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# Almost sure convergence of randomized urn models with application to elephant random walk

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ABSTRACT

We consider a randomized urn model with objects of finitely many colors. The replacement matrices are random, and are conditionally independent of the color chosen given the past. Further, the conditional expectations of the replacement matrices are close to an almost surely irreducible matrix. We obtain almost sure and $L^1$ convergence of the configuration vector, the proportion vector and the count vector. We show that first moment is sufficient for i.i.d. replacement matrices independent of past color choices. This significantly improves the similar results for urn models obtained in Athreya and Ney (1972) requiring $L\log_+ L$ moments. For more general adaptive sequence of replacement matrices, a little more than $L\log_+ L$ condition is required. Similar results based on $L^1$ moment assumption alone has been considered independently and in parallel in Zhang (2018). Finally, using the result, we study a delayed elephant random walk on the nonnegative orthant in $d$ dimension with random memory.



## 1. Introduction

We consider an urn with objects of $K$ colors indexed by $[K] = \{1, 2, \ldots, K\}$. At discrete time points, we select a color with probability proportional to its content in the urn, and add more amount of object of each color to the urn following a randomized rule. For $n \geq 0$, consider the configuration vector $\boldsymbol{C}_n := (C_{n1}, \ldots, C_{nK})$ where $C_{ni}$ is the amount of color $i$ in the urn at epoch $n$. Note that the amount added can be any nonnegative real value. Let $\boldsymbol{\chi}_n$ be the row vector indicating the color drawn at the $n$th trial, so that $\boldsymbol{\chi}_n = \boldsymbol{e}_i$, the $i$th coordinate vector in $\mathbb{R}^K$, if the selected color is $i$. The randomized rule for adding content to the urn is encoded by a sequence of $K \times K$ random matrices $(\boldsymbol{R}_n)_{n\geq 0}$, called the replacement matrices. If $\boldsymbol{\chi}_n = \boldsymbol{e}_i$ then $R_{n;ij}$ amount of color $j$ is added to the urn, for each $j \in [K]$. Therefore, for $n \geq 1$

$$\boldsymbol{C}_n = \boldsymbol{C}_{n-1} + \boldsymbol{\chi}_n \boldsymbol{R}_n. \tag{1}$$

The total content of the urn after $n$th trial is $S_n = \sum_{i=1}^{K} C_{ni}$. The count vector after $n$th trial, $\boldsymbol{N}_n$, giving the number of times each color is drawn, is $\boldsymbol{N}_n = \sum_{k=1}^{n} \boldsymbol{\chi}_k$. The vectors considered in this article are row vectors, and any column vector is indicated by transpose. We also use $\mathbf{1}$ for a row vector of 1's, whose dimension is clear from the context. In this article we discuss sufficient conditions for almost sure convergence of $\boldsymbol{C}_n$, $S_n$, $\boldsymbol{N}_n$ after suitable scaling.

For $n \geq 1$, let $\mathcal{F}_n$ be the $\sigma$-field generated by $\boldsymbol{C}_0$ and $\left(\boldsymbol{R}_m, \boldsymbol{\chi}_m\right)_{m=1}^{n}$. The basic assumptions about the evolution of the urn are the following.

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**Assumption 1.1.** The adapted sequence $((\boldsymbol{\chi}_n, \boldsymbol{R}_n), \mathcal{F}_n)_{n\geq 1}$ satisfies:

(i) The nonzero vector $\boldsymbol{C}_0$ has almost surely nonnegative entries and finite mean.
(ii) For $n \geq 1$ and $i \in [K]$, $\mathbb{P}\left(\boldsymbol{\chi}_n = \boldsymbol{e}_i | \mathcal{F}_{n-1}\right) = C_{n-1,i}/S_{n-1}$.
(iii) Each of the replacement matrices $(\boldsymbol{R}_n)_{n\geq 1}$ have finite mean.
(iv) The variables $\boldsymbol{\chi}_n$ and $\boldsymbol{R}_n$ are conditionally independent given $\mathcal{F}_{n-1}$.

We also assume that the replacement matrices are "close" to an irreducible replacement matrix in an appropriate sense. Recall that, a square matrix $\boldsymbol{A}$ with nonnegative entries is called *irreducible* if, for any $i, j \in [K]$, there exists a positive integer $N \equiv N_{ij}$, such that the $(i, j)$-th entry of $\boldsymbol{A}^N$ is positive. By Perron–Frobenius theory, the *dominant eigenvalue* of an irreducible matrix, namely the eigenvalue having the largest real part, is simple and positive and has corresponding left and right eigenvectors with all coordinates strictly positive.

For a matrix $\boldsymbol{A}$, we use the operator norm $\rho(\boldsymbol{A}) := \max_i \sum_j \left|A_{ij}\right|$. Further, for an event $A$, its indicator function is denoted by $\mathbb{1}_A$. The conditional expectations of the replacement matrices, denoted as $\boldsymbol{H}_{n-1} := \mathbb{E}(\boldsymbol{R}_n|\mathcal{F}_{n-1})$, are called the *generating matrices* and their truncated version, denoted as $\widetilde{\boldsymbol{H}}_{n-1} := \mathbb{E}\left(\boldsymbol{R}_n \mathbb{1}_{[\rho(\boldsymbol{R}_n)\leq n]} | \mathcal{F}_{n-1}\right)$, are called *truncated generating matrices*.

