On Kazhdan-Yom Din asymptotic Schur orthogonality for K-finite matrix coefficients

Anne-Marie Aubert, Alfio Fabio La Rosa

Abstract

In a recent article, D. Kazhdan and A. Yom Din conjectured the validity of an asymptotic form of Schur's orthogonality for tempered irreducible unitary representations of semisimple groups defined over local fields. In the non-Archimedean case, they established such an orthogonality for K-finite matrix coefficients. Building on their work, and exploiting the admissibility of irreducible unitary representations, we prove the analogous result in the Archimedean case.

Contents

1 Introduction

Let G be a semisimple group over a local field, let K be a maximal compact subgroup of G. We fix a Haar measure on G, denoted dg. If H is the Hilbert space underlying a unitary representation of G, let H_K denote the space of K-finite vectors and H^∞ the space of smooth vectors.

In their recent work [4], D. Kazhdan and A. Yom Din conjectured the validity of an asymptotic version of Schur's orthogonality relations. It should hold for matrix coefficients of tempered irreducible unitary representations of G, generalising Schur's well-known orthogonality relations for discrete series.

Following their article, we fix a norm, denoted $\|\cdot\|$, on the Lie algebra $\mathfrak{g}$ of G. We further define the function

$$\mathbf{r} : G \longrightarrow \mathbb{R}_{\geq 0}, \quad \mathbf{r}(g) = \log\left(\max\{\|\mathrm{Ad}(g)\|, \|\mathrm{Ad}(g^{-1})\|\}\right)$$

so that, given $r \in \mathbb{R}_{>0}$, we can introduce the corresponding ball

$$G_{<r} := \{g \in G | \mathbf{r}(g) < r\}.$$

Given this set-up, we are in position to state their conjecture.

Conjecture 1.1 (Kazhdan-Yom Din, Asymptotic Schur Orthogonality Relations)**.** *Let G be a semisimple group over a local field and let (π, H) be a tempered irreducible unitary representation of G. Then there exist $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$ and $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H$, the following holds:*

$$\lim_{r \to \infty} \frac{\int_{G_{<r}} \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)v_3, v_4 \rangle}\, dg}{r^{\mathbf{d}(\pi)}} = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle}.$$

Assuming that the matrix coefficients involved are K-finite, one has the following result:

Theorem 1.2 ([4], Theorem 1.7)**.** *Let (π, H) be a tempered irreducible unitary representation of G and K be a maximal compact subgroup of G. Then there exists $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$ such that:*

(1) *If G is non-Archimedean, there exists $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$, we have*

$$\lim_{r \to \infty} \frac{\int_{G_{<r}} \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)v_3, v_4 \rangle}\, dg}{r^{\mathbf{d}(\pi)}} = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle}.$$

(2) *If G is Archimedean, for any given non-zero $v_1, v_2 \in H_K$, there exists $C(v_1, v_2) > 0$ such that*

$$\lim_{r \to \infty} \frac{\int_{G_{<r}} |\langle \pi(g)v_1, v_2 \rangle|^2\, dg}{r^{\mathbf{d}(\pi)}} = C(v_1, v_2).$$

In the non-Archimedean case, the proof of (1) is achieved by first establishing the validity of the analogous version of (2). The polarisation identity allows the authors of [4] to define a form

$$D(\cdot, \cdot, \cdot, \cdot) : H_K \times H_K \times H_K \times H_K \longrightarrow \mathbb{C}$$

via the prescription

$$D(v_1, v_2, v_3, v_4) := \lim_{r \to \infty} \frac{\int_{G_{<r}} \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)v_3, v_4 \rangle}\, dg}{r^{\mathbf{d}(\pi)}}.$$

In [4], Section 4.1, this form is shown to be G-invariant and one would like to invoke an appropriate form of Schur's lemma to argue as in the standard proof of Schur's orthogonality relations. That is, for fixed $v_2, v_4 \in H_K$, one defines the form

$$\langle \cdot, \cdot \rangle_{v_2, v_4} := D(\cdot, v_2, \cdot, v_4)$$

and, for fixed $v_1, v_3 \in H_K$, the form

$$\langle\cdot,\cdot\rangle^{v_1,v_3} := D(v_1,\cdot,v_3,\cdot).$$

One applies Schur's lemma to these forms, which implies that each such form is a scalar multiple of the inner product on H. Upon comparing them, one obtains the desired orthogonality relations.

The appropriate version of Schur's lemma in the non-Archimedean case is provided by Dixmier's lemma, which can be applied since in the non-Archimedean setting the subspace of K-finite vectors H_K and the subspace of smooth vectors H^∞ coincide, the latter being equipped with the structure of a Fréchet representation of G, which is irreducible since H itself is irreducible.

The purpose of this article is to prove that the analogue of (1) in Theorem 3.3 holds in the Archimedean case. As explained in [4], Section 4.2, it suffices to prove the result for real semisimple groups (proved as Theorem 3.5).

Theorem 1.3. Let (π, H) be a tempered irreducible unitary representation of a real semisimple group G. Let K be a maximal compact subgroup of G. Then there exists $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\lim_{r\to\infty} \frac{\int_{G_{<r}} \langle\pi(g)v_1, v_2\rangle \overline{\langle\pi(g)v_3, v_4\rangle}\, dg}{r^{\mathbf{d}(\pi)}} = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

We need to modify the strategy above to account for the fact that the spaces of smooth and K-finite vectors of a Hilbert space representation (π, H) of a real semisimple group do not coincide.

