Cyclic A_∞-Algebras and Calabi–Yau Structures in the Analytic Setting

Okke van Garderen

Abstract. This paper considers A_∞-algebras whose higher products satisfy an analytic bound with respect to a fixed norm. We define a notion of right Calabi–Yau structures on such A_∞-algebras and show that these give rise to cyclic minimal models satisfying the same analytic bound. This strengthens a theorem of Kontsevich–Soibelman [KS08], and yields a flexible method for obtaining analytic potentials of Hua-Keller [HK19].
We apply these results to the endomorphism DGAs of polystable sheaves considered by Toda [Tod18], for which we construct a family of such right CY structures obtained from analytic germs of holomorphic volume forms on a projective variety. As a result, we can define a canonical cyclic analytic A_∞-structure on the Ext-algebra of a polystable sheaf, which depends only on the analytic-local geometry of its support. This shows that the results of [Tod18] can be extended to the quasi-projective setting, and yields a new method for comparing cyclic A_∞-structures of sheaves on different Calabi–Yau varieties.

Contents

1. Introduction

1.1. **Motivation.** In their seminal paper, Donaldson and Thomas [DT98] proposed a definition of gauge-theoretic invariants for a compact Calabi–Yau threefold X. Writing ν_X for the volume form on X, these invariants would count critical points of a holomorphic Chern-Simons functional

$$\mathsf{CS}\colon \alpha \mapsto \int_X \operatorname{tr}(D\alpha \wedge \alpha + \alpha \wedge \alpha \wedge \alpha) \wedge \nu_X$$

on spaces of connections on vector bundles on X, taken up to the action of a suitable gauge group. Defining such invariants directly in this infinite dimensional setup proved difficult, and this led Thomas [Tho00] to pursue an algebraic setup involving moduli spaces of finite type. In the study of Donaldson–Thomas type invariants that followed [KS08; Beh09; JS12] the moduli space of interest is the moduli stack of semistable sheaves $\mathcal{M}_X$, and the invariants are defined by expressing $\mathcal{M}_X$ locally as a critical locus of a finite type potential [Joy15].

The link between the gauge-theoretic and the algebraic setups was explored by Toda [Tod18], who shows that the moduli space $\mathcal{M}_X$ is locally controlled by a gauge problem associated to a polystable sheaf $\mathcal{F} \in \mathsf{coh}\, X$. Concretely, one takes a locally free resolution $\mathcal{E}^\bullet \to \mathcal{F}$ and considers the DG algebra

$$\mathfrak{g}_\mathcal{E} = \mathfrak{g}^\bullet_\mathcal{E} := \left(\Gamma(X, \mathscr{A}^{0,\bullet}(\mathcal{E}nd(\mathcal{E}))), D, \wedge\right),$$

of Dolbeault forms, whose Maurer-Cartan locus represents the critical locus of the Chern-Simons functional. The gauge problem is made finite dimensional, by transferring the DG algebra structure to a minimal model, given by an A_∞-structure on the Ext-algebra

$$\mathcal{H}_\mathcal{E} = (\operatorname{Ext}^\bullet_X(\mathcal{F},\mathcal{F}), \mu = (\mu_n)_{n\in\mathbb{N}}).$$

Toda [Tod18] shows that the Maurer-Cartan locus of $\mathcal{H}_\mathcal{E}$ is an analytic space, whose quotient by the gauge group presents an analytic neighbourhood of $\mathcal{F}$ in $\mathcal{M}_X$. Under the assumption that X is 3 Calabi–Yau, it is moreover a critical locus $\{\mathrm{d}\mathcal{W}_\mathcal{F} = 0\}$ of the associated analytic potential

$$\mathcal{W}_\mathcal{F}(\alpha) = \sum_n \sigma(\alpha)(\mu_n(\alpha, \dots, \alpha)),$$

where $\sigma\colon \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Ext}^{\dim X - \bullet}(\mathcal{F}, \mathcal{F})^\vee$ is the Serre duality pairing. The potential is the desired replacement for the Chern-Simons functional in this finite dimensional model, and is the correct function for defining refined versions of the DT invariants [KS08; MMNS12; Tod21].

These results suggest that it should be possible to study the enumerative geometry of X by directly manipulating the A_∞-algebra $\mathcal{H}_\mathcal{E}$ using algebraic methods. However, for this to work there are two additional structures which need to be considered:

- an *analytic structure* on $\mathfrak{g}_\mathcal{E}$ and $\mathcal{H}_\mathcal{E}$, in the form of a norm $\|\cdot\|$ for which (as shown by Tu [Tu14]) the higher products satisfy the following geometric series bound
$$\|\mu_n\| < C^n \quad \forall n \geq 1. \tag{1}$$
- a *cyclic structure* on $\mathcal{H}_\mathcal{E}$, in the form of the isomorphism $\sigma\colon \mathcal{H}_\mathcal{E} \to \mathcal{H}_\mathcal{E}^*$ which (as shown by Polishchuk [Pol01]) is compatible with the natural A_∞-bimodule structures on $\mathcal{H}_\mathcal{E}$ and $\mathcal{H}_E^*$.

Neither the analytic nor the cyclic structure is compatible with arbitrary A_∞-algebra morphisms. Instead one should consider morphisms of *cyclic A_∞-algebras* (defined in [Kaj07; KS09]), and morphisms of *analytic A_∞-algebras* (appearing implicitly in [Tu14; Tod18]) respectively.

The aim of this paper is to develop the theory of A_∞-algebras carrying both an analytic and cyclic structure. The main goal is to find a general method for constructing analytic cyclic minimal models for a given analytic A_∞-algebra, with the motivating example being the analytic DG algebra $\mathfrak{g}_\mathcal{E}$ and its analytic cyclic minimal model $(\mathcal{H}_\mathcal{E}, \sigma)$. To do this we study the behaviour of *right Calabi–Yau structures* on such analytic A_∞-algebras.

1.2. **Main result.** In what follows, A denotes an A_∞-algebra defined over a separable $\mathbb{C}$-algebra ℓ, which includes A_∞-structures on vector spaces and on quivers.

Recall that a right n-CY structure on A is a cocycle in the negative cyclic complex of A satisfying a certain nondegeneracy condition [BD19]. In this paper, we take the Connes complex $\mathsf{C}^\bullet_\lambda(A)$ as our model for negative cyclic cohomology $\mathrm{HC}^\bullet_\lambda(A)$, and an n-CY structure will be a cocycle in $\mathsf{C}^{-n}_\lambda(A)$. Given an analytic structure $\|\cdot\|\colon A \to \mathbb{R}$ as above we define the subcomplex

$$\mathsf{C}^{\mathrm{an},\bullet}_\lambda(A) := \{\ \phi \in \mathsf{C}^\bullet_\lambda(A) \mid \phi \text{ satisfies a geometric series bound in } \|\cdot\|\ \},$$

and call a right CY structure analytic if it lies in $\mathsf{C}^{\mathrm{an},\bullet}_\lambda(A)$. We show that the subcomplex $\mathsf{C}^{\mathrm{an},\bullet}_\lambda(A)$ admits pullbacks with respect to A_∞-morphisms satisfying a bound analogous to (1), and is therefore a good cohomology theory on a category of analytic A_∞-algebras. Working inside this category, we find the following main theorem.

Theorem A (Theorem 4.23)**.** *Let A be an analytic A_∞-algebra admitting an analytic minimal model $\mathrm{H}(A)$ which is finite dimensional and unital. Then every analytic right CY structure $\phi \in \mathsf{C}^{\mathrm{an},\bullet}_\lambda(A)$ defines an cyclic analytic minimal model*

$$(\mathrm{H}(A)^\phi, \sigma^\phi)$$

which depends only on the class $[\phi] \in \mathrm{HC}^\bullet_\lambda(A)$ up to an isomorphism of cyclic analytic A_∞-algebras. If B is a second such A_∞-algebra, then every analytic quasi-isomorphism $f\colon B \to A$ induces a cyclic analytic A_∞-isomorphism

$$(\mathrm{H}(B)^{f^*\phi}, \sigma^{f^*\phi}) \ \cong_{\mathrm{cyc,an}}\ (\mathrm{H}(A)^\phi, \sigma^\phi).$$

The proof of our main theorem follows the work of Amorim–Tu [AT22], proving a type of "analytic Darboux theorem" for a version of the homotopy-invariant version of cyclic structures [Cho08], which are related to the noncommutative symplectic structures of [KS09].

Cyclic A_∞-algebras are especially interesting in the 3-CY setting, because the cyclic A_∞-structure is captured by a potential: given such a 3-cyclic A_∞-algebra (D, σ) there is an associated noncommutative potential $W \in \widehat{\mathsf{T}}V := \prod_{n\geq 0} V^{\otimes n}$ in the completed tensor algebra over $V = (D^1)^\vee$. If D is moreover equipped with an analytic structure, then this potential lies in an analytic subring

$$\widetilde{\mathsf{T}}V := \left\{ (v_n)_{n\in\mathbb{N}} \in \widehat{\mathsf{T}}V \;\middle|\; \exists C \text{ such that } \|v_n\| < C^n \text{ for all } n > 0 \right\},$$

and is therefore an analytic potential in the framework of Hua–Keller [HK19]. Analytic morphisms induce algebra morphisms between these analytic subrings, so in the 3-CY setting our theorem can be interpreted as follows.

Corollary B (Corollary 4.24)**.** *In the situation of Theorem A suppose that ϕ is a right 3-CY structure, and write $V_A = \mathrm{H}^1(A)^\vee$. Then there is a canonical analytic potential*

$$W^\phi \in \widetilde{\mathsf{T}}V_A,$$

which is well-defined up to an automorphism of $\widetilde{\mathsf{T}}V_A$. Moreover, if $f : B \to A$ is a quasi-isomorphism and $V_B = \mathrm{H}^1(B)^\vee$ then there is an induced isomorphism $g : \widetilde{\mathsf{T}}V_A \to \widetilde{\mathsf{T}}V_B$ such that

$$g(W^\phi) = W^{f^*\phi}.$$

The main theorem and its corollary can be used to relate infinite dimensional setting, involving e.g. normed DG algebras, with finite dimensional settings, such as those considered in [HK19]. In particular, it applies to the gauge-theoretic setup mentioned above.

1.3. **Analytic CY structures in complex geometry.** We apply Theorem A to describe the cyclic analytic minimal models of Toda's DGA $\mathfrak{g}_\mathcal{E}$ for a perfect complex $\mathcal{E}$ of vector bundles on a smooth projective variety X of dimension n. If X is Calabi–Yau with holomorphic volume form $\nu \in \mathrm{H}^0(X, \Omega_X^n)$, then there is an induced right CY structure: the bounded linear functional

$$\mathfrak{g}_\mathcal{E} \to \mathbb{C}, \quad \alpha \mapsto \int_X \mathrm{tr}(\alpha) \wedge \nu \tag{2}$$

defines a cocycle $\phi^\nu \in \mathsf{C}_\lambda^{\mathrm{an},\bullet}(\mathfrak{g}_\mathcal{E})$ which is the nondegenerate in the appropriate sense. The associated cyclic analytic minimal model $(\mathcal{H}_\mathcal{E}^\nu, \sigma^\nu)$ recovers the A_∞-algebra of [Pol01; Tu14; Tod18].

There are in general many other choices of analytic right CY structures, each of which yields a cyclic analytic minimal model of $\mathfrak{g}_\mathcal{E}$. We identify a family of such CY structures corresponding to *holomorphic volume germs* along the support $Z = \mathsf{supp}\,\mathcal{E}$, by which we mean an equivalence class

$$\nu \in (\Omega_X^n)_Z := \operatorname*{colim}_{U\supset Z} \Gamma(U, \Omega_X^n)$$

of differential forms on analytic open neighbourhoods $U \supset Z$, which is nonvanishing when U is sufficiently small. Such a germ determines a bounded linear functional on the DG subalgebra

$$\mathfrak{g}_{\mathcal{E}|_U,c} := \{\xi \in \mathfrak{g}_\mathcal{E} \mid \mathsf{supp}\,\xi \subset U\} \subset \mathfrak{g}_{\mathcal{E},c},$$

of Dolbeault forms with compact support in any sufficiently small neighbourhood $U \supset Z$. This functional induces an analytic right CY structure on $\mathfrak{g}_{\mathcal{E}|_U,c}$, which we transfer to a right CY structure on $\mathfrak{g}_\mathcal{E}$ using an explicit A_∞-quasi-isomorphism $\mathfrak{g}_\mathcal{E} \to \mathfrak{g}_{\mathcal{E}|_U,c}$. This leads to the following theorem.

Theorem C (Theorem 5.9)**.** *Let X be smooth projective of dimension n, and $\mathcal{E} \in \mathsf{D}^{\mathrm{perf}}(X)$ a complex with support $Z \subset X$. Then every volume germ $\nu \in (\Omega_X^n)_Z$ determines a canonical class $[\phi^\nu]$ of an analytic right CY structure $\phi^\nu \in \mathsf{C}_\lambda^{\bullet,\mathrm{an}}(\mathfrak{g}_\mathcal{E})$, and hence determines a cyclic analytic minimal model*

$$(\mathcal{H}_\mathcal{E}^\nu, \sigma^\nu) = (\mathrm{Ext}_X^\bullet(\mathcal{E}, \mathcal{E}), \mu^\nu, \sigma^\nu),$$

which is analytically A_∞-isomorphic to $\mathcal{H}_\mathcal{E}$.

The above theorem applies to the setting of [Tod18] when $\mathcal{E}$ is a resolution of a polystable sheaf $\mathcal{F} \in \mathsf{coh}\, X$ whose support does not meet the canonical divisor K_X. Some examples considered in the enumerative geometry literature include point sheaves [BBS13], and sheaves on contractible curves [Sze08; Kat08]. If X is a threefold, then there is moreover a noncommutative analytic potential

$$W_{\mathcal{E}}^{\nu} \in \widetilde{\mathsf{T}}\mathrm{Ext}^1(\mathcal{E},\mathcal{E})^\vee,$$

whose abelianisation $\mathcal{W}_{\mathcal{E}}^{\nu} = (W_{\mathcal{E}}^{\nu})^{\mathrm{ab}}$ is an ordinary analytic function whose critical locus describes the moduli space of semistable sheaves around $\mathcal{F}$. Hence, the gauge-theoretic setup can be generalised from the projective CY-3 case in [Tod18] to open analytic neighbourhoods of projective CY-3 folds equipped with a choice of volume form.

In our setup, both the A_∞-algebra structure on the Ext-algebra of $\mathcal{E}$ and the volume germ ν are determined in terms of analytic-local geometry around the support of $\mathcal{E}$ in X. One would therefore expect that the cyclic A_∞-algebra $(\mathcal{H}_{\mathcal{E}}^{\nu}, \sigma^{\nu})$ is not sensitive to the global geometry of X. To show this we consider embeddings of such neighbourhoods into other projective varieties.

Theorem D (Theorem 5.12)**.** *Let X, X' be smooth projective varieties, and let Y be an open submanifold of X with an open embedding into X', as in the following diagram:*

$$X \xleftarrow{\ i\ } Y \xrightarrow{\ f\ } X'.$$

Let $\mathcal{E}'$ be a perfect complex on X' with cohomological support $f(Z) \subset f(Y)$ for some compact $Z \subset Y$, and let $\nu \in (\Omega^n_{X'})_Z$ be a volume germ. Then for any $\mathcal{E} \in \mathsf{D}^{\mathrm{perf}}(X)$ such that $\mathcal{E}|_Y \simeq f^*\mathcal{E}'$ there exists an analytic quasi-isomorphism

$$\mathfrak{g}_{\mathcal{E}} \simeq_{\mathrm{an}} \mathfrak{g}_{\mathcal{E}'}$$

which identifies the classes $[\phi^{\nu}]$ and $[\phi^{f^*\phi}]$ via pullback. In particular, there is an analytic cyclic A_∞-isomorphism between the cyclic analytic minimal models

$$(\mathcal{H}_{\mathcal{E}}^{f^*\nu}, \sigma^{f^*\nu}) \cong_{\mathrm{an,cyc}} (\mathcal{H}_{\mathcal{E}'}^{\nu}, \sigma^{\nu}).$$

Corollary E (Corollary 5.13)**.** *If X, X' are threefolds, then the noncommutative potentials $W_{\mathcal{E}'}^{\nu}$ and $W_{\mathcal{E}}^{f^*\nu}$ are related by an analytic change of variables $\widetilde{\mathsf{T}}\mathrm{Ext}^1(\mathcal{E},\mathcal{E})^\vee \xrightarrow{\sim} \widetilde{\mathsf{T}}\mathrm{Ext}^1(\mathcal{E}',\mathcal{E}')$.*

One implication of Theorem D is that there is a canonical cyclic A_∞-structure on the Ext-algebra of a compactly supported sheaf $\mathcal{F}$ on a quasi-projective Calabi–Yau threefold Y equipped with a fixed volume form. Indeed, applying Theorem C to a resolution of $\mathcal{F}$ on any compactification $X = \overline{Y}$, it follows from Theorem D that the cyclic A_∞-algebra obtained does not depend on these choices.

