Stability over cDV singularities and other complete local rings

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Abstract.

We characterise subcategories of semistable modules for noncommutative minimal models of compound Du Val singularities, including the non-isolated case. We find that the stability is controlled by an infinite polyhedral fan that stems from silting theory, and which can be computed from the Dynkin diagram combinatorics of the minimal models found in the work of Iyama–Wemyss. In the isolated case, we moreover find an explicit description of the deformation theory of the stable modules in terms of factors of the endomorphism algebras of 2-term tilting complexes. To obtain these results we generalise a correspondence between 2-term silting theory and stability, which is known to hold for finite dimensional algebras, to the much broader setting of algebras over a complete local Noetherian base ring.

§1. Introduction

Stability conditions are an important tool in algebra, as they provide an avenue for studying wild representation theory through moduli spaces of (semi)stable objects. In some cases one can extract noncommutative Donaldson–Thomas invariants [21] from these moduli spaces that give a "virtual count" of the semistable objects, which takes the deformation theory of these objects into account.

The aim of this paper is to understand stability condition from this perspective for algebras with geometric significance: noncommutative minimal models for *compound Du Val* singularities. These are a three-dimensional analogue of the noncommutative resolutions of ADE surface singularities encountered in the McKay correspondence, which are part of a noncommutative approach to the minimal model program for threefolds [24]. We work in a wide setting, which includes both Van

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den Bergh's NCCRs [22], as well as the more general noncommutative minimal models studied by Iyama–Wemyss [14, 13].

The (semi)stable modules for these noncommutative minimal models have only been classified in a limited number of cases, due to the complexity of the representation theory involved. Most notable is the work of Nagao–Nakajima [18], who show that the stability of the "conifold" is controlled by a *hyperplane arrangement*. As a consequence, the enumerative theory of these examples is now well-established [21, 5]. In our recent work [23] we were able to uncover the enumerative theory for a new family of examples, by showing that the stability is again controlled by a hyperplane arrangement, coming from the *silting theory* of the noncommutative minimal model. Moreover, we showed that these silting complexes determine the deformation theory of the stable modules.

In this paper we further develop this connection between silting and stability and thereby give a complete picture of the stability over *any* cDV singularity, via the characterisation of the 2-term silting theory in the work of Iyama-Wemyss [13]. In fact, by applying recent results of Kimura [15] we are able to work in a much more general setting of algebras over any complete local base ring. The paper is therefore split into two parts: the first containing general results about silting & stability for algebras over a complete local ring, and the second on the application of these results to cDV singularities.

1.1. Stability and silting over a complete local ring

We work over a complete local Noetherian commutative ring $(R, \mathfrak{m})$, and consider finite R-algebras Λ with R central. Our goal is to characterise semistable modules of such algebras for King stability conditions parametrised by K-theory vectors

$$\theta \in \mathrm{K}_0(\mathrm{proj}\,\Lambda)_\mathbb{R} := \mathrm{K}_0(\mathrm{proj}\,\Lambda) \otimes \mathbb{R}.$$

To do this we leverage a certain duality between King stability and silting theory, building on the work of various authors [2, 4, 26] in the setting of finite dimensional algebras. Concretely, we construct a *silting fan*

$$\mathrm{SFan}(\Lambda) = \bigcup_{T \in 2\text{-silt}\,\Lambda} \mathrm{Cone}(T)$$

as the union of closed cones in $\mathrm{K}_0(\mathrm{proj}\,\Lambda)_\mathbb{R}$ generated by basic 2-term silting complexes , and show that for each θ in this fan the category $\mathcal{S}_\theta \subset \mathrm{fl}\,\Lambda$ of finite length θ-semistable modules can be characterised via an orthogonality condition defined by a certain silting complex and its direct summands.

In Proposition 2.8 we show that the silting fan is indeed a fan, whose faces are the *strict* cones of basic 2-term *pre*silting complexes.

Proposition A (Proposition 2.8)**.** *Let* Λ *be a finite algebra over a complete local Noetherian ring. Then* $\mathrm{SFan}(\Lambda)$ *is a polyhedral fan with decomposition*

$$\mathrm{SFan}(\Lambda) = \bigsqcup_{T' \in 2\text{-presilt}\,\Lambda} \mathrm{Cone}^\circ(T'),$$

where each face $\mathrm{Cone}^\circ(T')$ *is the* strict *cone of 2-term presilting complexes* T'.

The above is well-known in the finite dimensional setting, where it follows from the work of Demonet–Iyama–Jasso [6]. To lift this to the complete local setting, we apply the a reduction theorem due to Kimura [15], which allows us to compare the silting theories of the algebra Λ and its finite dimensional fibre $\overline{\Lambda} := \Lambda \otimes_R R/\mathfrak{m}$ over the closed point in $\mathrm{Spec}\,R$. In particular, we find that the structure of $\mathrm{SFan}(\Lambda)$ depends only on the fibre $\overline{\Lambda}$, which yields the following.

Proposition B (Proposition 2.9)**.** *If* $I \subset \mathfrak{m}$ *is an ideal, the base-change over the quotient* $R \to R/I$ *identifies the the silting fans* $\mathrm{SFan}(\Lambda)$ *and* $\mathrm{SFan}(\Lambda/I\Lambda)$.

For finite dimensional algebras such as $\overline{\Lambda}$, the semistable objects for stability conditions in $\mathrm{SFan}(\overline{\Lambda})$ can be described via certain torsion pairs, as shown by Brüstle–Smith–Treffinger [4] in the language of τ-tilting theory, and by Yurikusa [26] and Asai [2] in terms of silting theory. One interpretation of these torsion pairs is as an orthogonality condition in the derived category between semistable modules and 2-term (pre)silting complexes. Our main technical result is that an analogous orthogonality condition also controls the stability in the complete local setting.

Theorem C (Theorem 2.12)**.** *Let* $T \in 2\text{-silt}\,\Lambda$ *be the Bongartz completion of a summand* $T' \subset T$. *For all* $\theta \in \mathrm{Cone}^\circ(T')$ *the* θ-*semistables form the subcategory*

$$\mathcal{S}_\theta = \{M \in \mathrm{fl}\,\Lambda \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T, M[1]) = 0,\ \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T', M) = 0\}.$$

Moreover, $\mathcal{S}_\theta$ *is an abelian length category containing* $|T| - |T'|$ *simple objects, which are precisely the* θ-*stable modules.*

The orthogonality condition gives a description of $\mathcal{S}_\theta$ as a subcategory inside the heart of a t-structure on the derived category $\mathrm{D}^b(\mathrm{fl}\,\Lambda)$, which is induced by induced by T. Via a derived Morita equivalence,

this heart can be identified with the module category $\mathrm{fl}\,\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T)$, leading to the following description of $\mathcal{S}_\theta$.

Theorem D (Theorem 2.13). *Let T be the Bongartz completion of a summand $T' \subset T$. Then for all $\theta \in \mathrm{Cone}^\circ(T')$ there is an equivalence of abelian categories*

$$\mathcal{S}_\theta \xrightarrow{\ \sim\ } \mathrm{fl}\,\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T)/(e),$$

where $e\colon T \to T' \to T$ is the idempotent projecting onto the summand $T' \subset T$.

One way to interpret this theorem is through the lens of noncommutative deformation theory: for each $\theta \in \mathrm{Cone}^\circ(T')$ the theorem implies that $\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T)/(e)$ pro-represent Laudal's [17] deformation functor for the set of θ-stable modules.

1.2. Results about cDV singularities

We now discuss the applications of the above results in the setting of cDV singularities. In [14] Iyama–Wemyss introduced a type of noncommutative minimal model for a cDV singularity R, in the form of endomorphism algebras

$$\Lambda_M := \mathrm{End}_R(M)$$

of a certain distinguished set of reflexive modules, which are related by mutations. In their follow up work [13] they moreover characterise the 2-term tilting theory of these minimal models by relating them to the 2-dimensional case: if $g \in R$ is a sufficiently generic section then the slice

$$\Lambda \to \Lambda/g\Lambda$$

is a noncommutative *partial* resolution of a simple surface singularity as in the McKay correspondence, and they show that the 2-term tilting theory of this slice is closely related to that of Λ. In particular, they are able to show that the 2-term tilting complexes of Λ correspond to chambers in an *intersection hyperplane arrangement*

$$X_{\Gamma,J} \subset \mathbb{R}^J,$$

which is determined by a Dynkin type (Γ, J) associated to the slice. Although this correspondence gives a strong grip on the tilting theory of Λ, it may fail to be a bijection if R is not an *isolated* singularity. Using Proposition 2.9, we are however able to strengthen their result in the non-isolated setting at the cost of moving to silting theory.

Proposition E (Proposition 3.4)**.** *Let* $\Lambda = \Lambda_M$ *be a noncommutative minimal model of a cDV singularity with Dynkin type* (Γ, J). *Then* $\mathrm{SFan}(\Lambda)$ *is identified with the intersection arrangement* $X_{\Gamma, J}$ *via an isomorphism* $\mathrm{K}_0(\mathrm{proj}\,\Lambda) \simeq \mathbb{R}^J$.

With the silting fan identified, Theorem 2.12 now directly yields a classification of all K-theory vectors for which there exists semistable modules. This holds both for isolated as well as non-isolated singularities.

Proposition F (Proposition 3.5)**.** *Let* $\Lambda = \Lambda_M$ *be a noncommutative minimal model of a cDV singularity with Dynkin type* (Γ, J). *For* θ *contained in a codimension* k *face of* $X_{\Gamma, J}$ *there are precisely* k *stable modules in* $\mathcal{S}_\theta \subset \mathrm{fl}\,\Lambda$. *In particular:*

- *if* θ *lies in a chamber, then* $\mathcal{S}_\theta = 0$,
- *if* θ *lies generically on a hyperplane, then there is a unique* θ-*stable module.*

This result gives a strong grip on the stability of finite length modules, because the intersection arrangements, though infinite, can be described concretely using Dynkin diagram combinatorics. As an example, one can consider the arrangement associated to a choice of two nodes in the extended D_4 Dynkin diagram:
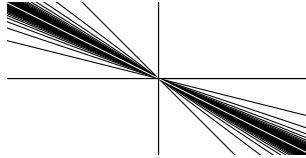
The proposition implies that for a Λ with this Dynkin-type, the subcategories $\mathcal{S}_\theta$ behave constructibly with respect to the above wall-and-chamber structure: $\mathcal{S}_\theta$ is trivial for θ in a chamber, $\mathcal{S}_\theta = \mathrm{fl}\,\Lambda$ if $\theta = 0$, and θ if lies in the complement of 0 inside a wall then $\mathcal{S}_\theta$ consists of self-extensions of a unique θ-stable module.