**Assumption 1.2.** There exists a (possibly random) almost surely irreducible matrix $\boldsymbol{H}$ with nonnegative entries, such that the truncated generating matrices $\widetilde{\boldsymbol{H}}_n$ converge to $\boldsymbol{H}$ in Cesaro sense in the operator norm almost surely, i.e., $\frac{1}{n}\sum_{k=0}^{n-1} \rho\left(\widetilde{\boldsymbol{H}}_k - \boldsymbol{H}\right) \to 0$. The dominant eigenvalue of $\boldsymbol{H}$ is $\lambda_{\boldsymbol{H}}$ and $\boldsymbol{\pi}_{\boldsymbol{H}}$ is the unique left eigenvector of $\boldsymbol{H}$ corresponding to $\lambda_{\boldsymbol{H}}$ such that $\boldsymbol{\pi}_{\boldsymbol{H}}$ has all coordinates strictly positive and is normalized to be a probability vector.

In Athreya and Ney (1972), the replacement matrices formed an i.i.d. sequence, with finite $L\log_+ L$ moments for the entries and irreducible mean matrix, which was additionally assumed to be aperiodic. Using branching process techniques, it was shown in Athreya and Ney (1972), Chapter V.9.3 that the normalized color count and composition vectors converge almost surely to the (normalized to probability) left eigenvector of the mean matrix corresponding to its dominant eigenvalue. In this article, we establish almost sure convergence for i.i.d. replacement matrices under $L^1$ conditions alone, as a significant relaxation of $L\log_+ L$ condition in Athreya and Ney (1972). The mean replacement matrix is taken to be irreducible alone, aperiodicity is not needed. We extend the result to an adapted sequence also, under $L\log_+ L$ condition, when the replacement matrices are suitably majorized. A referee has pointed out an unpublished manuscript (Zhang, 2018). It became available on the preprint server arXiv long after the initial submission of the present work. The manuscript Zhang (2018) obtained similar results independently and in parallel, using different stochastic approximation method. The stochastic approximation used here enables one to obtain a simpler proof where the associated differential equation is simpler. In the process, we develop Theorem 3.1 on Stochastic Approximation which uses only Cesaro negligibility of the error sequence and can be of independent interest.

In Section 2, we state the model and the main results. Theorem 2.3 extends the result for i.i.d. replacement matrices in Athreya and Ney (1972), Zhang (2012) under $L^1$ condition only. Corollary 2.5 studies adapted replacement matrices, majorized by a $L\log_+ L$ random variable. In Section 3, the main tool for the proof – an appropriate result on stochastic approximation – is provided and the main results are proven. Finally, in Section 4, we apply the results for the urn to study a delayed elephant random walk on $d$-dimensional nonnegative orthant with randomly reinforced memory.

## 2. Main results

To obtain the appropriate convergence results, we make further uniform integrability type conditions on the replacement matrices.

**Assumption 2.1.** The sequence $(\rho(\boldsymbol{R}_n))$ satisfy one of the following:

(a) The distributions of $\rho(\boldsymbol{R}_n)$ are majorized: there exists $c \in (0, \infty)$ and a positive random variable $R$ with finite expectation such that, for all $x > 0$,

$$\mathbb{P}\left(\rho(\boldsymbol{R}_n) > x\right) \leq c\mathbb{P}\left(R > x\right).$$

(b) For some nonnegative function $\phi$ on $[0, \infty)$ which is eventually positive and nondecreasing, satisfying $\sum 1/(n\phi(n)) < \infty$ and $x/\phi(x)$ eventually monotone nondecreasing, we have the bounded moment condition:

$$\sup_n \mathbb{E}\left(\rho(\boldsymbol{R}_n)\, \phi\left(\rho(\boldsymbol{R}_n)\right)\right) < \infty.$$

**Remark 2.1.** An example of $\phi(x) = (\log_+ x)^p$ for some $p > 1$, has been considered by Zhang (2012). Other choices include $\phi(x) = \log_+ x(\log_+ \log_+ x)^p$ for some $p > 1$.

**Remark 2.2.** It is interesting to note that the majorizing condition in Assumption 2.1(b) is on the unconditional distribution of the replacement matrices $(\boldsymbol{R}_n)$, in contrast to the conditions (3.3)–(3.5) of Theorem 3.1 of Zhang (2018).

**Remark 2.3.** Both Assumptions 2.1(a) and 2.1(b) imply uniform integrability of $(\rho(\boldsymbol{R}_n))$ and, hence, of $\left(\boldsymbol{R}_n \mathbb{1}_{[\rho(\boldsymbol{R}_n)\le n]}\right)$. Thus, under either condition, the convergence in Assumption 1.2 is also in $L^1$.