For the reader who is exclusively interesting in projective Calabi–Yau geometry, we stress that Theorem D also allows one to compare the cyclic analytic A_∞-structure (hence the enumerative geometry) of different projective CY varieties on a common analytic neighbourhood. As a motivating example, we compute the cyclic analytic A_∞-structure on the Ext-algebra of an arbitrary point sheaf.

Proposition F (Proposition 5.17)**.** *Let $p \in X$ be a point on a smooth projective variety X, and $\mathcal{F} \to \mathcal{O}_p$ a locally free resolution for the associated point sheaf. Then for every volume germ $\nu \in (\Omega^n_Y)_p$ there is an isomorphism of cyclic analytic A_∞-algebras*

$$\mathcal{H}_{\mathcal{F}}^{\nu} \cong_{\mathrm{an,cyc}} \left(\bigwedge^{\bullet} T_o\mathbb{A}^n,\ \sigma^{\lambda}\colon \xi \mapsto \lambda(\xi \wedge -) \right),$$

where the right hand side denotes the graded algebra of polyvectors at the origin $o \in \mathbb{A}^n$, with cyclic structure induced by some linear form $\lambda\colon \bigwedge^n T_o\mathbb{A}^n \xrightarrow{\sim} \mathbb{C}$.

1.4. **Structure of the paper.** In section §2 and we recall the definition of (analytic) A_∞-algebras defined over a separable commutative $\mathbb{C}$-algebra, and prove some new results about the existence of analytic (quasi-)inverses for analytic morphism that are used in later sections.

The subsequent section §3 contains some new definitions of A_∞-bimodules, A_∞-bimodules maps, and Hochschild cohomology. Again, we prove some new results about the invertibility of analytic A_∞-bimodule maps based on a tree formula, which seems to be new.

Section §4 contains the definition of cyclic structures and their homotopy invariant versions in the analytic setting, as well as the proof of Theorem A.

Finally, section §5 contains the application to the DG algebras of Dolbeault differential forms, which is used to prove Theorem C, Theorem D, and Proposition F

1.5. **Acknowledgements.** The author would like to thank Ben Davison and Louis Ioos for interesting discussions, and Sarah Scherotzke for general support.

2. Analytic A_∞-algebras

In this section we recall the definition of A_∞-algebras and morphisms, and introduce the relevant category of analytic A_∞-algebras based on the convergence conditions used in [Tu14].

In what follows let ℓ be a separable $\mathbb{C}$-algebra, and write $\mathsf{grMod}\,\ell^e$ for the graded $\mathbb{C}$-linear category of $\mathbb{Z}$-graded bimodules $V = V^\bullet$ which are nonzero in finitely many degrees. Given $V, W \in \mathsf{grMod}\,\ell^e$,

$$\hom^\bullet_{\ell^e}(V, W) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\ell^e}(V^n, V^{n+\bullet})$$

denotes the graded vector space of ℓ^e-linear maps, and $\mathrm{Hom}_{\ell^e}(V, W)$ the subspace of graded morphisms. Given a homogeneous element $f \in \hom^\bullet_{\ell^e}(V, W)$ we write $|f|$ for its degree. The category $\mathsf{grMod}\,\ell^e$ carries the usual graded tensor product $\otimes := \otimes_\ell$ and shift functor $[1]\colon \mathsf{grMod}\,\ell^e \to \mathsf{grMod}\,\ell^e$. We write $s \in \hom^{-1}_{\ell^e}(V, V[1])$ for the obvious shift map, acting as the identity $V^i \to V^i = (V[1])^{i-1}$ on each graded component.

2.1. A_∞**-algebras and morphisms.** A_∞-algebras are commonly as sequences of multilinear maps either of the form $V^{\otimes n} \to W$ with varying degrees or of the form $V[1]^{\otimes n} \to W[1]$ with all maps being homogeneous. In this section we adopt the latter convention, adopting the notation

$$\mathcal{M}^\bullet(V, W) := \prod_{n=0}^{\infty} \hom^\bullet_{\ell^e}(V[1]^{\otimes n}, W[1]),$$

for the graded vector space of these sequences $f = (f_n)_{n\in\mathbb{N}}$, and writing $\mathcal{M}_{>0}(V, W)$ for the subspace of elements with $f_0 = 0$. The exists various compositions between sequences in $\mathcal{M}(V, W)$, which are most cleanly described using the bar construction

$$\mathsf{B}V := \bigoplus_{n\in\mathbb{N}} V[1]^{\otimes n},$$

which is a counital cofree conilpotent coalgebra over ℓ with the decomposition coproduct [Tra08]. The coalgebra $\mathsf{B}V$ being cofree implies means that every sequence $f \in \mathcal{M}(V, V) \simeq \hom_{\ell^e}(\mathsf{B}V, V[1])$ induces a unique coderivation $\widetilde{f}\colon \mathsf{B}V \to \mathsf{B}V$ via the formula (see Tradler [Tra08])

$$\widetilde{f} := \sum_{k\geq 1} \sum_{i,j\geq 0} \mathrm{id}_{V[1]}^{\otimes i} \otimes f_k \otimes \mathrm{id}_{V[1]}^{\otimes j},$$

and likewise every coderivation defines a sequence in $\mathcal{M}(V, V)$ by composing with the obvious projection $\pi\colon \mathsf{B}V \to V[1]$. In terms of the bar construction, A_∞-algebras are defined as follows.

Definition 2.1. An A_∞-algebra is a pair (A, μ) of a graded bimodule $A \in \mathsf{grMod}\,\ell^e$ and a map $\mu \in \mathcal{M}^1_{>0}(A, A)$ satisfying the condition $\widetilde{\mu}^2 = 0$.

Given bimodules $V, W \in \mathsf{grMod}\,\ell^e$, every sequence $f \in \mathcal{M}(V, W) \simeq \hom_{\ell^e}(\mathsf{B}V, W[1])$ also induces a cohomomorphism $\widehat{f}\colon \mathsf{B}V \to \mathsf{B}W$ between the associated coalgebras, via the formula

$$\widehat{f} := \sum_{k\geq 1} \sum_{n_1,\dots,n_k} f_{n_1} \otimes \cdots \otimes f_{n_k},$$

and every cohomomorphism between bar coalgebras can again be described in this way (see again [Tra08]). These lifts are used to define (pre-)morphisms of A_∞-algebras.

Definition 2.2. A *pre-morphism* between A_∞-algebras (A, μ_A), (B, μ_B) is an element of the complex

$$\hom_{\mathsf{Alg}^\infty}(A, B) := (\mathcal{M}_{>0}(A, B),\ \delta), \quad \delta(f) := \mu_B \circ \widehat{f} - (-1)^{|f|} f \circ \widetilde{\mu}_A,$$

while a *morphism* is an element of the subspace $\mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B) := Z^0 \hom_{\mathsf{Alg}^\infty}(A, B)$.

By lifting pre-morphisms to cohomomorphisms, one obtains an associative composition between pre-morphisms which we will write as

$$\diamond\colon \hom_{\mathsf{Alg}^\infty}(B, C) \times \hom_{\mathsf{Alg}^\infty}(A, B) \to \hom_{\mathsf{Alg}^\infty}(A, C), \quad g \diamond f := g \circ \widehat{f},$$

for clarity. One verifies that this satisfies the Leibniz rule with respect to the differentials and A_∞-algebras therefore form a DG category. We will denote this DG category by Alg^∞.

Finally we recall the definition of (homotopy) units on an A_∞-algebra.

Definition 2.3. A *unit* for an A_∞-algebra $A = (A, \mu)$ is an injective map $\mathbb{1}\colon \ell[1] \to A[1]$ such that

$$\mu_2(\mathbb{1}(l), a) = la, \quad \mu_2(a, \mathbb{1}(l)) = (-1)^{|a|+1} al \quad \text{for all } l \in \ell[1],\ a \in A[1]$$
$$\mu_n(\dots, \mathbb{1}(l), \dots) = 0 \quad \text{for all } n \neq 2,$$

and a triple $(A, \mu, \mathbb{1})$ is called a *unital A_∞-algebra*. A *weak unit* is a map $\mathbb{1}\colon \ell[1] \to A[1]$ for which the image is μ_1-closed and which satisfies the following relations in μ_1-cohomology:

$$\mu_2([\mathbb{1}(l)], [a]) = [-la], \quad \mu_2([a], [\mathbb{1}(l)]) = al$$

If A and B are unital A_∞-algebras, with units $\mathbb{1}_A$ and $\mathbb{1}_B$, then the unital A_∞-pre-morphisms

$$\hom^{\mathrm{un}}_{\mathsf{Alg}^\infty}(A, B) := \left\{ f \in \hom_{\mathsf{Alg}^\infty}(A, B) \;\middle|\; \begin{array}{c} f_1 \circ \mathbb{1}_A = \mathbb{1}_B, \\ f_{i+j+1} \circ (\mathrm{id}^{\otimes i} \otimes \mathbb{1}_A \otimes \mathrm{id}^{\otimes j}) = 0 \quad \text{for all } i + j > 0 \end{array} \right\},$$

form a subcomplex of $\hom_{\mathsf{Alg}^\infty}(A, B)$, and unital morphisms $\mathrm{Hom}^{\mathrm{un}}_{\mathsf{Alg}^\infty}(A, B)$ likewise form a subspace of $\mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B)$. The composition of two unital pre-morphisms is again unital, so unital A_∞-algebras again form a DG category.

2.2. **Analytic algebras and morphisms.** We now consider the category $\mathsf{grNMod}\,\ell^e$ of *$\mathbb{Z}$-graded normed ℓ^e-bimodules*, by which we mean pairs $(V, \|\cdot\|_V)$ of a bimodule $V \in \mathsf{grMod}\,\ell^e$ and a norm $\|\cdot\|_V : V \to \mathbb{R}$ on the underlying ungraded vector space so that for any $l, r \in \ell$ the multiplication $v \mapsto lvr$ is bounded in $\|\cdot\|_\ell$. Between any $V, W \in \mathsf{grNMod}\,\ell^e$ there is a graded normed vector space

$$\hom^{\mathrm{cont}}_{\ell^e}(V, W) := \{f \in \hom_{\ell^e}(V, W) \mid \|f\|_{\mathrm{op}} < \infty\}$$

of morphisms whose operator norm $\|f\|_{\mathrm{op}} := \sup_{\|v\|_V = 1} \|f(v)\|_W$ is finite. The shift naturally extends to a functor $[1]\colon \mathsf{grNMod}\,\ell^e \to \mathsf{grNMod}\,\ell^e$, and for any pair of normed graded bimodules V, W the tensor product $V \otimes W$ is again normed via the projective tensor norm

$$\|z\|_\pi := \inf\{\textstyle\sum_i \|v_i\|_V \|w_i\|_W \mid z = \sum_i v \otimes_{\mathbb{C}} w\},$$

where the infimum is over all presentations of an element $z \in V \otimes W$ as a sum of pure tensors in the tensor product $V \otimes_{\mathbb{C}} W$ over $\mathbb{C}$. If $f_i\colon V_i \to W_i$ are continuous bimodule morphisms between normed bimodules $V_1, V_2, W_1, W_2 \in \mathsf{NMod}\,\ell^e$, then the tensor product $f_1 \otimes f_2$ has operator norm

$$\|f_1 \otimes f_2\|_{\mathrm{op}} = \|f_1\|_{\mathrm{op}} \|f_2\|_{\mathrm{op}} < \infty,$$

and is therefore a continuous bimodule map $V_1 \otimes V_2 \to W_1 \otimes W_2$. Henceforth, we will drop the subscripts on the above norms to reduce clutter.

Now given a normed ℓ-bimodule $V \in \mathsf{grNMod}\,\ell^e$ and $r \in \mathbb{R}$ with $r > 0$ we consider the following normed version of the bar construction:

$$\mathsf{B}(V, r) := (\mathsf{B}V, \|\cdot\|_r), \quad \|\textstyle\sum v_n\|_r := \sup\limits_{n \in \mathbb{N}} \|v_n\| r^{-n},$$

noting that the supremum in the definition of $\|\cdot\|_r$ is finite for any finite sum in $\mathsf{B}V$. Continuous maps out of these normed ℓ-bimodules are characterised by the following lemma.

Lemma 2.4. *Let $V, W \in \mathsf{grNMod}\,\ell^e$ and let $f \in \mathcal{M}(V,W)$. Then the following are equivalent:*

1. *there exists $C > 0$ such that $\|f_n\| < C^n$ for all $n \geq 1$,*
2. *there exists $r > 0$ such that $f\colon \mathsf{B}(V,r) \to W[1]$ is bounded,*
3. *for every $r' > 0$ there exists an $r > 0$ such that $\widehat{f}\colon \mathsf{B}(V,r) \to \mathsf{B}(W,r')$ is bounded,*

if additionally $V = W$ then this is also equivalent to:

4. *for every $r' > 0$ there exists an $r > 0$ such that $\widetilde{f}\colon \mathsf{B}(V,r) \to \mathsf{B}(V,r')$ is bounded.*

Proof. ($1 \implies 2$) Let $C > 0$ be such that $\|f_n\| < C^n$ for all $n \in \mathbb{N}$ and fix some $r < C^{-1}$. Then the map $f\colon \mathsf{B}(V,r) \to W[1]$ satisfies for every $v = \sum_{n\in\mathbb{N}} v_n \in \mathsf{B}(V,r)$

$$\|f(v)\| \leq \sum_{n\in\mathbb{N}} \|f_n(v_n)\| < \sum_{n\in\mathbb{N}} C^n \|v_n\| \leq \sum_{n\in\mathbb{N}} (Cr)^n \|v\|_r = \frac{\|v\|_r}{1 - Cr},$$

where the final equality for the geometric series holds because $Cr < 1$. It follows that f has operator norm bounded by $(1 - Cr)^{-1} < \infty$.

($2 \implies 1$) If $f\colon \mathsf{B}(V,r) \to W[1]$ is bounded in the operator norm by some constant $K > 1$, then for each $n \in \mathbb{N}$ and $v_n \in V[1]^{\otimes n}$ the map f_n satisfies

$$\|f_n(v_n)\| = \|f(v_n)\| \leq K\|v_n\| r^{-n} \leq (K/r)^n \|v_n\|.$$

Hence, picking any $C > K/r$ it follows that $\|f_n\| < C^n$ for all $n \in \mathbb{N}$, so $f \in \mathcal{A}(V,W)$ follows.

($1 \implies 3$) Let $C > 0$ be such that $\|f_n\| < C^n$ for all $n \in \mathbb{N}$, fix $r' > 0$ and let $K = \min\{r', 1\}$. Then expanding the formula for $\widehat{f}$, we find that for every $v = \sum v_n \in \mathsf{B}V$

$$\begin{aligned}\left\|\widehat{f}(v)\right\|_{r'} &\leq \sum_{n\in\mathbb{N}} \sum_{n_1+\cdots+n_k=n} \|f_{n_1} \otimes \cdots \otimes f_{n_k}(v_n)\| (r')^{-k} \\ &\leq \sum_{n\in\mathbb{N}} \sum_{n_1+\cdots+n_k=n} C^n K^{-k} \|v_n\| \\ &\leq \sum_{n\in\mathbb{N}} (\#\text{ partitions of n}) \cdot \left(\frac{C}{K}\right)^n \|v_n\|.\end{aligned}$$

The number of partitions of n is exponentially bounded by a function q^n for some $q > 0$. Picking $r < \frac{K}{qC}$ the geometric series in $\frac{rqC}{K}$ again converges, which shows that

$$\left\|\widehat{f}(v)\right\|_{r'} \leq \sum_{n\in\mathbb{N}} \left(\frac{rqC}{K}\right)^n \|v\|_r = \frac{K}{K - rqC}\|v\|_r.$$

As q and r can be chosen independently of v, it follows that $\widehat{f}\colon \mathsf{B}(V,r) \to \mathsf{B}(V,r')$ is bounded.

