MINIMAL DIFFEOMORPHISMS WITH L^1 HOPF DIFFERENTIALS

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ABSTRACT. We prove that for any two Riemannian metrics σ_1, σ_2 on the unit disk, a homeomorphism $\partial\mathbb{D} \to \partial\mathbb{D}$ extends to at most one quasiconformal minimal diffeomorphism $(\mathbb{D}, \sigma_1) \to (\mathbb{D}, \sigma_2)$ with L^1 Hopf differential. For minimal Lagrangian diffeomorphisms between hyperbolic disks, the result is known, but this is the first proof that does not use anti-de Sitter geometry. We show that the result fails without the L^1 assumption in variable curvature. The key input for our proof is the uniqueness of solutions for a certain Plateau problem in a product of trees.

1. Introduction

1.1. **Minimal diffeomorphisms.** Throughout, $\mathbb{D} \subset \mathbb{C}$ denotes the unit disk. A diffeomorphism between Riemannian surfaces $f : (\Sigma, \sigma_1) \to (\Sigma, \sigma_2)$ is a minimal diffeomorphism if its graph inside the product 4-manifold $(\Sigma^2, \sigma_1 \oplus \sigma_2)$ is a minimal surface. Schoen first proved that any diffeomorphism between closed hyperbolic surfaces can be deformed to a unique minimal one [25] (see also [30] and [21, Theorems B1 and C]). Bonsante-Schlenker then proved the following striking result [5].

Theorem 1.1 (Bonsante-Schlenker)**.** *Let $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ be a quasisymmetric map and let σ be a hyperbolic metric on $\mathbb{D}$. There exists a unique minimal diffeomorphism $f : (\mathbb{D}, \sigma) \to (\mathbb{D}, \sigma)$ that extends to φ on $\partial\mathbb{D}$. Moreover, f is quasiconformal.*

Note that any graph of a minimal diffeomorphism in $(\mathbb{D}^2, \sigma \oplus \sigma)$ is necessarily Lagrangian. Three more proofs of Theorem 1.1 have appeared in the literature [6], [15], [26]; including [5], all of the proofs have exploited a correspondence between minimal Lagrangian graphs in $(\mathbb{D}^2, \sigma \oplus \sigma)$ and spacelike maximal surfaces in $3d$ anti-de Sitter space, AdS^3. The AdS^3 perspective reveals a lot of interesting mathematics and deserves further exploration, but perhaps some of the geometry of minimal diffeomorphisms becomes obscured.

In this paper we give a short and purely Riemannian proof of the uniqueness statement, albeit under an additional integrability assumption, but which also applies in a broader setting. The proof is a manifestation of the basic fact that among all maps from the unit disk that fill up a Jordan domain $\Omega \subset \mathbb{C}$, the Riemann map minimizes the Dirichlet energy.

Before stating our main theorem, we should recall that Marković proved in [18] that any minimal diffeomorphism between closed Riemannian surfaces of genus at least 2 is unique in its homotopy class, interestingly with no requirement on the curvature of the metrics. The results of the current paper are inspired by the paper [18], along with the older paper [16] of Mateljević-Marković about harmonic maps of the disk.

Let σ_1, σ_2 be a pair of Riemannian metrics on the unit disk. Given the graph of a minimal map $G \subset (\mathbb{D}^2, \sigma_1 \oplus \sigma_2)$, the projections $p_i : G \to (\mathbb{D}, \sigma_i)$, $i = 1, 2$, are harmonic. Any harmonic map from a surface comes with a holomorphic quadratic differential called the Hopf differential, and the Hopf differentials of p_1 and p_2 necessarily sum to zero. Conversely, a pair of harmonic diffeomorphisms $h_i : \mathbb{D} \to (\mathbb{D}, \sigma_i)$ with opposite Hopf differentials determines a minimal diffeomorphism via $h_1 \circ h_2^{-1}$. We refer to the Hopf differential of the first harmonic projection as the Hopf differential of the minimal diffeomorphism.

Theorem A. *Let σ_1 and σ_2 be a pair of Riemannian metrics on $\mathbb{D}$ and let $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ be a homeomorphism. Then φ extends to at most one quasiconformal minimal diffeomorphism $(\mathbb{D}, \sigma_1) \to (\mathbb{D}, \sigma_2)$ with integrable Hopf differential.*

Theorem A has no curvature assumption, as in [18], and no completeness either. We do not treat the question of existence, although we expect existence to hold for complete metrics with pinched negative curvature. Existence results can be found in [7] and [28].

Remark 1.2. When the metric is complete and has pinched negative curvature, a harmonic map is quasiconformal, and hence admits quasisymmetric boundary extension, if and only if the Hopf differential has finite Bers norm [31], [27], which is weaker than integrability (see [12, Chapter 5.4]). So for σ_1, σ_2 hyperbolic, Theorem A is contained in Theorem 1.1.

Remark 1.3. The quasisymmetric assumption in Theorem 1.1 can be removed. The proof of existence in [5] doesn't appear to use quasisymmetry, although uniqueness certainly does. More recent work of Seppi-Smith-Toulisse [26] implies that any homeomorphism of $\partial\mathbb{D}$ extends uniquely to a minimal diffeomorphism of the hyperbolic disk whose associated maximal surface in AdS^3 is complete. Trebeschi shows in [29] that the required completeness assumption from [26] is automatic, thereby closing the uniqueness question.

We also observe that at the chosen level of generality, a boundedness assumption like L^1-ness is necessary.

Example A. *There exists a flat metric g on $\mathbb{D}$ and two quasiconformal minimal diffeomorphisms of $(\mathbb{D}, g)$ that extend to the same homeomorphisms of $\partial\mathbb{D}$. One of them has L^1 Hopf differential, while the other does not.*

So, interestingly, our assumption is a trade-off: by Remark 1.2 it is restrictive in hyperbolic space, but necessary in our general setting. Without such an assumption, our method still proves that minimal diffeomorphisms are stable (see Definition 4.1). Compare with [30] and [9, Proposition 1.7].

Theorem A'. *Let σ_1, σ_2 be a pair of Riemannian metrics on the unit disk. Any minimal diffeomorphism $(\mathbb{D}, \sigma_1) \to (\mathbb{D}, \sigma_2)$ is stable.*

The geometric meaning of the L^1 assumption will be explained in the coming subsections. For now, it allows us to invoke the new main inequality, introduced in [18] and [19] and studied further in [20].