Our approach relies crucially on the admissibility of irreducible, unitary representations of reductive groups, a foundational theorem proved by Harish Chandra. The theory of admissible $(\mathfrak{g}, K)$-modules then provides us with the appropriate version of Schur's lemma for $(\mathfrak{g}, K)$-invariant forms. Hence, we are reduced to verify that $\langle\cdot,\cdot\rangle_{v_2,v_4}$ and $\langle\cdot,\cdot\rangle^{v_1,v_3}$ are, indeed, $(\mathfrak{g}, K)$-invariant. Having established this, to conclude the proof of Theorem 1.3 we can argue as in [4], Section 4.

The article is organised as follows.

Section 2: We collect those notions and results on representation theory and $(\mathfrak{g}, K)$-modules that we need, including the admissibility of the $(\mathfrak{g}, K)$-modules associated to irreducible unitary representations, and the appropriate form of Schur's lemma.

Section 3: We establish the $(\mathfrak{g}, K)$-invariance of the forms introduced above and prove Theorem 1.3.

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2 Recollections on $(\mathfrak{g}, K)$-modules

Our presentation of the theory of $(\mathfrak{g}, K)$-modules, the core of this section, follows [7] very closely. To discuss even its basic features, we need to gather some results on unitary representations of compact groups. We begin by recalling the basic notions in the study of representations of topological groups, which we always assume to be Hausdorff.

First, following [7], Section 1.1, let G denote a second-countable, locally compact group, equipped with a left Haar measure dg, and let V denote a complex topological vector space. We denote by $\mathrm{GL}(V)$ the group of invertible continuous endomorphisms of V. A **representation** of G on V is a strongly continuous homomorphism $\pi : G \longrightarrow \mathrm{GL}(V)$. Let (π, V) denote the datum of a representation of G. A subspace of V which is stable under the action of G through π is called an **invariant subspace**. A representation is said to be **irreducible** if the only closed invariant subspaces are the trivial subspace and V itself.

If $(H, \langle \cdot, \cdot \rangle)$ is a separable Hilbert space, a representation π of G on H is termed a **Hilbert representation**. If, in addition, G acts by unitary operators through π, the representation is said to be **unitary**.

Next, following [6], Section 10, we introduce the basic features of the theory of vector-valued integration.

Let (X, dx) be a Radon measure space, let H be a Hilbert space and assume that

$$f : X \longrightarrow H$$

is measurable. The function f is **integrable** if it satisfies the following two conditions:

(1) For all $v \in H$,

$$\int_X |\langle f(x), v \rangle| \, dx < \infty.$$

(2) The map

$$v \mapsto \int_X |\langle f(x), v \rangle| \, dx$$

is a bounded conjugate-linear functional.

If $f : X \longrightarrow H$ is integrable, then, by the Riesz' representation theorem, there exists a unique element in H, denoted

$$\int_X f(x)\, dx,$$

such that, for all $v \in H$, we have

$$\left\langle \int_X f(x)\, dx, v \right\rangle = \int_X \langle f(x), v \rangle\, dx.$$

Proposition 2.1. Let $(X,\, dx)$ be as above. Let H, E be Hilbert spaces, $f : X \longrightarrow H$ a measurable function and $T : H \longrightarrow E$ a bounded linear operator. Then the following holds:

(1) If

$$\int_X \|f(x)\|\, dx < \infty,$$

then $f : X \longrightarrow H$ is integrable.

(2) If $f : X \longrightarrow H$ is integrable, then so is $Tf : X \longrightarrow E$. Moreover,

$$T\left(\int_X f(x)\, dx \right) = \int_X Tf(x)\, dx.$$

Proof. See [6], Proposition 10.8 and Proposition 10.9. □

Now, let (π, H) be a unitary representation of G. Let $v \in H$ and $f : G \longrightarrow H$ be such that the map

$$g \mapsto f(g)\pi(g)v$$

is integrable. Let $\pi(f)v$ denote the unique element in H such that, for all $w \in H$, we have

$$\langle \pi(f)v, w \rangle = \int_G f(g) \langle \pi(g)v, w \rangle\, dg.$$

Proposition 2.2. Let (π, H) be as above. If $f \in L^1(G)$, then, for all $v \in H$, the map $g \mapsto f(g)\pi(g)v$ is integrable and the prescription

$$\pi(f) : H \longrightarrow H,\ \ v \mapsto \pi(f)v$$

defines a bounded linear operator.

Proof. See [6], Proposition 10.20. □

With the integral operators introduced in Proposition 2.2 at our disposal, we have all the tools needed to state the main results on the unitary representations of compact groups.

Let K be a compact group. Let $\widehat{K}$ denote the set of equivalence classes of irreducible unitary representations of K. If (π, H) is a unitary representation, for each $\gamma \in \widehat{K}$ let $H(\gamma)$ denote the closure of the sum of all the closed invariant subspaces of H in the equivalence class of γ. We refer to $H(\gamma)$ as the γ-isotypic component of H.

Proposition 2.3. Let K be a compact group. Let (π, H) be an irreducible unitary representation of K. Then H is finite-dimensional.

Proof. See [7], Proposition 1.4.2. $\square$

Given Proposition 2.3, we can associate, to each $\gamma \in \widehat{K}$, the function

$$\chi_\gamma : K \longrightarrow \mathbb{C},\ \chi_\gamma(g) := \mathrm{tr}\gamma(g),$$

the **character** of γ.

Recall that if $\{(\pi_i, H_i)|i \in I\}$ is a countable family of unitary representations of a topological group G, we can construct a new unitary representation of G, the **direct sum**, on the Hilbert space completion of the algebraic direct sum of the H_i's. We refer the reader to [7], Section 1.4.1, for the details of this construction. We let

$$\bigoplus_{i \in I} H_i$$

denote the direct sum of the family $\{(\pi_i, H_i)|i \in I\}$, dropping explicit reference to the π_i's.