The constructible behaviour of $\mathcal{S}_\theta$ also holds for higher dimensional arrangements, such as those illustrated in [13], and implies that the K-theory classes of semistable modules in $\mathrm{K}_0(\mathrm{fl}\,\Lambda)$ are contained in a lattice that is *dual* to the intersection arrangement. This has implications for enumerative theories, because it determines which classes in $\mathrm{K}_0(\mathrm{fl}\,\Lambda)$ will contribute to the enumerative invariants of Λ.

In the isolated setting, the correspondence of [13] yields an explicit tilting complex for each chamber in $X_{\Gamma, J}$, and the endomorphism algebras are again noncommutative minimal models. Therefore Theorem

2.13 can be used in this setting to give an explicit description of the subcategories of semistable modules.

Theorem G (Theorem 3.7). *Let* $\Lambda = \Lambda_M$ *be a noncommutative minimal model of an* isolated *cDV singularity. Then for all* $\theta \in X_{\Gamma,J}$ *there is an equivalence of abelian categories*

$$\mathcal{S}_\theta \simeq \mathrm{fl}\, \Lambda_N/(e),$$

for some noncommutative minimal model Λ_N *and idempotent* $e \in \Lambda_N$.

Hence, the deformation theory of θ-stable modules of a fixed minimal model Λ_M is pro-represented by the factor algebras $\Lambda_N/(e)$ of the other noncommutative minimal models, at least for θ contained in the fan. This has strong implications for the enumerative geometry of flops, which we will cover in a separate paper.

1.3. Acknowledgements

§2. Stability and silting

In this section we fix a complete local Noetherian commutative ring $(R, \mathfrak{m})$ with residue field $\kappa = R/\mathfrak{m}$, and an R-algebra Λ such that R is central in Λ and Λ is finite as an R-module. Equivalently, Λ is a coherent sheaf of algebras over $\operatorname{Spec} R$, whose fibre over the unique closed point $\operatorname{Spec} \kappa$ is the finite dimensional algebra

$$\overline{\Lambda} := \Lambda/\mathfrak{m}\Lambda \simeq \Lambda \otimes_R \kappa.$$

In what follows we compare the stability conditions and silting theory for Λ and $\overline{\Lambda}$ via functors induced by the quotient map $\Lambda \to \overline{\Lambda}$.

2.1. King stability over a complete local base

King [16] defined stability conditions on any abelian category $\mathcal{A}$ in terms of *linear forms* $\mathrm{K}_0(\mathcal{A}) \to \mathbb{R}$ on its Grothendieck group. Here we want to define such stability conditions on the abelian categories $\mathcal{A} = \mathrm{fl}\,\Lambda$ and $\mathcal{A} = \mathrm{fl}\,\overline{\Lambda}$ of finite length modules over Λ and $\overline{\Lambda}$, in a compatible way. To do this we use a pairing of $\mathrm{K}_0(\mathrm{fl}\,\Lambda)$ with the real K-theory space

$$\mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}} := \mathrm{K}_0(\mathrm{proj}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R},$$

where $\operatorname{proj}\Lambda$ denotes the category of projectives, and similarly for $\overline{\Lambda}$. Before defining this pairing we need the following elementary lemma.

In what follows, let $\operatorname{length}_R M$ denote the length of a module M over R.

Lemma 2.1. *Let $P \in \operatorname{proj}\Lambda$ and $M \in \operatorname{fl}\Lambda$, then $\operatorname{Hom}_\Lambda(P, M)$ is a finite length R-module.*

Proof. Because P is a finitely generated projective, it is a direct summand of $\Lambda^{\oplus n}$ for some n. Because $\operatorname{Hom}_\Lambda(P, M)$ is a direct summand of $\operatorname{Hom}_\Lambda(\Lambda^{\oplus n}, M)$, the additivity of the length therefore implies

$$\operatorname{length}_R \operatorname{Hom}_\Lambda(P, M) \leq \operatorname{length}_R \operatorname{Hom}_\Lambda(\Lambda^{\oplus n}, M) = n \cdot \operatorname{length}_R M.$$

Because Λ is finite as an R-module, a finitely generated Λ-module has finite length over Λ if and only if it has finite length over the base ring R. In particular, $M \in \operatorname{fl}\Lambda$ implies $\operatorname{length}_R M < \infty$, and the result follows. Q.E.D.

Lemma 2.1 implies that there is a well-defined mapping $\operatorname{proj}\Lambda \times \operatorname{fl}\Lambda \to \mathbb{N}$ which sends a pair $(P, M) \in \operatorname{proj}\Lambda \times \operatorname{fl}\Lambda$ to $\operatorname{length}_R \operatorname{Hom}_\Lambda(P, M)$. Because the length is additive over short exact sequences, it induces the following pairing on K-theory.

Lemma 2.2. *The assignment $(P, M) \mapsto \operatorname{length}_R \operatorname{Hom}_\Lambda(P, M)$ extends to a pairing*

$$\langle -, - \rangle_\Lambda \colon \operatorname{K}_0(\operatorname{proj}\Lambda)_\mathbb{R} \otimes \operatorname{K}_0(\operatorname{fl}\Lambda) \to \mathbb{R}.$$

Proof. The Grothendieck groups $\operatorname{K}_0(\operatorname{proj}\Lambda)_\mathbb{R}$ and $\operatorname{K}_0(\operatorname{fl}\Lambda)$ are generated by the symbols $[P]$ for $P \in \operatorname{proj}\Lambda$, and $[M]$ for $M \in \operatorname{fl}\Lambda$ respectively. Therefore, we can define $\langle -, - \rangle_\Lambda$ by extending bi-linearly:

$$\left\langle \sum_i \theta_i [P_i], \sum_j n_j [M_j] \right\rangle = \sum_{ij} \theta_i n_j \operatorname{length}_R \operatorname{Hom}_\Lambda(P_i, M_j),$$

and it suffices to check that this form $\langle -, - \rangle_\Lambda$ respects the relations in the Grothendieck group. For $P \in \operatorname{proj}\Lambda$, the functor $\operatorname{Hom}_\Lambda(P, -)$ is exact and therefore maps every short exact sequence $0 \to N \to M \to M/N \to 0$ in $\operatorname{fl}\Lambda$ to a short exact sequence in $\operatorname{fl}R$, and hence by the additivity of the length:

$$\operatorname{length}_R \operatorname{Hom}_\Lambda(P, M) = \operatorname{length}_R \operatorname{Hom}_\Lambda(P, N) + \operatorname{length}_R \operatorname{Hom}_\Lambda(P, M/N).$$

It follows that $\langle [P], [N] - [M] + [M/N] \rangle_\Lambda = 0$. Likewise, every admissible exact sequence $Q \hookrightarrow P \twoheadrightarrow P/Q$ in the exact category $\operatorname{proj}\Lambda$ induces an

exact sequence for every $M \in \mathrm{fl}\,\Lambda$

$$0 \longrightarrow \mathrm{Hom}_\Lambda(P/Q, M) \longrightarrow \mathrm{Hom}_\Lambda(P, M)$$
$$\hookrightarrow \mathrm{Hom}_\Lambda(Q, M) \longrightarrow \mathrm{Ext}^1_\Lambda(P/Q, M) = 0$$

which implies that $\langle [Q] - [P] + [P/Q], [M] \rangle_\Lambda = 0$.   Q.E.D.

Given any vector $\theta \in \mathrm{K}_0(\mathrm{proj}\,\Lambda)_\mathbb{R}$, the pairing now yields a linear form

$$\langle \theta, - \rangle \colon \mathrm{K}_0(\mathrm{fl}\,\Lambda) \to \mathbb{R},$$

and therefore a King stability condition on $\mathrm{fl}\,\Lambda$. Following King's definition [16, Def 1.1], a module $M \in \mathrm{fl}\,\Lambda$ is θ-*semistable* if $\langle \theta, [M] \rangle = 0$ and $\langle \theta, [N] \rangle \leq 0$ for every submodule $N \subset M$, and is additionally called $\theta-$*stable* if[1] $M \neq 0$ and the inequality is strict for all proper nonzero submodules. We write $\mathcal{S}_\theta \subset \mathrm{fl}\,\Lambda$ for the full subcategory of semistable modules:

$$\mathcal{S}_\theta = \{ M \in \mathrm{fl}\,\Lambda \mid M \text{ is } \theta\text{-semistable} \}.$$

In the finite dimensional setting discussed in [4, 2] these subcategories are known to be *wide* subcategories, i.e. a finite length subcategory which is closed under kernels, cokernels, and extensions. In our setting the analogous result holds.

Lemma 2.3. *For all $\theta \in \mathrm{K}_0(\mathrm{proj}\,\Lambda)_\mathbb{R}$ the subcategory $\mathcal{S}_\theta \subset \mathrm{fl}\,\Lambda$ is a wide subcategory, and its simple objects are exactly the θ-stable modules.*

Proof. This follows directly from [3, Prop 2.20], after translating the King stability condition $\langle \theta, - \rangle$ to the associated stability function of Rudakov [19].   Q.E.D.

One similarly obtains a pairing $\langle -, - \rangle_{\overline{\Lambda}}$ for the finite dimensional fibre $\overline{\Lambda}$, and each vector $\overline{\theta} \in \mathrm{K}_0(\mathrm{proj}\,\overline{\Lambda})$ induces a stability condition on $\mathrm{fl}\,\overline{\Lambda}$, with an associated wide subcategory $\mathcal{S}_{\overline{\theta}} \subset \mathrm{mod}\,\overline{\Lambda}$. The pairings for Λ and $\overline{\Lambda}$ are compatible with the extension/restriction of scalars functors

$$\overline{(-)} := (-) \otimes_\Lambda \overline{\Lambda} \colon \mathrm{proj}\,\Lambda \to \mathrm{proj}\,\overline{\Lambda}, \quad (-)_\Lambda \colon \mathrm{fl}\,\overline{\Lambda} \to \mathrm{fl}\,\Lambda$$

via the adjunction $\mathrm{Hom}_\Lambda(-, (-)_\Lambda) \simeq \mathrm{Hom}_{\overline{\Lambda}}(- \otimes_\Lambda \overline{\Lambda}, -)$. This results in the following compatibility between stability conditions on $\mathrm{fl}\,\Lambda$ and stability conditions on $\mathrm{fl}\,\overline{\Lambda}$.