1.2. **The new main inequality.** Let S be a Riemann surface, $\phi_1, \dots, \phi_n$ integrable holomorphic quadratic differentials on S summing to zero, and $f_1, \dots, f_n : S \to S'$ mutually homotopic quasiconformal maps to another Riemann surface with Beltrami forms $\mu_1, \dots, \mu_n$. If ∂S is non-empty, we ask that $f_1, \dots, f_n$ are mutually homotopic relative to ∂S.

Definition 1.4. We say that the new main inequality holds if:

$$\mathrm{Re} \sum_{i=1}^{n} \int_S \phi_i \cdot \frac{\mu_i}{1 - |\mu_i|^2} \leq \sum_{i=1}^{n} \int_S |\phi_i| \cdot \frac{|\mu_i|^2}{1 - |\mu_i|^2}. \tag{1}$$

Taking $n = 2$, f_1 (relatively) homotopic to the identity, and f_2 the identity, (1) becomes the classical main inequality for quasiconformal maps, a key ingredient in the proof of Teichmüller's uniqueness theorem. On closed surfaces, (1) always holds for $n = 1, 2$ [18], but fails for $n \geq 3$ [19]. We prove

Theorem B. *For $n = 2$ and $S = \mathbb{D}$, the new main inequality always holds.*

1.3. **Outline of the argument.** The proof of Theorem A follows the methodology of a series of papers on the uniqueness and non-uniqueness of minimal surfaces in products of closed surfaces [18], [19], [21]. These papers led to the paper [24], which resolved a conjecture about minimal surfaces in more general locally symmetric spaces.

In [18], Marković proves uniqueness of minimal diffeomorphisms using the new main inequality for $n = 2$. The main difference between our argument and that of [18] is the proof of the new main inequality. In [18], (1) is proved as a consequence of the known uniqueness of minimal Lagrangians between closed hyperbolic surfaces. Our proof, on the other hand, does not assume any other uniqueness result. One of our motivations for writing this paper is to share what we think is the most natural proof.

1.3.1. *Theorem A.* There is an established theory of harmonic maps from Riemannian manifolds to complete and non-positively curved (NPC) length spaces, such as $\mathbb{R}$-trees. A holomorphic quadratic differential on a Riemann surface gives rise to an $\mathbb{R}$-tree and a harmonic map from (a cover of) the Riemann surface to that $\mathbb{R}$-tree called a leaf-space projection.

Given a minimal diffeomorphism, we get a minimal map h into a product of two disks. As well, the Hopf differentials determine a minimal map π into a product of two $\mathbb{R}$-trees. The L^1 assumption is equivalent to demanding that the image of π has finite total energy (see the definition in Section 2.1). If $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, we show that the difference between the energy of $h|_{\mathbb{D}_r}$ and that of any other map with the same asymptotic boundary data is strictly bounded below by the difference in energies between π and some other map to the same product of $\mathbb{R}$-trees (we're lying a little bit to keep things simpler here). Theorem B implies that the total energy of π is less than that other map. We deduce that for r large, any such h strictly minimizes the energy over $\mathbb{D}_r$, and hence h is unique.

1.3.2. *Theorem B.* To prove Theorem B, we set up a Plateau problem in a product of $\mathbb{R}$-trees, asking for maps from $\overline{\mathbb{D}}$ that take $\partial\mathbb{D}$ onto a fixed embedded circle and minimize a certain energy or area. We will observe that if a product of leaf-space projections solves the Plateau problem, then (1) holds for their associated quadratic differentials. Thus, Theorem B amounts to studying this Plateau problem for $n = 2$.

Let $\mathbb{A}^1(\mathbb{D})$ be the Banach space of integrable holomorphic quadratic differentials on the unit disk equipped with the L^1 norm. Since (1) is a closed inequality, it is routine to prove

Proposition 1.5. *Let $B \subset \mathbb{A}^1(\mathbb{D})$ be a dense subset, and suppose that for all quasiconformal maps that agree on $\partial\mathbb{D}$, (1) holds for every $\phi \in B$. Then (1) holds for all $\phi \in \mathbb{A}^1(\mathbb{D})$.*

Thus, while $\mathbb{R}$-trees can be quite wild (see the images in [2]), we only need to study the Plateau problem for $\mathbb{R}$-trees arising from a generic class in $\mathbb{A}^1(\mathbb{D})$. Choosing our quadratic differentials to be polynomials, the $\mathbb{R}$-trees become simplicial trees, and the Plateau problem becomes nearly identical to the (more or less trivial) Plateau problem in the plane.

1.4. **Future perspectives.** In view of Theorem 1.1 and Remark 1.2, it's natural to pose the following problem, which we'll say more about in Section 4.1.

Problem A. *For σ_1 and σ_2 complete and with pinched negative curvature, prove uniqueness of minimal diffeomorphisms $(\mathbb{D}, \sigma_1) \to (\mathbb{D}, \sigma_2)$ with quasisymmetric boundary data.*

In another direction, the ideas here might be applicable for studying minimal disks in rank 2 symmetric spaces. For all of the rank 2 Lie groups that admit higher Teichmüller spaces, except the split real form of G_2, various authors have established existence and uniqueness results for quasi-isometric minimal disks in associated pseudo-Riemannian homogeneous spaces (see [4], [15], and [14]). Through some form of a twistor correspondence, akin to using AdS^3 to prove Theorem 1.1, these results imply existence and uniqueness results for minimal surfaces in symmetric spaces. One could try to push further the ideas from this paper to give a unified set of uniqueness statements, without passing through pseudo-Riemannian geometry. In the symmetric space context, products of trees are replaced by rank 2 Euclidean buildings modeled on Cartan subalgebras.

The L^1 condition in fact generalizes nicely. As explained in [24] (for closed surfaces), a minimal map to a symmetric space of rank k gives rise to a minimal map to $\mathbb{R}^k$ with special properties, in a rather explicit way. For $\mathbb{D}^2$, the construction is less complicated and we can explain it here. Given a minimal diffeomorphism from $\mathbb{D}$ to $\mathbb{D}$ (with any metrics), assume that the Hopf differential ϕ is the square of an abelian differential α. Then the differentials α and $i\alpha$ determine Weierstrass-Enneper data for a minimal map to $\mathbb{R}^2$. The energy of this map, which dominates the area of the image, is equal to the L^1 norm of ϕ. If ϕ is not a square, there is a degree 2 branched cover of $\mathbb{D}$ on which it lifts to one, and after passing to the universal cover of that branched cover, we can repeat the construction. Going back to symmetric spaces, one could look at minimal maps that give finite area maps to $\mathbb{R}^2$.