($1 \implies 4$) Let $C > 1$ be such that $\|f_n\| < C^n$ for all $n \in \mathbb{N}$, fix $r' > 0$ and let $K = \min\{r', 1\}$. Then expanding the formula for $\widetilde{f}$ yields for every $v = \sum v_n \in \mathsf{B}V$

$$\begin{aligned}\left\|\widetilde{f}(v)\right\|_{r'} &\leq \sum_{n\in\mathbb{N}} \sum_{i+k+j=n} \left\|(\mathrm{id}^{\otimes i} \otimes f_k \otimes \mathrm{id}^{\otimes j})(v_n)\right\| (r')^{-(i+j+1)} \\ &\leq \sum_{n\in\mathbb{N}} \sum_{i+k+j=n} C^k K^{-(i+j+1)} \|v_n\| \\ &\leq \sum_{n\in\mathbb{N}} n^3 \left(\frac{C}{K}\right)^n \|v_n\|.\end{aligned}$$

Picking $q > 0$ such that $n^3 < q^n$ for all n and an $r < \frac{K}{qC}$, it again follows that

$$\left\|\widetilde{f}(v)\right\|_{r'} \leq \sum_{n\in\mathbb{N}} \left(\frac{rqC}{K}\right)^n \|v\|_r = \frac{K}{K - rqC}\|v\|_r,$$

hence $\widetilde{f}$ is bounded when considered as a linear map $\mathsf{B}(V,r) \to \mathsf{B}(V,r')$.

($3,4 \implies 2$) For any r', the projection $\pi\colon \mathsf{B}(V,r') \to V[1]$ is obviously bounded, so if a map $\widehat{f}$ or $\widetilde{f}$ is bounded it follows that $f = \pi \circ \widehat{f} = \pi \circ \widetilde{f}$ is again bounded. □

We define the space of *analytic sequences of multilinear maps* between V and W as

$$\mathcal{A}(V,W) = \bigcup_{r>0} \hom_{\ell^e}^{\text{cont}}(\mathsf{B}(V,r), W[1]) \subset \mathcal{M}(V,W),$$

and note that its elements are sequences $f = (f_n)_{n\in\mathbb{N}}$ satisfying one of the equivalent conditions in the above lemma. The sequences with $f_0 = 0$ are again denoted by the subscript $\mathcal{A}_{>0}(V,W)$.

Definition 2.5. An *analytic A_∞-algebra* is a pair (A,μ) of a graded normed bimodule $A \in \mathsf{grNMod}\,\ell^e$ and an analytic sequence $\mu \in \mathcal{A}^1_{>0}(A,A)$ which satisfies $\widetilde{\mu}^2 = 0$.

Example 2.6. Let (A,d) be a DG algebra, and suppose $\|\cdot\|\colon A \to \mathbb{R}$ is any norm on the underlying vector space. Then the differential and multiplication defines an analytic A_∞-algebra structure if and only if both are bounded. In particular, this includes the case of Banach DG algebras.

Example 2.7. Let (A,μ) be a finite dimensional A_∞-algebra such that $\mu_n = 0$ for $n \gg 0$. Then (A,μ) is analytic for any choice of norm on A.

Remark 2.8. It follows from Lemma 2.4 that the definition given above is equivalent to the A_∞-algebras satisfying a convergence condition used in the literature [Tu14; Tod18].

Given two analytic A_∞-algebras, we can impose the convergence condition on (pre-)morphism $f \in \hom_{\mathsf{Alg}^\infty}(A,B)$. This defines a graded subspace

$$\hom^{\text{an}}_{\mathsf{Alg}^\infty}(A,B) := \mathcal{A}_{>0}(A,B) \cap \hom_{\mathsf{Alg}^\infty}(A,B),$$

of analytic pre-morphisms. Lemma 2.4 now implies the following.

Lemma 2.9. *Let A,B be analytic A_∞-algebras. Then $\hom^{\text{an}}_{\mathsf{Alg}^\infty}(A,B)$ is a subcomplex of $\hom_{\mathsf{Alg}^\infty}(A,B)$.*

Proof. Suppose μ_A, μ_B are analytic A_∞-structures on A and B and consider an analytic map $f \in \hom^{\text{an}}_{\mathsf{Alg}^\infty}(A,B)$. By Lemma 2.4 there exist $r_1, r_2, r_3 > 0$ such that the maps

$$\mu_B\colon \mathsf{B}(B,r_1) \to B[1], \quad \widehat{f}\colon \mathsf{B}(A,r_2) \to \mathsf{B}(B,r_1), \quad \widetilde{\mu}_A\colon \mathsf{B}(A,r_3) \to \mathsf{B}(A,r_2),$$

are bounded linear maps. Because compositions and sums of bounded maps are bounded, it follows that the differential of f is bounded:

$$\delta(f) = \mu_B \circ \widehat{f} + f \circ \widetilde{\mu}_A\colon \mathsf{B}(A,r_3) \to B[1].$$

It follows that $\hom^{\text{an}}_{\mathsf{Alg}^\infty}(A,B) \subset \hom_{\mathsf{Alg}^\infty}(A,B)$ is a subcomplex. □

Lemma 2.10. *For any three analytic A_∞-algebras A,B,C, the composition restricts to a map*

$$\diamond\colon \hom^{\text{an}}_{\mathsf{Alg}^\infty}(B,C) \times \hom^{\text{an}}_{\mathsf{Alg}^\infty}(A,B) \to \hom^{\text{an}}_{\mathsf{Alg}^\infty}(A,C).$$

Proof. Given $f \in \hom^{\text{an}}_{\mathsf{Alg}^\infty}(A,B)$ and $g \in \hom^{\text{an}}_{\mathsf{Alg}^\infty}(B,C)$, it follows from Lemma 2.4 that there exist $r_1, r_2 > 0$ such that the maps

$$g\colon \mathsf{B}(B,r_1) \to C[1], \quad \widehat{f}\colon \mathsf{B}(A,r_2) \to \mathsf{B}(B,r_1)$$

are bounded. Then the composition $g \diamond f = g \circ \widehat{f}$ is also bounded. □

We conclude that analytic A_∞-algebras again form a DG category $\mathsf{Alg}^{\infty,\text{an}}$, with an obvious forgetful DG functor $\mathsf{Alg}^{\infty,\text{an}} \to \mathsf{Alg}^\infty$ to the ordinary DG category of A_∞-algebras.

2.3. **Analytic minimal models.** Given $A, B \in \mathsf{Alg}^\infty$ recall that a morphism $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B)$ is a *quasi-isomorphism* if f_1 is a quasi-isomorphism of chain complexes $f_1\colon A[1] \to B[1]$ with respect to the differential $\mu_{A,1}$ and $\mu_{B,1}$. A morphism $g \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(B, A)$ is a *quasi-inverse* if

$$f \circ g = \delta(h), \quad g \circ f = \delta(h')$$

for some homotopies $h \in \mathrm{hom}^{-1}_{\mathsf{Alg}^\infty}(B, B)$ and $h' \in \mathrm{hom}^{-1}_{\mathsf{Alg}^\infty}(A, A)$. It is well known that any A_∞-algebra $A \in \mathsf{Alg}^\infty$ is quasi-isomorphic to a *minimal model*: there exists an A_∞-algebra $\mathrm{H}(A) \in \mathsf{Alg}^\infty$ with trivial differential and a diagram

$$\mathrm{H}(A) \underset{P}{\overset{I}{\rightleftarrows}} A \circlearrowleft Q \tag{3}$$

of quasi-isomorphisms P and I such that $P \diamond I = \mathrm{id}_{\mathrm{H}(A)}$, and a degree -1 pre-morphism Q such that $I \diamond P = \mathrm{id}_A + \delta(Q)$. The minimal model can be constructed explicitly using a homotopy transfer formula [Kad80]. We recall that the A_∞-algebra A is called *compact* (or proper) if the minimal model is finite-dimensional over $\mathbb{C}$.

In the analytic setting the construction of minimal models is more subtle: if the quasi-isomorphisms I and P are analytic then the composition $I \diamond P$ is a continuous splitting of $\mathrm{H}(A)$ as a direct summand of A, which does not exist in general. Even if such a splitting exists, there may not be an analytic homotopy Q witnessing the fact that I and P are quasi-inverse. For this reason we work with two notions of minimal model in the analytic setting.

Definition 2.11. Let $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$ be an analytic A_∞-algebra, then a *weak analytic minimal model* is an analytic A_∞-algebra $\mathrm{H}(A) \in \mathsf{Alg}^{\infty,\mathrm{an}}$ with trivial differential together with a diagram

$$\mathrm{H}(A) \underset{P}{\overset{I}{\rightleftarrows}} A,$$

for quasi-isomorphisms $I \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathrm{H}(A), A)$ and $P \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, \mathrm{H}(A))$ satisfying $P \diamond I = \mathrm{id}_{\mathrm{H}(A)}$. A minimal model is *strong* if in addition, there exists an analytic pre-morphism $Q \in \mathrm{hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, A)$ such that $I \diamond P = \mathrm{id}_A + \delta(Q)$.

We will not give a construction of analytic minimal models, but note that they can be constructed via a homotopy transfer formula given a suitable homotopy operator as in [Tu14].

2.4. **Analytic (quasi-)inverses.** We conclude this section with some technical results about inverses for analytic A_∞-(pre-)morphisms, used in the rest of the paper. The definition of these inverses uses common "tree-formulas" based on the following notion of planar trees.

Definition 2.12. $\mathcal{O}(n)$ denotes the set of *planar rooted trees on n leaves*, in which every internal node has valency ≥ 3, and $\mathcal{O}(n, d) \subset \mathcal{O}(n)$ denotes the subset of trees with d internal nodes.

Under the above assumptions there is a unique tree in $\mathcal{O}(1) = \mathcal{O}(1, 0)$, and for $n > 1$ each tree $T \in \mathcal{O}(n)$ is uniquely determined by the ordered list of rooted subtrees $T_1, \dots, T_k \subset T$ starting in the internal node of T that is connected to the root. For a fixed morphism $g_1\colon B[1] \to A[1]$ we define a map g_T for all $T \in \mathcal{O}(n)$ as follows: for the unique tree $T \in \mathcal{O}(1)$ we set $g_T = g_1$ and for any $T \in \mathcal{O}(n)$ with $n > 1$ we define recursively

$$g_T = -g_1 \circ f_k \circ (g_{T_1} \otimes \cdots \otimes g_{T_k}).$$

where $T_1, \dots, T_k$ are the subtrees starting in the internal node in T connected to the root. The following result is classical, and can be easily verified using the recursive formula.

Proposition 2.13. *Let $f \in \mathrm{hom}_{\mathsf{Alg}^\infty}(A, B)$ be a pre-morphism such that $f_1\colon A[1] \to B[1]$ admits a left (right) inverse $g_1\colon B[1] \to A[1]$. Then the pre-morphism $g \in \mathrm{hom}_{\mathsf{Alg}^\infty}(B, A)$ defined by*

$$g_n = \sum_{T \in \mathcal{O}(n)} g_T,$$

is a left (right) inverse to f in Alg^∞. If f is moreover a morphism and g_1 is both a left and right inverse, then g is again a morphism.

We now claim that the analogous result holds in $\mathsf{Alg}^{\infty,\mathrm{an}}$, assuming the inverse of the map f_1 is bounded. This follows easily from the recursive formula, as we show below.

Lemma 2.14. *Let $f \in \hom^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A,B)$ be an* analytic *pre-morphism such that $f_1\colon A[1] \to B[1]$ admits a continuous left (right) inverse g_1. Then the left (right) inverse g is again analytic.*

Proof. By assumption there exists a common constant $C > 1$ such that $\|g_1\| < C$ and $\|f_n\| < C^n$ for all $n \in \geq 1$. We now claim that for any tree $T \in \mathcal{O}(n,d)$ the norm of g_T is bounded by $\|g_T\| < C^{2n+2d-1}$. This follows by induction: for the base case $T \in \mathcal{O}(1) = \mathcal{O}(1,0)$, as

$$\|g_T\| = \|g_1\| < C = C^{2+0-1},$$

while for $n > 1$ any tree $T \in \mathcal{O}(n)$ determined by subtrees $T_i \in \mathcal{O}(n_i,d_i)$ such that $\sum n_i = n$ and $\sum d_i = d-1$, we have

$$\|g_T\| \leq \|g_1\|\|f_k\|\|g_{T_1} \otimes \cdots \otimes g_{T_k}\| < C \cdot C^k \cdot C^{2\sum n_i + 2\sum d_i - k} = C^{2n+2d-1},$$

assuming the claim holds for all $n_i < n$ and $d_i < d$. Now we note that a tree $T \in \mathcal{O}(n,d)$ has $n+d$ total vertices, so the cardinality of $\mathcal{O}(n,d)$ is bounded by the Catalan number $\mathcal{C}_{n+d}$. Because internal nodes have valency ≥ 3 it follows that any tree in $\mathcal{O}(n)$ has at most $d = n-1$ internal nodes, so that the total cardinality is bounded by the sum $\mathcal{C}_1 + \cdots + \mathcal{C}_{2n-1}$ of Catalan numbers. It is well-known that the function $n \mapsto \mathcal{C}_1 + \cdots + \mathcal{C}_{2n-1}$ can be bounded by q^n for all n, for some sufficiently large constant $q \gg 0$. We therefore find the following bound on the norm of g_n:

$$\|g_n\| \leq \sum_{d=0}^{n-1} \sum_{T \in \mathcal{O}(n,d)} \|g_T\| < \sum_{d=0}^{n-1} \sum_{T \in \mathcal{O}(n,d)} C^{2n+2d-1} \leq |\mathcal{O}(n)| \cdot C^{4n} < (qC^4)^n.$$

Because the constant qC^4 is independent of n, it follows that g is analytic. □

Note that if the A_∞-algebras A and B are finite-dimensional, then the continuity of g_1 is automatic. The above now implies that the morphism between minimal models induced by an analytic quasi-isomorphism is invertible in the analytic category.

Lemma 2.15. *Suppose $A, B \in \mathsf{Alg}^{\infty,\mathrm{an}}$ are compact and admit analytic minimal models* $\mathrm{H}(A)$*,* $\mathrm{H}(B)$*, and let $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A,B)$ be an analytic quasi-isomorphism. Then there is an induced analytic isomorphism* $\mathrm{H}(f) \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathrm{H}(A),\mathrm{H}(B))$*.*

Proof. Composing f with the quasi-isomorphisms $I_A \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathrm{H}(A),A)$, $P_A \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(B,\mathrm{H}(B))$, yields an analytic morphism

$$\mathrm{H}(f) := P_B \diamond f \diamond I_A \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathrm{H}(A),\mathrm{H}(B)),$$

with $\mathrm{H}(f)_1\colon \mathrm{H}(A)[1] \to \mathrm{H}(B)[1]$ an isomorphism of graded ℓ-bimodules. Because $\mathrm{H}(A)$ and $\mathrm{H}(B)$ are finite-dimensional, the inverse of $\mathrm{H}(f)_1$ is continuous, and Lemma 2.14 then implies that $\mathrm{H}(f)$ is an isomorphism in $\mathsf{Alg}^{\infty,\mathrm{an}}$. □

Corollary 2.16. *Analytic minimal models are unique up to analytic isomorphism.*

Proof. This follows directly from Lemma 2.15 when taking $\mathrm{H}(A)$ and $\mathrm{H}(B)$ to be two different minimal models of $A = B$ with morphism $f = \mathrm{id}_A$. □

One can lift the inverse $\mathrm{H}(f)^{-1}$ of the induced map between the minimal models $\mathrm{H}(A)$ and $\mathrm{H}(B)$ back up to a map $I_A \diamond \mathrm{H}(f)^{-1} \diamond P_B$ between B and A. If the minimal model of either A or B is strong, the result will be quasi-inverse to the original morphism.

Lemma 2.17. *Let $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A,B)$ be an analytic quasi-isomorphism between analytic A_∞-algebras admitting minimal models, then:*

- *if A admits a strong minimal model then f admits a left quasi-inverse in $\mathsf{Alg}^{\infty,\mathrm{an}}$,*
- *if B admits a strong minimal model then f admits a right quasi-inverse in $\mathsf{Alg}^{\infty,\mathrm{an}}$,*

- *if both admit a strong minimal model then f admits a quasi-inverse in* $\mathsf{Alg}^{\infty,\mathrm{an}}$.

