Free entry in successive oligopolies*

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Abstract

In this paper, we propose an extension of Spengler's (1950) analysis to successive oligopolies, to study the effects of entry in the downstream and upstream markets. Free entry is analysed using replica economies à la Debreu and Scarf (1963). We find that free entry may have different effects in the upstream and in the downtream market. Namely, the usual convergence of the price to the correspondoing marginal cost only occurs in the downstream market.

Keywords: successive oligopolies, free entry

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1 Introduction

In this paper, we analyze the effects of entry in a setting where there is an interaction between upstream and downstream firms, i.e. successive markets. In a seminal paper, Spengler (1950) introduces the basic idea of successive markets, but he analyzes the simplest possible case to capture the interlink between the downstream and upstream markets: namely, the case of bilateral monopoly. And, he ignores the phenomenon of free entry in the two markets. In fact, his interests is mainly invested in the so called double marginalization phenomenon and the resulting effects of vertical collusive agreements between the upstream and downstream monopolist.

In order to analyse entry in successive markets, we consider here more complex structures in both markets than a simply bilateral monopoly. In particular, the concept of industry equilibrium is extended to frameworks embodying an arbitrary number of firms both in the upstream and downstream markets. This extension has been already considered in the existing literature after Spengler, but with the exclusive purpose of analysing vertical collusive agreements, as in Salop and Scheffman (1987), Salinger (1988), Ordover, Salop and Scheffman (1990), and Gaudet and Van Long (1996) among others.

In the present paper, we neglect the analysis of vertical agreements and double marginalization but rather examine the effects of free entry when there are interactions between upstream and downstream markets. Upstream firms select non cooperatively the quantities of their output, but the output of the upstream firms serves as input in the production of the final good in the downstream market. Hence, the link between the two markets follows from the fact that the downstream firms'unit cost appears as the unit revenue for the upstream ones : the price paid for a unit of input for the firms in the former constitutes the unit receipt for the firms in the latter. This gives rise to two games. In the upstream game, input firms declare the amount of input they supply; in the downstream game, downstream firms select the amount of input to use in the production of the output. Thus, ultimately they select the level of the final good to supply to the final consumers. The input price in equilibrium makes its demand and supply equal.

The main finding of the paper can be summarized as follows. Free entry of firms in both markets does not always entail the usual convergence for the input price to adjust to its marginal cost. While, the convergence towards the marginal cost is always obtained in the downstream market. Our result is in line with Cournot (1838) and Gabszewicz and Vial (1972), because the quantities of input corresponding to Cournot equilibria in our model converge to the quantity of input corresponding to the competitive equilibrium. The novelty brought by our analysis is that the corresponding sequence of Cournot equilibrium prices does not converge to the input marginal cost. The reason is that in successive markets the marginal cost for the firms who produce the final good is not the fixed marginal cost to produce the input, as in Cournot (1838). Here, their marginal cost is determined by the input price at the industry equilibrium, which is a conseguence of the market power in both the downstream and the upstream markets. Free entry *with the same speed* of firms in both markets affects market power in each market differently, therefore, the input demand and the input supply are affected differently by the entry of new firms. As a conseguence, the input price, that clears the input market, does not necessarly lead to an input price that is equal to the technological marginal cost to produce the input. Nevertheless, this input price does not preclude the limit economy to be in a Pareto optimal state simultaneously in both markets. This discrepancy between marginal cost and input price may disappear when the upstream market is replicated infinitely faster than the downstream one. In other words, the market power of upstream firms who fix the input price should be diluted much faster than the downstream firms' one in order to force the competitive input price! When free entry takes place only in the upstream market, the input demand is determined by a given fixed number of firms. While, the input supply is a conseguence of the diluted market power of upstream firms. As a result, the input price does converge to its marginal cost.

The rest of the paper is organized as follows. In the next section we present the model, assuming a given number of firms in the upstream and downstream markets. In Section 3, explores the industry equilibria and the effects of free entry. Section 4 concludes.