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[1]Note that this definition guarantees that the zero module is semistable but not stable, just as it is semisimple but not simple.

Proposition 2.4. *Let* $\theta \in K_0(\text{proj}\,\Lambda)_{\mathbb{R}}$ *with image* $\overline{\theta} \in K_0(\text{proj}\,\overline{\Lambda})_{\mathbb{R}}$, *then the functor* $(-)_{\Lambda}$ *defines an exact embedding* $\mathcal{S}_{\overline{\theta}} \longrightarrow \mathcal{S}_{\theta}$ *which identifies the* $\overline{\theta}$-*stable modules with the* θ-*stable modules. In particular* $\mathcal{S}_{\theta}$ *is the extension closure*

$$\mathcal{S}_{\theta} = \langle (\mathcal{S}_{\overline{\theta}})_{\Lambda} \rangle.$$

Proof. Write θ as a linear combination $\theta = \sum_i \theta_i [P_i]$ for $P_i \in \text{proj}\,\Lambda$, so that it image is $\overline{\theta} = \theta_i[\overline{P}_i]$. For any $\overline{M} \in \text{fl}\,\overline{\Lambda}$, the adjunction yields

$$\langle \theta, [\overline{M}_{\Lambda}] \rangle_{\Lambda} = \sum_i \theta_i \cdot \text{length}_R \text{Hom}_{\Lambda}(P_i, \overline{M}_{\Lambda})$$
$$= \sum_i \theta_i \cdot \text{length}_R \text{Hom}_{\overline{\Lambda}}(\overline{P}_i, \overline{M}) = \langle \overline{\theta}, [\overline{M}] \rangle_{\overline{\Lambda}}.$$

Because the submodules of the restriction $\overline{M}_{\Lambda}$ are precisely the restrictions of submodules of $\overline{M}$, it then follows that that $\overline{M}$ is $\overline{\theta}$-(semi)stable if and only if its image $\overline{M}_{\Lambda}$ is θ-(semi)stable. Hence $(-)_{\Lambda}$ restricts to an embedding $\mathcal{S}_{\overline{\theta}} \to \mathcal{S}_{\theta}$.

To see that every θ-stable module is in the image, recall (see e.g. [19, Theorem 1]) that the endomorphism ring of a stable object in an abelian category is a division ring. Therefore, if $M \in \mathcal{S}_{\theta}$ is a stable module, then for any $z \in \mathfrak{m}$ the homomorphism $z\cdot: M \to M$ of multiplication by z is either 0 or an isomorphism. Because $\mathfrak{m}$ maps into the Jacobson radical of Λ, Nakayama's lemma implies $z\cdot$ is the zero map for all $z \in \mathfrak{m}$, which shows that $M \simeq \overline{M}_{\Lambda}$ is in the image of $(-)_{\Lambda}$.

By Lemma 2.3 the category $\mathcal{S}_{\theta}$ is the extension closure of the set of θ-stable modules. As these are all in the image $(\mathcal{S}_{\overline{\theta}})_{\Lambda}$ it then follows that $\mathcal{S}_{\theta} = \langle (\mathcal{S}_{\overline{\theta}})_{\Lambda} \rangle$.   Q.E.D.

2.2. The silting fan

Let $\mathsf{K}^b(\Lambda) := \mathsf{K}^b(\text{proj}\,\Lambda)$ denote the homotopy category of complexes of projectives. Recall that $T \in \mathsf{K}^b(\Lambda)$ is *presilting* if

$$\text{Hom}_{\mathsf{K}^b(\Lambda)}(T, T[i]) = 0 \quad \forall i > 0,$$

and is *silting* if it additionally generates $\mathsf{K}^b(\Lambda)$ as a triangulated category. Because Λ is finite over a complete local ring, $\mathsf{K}^b(\Lambda)$ is known to be Krull-Schmidt: every complex $T \in \mathsf{K}^b(\Lambda)$ splits as a direct sum $T = T_1 \oplus \ldots \oplus T_k$ of indecomposables. We write $|T|$ for the number of indecomposables in the Krull-Schmidt decomposition. A complex is *basic* if it has no repeated indecomposable summands, and is a *2-term* complex if it is supported in degrees -1 and 0. The sets of isomorphism classes of basic 2-term presilting complexes and silting complexes will be denote by

$$2\text{-presilt}\,\Lambda, \quad 2\text{-silt}\,\Lambda$$

respectively, and in what follows we will take the liberty to identify the elements of 2-presilt Λ and 2-silt Λ with a choice of representatives $T \in \mathsf{K}^b(\Lambda)$. (Pre)silting complexes for $\overline{\Lambda}$ are defined in a similar way.

The 2-term silting theory of finite dimensional algebras is well-studied and known to induce a wall-and-chamber structure inside the K-theory space, which can be used to characterise stability conditions [4, 2]. To find a similar wall-and-chamber structure for $\mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}} = \mathrm{K}_0(\mathsf{K}^b(\Lambda))_{\mathbb{R}}$ we will apply the following reduction result of Kimura [15] for algebras over complete local Noetherian rings.

Proposition 2.5 ([15, Prop 4.2(b), Prop 4.5(a)]). *Let $I \subset \mathfrak{m}$ be an ideal, then the functor $- \otimes_\Lambda \Lambda/I\Lambda$ defines a surjection*

$$2\text{-presilt}\,\Lambda \to 2\text{-presilt}\,\Lambda/I\Lambda,$$

which restricts to an equivalence of partially ordered sets

$$2\text{-silt}\,\Lambda \xrightarrow{\sim} 2\text{-silt}\,\Lambda/I\Lambda.$$

Clearly, this result applies to the case $I = \mathfrak{m}$, given a bijection 2-silt $\Lambda \simeq$ 2-silt $\overline{\Lambda}$. However, we require a slightly stronger statement, as we will also need to compare presilting complexes to find the wall-and-chamber structure. For this we use a result of Aihara [1, Proposition 2.16], which shows that every $T' \in$ 2-presilt Λ has a *Bongartz completion* in the form of a silting complex $T = T' \oplus T'' \in$ 2-silt Λ.

Lemma 2.6. *The map 2-presilt $\Lambda \to$ 2-presilt $\overline{\Lambda}$ is a bijection.*

Proof. Suppose $T, T' \in$ 2-presilt Λ are basic 2-term presilting complexes such that $\overline{T} \simeq \overline{T}'$. Let $T \oplus U \in$ 2-silt Λ be the Bongartz completion of T, then $\overline{T} \oplus \overline{U} \simeq \overline{T}' \oplus \overline{U}$ is also the Bongartz completion of $\overline{T}'$, which lifts to $T' \oplus U \in$ 2-silt Λ. Because the map 2-silt $\Lambda \to$ 2-silt $\overline{\Lambda}$ is bijective, the isomorphism $\overline{T} \oplus \overline{U} \simeq \overline{T}' \oplus \overline{U}$ lifts to an isomorphism $T \oplus U \simeq T' \oplus U$, and it then follows that $T \simeq T'$ because $\mathsf{K}^b(\Lambda)$ is Krull-Schmidt. It follows that the surjective map 2-presilt $\Lambda \to$ 2-presilt $\overline{\Lambda}$ is injective and hence also bijective. Q.E.D.

Given a presilting complex $T \in$ 2-presilt Λ with Krull-Schmidt decomposition of the form $T = T_1 \oplus \ldots \oplus T_k$, let

$$\mathrm{Cone}(T) := \left\{ \sum_{i=1}^k \lambda_i [T_i] \mid \lambda_i \geq 0 \right\} \subset \mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}},$$
$$\mathrm{Cone}^\circ(T) := \left\{ \sum_{i=1}^k \lambda_i [T_i] \mid \lambda_i > 0 \right\} \subset \mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}},$$

denote respectively the cone and strict cone of T in $K_0(\text{proj}\,\Lambda)_{\mathbb{R}}$. The cone of T is polyhedral with faces given by the strict cones of its summands, i.e. $\text{Cone}(T)$ decomposes as a disjoint union

$$\text{Cone}(T) = \bigsqcup_{T' \subset T} \text{Cone}^\circ(T').$$

In particular, $\text{Cone}^\circ(T)$ is the interior of $\text{Cone}(T)$ inside its linear span, and the boundary is a union of faces $\text{Cone}(T \setminus T_i)$ of the presilting complexes obtained by removing one of the indecomposable summands.

Similarly, every presilting complex $\overline{T} \in 2\text{-presilt}\,\overline{\Lambda}$ defines a cone in $K_0(\text{proj}\,\Lambda)$, and we claim that this construction is compatible with the reduction map.

Lemma 2.7. *For every $T \in 2\text{-presilt}\,\Lambda$, the strict cones $\text{Cone}^\circ(T)$ and $\text{Cone}^\circ(\overline{T})$ are identified by $\overline{(-)}: K_0(\text{proj}\,\Lambda)_{\mathbb{R}} \xrightarrow{\sim} K_0(\text{proj}\,\overline{\Lambda})_{\mathbb{R}}$.*

Proof. Let $T = T_1 \oplus \ldots \oplus T_k$ be the Krull-Schmidt decomposition of some $T \in 2\text{-presilt}\,\Lambda$, then $\overline{T}$ has the Krull-Schmidt decomposition $\overline{T} = \overline{T}_1 \oplus \ldots \oplus \overline{T}_k$ as the $\overline{T}_i$ are indecomposable by Proposition 2.5. Hence

$$\overline{\text{Cone}^\circ(T)} = \left\{ \sum_{i=1}^k \lambda_i [\overline{T}_i] \mid \lambda > 0 \right\} = \text{Cone}^\circ(\overline{T}),$$

as claimed.   Q.E.D.

If $T \in 2\text{-silt}\,\Lambda$ the classes $[T_i]$ of its indecomposable summands form a basis for $K_0(\text{proj}\,\Lambda)_{\mathbb{R}}$, and $\text{Cone}^\circ(T)$ is therefore an open subspace of $K_0(\text{proj}\,\Lambda)_{\mathbb{R}}$. In the finite dimensional setting it is known that these open subspaces form the chambers of a wall-and-chamber structure, with walls given by the cones of a common summand. Here we find a similar result.