**Theorem 2.2.** *Under Assumptions 1.1, 1.2 and 2.1, we have, almost surely, as well as in $L^1$,*

$$\frac{\boldsymbol{C}_n}{S_n} \to \boldsymbol{\pi_H}; \quad \frac{S_n}{n} \to \lambda_{\boldsymbol{H}}; \quad \frac{\boldsymbol{C}_n}{n} \to \lambda_{\boldsymbol{H}}\boldsymbol{\pi_H}; \quad \text{and} \quad \frac{\boldsymbol{N}_n}{n} \to \boldsymbol{\pi_H}. \tag{2}$$

**Remark 2.4.** It is interesting to compare the assumptions in this article with those required to establish convergence in probability and in $L^1$ in Gangopadhyay and Maulik (2019). Note that Assumption 1.1 is same as Assumption 3.1 of Gangopadhyay and Maulik (2019). Assumption 1.2 corresponds to Assumptions 3.2 and 3.3 of Gangopadhyay and Maulik (2019). Assumption 3.2 of Gangopadhyay and Maulik (2019) on the properties of the matrix $\boldsymbol{H}$ remain unchanged. However, given the almost sure convergence in conclusion, we naturally strengthen the assumption to almost sure convergence. The methods of proof differ however. We further require the truncated generating matrices $\widetilde{\boldsymbol{H}}_n$ to converge to $\boldsymbol{H}$ instead of the generating matrices, as considered in Gangopadhyay and Maulik (2019). In Remark 2.5 sufficient conditions will be provided for convergence of $\boldsymbol{H}_n$, as in Gangopadhyay and Maulik (2019), to work. Finally, uniform integrability condition in Assumption 3.4 of Gangopadhyay and Maulik (2019) has been appropriately strengthened in Assumption 2.1.

The case of i.i.d. replacement matrices have been considered in Athreya and Ney (1972), Zhang (2012). Assumption 2.1(a) holds, when $(\boldsymbol{R}_n)$ is i.i.d. with finite mean. Further, since $R_n$ is independent of $\mathcal{F}_{n-1}$ and i.i.d. with finite mean, Assumption 1.2 holds. Thus, we relax $L\log_+ L$ condition of Athreya and Ney (1972), Zhang (2012) to existence of first moment alone and obtain the following theorem as a corollary to Theorem 2.2. As has been noted before, a referee has informed us about the unpublished manuscript (Zhang, 2018), which was developed independently and in parallel, and considered the following theorem in its Corollary 2.1. However, a different stochastic approximation has been used here, which has significantly simplified the proof.

**Theorem 2.3.** *We consider an urn model with nonzero initial configuration $\boldsymbol{C}_0$ having almost surely nonnegative entries and finite mean; as well as an i.i.d. sequence of replacement matrices $(\boldsymbol{R}_n)$, independent of $\boldsymbol{C}_0$ with irreducible mean matrix and $\mathbb{E}(\rho(\boldsymbol{R}_1)) < \infty$. Also, let*

$$\mathbb{P}\left(\boldsymbol{\chi}_n = \boldsymbol{e}_i | \boldsymbol{C}_0, (\boldsymbol{R}_m)_{m=1}^{n-1}, (\boldsymbol{\chi}_m)_{m=1}^{n-1}\right) = C_{n-1,i}/S_{n-1}$$

*hold for $n \ge 1$ and $i \in [K]$, and let $\boldsymbol{\chi}_n$ be independent of $(\boldsymbol{R}_m)_{m\ge n}$. Then the convergence (2) in Theorem 2.2 holds both almost surely and in $L^1$.*

It is natural to also consider the following variant of Assumption 1.2.

**Assumption 2.4.** There exists a (possibly random) matrix $\boldsymbol{H}$ with nonnegative entries, which is almost surely irreducible, such that the generating matrices $\boldsymbol{H}_n$ converge to $\boldsymbol{H}$ in Cesaro sense in the operator norm almost surely i.e., $\frac{1}{n}\sum_{k=0}^{n-1}\rho(\boldsymbol{H}_k - \boldsymbol{H}) \to 0$ almost surely.

**Remark 2.5.** By Kronecker's lemma, Assumption 2.4 and the condition

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathbb{E}\left(\rho(\boldsymbol{R}_n)\mathbb{1}_{[\rho(\boldsymbol{R}_n)>n]} | \mathcal{F}_{n-1}\right) < \infty \text{ almost surely,} \tag{3}$$

imply Assumption 1.2.

**Remark 2.6.** It may be noted that the analysis in Zhang (2018) is done under its assumption (3.1) alone, which is same as Assumption 2.4. The additional truncated moment condition (3) corresponds to (3.4) in Zhang (2018).

Corollary 2.5 extends the result on i.i.d. sequence of replacement matrices given in Athreya and Ney (1972) to adapted sequence with $L\log_+ L$ moment for the majorizing random variable. The moment condition on the majorizing random variable in Assumption 2.1(a) is strengthened to replace Assumption 1.2 by Assumption 2.4.

**Corollary 2.5.** *Consider an adapted sequence $\left((\boldsymbol{\chi}_n, \boldsymbol{R}_n), \mathcal{F}_n\right)$ satisfying Assumptions 1.1, 2.1(a) and 2.4, where the majorizing random variable $R$ in Assumption 2.1(a) additionally satisfies $\mathbb{E}(R\log_+ R) < \infty$. Then the convergence (2) in Theorem 2.2 holds both almost surely and in $L^1$.*

**Remark 2.7.** It may be noted that Assumption 2.1(a) involves majorization of the unconditional distribution of the replacement matrices in contrast to the conditional distribution in Zhang (2018).