Finally, this paper also opens up some new discussions on harmonic maps from the disk to $\mathbb{R}$-trees that we would like to highlight (see Remarks 2.8 and 3.7).

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2. Harmonic maps

We give all definitions specialized to open subsets of $\mathbb{C}$. Throughout, let $z = x + iy$ be the standard coordinate on $\mathbb{C}$. For this section, fix a simply connected domain $\Omega \subset \mathbb{C}$.

2.1. **Harmonic maps.** Let (M, d) be a complete and NPC length space and $h : \Omega \to (M, d)$ a Lipschitz map. By work of Korevaar-Schoen [13, Theorem 2.3.2], we can associate a locally L^1 measurable metric $g(h)$, defined on pairs of Lipschitz vector fields. If h is a C^1 map to a smooth manifold M and the distance d is induced by a Riemannian metric σ, then $g(h)$ is represented by the pullback metric $h^*\sigma$. The energy density form is the locally L^1 form

$$e(h) = \frac{1}{2}(\mathrm{trace} g(h))dxdy, \tag{2}$$

where the trace is with respect to the flat metric on Ω. The total energy is

$$\mathcal{E}(\Omega, h) = \int_\Omega e(h).$$

We want to consider boundary value problems and to allow $\mathcal{E}(\Omega, h) = \infty$. Hence we make

Definition 2.1. h is harmonic if for all relatively compact domains $U \subset \Omega$ with compact closure contained in Ω, $h|_U$ is a critical point of $f \mapsto \mathcal{E}(U, f)$ with respect to variations compactly supported on U.

There are many existence and uniqueness results that extend the Riemannian theory. We recall just one.

Theorem 2.2 (Theorem 2.2 in [13])**.** *Let $U \subset \Omega$ be a relatively compact and Lipschitz domain with compact closure contained in Ω. Given a Lipschitz map $f : \overline{U} \to (X, d)$, there exists a unique harmonic map $h : \overline{U} \to X$ such that $h = f$ on ∂U. Moreover, h minimizes $\mathcal{E}(U, \cdot)$ relative to its boundary values.*

Writing $x_1 = x$ and $x_2 = y$, let $g_{ij}(h)$ be the components of $g(h)$.

Definition 2.3. The Hopf differential of h is the measurable quadratic differential given by

$$\phi(h)(z) = \frac{1}{4}(g_{11}(h)(z) - g_{22}(h)(z) - 2ig_{12}(h)(z))dz^2. \tag{3}$$

When h is harmonic, the Hopf differential is represented by a holomorphic quadratic differential. In the Riemannian setting, (3) is

$$\phi(h)(z) = h^*\sigma\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big)(z)dz^2,$$

and hence h is weakly conformal if and only if $\phi(h) = 0$. Recall that the image of a harmonic and conformal immersion inside a Riemannian manifold defines a minimal surface. Thus, the following definition is appropriate for the singular setting.

Definition 2.4. $h : \Omega \to (M, d)$ is minimal if it is harmonic and $\phi(h) = 0$.

Directly from the definitions (2) and (3), the energy density and Hopf differential of a map into a product of spaces is the sum of the energies and Hopf differentials of the component maps. Consequently, a map into a product of disks

$$h = (h_1, h_2) : \Omega \to (\mathbb{D}^2, \sigma_1 \oplus \sigma_2)$$

is harmonic if both components are harmonic, and minimal when $\phi(h_1) = -\phi(h_2)$. It should thus be clear that minimal diffeomorphisms yield minimal graphs in the product 4-manifold.

2.2. **Harmonic maps to $\mathbb{R}$-trees.** For an introduction to harmonic maps to $\mathbb{R}$-trees in the more usual context of closed surfaces, see [33].

Definition 2.5. An $\mathbb{R}$-tree is a length space (T, d) such that any two points are connected by a unique arc, and every arc is a geodesic, isometric to a segment in $\mathbb{R}$.

Let ϕ be a holomorphic quadratic differential on $\mathbb{C}$. The vertical (resp. horizontal) foliation of ϕ is the singular foliation on $\mathbb{C}$ whose non-singular leaves are the integral curves of the line field on $\Omega\backslash\phi^{-1}(0)$ on which ϕ returns a negative (resp. positive) real number. The singularities are standard prongs at the zeros. Both foliations come with transverse measures $|\mathrm{Im}\sqrt{\phi}|$ and $|\mathrm{Re}\sqrt{\phi}|$ respectively (see [11, Exposé 5] for precise definitions).

We define an equivalence relation on $\mathbb{C}$ under which two points $x, y \in \mathbb{C}$ are identified if they lie on the same leaf of the vertical foliation of ϕ. We denote the quotient by T, and the quotient projection as $\pi : \mathbb{C} \to T$. The transverse measure pushes down via π to a distance function d that makes (T, d) into an $\mathbb{R}$-tree. The same construction applies for the horizontal foliation, which is also the vertical foliation of $-\phi$. We work with the leaf space for the vertical foliation, unless specified otherwise.

The energy density and Hopf differential of π can be described explicitly: at a point $p \in \mathbb{C}$ on which $\phi(p) \neq 0$, π locally isometrically factors through a segment in $\mathbb{R}$. In a small neighbourhood around that point, $g(h)$ is represented by the pullback metric of the locally defined map to $\mathbb{R}$. Therefore, it can be computed that

$$e(\pi) = |\phi|/2, \ \phi(\pi) = \phi/4. \tag{4}$$

In view of (4), we will always rescale the metric on T from (T, d) to $(T, 2d)$.

Now, recall that we've been working on a simply connected domain $\Omega \subset \mathbb{C}$. We add the assumption that Ω is bounded and has Lipschitz boundary, and restrict π from $\mathbb{C}$ to $\overline{\Omega} \subset \mathbb{C}$. In our normalization, by the first formula in (4) the total energy of π on Ω is

$$\mathcal{E}(\Omega, \pi) = \int_\Omega |\phi| < \infty,$$

the $L^1(\Omega)$-norm of ϕ. We will think of $\pi : \overline{\Omega} \to (T, 2d)$ as solving a boundary value problem, which is justified by the proposition below.