Proposition 2.4. Let K be a compact group. Let (π, H) be a unitary representation of K. Then (π, H) is the direct sum representation of its K-isotypic components; that is,

$$H = \bigoplus_{\gamma \in \widehat{K}} H(\gamma).$$

Moreover, let α_γ denote the function

$$\alpha_\gamma(k) := \dim(\gamma)\overline{\chi_\gamma}(k).$$

Then the following holds:

$$H(\gamma) = \pi(\alpha_\gamma)H.$$

Proof. See [7], Lemma 1.4.7. $\square$

Proposition 2.5. Let K be a compact group. If (π, H) is a Hilbert space representation of K, then there exists an inner product on H that induces the original topology on H and for which K acts unitarily through π.

Proof. See [7], Lemma 1.4.8. $\square$

We are finally ready to introduce $(\mathfrak{g}, K)$-modules. Although they can be defined for general Lie groups, as in [7], Section 3.3, we will assume at the outset that our group G is a **reductive** group. We adopt the definition of reductive group that is given in [7], Chapter 2. Similarly for **semisimple** groups.

Definition 2.6. Let G be a real reductive group with Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. A vector space V, equipped with the structure of $\mathfrak{g}$-module and K-module, is called a $(\mathfrak{g}, K)$-**module** if the following conditions hold:

(1) For all $v \in V$, for all $X \in \mathfrak{g}$, for all $k \in K$,

$$kXv = Ad(k)Xkv$$

(2) For all $v \in V$, the span of the set

$$Kv := \{kv | k \in K\}$$

is a finite-dimensional subspace of V, on which the action of K is continuous.

(3) For all $v \in V$, for all $Y \in \mathfrak{k}$,

$$\frac{d}{dt} \exp(tY)v|_{t=0} = Yv.$$

We remark that (3) implicitly uses the smoothness of the action of K on the span of Kv. This follows from the fact that a continuous group homomorphism between Lie groups is automatically smooth.

Let V and W be $(\mathfrak{g}, K)$-modules and let $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ denote the space of $\mathfrak{g}$-morphisms that are also K-equivariant. Then V and W are said to be **equivalent** if $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ contains an invertible element.

A $(\mathfrak{g}, K)$-module V is called **irreducible** if the only subspaces that are invariant under the actions of $\mathfrak{g}$ and K are the trivial subspace and V itself. In this case, we have the following theorem:

Theorem 2.7. Let V be an irreducible $(\mathfrak{g}, K)$-module. Then the space $\mathrm{Hom}_{\mathfrak{g},K}(V, V)$ is 1-dimensional.

Proof. This is the result actually proved in [7], Lemma 3.3.2, although the statement there says $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$, for an unspecified W. We believe it is a typo. $\square$

Let V be a $(\mathfrak{g}, K)$-module. Since, given each $v \in V$, the span of Kv, say W_v, is a finite-dimensional continuous representation of K, we can use Proposition 2.5 and then apply Proposition 2.4, thus decomposing W_v into a finite sum of finite-dimensional K-invariant subspaces of V. For $\gamma \in \widehat{K}$, we let $V(\gamma)$ denote the sum of all the K-invariant finite dimensional subspaces in the equivalence class of γ. Then the discussion above implies that

$$V = \bigoplus_{\gamma \in \widehat{K}} V(\gamma)$$

as a K-module, with the direct sum indicating the algebraic direct sum. A $(\mathfrak{g}, K)$-module V is called **admissible** if, for all $\gamma \in \widehat{K}$, $V(\gamma)$ is finite-dimensional.

Given a unitary representation (π, H), there exists a $(\mathfrak{g}, K)$-module naturally associated to it. To define it, recall that a vector $v \in H$ is called **smooth** if the map

$$g \mapsto \pi(g)v$$

is smooth. Let H^∞ denote the subspace of smooth vectors of H. It is a standard fact that the prescription

$$\dot{\pi}(X) := \frac{d}{dt}\pi(\exp(tX))v|_{t=0},$$

for $v \in H^\infty$ and $X \in \mathfrak{g}$, defines an action of $\mathfrak{g}$ on H^∞. Recall that a vector $v \in H$ is K-**finite** if the span of the set

$$\pi(K)v := \{\pi(k)v | k \in K\}$$

is finite-dimensional. Let H_K denote the subspace of K-finite vectors of H. By [7], Lemma 3.3.5, with the action of $\mathfrak{g}$ so defined and with the action of K through π, the space $H_K \cap H^\infty$ is a $(\mathfrak{g}, K)$-module. The representation (π, H) is said to be **admissible** if $H_K \cap H^\infty$ is admissible as a $(\mathfrak{g}, K)$-module and (π, H) is called **infinitesimally irreducible** if $H_K \cap H^\infty$ is irreducible as a $(\mathfrak{g}, K)$-module. It is in general not true that a K-finite vector is smooth. However, if (π, H) is admissible, we have the following result:

Theorem 2.8. Let G be a real reductive group and (π, H) an admissible representation of G. Then every K-finite vector is smooth.

Proof. See [8], Theorem 4, part 1. $\square$

In light of the following fundamental result of Harish Chandra, Theorem 2.8 will play an important role in this article.

Theorem 2.9. Let (π, H) be an irreducible unitary representation of G. Then (π, H) is admissible.

Proof. See [7], Theorem 3.4.10. $\square$

As in [7], Section 3.3.6, given a $(\mathfrak{g}, K)$-module V, we can equip its algebraic dual V^* with the following actions of K and $\mathfrak{g}$. For all $k \in K$, for all $\mu \in V^*$, we define

$$\widetilde{k}\mu(v) := \mu(k^{-1}v),$$

for all $v \in V$. For all $X \in \mathfrak{g}$, for all $\mu \in V^*$, we define

$$\widetilde{X}\mu(v) := -\mu(Xv),$$

for all $v \in V$. The subspace of V^* consisting of the K-finite elements with respect to the action of K so defined is a $(\mathfrak{g}, K)$-module: the dual, or contragredient, $\widetilde{V}$, of V. Similarly, we can equip the space of conjugate-linear functionals on V with analogously defined actions. If μ is a conjugate-linear functional on V, for all $k \in K$ and for all $v \in V$, we define

$$\overline{k}\mu(v) := \mu(k^{-1}v).$$

For all $X \in \mathfrak{g}$ and for all $v \in V$, we define

$$\overline{X}\mu(v) := -\mu(Xv).$$

The subspace of K-finite conjugate-linear functionals on V, equipped with these actions, is a $(\mathfrak{g}, K)$-module: the conjugate-dual, $\overline{V}$, of V.