Proof. By Lemma 2.15 the morphism $\mathrm{H}(f) = P_B \diamond f \diamond I_A$ admits an analytic inverse, which can be used to define an analytic morphism $g = I_A \diamond \mathrm{H}(f)^{-1} \diamond P_B$ from B to A. Now suppose the minimal model $\mathrm{H}(A)$ of A is strong, with homotopy operator Q_A such that $I_A \diamond P_A = \mathrm{id}_A + \delta(Q_A)$. Then

$$\begin{aligned} g \diamond f &= g \diamond f \diamond (I_A \diamond P_A - \delta(Q_A)) \\ &= I_A \diamond \mathrm{H}(f)^{-1} \diamond \underbrace{P_B \diamond f \diamond I_A}_{\mathrm{H}(f)} \diamond P_A - \delta(g \diamond f \diamond Q_A) \\ &= I_A \diamond P_A - \delta(g \diamond f \diamond Q_A), \\ &= \mathrm{id}_A + \delta(Q_A - g \diamond f \diamond Q_A), \end{aligned}$$

and it follows that g is a left quasi-inverse of f. Similarly, it follows that g is a right quasi-inverse if the minimal model of B is strong, and that g is a two-sided quasi-inverse if both minimal models are strong. □

Finally we show how to transfer minimal models between quasi-isomorphic analytic A_∞-algebras.

Lemma 2.18. *Let $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$ admit a strong analytic minimal model $\mathrm{H}(A)$, let $B \in \mathsf{Alg}^{\infty,\mathrm{an}}$ be arbitrary, and let $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, B)$ and $g \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(B, A)$ be two quasi-isomorphisms. Then there is a quasi-automorphism $h \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, A)$ such that:*

- *$h \diamond g$ is a left quasi-inverse for f*
- *$f \diamond h$ is a right quasi-inverse for g*
- *the analytic A_∞-algebra $\mathrm{H}(A)$ is an analytic minimal model for B via either of the two diagrams:*

$$\mathrm{H}(A) \underset{P_A}{\overset{I_A}{\rightleftarrows}} A \underset{h\diamond g}{\overset{f}{\rightleftarrows}} B \ , \qquad \mathrm{H}(A) \underset{P_A}{\overset{I_A}{\rightleftarrows}} A \underset{g}{\overset{f\diamond h}{\rightleftarrows}} B$$

Proof. Because f and g are quasi-isomorphisms, so is the composition $g \diamond f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, A)$. Because A admits $\mathrm{H}(A)$ as a strong minimal model, it then follows from Lemma 2.17 that $g \diamond f$ admits a two-sided quasi-inverse $h = I_A \diamond \mathrm{H}(g \diamond f)^{-1} \diamond P_A$. It is then immediate that

$$(h \diamond g) \diamond f \sim_{\mathrm{an}} \mathrm{id}_A, \quad g \diamond (f \diamond h) \sim_{\mathrm{an}} \mathrm{id}_A,$$

so $h \diamond g$ is a left quasi-inverse for f and $f \diamond h$ is right quasi-inverse to g. To show that these maps exhibit $\mathrm{H}(A)$ as a minimal model for B, it now suffices to check that $P_A \diamond (h \diamond g)$ is a left inverse for $f \diamond I_A$. This follows readily from the definition:

$$P_A \diamond (h \diamond g) \diamond f \diamond I_A = \underbrace{P_A \diamond I_A}_{\mathrm{id}_A} \diamond \mathrm{H}(g \diamond f)^{-1} \diamond \underbrace{P_A \diamond g \diamond f \diamond I_A}_{\mathrm{H}(g\diamond f)} = \mathrm{id}_A.$$

Similarly, one shows that the second diagram exhibits $\mathrm{H}(A)$ as a minimal model for B. □

3. Analytic A_∞-bimodules

In this section we define A_∞-bimodules and morphisms over an analytic A_∞-algebra by imposing an analogous condition on sequences of multilinear maps. We start by recalling the definition of such bimodules using a bar construction, for which we again follow Tradler [Tra08].

3.1. A_∞-**bimodules.** Given an A_∞-algebra $A \in \mathsf{Alg}^\infty$, A-bimodules and the morphisms between them can be defined via double sequences of morphisms in the space

$$\mathcal{M}_A(M, N) := \prod_{i,j=0}^{\infty} \mathrm{hom}_{\ell^e}(A[1]^{\otimes i} \otimes M[1] \otimes A[1]^{\otimes j} . N[1]).$$

We will use the notation $\rho = (\rho_{i,j})_{i,j\in\mathbb{N}}$ for homogeneous elements of $\mathcal{M}_A(M, N)$. As before, this space of sequences can be identified with the morphisms out of a bar construction

$$\mathsf{B}_A M := \mathsf{B}A \otimes M[1] \otimes \mathsf{B}A = \bigoplus_{i,j=0}^{\infty} A[1]^{\otimes i} \otimes M[1] \otimes A[1]^{\otimes j},$$

which is naturally a cofree cobimodule over the coalgebra $\mathsf{B}A$. In [Tra08] it is shown that one can lift any map $\rho \in \mathcal{M}_A(M,N)$ to a cohomomorphism $\widehat{\rho}\colon \mathsf{B}_A M \to \mathsf{B}_A N$, and to a coderivation $\widetilde{\rho}\colon \mathsf{B}_A M \to \mathsf{B}_A M$ if $M = N$. These maps are explicitly given by the following formulas:

$$\begin{aligned}
\widehat{\rho} &:= \sum_{i,j\geq 0} \mathrm{id}^{\otimes i} \otimes \underline{\rho} \otimes \mathrm{id}^{\otimes j} \\
\widetilde{\rho} &:= \sum_{i,j} \mathrm{id}^{\otimes i} \otimes \underline{\rho} \otimes \mathrm{id}^{\otimes j} + \sum_{i,j,k\geq 0} \mathrm{id}^{\otimes i} \otimes \mu_A \otimes \mathrm{id}^{\otimes k} \otimes \underline{\mathrm{id}}_{M[1]} \otimes \mathrm{id}^{\otimes j} \\
&\quad + \sum_{i,j,k\geq 0} \mathrm{id}^{\otimes i} \otimes \underline{\mathrm{id}}_{M[1]} \otimes \mathrm{id}^{\otimes k} \otimes \mu_A \otimes \mathrm{id}^{\otimes j},
\end{aligned}$$

These lifts can again be used to define A_∞-bimodule structures and their morphisms in terms of differentials.

Definition 3.1. An A_∞-bimodule over an A_∞-algebra A is pair (M, ν) of a bimodule $M \in \mathsf{grMod}\,\ell^e$ and a map $\nu \in \mathcal{M}_A(M,M)$ satisfying $\widetilde{\nu}^2 = 0$.

The main examples of bimodules which we will consider are the following.

Example 3.2. The *diagonal bimodule* $A_\Delta = (A, \nu_\Delta)$ is defined by the sequence of maps

$$\nu_{\Delta,i,j}\colon A[1]^{\otimes i} \otimes A[1] \otimes A[1]^{\otimes j} \xrightarrow{\sim} A[1]^{\otimes i+j+1} \xrightarrow{\mu_{i+j+1}} A[1]$$

Example 3.3. Writing $A^\vee = \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$, the pair $A^\vee_\Delta := (A^\vee, \nu^\vee_\Delta)$ defined by the sequence of maps

$$\nu^\vee_{\Delta,i,j}(a_1, \ldots, a_i, \underline{f}, a_{i+1}, \ldots a_{i+j})(a_0) = (-1)^K f(\mu_{i+j+1}(a_{i+1}, \ldots, a_{i+j}, a_0, a_1, \ldots, a_i)),$$

is a bimodule, called the *dual bimodule* [Tra08].

Bimodules over an A_∞-algebra A again form a DG category with the following morphism complexes.

Definition 3.4. A pre-morphisms between A_∞-bimodules (M, ν_M) and (N, ν_N) is an element of

$$\mathrm{hom}_{A-A}(M,N) := (\mathcal{M}_A(M,N), \delta)\,, \quad \delta(\rho) = \nu_N \circ \widehat{\rho} - (-1)^{|\rho|} \rho \circ \widetilde{\nu}_M.$$

The space of morphisms, i.e. degree 0 cocycles is denoted $\mathrm{Hom}_{A-A}(M,N) \subset \mathrm{hom}^0_{A-A}(M,N)$.

Given $\rho \in \mathrm{hom}_{A-A}(M,N)$ and $\tau \in \mathrm{hom}_{A-A}(L,M)$ the composition is defined as $\rho \diamond \tau = \rho \circ \widehat{\tau}$. This composition then yields a well-defined DG category $\mathsf{Mod}^\infty_{A-A}$ of A_∞-bimodules.

GIven a morphism $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A,B)$ of A_∞-algebras $A, B \in \mathsf{Alg}^\infty$ there is a DG functor

$$f^\sharp\colon\ \mathsf{Mod}^\infty_{B-B} \to \mathsf{Mod}^\infty_{A-A},$$

mapping a B-bimodule (M,ν) to $f^\sharp(M,\nu) = (M, \nu \circ (\widehat{f} \otimes \underline{\mathrm{id}}_{M[1]} \otimes \widehat{f}))$ and mapping a B-bimodule pre-morphism ρ to $f^\sharp\rho = \rho \circ (\widehat{f} \otimes \underline{\mathrm{id}}_{M[1]} \otimes \widehat{f})$, where

$$\widehat{f} \otimes \underline{\mathrm{id}}_{M[1]} \otimes \widehat{f}\colon \mathsf{B}_A M = \mathsf{B}A \otimes M[1] \otimes \mathsf{B}A \to \mathsf{B}B \otimes M[1] \otimes \mathsf{B}B = \mathsf{B}_B M$$

is the induced map on the relative bar constructions. A tedious but straightforward computation verifies that $f^\sharp\nu$ again lifts to a codifferential, that $\delta(f^\sharp\rho) = f^\sharp\delta(\rho)$, and that $f^\sharp$ commutes with the differential δ and composition $\diamond$; see [Gan12, §2.8].

Maps between the diagonal and dual bimodule satisfy an additional functoriality: given an A_∞-morphism $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A,B)$ and a pre-morphism $\rho \in \mathrm{hom}_{B-B}(B_\Delta, B^\vee_\Delta)$ there is a pre-morphism

$$f^*\rho\colon A_\Delta \xrightarrow{f_\Delta} f^\sharp B \xrightarrow{f^\sharp\rho} f^\sharp B^\vee_\Delta \xrightarrow{f^\vee_\Delta} A^\vee_\Delta,$$

where the map $f_\Delta \in \mathrm{Hom}_{A-A}(A_\Delta, f^\sharp B_\Delta)$ has components $(f_\Delta)_{i,j} = f_{i+1+j}$ and the dual version $f^\vee_\Delta \in \mathrm{Hom}_{A-A}(f^\sharp B^\vee_\Delta, A^\vee_\Delta)$ is defined in components as

$$(f^\vee_\Delta)_{i,j}(a_1, \ldots, a_i, \underline{b}, c_1, \ldots, c_j)(d) = (-1)^K b(f_{i+1+j}(c_1, \ldots, c_j, d, a_1, \ldots, a_i)),$$

where $K = (|a_1| + \cdots + |a_i|)(|b| + |d| + |c_1| + \cdots + |c_j)$ is the appropriate Koszul sign. This gives rise to an additional pull-back

$$f^* \colon \hom_{B-B}(B_\Delta, B_\Delta^\vee) \to \hom_{A-A}(A_\Delta, A_\Delta^\vee).$$

One checks that $(g \diamond f)^* \rho = f^*(g^* \rho)$ holds, so that the assignment $A \mapsto \hom_{A-A}(A_\Delta, A_\Delta^\vee)$ is contravariantly DG-functorial.

3.2. **Hochschild cohomology.** Given an A_∞-algebra A, one can define the Hochschild cohomology with values in any A_∞-bimodule M. The Hochschild complex can again be modeled using the bar construction: following Tradler [Tra08, Lemma 2.3] one can define for each $\xi \in \mathcal{M}(A, M)$ a coderivation $\overline{\xi} \colon \mathsf{B}A \to \mathsf{B}_A M$ by the formula

$$\overline{\xi} := \sum_{i,j,n \geq 0} \mathrm{id}_{A[1]}^{\otimes i} \otimes \underline{\xi_n} \otimes \mathrm{id}_{A[1]}^{\otimes j},$$

where the underline again signifies the component mapping to the factor M. Using this lift, the Hochschild cochain complex can be defined as follows.

Definition 3.5. Let (A, μ) be an A_∞-algebra and (M, ν) an A-bimodule. Then the *M-valued Hochschild cochain complex* is the complex

$$\mathsf{C}^\bullet(A, M) := (\mathcal{M}(A, M)[-1], \ \mathbf{b}), \quad \mathbf{b}(\xi) := \nu \circ \overline{\xi} - (-1)^{|\xi|} \xi \circ \widetilde{\mu}.$$

If $M = A_\Delta$ then the Hochschild complex is abbreviated as $\mathsf{C}^\bullet(A) := \mathsf{C}^\bullet(A, A_\Delta)$.

The Hochschild complex is functorial in each argument: given an A_∞-algebra morphism $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B)$ and a B-bimodule M there is a chain map

$$\mathsf{C}^\bullet(B, M) \to \mathsf{C}^\bullet(A, f^\sharp M) \quad \xi \mapsto \xi \circ \widehat{f},$$

and likewise every bimodule morphism $\rho \in \mathrm{Hom}_{A-A}(M, N)$ induces a chain map

$$\mathsf{C}^\bullet(A, M) \to \mathsf{C}^\bullet(A, N), \quad \xi \mapsto \rho \circ \overline{\xi}.$$

In particular, there is a pullback for Hochschild cohomology with values in the dual bimodule: given any morphism $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B)$ there is a chain map

$$f^* \colon \mathsf{C}^\bullet(B, B_\Delta^\vee) \xrightarrow{-\circ \widehat{f}} \mathsf{C}^\bullet(A, f^\sharp B_\Delta^\vee) \xrightarrow{f_\Delta^\vee \circ \overline{-}} \mathsf{C}^\bullet(A, A_\Delta^\vee),$$

where $f_\Delta^\vee \in \mathrm{Hom}_{A-A}(f^\sharp B_\Delta^\vee, A_\Delta^\vee)$ is the morphism defined in the previous section. One checks that this satisfies $(g \diamond f)^* \xi = f^*(g^* \xi)$, making the assignment $A \mapsto \mathsf{C}^\bullet(A, A_\Delta^\vee)$ functorial.

Lastly we consider *negative cyclic cohomology* of an A_∞-algebra. We will realise negative cyclic cohomology using Connes' complex, which can be defined as follows. An element $\xi = (\xi_n)_{n \in \mathbb{N}} \in \mathsf{C}^\bullet(A, A_\Delta^\vee)$ is called *cyclic* if for each $n \in \mathbb{N}$ and all $a_0, \ldots, a_n \in A[1]$ the following relation holds:

$$\xi_n(a_1, \ldots, a_n)(a_0) = (-1)^{|a_0|(|a_1| + \cdots + |a_n|)} \xi_n(a_0, \ldots, a_{n-1})(a_n)$$

One checks that the Hochschild differential preserves cyclic cochains, yielding a subcomplex

$$\mathsf{C}_\lambda^\bullet(A) := (\{ \ \xi \in \mathsf{C}^\bullet(A, A_\Delta^\vee) \mid \xi \text{ is cyclic } \}, \mathbf{b}),$$

called Connes' complex. We will denote the cohomology of this complex by $\mathrm{HC}_\lambda^\bullet(A)$. We remark that the pullback along ant morphism $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B)$ preserves cyclic cocycles, and therefore restricts to a map $f^* \colon \mathsf{C}_\lambda^\bullet(B) \to \mathsf{C}_\lambda^\bullet(A)$.

3.3. **Analytic A_∞-bimodules.** Given an analytic A_∞-algebra A and normed ℓ-bimodule $M \in \mathsf{NMod}^{\mathbb{Z}} \, \ell^e$ we consider a family of norms as in §2.2: for each $r > 0$ we define

$$\mathsf{B}_A(M, r) := (\mathsf{B}_A M, \|\cdot\|_r), \quad \|m\|_r = \sup_{i,j} \|m_{i,j}\| r^{-i-j-1},$$

where $m = \sum_{i,j} m_{i,j} \in \mathsf{B}_A M$ is a decomposition into elements $m_{i,j} \in A[1]^{\otimes i} \otimes M[1] \otimes A[1]^{\otimes j}$. Boundedness of linear maps out of these normed bar constructions can again be characterised in various ways, analogously to Lemma 2.4.