Proposition 2.6. *$\pi : \overline{\Omega} \to (T, 2d)$ is harmonic.*

Proof. From (4), $\pi|_{\overline{\Omega}}$ is Lipschitz. Thus, we can appeal to the existence result of Korevaar-Schoen, Theorem 2.2, which produces a harmonic map $h : \overline{\Omega} \to (T, 2d)$ that agrees with π on $\partial\Omega$. It is proved in [10, Proposition 3.2] that harmonic maps to $\mathbb{R}$-trees pull back germs of convex functions to germs of subharmonic functions. On the other hand, the local argument from [33, Section 4] shows that π pulls back germs of convex functions on $(T, 2d)$ to germs of subharmonic functions. As the distance function d is convex, the map $p \mapsto d(h(p), \pi(p))$ is subharmonic on Ω. Since $d(h(p), \pi(p)) = 0$ on $\partial\Omega$, we deduce that $h = \pi$. □

Later on, we will add a condition that will simplify the proofs: that ϕ is a polynomial. Through basic geometric considerations, the following should be clear. Details are explained in [3, Proposition 2.2].

Proposition 2.7. *When ϕ is a polynomial of degree n, the leaf spaces of the vertical and horizontal foliations are complete simplicial $\mathbb{R}$-trees with $n + 2$ infinite edges, which can be properly and geodesically embedded in $\mathbb{R}^2$.*

See Figure 1. In fact, the converse to Proposition 2.7 is the subject of the paper [3].

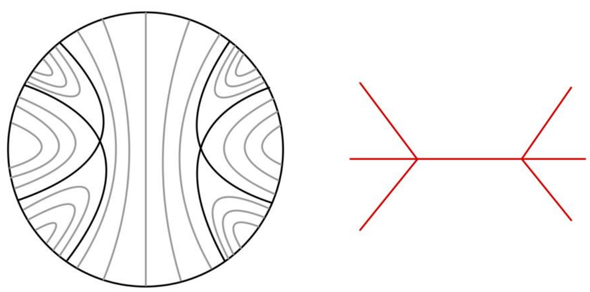

FIGURE 1. The foliation and $\mathbb{R}$-tree of a polynomial with two double order zeros, restricted to $\overline{\mathbb{D}}$.

Remark 2.8. Since our theorem concerns $\phi \in \mathbb{A}^1(\mathbb{D})$, it might appear strange to begin with a quadratic differential on $\mathbb{C}$. Our main reason for making this choice is that the $\mathbb{R}$-tree construction requires more care on the disk, and it's unnecessarily complicated to define a Plateau problem as below for quadratic differentials that don't extend holomorphically past $\partial\mathbb{D}$. It would be interesting to develop the theory of harmonic maps from $\mathbb{D}$ to $\mathbb{R}$-trees more thoroughly.

3. PROOFS

3.1. **The Reich-Strebel formula.** The Reich-Strebel formula (Proposition 5 and originally equation 1.1 in [23]) will play an important role in the main proofs.

As above, let $\Omega, \Omega' \subset \mathbb{C}$ be open. Recall that the Beltrami form of a locally quasiconformal map $f : \Omega \to \Omega' \subset \mathbb{C}$ is the measurable tensor

$$\mu(z) = \frac{f_{\overline{z}}(z)d\overline{z}}{f_z(z)dz},$$

where derivatives are taken in the weak sense. Let (M,d) be a complete NPC space, and $h:\Omega\to(M,d)$ a Lipschitz map with finite total energy. Let μ_f be the Beltrami form of f, $J_{f^{-1}}$ the Jacobian of f^{-1}, and ϕ the Hopf differential of h, which need not be holomorphic. One can verify the identity

$$e(h\circ f^{-1}) = (e(h)\circ f^{-1})J_{f^{-1}} + 2(e(h)\circ f^{-1})J_{f^{-1}}\frac{(|\mu_f|^2\circ f^{-1})}{1-(|\mu_f|^2\circ f^{-1})}$$
$$- 4\mathrm{Re}\Big((\phi(h)\circ f^{-1})J_{f^{-1}}\frac{(\mu_f\circ f^{-1})}{1-(|\mu_f|^2\circ f^{-1})}\Big)$$

from [23]. Integrating, we obtain the proposition below.

Proposition 3.1. *We have the formula*

$$\mathcal{E}(\Omega', h\circ f^{-1}) - \mathcal{E}(\Omega,h) = \int_\Omega -4Re\Big(\phi(h)\cdot\frac{\mu}{1-|\mu|^2}\Big) + 2e(h)\cdot\frac{|\mu|^2}{1-|\mu|^2}. \tag{5}$$

When the target (M,d) is an $\mathbb{R}$-tree, we replace $e(h)$ by $2|\phi(h)|$, so the formula (5) involves only ϕ and μ.

3.2. **The Plateau problem in a product of trees.** See [22] or [8, Chapter 4] for exposition on the classical Plateau problem about minimal surfaces in $\mathbb{R}^n$ (among many other excellent sources).

For $n\geq 2$ and $i=1,\dots,n$, let (T_i,d_i) be a geodesically complete $\mathbb{R}$-tree and let (X,d) be the product $(X,d)=\prod_{i=1}^n(T_i,d_i)$. Let $\gamma\subset(X,d)$ be an embedded circle.

Definition 3.2. Let $h:\overline{\mathbb{D}}\to(X,d)$ be a Lipschitz map such that $h(\partial\mathbb{D})=\gamma$. We say that h solves the Plateau problem for γ if for all Lipschitz maps $g:\overline{\mathbb{D}}\to(X,d)$ with $g(\partial\Omega)=\gamma$,

$$\mathcal{E}(\mathbb{D},h)\leq\mathcal{E}(\mathbb{D},g).$$

The proposition below explains the relation between our Plateau problem and the new main inequality.