Both the following statements are extracted from [7]. The first appears at the end of the proof of Lemma 4.3.2, while the second is essentially given in Lemma 4.3.3. Since we shall need them in the third section, we decided to fill in the details of the proofs. The first part of the argument in the next proposition is an adaption of [3], Proposition 8.5.7. The key idea is to exploit the local finiteness of $(\mathfrak{g}, K)$-modules to reduce to a setting where the projection to a given K-isotypic component of an arbitrary element is given by an appropriate integral operator, as in Theorem 2.4.

Proposition 2.10. Let V be an admissible $(\mathfrak{g}, K)$-module. Then

$$\widetilde{V} = \bigoplus_{\gamma \in \widehat{K}} (V(\gamma)^*).$$

In particular, $\widetilde{V}$ is admissible.

Proof. Let $\mu \in \widetilde{V}$, let W be defined as

$$W := \mathrm{Span}\{\widetilde{K}\mu\}.$$

Then W is a finite-dimensional representation of K, which, by Proposition 2.5, we can assume unitary. By decomposing μ according to its K-isotypic components, we can reduce to the case that $\mu \in W(\gamma)$ and, by Proposition 2.4, that

$$\mu = \widetilde{k}(\alpha_{\widetilde{\gamma}})\mu',$$

for some $\mu' \in W$. Here, $\widetilde{k}(\alpha_{\widetilde{\gamma}})\mu'$ denotes the functional obtained as the W-valued integral of the function

$$k \mapsto \alpha_{\widetilde{\gamma}}(k)\widetilde{k}\mu'.$$

We want to show that μ vanishes identically outside $V(\gamma)$. So, let $v \in V$ and let E be defined as

$$E := \mathrm{Span}\{Kv\}.$$

Then, as above, E is a finite-dimensional unitary representation of K and we have

$$E(\gamma) = k(\alpha_\gamma)E$$

We can compute as follows:

$$\begin{aligned}
\mu(v) &= \widetilde{k}(\alpha_{\widetilde{\gamma}})\mu'(v) \\
&= \int_K \alpha_{\widetilde{\gamma}}(k)\widetilde{k}\mu'(v)\,dk \\
&= \int_K \alpha_{\widetilde{\gamma}}(k)\mu'(k^{-1}v)\,dk \\
&= \mu'\left(\int_K \alpha_{\widetilde{\gamma}}(k)k^{-1}v\,dk\right) \\
&= \mu'\left(\int_K \alpha_{\gamma}(k^{-1})k^{-1}v\,dk\right) \\
&= \mu'(k(\alpha_\gamma)v).
\end{aligned}$$

Here, the second equality is justified by a reasoning that we have learnt from P. Garrett: we view evaluation at $v \in V$ as a linear functional on W. Since W is finite-dimensional, this functional is continuous and we can interchange it with the integral

$$\int_K \alpha_{\widetilde{\gamma}}(k)\widetilde{k}\mu'\,dk,$$

which is W-valued. The fourth equality follows from the fact that μ' is continuous on the finite-dimensional vector space E, hence we can interchange it with an integral valued in E. The fifth equality follows from

$$\chi_{\widetilde{\gamma}}(k) = \chi_\gamma(k^{-1}),$$

a fact proved in [2], Chapter II, Proposition 4.10, (vi). Finally, we used the unimodularity of K. Hence, we showed that $\mu \in \widetilde{V}(\gamma)$ acts on $v \in V$ through the projection of v to its γ-isotypic component. This implies that μ vanishes identically outside $V(\gamma)$ and gives an element in $V(\gamma)^*$. To complete the proof, it is enough to observe that we can decompose any element in $\widetilde{V}$ uniquely according to its K-isotypic components and apply the discussion above to each resulting summand.

For the reverse inclusion, Let μ be an element of the right-hand side of the display in the statement. Then there exist $\mu_{\gamma_1}, \dots, \mu_{\gamma_n}$, with $\mu_{\gamma_i} \in V(\gamma_i)^*$ for each i, such that

$$\mu = \sum_{i=1}^{n} \mu_{\gamma_i}.$$

Admissibility of V implies that $V(\gamma)$ is finite-dimensional for every $\gamma \in \widehat{K}$, hence so is $V(\gamma)^*$. Since $\widetilde{K}\mu_{\gamma_i}$ is contained in $V(\gamma_i)^*$, it follows that, for each i, the span of $\widetilde{K}\mu_{\gamma_i}$ is finite-dimensional. Hence, the same holds for the span of $\widetilde{K}\mu$, proving that the right-hand side of the above display is contained in $\widetilde{V}$. □