Proposition 2.8. *For $T, U \in 2\text{-presilt}\,\Lambda$ their cones intersect in*

$$\text{Cone}(T) \cap \text{Cone}(U) = \text{Cone}(V)$$

for some $V \in 2\text{-presilt}\,\Lambda$ which is a summands of both T and U. In particular, if $T, U \in 2\text{-silt}\,\Lambda$ are distinct silting complexes, then

$$\text{Cone}^\circ(T) \cap \text{Cone}^\circ(U) = \emptyset.$$

Proof. This is well-known in the finite dimensional setting: for 2-term tilting complexes it is a result by Hille [9], for 2-term silting complexes by Demonet–Iyama–Jasso [6, Cor 6.7], and analogously for 2-term presilting complexes as in [2, Prop 3.9]. Because the fibre $\overline{\Lambda}$ is

finite dimensional, the reductions $\overline{T}, \overline{U} \in$ 2-presilt $\overline{\Lambda}$ of a pair of presilting complexes $T, U \in$ 2-presilt Λ satisfy

$$\mathrm{Cone}(\overline{T}) \cap \mathrm{Cone}(\overline{U}) = \mathrm{Cone}(\overline{V}),$$

for some $\overline{V} \in$ 2-presilt $\overline{\Lambda}$ which is then a summand of $\overline{T}$ and $\overline{U}$. By Lemma 2.6 the object $\overline{V}$ lifts uniquely to some $V \in$ 2-presilt Λ which is a summand of T and U. By Lemma 2.7 it then also follows that

$$\mathrm{Cone}(T) \cap \mathrm{Cone}(U) = \mathrm{Cone}(V).$$

If T and U are silting then they have the same number $|T| = |U|$ of indecomposable summands, so if $T \not\simeq U$ then any shared summand V has $|V| < |T|$ and $|V| < |U|$. It follows that $\mathrm{Cone}(V)$ is disjoint from $\mathrm{Cone}^\circ(T)$ and $\mathrm{Cone}^\circ(U)$.   Q.E.D.

The proposition shows that the union

$$\mathrm{SFan}(\Lambda) := \bigcup_{T \in 2\text{-silt}\,\Lambda} \mathrm{Cone}(T)$$

of the cones of 2-term silting complex forms a polyhedral fan inside $\mathrm{K}_0(\mathrm{proj}\,\Lambda)$, which we will refer to as the *silting fan* of Λ. The fan decomposes into a disjoint union

$$\mathrm{SFan}(\Lambda) = \bigsqcup_{T \in 2\text{-presilt}\,\Lambda} \mathrm{Cone}^\circ(T),$$

of its faces, which are precisely the strict cones of 2-term *pre*silting complexes. In the following section it will be useful to stratify the silting fan by codimension. If $d = \dim_{\mathbb{R}} \mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}}$ denotes the rank of the K-theory, we define for each $k = 0, 1, \ldots, d$ the codimension k stratum as the disjoint union

$$\mathrm{SFan}_k(\Lambda) := \bigsqcup_{|T| = d-k} \mathrm{Cone}^\circ(T),$$

over all $T \in$ 2-presilt Λ with $d - k$ indecomposable summands. As a consequence of Kimura's theorem, we find that the construction of the silting fan is compatible with any central reduction. More precisely, we have the following.

Proposition 2.9. *For every ideal $I \subset \mathfrak{m}$, the silting fans $\mathrm{SFan}(\Lambda)$ and $\mathrm{SFan}(\Lambda/I\Lambda)$ are identified by $\mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}} \xrightarrow{\sim} \mathrm{K}_0(\mathrm{proj}\,\Lambda/I\Lambda)_{\mathbb{R}}$.*

Proof. By Lemma 2.7 the faces $\mathrm{Cone}^{\circ}(T)$ of the silting fan $\mathrm{SFan}(\Lambda)$ are mapped to the the faces $\mathrm{Cone}^{\circ}(\overline{T})$ of the silting fan $\mathrm{SFan}(\overline{\Lambda})$ of the fibre, and the faces are moreover in bijection by 2.6. Hence, $\mathrm{SFan}(\Lambda)$ and $\mathrm{SFan}(\overline{\Lambda})$ are identified as polyhedral fans. Likewise, if $I \subset \mathfrak{m}$ then the reduction $\Lambda/I\Lambda \to \overline{\Lambda}$ identifies $\mathrm{SFan}(\Lambda/I\Lambda)$ with $\mathrm{SFan}(\overline{\Lambda})$. Because the reduction of Λ factors as $\Lambda \to \Lambda/I\Lambda \to \overline{\Lambda}$, it follows that the natural map $\mathrm{K}_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}} \xrightarrow{\sim} \mathrm{K}_0(\mathrm{proj}\,\overline{\Lambda}/I\Lambda)$ identifies $\mathrm{SFan}(\Lambda)$ with $\mathrm{SFan}(\Lambda/I\Lambda)$, as both fans map to $\mathrm{SFan}(\overline{\Lambda})$ in $\mathrm{K}_0(\mathrm{proj}\,\overline{\Lambda})$. Q.E.D.

2.3. Stability conditions inside the silting fan

In the finite dimensional setting, a connection between silting theory and stability conditions has been established in the work of several authors [4, 26, 2]: for a finite dimensional algebra A and a stability condition in $\theta \in \mathrm{SFan}(A)$ it is shown that

- there are a finite number of θ-stable modules,
- the subcategory $\mathcal{S}_\theta$ depends only on the face $\mathrm{Cone}^{\circ}(T)$ of the fan in which θ lies,
- $\mathcal{S}_\theta$ is equivalent as an abelian category to a module category for an algebra determined by $T \in 2\text{-presilt}\,A$ and its Bongartz completion.

In this section we show how to derive similar result for the algebra Λ by reducing to the finite dimensional setting along the quotient $\Lambda \to \overline{\Lambda}$. Our strategy follows the approach of Asai [2]: we recover the subcategories $\mathcal{S}_\theta \subset \mathrm{fl}\,\Lambda$ by identifying the θ-stable objects with simples in the heart of a t-structure associated to a silting complex. This t-structure can be identified as follows.

Let $T \in 2\text{-silt}\,\Lambda$, then T can be viewed as a chain complex in $\mathrm{C}^b(\mathrm{proj}\,\Lambda)$, and therefore has a well-defined endomorphism DG algebra $\mathcal{E}_T := (\mathrm{End}^{\bullet}_{\Lambda}(T), \partial)$. Because T is silting, it is by definition also a generator for the derived category $\mathrm{D}(\Lambda)$ and therefore (see e.g. [25, Thm 14.2.29]) induces a *derived Morita equivalence*

$$\mathbf{R}\mathrm{Hom}_{\Lambda}(T, -) \colon \mathrm{D}(\Lambda) \longrightarrow \mathrm{D}(\mathcal{E}_T)$$

between $\mathrm{D}(\Lambda)$ and the derived category of DG modules over $\mathcal{E}_T$. Moreover, it follows from Lemma 2.1 that this restricts to an equivalence on finite length objects

$$\mathbf{R}\mathrm{Hom}_{\Lambda}(T, -) \colon \mathrm{D}^b(\mathrm{fl}\,\Lambda) \longrightarrow \mathrm{D}^b_{\mathrm{fl}}(\mathcal{E}_T).$$

Because T is silting, the cohomology $\mathrm{H}^i \mathcal{E}_T = \mathrm{Hom}_{\mathrm{K}^b(\Lambda)}(T, T[i])$ of $\mathcal{E}_T$ vanishes in positive degrees, and the derived category $\mathrm{D}^b_{\mathrm{fl}}(\mathcal{E}_T)$ has a

standard t-structure with heart given by the complexes concentrated in degree 0:

$$\mathrm{H}^0(\mathrm{D}^b_{\mathrm{fl}}(\mathcal{E}_T)) \simeq \mathrm{fl}\,\mathrm{H}^0\mathcal{E}_T \simeq \mathrm{fl}\,\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T).$$

Pulling this t-structure back along the derived Morita equivalence, one obtains a t-structure on $\mathrm{D}^b(\mathrm{fl}\,\Lambda)$ with heart

$$\mathcal{H}_T = \{E \in \mathrm{D}^b(\mathrm{fl}\,\Lambda) \mid \mathbf{R}\mathrm{Hom}_\Lambda(T, E) \in \mathrm{fl}\,\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T) \subset \mathrm{D}^b_{\mathrm{fl}}(\mathcal{E}_T)\}.$$

This heart $\mathcal{H}_T \subset \mathrm{D}^b(\mathrm{fl}\,\Lambda)$ is a tilt of the standard heart $\mathrm{fl}\,\Lambda \subset \mathrm{D}^b(\mathrm{fl}\,\Lambda)$ at a torsion pair, as defined by Happel-Reiten-Smalø [8].

Lemma 2.10. *Let $T \in$ 2-silt Λ, then the heart $\mathcal{H}_T$ is the HRS-tilt*

$$\mathcal{T} * \mathcal{F}[1] := \{E \in \mathrm{D}^b(\mathrm{fl}\,\Lambda) \mid \mathrm{H}^0E \in \mathcal{T},\ \mathrm{H}^{-1}E \in \mathcal{F},\ \mathrm{H}^iE = 0 \text{ otherwise}\}$$

associated to the torsion pair

$$\mathcal{T} = \{M \in \mathrm{fl}\,\Lambda \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T, M[1]) = 0\},$$
$$\mathcal{F} = \{M \in \mathrm{fl}\,\Lambda \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T, M) = 0\}.$$

Proof. By construction $\mathcal{H}_T$ is the heart of the t-structure $(T[< 0]^\perp, T[> 0]^\perp)$, where

$$T[< 0]^\perp := \{E \in \mathrm{D}^b(\mathrm{fl}\,\Lambda) \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T, E[i]) = 0\ \forall i > 0\},$$
$$T[> 0]^\perp := \{E \in \mathrm{D}^b(\mathrm{fl}\,\Lambda) \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T, E[i]) = 0\ \forall i < 0\}.$$

It was shown in [11] in a more general setting that this t-structure is induced by the torsion pair $(\mathcal{T}, \mathcal{F})$, and hence $\mathcal{H}_T = \mathcal{T} * \mathcal{F}[1]$.   Q.E.D.