Proposition 3.3. *Suppose that $h=(h_1,\dots,h_n):\overline{\mathbb{D}}\to(X,d)$ solves the Plateau problem for γ. For $i=1,\dots,n$, let ϕ_i be the Hopf differential of h_i. Then, for any choice of quasiconformal maps $f_i:\mathbb{D}\to\mathbb{D}$ that all extend to the same map on $\partial\mathbb{D}$, (1) holds.*

Proof. For each i, let μ_i be the Beltrami form of f_i. Since every f_i agrees on $\partial\mathbb{D}$, postcomposing each h_i with every f_i^{-1} provides a reparametrization of γ. In symbols,

$$(h_1\circ f_1^{-1},\dots,h_n\circ f_n^{-1})(\partial\mathbb{D})=\gamma.$$

The hypothesis that h solves the Plateau problem thus yields that

$$\mathcal{E}(\mathbb{D},h)\leq\sum_{i=1}^n\mathcal{E}(\mathbb{D},h_i\circ f_i^{-1}).$$

Splitting $\mathcal{E}(\mathbb{D},h)$ into a sum of energies,

$$\sum_{i=1}^n\mathcal{E}(\mathbb{D},h_i)-\sum_{i=1}^n\mathcal{E}(\mathbb{D},h_i\circ f_i^{-1})\leq 0. \tag{6}$$

For each i, we apply the Reich-Strebel formula to (6) and insert (4) for $e(h_i)$. Dividing by 4 returns

$$\sum_{i=1}^n\mathrm{Re}\int_S\phi(h)\cdot\frac{\mu}{1-|\mu|^2}\leq\sum_{i=1}^n\int_S|\phi_i|\cdot\frac{|\mu|^2}{1-|\mu|^2}.$$

□

Remark 3.4. A version of Proposition 3.3 holds for maps between closed surfaces. In that context, we proved the converse statement [20, Theorem A].

Define the Korevaar-Schoen area as

$$A(h) = \int_{\mathbb{D}} \sqrt{\det g(h)}.$$

When (X,d) can be isometrically embedded in a manifold, h is injective, and $g(h)$ is smooth and non-degenerate on a residual set, then $A(h)$ is the area of the image of h. By Cauchy-Schwarz, $\sqrt{\det g(h)} \leq e(h)$ with equality if and only if $\phi(h) = 0$. Here, the inequality of tensors means that over any point p, for any non-zero tangent vector v, $\sqrt{\det g(h)}_p(v,v) \leq e(h)_p(v,v)$. Integrating,

$$A(h) \leq \mathcal{E}(\mathbb{D}, h),$$

with the same equality condition. We now set $n = 2$. In view of Proposition 1.5, we concern ourselves only with the Plateau problem for products of simplicial trees.

Proposition 3.5. *Let ϕ be a polynomial on $\mathbb{C}$ with simplicial trees $(T_1, 2d_1)$ and $(T_2, 2d_2)$ arising from the vertical and horizontal foliations, and let*

$$\pi : \overline{\mathbb{D}} \to (X,d) = (T_1, 2d_1) \times (T_2, 2d_2)$$

be the product of the leaf-space projections, restricted to $\overline{\mathbb{D}}$. Then π solves the Plateau problem for $\pi(\partial\mathbb{D})$.

Proof. As a product of simplicial trees, (X,d) is, apart from at finitely many points, locally isometric to $\mathbb{R}^2$. By basic 2-dimensional Euclidean geometry, any region in (X,d) containing $\pi(\overline{\partial\mathbb{D}})$ and intersecting the image $\pi(\overline{\mathbb{D}})$ must contain $\pi(\overline{\mathbb{D}})$. One checks by hand that π is injective, and hence if g is another candidate for the Plateau problem, it has a larger Korevaar-Schoen area:

$$A(h) \leq A(g).$$

We then use that energy dominates area, with equality only for minimal maps:

$$\mathcal{E}(\mathbb{D}, h) = A(h) \leq A(g) \leq \mathcal{E}(\mathbb{D}, g).$$

□

Remark 3.6. Plateau problem solutions are minimal maps and minimize $A(\cdot)$. Indeed, by local isometric factoring, the Korevaar-Schoen metric of a harmonic map to (X,d) is smooth off of a discrete set. From that observation, the proof for maps to $\mathbb{R}^n$ goes through (see [8, Chapter 4.1]).

Remark 3.7. Solutions for our Plateau problem do not always exist (take n copies of the same tree). In view of Theorem 2.2, one could hope to solve the Plateau problem by adapting the Douglas-Rado strategy for the classical problem. We expect this method to work when the trees are dual to foliations that have a "filling" property (see [32]).

3.3. **Proof of the new main inequality.** Combining Proposition 3.3 and Proposition 3.5, we obtain

Proposition 3.8. *Let ϕ be a polynomial quadratic differential on $\mathbb{C}$. Then, for $\phi, -\phi$ and any choice of quasiconformal maps from $\mathbb{D} \to \mathbb{D}$ that agree on $\partial\mathbb{D}$, (1) holds.*

To promote Proposition 3.8 to Theorem B we use the following basic fact.

Lemma 3.9. *Polynomials are dense in $A^1(\mathbb{D})$.*

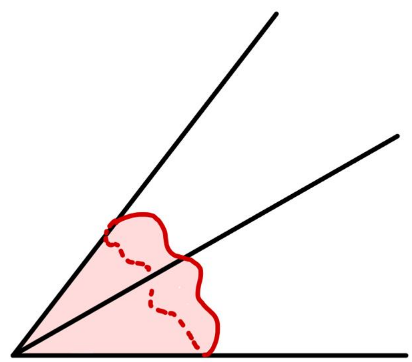

FIGURE 2. An embedded circle bounding a disk in a particularly simple product of simplicial complexes.

Proof. By Stone-Weierstrass, polynomials are dense in the Banach space of continuous functions on $\overline{\mathbb{D}}$ with the sup norm. A simple $\frac{\epsilon}{3}$-argument then shows density in $\mathbb{A}^1(\mathbb{D})$. □

It's now time to prove Proposition 1.5. Recall the statement is that if (1) holds on an arbitrary dense subset $B \subset \mathbb{A}^1(\mathbb{D})$, then (1) holds on all of $\mathbb{A}^1(\mathbb{D})$.