Proposition 2.11. Let (π, H) be an irreducible unitary representation. For $w \in H_K$, let ϕ_w denote the linear functional on H_K defined by $\phi_w(v) = \langle v, w \rangle$, and set

$$\Phi(H_K) := \{\phi_w | w \in H_K\}.$$

Then the following equality holds:

$$\widetilde{H_K} = \Phi(H_K).$$

Proof. By Theorem 2.9 and by Proposition 2.10, we have

$$\widetilde{H_K} = \bigoplus_{\gamma \in \widehat{K}} (H_K(\gamma)^*)$$

with $H_K(\gamma)$ finite-dimensional for each $\gamma \in \widehat{K}$. Hence, each $H_K(\gamma)$ inherits a Hilbert space structure from H. Let $\mu \in \widetilde{H_K}$ and let $\mu_{\gamma_1}, \ldots, \mu_{\gamma_n}$, with $\mu_{\gamma_i} \in H_K(\gamma_i)^*$, be such that

$$\mu = \sum_{i=1}^{n} \mu_{\gamma_i}.$$

For each i, by Riesz representation theorem, we can find $w_i \in H_K$ such that, for all $v \in H_K$,

$$\mu_{\gamma_i}(v) = \langle v, w_i \rangle.$$

Then, setting

$$w := \sum_{i=1}^{n} w_i \in H_K,$$

we obtain, for all $v \in H_K$,

$$\mu(v) = \langle v, w \rangle,$$

proving that $\widetilde{H_K}$ is a subset of $\Phi(H_K)$.

For the reverse inclusion, let $\phi_w \in \Phi(H_K)$. Let $w_1, \ldots, w_m \in H_K$ be a basis for the space spanned by $\pi(K)w$ and, for $k \in K$, let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ be such that

$$\pi(k)w = \sum_{i=1}^{m} \lambda_i w_i.$$

Since

$$\widetilde{k}\phi_w(v) = \langle \pi(k)^{-1}v, w \rangle = \langle v, \pi(k)w \rangle = \sum_{i=1}^{m} \overline{\lambda_i}\phi_{w_i}(v),$$

it follows that the space spanned by $\widetilde{K}\phi_w$ is contained in the subspace spanned by $\phi_{w_1}, \ldots, \phi_{w_m}$. Hence, $\phi_w \in \widetilde{H_K}$, concluding the proof. □

We conclude this quick (and far from complete) review of the theory of $(\mathfrak{g}, K)$-modules by proving the version of Schur's lemma for sesquilinear forms that we will use in Section 3. This is given by Corollary 2.14 below. First, we need:

Theorem 2.12. Let (π, H) be an admissible representation. Then (π, H) is irreducible if and only if it is infinitesimally irreducible.

Proof. See [7], Theorem 3.4.11. □

Theorem 2.13. Let V be an admissible $(\mathfrak{g}, K)$-module. Suppose that there exist a $(\mathfrak{g}, K)$-module W and a non-degenerate $(\mathfrak{g}, K)$-invariant sesquilinear form

$$B(\cdot, \cdot) : V \times W \longrightarrow \mathbb{C}.$$

Then W is $(\mathfrak{g}, K)$-isomorphic to $\overline{V}$.

Proof. This is [7], Lemma 4.5.1, except for the fact that our form is sesquilinear. To account for it, we modify the definition of the map T in the reference by setting, for a given $w \in W$, $T(w)(v) = B(w, v)$ for all $v \in V$. This defines a map from W to $\overline{V}$ obtained by sending w to $T(w)$ which, by the argument in the reference, is a $(\mathfrak{g}, K)$-isomorphism. □

The next corollary is proved by adapting to our case the argument in [3], Proposition 8.5.12 and using the beginning of the proof of [5], Proposition 9.1.

Corollary 2.14. Let (π, H) be an irreducible admissible representation. Then, up to a constant, there exists at most one non-zero (g, K)-invariant sesquilinear form on H_K. In particular, if (π, H) is irreducible unitary, then every such form is a constant multiple of $\langle \cdot, \cdot \rangle$.

Proof. The irreducibility of H implies that of H_K, by Theorem 2.12 and by Theorem 2.8. Let $B(\cdot, \cdot)$ be a $(\mathfrak{g}, K)$-invariant sesquilinear form. Consider the linear subspace V_0 of H_K defined as

$$V_0 := \{v \in H_K | B(v, w) = 0 \text{ for all } w \in H_K\}.$$

Since $B(\cdot, \cdot)$ is non-zero, V_0 is a proper subspace of H_K. Since $B(\cdot, \cdot)$ is moreover $(\mathfrak{g}, K)$-invariant, it follows that V_0 is a $(\mathfrak{g}, K)$-invariant subspace of H_K, hence, by the irreducibility of H_K, it must be zero. Analogous considerations for the subspace

$$V^0 := \{w \in H_K | B(v, w) = 0 \text{ for all } v \in H_K\}$$

imply that $B(\cdot, \cdot)$ is non-degenerate. By Theorem 2.13, the map $v \mapsto T(v)$, $T(v)(\cdot) := B(v, \cdot)$, is a $(\mathfrak{g}, K)$-isomorphism. Since H_K is irreducible, the space $\mathrm{Hom}_{\mathfrak{g},K}(H_K, H_K)$ is 1-dimensional by Theorem 2.7. Now, let $B'(\cdot, \cdot)$ be another such form, with associated isomorphism T'. Then $T(T')^{-1} = cI$, for some $c \in \mathbb{C}$. For the last statement, the unitarity of (π, H) implies that $\langle \cdot, \cdot \rangle$ is a $(\mathfrak{g}, K)$-invariant non-degenerate sesquilinear form and Theorem 2.9, with the discussion above, implies the result. □

3 Schur orthogonality

We fix the datum of a real semisimple group G and a maximal compact subgroup K of G.

For a fixed $d \in \mathbb{Z}_{\geq 0}$, let $\mathcal{L}_d$ be the set of measurable functions $f : G \longrightarrow \mathbb{C}$ such that

$$\lim_{r\to\infty} \frac{\int_{G_{<r}} |f(g)|^2 \, dg}{r^d} < \infty.$$

Let $\|\cdot\|_d$ denote the following prescription:

$$\|f(g)\|_d := \left(\lim_{r\to\infty} \frac{\int_{G_{<r}} |f(g)|^2 \, dg}{r^d}\right)^{\frac{1}{2}}.$$

Recall that for all $x \in G$, we can define the right and left action of G on a function f as $R(x)f(g) := f(gx)$ and $L(x)f(g) := f(x^{-1}g)$, respectively.