2 The model

Consider two successive markets, the downstream and upstream market, with n *downstream* firms $i, i = 1, ...n$, in the first, producing the final good, and m *upstream* firms j, $j = 1, ...m$, in the second, producing and selling the input. The n downstream firms face a demand function $\pi(Q)$ in the downstream market, with Q denoting the aggregate quantity of the final good. Firm i owns technology $f_i(z)$ to produce the final good, with z_i denoting the quantity of the sole input used in the production process and bought by firm i in the upstream market. The m *upstream* firms each produce the input z at a total cost $C_j(z), j = 1, .., m$. We assume that this situation gives rise to two games. The players in the first game, *the downstream game*, are the n downstream firms who choose their input strategies $z_i(p)$ in order to reach a Cournot equilibrium in the downstream market, while the players in the second, *the upstream game*, are the upstream firms with input strategies $s_j(p)$ conditional on the input price p. The two markets are linked to each other as follows. In the downstream game, firms select strategically the input levels $z_i(p)$ which determines their individual output level $f(z_i(p))$ of input *via* the production function f. Consequently, the downstream firms while behaving strategically in the final good market, are assumed to be price takers in the input market. Faced with the input demand schedule $\Sigma_{i=1}^n z_i(p)$ resulting from aggregating individual demands, firms in the upstream game select non cooperatively the quantities of input. The choice of input quantities, given the input demand $\Sigma_{i=1}^n z_i(p)$, determines the input market price p^* that satisfies

$$\Sigma_{i=1}^n z_i(p^*) = \Sigma_{j=1}^m s_j(p^*).$$

The payoff in the downstream game for the i_{th} firm at the vector of strategies $(z_i(p), z_{-i}(p))$ obtains as

$$\Pi_i(z_i, z_{-i}; p) = \pi\left(f_i(z_i) + \Sigma_{k\neq i} f_k(z_k)\right) f_i(z_i)) - pz_i.$$

Given these payoffs, the best reply, $z_i(z_{-i}; p)$ of firm i in the downstream game, obtains as a solution (whenever it exists) to the problem

$$\underset{z_i}{Max}\ \Pi_i(z_i, z_{-i}; p).$$

A Nash equilibrium in the downstream game (whenever it exists) writes as an input vector $(z_1^*(p)), ..., z_n^*(p))$, where $z_i^*(p)$ solves

$$\underset{z_i}{Max}\ \Pi_i(z_i, z_{-i}^*(p)); p)$$

for all $i, i = 1, .., n$.

In the upstream game, firms select their selling strategies $s_j(p), j = 1, .., m$. Assuming a Cournot equilibrium in the downstream game, they face a total demand $\Sigma_{i=1}^n z_i^*(p)$ of input. Accordingly, from the market clearing condition $\Sigma_{i=1}^n z_i(p^*) = \Sigma_{j=1}^m s_j(p^*)$, we obtain the inverse input demand function $p(\Sigma_{j=1}^m s_j)$ and thus, write the payoff function $\Gamma_j(s_j, s_{-j})$ of firm j in the upstream game as

$$\Gamma_j(s_j, s_{-j}) = p(\Sigma_{j=1}^m s_j)s_j - C_j(s_j)$$

whenever it is defined for all admissible values of p. Denote by $(s_1^*(p^*), ..., s_m^*(p^*))$ the vector of input quantities supplied at the Nash equilibrium in the upstream game (whenever it exists). Thus, finally we obtain

Definition *An industry equilibrium is a (m+n)-tuple vector* $(z_1^*(p^*), ..., z_n^*(p^*)); s_1^*(p^*), ..., s_m^*(p^*))$ *and an input price* p^* *such that (i)* $(z_1^*(p^*), ..., z_n^*(p^*))$ *is a Nash equilibrium in the downstream game (ii)* $(s_1^*(p^*), ..., s_m^*(p^*))$ *is a Nash equilibrium in the upstream one, and (iii)* p^* *satisfies* $\Sigma_{i=1}^n z_i^*(p^*) = \Sigma_{j=1}^m s_j^*(p^*)$.