Proof of Proposition 1.5. Let $\phi \in \mathbb{A}^1(\mathbb{D})$, and let $(p_n)_{n=1}^\infty \subset B$ be a sequence approximating ϕ in $\mathbb{A}^1(\mathbb{D})$. Let μ_1 and μ_2 be Beltrami forms of quasiconformal maps on the unit disk with the same boundary values, and let $k = \max\{||\mu_1||_{L^\infty(\mathbb{D})}, ||\mu_2||_{L^\infty(\mathbb{D})}\} < 1$. By our hypothesis, for μ_1, μ_2 and every p_n:

$$\mathrm{Re} \int_{\mathbb{D}} \Big(p_n \frac{\mu_1}{1-|\mu_1|^2} - p_n \frac{\mu_2}{1-|\mu_2|^2} \Big) \leq \int_{\mathbb{D}} |p_n| \Big(\frac{|\mu_1|^2}{1-|\mu_1|^2} + \frac{|\mu_2|^2}{1-|\mu_2|^2} \Big).$$

We check that for both $i = 1, 2$,

$$\Big| \mathrm{Re} \int_{\mathbb{D}} \phi \frac{\mu_i}{1-|\mu_i|^2} - \mathrm{Re} \int_{\mathbb{D}} p_n \frac{\mu_i}{1-|\mu_i|^2} \Big| \leq \frac{k}{1-k^2} \int_{\mathbb{D}} |\phi - p_n| \to 0$$

as $n \to \infty$. Consequently,

$$\mathrm{Re} \int_{\mathbb{D}} \Big(\phi \frac{\mu_1}{1-|\mu_1|^2} - \phi \frac{\mu_2}{1-|\mu_2|^2} \Big) \leq \liminf_{n\to\infty} \int_{\mathbb{D}} |p_n| \Big(\frac{|\mu_1|^2}{1-|\mu_1|^2} + \frac{|\mu_2|^2}{1-|\mu_2|^2} \Big). \tag{7}$$

For the right hand side,

$$\int_{\mathbb{D}} |p_n| \Big(\frac{|\mu_1|^2}{1-|\mu_1|^2} + \frac{|\mu_2|^2}{1-|\mu_2|^2} \Big) \leq \int_{\mathbb{D}} |\phi| \Big(\frac{|\mu_1|^2}{1-|\mu_1|^2} + \frac{|\mu_2|^2}{1-|\mu_2|^2} \Big) + \frac{2k^2}{1-k^2} \int_{\mathbb{D}} |\phi - p_n|.$$

Taking $n \to \infty$ in (7) thus yields

$$\mathrm{Re} \int_{\mathbb{D}} \Big(\phi \frac{\mu_1}{1-|\mu_1|^2} - \phi \frac{\mu_2}{1-|\mu_2|^2} \Big) \leq \int_{\mathbb{D}} |\phi| \Big(\frac{|\mu_1|^2}{1-|\mu_1|^2} + \frac{|\mu_2|^2}{1-|\mu_2|^2} \Big),$$

which is exactly (1) for $n = 2$. □

Putting everything together, we obtain Theorem B.

Proof of Theorem B. By Proposition 3.8 and Lemma 3.9, the new main inequality holds on a dense subset of $\mathbb{A}^1(\mathbb{D})$. We then apply Proposition 1.5. □

3.4. **Proof of uniqueness.** We're now prepared to prove Theorem A. We do so following the argument of Marković for closed surfaces in [18]. We have to deal with a minor complication, which is that our harmonic maps might have infinite total energy. Indeed this is often the case: the harmonic diffeomorphisms in the hyperbolic case always have infinite energy (from [31, Proposition 10]). Accordingly, we don't want to globally integrate any term that involves an energy density.

Before we begin, we record a consequence of the definitions (2) and (3): for any immersion h to a Riemannian manifold of dimension at least 2,

$$e(h) > 2|\phi(h)| \tag{8}$$

everywhere, in the sense of tensors as before.

Proof of Theorem A. Let (h_1, h_2) and (g_1, g_2) be two quasiconformal minimal maps to $(\mathbb{D}^2, \sigma_1 \oplus \sigma_2)$ that induce minimal diffeomorphisms with L^1 Hopf differentials and that extend to the boundary as the same map $h_1 \circ h_2^{-1} = g_1 \circ g_2^{-1} = \varphi$. For $i = 1, 2$, set $f_i = g_i^{-1} \circ h_i$. The extensions of f_1 and f_2 to $\partial\mathbb{D}$ agree, so the new main inequality can be used on the Beltrami forms μ_i of f_i. Suppose for the sake of contradiction that for at least one i, f_i is not the identity. Let $\phi = \phi_1$ be the Hopf differential of h_1, so that $-\phi = \phi_2$ is the Hopf differential of h_2. To simplify notation, for each $0 < r < 1$, set $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. Applying the Reich-Strebel formula (5) over the subsurface $\mathbb{D}_r$,

$$\begin{aligned}
\sum_{i=1}^2 \mathcal{E}(f_i(\mathbb{D}_r), g_i) - \sum_{i=1}^2 \mathcal{E}(\mathbb{D}_r, h_i) &= \sum_{i=1}^2 \mathcal{E}(f_i(\mathbb{D}_r), h_i \circ f_i^{-1}) - \sum_{i=1}^2 \mathcal{E}(\mathbb{D}_r, h_i) \\
&= \sum_{i=1}^2 -4\mathrm{Re} \int_{\mathbb{D}_r} \phi_i \frac{\mu_i}{1 - |\mu_i|^2} + \sum_{i=1}^2 2 \int_{\mathbb{D}_r} e(h_i) \frac{|\mu_i|^2}{1 - |\mu_i|^2} \\
&> \sum_{i=1}^2 -4\mathrm{Re} \int_{\mathbb{D}_r} \phi_i \frac{\mu_i}{1 - |\mu_i|^2} + \sum_{i=1}^2 4 \int_{\mathbb{D}_r} |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2},
\end{aligned}$$

where in the last line we used (8). Note that difference between the last two lines is increasing with r, so in particular once r is at least $1/2$, there is an $\epsilon > 0$ independent of r such that

$$\sum_{i=1}^2 \mathcal{E}(f_i(\mathbb{D}_r), g_i) - \sum_{i=1}^2 \mathcal{E}(\mathbb{D}_r, h_i) \geq \sum_{i=1}^2 -4\mathrm{Re} \int_{\mathbb{D}_r} \phi_i \frac{\mu_i}{1 - |\mu_i|^2} + \sum_{i=1}^2 4 \int_{\mathbb{D}_r} |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2} + \epsilon.$$