Proposition 3.1. Let $f \in \mathcal{L}_d$. Then the following holds:

(1) Let $x \in G$ be such that both $R(x)f$ and $R(x^{-1})f$ belong to $\mathcal{L}_d$. Then $\|R(x)f\|_d = \|f\|_d$. The analogous statement is true for $L(x)f$.

(2) $\|f(g^{-1})\|_d = \|f(g)\|_d$

Proof. For (1), we observe that, for a given $x \in G$, the right translate $(G_{<r})x$ of $G_{<r}$ is contained in $G_{<r+\mathbf{r}(x)}$. Indeed, by definition, $g \in G_{<r}$ implies $\log\|\mathrm{Ad}(g)\| < r$ and $\log\|\mathrm{Ad}(g^{-1})\| < r$. Thus, $\log\|\mathrm{Ad}(gx)\| < r + \log\|\mathrm{Ad}(x)\|$ and $\log\|\mathrm{Ad}((gx)^{-1})\| < r + \log\|\mathrm{Ad}(x^{-1})\|$, which implies

$$\begin{aligned}
\mathbf{r}(gx) &< r + \max\{\log\|\mathrm{Ad}(x)\|, \log\|\mathrm{Ad}(x^{-1})\|\} \\
&= r + \log\big(\max\{\|\mathrm{Ad}(x)\|, \|\mathrm{Ad}(x^{-1})\|\}\big) \\
&= r + \mathbf{r}(x).
\end{aligned}$$

Now, for f an element in $\mathcal{L}_d$ as in the statement, we compute as follows:

$$\begin{aligned}
\|R(x)f(g)\|_d^2 &= \|f(gx)\|_d^2 \\
&= \lim_{r\to\infty} \frac{\int_{G_{<r}} |f(gx)|^2 \, dg}{r^d} \\
&= \lim_{r\to\infty} \frac{\int_{G} |f(gx)|^2 \mathbb{1}_{G_{<r}}(g) \, dg}{r^d} \\
&= \lim_{r\to\infty} \frac{\int_{G} |f(g)|^2 \mathbb{1}_{G_{<r}}(gx^{-1}) \, dg}{r^d} \\
&= \lim_{r\to\infty} \frac{\int_{(G_{<r})x} |f(g)|^2 \, dg}{r^d} \\
&\leq \lim_{r\to\infty} \frac{\int_{G_{<r+\mathbf{r}(x)}} |f(g)|^2 \, dg}{r^d} \\
&= \|f(g)\|_d^2.
\end{aligned}$$

where, in the fourth equality, we used the unimodularity of G.
We remark that, without the assumption on $R(x)f$, we could only prove the weaker inequality

$$\limsup_{r\to\infty} \frac{\int_{G_{<r+\mathbf{r}(x)}} |f(g)|^2\, dg}{r^d} \leq \|f(g)\|_d.$$

Coming back to our proof, we obtained

$$\|R(x)f\|_d \leq \|f\|_d.$$

Similarly, we have

$$\|R(x^{-1})f\|_d \leq \|f\|_d.$$

In this last inequality, substituting $R(x)f$ with f, we obtain $\|f\|_d \leq \|R(x)f\|_d$, which implies $\|f\|_d \leq \|R(x)f\|_d \leq \|f\|_d$. The argument for $L(x)f$ runs analogously. For (2), it suffices to observe that $(G_{<r})^{-1} = G_{<r}$ and to use unimodularity. $\square$

The set $\mathcal{L}_d$ is **not** a vector space and $\|\cdot\|_d$ is **not** a seminorm on it. As A. Yom Din pointed out, following [1], one could remedy the issue by using a limit superior in the definition of $\|\cdot\|_d$. However, we will not need anything but Proposition 3.1 in this article.

For a tempered irreducible unitary representation (π, H) of G, for given $v, w \in H$, let

$$\phi_{v,w}(g) := \langle \pi(g)v, w\rangle$$

denote the associated matrix coefficient. By (2) of Theorem 1.2, there exists $d(\pi) \in \mathbb{Z}_{\geq 0}$ such that $\phi_{v,w} \in \mathcal{L}_{d(\pi)}$ for all $v, w \in H_K$.

As in Section 4.1 of [4], by the polarisation identity and by (2) of Theorem 1.2, the prescription

$$D(v_1, v_2, v_3, v_4) := \lim_{r\to\infty} \frac{\int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg}{r^{\mathbf{d}(\pi)}}$$

is a well-defined form on H_K that is linear in the first and fourth variable, conjugate-linear in the second and the third.

As explained in the introduction, we consider the following family of forms. For fixed $v_2, v_4 \in H_K$, we define

$$\langle\cdot,\cdot\rangle_{v_2,v_4} := D(\cdot, v_2, \cdot, v_4),$$

which is linear in the first variable and conjugate linear in the second. For fixed $v_1, v_3 \in H_K$, we define

$$\langle\cdot,\cdot\rangle^{v_1,v_3} := D(v_1, \cdot, v_3, \cdot),$$

which is conjugate-linear in the first variable and linear in the second.

As in [4], Section 4.1, showing that for all $k_1, k_2, k_3, k_4 \in K$ and for all $v_2, v_2, v_3, v_4 \in H_K$ we have

$$D(\pi(k_1)v_1, \pi(k_2)v_2, \pi(k_3)v_3, \pi(k_4)v_4) = D(v_1, v_2, v_3, v_4),$$

can be reduced, by the polarisation identity, to prove that, with the notation of (2) of Theorem 1.2, we have

$$C(\pi(k_1)v, \pi(k_2)w) = C(v, w)$$

for all $k_1, k_2 \in K$ and for all $v, w \in H_K$. This will imply the K-invariance of $\langle \cdot, \cdot \rangle^{v_1, v_3}$ and $\langle \cdot, \cdot \rangle_{v_2, v_4}$.