Lemma 3.6. *For $\rho \in \mathcal{M}_A(M,N)$ the following are equivalent:*

1. *there exists $C > 0$ such that $\|\rho_{i,j}\| \le C^{i+j+1}$ for all $i,j \in \mathbb{N}$*
2. *there exists $r > 0$ such that $\rho\colon \mathsf{B}_A(M,r) \to N[1]$ is bounded,*
3. *for every $r' > 0$ there exists $r > 0$ such that $\widehat{\rho}\colon \mathsf{B}_A(M,r) \to \mathsf{B}_A(N,r')$ is bounded*

if moreover $M = N$, then this is also equivalent to:

4. *for every $r' > 0$ there exists $r > 0$ such that $\widetilde{\rho}\colon \mathsf{B}_A(M,r) \to \mathsf{B}_A(M,r')$ is bounded.*

Proof. ($1 \implies 2$) Suppose there exists $C > 0$ such that $\|\rho_{i,j}\| \le C^{i+j+1}$ for all $i,j \in \mathbb{N}$ and fix $0 < r < C^{-1}$. Then an element $m = \sum_{i,j} m_{i,j} \in \mathsf{B}_A(M,r)$ of norm $\|m\|_r = 1$ satisfies $\|m_{i,j}\| \le r^{i+j+1}$ for all $i,j \in \mathbb{N}$ and therefore

$$\|\rho(m)\| \le \sum_{n\in\mathbb{N}} \sum_{i+j=n} \|\rho_{i,j}(m_{i,j})\| \le \sum_{n\in\mathbb{N}} \sum_{i+j=n} C^{i+j+1} r^{i+j+1} = \sum_{n\in\mathbb{N}} n(Cr)^{n+1}.$$

Because the power series $\sum_n nz^{n+1}$ has radius of convergence 1, it converges at $z = Cr < 1$. In particular, the norm ρ is bounded.

($2 \implies 1$) Let $K > 1$ be a constant bounding the norm of $\rho\colon \mathsf{B}_A(M,r) \to N[1]$ and pick $C \ge \frac{K}{r}$, then for all $i,j \in \mathbb{N}$ and $m_{i,j} \in A[1]^{\otimes i} \otimes M[1] \otimes A[1]^{\otimes j}$ of norm $\|m_{i,j}\| = 1$ the map $\rho_{i,j}$ satisfies

$$\|\rho_{i,j}(m_{i,j})\| = \|\rho(m_{i,j})\| \le K\|m_{i,j}\| r^{-i-j-1} \le \left(\frac{K}{r}\right)^{i+j+1} \le C^{i+j+1},$$

which shows that in particular that $\|\rho_{i,j}\| \le C^{i+j+1}$.

($1 \implies 3$) Suppose there exists $C > 1$ such that $\|\rho_{i,j}\| \le C^{i+j+1}$ for all $i,j \in \mathbb{N}$, and fix $r' > 0$. Then we choose a constant $q > 0$ such that $ij < q^{i+j+1}$ for all $i,j \in \mathbb{N}$, a constant $K = \min\{r', 1\}$, and a radius $r < \frac{K}{qC}$. Then for any element $m = \sum m_{i,j} \in \mathsf{B}_A M$ of norm $\|m\|_r = 1$ one has $\|m_{i,j}\| \le r^{i+j+1}$ and hence

$$\begin{aligned}
\|\widehat{\rho}(m)\|_{r'} &\le \sum_{i,j\ge 0} \sum_{\substack{0\le i\le n\\ 0\le j\le m}} \big\|(\mathrm{id}^{\otimes i} \otimes \rho_{n-i,m-j} \otimes \mathrm{id}^{\otimes j})(m_{i,j})\big\| (r')^{-i-j-1} \\
&\le \sum_{i,j\ge 0} \sum_{\substack{0\le i\le n\\ 0\le j\le m}} C^{i+j-i-j+1} K^{-i-j-1} r^{i+j+1} \\
&\le \sum_{i,j\ge 0} nm \left(\frac{rC}{K}\right)^{i+j+1} \\
&\le \sum_{N\ge 1} (N-1) \left(\frac{rqC}{K}\right)^{N}
\end{aligned}$$

Then the norm is bounded by the value of the power series $\sum_N (N-1)z^N$ at $z = \frac{rqC}{K} < 1$.

($1 \implies 4$) This follows in an analogous way to previous implication. We leave the details up to the reader.

($3,4 \implies 2$) The projection $\pi\colon \mathsf{B}_A(N,r') \to N[1]$ is clearly bounded for any $r' > 0$, so boundedness of $\widehat{\rho}\colon \mathsf{B}_A(M,r) \to \mathsf{B}_A(N,r')$ implies that $\rho = \pi \circ \widehat{\rho}$ is bounded. Likewise, $\rho = \pi \circ \widetilde{\rho}$ is bounded when $\widetilde{\rho}\colon \mathsf{B}_A(M,r) \to \mathsf{B}_A(M,r')$ is bounded. ☐

Hence, as in the algebra case we find that the union over the morphism spaces $\hom^{\mathrm{cont}}_{\ell^e}(\mathsf{B}_A(M,r), N[1])$ is given by a graded vector space of double sequences

$$\mathcal{A}_A(V,W) := \left\{\, f \in \mathcal{M}_A(V,W) \;\middle|\; \exists C > 0 \text{ such that } \|f_{i,j}\| \le C^{i+j+1} \text{ for all } i,j \in \mathbb{N} \,\right\},$$

whose elements we again call *analytic*. In particular, we obtain a definition of analytic A_∞-bimodules over an analytic A_∞-algebra.

Definition 3.7. Let (A, μ) be an analytic A_∞-algebra, then an *analytic* A-bimodule is a pair (M, ν) of $M \in \mathsf{NMod}^{\mathbb{Z}}(\ell^e)$ and $\nu \in \mathcal{A}^1_A(M, M)$ satisfying $\widetilde{\nu}^2 = 0$.

Example 3.8. The diagonal bimodule A_Δ is analytic for any analytic A_∞ algebra when endowed with the induced norm, as $\|\nu_{\Delta,i,j}\| = \|\mu_{i+j+1}\|$ satisfies the appropriate bound.

Example 3.9. The dual bimodule $A^\vee$ is not analytic in general, as the dual space is not endowed with a natural norm. However, one can consider the *continuous dual bimodule*

$$A'_\Delta := \left(\hom^{\mathrm{cont}}_{\mathbb{C}}(A, \mathbb{C}), \nu'_\Delta\right),$$

where the components of ν'_Δ are defined by the same formula as the components of $\nu^\vee_\Delta$. It is again straightforward to check that $\left\|\nu^\vee_{\Delta,i,j}\right\| = \|\mu_{i+j+1}\|$ is bounded by a geometric series. If A is finite dimensional, then A'_Δ is equal to $A^\vee_\Delta$ in $\mathsf{Mod}^\infty_{A-A}$, but in general it is only a submodule.

As before we call a (pre-)morphism $\rho \in \hom^\bullet_{A-A}(M, N)$ between A-bimodules (M, ν_M) and (N, ν_N) analytic if it lies in the subspace $\mathcal{A}_A(M, N) \subset \mathcal{M}_A(M, N)$. It follows from Lemma 3.6 that

$$\delta = \nu_N \circ \widehat{(-)} - (-1)^{|\rho|}(-) \circ \widetilde{\nu}_M$$

is a bounded operator on this space, and hence gives rise to a complex of analytic pre-morphisms

$$\hom^{\bullet,\mathrm{an}}_{A-A}(M, N) := (\mathcal{A}_A(M, N), \delta),$$

which is a subcomplex of $\hom^\bullet_{A-A}(M, N)$. As before we denote the subspace of analytic morphisms by $\mathrm{Hom}^{\bullet,\mathrm{an}}_{A-A}(M, N) \subset \mathrm{Hom}^\bullet_{A-A}(M, N)$. It follows from Lemma 3.6 that the composition of analytic bimodule pre-morphisms is again analytic, and therefore defines morphism complexes for a DG category $\mathsf{NMod}^{\infty,\mathrm{an}}_{A-A}$ of analytic bimodules over A.

Given an analytic A_∞-morphism f one can again consider the functor f^*, and the pullback map $f^\sharp$ on the Hom-spaces between the diagonal bimodule and its continuous dual.

Lemma 3.10. *Let $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, B)$ be an analytic morphism, then there is a well-defined functor*

$$f^\sharp \colon \mathsf{NMod}^{\infty,\mathrm{an}}_{B-B} \to \mathsf{NMod}^{\infty,\mathrm{an}}_{A-A},$$

which induces a pullback $f^* \colon \hom^{\mathrm{an}}_{B-B}(B_\Delta, B'_\Delta) \to \hom^{\mathrm{an}}_{A-A}(A_\Delta, A'_\Delta)$.

Proof. Given $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, B)$, for any $N, M \in \mathsf{NMod}^{\mathbb{Z}}\,\ell^e$ and $\rho \in \mathcal{A}_B(M, N)$, there exists a common constant $C > 0$ such that $\|f_n\| < C^n$ for all $n \in \mathbb{N}$ and $\|\rho_{i,j}\| < C^{i+j+1}$ for all $i, j \in \mathbb{N}$. Then the construction $f^\sharp\rho$ has components satisfying

$$\begin{aligned}\left\|(f^\sharp\rho)_{i,j}\right\| &\le \sum_{\substack{i_1+\dots+i_n=i\\ j_1+\dots+j_n=j}} \|\rho_{n,m}\|\|f_{i_1}\|\cdots\|f_{i_n}\|\|f_{j_1}\|\cdots\|f_{j_m}\| \\ &< \sum_{\substack{i_1+\dots+i_n=i\\ j_1+\dots+j_n=j}} C^{n+m+i+j+1} \\ &\le (\#\text{ of partitions of } i)\cdot(\#\text{ of partitions of } j))\cdot C^{2(i+j)+1}.\end{aligned}$$

Picking some constant $q > 1$ such that q^n bounds the number of partitions of n, it follows that $\left\|(f^\sharp\rho)_{i,j}\right\| < q^{i+j}C^{2(i+j)+1} \le (qC^2)^{i+j+1}$, which shows that $f^\sharp\rho \in \mathcal{A}_A(M, N)$. If follows that $f^\sharp$ maps every analytic bimodule structure ν to an analytic bimodule structure $f^\sharp\nu$, and an analytic (pre-)morphism $\rho \in \hom^{\mathrm{an}}_{A-A}(M, N)$ between analytic bimodules to an analytic bimodule map $f^\sharp\rho$. Hence $f^\sharp$ is a well-defined functor between categories of analytic bimodules.

It follows that $f^\sharp B_\Delta$ is an analytic bimodule, and it follows directly from the definition that the map $f_\Delta \colon A \to f^\sharp B_\Delta$ is analytic. The bimodule $f^\sharp B'_\Delta$ is likewise analytic, and it follows easily that the formula for $f^\vee_\Delta$ defines an analytic map $f'_\Delta \colon f^\sharp B'_\Delta \to A'_\Delta$. Hence, if $\rho \in \hom^{\mathrm{an}}_{B-B}(B_\Delta, B'_\Delta)$ is an analytic bimodule map, the composition $f^*\rho = f'_\Delta \diamond f^\sharp\rho \diamond f_\Delta$ is again analytic. □

3.4. **Analytic Hochschild cohomology.** Given an analytic A_∞-algebra (A, μ) and an analytic A-bimodule $(M, \nu) \in \mathsf{NMod}^{\infty,\mathrm{an}}_{A-A}$, we call a Hochschild cochain $\xi \in \mathsf{C}^\bullet(A, M) = (\mathcal{M}(A, M)[-1], \mathbf{b})$ analytic if it lies in the subspace $\mathcal{A}(A, M)[-1] \subset \mathcal{M}(A, M)[-1]$. This analytic condition can again be characterised in multiple ways.

Lemma 3.11. *Let $\xi \in \mathsf{C}^\bullet(A, M)$, then the following are equivalent:*

1. *ξ is analytic*
2. *for every $r' > 0$ there exists $r > 0$ such that $\overline{\xi}\colon \mathsf{B}(A, r) \to \mathsf{B}_A(M, r')$ is bounded*

Proof. The implication (2) $\implies$ (1) is obvious, hence we only check the other implication. Given $\xi \in \mathsf{C}^{\bullet,\mathrm{an}}(A, M)$ there exists by Lemma 2.4 a $C > 0$ such that $\|\xi_n\| < C^n$. Fix r', let $K = \min r', 1$ and pick $r < \frac{K}{2C}$. Then for every element $\sum a_n \in \mathsf{B}(A, r)$ of norm $\|a\|_r \leq 1$

$$\begin{aligned}\left\|\overline{\xi}(a)\right\|_{r'} &\leq \sum_{n\geq 0}\sum_{i+1+j=n}\left\|(\mathrm{id}^i \otimes \xi_n \otimes \mathrm{id}^{\otimes j})(a_n)\right\|(r')^{-n} \\ &\leq \sum_{n\geq 0}(n-1)C^n \cdot \|a_n\| \cdot (r')^{-n} \\ &\leq \sum_{n\geq 0}\left(\frac{2Cr}{r'}\right)^n =: M\end{aligned}$$

is a finite bound as the sum is a convergent geometric series. Hence, $\left\|\overline{\xi}\right\| < M$ and $\overline{\xi}$ is bounded. ☐

As a corollary, it follows that the differential maps an analytic cochain $\xi \in \mathsf{C}^{\bullet,\mathrm{an}}(A, M)$ to a cochain

$$\mathbf{b}(\xi) = \nu \circ \overline{\xi} - (-1)^{|\xi|}\xi \circ \widetilde{\mu}$$

which is bounded as a map $\mathsf{B}(A, r) \to M$ for some r, hence analytic by Lemma 2.4. It follows that the analytic cochains form a well-defined subcomplex which we denote by

$$\mathsf{C}^{\bullet,\mathrm{an}}(A, M) := (\mathcal{A}(A, M)[-1], \mathbf{b}).$$

For $M = A_\Delta$ we again abbreviate $\mathsf{C}^{\bullet,\mathrm{an}}(A) := \mathsf{C}^{\bullet,\mathrm{an}}(A, A_\Delta)$. Similarly, we define the analytic version of Connes' complex as the intersection

$$\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A) := \mathsf{C}^\bullet_\lambda(A) \cap \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta),$$

which will play the role of the negative cyclic cohomology in the analytic setting.

The analytic Hochschild complex is again functorial in each argument with respect to analytic maps: given $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, B)$ it follows directly from Lemma 2.4 that there is a well-defined map

$$-\circ \widehat{f}\colon \mathsf{C}^{\bullet,\mathrm{an}}(B, M) \to \mathsf{C}^{\bullet,\mathrm{an}}(A, f^\sharp M),$$

and likewise for any $\rho \in \mathrm{Hom}^{\mathrm{an}}_{A-A}(M, N)$ it follows from Lemma 3.11 that

$$\rho \circ \overline{-}\colon \mathsf{C}^{\bullet,\mathrm{an}}(A, M) \to \mathsf{C}^{\bullet,\mathrm{an}}(A, N),$$

is well-defined. In particular, composition with $\widehat{f}$ and f'_Δ induces pull-back maps $f^*\colon \mathsf{C}^{\bullet,\mathrm{an}}(B, B'_\Delta) \to \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta)$ and $f^*\colon \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(B) \to \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A)$ for any analytic A_∞-morphism $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, B)$.

3.5. **Analytic inverses for bimodule morphisms.** In the remained of this section we show the analogue of Lemma 2.14 for bimodule morphisms: given $\rho \in \mathrm{Hom}^{\mathrm{an}}_{A-A}(M, N)$ such that $\rho_{0,0}$ is invertible in $\mathsf{NMod}^{\mathbb{Z}}\,\ell^e$, we show that it admits an analytic inverse $\tau \in \mathrm{Hom}^{\mathrm{an}}_{A-A}(N, M)$. To check the growth condition on τ, we define it explicitly using a tree formula using a special kind of planar tree.

Definition 3.12. A *caterpillar* is a rooted planar tree T with one of the leaves marked, such that all internal nodes have valency ≥ 3 and lie on a *central path* between the root and the marked leaf. The set of caterpillars for which n_1 unmarked leaves lie to the left of this central path and n_2 unmarked leaves lie to the right of the central path is denoted $\mathsf{Catp}(n_1, n_2)$. The subset of caterpillars with d internal nodes is denoted $\mathsf{Catp}(n_1, n_2, d)$.