An industry equilibrium is a situation in which both the downstream and upstream markets exhibit Cournot equilibria, and where the quantity of input demanded at equilibrium in the first market exactly balances the quantity supplied in the second.

3 Exploring industry equilibria

It is difficult to analyze industry equilibria at the full level of generality. This is why we try to get some insights into the effects of entry on industry equilibria by considering explicit functions. Downstream firms share the same technology $f(z_i)$ to produce the final good, namely

$$f(z_i) = \sqrt{z_i}.$$

Moreover, we assume that the n *downstream* firms face a linear demand $\pi(Q) = 1 - Q$ in the downstream market. We assume a linear demand function in the downstream market, as usually in the literature on successive markets (see for instance Salinger (1988) and Gaudet and Van Long (1996)). As for the m upstream firms, each produces the input z using the same linear technology characterized by the same linear total cost $C_j(s_j) = \beta s_j$, $j = 1, .., m$. As in the general formulation above, we assume that this situation gives rise to two games. The players in the first game are the n downstream firms with strategies $z_i(p)$, while the players in the second are the m upstream firms with input strategies $s_j(p)$.

The profits of the i_{th} downstream firm at the vector of strategies (z_i, z_{-i}) obtains as

$$\Pi_i(z_i, z_{-i}) = (1 - \sqrt{z_i} - \Sigma_{k \neq i}\sqrt{z_k})\sqrt{z_i} - pz_i.$$

From profit maximization with respect to z_i, we get the demand function for the input

$$\Sigma_{i=1}^{n} z_i^*(p) = nz^*(p) = \frac{n}{(n + 2p + 1)^2};\ i = 1...n.$$

At a given n-tuple $(s_1(p), ..., s_j(p), ..s_m(p))$ of input strategies chosen by the upstream firms in the upstream game, the input price clearing the upstream market must satisfy

$$nz^*(p) = \frac{n}{(n + 2p + 1)^2} = \Sigma_{j=1}^{m} s_j(p)$$

so that we get the inverse input demand $p(\Sigma_{j=1}^{m} s_j)$ as

$$p(\Sigma_{j=1}^{m} s_j) = \sqrt{\frac{n}{4\Sigma_{j=1}^{m} s_j}} - \frac{n+1}{2}. \tag{1}$$

It follows that substituting (2) into (1), the payoff function of the upstream firm j in the upstream game is

$$\Gamma_j(s_j, s_{-j}) = (\sqrt{\frac{n}{4\Sigma_{j=1}^{m} s_j}} - \frac{n+1}{2})s_j - \beta s_j,$$

Notice that the profit function $\Gamma_j(s_j, s_{-j})$ is concave in the input quantity so that we can use the first order necessary and sufficient conditions to characterize an equilibrium. Accordingly, at the symmetric Nash equilibrium of the upstream game, we obtain

$$s^*(m, n) = \frac{n\,(2m - 1)^2}{4m^3\,(2\beta + 1 + n)^2}.$$

Hence the profit $\Gamma_j(m, n)$ of an upstream firm at the symmetric equilibrium of the upstream game obtains as

$$\Gamma_j(m, n) = \frac{n(2m - 1)}{8\,(n + 1)\,m^3}.$$

Finally, the equilibrium price $p^*(m,n)$ in the input market obtains as

$$p^*(m,n) = \frac{n+1+4m\beta}{2(2m-1)}.$$

Consequently, substituting this equilibrium price into the equilibrium quantities z^* of input bought by each downstream firm, we get

$$z^*(m,n) = \frac{(2m-1)^2}{4m^2(2\beta+n+1)^2}$$

so that, the optimal final good quantity obtains as

$$f(z_i^*(m,n)) = f(z^*(m,n)) = \frac{2m-1}{2m(2\beta+n+1)}.$$

Therefore, the resulting final good price $\pi^*(m,n)$ in the downstream market obtains as