Since each ϕ_i is integrable,

$$\sum_{i=1}^2 4\mathrm{Re} \int_{\mathbb{D}_r} \phi_i \frac{\mu_i}{1 - |\mu_i|^2} \to \sum_{i=1}^2 4\mathrm{Re} \int_{\mathbb{D}} \phi_i \frac{\mu_i}{1 - |\mu_i|^2}$$

and

$$\sum_{i=1}^2 4 \int_{\mathbb{D}_r} |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2} \to \sum_{i=1}^2 4 \int_{\mathbb{D}} |\phi_i| \frac{|\mu_i|^2}{1 - |\mu_i|^2}$$

as r increases to 1. Using Theorem B, we deduce that

$$\sum_{i=1}^{2}\mathcal{E}(f_i(\mathbb{D}_r),g_i)-\sum_{i=1}^{2}\mathcal{E}(\mathbb{D}_r,h_i)\geq\epsilon$$

for r sufficiently close to 1. Reversing the roles of h_i and g_i, we repeat the argument (while choosing our domains carefully): if ψ_i is the Hopf differential of g_i and ν_i is the Beltrami form of f_i^{-1}, there is a $\delta>0$ such that for $r\geq 1/2$,

$$\begin{aligned}\sum_{i=1}^{2}\mathcal{E}(\mathbb{D}_r,h_i)-\sum_{i=1}^{2}\mathcal{E}(f_i(\mathbb{D}_r),g_i)&=\sum_{i=1}^{2}\mathcal{E}(\mathbb{D}_r,g_i\circ f_i)-\sum_{i=1}^{2}\mathcal{E}(f_i(\mathbb{D}_r),g_i)\\&\geq\sum_{i=1}^{2}-4\mathrm{Re}\int_{f_i(\mathbb{D}_r)}\psi_i\frac{\nu_i}{1-|\nu_i|^2}+\sum_{i=1}^{2}4\int_{f_i(\mathbb{D}_r)}|\psi_i|\frac{|\nu_i|^2}{1-|\nu_i|^2}+\delta.\end{aligned}$$

Since f_i is quasiconformal, it is not hard to justify that

$$\sum_{i=1}^{2}4\mathrm{Re}\int_{f_i(\mathbb{D}_r)}\psi_i\frac{\nu_i}{1-|\nu_i|^2}\to\sum_{i=1}^{2}4\mathrm{Re}\int_{\mathbb{D}}\psi_i\frac{\nu_i}{1-|\nu_i|^2}$$

as $r\to 1$, and likewise for the other term. We deduce that

$$\sum_{i=1}^{2}\mathcal{E}(\mathbb{D}_r,h_i)-\sum_{i=1}^{2}\mathcal{E}_r(f_i(\mathbb{D}_r),g_i)\geq\delta$$

for r close to 1, which gives the contradiction. We conclude that both f_i are the identity map, so that $(h_1,h_2)=(g_1,g_2)$. □

4. The general situation

4.1. **Counterexample without integrability.** We now present Example A, showing that Theorem A is not true without the L^1 condition. The minimal diffeomorphisms are simple enough to write out by hand.

Example A. Let m be the flat Euclidean metric on the upper halfspace $\mathbb{H}\subset\mathbb{C}$. We give two minimal maps $(\mathbb{H},m)\to(\mathbb{H},m)$ with the same boundary data, one of which has Hopf differential ϕ with $\int_{\mathbb{H}}|\phi|=\infty$. Conjugating via the Cayley transform $\mathcal{C}$ gives an example of non-uniqueness on the unit disk with the flat metric $|\mathcal{C}'(z)||dz|^2$.

Let $\zeta=\xi+i\eta$ be the coordinate on $\mathbb{H}$. For $k>0$ and $k\neq 1$, set

$$h_k(\zeta)=\xi+ik\eta,\ g_k(\zeta)=k\xi+i\eta.$$

Then, $\phi(h_k)=(1-k^2)d\zeta^2$ and $\phi(g_k)=(k^2-1)d\zeta^2$, so that $f_k=h_k\circ g_k^{-1}$ is a minimal diffeomorphism, explicitly given by

$$f_k=h_k\circ g_k^{-1}(\zeta)=k^{-1}\xi+ik\eta.$$

$f_k|_{\partial\mathbb{H}}$ is the map $\xi\mapsto k^{-1}\xi$. Next, consider the conformal maps

$$\varphi_k(\zeta)=k^{-1/2}(\xi+i\eta),\ \psi_k(\zeta)=k^{1/2}(\xi+i\eta).$$

The map $\tau_k=\varphi_k\circ\psi_k^{-1}$ is simply multiplication by k^{-1}, both conformal and a minimal diffeomorphism, and it agrees with f_k on $\partial\mathbb{H}$. □

For the Hopf differentials $\phi(h_k)$ and $\phi(g_k)$ of f_k, the leaf space projections identify the product of $\mathbb{R}$-trees with the upper halfspace $\mathbb{H}$ equipped with the flat metric $4|k^2-1|m$. In contrast to the $\mathbb{R}$-trees from Section 3.2, this space admits lots of minimal maps that extend to the boundary.

Analytically, we take note of the following aspects. Let $\rho = \rho(z)|dz|^2$ be the hyperbolic metric on $\mathbb{D}$ of constant curvature -1. The Bers norm of a holomorphic quadratic differential $\phi = \phi(z)dz^2$ on $\mathbb{D}$ is

$$||\phi||_\rho^2 = \sup_{z\in\mathbb{D}} |\phi(z)|^2\rho^{-2}(z).$$

As we mentioned in Remark 1.2, if σ is a complete metric on $\mathbb{D}$ with pinched negative curvature and $f:\mathbb{D}\to(\mathbb{D},\sigma)$ is a quasiconformal harmonic diffeomorphism, then the Hopf differential of f has finite Bers norm.

Turning to the example above, the Bers norm of $\phi(h_k)$ blows up as $z\to -i$, since the Cayley transform pulls back $y^{-2}|d\zeta|^2$ to $\rho(z)|dz|^2$, and

$$|\phi(h_k)|y^2 \to \infty$$

as $y\to\infty$. Thus, our example reflects the failure of Remark 1.2 in general. It is proved in [1] that a quadratic differential has finite Bers norm if and only if its maximal ϕ-disks (see [1] for the definition) have bounded radius, which suggests to us that products of $\mathbb{R}$-trees arising from such differentials are "thinner." These observations provide evidence for a positive resolution to Problem A.