Next corollary uses a generalization of Assumption 2.2(a) in Zhang (2012). See also Remark 3.1 of Zhang (2018). No additional condition is required to replace Assumption 1.2 by Assumption 2.4.

**Corollary 2.6.** *Assumptions 1.1, 2.1(b) and 2.4 imply the convergence (2) in Theorem 2.2 both almost surely and in $L^1$.*

## 3. Proofs of the main results

In the first subsection, we provide the relevant result on stochastic approximation and another technical result. In the last two subsections, we prove the main theorem and the corollaries of Section 2.

### 3.1. Stochastic approximation and other results

The main tool for proving Theorem 2.2 is Stochastic Approximation. We shall use the following theorem on Stochastic Approximation, which follows from a special case of Theorem 9.2.8 of Duflo (1997), which in turn was adapted from Kushner and Clark (1978).

**Theorem 3.1.** *Let $S$ be a compact and convex subset of $\mathbb{R}^K$. Let $h : S \to \mathbb{R}^K$ be a continuous function. Let $(a_n)_{n=0}^{\infty}$ be a nonincreasing sequence of positive real numbers, called the step sizes, satisfying*

$$0 < \liminf na_n \le \limsup na_n < \infty. \tag{4}$$

*Let $(\boldsymbol{\gamma}_n)_{n=0}^{\infty}$ be a sequence, called the error sequence, which is Cesaro negligible, i.e., $\frac{1}{n}\boldsymbol{\Gamma}_n \to \mathbf{0}$, where $\boldsymbol{\Gamma}_n := \sum_{i=0}^{n} \boldsymbol{\gamma}_i$. Let $(\boldsymbol{x}_n)_{n=0}^{\infty}$ be a sequence that takes values in $S$ and evolves as*

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + a_n h(\boldsymbol{x}_n) + a_n \boldsymbol{\gamma}_n. \tag{5}$$

*If the ODE $\dot{\boldsymbol{x}} = h(\boldsymbol{x})$ has a unique solution $\boldsymbol{z} : \mathbb{R} \to S$, then $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{z}(0)$.*

The above results uses only Cesaro negligibility of the error sequence, rather than its summability or negligibility. The result can be of independent interest.

It should also be noted that general form of step sizes allows us to develop stochastic approximation for $(\boldsymbol{C}_n/S_n)$ in contrast to $(C_n/n)$ considered in Zhang (2018). As $(\boldsymbol{C}_n/S_n)$ already lies in the probability simplex and the corresponding differential equation is of the standard Lotka–Volterra type, our analysis becomes significantly simpler.

**Proof of Theorem 3.1.** It is immediate from (4) that $a_n \to 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then, using Theorem 9.2.8 of Duflo (1997), it is enough to show that, for all $T > 0$,

$$\lim_{n\to\infty} \sup_{n\le j\le \tau_n(T)} \left\| \sum_{k=n}^{j} a_k \boldsymbol{\gamma}_k \right\| = 0,$$

where $\tau_n(T) := \inf\{k \ge n : a_n + \cdots + a_k \ge T\}$, and $\|\cdot\|$ is any norm in $\mathbb{R}^K$. Fix $T > 0$. First observe that

$$a_k \boldsymbol{\gamma}_k = (a_k \boldsymbol{\Gamma}_k - a_{k-1}\boldsymbol{\Gamma}_{k-1}) + a_k a_{k-1} \boldsymbol{\Gamma}_{k-1} \left( \frac{1}{a_k} - \frac{1}{a_{k-1}} \right).$$

Since $(a_n)$ is nonincreasing, for $n \le j \le \tau_n(T)$, we have,

$$\left\| \sum_{k=n}^{j} a_k \boldsymbol{\gamma}_k \right\| \le \left\| a_j \boldsymbol{\Gamma}_j \right\| + \left\| a_{n-1} \boldsymbol{\Gamma}_{n-1} \right\| + a_n \sup_{k\ge n-1} \left\| a_k \boldsymbol{\Gamma}_k \right\| \frac{1}{a_j}.$$

Hence, we have

$$\sup_{n\le j\le \tau_n(T)} \left\| \sum_{k=n}^{j} a_k \boldsymbol{\gamma}_k \right\| \le \sup_{j\ge n-1} \left\| a_j \boldsymbol{\Gamma}_j \right\| \left( 2 + na_n \cdot \frac{\tau_n(T)}{n} \cdot \frac{1}{\tau_n(T)\, a_{\tau_n(T)}} \right).$$

Using (4) and $\frac{1}{n}\boldsymbol{\Gamma}_n \to 0$, it is enough to show that $\tau_n(T)/n$ is bounded for any $T > 0$. Since $\liminf na_n > 0$, choose $a^* > 0$ such that $na_n > a^*$ for large enough $n$. Also $a_n < T$ for large enough $n$. Hence, for large enough $n$, $\tau_n(T) > n$ and

$$a^* \left[ \log \frac{\tau_n(T) - 1}{n-1} - 1 \right] \le \frac{a^*}{n} + \cdots + \frac{a^*}{\tau_n(T) - 1} \le a_n + \cdots + a_{\tau_n(T)-1} \le T.$$

Thus, $\tau_n(T)/n \le (\tau_n(T) - 1)/(n-1) \le \exp(T/a^* + 1)$, as required. □

We also need the following technical result for the average of a dominated adapted sequence, with the terms appropriately centered.