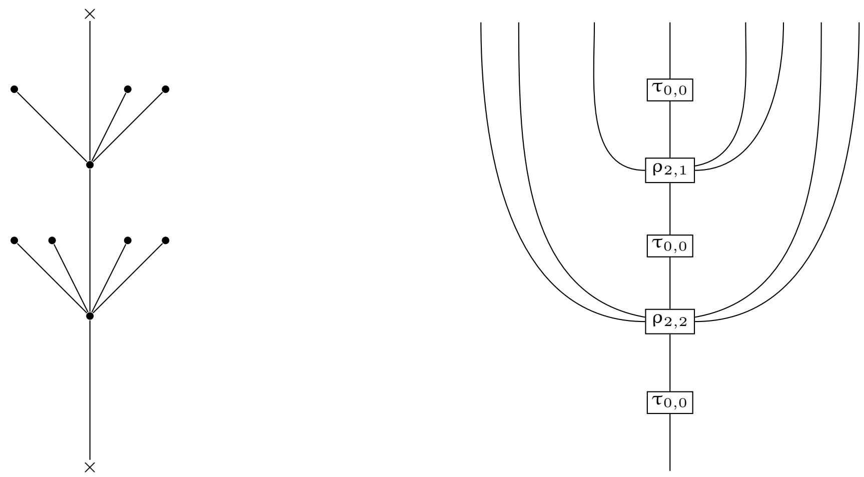
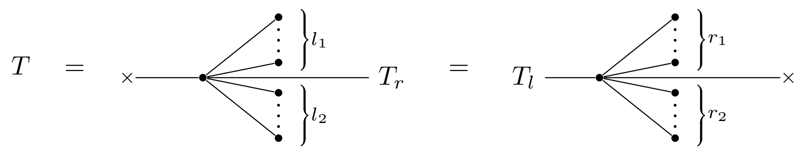


Figure 1: Left: an example of a tree $T \in \mathsf{Catp}(3,4,2)$ with the root at the bottom. Right: the string diagram for the corresponding map $\tau_T \colon A[1]^{\otimes 3} \otimes M[1] \otimes A[1]^{\otimes 4} \to N[1]$.

A typical example of a caterpillar tree is given in Figure 1. There is a unique tree $T \in \mathsf{Catp}(0,0)$ which has $d = 0$ internal nodes. For $n > 1$ one has $d > 1$ and every tree $T \in \mathsf{Catp}(n_1, n_2, d)$ can be uniquely decomposed into an internal node and a subtree in two ways:



so T is unique determined by a pair of numbers (l_1, l_2) and a tree $T_r \in \mathsf{Catp}(n_1 - l_1, n_2 - l_2, d-1)$, and also by a pair of numbers (r_1, r_2) and a tree as $T_l \in \mathsf{Catp}(n_1 - r_1, n_2 - r_2, d-1)$. Hence we have bijections

$$\begin{aligned}\mathsf{Catp}(n_1, n_2) &\cong \{(l_1, l_2, T_r) \mid l_1 + l_2 > 0,\ T_r \in \mathsf{Catp}(n_1 - l_1, n_2 - l_2)\} \\ &\cong \{(T_l, r_1, r_2) \mid r_1 + r_2 > 0,\ T_l \in \mathsf{Catp}(n_1 - r_1, n_2 - r_2)\}\end{aligned}$$

Given $\rho \in \hom_{A-A}(M,N)$ and $\tau_{0,0} \in \hom^{\mathrm{cont}}_{\ell^e}(N,M)$ we then define for every tree $T \in \mathsf{Catp}(n_1, n_2)$ a bimodule pre-morphism as follows. For $n_1 = n_2 = 0$ we set $\tau_T = \tau_{0,0}$, and otherwise we define τ_T via the following equivalent recursive formulas:

$$\begin{aligned}\tau_T &= -\tau_{0,0} \circ \rho_{l_1,l_2} \circ (\mathrm{id}^{\otimes l_1} \otimes \underline{\tau_{T_r}} \otimes \mathrm{id}^{\otimes l_2}) \\ &= -\tau_{T_l} \circ (\mathrm{id}^{\otimes (n_1 - r_1)} \otimes \underline{(\rho_{r_1,r_2} \circ \tau_{0,0})} \otimes \mathrm{id}^{\otimes (n_2 - r_2)}).\end{aligned} \tag{4}$$

See again figure Figure 1 for an example of the map τ_T. If $\tau_{0,0}$ is an inverse for $\rho_{0,0}$ then the sum of these maps determines an inverse bimodule map.

Lemma 3.13. *Let $\rho \in \hom_{A-A}(M,N)$ be a pre-morphism such that $\rho_{0,0}$ admits a left-/right inverse $\tau_{0,0}$. Then ρ admits a left-/right inverse $\tau \in \hom_{A-A}(N,M)$ with components*

$$\tau_{i,j} = \sum_{T \in \mathsf{Catp}(i,j)} \tau_T,$$

where the maps τ_T are defined as above.

Proof. Suppose $\tau_{0,0}$ is a right-inverse for $\rho_{0,0}$, then we claim that τ is a right-inverse for ρ. The base case $i = j = 0$ is trivial as

$$(\rho \diamond \tau)_{0,0} = \rho_{0,0} \circ \sum_{T \in \mathsf{Catp}(0,0)} \tau_T = \rho_{0,0} \circ \tau_{0,0} = \mathrm{id}_{M[1]}.$$

For $i+j>0$ it follows from the recursive definition of the maps τ_T that

$$\begin{aligned}(\rho\diamond\tau)_{i,j} &= \rho_{0,0}\circ\tau_{i,j} + \sum_{l_1+l_2>0}\sum_{T_r\in\mathsf{Catp}(i-l_1,j-l_2)} \rho_{l_1,l_2}\circ(\mathrm{id}^{\otimes l_1}\otimes\rho_{T_r}\otimes\mathrm{id}^{\otimes l_2})\\ &= \rho_{0,0}\circ\tau_{i,j} + \sum_{l_1+l_2>0}\sum_{T_r\in\mathsf{Catp}(i-l_1,j-l_2)} \rho_{0,0}\circ\tau_{0,0}\circ\rho_{l_1,l_2}\circ(\mathrm{id}^{\otimes l_1}\otimes\rho_{T_r}\otimes\mathrm{id}^{\otimes l_2})\\ &= \rho_{0,0}\circ\tau_{i,j} - \sum_{T\in\mathsf{Catp}(i,j)}\rho_{0,0}\circ\tau_T = 0,\end{aligned}$$

where the final line uses the bijection between trees in $\mathsf{Catp}(i,j)$ and triples (l_1,l_2,T_r) discussed above. It follows that $\rho\diamond\tau=\mathrm{id}$, so τ is a right inverse. The case where $\tau_{0,0}$ is a left-inverse is similar, using the other decomposition into triples (T_l,r_1,r_2). In particular, τ is a two-sided inverse if $\tau_{0,0}$ is. □

We remark that if ρ is a morphism and $\tau=\rho^{-1}$ is a two-sided inverse, then the latter is again a morphism by the equality

$$0=\tau\diamond\delta(\mathrm{id})=\tau\diamond\delta(\rho)\diamond\tau+\delta(\tau)=\delta(\tau).$$

We claim that the map τ is moreover analytic when ρ is analytic.

Proposition 3.14. *Let $\rho\in\hom^{\mathrm{an}}_{A-A}(M,N)$ be an analytic pre-morphism, and suppose $\rho_{0,0}$ admits a continuous left-/right-inverse $\tau_{0,0}$. Then ρ admits an analytic left-/right-inverse.*

Proof. Because ρ is analytic and $\tau_{0,0}$ is continuous, there exists $C>0$ such that $\|\tau_{0,0}\|<C$ and $\|\rho_{i,j}\|<C^{i+j+1}$ for all $i,j\in\mathbb{N}$. We claim that for $T\in\mathsf{Catp}(n_1,n_2,d)$ the map τ_T satisfies $\|\tau_T\|<C^{n_1+n_2+2d+1}$. For $n_1=n_2=0$ and $d=0$ one has

$$\|\tau_T\|=\|\tau_{0,0}\|<C=C^{0+0+0+1}.$$

Now let n_1,n_2 and $d>0$ be arbitrary, then assuming the bound holds for all $d'<d$ it follows by the recursive formula that there exists (l_1,l_2) and a tree $T_r\in\mathsf{Catp}(n_1-l_1,n_2-l_2,d-1)$ such that

$$\|\tau_T\|=\|\tau_{0,0}\|\|\rho_{l_1,l_2}\|\|\tau_{T_r}\|<C\cdot C^{l_1+l_2+1}\cdot C^{(n_1-l_1)+(n_1-l_2)+2(d-1)+1}=C^{n_1+n_2+2d+2}.$$

As the number of internal nodes satisfies $d\le i+j$, it follows that $\|\tau_T\|<C^{3(i+j)+3}$ for $i+j>0$. Hence, taking the sum over all trees in $\mathsf{Catp}(i,j)$ one finds that

$$\|\tau_{i,j}\|\le\sum_{T\in\mathsf{Catp}(i,j)}\|\tau_T\|<\sum_{T\in\mathsf{Catp}(i,j)}C^{3(i+j)+3}=|\mathsf{Catp}(i,j)|(C^3)^{i+j+1}.$$

Each $T\in\mathsf{Catp}(i,j)$ corresponds to a unique planar tree with less than $3(i+j)+3$ total nodes, so the cardinality of the set $\mathsf{Catp}(i,j)$ is clearly bounded by the Catalan number $\mathcal{C}_{3(i+j)+3}$. Taking again a constant $q>1$ such that $\mathcal{C}_n<q^n$ for $n\gg0$, it follows that $\|\tau_{i,j}\|<(qC)^{i+j+1}$ for $i+j\gg0$. Hence τ is analytic. □

4. Cyclic structures and analytic Calabi-Yau structures

The notion of a cyclic structure on an A_∞-algebra was introduced in [Kaj07; KS09] as a non-degenerate skew-symmetric pairing $\langle -,-\rangle\colon A[1]\otimes A[1]\to\ell[2-d]$ satisfying a compatibility condition

$$\langle\mu_n(a_1,\ldots,a_n),a_{n+1}\rangle=(-1)^K\langle a_1,\mu_n(a_2,\ldots,a_{n+1})\rangle,\quad\forall n\in\mathbb{N}$$

for some appropriate Koszul sign K. In particular, a cyclic A_∞-algebra is always finite dimensional. In practice, it is easier to work with an alternative characterisation of the cyclic structures as a bimodule map [Tra08; Cho08] which is as follows.

Definition 4.1. A *d-cyclic structure* on an A_∞-algebra A is a morphism $\sigma\in\mathrm{Hom}_{A-A}(A_\Delta,A_\Delta^\vee[-d])$ such that $\sigma_{0,0}$ is a skew-symmetric isomorphism and $\sigma_{i,j}=0$ for $i+j>0$.

A pair (A, σ_A) of an A_∞-algebra and a cyclic A_∞-structure is called a cyclic A_∞-algebra, and given two such cyclic A_∞-algebras (A, σ_A) and (B, σ_B) a morphism $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B)$ is said to be cyclic precisely if $f^*\sigma_B = \sigma_A$.

A case of special interest is when A is d-cyclic for $d = 3$, and $A^0 \cong \ell$. In this case, the structure of the cyclic A_∞-algebra (A, σ_A) can be recovered from a *potential*: setting $V_A = (A^1)^*$ the cyclic structure defines an element

$$W_A := \sum_{n\in\mathbb{N}} \sigma(\mu_n(-,\dots,-))(-) \in \widehat{\mathsf{T}}_\ell V_A$$

in the *completed tensor algebra* $\widehat{\mathsf{T}}_\ell V_A := \prod_{n\geq 0} V_A^{\otimes n}$ on V_A, which can be used to recover the A_∞-structure (see e.g. [VdB15]). Likewise, given an ℓ-bimodule V and an element $W = \sum_n W_n \in \widehat{\mathsf{T}}_\ell V$ such that each W_n is invariant under cyclic permutation, there is a well-defined cyclic A_∞-algebra structure on

$$A_V = \ell \oplus V^*[-1] \oplus V[-2] \oplus \ell^*[-3].$$

Any (pre-)morphism $f \in \mathrm{hom}^0_{\mathsf{Alg}^\infty}(A, B)$ induces a graded algebra morphism $f^* \colon \widehat{\mathsf{T}}_\ell V_B \to \widehat{\mathsf{T}}_\ell V_A$, and a morphism f is cyclic if and only if $f^*(W_B) = W_A$ by a result of Kajiura [Kaj07].

In this section we consider cyclic analytic A_∞-algebras, by which we mean a pair (A, σ_A) with $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$ and σ_A a cyclic structure in the above sense. In the $d = 3$ case the analytic requirement yields an *analytic potential*, as defined in [HK19].

Proposition 4.2. *Let (A, σ) be a 3-cyclic analytic A_∞-algebra with $A^0 \cong \ell$ and pick a basis $v_1, \dots v_m$ for V_A, then the potential W_A lies in the analytic subring*

$$\widetilde{\mathsf{T}}_\ell V_A := \left\{ \sum_{n=0}^{\infty} \sum_{i_1,\dots,i_n} c_{i_1,\dots,i_n} \cdot v_{i_1} \otimes \dots \otimes v_{i_n} \in \widehat{\mathsf{T}}_\ell V_A \;\middle|\; \begin{array}{c} \text{there exists } C > 0 \text{ such that} \\ |c_{i_1,\dots,i_n}| < C^n \text{ for all } i_1,\dots,i_n \end{array} \right\}.$$

Proof. Let $a_1, \dots, a_m \in A^1$ be a dual basis to $v_1, \dots, v_m \in V_A = (A^1)^*$. Then the coefficient $c_{i_1,\dots,i_n}$ of W_A in $\widehat{\mathsf{T}}_\ell V_A$ is given by $c_{i_1,\dots,i_n} = \sigma(\mu_{n-1}(a_{i_1}, \dots, a_{i_{n-1}}))(a_{i_n})$. Because μ is analytic by assumption, there exists C_0 such that $\|\sigma\| < C_0$ and $\|\mu_n\| < C^n$ for all n, and $\|a_i\| < C_0$ for each basis vector. It follows that there is a bound for all $i_1, \dots, i_n$:

$$|c_{i_1,\dots,i_n}| = |\sigma(\mu_{n-1}(a_{i_1}, \dots, a_{i_{n-1}}))(a_{i_n})| \leq \|\sigma\|\|\mu_{n-1}\|\|a_{i_1}\| \cdots \|a_{i_n}\| < C_0 \cdot C_0^{2n-1} \cdot C_0^n = (C_0^2)^n,$$

so setting $C = C_0$, the result follows. □

Remark 4.3. Hua–Keller [HK19] use the language of quivers to define their potential. Here the base is $\ell = \prod_{v\in Q_0} \mathbb{C}v$ is a product over the nodes of a quiver Q, and the ℓ-bimodule V should be seen as the span of the arrows in Q. This identifies $\widehat{\mathsf{T}}_\ell V$ with the completed path algebra $\widehat{\mathbb{C}Q}$, and identifies the subring $\widetilde{\mathsf{T}}_\ell V_A$ with the analytic path algebra $\widetilde{\mathbb{C}Q} \subset \widehat{\mathbb{C}Q}$ in [HK19].

In the rest of this section we will show how to obtain a cyclic structure on analytic minimal models of analytic A_∞-algebras that satisfy a Calabi–Yau property. This requires us to consider a "homotopic" generalisation of cyclic structures, called a *strong homotopy inner product*.

4.1. Strong homotopy inner products.

If σ is a cyclic structure on an A_∞-algebra B then the pull-back $f^*\sigma$ along a quasi-morphism $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B)$ will, in general, not be a cyclic structure on A because both the condition

$$(f^*\sigma)_{i,j} = 0 \text{ if } i + j > 0,$$

as well as the requirement that $(f^*\sigma)_{0,0}$ is an isomorphism may be violated. One says that cyclic structures are *strict*. Because A_∞-algebras are most useful when viewed in a derived setting, it is better to use a homotopy-invariant alternative which relaxes these conditions. One such alternative was define by Cho [Cho08], which we recall here following the setup of [AT22].