For any silting complex $T \in$ 2-silt Λ the reduction $\overline{T} \in$ 2-silt $\overline{\Lambda}$ again induces a t-structure on the derived category $\mathrm{D}^b(\mathrm{fl}\,\overline{\Lambda})$ of the finite dimensional algebra $\overline{\Lambda}$, with heart

$$\mathcal{H}_{\overline{T}} = \{\overline{E} \in \mathrm{D}^b(\mathrm{fl}\,\overline{\Lambda}) \mid \mathrm{Hom}_{\mathrm{D}(\overline{\Lambda})}(\overline{T}, \overline{E}) = 0\ \forall i \neq 0\}$$
$$= \{\overline{E} \in \mathrm{D}^b(\mathrm{fl}\,\overline{\Lambda}) \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T, \overline{E}_\Lambda) = 0\ \forall i \neq 0\}.$$

Via the exact embedding $(-)_\Lambda \colon \mathrm{D}^b(\mathrm{fl}\,\overline{\Lambda}) \to \mathrm{D}^b(\mathrm{fl}\,\Lambda)$ we can view this heart as an abelian subcategory $(\mathcal{H}_{\overline{T}})_\Lambda = \mathcal{H}_T \cap \mathrm{D}^b(\mathrm{fl}\,\overline{\Lambda})$ of $\mathcal{H}_T$. Using Lemma 2.10 we can now show that its extension closure in $\mathrm{D}^b(\mathrm{fl}\,\Lambda)$ is precisely $\mathcal{H}_T$.

Lemma 2.11. *The heart $\mathcal{H}_T$ is the extension closure of the subcategory $(\mathcal{H}_{\overline{T}})_\Lambda \subset \mathrm{D}^b(\mathrm{fl}\,\Lambda)$.*

Proof. By Lemma 2.10 the heart $\mathcal{H}_T$ is generated under extension by $\mathcal{T}$ and $\mathcal{F}[1]$, and hence it suffices to show that $M[1], N \in \langle (\mathcal{H}_{\overline{T}})_\wedge \rangle$ for all $N \in \mathcal{T}$ and $M \in \mathcal{F}$.

For $M \in \mathcal{F}$, the $\mathfrak{m}^i$-torsion submodules $M_i = M[\mathfrak{m}^i]$ yield a finite filtration

$$M_1 \subset M_2 \subset \ldots \subset M_n = M,$$

with subquotients $M_i/M_{i-1} \simeq M[\mathfrak{m}^i]/M[\mathfrak{m}^{i-1}] \in \mathrm{fl}\,\overline{\Lambda} \subset \mathrm{fl}\,\Lambda$. Recall that a torsion free class is closed under submodules, so that M_i are again in $\mathcal{F}$, and their shifts $M_i[1]$ therefore give elements in $\mathcal{F}[1] \subset \mathcal{H}_T$. Because $\mathcal{H}_T$ is abelian, it then also contains the quotients

$$M_i[1]/M_{i-1}[1] \simeq (M_i/M_{i-1})[1] \in \mathcal{H}_T \cap (\mathrm{fl}\,\overline{\Lambda}[1])$$

Hence, $M[1]$ is filtered by objects in $\mathcal{H}_{\overline{T}}$, and therefore lies in its extension closure.

The statement for $N \in \mathcal{T}$ can be proven dually, by using the cofiltration

$$N = N/\mathfrak{m}^n N \to \ldots \to N/\mathfrak{m}^2 N \to N/\mathfrak{m}N.$$

Each quotient module N_i is again contained in $\mathcal{T}$, because $\mathcal{T}$ is a torsion class. Hence N_i are contained in $\mathcal{H}_T$, and the subkernels $\ker(N_i \to N_{i-1})$ are therefore objects in $\mathcal{H}_T \cap \mathrm{fl}\,\overline{\Lambda} \subset (\mathcal{H}_{\overline{T}})_\wedge$. It follows that N is again in the extension closure of $(\mathcal{H}_{\overline{T}})_\wedge$, completing the proof.   Q.E.D.

Let $T \in 2\text{-silt}\,\Lambda$ with Krull-Schmidt decomposition $T = T_1 \oplus \ldots \oplus T_n$ and reduction $\overline{T} \in 2\text{-silt}\,\overline{\Lambda}$. Then, following [2, Def. 3.8], the heart $\mathcal{H}_{\overline{T}}$ is a finite length abelian category of which the simple objects are dual to the summands $\overline{T}_i$ of $\overline{T}$: for each $i = 1, \ldots, n$ there is a unique simple object $\overline{X}_i \in \mathcal{H}_{\overline{T}}$ for which

$$\mathrm{Hom}_{\mathrm{D}(\overline{\Lambda})}(\overline{T}_j, \overline{X}_i) = 0 \quad \Longleftrightarrow \quad i \neq j.$$

By Lemma 2.11 the heart now has a similar structure: $\mathcal{H}_T$ is generated by the image of $\mathcal{H}_{\overline{T}}$ under the exact embedding $(-)_\wedge \colon \mathrm{D}^b(\mathrm{fl}\,\overline{\Lambda}) \to \mathrm{D}^b(\mathrm{fl}\,\Lambda)$, and so

$$\mathcal{H}_T = \langle (\mathcal{H}_{\overline{T}})_\wedge \rangle = \langle X_1, \ldots, X_n \rangle,$$

where $X_i = (\overline{X}_i)_\wedge$ are the images of the simples in $\mathcal{H}_{\overline{T}}$. In particular, $\mathcal{H}_T$ is again a finite length abelian category with simples X_i characterised by the condition

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(T_j, X_i) \simeq \mathrm{Hom}_{\mathrm{D}(\overline{\Lambda})}(\overline{T}_j, \overline{X}_i) = 0 \quad \Longleftrightarrow \quad i \neq j.$$

Now suppose $\overline{T}$ is the Bongartz completion of a summand $\overline{T}' \subset \overline{T}$. Then it is shown in [2] that for any $\overline{\theta} \in \mathrm{Cone}^\circ(\overline{T}')$ the $\overline{\theta}$-stable modules in $\mathrm{fl}\,\overline{\Lambda}$ are all given by some $\overline{X}_i$, and in particular $\mathcal{S}_{\overline{\theta}}$ is a subcategory of $\mathcal{H}_{\overline{T}}$. Using this fact, we can now also characterise $\mathcal{S}_\theta$ as a subcategory of the heart $\mathcal{H}_T$.

Proposition 2.12. *Let $T' \in 2\text{-presilt}\,\Lambda$ with Bongartz completion $T \in 2\text{-silt}\,\Lambda$, then for all $\theta \in \mathrm{Cone}^\circ(T')$ the subcategory $\mathcal{S}_\theta$:*

(1)   *contains exactly $|T| - |T'|$ stable modules.*
(2)   *embeds into $\mathrm{D}^b(\mathrm{fl}\,\Lambda)$ as the subcategory*

$$\mathcal{H}_T \cap (T')^\perp := \{E \in \mathcal{H}_T \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T', E) = 0\}.$$

(3)   *is the following subcategory of $\mathrm{fl}\,\Lambda$*

$$\{M \in \mathrm{fl}\,\Lambda \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T, M[1]) = \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T', M) = 0\}.$$

Proof.   We may number the summands Krull-Schmidt decomposition $T = T_1 \oplus \ldots \oplus T_n$ of T, such that $T' = T_{k+1} \oplus \ldots \oplus T_n$ for $k = |T| - |T'|$, and number the simples $X_i = \overline{X}_i$ in the heart accordingly.

(1) Let $\theta \in \mathrm{Cone}^\circ(T')$, then it follows from Lemma 2.7 that the reduction $\overline{\theta}$ lies in $\mathrm{Cone}^\circ(\overline{T}')$. As $\overline{T} = \overline{T}_1 \oplus \ldots \oplus \overline{T}_n$ is the Bongartz completion of $\overline{T}_{k+1} \oplus \ldots \oplus \overline{T}_n$ and $\overline{\Lambda}$ is finite dimensional, it follows from [2, Prop 4.1] that

$$\mathcal{S}_{\overline{\theta}} = \langle \overline{X}_1, \ldots, \overline{X}_k \rangle \subset \mathrm{fl}\,\overline{\Lambda},$$

and the objects $\overline{X}_1, \ldots, \overline{X}_k$ are precisely the θ-stable modules. By Proposition 2.4 the θ-stable modules in $\mathcal{S}_\theta \subset \mathrm{fl}\,\Lambda$ are exactly their images $X_i := (\overline{X}_i)_\Lambda$. There are precisely $k = |T| - |T'|$ such objects, so the result follows.

(2) Because the objects $X_1, \ldots, X_n$ generate the heart $\mathcal{H}_T$, it follows that $\mathcal{S}_\theta = \langle X_1, \ldots, X_k \rangle$ is a subcategory of $\mathcal{H}_T$. Moreover, the objects $X_1, \ldots, X_k$ are precisely those simple objects in $\mathcal{H}_T$ such that

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(T', X_i) \simeq \mathrm{Hom}_{\mathrm{D}(\overline{\Lambda})}(\overline{T}_{k+1} \oplus \ldots \oplus \overline{T}_n, \overline{X}_i) = 0,$$

which implies that $\mathcal{S}_\theta$ is precisely the subcategory $\mathcal{H}_T \cap (T')^\perp \subset \mathcal{H}_T$.

(3) As $\mathcal{S}_\theta$ is contained in both $\mathrm{fl}\,\Lambda$ and $\mathcal{H}_T$ it follows from Lemma 2.10 that $\mathcal{S}_\theta$ is the intersection

$$
\begin{aligned}
(\mathrm{fl}\,\Lambda) \cap \mathcal{H}_T \cap (T')^\perp &= \{M \in \mathrm{fl}\,\Lambda \mid M \in \mathcal{T},\ \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T', M) = 0\} \\
&= \{M \in \mathrm{fl}\,\Lambda \mid \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T, M[1]) \\
&\qquad\qquad = \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T', M) = 0\},
\end{aligned}
$$

as claimed. Q.E.D.

Having established the subcategories $\mathcal{S}_\theta \subset \mathrm{fl}\,\Lambda$ as embedded in a heart $\mathcal{H}_T$ of some silting object, we can apply the derived Morita equivalence to give an explicit description of $\mathcal{S}_\theta$ as a module category.