**Lemma 3.2.** *Let $(X_n, \mathcal{F}_n)$ be adapted, and $|X_n| \le V_n$, where $(V_n)$ satisfies*

- *either a majorization condition as in Assumption 2.1(a), namely there exists $c' \in (0, \infty)$ and a positive random variable $V$ with finite expectation satisfying $\mathbb{P}(V_n > x) \le c'\mathbb{P}(V > x)$ for all $x > 0$,*

- *or a bounded moment condition as in Assumption 2.1(b), namely a function* $\phi$ *as in Assumption 2.1(b), satisfying* $\sup_n \mathbb{E}(V_n\phi(V_n)) < \infty$,

*then,* $\frac{1}{n}\sum_{k=1}^{n}\left(X_k - \mathbb{E}\left(X_k \mathbb{1}_{[V_k\le k]}|\mathcal{F}_{k-1}\right)\right) \to 0$ *almost surely.*

*Further, if* $(V_n)$ *is identically distributed with finite mean and if* $V_n$ *is independent of* $\mathcal{F}_{n-1}$ *for all* $n$*, then* $\frac{1}{n}\sum_{k=1}^{n}\left(X_k - \mathbb{E}\left(X_k|\mathcal{F}_{k-1}\right)\right) \to 0$ *almost surely.*

**Proof.** The first part of the proof, where centering is done by the truncated moment, is similar to the derivation of (2.20) of Theorem 2.19 of Hall and Heyde (1980) under majorization condition. Under bounded moment condition, we additionally use, for all large enough $n$, $\mathbb{P}(V_n > n) \le \frac{1}{n\phi(n)}\mathbb{E}(V_n\phi(V_n))$ and $\mathbb{E}\left(V_n^2\mathbb{1}_{[V_n\le n]}|\mathcal{F}_{n-1}\right) \le \frac{n}{\phi(n)}\mathbb{E}\left(V_n\phi(V_n)|\mathcal{F}_{n-1}\right)$.

Finally, when $(V_n)$ is identically distributed with finite mean and $V_n$ is independent of $\mathcal{F}_{n-1}$, the majorization condition holds. Also

$$\mathbb{E}\left(X_n\mathbb{1}_{[V_n>n]}|\mathcal{F}_{n-1}\right) \le \mathbb{E}V_n\mathbb{1}_{[V_n|>n]} \to 0 \qquad \text{almost surely,}$$

by independence of $V_n$ from $\mathcal{F}_{n-1}$, which takes care of the remaining term in the centering. □

*3.2. Proof of Theorem 2.2*

First we establish convergence of $(\boldsymbol{C}_n/S_n)$ to $\boldsymbol{\pi_H}$ by rewriting the evolution equation (1) in the form of (5) and checking the conditions of Theorem 3.1. Define $Y_0 := S_0$ and for $n \ge 1$, $Y_n := S_n - S_{n-1} = \boldsymbol{\chi}_n\boldsymbol{R}_n\boldsymbol{1}^T$. So $Y_n$ is the total amount added to the urn after the $n$th draw. Observe that, for $n \ge 1$,

$$0 \le Y_n = \boldsymbol{\chi}_n\boldsymbol{R}_n\boldsymbol{1}^T \le \rho\left(\boldsymbol{R}_n\right). \tag{6}$$

We rewrite the evolution Eq. (1) as

$$\frac{\boldsymbol{C}_n}{S_n} = \frac{\boldsymbol{C}_{n-1}}{S_{n-1}} + \frac{1}{S_n}h_{\boldsymbol{H}}\left(\frac{\boldsymbol{C}_{n-1}}{S_{n-1}}\right) + \frac{\boldsymbol{\delta}_n}{S_n} + \frac{\boldsymbol{\xi}_n}{S_n},$$

where $h_{\boldsymbol{H}}$, $\boldsymbol{\delta}_n$, $\boldsymbol{\xi}_n$ are defined as follows. The drift $h_{\boldsymbol{H}}$, indexed by $K\times K$ matrices $\boldsymbol{H}$, is defined as $h_{\boldsymbol{H}}\left(\boldsymbol{X}\right) := \boldsymbol{XH} - \boldsymbol{X}\left(\boldsymbol{XH1}^T\right)$. For each $n \ge 1$,

$$\boldsymbol{\delta}_n := \left(\boldsymbol{\chi}_n\boldsymbol{R}_n - \frac{\boldsymbol{C}_{n-1}}{S_{n-1}}Y_n\right) - \mathbb{E}\left(\left(\boldsymbol{\chi}_n\boldsymbol{R}_n - \frac{\boldsymbol{C}_{n-1}}{S_{n-1}}Y_n\right)\mathbb{1}_{[\rho(\boldsymbol{R}_n)\le n]}|\mathcal{F}_{n-1}\right),$$