4.2. **Stability.** In the general case we can prove a stability result, Theorem A'. Given a surface Σ with Riemannian metrics σ_1,σ_2, a diffeomorphism f of Σ, and a relatively compact open subset U of Σ, let $A(f(U))$ be the area of the graph of f in the product $(\Sigma^2,\sigma_1\oplus\sigma_2)$.

Definition 4.1. A minimal diffeomorphism $f:(\Sigma,\sigma_1)\to(\Sigma,\sigma_2)$ is stable if for every relatively compact open subset U of Σ and C^∞ path of diffeomorphisms $f_t:(\Sigma,\sigma_1)\to(\Sigma,\sigma_2)$ such that $f_0=f$ and $f_t=f$ on the complement of U,

$$\frac{d^2}{dt^2}|_{t=0}A(f_t(U))\geq 0.$$

To prove Theorem A', we take the second derivative of the new main inequality. Given a C^∞ path of quasiconformal maps $t\mapsto v_t$ on $\mathbb{D}$ with compact support, we obtain a path of Beltrami forms

$$t\mapsto \mu_t = \mu + t\dot{\mu} + t^2\ddot{\mu}.$$

We interpret $\dot{\mu}$ as another Beltrami form, although its norm is allowed to be larger than 1, and call it the tangent vector to the path. Two such forms $\dot{\mu},\dot{\nu}$ are infinitesimally equivalent if for all $\phi\in\mathbb{A}^1(\mathbb{D})$,

$$\int\dot{\mu}\phi = \int\dot{\mu}\phi.$$

If two paths of quasiconformal maps have infinitesimally equivalent tangent vectors, then they are tangent in universal Teichmüller space (see [20, Section 4.1] for more explanation).

Proposition 4.2 (Infinitesimal new main inequality for $n=2$)**.** *Let $t\mapsto f_t^1, f_t^2$ be two paths of quasiconformal maps with infinitesimally equivalent tangent vectors $\dot{\mu}_1,\dot{\mu}_2$ and second variations $\ddot{\mu}_1$ and $\ddot{\mu}_2$. For any $\phi\in\mathbb{A}^1(\mathbb{D})$, the infinitesimal new main inequality holds:*

$$Re\int_{\mathbb{D}}\phi(\ddot{\mu_1}-\ddot{\mu_2}) \leq \int_{\mathbb{D}}|\phi|(|\dot{\mu_1}|^2+|\dot{\mu_2}|^2). \quad (9)$$

Proof. By a procedure explained in [19, Section 4.3], we can modify f_t^1 and f_t^2 so that they agree on the boundary at each time t and continue to have the same tangent vectors. It follows from the computation in [19] that the integral on the left depends only on the tangent vectors and hence does not change when we do this modification.

At each time t, we apply Theorem B to ϕ and f_t^1, f_t^2, and take second derivatives of the left and right hand sides. A computation of second derivatives is contained in [19, Section 4.3], and returns exactly (9). □

Remark 4.3. Assuming ϕ is the square of an abelian differential, Marković found a beautiful proof of the infinitesimal new main inequality for $n = 2$ using singular integral operators (unpublished, see slides [17]).

Proof of Theorem A'. Let $f_t : (\mathbb{D}, \sigma_1) \to (\mathbb{D}, \sigma_2)$ be a path of diffeomorphisms with $f_0 = f$ and $f_t = f$ on the complement of a relatively compact subset U. It does no harm to enlarge U, so let's assume it is a disk centered at the origin. Let (h_1, h_2) be the initial minimal map to the product of disks, and $t \mapsto (h_1^t, h_2^t)$ the path of minimal maps engendered by the path of diffeomorphisms $t \mapsto f_t$. By an application of the measurable Riemann mapping theorem, explained in the proof of [20, Proposition 4.13], stability is equivalent to energy stability: by this we mean that it suffices to show that

$$\frac{d^2}{dt^2}|_{t=0} \sum_{i=1}^{2} \mathcal{E}(U, h_i^t) \geq 0.$$

For each $i = 1, 2$ and $t > 0$, set $v_i^t = (h_i^t) \circ h_i^{-1}$ and let μ_i^t be the Beltrami form of v_i^t, which vanishes outside of U. Both paths $t \mapsto \mu_i^t$ are tangent to 0 on ∂U, and thus the infinitesimal new main inequality (9) applies to ϕ and μ_1^t, μ_2^t in restriction to $\overline{U}$.

We now compute in the same manner as the proof of Theorem A. Let $\phi_1 = \phi$ be the Hopf differential of h_1 and $\phi_2 = -\phi$ the differential of h_2. Using $h_i^t = h_i \circ (v_i^t)^{-1}$, (5), and (8),

$$\sum_{i=1}^{2} \mathcal{E}(U, h_i^t) - \sum_{i=1}^{2} \mathcal{E}(U, h_i) \geq \sum_{i=1}^{2} -4\mathrm{Re} \int_U \phi_i \frac{\mu_i^t}{1 - |\mu_i^t|^2} + \sum_{i=1}^{2} 4 \int_U |\phi_i| \frac{|\mu_i^t|^2}{1 - |\mu_i^t|^2}, \tag{10}$$

with equality if and only if both μ_i^t are zero almost everywhere. Since the initial map (h_1, h_2) is minimal, or really just because $\phi_1 + \phi_2 = 0$, the first derivatives of both sides of (10) are zero. The infinitesimal new main inequality (9) is equivalent to the statement that the second derivative at zero of the right hand side of equation (10) is non-negative. It follows that the second derivative at zero of the left hand side of (10) has non-negative second derivative:

$$\frac{d^2}{dt^2}|_{t=0} \Big(\sum_{i=1}^{2} \mathcal{E}(U, h_i^t) - \sum_{i=1}^{2} \mathcal{E}(U, h_i) \Big) \geq 0. \tag{11}$$

We conclude by noting that

$$\frac{d^2}{dt^2}|_{t=0} \sum_{i=1}^{2} \mathcal{E}(U, h_i^t) = \frac{d^2}{dt^2}|_{t=0} \Big(\sum_{i=1}^{2} \mathcal{E}(U, h_i^t) - \sum_{i=1}^{2} \mathcal{E}(U, h_i) \Big)$$

and applying (11).

□

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