Definition 4.4. Given $A \in \mathsf{Alg}^\infty$, a *closed 2-form* is an element of the subcomplex $\Omega^{2,\mathrm{cl}}(A) \subset \mathrm{hom}^\bullet_{A-A}(A_\Delta, A_\Delta^\vee)[-2]$ of shifts of bimodule morphisms ρ satisfying:

- *skew-symmetry*: for any $i, j \in \mathbb{N}$ and $a_0 \otimes \cdots \otimes a_{i+j+1} \in (A[1])^{\otimes i+j+1}$ the relation
$$\rho(a_1 \otimes \cdots \otimes \underline{a}_{i+1} \otimes \cdots \otimes a_n)(a_0) = (-1)^K \cdot \rho(a_{i+2} \otimes \cdots \otimes \underline{a}_0 \otimes \cdots \otimes a_i)(a_{i+1})$$
holds, where K is obtained from the Koszul sign rule after cyclic permutation
- *closedness*: for any $a_0 \otimes \cdots \otimes a_n \in (A[1])^{\otimes n}$ and indices $1 \leq i < j < k \leq n$ the relation
$$(-1)^{K_i} \cdot \rho(a_{j+1} \otimes \cdots \otimes \underline{a}_i \otimes \cdots \otimes a_{j-1})(a_j) + (-1)^{K_j} \cdot \rho(a_{k+1} \otimes \cdots \otimes \underline{a}_j \otimes \cdots \otimes a_{k-1})(a_k) + (-1)^{K_k} \rho(a_{i+1} \otimes \cdots \otimes \underline{a}_k \otimes \cdots \otimes a_{i-1})(a_i) = 0$$
holds, where K_i, K_j, K_k are again obtained from the Koszul sign rule after cyclic permutation.

Definition 4.5. A *strong homotopy inner product* (SHIP) is a cocycle $\rho \in \Omega^{2,\mathrm{cl}}(A)$ of some degree d such that the map $\rho_{0,0} \colon A_\Delta[1] \to A_\Delta^\vee[d-1]$ is a quasi-isomorphism.

Any cyclic structure defines a SHIP, but the latter class is much better behaved under A_∞-morphisms: given $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A, B)$ the pullback f^* on bimodule morphisms restricts to a chain map
$$f^* \colon \Omega^{2,\mathrm{cl}}(B) \to \Omega^{2,\mathrm{cl}}(A),$$
which maps SHIPs to SHIPs when f is a quasi-isomorphism. Following the philosophy of [KS09], a SHIP can be viewed as a type of noncommutative shifted-symplectic structure: one can view the Hochschild cohomologies $\mathsf{C}^\bullet(A)$ and $\mathsf{C}^\bullet(A, A^\vee)$ as vector fields and differential 1-forms on a noncommutative space, and any cocycle $\rho \in \Omega^{2,\mathrm{cl}}(A)$ defines a *contraction map*
$$\iota_{-}\rho \colon \mathsf{C}^\bullet(A) = \mathsf{C}^\bullet(A, A) \to \mathsf{C}^\bullet(A, A_\Delta^\vee), \quad \alpha \mapsto \rho \circ \overline{\alpha},$$
which is a quasi-isomorphism if ρ is a SHIP. We now generalise the notion of strong homotopy inner product to analytic A_∞-algebra, as follows.

For an analytic A_∞-algebra $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$, we define the complex of *analytic closed 2-forms*
$$\Omega^{2,\mathrm{cl,an}}(A) := \Omega^{2,\mathrm{cl}}(A) \cap \mathcal{A}(A_\Delta, A'_\Delta)[-2],$$
where we view $\mathcal{A}(A_\Delta, A'_\Delta)$ as a subspace of $\mathcal{M}(A_\Delta, A_\Delta^\vee)$ via the inclusion $A'_\Delta \hookrightarrow A_\Delta^\vee$. The assignment $A \mapsto \Omega^{2,\mathrm{cl,an}}(A)$ is again functorial, as Lemma 3.10 guarantees that there is a chain map
$$f^* \colon \Omega^{2,\mathrm{cl,an}}(B) \to \Omega^{2,\mathrm{cl,an}}(A),$$
for every analytic A_∞-morphism $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, B)$. We define the following analogue of a SHIP for analytic A_∞-algebras admitting an analytic minimal model.

Definition 4.6. Let A be an analytic A_∞-algebra A with minimal model $\mathrm{H}(A)$. Then an *analytic SHIP* is a cocycle $\rho \in \Omega^{2,\mathrm{cl,an}}(A)$ such that the map
$$\mathrm{H}(A)[1] \xrightarrow{I} A[1] \xrightarrow{\rho_{0,0}} A^\vee[1] \xrightarrow{I^\vee} \mathrm{H}(A)^\vee[1]$$
admits a continuous inverse.

As before, an analytic SHIP can be though of as a noncommutative shifted symplectic structure with contraction map
$$\iota_{-}\rho \colon \mathsf{C}^{\bullet,\mathrm{an}}(A) \to \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta),$$
which is well-defined by the discussion in §3.4. If $\mathrm{H}(A)$ is a strong minimal model, one can again show that this map is a quasi-isomorphism. In what follows we consider finite dimensional minimal A_∞-algebras, for which it suffices that $\rho_{0,0}$ is invertible and the contraction map is an isomorphism.

Proposition 4.7. *Suppose A is a finite dimensional minimal A_∞-algebra and $\rho \in \Omega^{2,\mathrm{cl,an}}(A)$ a cocycle with $\rho_{0,0}$ invertible in $\mathsf{grMod}\,\ell^e$. Then ρ is an analytic SHIP and $\iota_{-}\rho$ is an isomorphism.*

Proof. Continuity of $\rho_{0,0}^{-1}$ is automatic if A is finite dimensional, hence ρ is an analytic SHIP. It then follows from Proposition 3.14 that ρ admits an analytic inverse $\tau \in \mathrm{Hom}^{\mathrm{an}}_{A-A}(A'_\Delta, A_\Delta)$, and the induced map
$$\mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta) \to \mathsf{C}^{\bullet,\mathrm{an}}(A) \quad \xi \mapsto \tau \circ \overline{\xi}$$
is then inverse to the contraction map by bi-functoriality of the Hochschild cochain complex. □

4.2. **An analytic Darboux lemma.** Kontsevich–Soibelman [KS09] showed that when interpreted in the language of noncommutative symplectic geometry, any SHIP can be put into a standard *Darboux form*. In the algebraic setup we use here, this means that any SHIP $\rho \in \Omega^{2,\mathrm{cl}}(A)$ on a minimal A_∞-algebra $A = (A^\bullet, \mu)$ can be strictified to a genuine cyclic structure

$$\sigma = f^*\rho \in \Omega^{2,\mathrm{cl}}(A')$$

where $A' = (A^\bullet, \mu')$ is a perturbation of the A_∞-structure and $f \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(A', A)$ is an A_∞-morphism with first order term $f_1 = \mathrm{id}_A$. A Darboux lemma of this form was first shown by Cho–Lee [CL11]. More recently, Amorim–Tu [AT22] show that the morphism f can be obtained as $f = f^1$ from a solution of a differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} f^t = \alpha^t \diamond f^t \\ \quad f^0 = \mathrm{id}, \end{cases} \tag{5}$$

where $t \mapsto \alpha^t$ is certain one-parameter family of Hochschild cocycles interpolating between ρ and σ, chosen such that the solution satisfies $f^1 = f$. In what follows we give an analytic version of this result: we construct an explicit tree expression for the solution $t \mapsto f^t$ to show that $f = f^1$ and μ' are analytic if ρ and μ are analytic. We first specify the type of families we wish to use.

Let $V, W \in \mathsf{grNMod}\,\ell^e$ be normed bimodules, then a *family of multilinear maps from V to W* is a sequence $f = (f_n)_{n\in\mathbb{N}}$ of continuous functions over $t \in [0,1]$ in the space

$$\mathcal{M}_{[0,1]}(V, W) := \prod_{n\geq 0} C^0\left([0,1], \hom^{\mathrm{cont}}_{\ell^e}(V[1]^{\otimes n}, W[1])\right).$$

In particular, the values f_n^t of the f_n at each $t \in [0,1]$ define a sequence map $f^t = (f_n^t)_{n\in\mathbb{N}} \in \mathcal{M}(V, W)$. The composition $\diamond$ extends to families, as the pointwise formula

$$(f \diamond g)_n^t := (f^t \diamond g^t)_n = \sum_k \sum_{n_1+\ldots+n_k} f_k^t \circ (g_{n_1}^t \otimes \cdots \otimes g_{n_k}^t)$$

defines again a sequence of continuous functions. We say a family is differentiable if each f_n is a differentiable function, and note that the differential equation (5) makes sense for a differentiable family f and any continuous α.

Fix a finite dimensional minimal $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$. Given a family $\alpha \in \mathcal{M}_{[0,1]}(A, A)[-1]$ parametrising Hochschild cochains $\alpha^t \in \mathsf{C}_\bullet(A)$, we define for every rooted tree $T \in \mathcal{O}(n)$ a differentiable family

$$f_T \in C^0([0,1], \hom^{\mathrm{cont}}_{\ell^e}(A[1]^{\otimes n}, A[1]))$$

as follows. The unique tree $T \in \mathcal{O}(1)$ determines the constant map $f_T^t = \mathrm{id}_{A[1]}$, and for $T \in \mathcal{O}(n)$ with $n > 1$ the map f_T is given by the recursive formula:

$$f_T^t = \int_0^t \alpha_k^\tau \circ (f_{T_1}^\tau \otimes \cdots \otimes f_{T_k}^\tau)\mathrm{d}\tau,$$

where $(T_1, \ldots, T_k)$ are the subtrees emanating from the first internal node as in §2.4. The following lemma shows that these functions can be combines to a solution for the differential equation of [AT22].

Lemma 4.8. *Suppose $\alpha_0^t = \alpha_1^t = 0$. Then the family of pre-morphisms $f \in \mathcal{M}_{[0,1]}(A, A)$ defined by*

$$f_n = \sum_{T\in\mathcal{O}(n)} f_T,$$

is a solution to the differential equation (5).

Proof. For $t = 0$ we have $f_T^0 = \mathrm{id}_{A[1]}$ for the unique tree $T \in \mathcal{O}(1)$ and $f_T^0 = 0$ for $T \in \mathcal{O}(n)$ with $n > 1$, as the latter are given by an empty integral. Hence f satisfies the initial condition $f^0 = \mathrm{id}$ in (5) and just remains to check that it satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f_n^t = (\alpha^t \diamond f^t)_n := \sum_{k=1}^n \sum_{n_1+\cdots+n_k=n} \alpha_k^t \circ (f_{n_1} \otimes \cdots \otimes f_{n_k}), \tag{6}$$

which is well-defined because $\alpha_0^t = 0$. Applying the recursive formula for f_T it follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f_n^t &= \sum_{T\in\mathcal{O}(n)} \frac{\mathrm{d}}{\mathrm{d}t} f_T^t \\
&= \sum_{k=2}^{n} \sum_{n_1+\cdots+n_k=n} \sum_{T_i\in\mathcal{O}(n_i)} \alpha_k^t \circ (f_{T_1}^t \otimes \cdots \otimes f_{T_k}^t) \\
&= \sum_{k=2}^{n} \sum_{n_1+\cdots+n_k=n} \alpha_k^t \circ \left(\left(\sum_{T_1\in\mathcal{O}(n_1)} f_{T_1}^t \right) \otimes \cdots \otimes \left(\sum_{T_k\in\mathcal{O}(n_k)} f_{T_k}^t \right) \right) \\
&= \sum_{k=2}^{n} \sum_{n_1+\cdots+n_k=n} \alpha_k^t \circ (f_{n_1} \otimes \cdots \otimes f_{n_k}),
\end{aligned}$$

which is equal to (6) under the assumption $\alpha_1^t = 0$. Hence f is a solution to (5) as claimed. □

For each n we endow the space $C^0([0,1], \hom_{\ell^e}^{\mathrm{cont}}(V[1]^{\otimes n}, W))$ with the supremum norm

$$\|g\|_\infty = \sup_{t\in[0,1]} \|g^t\|.$$

and say that a family $f = (f_n)_{n\in\mathbb{N}} \in \mathcal{M}_{[0,1]}(V,W)$ is *uniformly analytic* if there exists $C > 0$ such that the components f_n satisfy the following bound for all $n \geq 1$

$$\|f_n\|_\infty < C^n.$$

If a family f of morphisms is uniformly analytic, then for all $t \in [0,1]$ and $n \geq 1$

$$\|f_n^t\| \leq \sup_{\tau\in[0,1]} \|f_n^\tau\| \leq \|f_n\|_\infty < C^n,$$

so $f^t \in \mathcal{A}(V,W)$ is analytic for each $t \in [0,1]$, with a uniform choice of bounded constant C. We have the following.

Lemma 4.9. *Suppose α^t is uniformly analytic, then f^t is also uniformly analytic.*

Proof. Let $C > 0$ be such that $\|\alpha_n\|_\infty < C^n$ for all $n \in \mathbb{N}$. We will show that $\|f_T\|_\infty \leq C^{n+d-1}$ holds for each $T \in \mathcal{O}(n,d)$ by induction over n. For the unique tree $T \in \mathcal{O}(1) = \mathcal{O}(1,0)$ then

$$\|f_T\|_\infty = \sup_{t\in[0,1]} \|\mathrm{id}\| = 1 = C^{1-0-1},$$

which establishes the base case. For the induction step let $n > 1$ and assume the statement holds for all $T \in \mathcal{O}(m,d)$ with $m < n$. Then $T \in \mathcal{O}(n,d)$ is built out of $l \geq 2$ subtrees $T_1, \ldots, T_l \subset T$ with $T_i \in \mathcal{O}(n_i, d_i)$ satisfying $n_i \neq 0$, $\sum n_i = n$, and $\sum_{i=1}^k d_i = d - 1$. The because each n_i is less than n it follow by the induction hypothesis that

$$\begin{aligned}
\|f_T\|_\infty &= \sup_{t\in[0,1]} \left\| \int_0^t \alpha_k^\tau \circ (f_{T_1}^\tau \otimes \cdots \otimes f_{T_l}^\tau) \mathrm{d}\tau \right\| \\
&\leq \sup_{\tau\in[0,1]} \|\alpha_k^\tau\| \|f_{T_1}^\tau\| \cdots \|f_{T_k}^\tau\| \\
&\leq \|\alpha_k\|_\infty \|f_{T_1}\|_\infty \cdots \|f_{T_k}\|_\infty \\
&\leq C^k \cdot C^{n_1+d_1-1} \cdots C^{n_k+d_k-1} = C^{n+d-1}
\end{aligned}$$

which verifies the induction. Because the valency of every internal node of a tree in $\mathcal{O}(n)$ is at least 3, one easily verifies that the maximal number of internal nodes is $d \leq n$. Hence, the sum over all trees is bounded by

$$\|f_n\|_\infty \leq \sum_{T\in\mathcal{O}(n)} \|f_T\|_\infty \leq |\mathcal{O}(n)| \cdot C^{2n-1}$$

where $|\mathcal{O}(n)|$ denotes the cardinality of $\mathcal{O}(n)$, which is bounded by the Catalan number $\mathcal{C}_{2n}$. These Catalan numbers are bounded by q^n for some fixed $q > 0$, so we obtain a bound

$$\|f_n\|_\infty < \mathcal{C}_{2n} \cdot C^{2n} < q^n C^{2n} = (qC^2)^n.$$

which shows that f is uniformly analytic. □

With the above convergence and the results of the previous section we arrive at an analytic generalisation of the Darboux lemma appearing in [KS08; CL11; AT22].

Lemma 4.10 (Analytic Darboux)**.** *Let $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$ be minimal and finite dimensional and suppose $\rho \in \Omega^{2,\mathrm{cl},\mathrm{an}}(A)$ is a SHIP of degree $2-d$. Then:*

1. *there exists an A_∞-algebra A^ρ and an analytic A_∞-isomorphism $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A^\rho, A)$ such that*
$$\sigma^\rho := f^*\rho$$
is a cyclic structure.
2. *if $\tau \in \Omega^{2,\mathrm{cl},\mathrm{an}}(A)$ is another SHIP such that the image $\tau - \rho$ along the inclusion $\Omega^{2,\mathrm{cl},\mathrm{an}}(A) \to \Omega^{2,\mathrm{cl}}(A)$ is exact, then there is a cyclic analytic A_∞-isomorphism*
$$(A^\rho, \sigma^\rho) \cong_{\mathrm{an,cyc}} (A^\tau, \sigma^\tau).$$

Effectively, every class in $\mathrm{H}^{2-d}\Omega^{2,\mathrm{cl}}(A)$ admitting an analytic lift defines a cyclic analytic model.