is the martingale difference term, and $\boldsymbol{\xi}_n := h_{\widetilde{\boldsymbol{H}}_{n-1}-\boldsymbol{H}}\left(\frac{\boldsymbol{C}_{n-1}}{S_{n-1}}\right)$ is the adjusted truncated conditional expectation term. The $(\boldsymbol{C}_n/S_n)$ takes values in the closed bounded convex set of probability vectors in $\mathbb{R}^K$. The corresponding differential equation is $\dot{\boldsymbol{x}} = h_{\boldsymbol{H}}\left(\boldsymbol{x}\right)$ with $h_{\boldsymbol{H}}(\boldsymbol{x})$, a quadratic polynomial in $\boldsymbol{x}$, is continuous. Further, from Proposition 3.3 of Gangopadhyay and Maulik (2019), the differential equation has unique solution in the probability simplex given by $\boldsymbol{x}(t) = \boldsymbol{\pi_H}$ for all $t$. Now we check the conditions on the step sizes. Since $S_n$ is partial sum of nonnegative $(Y_n)$, the step sizes are nonincreasing. Now we check (4). Define

$$\begin{aligned}\eta_n &:= \frac{S_n}{n} - \frac{1}{n}\sum_{m=1}^{n}\frac{\boldsymbol{C}_{m-1}}{S_{m-1}}\boldsymbol{H1}^T \\ &= \left[\frac{S_n}{n} - \frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left(Y_m\mathbb{1}_{[\rho(\boldsymbol{R}_m)\le m]}|\mathcal{F}_{m-1}\right)\right] + \frac{1}{n}\sum_{m=1}^{n}\frac{\boldsymbol{C}_{m-1}}{S_{m-1}}\left(\widetilde{\boldsymbol{H}}_{m-1} - \boldsymbol{H}\right)\boldsymbol{1}^T.\end{aligned}$$

Using the bound (6) and Lemma 3.2, $\frac{S_n}{n} - \frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left(Y_m\mathbb{1}_{[\rho(\boldsymbol{R}_m)\le m]}|\mathcal{F}_{m-1}\right) \to 0$ almost surely. Also, $\frac{1}{n}\sum_{m=1}^{n}\frac{\boldsymbol{C}_{m-1}}{S_{m-1}}\left(\widetilde{\boldsymbol{H}}_{m-1} - \boldsymbol{H}\right)\boldsymbol{1}^T \to 0$, using Assumption 1.2. Thus $\eta_n \to 0$ almost surely. Let $\sigma(\boldsymbol{H})$ be the least absolute row sum of $\boldsymbol{H}$. Since $\boldsymbol{H}$ is irreducible, no row can be the zero vector and hence $\sigma(\boldsymbol{H}) > 0$. Using $\eta_n \to 0$ and $0 < \sigma\left(\boldsymbol{H}\right) \le \frac{1}{n}\sum_{m=1}^{n}\frac{\boldsymbol{C}_{m-1}}{S_{m-1}}\boldsymbol{H1}^T \le \rho\left(\boldsymbol{H}\right) < \infty$ we get that the step sizes $(1/S_n)$ satisfy condition (4). Finally, each coordinate of $(\boldsymbol{\delta}_n)$ is Cesaro negligible using Lemma 3.2 and the bound $\left\|\boldsymbol{\chi}_n\boldsymbol{R}_n - \frac{\boldsymbol{C}_{n-1}}{S_{n-1}}Y_n\right\| \le 2\rho\left(\boldsymbol{R}_n\right)$. Also, $(\boldsymbol{\xi}_n)$ is Cesaro negligible by Assumption 1.2.

Then from Theorem 3.1 we have the almost sure convergence of the proportion vector $\boldsymbol{C}_n/S_n$ to $\boldsymbol{\pi_H}$. Using $\boldsymbol{C}_n/S_n \to \boldsymbol{\pi_H}$ we get $S_n/n - \eta_n = \frac{1}{n}\sum_{m=1}^{n}\frac{\boldsymbol{C}_{m-1}}{S_{m-1}}\boldsymbol{H1}^T \to \boldsymbol{\pi_H}\boldsymbol{H1}^T = \lambda_{\boldsymbol{H}}$. Since $\eta_n \to 0$, we get $S_n/n \to \lambda_{\boldsymbol{H}}$, and hence $\boldsymbol{C}_n/n \to \lambda_{\boldsymbol{H}}\boldsymbol{\pi_H}$ almost surely. Now $\frac{\boldsymbol{N}_n}{n} - \frac{1}{n}\sum_{m=0}^{n-1}\frac{\boldsymbol{C}_m}{S_m} = \frac{1}{n}\sum_{m=1}^{n}\left(\boldsymbol{\chi}_m - \frac{\boldsymbol{C}_{m-1}}{S_{m-1}}\right)$, being a scaled $L^2$-bounded martingale, is negligible. The almost sure convergence of $\boldsymbol{N}_n/n$ to $\boldsymbol{\pi_H}$ then follows from that of $\boldsymbol{C}_n/S_n$.