$$\pi^*(m,n) = 1 - \frac{n(2m-1)}{2m(2\beta+n+1)}. \tag{2}$$

The profit $\Pi_i(m,n)$ of a downstream firm at equilibrium in the corresponding game is thus equal to

$$\Pi_i(m,n) = \frac{(4m\beta+4m+n-1)(2m-1)}{8m^2(2\beta+n+1)^2}.$$

Notice that the profit of a downstream firm can decrease as entry of new firms takes place in the upstream market, $\frac{\partial \Pi_i(m,n)}{\partial m} \lesseqgtr 0$. Conventional wisdom suggests that with more firms in the upstream market, input price must decline, leading to a decline in costs of final good. Consequently, per-firm profit in the downstream market should increase. But in successive oligopolies, an increase of the number of upstream firms m, has an indirect strategic effect on revenue of each downstream firm and a direct effect on the input cost. Thus, the overall effect on the downstream profit can be of either sign.

3.1 Free entry

In this section, we examine the effects of entry on industry equilibria in successive markets. We choose to model entry by replicating r-times the *basic economy*, as in Debreu and Scarf (1963). In the r-th replica, downstream market demand is given by $r(1-Q)$ and there are rn downstream and rm upstream firms. Notice that, in the r_{th}-replica, the prices at which demand is equal to supply both in the downstream and upstream markets, do not depend on the number r, but depend only on m and n. Indeed, at the symmetric equilibrium in the upstream market, the input quantities supplied by the m upstream firms have to be multiplied by r in the r_{th}-replica; similarly for the quantities demanded by the n downstream firms in the downstream market. Consequently, the equality of supply and demand in the upstream market eliminates the $r-$ factor in each side of the equality. A similar reasoning applies for the symmetric price equilibrium in the downstream market. It follows that the study of the behavior of the upstream and downstream markets when the number of replications increases is equivalent to the study of the limit equilibrium prices and quantities obtained in the previous section when the number of firms is rn and rm, instead of n and m, in each market, respectively.

We consider successively the following situations.

3.1.1 Perfect competition

We compute

$$\lim_{r\to\infty} \pi^*(rm, rn) = 0$$

and

$$\lim_{r\to\infty} p^*(rm, rn) = \frac{1}{4}\frac{n}{m} + \beta.$$

Furthermore we get

$$\lim_{r\to\infty} f(z^*(rm, rn)) = 0.$$

Therefore,

Proposition 1 *Free entry in both markets does not make the input prices, corresponding to the industry equilibria, to converge to the upstream firms' marginal cost. However, the sequence of the corresponding prices of the final good does converge to the competitive price.*

The usual practice when increasing the number of firms in the market consists in comparing the resulting price with a *fixed* marginal cost. The novelty here is that the marginal cost of the downstream firms does not remain fixed when increasing the number of firms in the downstream and upstream markets simultaneously. Importantly, notice that, whatever r, the marginal cost of producing the input, which is equal to β, is lower than the input price by an amount of $\frac{1}{4}\frac{n}{m}$. This looks as a surprise since this context, for large values of r, corresponds closely to perfect competition. It is as if the downstream firms would be charged a constant tax per unit of input over the marginal cost of producing the input, β. In fact, when r is close to ∞, $f(z^*(rm, rn))$ is close to zero, implying an infinitesimal *individual* demand for input from each downstream firm and, accordingly, a marginal product of the input which tends to infinity with r. In particular, if the price of input were set at the marginal cost β, the quantity of input demanded by the downstream firms would exceed the quantity which would be offered by the upstream firms at the same price, preventing thereby the equality of supply and demand, as required by the definition of a competitive equilibrium.The total quantity demanded by the downstream firms at the downstream Cournot equilibrium if $p = \beta$ obtains from the solution of the problem $\underset{z_i}{Max}(1 - \sqrt{z_i} - \Sigma_{k\neq i}\sqrt{z_k})\sqrt{z_i} - pz_i$ from which we easily obtain:

$nz^* = \frac{n^2}{(n+2\beta+2)^2}$.Thus $\underset{r\to\infty}{lim}$ $\{rnz^*\} = 1$. On the other hand, the amount of input offered by the upstream firms at price β is $\frac{n}{2(\frac{1}{2}+\beta+\frac{1}{2})^2}$. This amount tends to zero when rm and rn tend to inifnity, and not to 1.