Theorem 2.13. *Let $T' \in 2\text{-presilt}\,\Lambda$ with Bongartz completion $T \in 2\text{-silt}\,\Lambda$. Then for all $\theta \in \mathrm{Cone}^\circ(T')$, the derived Morita equivalence restricts to an exact equivalence*

$$\mathcal{S}_\theta \xrightarrow{\sim} \mathrm{fl}\,\mathrm{End}_{\mathrm{D}(\Lambda)}(T)/(e) \subset \mathrm{fl}\,\mathrm{End}_{\mathrm{D}(\Lambda)}(T)$$

of abelian categories, where e is the idempotent $e \colon T \to T' \to T$. In particular, every θ-stable module corresponds to a simple $\mathrm{End}_{\mathrm{D}(\Lambda)}(T)$-module.

Proof. By Lemma 2.12, for every $\theta \in \mathrm{Cone}^\circ(T')$ the subcategory $\mathcal{S}_\theta$ can be identified with $\mathcal{H}_T \cap (T')^\perp$. By construction, the derived Morita equivalence

$$\mathbf{R}\mathrm{Hom}_\Lambda(T, -) \colon \mathrm{D}^b(\mathrm{fl}\,\Lambda) \xrightarrow{\sim} \mathrm{D}^b_{\mathrm{fl}}(\mathcal{E}_T),$$

maps the heart $\mathcal{H}_T$ to heart standard heart $\mathrm{fl}\,\mathrm{End}_{\mathrm{K}^b(\Lambda)}(T)$ and it therefore suffices to show that the subcategory $\mathcal{H}_T \cap (T')^\perp \subset \mathcal{H}_T$ is mapped to the subcategory

$$\mathrm{fl}\,\mathrm{End}_{\mathrm{K}^b(\Lambda)}(T)/(e) \subset \mathrm{fl}\,\mathrm{End}_{\mathrm{K}^b(\Lambda)}(T)$$

consisting of those $\mathrm{End}_{\mathrm{K}^b(\Lambda)}(T)$-modules which are annihilated by the idempotent.

Suppose $N \in \mathrm{fl}\,\mathrm{End}_{\mathrm{K}^b(\Lambda)}(T)$, then $N \simeq \mathbf{R}\mathrm{Hom}_\Lambda(T, E)$ for some object E and therefore N is in the image of $\mathcal{H}_T \cap (T')^\perp$ if and only if

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}(\mathcal{E}_T)}(\mathbf{R}\mathrm{Hom}_\Lambda(T, T'), N) \\
&\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{E}_T)}(\mathbf{R}\mathrm{Hom}_\Lambda(T, T'), \mathbf{R}\mathrm{Hom}_\Lambda(T, E)) \\
&\simeq \mathrm{Hom}_{\mathrm{D}(\Lambda)}(T', E) = 0.
\end{aligned}$$

Because N is concentrated in degree 0, while the complex $\mathbf{R}\mathrm{Hom}_\Lambda(T, T')$ has cohomology concentrated in negative degrees it follows that

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(\Lambda)}(\mathbf{R}\mathrm{Hom}_\Lambda(T, T'), N) &\simeq \mathrm{Hom}_{\mathrm{D}(\Lambda)}(\tau_{\geq 0}\mathbf{R}\mathrm{Hom}_\Lambda(T, T'), N) \\
&\simeq \mathrm{Hom}_{\mathrm{D}(\Lambda)}(\mathrm{H}^0\mathbf{R}\mathrm{Hom}_\Lambda(T, T'), N) \\
&= \mathrm{Hom}_{\mathrm{End}_{\mathrm{K}^b(\Lambda)}(T)}(\mathrm{Hom}_{\mathrm{K}^b(T)}(T, T'), N)
\end{aligned}$$

where $\tau_{\geq 0}$ denotes the canonical truncation. By inspection, the module $\mathrm{Hom}_{\mathsf{K}^b(\Lambda)}(T, T')$ is precisely the direct summand

$$\mathrm{Hom}_{\mathsf{K}^b(\Lambda)}(T, T') \simeq e \circ \mathrm{End}_{\mathsf{K}^b(\Lambda)}(T) \subset \mathrm{End}_{\mathsf{K}^b(\Lambda)}(T),$$

associated to the idempotent $e\colon T \to T' \to T$, so it follows that N lies in the image of $\mathcal{H}_T \cap (T')^\perp$ if and only if $N \in \mathrm{fl}\,\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T)/(e)$.

Hence, $\mathbf{R}\mathrm{Hom}_\Lambda(T, -)$ maps $\mathcal{S}_\theta = \mathcal{H}_T \cap (T')^\perp$ to $\mathrm{fl}\,\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T)/(e)$, and therefore restricts to an equivalence of abelian categories

$$\mathrm{Hom}_{\mathsf{D}(\Lambda)}(T, -)\colon \mathcal{S}_\theta \xrightarrow{\sim} \mathrm{fl}\,\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T)/(e)$$

$$\text{Q.E.D.}$$

§3. Application to cDV singularities

We now apply the results from the previous section to the non-commutative minimal models of cDV singularities, using the relation between their tilting theory and the combinatorics of Dynkin diagrams described in Iyama–Wemyss [13]. The section is split up as follows.

We start by recalling the construction of the "intersection arrangements" in [13] which associates a hyperplane arrangement to a pair (Γ, J) of an extended Dynkin diagram and a set of vertices.

Next we recall the bijective correspondence found in [13] between the chambers of the intersection arrangement and 2-term tilting complexes of certain subalgebras of the preprojective algebra associated to (Γ, J).

With this background we are then able to derive several results about noncommutative minimal models of cDV singularities, both isolated and non-isolated, and give an explicit exposition using a small example.

3.1. Intersection arrangements of extended Dynkin type

In what follows Γ denotes one of the extended ADE Dynkin graphs:

$$\widetilde{A}_n \qquad\qquad \widetilde{D}_n \qquad\qquad \widetilde{E}_6$$

$$\widetilde{E}_7 \qquad\qquad\qquad \widetilde{E}_8$$
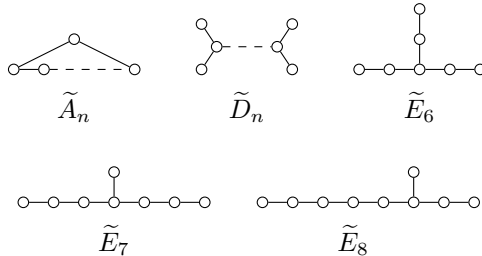
Each extended Dynkin graph has an associated hyperplane arrangement inside the vectorspace of real functions on Γ, with an associated

dense polyhedral fan. We recall here the construction, for which a full overview can be found in [12].

Let $R^\Gamma := \{f \colon \Gamma \to \mathbb{R}\}$ denote the space of real functions on the vertices of Γ. This vectorspace contains a standard cone $C \subset \mathbb{R}^\Gamma$ of functions $f \colon \Gamma \to \mathbb{R}$ which are nonnegative: $f(v) \geq 0$ for all $v \in V$. The cone is a union of its faces, which are the open subcones

$$C_V := \{f \colon \Gamma \to \mathbb{R} \mid f(v) = 0 \text{ if } v \in V, \ f(v) > 0 \text{ if } v \notin V\},$$

where $V \subset \Gamma$ ranges over all subsets of vertices in Γ. The cone C is a fundamental domain for the action of the affine Weyl group W_Γ associated to the extended Dynkin diagram, and the orbits form a subspace $\mathrm{TCone}(\Gamma) = \bigcup_{w \in W_\Gamma} w \cdot C$ called the *Tits cone* of Γ. The Tits cone is a polyhedral fan with decomposition

$$\mathrm{TCone}(\Gamma) = \bigsqcup_{V \subset \Gamma} \ \bigsqcup_{w \in W_\Gamma / W_V} w \cdot C_V,$$

with faces labeled by a subset V of vertices and a coset w of the stabiliser subgroup $W_V \subset W_\Gamma$ of C_V. In particular, the images $w \cdot C_\varnothing$ of the interior $C_\varnothing \subset C$ are the Weyl chambers of the hyperplane arrangement. If one removes the origin from the Tits cone, the resulting space $\mathrm{TCone}(\Gamma) \setminus \{0\}$ forms an open halfspace in $\mathbb{R}^\Gamma$, and has a boundary given by certain hyperplane H_∞. Hence, $\mathbb{R}^\Gamma$ decomposes as

$$\mathbb{R}^\Gamma = (\mathrm{TCone}(\Gamma) \setminus \{0\}) \sqcup H_\infty \sqcup (-\mathrm{TCone}(\Gamma) \setminus \{0\}),$$

where $-\mathrm{TCone}(\Gamma)$ denotes the reflection of $\mathrm{TCone}(\Gamma)$ in the origin. The following example for $\Gamma = \widetilde{A}_1$ illustrates this decomposition:
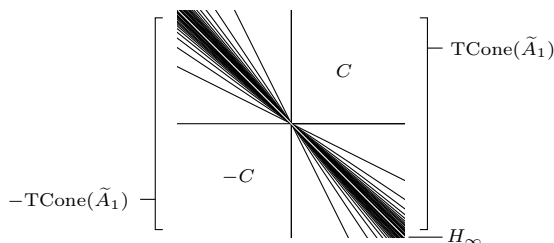
Given a fixed subset $J \subset \Gamma$ of vertices, we consider the linear subspace $\mathbb{R}^J \subset \mathbb{R}^\Gamma$ of dimension $|J|$ that consists of all functions $f \colon \Gamma \to \mathbb{R}$ which vanish on the complement $\Gamma \setminus J$. Following [13] we define the Tits cone

$$\mathrm{TCone}(\Gamma, J) := \bigsqcup_{\substack{V \subset \Gamma \\ }} \ \bigsqcup_{\substack{w \in W / W_V \\ w \cdot C_V \subset \mathbb{R}^J}} w \cdot C_V,$$

which is the subfan of $\mathrm{TCone}(\Gamma)$, consisting of all faces that lie inside the subspace $\mathbb{R}^J$. The vectorspace $\mathbb{R}^J$ now has an analogous decomposition

$$\mathbb{R}^J = (\mathrm{TCone}(\Gamma, J) \setminus \{0\}) \sqcup H_\infty \sqcup (-\mathrm{TCone}(\Gamma, J) \setminus \{0\})$$

into the positive and negative Tits cones, separated by a hyperplane $H_\infty \subset \mathbb{R}^J$. The *intersection arrangement* of the pair (Γ, J) is defined as the union of the positive and negative cone:

$$X_{\Gamma, J} = \mathrm{TCone}(\Gamma, J) \cup -\mathrm{TCone}(\Gamma, J).$$

In what follows we also write X_k for the codimension k stratum of a given $X = X_{\Gamma, J}$, which consists of the faces $w \cdot C_V$ and $w \cdot (-C_V)$ for which V satisfies $|V| = k + |J|$.