Being bounded, $\boldsymbol{C}_n/S_n$ and $\boldsymbol{N}_n/n$ converge in $L^1$. We next consider $S_n/n$. From (6), $0 \le Y_n \le \rho\left(\boldsymbol{R}_n\right)$ for $n \ge 1$ and from Assumption 1.1 (i))$Y_0 = \boldsymbol{C}_0\boldsymbol{1}^T$ is integrable. As noted in Remark 2.3, Assumption 2.1 implies uniform integrability of $(\rho\left(\boldsymbol{R}_n\right))$. Hence $(Y_n)$ and $(S_n/n)$ are also uniformly integrable. Hence $S_n/n$ converges $L^1$. Also, so does $\boldsymbol{C}_n/n$ using Lemma 3.5 of Gangopadhyay and Maulik (2019).

*3.3. Proofs of Corollaries 2.5 and 2.6*

Assumption 2.1(a) with $L\log_+ L$ moment condition gives

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}\left(\rho\left(\boldsymbol{R}_n\right)\mathbb{1}_{[\rho(\boldsymbol{R}_n)>n]}\right)\le c\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}\frac{1}{n}\mathbb{E}\left(R\mathbb{1}_{[R\in(j,j+1]]}\right)$$
$$\le c\sum_{j=1}^{\infty}\mathbb{E}\left(R\mathbb{1}_{[R\in(j,j+1]]}\right)\sum_{n=1}^{j}\frac{1}{n}\le c\mathbb{E}\left(R(1+\log_+ R)\right)<\infty.$$

This implies (3) and, using Remark 2.5 and Theorem 2.2, proves Corollary 2.5.

Similarly, Assumption 2.1(b) gives, for large enough $m$

$$\sum_{n=m}^{\infty}\frac{1}{n}\mathbb{E}\left(\rho\left(\boldsymbol{R}_n\right)\mathbb{1}_{[\rho(\boldsymbol{R}_n)>n]}\right)\le\sum_{n=m}^{\infty}\frac{1}{n\phi(n)}\mathbb{E}\left(\rho\left(\boldsymbol{R}_n\right)\phi\left(\rho\left(\boldsymbol{R}_n\right)\right)\right)<\infty.$$

This again implies (3) and proves Corollary 2.6.

## 4. Applications to elephant random walk

We consider a delayed elephant random walk (ERW) on nonnegative integer lattice of dimension $d$ with randomly reinforced memory. ERW was introduced in Schütz and Trimper (2004). We analyze the model using the results obtained in this article utilizing an interesting connection between the urn model and ERW discovered in Baur and Bertoin (2016).

The random walk is parametrized by three parameters, namely, mean memory reinforcement parameter $a>0$ and two mixing parameters $p,q\in(0,1)$ for delay and shift respectively. At every epoch, a past epoch is selected with probability proportional to its memory and the memory of the selected epoch is randomly reinforced. If there was a delay at the selected epoch, then the current epoch is also delayed with probability $1-p$ or else there is a unit movement in a randomly selected direction. If there was a shift at the selected epoch, it is repeated with probability $1-q$ or else there is a delay.

To construct the random walk, we consider three mutually independent i.i.d. sequences $(\boldsymbol{A}_n)_{n>1} := ((A_{0n},A_{1n},\ldots,A_{dn}))_{n\ge1}$, $(I_n)_{n\ge1}$ and $(J_n)_{n\ge1}$. The memory reinforcement $\boldsymbol{A}_n$ has nonnegative coordinates. The coordinates have common finite first moment $a$, but may have different distributions. The variable $I_n$ takes values $0,1,\ldots,d$ with probabilities $1-p,p/d,\ldots,p/d$ respectively, while $J_n$ is a Bernoulli $(1-q)$ random variable. Independent of these sequences, $\boldsymbol{U}$ is a random vector, uniform over $\{\boldsymbol{e}_0,\ldots,\boldsymbol{e}_d\}$, where $\boldsymbol{e}_0$ is the zero vector in $\mathbb{R}^d$ and, as before, for each $i\ge1$, $\boldsymbol{e}_i$ is the $i$th coordinate vector in $\mathbb{R}^d$.

At epoch $n$, the elephant is at $\boldsymbol{L}_n$ and the step taken is $\boldsymbol{X}_n$, giving, $\boldsymbol{L}_n=\boldsymbol{L}_{n-1}+\boldsymbol{X}_n$. The step $\boldsymbol{X}_n$ takes values $\boldsymbol{e}_0,\boldsymbol{e}_1,\ldots,\boldsymbol{e}_d$, corresponding to a delay or a unit step shift along one of the $d$ coordinate axes respectively. The elephant starts at the origin, i.e., $\boldsymbol{L}_0=\boldsymbol{e}_0$, and the first step $\boldsymbol{X}_1$ is taken to be $\boldsymbol{U}$.

For $n\ge1$, let $\mathcal{F}_n$ denote the $\sigma$-field generated by $\boldsymbol{U}$, $(I_m)_{m\le n}$, $(J_m)_{m\le n}$. An adapted sequence $((M_1^{(n)},\ldots,M_n^{(n)}),\mathcal{F}_n)_{n\ge1}$ denotes the memory of the elephant about the past epochs evolving over time. The memory sequence is initiated by taking $M_1^{(1)}=1$. At epoch $n>1$, the elephant chooses $\tau_n$, one of the past epochs, such that,

$$\mathbb{P}\left(\tau_n=u|\mathcal{F}_{n-1}\right)=\frac{M_u^{(n-1)}}{M_1^{(n-1)}+\cdots+M_{n-1}^{(n-1)}},\ \text{for } u=1,\ldots,n-1.$$

If the step $\boldsymbol{X}_{\tau_n}$, at the selected epoch $\tau_n$, was $\boldsymbol{e}_i$ for some $i=0,1,\ldots,d$, then the memory $M_{\tau_n}^{(n-1)}$ associated with the selected epoch is reinforced by $A_{in}$, that is $M_{\tau_n}^{(n)}=M_{\tau_n}^{(n-1)}+A_{in}$. Other memories remain unchanged.