Free entry *with the same speed* of firms in both markets affects market power in each market differently, therefore, the input demand and the input supply are affected differently by the entry of new firms. As a conseguence, the input price, that clears the input market, does not necessarly lead to an input price that is equal to the technological marginal cost to produce the input. Notice however that, even though upstream firms get the amount of the tax, it *does not* prevent the quantity of input exchanged in the input market to correspond exactly to the quantity required to produce an aggregate final good corresponding to the competitive equilibrium final good. More than that: the burden of this tax is even required in order to induce downstream firms to reduce their input demand in order to produce exactly the competitive equilibrium final good level! The presence of this subsidy does not bring any extra profits to the upstream firms themselves: their profit tends to zero when r tends to infinity. Consequently, this limit value of the input price, including the existence of the subsidy, does not preclude the limit economy to be in a Pareto optimal state simultaneously in both markets. *The existence of this transfer, through the input price from the downstream to the upstream firms, reveals the interlinkage between markets resulting from the simultaneous increase in the number of firms in both of them.*

Finally notice that if the economy would be replicated *at a different speed* in the downstream and upstream markets, this discrepancy between marginal cost and input price may disappear. In fact, when the upstream market is replicated infinitely faster than the downstream one, this discrepancy disappears at the limit. For instance, when the downstream market is replicated at speed r, while the upstream market is replicated at speed r^2, the limit input price is equal to the marginal cost β. In other words, the power of upstream firms should be diluted much faster than the downstream firms' one in order to force the competitive outcome !

This discrepancy between marginal cost and input price may disappear when the upstream market is replicated infinitely faster than the downstream one. In other words, the market power of upstream firms who fix the input price should be diluted much faster than the downstream firms' one in order to force the competitive input price!

3.1.2 Upstream competition and downstream oligopoly

We compute

$$\lim_{r\to\infty} \{\pi^*(rm, n)\} = \frac{1+2\beta}{n+1+2\beta}$$

and

$$\lim_{r\to\infty} \{p^*(rm, n)\} = \beta.$$

Proposition 2 *Free entry only in the upstream market yields*

(i) the equilibrium input price converges to upstream firms' marginal cost;

(ii) the equilibrium price of the final good converges to the final good price $\frac{1+2\beta}{1+n+2\beta}$ *corresponding to the Cournot equilibrium with* n *downstream firms producing the good at a unit cost* β*.*

Thus, differently from proposition 1, proposition 2 fits the standard asymptotic results obtained in the usual Cournot framework of a single market. In fact, the effects that are present when n and m tend simultaneously to infinity, disappear when n is fixed: the production level of each downstream firm does not tend to zero, so that, whatever m, the marginal product of the input remains bounded away from infinity. Then no tax is needed to dampen the incentive to overproduce the final good.

3.1.3 Downstream competition and upstream oligopoly

We compute

$$\lim_{r\to\infty}\{\pi^*(m,rn)\}=\frac{1}{2m}$$

and

$$\lim_{r\to\infty}\{p^*(m,rn)\}=\infty.$$

Proposition 3 *Free entry in downstream market makes the price of the final good to converge to the marginal cost of producing the final good when* m *firms are operating in the input market. In this case, the input price gets arbitrary large.*

This immediately follows from the fact that the marginal cost of producing the final good at the equilibrium in the downstream market when m firms operate in the upstream market is equal to $2pf(z)$, with $p=p^*(m,n)$ and $f(z)=f(z^*(m,n))$. The price of the final good exactly reflects the market power existing in the upstream market, which is transferred in the downstream market through its dependence on the number of upstream firms, m. This sheds some further light on the interaction between the two succesuve markets under Cournot competition. Even if the competitive conditions are met in the downstream market, since $MC\cong\pi^*(m,n)$, the final good price encompasses the non competitiveness in the input market. The usual analysis of Cournot competition in a market does not allow this type of consideration because the relationship of costs to market power in the input market cannot be taken into account when the cost function is exogenous.