Proposition 3.2. Let (π, H) be a tempered irreducible unitary representation of a real semisimple group G. Then, for all $k_1, k_2 \in K$, for all $v, w \in H_K$, we have

$$\lim_{r\to\infty} \frac{\int_{G_{<r}} |\langle \pi(g)\pi(k_1)v, \pi(k_2)w\rangle|^2 \, dg}{r^{\mathbf{d}(\pi)}} = \lim_{r\to\infty} \frac{\int_{G_{<r}} |\langle \pi(g)v, w\rangle|^2 \, dg}{r^{\mathbf{d}(\pi)}}.$$

Hence, for all $k \in K$, and for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\langle \pi(k)v_1, \pi(k)v_3\rangle_{v_2, v_4} = \langle v_1, v_3\rangle_{v_2, v_4}$$

and

$$\langle \pi(k)v_2, \pi(k)v_4\rangle^{v_1, v_3} = \langle v_2, v_4\rangle^{v_1, v_3}.$$

Proof. We observe that

$$\langle \pi(g)\pi(k_1)v, \pi(k_2)w\rangle = \phi_{v,w}(k_2^{-1}gk_1) = L(k_2)R(k_1)\phi_{v,w}(g).$$

Since $\pi(k_1)v \in H_K$, by (2) of Theorem 1.2 it follows that $R(k_1)\phi_{v,w} \in \mathcal{L}_{d(\pi)}$. Therefore, we also have $L(k_2)R(k_1)\phi_{v,w} \in \mathcal{L}_{d(\pi)}$. Using (1) of Proposition 3.1, we compute

$$\begin{aligned} \lim_{r\to\infty} \frac{\int_{G_{<r}} |\langle \pi(g)\pi(k_1)v, \pi(k_2)w\rangle|^2 \, dg}{r^{\mathbf{d}(\pi)}} &= \|L(k_2)R(k_1)\phi_{v,w}(g)\|^2_{\mathbf{d}(\pi)} \\ &= \|\phi_{v,w}(g)\|^2_{\mathbf{d}(\pi)}, \end{aligned}$$

as required. The K-invariance of the forms $\langle \cdot, \cdot \rangle_{v_2, v_4}$ and $\langle \cdot, \cdot \rangle^{v_1, v_3}$ now follows from the polarisation identity. □

Next, we proceed to prove that the forms we consider are $\mathfrak{g}$-invariant. First, we observe that if $v_2, v_3, v_4 \in H_K$, the linear functional

$$\langle \cdot, v_3\rangle_{v_2, v_4} : H_K \longrightarrow \mathbb{C}$$

is K-finite, hence it belongs to the contragredient $(\mathfrak{g}, K)$-module, $\widetilde{H_K}$, of H_K.

Lemma 3.3. Let $v_2, v_3, v_4 \in H_K$. Then the linear functional

$$\langle \cdot, v_3 \rangle_{v_2,v_4} : H_K \longrightarrow \mathbb{C}$$

is K-finite; that is, it belongs to $\widetilde{H_K}$.

Proof. Since $v_3 \in H_K$, the span of $\pi(K)v_3$ is finite-dimensional and we let $w_1, \dots, w_n \in H_K$ be a basis for it. For $k \in K$, let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be such that

$$\pi(k)v_3 = \sum_{i=1}^{n} \lambda_i w_i.$$

We compute

$$\begin{aligned} \widetilde{k}\langle v, v_3 \rangle_{v_2,v_4} &= \langle \pi(k)^{-1} v, v_3 \rangle_{v_2,v_4} \\ &= \langle v, \pi(k) v_3 \rangle_{v_2,v_4} \\ &= \sum_{i=1}^{n} \overline{\lambda_i} \langle v, w_i \rangle_{v_2,v_4}, \end{aligned}$$

where the second equality follows by K-invariance of $\langle \cdot, \cdot \rangle_{v_2,v_4}$.
Therefore, the span of $\widetilde{K}\langle \cdot, v_3 \rangle_{v_2,v_4}$ is contained in the span of the functionals $\langle \cdot, w_1 \rangle_{v_2,v_4}, \dots, \langle \cdot, w_n \rangle_{v_2,v_4}$. □

Proposition 3.4. Let (π, H) be a tempered irreducible unitary representation of a real semisimple group G. Then, for all $X \in \mathfrak{g}$, and for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\langle \dot{\pi}(X) v_1, v_3 \rangle_{v_2,v_4} = -\langle v_1, \dot{\pi}(X) v_3 \rangle_{v_2,v_4}$$

and

$$\langle \dot{\pi}(X) v_2, v_4 \rangle^{v_1,v_3} = -\langle v_2, \dot{\pi}(X) v_4 \rangle^{v_1,v_3},$$

where $\dot{\pi}$ denotes the action of $\mathfrak{g}$ through π introduced in Section 2, page 8.