Example 3.1. Let $\Gamma = \widetilde{D}_4$ and choose $J \subset \Gamma$ to consist of the extended and middle vertex (indicated by the black nodes below), then $X_{\Gamma, J}$ has the following structure:

$$V \subset D_4$$
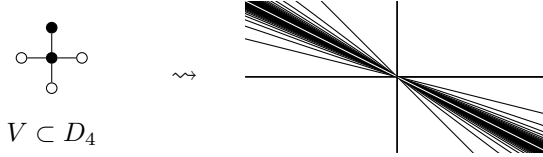
The stratum X_0 consists of the Weyl chambers, while X_1 is the union of the strict rays emanating from the origin, and X_2 is the origin. The hyperplane H_∞ is the line of slope $-\frac{1}{2}$ through the origin.

3.2. Contracted preprojective algebras

Let $Q = (Q_0, Q_1)$ be the quiver which has a vertex $v \in Q_0$ for every vertex in Γ, and a pair of arrows $a\colon v \to w$, $a^*\colon w \to v$ in Q_1 for every edge between v and w in Γ. For a given field $\mathbb{K}$, consider the preprojective algebra

$$\mathbb{K}Q/(\textstyle\sum_a aa^* - a^*a).$$

It is well known that the preprojective algebra is finite as a module over its centre, which is the coordinate ring of an ADE surface singularity of the corresponding Dynkin type. We let $\Pi = \Pi_\Gamma$ denote the completion of the preprojective algebra at this singularity, which is then a finite algebra over the complete local ring $Z(\Pi)$.

There is an idempotent $e_v \in \Pi$ for every vertex $v \in \Gamma$ and together the e_v form a complete set of primitive orthogonal idempotents for Π.

Hence, for each pair (Γ, J) as in the previous section there is an idempotent $e = e_J = \sum_{v \in J} e_v$ and, as in [13], define the *contracted preprojective algebra*

$$e\Pi e = e_J \Pi e_J \subset \Pi,$$

generated by the paths starting and ending in a vertex of J. In [13] it is shown that the hyperplane arrangement $X_{\Gamma, J}$ of a pair (Γ, J) is related to the silting fan of these algebras via the the natural identification

$$(1) \qquad \mathrm{K}_0(\mathrm{proj}\, e\Pi e)_{\mathbb{R}} \simeq \bigoplus_{v \in J} \mathbb{R}[e_v \Pi e] \simeq \mathbb{R}^J,$$

of the classes of indecomposable projectives $e_v \Pi e$ with the corresponding basis vectors in $\mathbb{R}^J$. We recall this theorem here in our current notation.

Theorem 3.2 ([13, Thm 7.24])**.** *For each preprojective algebra $e\Pi e$ associated to a pair (Γ, J), the isomorphism* (1) *identifies the fan $X_{\Gamma, J}$ with the silting fan $\mathrm{SFan}(e\Pi e)$. In particular, for every $T \in$ 2-presilt $e\Pi e$ the strict cone is given by*

$$\mathrm{Cone}^\circ(T) = w \cdot C_V,$$

for some subset $V \subset \Gamma$ and $w \in W_\Gamma / W_V$, such that $w \cdot C_V$ lies in $\mathbb{R}^J$.

In particular, Proposition 2.12 now yields the number of stable $e\Pi e$-modules for stability conditions in the fan.

Proposition 3.3. *Let (Γ, J) be a Dynkin type with fan $X = X_{\Gamma, J}$ as above. Then for $\theta \in X_k \subset \mathbb{R}^J$ the subcategory $\mathcal{S}_\theta \subset \mathrm{fl}\, e\Pi e$ is generated by k stable modules, and only depends on the face $w \cdot C_V$ in which θ lies.*

The above proposition includes the case where $e\Pi e = \Pi$ is the entire preprojective algebra, for which this result was already established by Sekiya–Yamaura [20] via a similar tilting method.

3.3. Compound Du Val singularities

In dimension three, the natural analogue of Du Val singularities are the *compound Du Val (cDV) singularities*, which have a coordinate ring that is (up to isomorphism) of the form

$$R = \mathbb{C}[[x, y, z, t]]/(f(x, y, z) + t \cdot g(x, y, z, t)),$$

where f defines the Du Val surface singularity $\mathrm{Spec}\, \mathbb{C}[x, y, z]/(f) = \mathrm{Spec}\, R/(t)$. These types of singularities are a basic building block of the minimal model program, as they form the base of various curve contractions.

In [14] Iyama–Wemyss characterise a type of *noncommutative* minimal model for cDV singularities. Concretely, they define a set of R-modules $\mathrm{MM}\,R$ called the *maximal modifying modules*, which are basic reflexive R-modules whose endomorphism algebras $\Lambda_M := \mathrm{End}_R(M)$ have similar homological properties to the geometric minimal models. Moreover, these endomorphism algebras are deformations of a contracted preprojective algebra: by [13, Prop 9.3(a)] there are isomorphisms

$$(2) \qquad\qquad \Lambda_M/t\Lambda_M \simeq e\Pi e = e_J \Pi_\Gamma e_J,$$

for each $M \in \mathrm{MM}\,R$, where Γ is the extended Dynkin diagram of the Du Val singularity $\mathrm{Spec}\,R/(t)$ and $J \subset \Gamma$ a subset which may depend on M.

By applying Proposition 2.9 to the quotient $\Lambda_M \to e\Pi e$, we recover the silting fan of any minimal model Λ_M, yielding a refinement of [13, Theorem 9.6].

Proposition 3.4. *Let $M \in \mathrm{MM}\,R$ with associated Dynkin type (Γ, J). Then there is an isomorphism $\mathrm{K}_0(\mathrm{proj}\,\Lambda_M) \xrightarrow{\sim} \mathbb{R}^J$, which identifies $\mathrm{SFan}(\Lambda_M)$ with $X_{\Gamma,J}$.*

Proof. The endomorphism algebra $\Lambda_M = \mathrm{End}_R(M)$ is finite as a module over the complete local ring R, so it follows from Lemma 2.9 that the isomorphism

$$\mathrm{K}_0(\mathrm{proj}\,\Lambda_M)_\mathbb{R} \xrightarrow{\sim} \mathrm{K}_0(\mathrm{proj}\,\Lambda_M/t\Lambda_M)_\mathbb{R} \simeq \mathrm{K}_0(\mathrm{proj}\,e\Pi e)_\mathbb{R}$$

induced by the quotient $\Lambda_M \to \Lambda_M/t\Lambda_M \simeq e\Pi e$ onto the contracted preprojective algebra associated to (Γ, J) identifies the fans $\mathrm{SFan}(\Lambda_M)$ and $\mathrm{SFan}(e\Pi e)$. The result now follows directly from Theorem 3.2, as the isomorphism $\mathrm{K}_0(\mathrm{proj}\,e\Pi e)_\mathbb{R} \xrightarrow{\sim} \mathbb{R}^J$ identifies $\mathrm{SFan}(e\Pi e)$ with $X_{\Gamma,J}$.
                                                                    Q.E.D.

In what follows we identify $\mathrm{K}_0(\mathrm{proj}\,\Lambda_M)$ with $\mathbb{R}^J$ and $\mathrm{SFan}(\Lambda_M)$ with the fan $X_{\Gamma,J}$ of the intersection arrangement. Proposition 2.12 now yields the following.

Proposition 3.5. *Let $M \in \mathrm{MM}\,R$ with fan $X = X_{\Gamma,J}$. Then for $\theta \in X_k$ the subcategory $\mathcal{S}_\theta \subset \mathrm{fl}\,\Lambda_M$ is generated by k stable modules and depends only on the face $w \cdot C_V$ in which θ lies. In particular:*

- *there are no θ-semistable modules if θ lies in a Weyl chamber,*
- *there is a unique θ-stable module if θ lies generically on a hyperplane.*

3.4. The isolated case

Now assume that R is an isolated cDV singularity, and fix a minimal model $\Lambda = \Lambda_M$ for some $M \in \mathrm{MM}\,R$ with Dynkin type (Γ, J).

For isolated cDV singularities Iyama-Wemyss [13] give a complete classification of the 2-term silting complexes, which are in fact all *tilting*, via an Auslander-McKay type correspondence: by [14, Theorem 4.17] there is a bijection

$$(3) \qquad \{N \in \mathrm{MM}\,R\} \xrightarrow[\sim]{\mathrm{Hom}_R(M,-)} \{\text{basic tilting } \Lambda\text{-modules}\},$$

which assigns a tilting module $\mathrm{Hom}_R(M,N) \in \mathrm{mod}\,\Lambda$ to each $N \in \mathrm{MM}\,R$ with endomorphism algebra

$$\mathrm{End}_\Lambda(\mathrm{Hom}_R(M,N)) \simeq \mathrm{End}_R(N) = \Lambda_N.$$

The modules $\mathrm{Hom}_R(M,N)$ define 2-term tilting complex $P_N \in 2\text{-silt}\,\Lambda$ via its projective resolution

$$\ldots \to 0 \to P_N^{-1} \to P_N^0 \to \mathrm{Hom}_R(M,N).$$

And according to [13] the cones of these complexes fill out the Tits cone. The negative cone can be obtained by dualising: the dual statement of (3) yields a tilting module $\mathrm{Hom}_R(N,M) \in \mathrm{mod}\,\Lambda^{\mathrm{op}}$ for the opposite algebra and the resolution

$$\ldots \to 0 \to Q_N^{-1} \to Q_N^0 \to \mathrm{Hom}_R(N,M)$$

yields an object $Q_N \in 2\text{-silt}\,\Lambda^{\mathrm{op}}$. By applying the R-linear dual $(-)^* = \mathrm{Hom}(-,R)$ and shifting, one then obtains a new 2-term silting complex for Λ.