In fact, as in the case of pure competition considered above, when n is close to ∞, $f(z^*(m,n))$ is again close to zero, implying an infinitesimal individual demand of input from each downstream firm and, accordingly, a marginal product of the input which tends to infinity with m and n. This leads downstream firms' demand to increase beyond any limit, forcing in turn the input price to increase itself beyond any limit when the number of upstream firms remains fixed.

4 Conclusion

Our exploration of industry equilibria deserves to be continued. First, as in the existing literature, we have kept the assumption of price taking agents in the demand side of the markets. This assumption is not very satisfactory because it is difficult to justify the fact that an economic agent behaves strategically in one market but not in another. A full treatment would require downstream firms behaving strategically simultaneously in the downstream and upstream markets. This constitutes the next point on our research agenda. Another avenue for potential research would consist in analyzing the stability of collusive agreements, as in d'Aspremont *et al.* (1983), using the framework identified in the present paper. Furthermore, the analysis could be extended to chains of technology-linked markets and to technological contexts involving more than one factor. Finally, it would be very natural to study in depth the effects of collusive agreements among downstream and upstream firms, in the framework of successive markets. The above analysis provides a good setup to cast some important issues studied in the literature. All this looks like a promising research territory for a better understanding of industry equilibria in technology linked markets.

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5 Appendix: Equilibrium input supply

The profit of an upstream firm Γ_j at the vector of strategies writes as

$$\Gamma_j(s_j, s_{-j}) = p(s_j, s_{-j})s_j - \beta s_j,$$

with $p(s_j, s_{-j})$ such that $\sum_j s_j(p) = \sum_i z_i(p)$, namely

$$p(s_j, s_{-j}) = \frac{1}{4(\beta+1)} \left(\begin{array}{c} k - n - 2\beta + 2k\beta - 2n\beta - 2+ \\ \frac{\sqrt{(n-k)}\sqrt{\sum_{k\neq j} s_k}}{\sum_{k\neq j} s_k}(2\beta+1) \end{array} \right).$$

Denote by a the value $\frac{k-n-2\beta+2k\beta-2n\beta-2}{4(\beta+1)}$ and by b the expression $\frac{\sqrt{(n-k)}(2\beta+1)}{4(\beta+1)}$. Accordingly, the payoff of the j-th upstream firm writes as

$$\Gamma_j(s_j, s_{-j}) = \left(a + b\frac{\sqrt{\sum_{k\neq j} s_k}}{\sum_{k\neq j} s_k} \right) s_j - \beta s_j.$$

Taking the first derivative with respect to s_j, we get

$$a-\beta+\frac{b}{\sqrt{s(m-h)}}-\frac{1}{2}b\frac{s}{s\sqrt{s(m-h)}+s\,(m-h-1)\,\sqrt{s(m-h)}}=0$$

in a symmetric equilibrium. Let us make the change of variable $\sqrt{s\,(m-h)}=x$, and let us solve the equation $a-\beta+\frac{b}{x}-\frac{s}{2}\frac{b}{sx+s(m-h-1)x}=0$ in two steps. The solution in x obtains as

$$x=\frac{(2m-2h-1)\,b}{2\,(\beta-a)\,(m-h)}.$$

Thus, using the definition of x, we can solve the equation in s, namely,

$$s^{*}=\frac{1}{4}\frac{b^{2}\,(2m-2h-1)^{2}}{(m-h)^{3}\,(\beta-a)^{2}}.$$

substituting back for $a=\frac{k-n-2\beta+2k\beta-2n\beta-2}{4(\beta+1)}$and $b=\frac{\sqrt{(n-k)}(2\beta+1)}{4(\beta+1)}$ in the expression of $s^{*}(n,m)$, we get $s^{*}(n,m)$ as in the paper.