frac{1}{2}$: there exist random variables η and η' which are almost surely positive and finite such that*

$$\sup_{\xi} |L(\xi, y+h) - L(\xi, y)| \leq \eta' |h|^{\gamma}, \tag{54}$$

for all y, $y+h \in [a,b]$ and all $|h| < \eta$.

We also have, by [9, Lemma 2.1.], the following Besov regularity of the local time $L(\xi, y)$ in the space variable y

Theorem 3.20. *For all $\lambda > 0$ and $p > \frac{1}{\lambda}$,*

$$\mathbb{P}\left(L(\xi, .) \in \mathcal{B}_p^{\omega_\lambda}\right) = 1,$$

where $\omega_\lambda(t) = t^{1/2}(\log(1/t))^{\lambda}$ and $L(\xi, .)$ is the sample paths $y \to L(\xi, y)$, $y \in [0,1]$.

For fixed $t > 0$, let $M_t(\xi) = \{x \in [a,b] \ : \ u(t,x) = \xi\}$ be the ξ-level set of the process $(u(t,x)\,,\, x \in [a,b])$. Proceeding in the same way as for $(u(t,x)\,,\, t \in [0,T])$, we have

Corollary 3.21. *For all $t > 0$, we have for almost every x_0*

$$\dim_H(M_t(u(t, x_0))) = \frac{1}{2} \quad a.s.$$

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