Lemma 3.6. *For every $T \in 2\text{-silt}\,\Lambda$ there exists some $N \in \mathrm{MM}\,R$ such that T is either isomorphic to P_N or isomorphic to $Q_N^*[1]$.*

Proof. Let (Γ, J) denote the Dynkin type of Λ, then [13, Corollary 9.8] shows that the cones of P_N form the Tits cone:

$$\bigcup_{N \in \mathrm{MM}\,R} \mathrm{Cone}(P_N) = \mathrm{TCone}(\Gamma, J).$$

Hence if $T \in 2\text{-silt}\,\Lambda$ with $\mathrm{Cone}(T) \subset \mathrm{TCone}(\Gamma, J)$, then Proposition 2.8 shows that T must be isomorphic to P_N for some $N \in \mathrm{MM}\,R$.

Consider now the dual tilting complexes $Q_N \in 2\text{-silt}\,\Lambda^{\mathrm{op}}$. The R-linear dual induces an exact anti-equivalence $(-)^* : \mathsf{K}^b(\Lambda^{\mathrm{op}})^{\mathrm{op}} \to \mathsf{K}^b(\Lambda)$. Hence for every the $N \in \mathrm{MM}\,R$ there is a tilting complex

$$Q_N^* = \ldots \to 0 \to \mathrm{Hom}(Q_N^0, R) \to \mathrm{Hom}(Q_N^{-1}, R) \to 0 \to \ldots$$

which is concentrated in degrees $0, 1$. Hence the objects $Q_N^*[1]$ are indeed 2-term silting complexes in 2-silt Λ. It is shown in [13, Theorem 9.17] that the K-theory classes of the summands of Q_N^* are equal to those of the summands of P_N, which implies that $\mathrm{Cone}(Q_N^*[1]) = -\mathrm{Cone}(P_N)$. Hence, the union of these cones fill out the negative Tits cone:

$$\bigcup_{N \in \mathrm{MM}\,R} \mathrm{Cone}(Q_N^*[1]) = \bigcup_{N \in \mathrm{MM}\,R} -\mathrm{Cone}(P_N) = -\mathrm{TCone}(\Gamma, J).$$

If $\mathrm{Cone}(T)$ is contained in $-\mathrm{TCone}(T)$, then it again follows that $T \simeq Q_N^*[1]$ for some $N \in \mathrm{MM}\,R$. As $\mathrm{SFan}(\Lambda) = X_{\Gamma,J} = \mathrm{TCone}(\Gamma, J) \cup -\mathrm{TCone}(\Gamma, J)$, there are no other 2-term silting complexes.   Q.E.D.

Using Theorem 2.13 we can now derive an explicit form for the subcategories $\mathcal{S}_\theta$ for all θ in the intersection arrangement, as module categories for quotients of noncommutative minimal models.

Theorem 3.7. *Suppose $\Lambda = \Lambda_M$ is a noncommutative minimal model of an isolated cDV singularity R as above. Then for each $\theta \in X_{\Gamma,J}$ there exists an $N \in \mathrm{MM}\,R$ and an idempotent $e \in \Lambda_N$ such that*

$$\mathcal{S}_\theta \simeq \mathrm{fl}\,\Lambda_N/(e),$$

as abelian categories.

Proof. Let $\theta \in X_{\Gamma,J} = \mathrm{SFan}(\Lambda)$, Theorem 2.13 implies that the subcategory

$$\mathcal{S}_\theta \simeq \mathrm{fl}\,\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T)/(e)$$

for some $T \in$ 2-silt Λ and the idempotent $e \colon T \to T' \to T$ associated to a summand $T' \subset T$. By Lemma 3.6 any $T \in$ 2-silt Λ is isomorphic to either some P_N, which has an endomorphism algebra

$$\mathrm{End}_{\mathsf{K}^b(\Lambda)}(P_N) \simeq \mathrm{End}_\Lambda(\mathrm{Hom}_R(M, N)) \simeq \Lambda_N,$$

or T is isomorphic to $Q_N^*[1]$, which has endomorphism algebra

$$\begin{aligned}
\mathrm{End}_{\mathsf{K}^b(\Lambda)}(Q_N^*[1]) &\simeq \mathrm{End}_{\mathsf{K}^b(\Lambda^{\mathrm{op}})}(Q_N)^{\mathrm{op}} \\
&\simeq \mathrm{End}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_R(N, M))^{\mathrm{op}} \simeq \Lambda_N.
\end{aligned}$$

Hence $\mathrm{End}_{\mathsf{K}^b(\Lambda)}(T) \simeq \Lambda_N$ for some $N \in \mathrm{MM}\,R$ and the result follows.
                                                                    Q.E.D.

Finally we consider the special case where R is the base of a three-fold flopping contraction $f \colon Y \to \mathrm{Spec}\,R$. In this setting, Hirano–Wemyss [10] show that the set $\mathrm{MM}\,R$ is freely acted on by a group

$\mathbb{Z}^n = \langle L_1, \ldots, L_n \rangle \subset \mathrm{Cl}(R)$ of invertible ideals, preserving the endomorphism algebras: for each $N \in \mathrm{MM}\,R$ there are isomorphisms

$$\Lambda_{L_i \cdot N} = \mathrm{End}_R(L_i \cdot N) \simeq \mathrm{End}_R(N) = \Lambda_N$$

for each generator L_i, where $N \mapsto L_i \cdot N \in \mathrm{MM}\,R$ denotes the action. They moreover show that the action partitions $\mathrm{MM}\,R$ into finitely many orbits, so that there are only finitely many noncommutative minimal models Λ_N up to isomorphism. We can therefore give the following strengthening of Theorem 3.7 for flops.

Proposition 3.8. *Suppose R is the base of a threefold flopping contraction. Then there is a finite set $\{N^1, \ldots, N^k\} \subset \mathrm{MM}\,R$ such that: for each $\theta \in X_{\Gamma,J}$ there is an equivalence $\mathcal{S}_\theta \simeq \mathrm{fl}\,\Lambda_{N^i}/(e)$ for some $i = 1, \ldots, k$ and idempotent $e \in \Lambda_{N^i}$.*

We finish with the example of an A_2 flop, where we can explicitly determine the set $\{N^1, \ldots, N^k\}$ and the quotients of the endomorphism algebras Λ_{N^i}.

Example 3.9. We recall the example [7, Ex 6.1] where R is the base of an A_2 flopping contraction, in which two rational curves in a smooth threefold are contracted to the cDV singularity

$$R = \mathbb{C}[[x, y, z, t]]/(xy(x + y) - tz)$$

As in [7] we may pick the MM module $M = R \oplus (u, x) \oplus (u, xy)$, which has an endomorphism algebra isomorphic to the Jacobi algebra $\mathbb{C}(Q, W)$ of the quiver with potential

$Q:$

$W = \frac{1}{2}\sum_{i=1}^3 a_i b_i a_i b_i + a_2 a_1 b_1 b_2 - a_1 a_3 b_3 b_1 - a_3 a_2 b_2 b_3.$
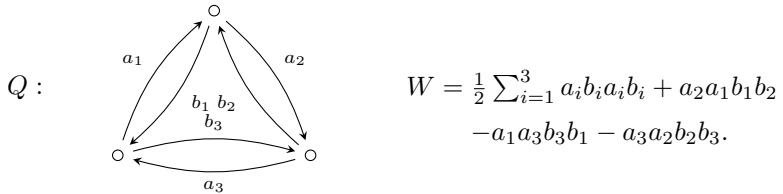
The Dynkin type (Γ, J) of M is the diagram $\Gamma = A_2$ with J the full set of its vertices. In particular, the intersection arrangement is just the ordinary $\widetilde{A}_2$ hyperplane arrangement $X_{\Gamma,J)} \subset \mathbb{R}^3$, and the Tits cone $\mathrm{TCone}(\Gamma, J)$ fills the halfspace $\{r \in \mathbb{R}^3 \mid r_1 + r_2 + r_3 > 0\}$. As in [10] we can visualise the Tits cone by intersecting it with a level $\{r \in \mathbb{R}^3 \mid$

$r_1 + r_2 + r_3 = 1\}$, which yields the affine A_2 lattice
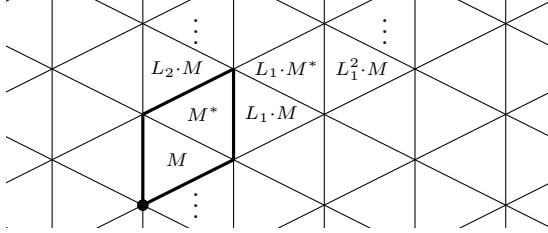
Here the thick border indicates a fundamental region for the action $\mathbb{Z}^2 \simeq \langle L_1, L_2 \rangle$ on the level, which contains precisely two chambers: one for M itself and one for its dual M^*. There are thus two endomorphism algebras to consider:

$$\operatorname{End}_R(M) \simeq \mathbb{C}(Q, W) \quad \text{and} \quad \operatorname{End}_R(M^*) \simeq \mathbb{C}(Q, W)^{\mathrm{op}}.$$

From the presentation of the quiver above, one can however observe that $\mathbb{C}(Q, W)^{\mathrm{op}}$ and $\mathbb{C}(Q, W)$ are isomorphic.

If $\theta \in X_{\Gamma, J}$ is nonzero, it therefore follows from Proposition 3.8 above that $\mathcal{S}_\theta \subset \operatorname{fl} \Lambda$ is equivalent to $\operatorname{fl} \operatorname{End}_R(M)/(e)$ as an abelian category, where e is the idempotent associated to a nontrivial summands of M. The isomorphism $\mathbb{C}(Q, W) \simeq \operatorname{End}_R(M)$ identifies this quotient with the Jacobi algebra $\mathbb{C}(Q', W)$, where Q' is obtained from Q by deleting the vertices corresponding to the summand of M. By inspection, $\mathcal{S}_\theta$ is then the module category of one of the following Jacobi algebras:

$$\mathbb{C}(\, \circ \,, 0) \simeq \mathbb{C}, \quad \mathbb{C}\left(\circ \underset{b}{\overset{a}{\underset{\longleftarrow}{\longrightarrow}}} \circ \,, \tfrac{1}{2} abab \right).$$

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