Proof. Since the functional $\langle \cdot, v_3 \rangle_{v_2,v_4}$ belongs to $\widetilde{H_K}$ by Lemma 3.3 and since H_K is admissible by Theorem 2.9, it follows from Proposition 2.11 that there exists $w \in H_K$ such that

$$\langle v, v_3 \rangle_{v_2,v_4} = \langle v, w \rangle$$

for all $v \in H_K$. For $X \in \mathfrak{g}$, we compute

$$\begin{aligned}
\langle \dot\pi(X)v_1, v_3\rangle_{v_2,v_4} &= \langle \dot\pi(X)v_1, w\rangle \\
&= \frac{d}{dt}\langle \pi(\exp(tX))v_1, w\rangle|_{t=0} \\
&= \frac{d}{dt}\langle v_1, \pi(\exp(-tX))w\rangle|_{t=0} \\
&= \frac{d}{dt}\overline{\langle \pi(\exp(-tX))w, v_1\rangle}|_{t=0} \\
&= \overline{\langle -\dot\pi(X)w, v_1\rangle} \\
&= \overline{X}\,\overline{\langle w, v_1\rangle} \\
&= \overline{X}\,\overline{\langle v_3, v_1\rangle}_{v_4,v_2} \\
&= -\overline{\langle \dot\pi(X)v_3, v_1\rangle}_{v_4,v_2} \\
&= -\langle v_1, \dot\pi(X)v_3\rangle_{v_2,v_4}
\end{aligned}$$

Here, the notation $\overline{X}$ denotes the action of $\mathfrak{g}$ on the conjugate-dual $(\mathfrak{g}, K)$-module $\overline{H_K}$, introduced in Section 2, page 9. The sixth equality follows from viewing $\overline{\langle \cdot, v_1\rangle}$ as an element in $\overline{H_K}$, the conjugate-dual of H_K (See Section 2, after Theorem 2.9). For the seventh equality, we observe that

$$\langle v_1, w\rangle = \langle v_1, v_3\rangle_{v_2,v_4}$$

implies that

$$\langle w, v_1\rangle = \overline{\langle v_1, w\rangle} = \overline{\langle v_1, v_3\rangle}_{v_2,v_4} = \langle v_3, v_1\rangle_{v_4,v_2}.$$

Finally, the eighth equality follows from viewing $\overline{\langle \cdot, v_1\rangle}_{v_4,v_2}$ as an element in $\overline{H_K}$. For the $\mathfrak{g}$-invariance of the form $\langle \cdot, \cdot\rangle_{v_1,v_3}$, the computation

$$\begin{aligned}
\langle v_2, v_4\rangle^{v_1,v_3} &= \lim_{r\to\infty} \frac{\int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg}{r^{\mathbf{d}(\pi)}} \\
&= \lim_{r\to\infty} \frac{\int_{G_{<r}} \langle v_1, \pi(g^{-1})v_2\rangle \overline{\langle v_3, \pi(g^{-1})v_4\rangle}\, dg}{r^{\mathbf{d}(\pi)}} \\
&= \lim_{r\to\infty} \frac{\int_{G_{<r}} \langle v_1, \pi(g)v_2\rangle \overline{\langle v_3, \pi(g)v_4\rangle}\, dg}{r^{\mathbf{d}(\pi)}} \\
&= \lim_{r\to\infty} \frac{\int_{G_{<r}} \langle \pi(g)v_4, v_3\rangle \overline{\langle \pi(g)v_2, v_1\rangle}\, dg}{r^{\mathbf{d}(\pi)}} \\
&= \langle v_4, v_2\rangle_{v_3,v_1},
\end{aligned}$$

where the third equality follows from (2) of Proposition 3.1, reduces the statement to the previous case. $\square$

We can now complete the strategy outlined in the introduction.

Theorem 3.5. Let (π, H) be a tempered irreducible unitary representation of a real semisimple group G and let K be a maximal compact subgroup of G. Then there exists $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$, we have

$$\lim_{r\to\infty} \frac{\int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg}{r^{\mathbf{d}(\pi)}} = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

Proof. Fix $v_2, v_4 \in H_K$. By Proposition 3.4, we can apply Corollary 2.14 to the form $\langle \cdot, \cdot \rangle_{v_2,v_4}$. Hence there exists $c_{v_2,v_4} \in \mathbb{C}$ such that for all $v_1, v_3 \in H_K$ we have

$$\langle v_1, v_3\rangle_{v_2,v_4} = c_{v_2,v_4} \langle v_1, v_3\rangle.$$

Similarly, fixing $v_1, v_3 \in H_K$, there exists a $d_{v_1,v_3} \in \mathbb{C}$ such that

$$\overline{\langle v_4, v_2\rangle}^{v_3,v_1} = d_{v_1,v_3} \langle v_4, v_2\rangle,$$

since the left-hand side is conjugate-linear in the first variable. Hence, since

$$\overline{\langle v_4, v_2\rangle}^{v_3,v_1} = \langle v_2, v_4\rangle^{v_1,v_3},$$

we obtain

$$\langle v_2, v_4\rangle^{v_1,v_3} = d_{v_1,v_3} \overline{\langle v_2, v_4\rangle}$$

By definition, we have

$$D(v_1, v_2, v_3, v_4) = \langle v_1, v_3\rangle_{v_2,v_4} = \langle v_2, v_4\rangle^{v_1,v_3},$$

so, for a vector $v_0 \in H_K$ of norm 1, using (2) of Theorem 1.2, we obtain a real number $C(v_0, v_0) > 0$ such that

$$D(v_0, v_0, v_0, v_0) = C(v_0, v_0) = c_{v_0,v_0} = d_{v_0,v_0}.$$

Computing $D(v_1, v_0, v_3, v_0)$, we have

$$d_{v_1,v_3} = c_{v_0,v_0} \langle v_1, v_3\rangle.$$

Therefore, we obtained

$$D(v_1, v_2, v_3, v_4) = c_{v_0,v_0} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle},$$

showing that $\mathbf{f}(\pi) := \frac{1}{C(v_0,v_0)}$ does not depend on the choice of v_0, as required. ∎

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(Anne-Marie Aubert) Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France.

Email address: anne-marie.aubert@imj-prg.fr

(Alfio Fabio La Rosa) Department of Mathematics, University of Luxembourg, Maison du Nombre, 6 Avenue de la Fonte, L-4364, Esch-sur-Alzette, Luxembourg.

Email address: fabio.larosa@uni.lu