For $n>1$, the current step $\boldsymbol{X}_n$ is chosen as follows. If $\boldsymbol{X}_{\tau_n}$ was $\boldsymbol{e}_0$, then $\boldsymbol{X}_n$ becomes $\boldsymbol{e}_{I_n}$, that is, it is $\boldsymbol{e}_0$ with probability $1-p$, and, for $i=1,\ldots,d$, it is $\boldsymbol{e}_i$ with probability $p/d$. If, on the other hand, $\boldsymbol{X}_{\tau_n}$ was $\boldsymbol{e}_i$ for some $i=1,\ldots,d$, then $\boldsymbol{X}_n$ is $\boldsymbol{e}_{i\cdot J_n}$, that is, it takes values $\boldsymbol{e}_0$ and $\boldsymbol{e}_i$ with probability $q$ and $1-q$ respectively. Finally, the current epoch $n$ is assigned memory 1, i.e., $M_n^{(n)}=1$.

Theorem 2.3 and the connection between ERW and urn models from Baur and Bertoin (2016) yield the strong law of $(\boldsymbol{L}_n)_{n\ge0}$. It depends on the mixing parameters $p,q$, but not on the mean memory reinforcement parameter $a$.

**Theorem 4.1.** *Consider the delayed elephant random walk on the positive orthant in dimension d with random reinforcement of memory, parametrized by a, p and q as described above. Then* $\frac{1}{n}\boldsymbol{L}_n\to\frac{p}{d(p+q)}\mathbf{1}$ *almost surely and in* $L^1$.

**Proof.** It was noted in Baur and Bertoin (2016) that the evolution of the ERW depends on the moves at the selected past epoch rather than the selected epoch itself. Thus we consider an urn model with memory as objects categorized by the types of moves. Consider the vector of memory content of each type of step at epoch $n\ge1$, denoted by $\boldsymbol{W}^{(n)}:=(W_0^{(n)},\ldots,W_d^{(n)})$, where $W_i^{(n)}:=\sum_{1\le k\le n}M_k^{(n)}\mathbb{1}_{[\boldsymbol{X}_k=\boldsymbol{e}_i]}$ for $i=0,1,\ldots,d$. At epoch $n$, memory of the type of $\boldsymbol{X}_{\tau_n}$

is increased by the random reinforcement, while, memory of the type of $\boldsymbol{X}_n$ is increased by 1. Thus $\left(\boldsymbol{W}^{(n)}\right)_{n\geq 0}$ behaves as an urn model of $(d+1)$ types (indexed by $0, 1, \ldots, d$) with i.i.d. replacement matrices having common mean

$$\boldsymbol{R} = \begin{pmatrix} 1-p+a & \frac{p}{d} & \frac{p}{d} & \cdots & \frac{p}{d} \\ q & 1-q+a & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q & 0 & 0 & \cdots & 1-q+a \end{pmatrix}.$$

Clearly, the dominant eigenvalue of $\boldsymbol{R}$ is the common row sum $(1+a)$ and the corresponding left eigenvector normalized to probability is $\frac{1}{d(p+q)}(dq, p, \ldots, p)$. Finally, note that, for $i = 1, \ldots, d$, the memory of the step at epoch $k$ can be reinforced by 1 at epoch $k$ and, further in future epochs by a random amount if selected, and, hence, for $i = 1, \ldots, d$

$$\begin{aligned} \frac{1}{n}W_i^{(n)} &= \frac{1}{n}\sum_{k=1}^{n}\left(1+\sum_{l=k+1}^{n}A_{il}\mathbb{1}_{[\tau_l=k]}\right)\mathbb{1}_{[\boldsymbol{X}_k=\boldsymbol{e}_i]} \\ &= \frac{1}{n}L_{ni} + \frac{a}{n}\sum_{l=2}^{n}\mathbb{1}_{[\boldsymbol{X}_{\tau_l}=\boldsymbol{e}_i]} + \frac{1}{n}\sum_{l=2}^{n}(A_{il}-a)\mathbb{1}_{[\boldsymbol{X}_{\tau_l}=\boldsymbol{e}_i]}. \end{aligned} \tag{7}$$

By Theorem 2.3, the composition vector $\frac{1}{n}W_i^{(n)} \to \frac{(1+a)p}{d(p+q)}$ and the count vector $\frac{1}{n}\sum_{l=2}^{n}\mathbb{1}_{[\boldsymbol{X}_{\tau_l}=\boldsymbol{e}_i]} \to \frac{p}{d(p+q)}$. Finally, by Lemma 3.2, the last term of (7) is negligible, with $(A_{il}+a)$ as the dominator. Hence the result follows. □

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