Proof. (1) Let $\sigma := \rho_{0,0}$ and consider for each $t \in [0,1]$ the analytic closed 2-form

$$\rho^t := \sigma + t(\rho - \sigma) \in \Omega^{2,\mathrm{cl},\mathrm{an}}(A), \tag{7}$$

and use Lemma 4.17 to fix an analytic Hochschild cocycle $\xi \in \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta)$ with image $S(\xi) = \rho - \sigma$ along the surjective map $S\colon \to \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta) \to \Omega^{2,\mathrm{cl},\mathrm{an}}(A)[1]$. It follows from the formula (10) that ξ_1 is given on generators by

$$\xi_1(x_1)(x_0) = \tfrac{1}{2}(\rho - \sigma)(\underline{x}_0)(x_1) = 0,$$

so $\xi_1 = 0$, and likewise $\xi_0 = 0$. Because A is minimal the component $\rho^t_{0,0} = \rho_{0,0}$ is invertible and hence ρ^t is an isomorphism by Proposition 3.14. Hence, for every t the Hochschild cocycle $\alpha^t \in \mathsf{C}^{\bullet,\mathrm{an}}(A)$ defined by the equation

$$\alpha^t := (\rho^t)^{-1} \circ \overline{\xi},$$

satisfies the equation $\iota_{\alpha^t}\rho^t = \xi$. It is straightforward to verify that the sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ of functions $\alpha_n\colon t \mapsto \alpha^t_n$ is in $\mathcal{M}_{[0,1]}(A, A)$. Indeed, each component ρ^t_n varies linearly in t, and it is easy to verify from the recursive identity (7) that the function $t \mapsto (\rho^t)^{-1}_n$ is a polynomial of degree at most n in t. Hence, α is a well-defined family of Hochschild cocycles, which satisfies

$$\alpha^t_1 = (\rho^t)^{-1}_{0,0} \circ \xi_1 = 0$$

because $\xi_1 = 0$, and likewise $\alpha^t_0 = ((\rho^t)^{-1} \circ \overline{\xi})_0 = 0$ by definition. To show that it is analytic we note that for each $t \in [0,1]$ we have

$$\left\|\rho^t_{0,0}\right\| = \|\rho_{0,0}\|, \quad \left\|\rho^t_{i,j}\right\| = t\left\|\rho^t_{i,j}\right\| \le \|\rho_{i,j}\| \quad (i + j > 0),$$

so that $\|\rho^t_n\| < C_1^n$ for a uniform choice of constant $C_1 > 0$ independent of t. It then follows from the proof of Proposition 3.14 that there is a uniform constant $C_2 > 0$ such that $\left\|(\rho^t)^{-1}_n\right\| < C_2^n$ for all t, and because ξ is analytic there is some $C_3 > 0$ bounding the composition

$$\left\|\alpha^t_n\right\| = \left\|((\rho^t)^{-1} \circ \overline{\xi})_n\right\| < C_3^n,$$

for all t. Therefore $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is uniformly analytic, and it follows from Lemma 4.8 and Lemma 4.9 that there exists a uniformly analytic family of pre-morphisms $f = (f_n)_{n\in\mathbb{N}}$ satisfying the differential equation (5). It then follows from [AT22, Proposition 2.32] that for each t the pullback $(f^t)^*\colon \Omega^{2,\mathrm{cl},\mathrm{an}}(A) \to \Omega^{2,\mathrm{cl},\mathrm{an}}(A)$ satisfies

$$(f^t)^*\rho^t = \sigma,$$

so in particular f^1 pulls back $\rho^1 = \rho$ to the constant 2-form $(f^1)^*\rho = \sigma$. Because $f^1_1 = \mathrm{id}$, the pre-morphism f^1_1 admits an analytic inverse by Lemma 2.14. Hence if we define A^ρ as the A_∞-algebra with product defined by the analytic Hochschild cocycle

$$\mu^\rho := (f^1)^{-1} \circ \overline{\mu}_A \circ \widehat{f^1} \in \mathsf{C}^{\bullet,\mathrm{an}}(A),$$

it follows as in [AT22, Proposition 2.32] that $f^1 \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A^\rho, A)$ is an isomorphism and $\sigma^\rho = \sigma = (f^1)^*\rho$ is a cyclic structure as claimed.

(2) Now let τ be another analytic SHIP defining a cyclic analytic A_∞-algebra (A^τ, σ^τ), and let $g \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A^\tau, A)$ denote the isomorphism for which $\sigma^\tau = g^*\tau$ is cyclic. Suppose that $\tau - \rho$ is exact in the complex $\Omega^{2,\mathrm{cl}}(A)$, then as in [AT22, Lemma 2.33] we note that $\tau_{0,0} = \rho_{0,0}$, so that for every $t \in [0,1]$ the analytic closed 2-form

$$\rho^t := \tau + t(\rho - \tau) \in \Omega^{2,\mathrm{cl},\mathrm{an}}(A),$$

has $\rho^t_{0,0} = \tau_{0,0} = \rho_{0,0}$ an isomorphism, and is therefore a SHIP. Fixing a cocycle $\xi \in \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta)$ such that $S(\xi) = \rho - \tau$ as before, there again exists a family of Hochschild cocycles α which satisfy the equation $\iota_{\alpha^t}\rho^t = \xi$ for each $t \in [0,1]$. The norms of ρ^t_n are bounded by $\max\{\|\tau_n\|, \|\rho_n\|\}$ for all t, so it again follows that α is uniformly analytic. Hence there exists a family of pre-morphisms h with $h^0 = \mathrm{id}$ solving the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} h^t = \alpha^t \diamond h^t.$$

Because $\rho - \tau$ is exact in $\Omega^{2,\mathrm{cl}}(A)$ it follows that ξ is exact in $\mathsf{C}^\bullet(A, A^\vee)$ Therefore [AT22, Lemma 2.33] implies that the analytic pre-morphism h^1 is an A_∞-morphism such that $(h^1)^*\tau = \rho$, hence lies in $h^1 \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, A)$. Now the composition $g^{-1} \circ h^1 \circ f^1 \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A^\tau, A^\rho)$ is an analytic A_∞-isomorphism satisfying

$$(g^{-1} \circ h^1 \circ f^1)^*\sigma^\tau = (h^1 \circ f^1)^*\tau = (f^1)^*\rho = \sigma^\rho. \qquad \square$$

4.3. **Analytic Calabi–Yau structures and SHIPs.** Cyclic structures express a type of Calabi–Yau symmetry, which can be formalised using the notion of a *right Calabi–Yau structure* in the terminology of [BD19]. For ordinary A_∞-algebras, these structures can be defined as follows.

Definition 4.11. A *right d-Calabi–Yau structure* on an A_∞-algebra $A = (A, \mu) \in \mathsf{Alg}^\infty$ is a cocycle $\phi \in Z^{-d}\mathsf{C}^\bullet_\lambda(A)$ for which the map $A \to A^\vee[-d] \colon a \mapsto \phi_0(\mu_2(a, -))$ is a quasi-isomorphism.

It was shown by Cho–Lee [CL11] (and [KS08] in the symplectic language) that right CY structures and SHIPs on an A_∞-algebra A are equivalent notions. More precisely, the two structures are identified up to homotopy by a map $\Omega^{2,\mathrm{cl}}(A)[1] \to \mathsf{C}^\bullet_\lambda(A)$. This map can be constructed in two steps.

Proposition 4.12 (see [CL11][AT22]). *The map* $S \colon \mathsf{C}^\bullet(A, A^\vee_\Delta) \to \Omega^{2,\mathrm{cl}}(A)[1]$ *defined by*

$$(S\xi)(\mathbf{x} \otimes \underline{v} \otimes \mathbf{y})(w) = \xi(\mathbf{x} \otimes v \otimes \mathbf{y})(w) - \xi(\mathbf{y} \otimes w \otimes \mathbf{x})(v),$$

defines a chain map fitting into a short exact sequence of chain complexes

$$0 \longrightarrow \mathsf{C}^\bullet_\lambda(A) \longrightarrow \mathsf{C}^\bullet(A, A^\vee_\Delta) \xrightarrow{S} \Omega^{2,\mathrm{cl}}(A)[1] \longrightarrow 0 \tag{8}$$

where the first map is the inclusion of the cyclic cochains. Moreover, this short exact sequence is natural over $A \in \mathsf{Alg}^\infty$ *along the induced pullbacks on Hochschild cochains and bimodule maps.*

The above shows that the map S induces an isomorphism between $\Omega^{2,\mathrm{cl}}(A)[1]$ and the quotient complex $\mathsf{C}^\bullet(A, A^\vee_\Delta)/\mathsf{C}^\bullet_\lambda(A)$. As observed by Kontsevich–Soibelman, the latter complex can be related to the negative cyclic complex via the following short exact sequence.

Proposition 4.13 ([KS08][AT22]). *There is a short exact sequence of chain complexes*

$$0 \longrightarrow \mathsf{C}^\bullet(A, A^\vee)/\mathsf{C}^\bullet_\lambda(A) \xrightarrow{\mathrm{id}-t} (\mathsf{B}A/\ell)^\vee \xrightarrow{N} \mathsf{C}^\bullet_\lambda(A) \longrightarrow 0. \tag{9}$$

where the maps are the compositions of the norm operator $(N\xi)_n = (1 + t + \cdots + t^n)\xi_n$ *and the operator* $\mathrm{id} - t$ *via the identification*

$$(\mathsf{B}A/\ell)^\vee \cong \prod_{n\geq 0} \hom_{\ell^e}(A[1]^{\otimes n+1}, \ell)[1] \cong \prod_{n\geq 0} \hom_{\ell^e}(A[1]^{\otimes n}, A^\vee) \cong \mathcal{M}(A, A^\vee)[-1].$$

The short exact sequence is moreover natural in $A \in \mathsf{Alg}^\infty$ *via the induced maps.*

It is well known that the normalised bar complex $\mathsf{B}A/\ell$ is acyclic if A is unital, hence so is its dual. Choosing a contracting homotopy $s\colon (\mathsf{B}A/\ell)^\vee[1] \to (\mathsf{B}A/\ell)^\vee$, one obtains a quasi-isomorphism

$$\Omega^{2,\mathrm{cl}}(A)[2] \cong (\mathsf{C}^\bullet(A, A^\vee_\Delta)/\mathsf{C}^\bullet_\lambda(A))[1] \xrightarrow{N\circ s\circ(\mathrm{id}-t)} \mathsf{C}^\bullet_\lambda(A),$$

after composing with the inverse of S. One can show that this map identifies right CY structures with SHIPs, leading to the following statement for unital A_∞-algebras.

Proposition 4.14 ([AT22, Corollary 2.24])**.** *The map* $\Omega^{2,\mathrm{cl}}(A)[2] \xrightarrow{\sim} \mathsf{C}^\bullet_\lambda(A)$ *is a quasi-isomorphism, is natural with respect to* A_∞*-morphisms, and identifies right CY structures with SHIPs.*

We now want to show a version of the above for analytic A_∞-algebras, involving the analytic SHIPs and analytic right Calabi–Yau structures, by which we mean the following.

Definition 4.15. Let $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$, then a right Calabi–Yau structure ϕ is *analytic* if it lies in the subcomplex $\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A) \subset \mathsf{C}^\bullet_\lambda(A)$.

Example 4.16. Let $(A, D, \|\cdot\|)$ be a normed DG algebra and $\lambda\colon A^n \to \mathbb{C}$ a $\|\cdot\|$-bounded linear functional such that

$$\lambda(Da) = 0, \quad \lambda(a_1 \cdot a_2) = (-1)^{|a_1||a_2|}\lambda(a_2 \cdot a_1).$$

Then the element $\phi \in \mathsf{C}^{-n,\mathrm{an}}_\lambda(A)$ defined by $\phi_0 = \lambda$ and $\phi_n = 0$ for $n > 0$ is a cocycle, and it is a right n-Calabi–Yau structure if the cohomology pairing $([a_1], [a_2]) \mapsto \lambda([a_1 \cdot a_2])$ is nondegenerate.

We will show that the map $\Omega^{2,\mathrm{cl}}(A)[2] \to \mathsf{C}^\bullet_\lambda(A)$ restricts to a map on analytic elements by showing that the short exact sequences (8) and (9) restrict to the relevant analytic subspaces. The analytic version of Proposition 4.12 is as follows.

Lemma 4.17. *Let* A *be an analytic* A_∞*-algebra, then the map* S *fits into a short-exact sequence*

$$0 \longrightarrow \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A) \longrightarrow \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta) \xrightarrow{S} \Omega^{2,\mathrm{cl},\mathrm{an}}(A)[1] \longrightarrow 0$$

which is moreover natural with respect to analytic A_∞*-morphisms* $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, B)$.

Proof. By construction $\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A) = \mathsf{C}^\bullet_\lambda(A) \cap \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta)$ is a subcomplex of $\mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta)$, and this subcomplex is the kernel of the restriction of S to $\mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta)$ by Proposition 4.12. Hence it suffices to show that S restricts to a surjective map $S\colon \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta) \to \Omega^{2,\mathrm{cl},\mathrm{an}}(A)[1]$.

Suppose $\phi \in \mathsf{C}^{\bullet,an}(A, A'_\Delta)$, and pick $C > 0$ such that $\|\phi_n\| < C^n$ for all $n \geq 1$. Then for any element $a \in A[1]^{\otimes i} \otimes A[1] \otimes A[1]^{\otimes j}$ and any decomposition $a = \sum_k \mathbf{x}_k \otimes \underline{v}_k \otimes \mathbf{y}_k$

$$\begin{aligned}\|(S\phi)_{i,j}(a)\| &\leq \sum_k \sup_{\|w\|=1} \|(S\phi)_{i,j}(\mathbf{x}_k \otimes \underline{v}_k \otimes \mathbf{y}_k)(w)\| \\ &\leq \sum_k \sup_{\|w\|=1} (\|\phi_{i+j+1}(\mathbf{x}_k \otimes \underline{v}_k \otimes \mathbf{y}_k)(w)\| + \|\phi_{i+j+1}(\mathbf{y}_k \otimes \underline{w} \otimes \mathbf{x}_k)(v_k)\|) \\ &\leq 2\|\phi_{i+j+1}\| \cdot \sum_k \|\mathbf{x}_k\|\|v_k\|\|\mathbf{y}_k\|.\end{aligned}$$

Taking the infimum over all decompositions of a, it follows that $\|(S\phi)_{i,j}(a)\| \leq \|\phi_{i+j+1}\| \cdot \|a\|$, so that $\|(S\phi)_{i,j}\| \leq 2\|\phi_{i+j+1}\| < (2C)^{i+j+1}$. It follows that by Lemma 3.6 that $S\phi$ is analytic.

To show the surjectivity of S we use the map[1] $h\colon \Omega^{2,\mathrm{cl}}(A)[1] \to \mathsf{C}^\bullet(A, A^\vee)$ defined in [AT22, Equation 11], which is defined by the equations

$$(h\rho)(x_1 \otimes \cdots \otimes x_n)(x_0) = \sum_{i=1}^n \frac{(-1)^\#}{n+1} \rho(x_{i+1}, \ldots, x_n, \underline{x}_0, \ldots, x_{i-1})(x_i), \tag{10}$$

[1]Note that this is not a chain map, hence does not split the exact sequence of chain complexes.

which satisfies $S \circ h = \mathrm{id}$. Now suppose $\rho \in \Omega^{2,\mathrm{cl},an}(A)$ is an analytic closed 2-form, and let $C > 0$ be a constant such that $\|\rho_{i,j}\| < C^{i+j+1}$ for all $i, j \in \mathbb{N}$. Then for any element $a \in A[1]^{\otimes n}$ and decomposition $a = \sum_k a_{k,1} \otimes \cdots \otimes a_{k,n}$ there is an inequality

$$\begin{aligned}\|(h\rho)_n(a)\| &\leq \sup_{\|a_0\|=1} \sum_k \|(h\rho)_n(a_{k,1} \otimes \cdots \otimes a_{k,n})(a_0)\| \\ &\leq \frac{1}{n+1} \cdot \sum_k \sum_{i=1}^n \sup_{\|a_0\|=1} \|\rho_{n-i,i-1}(a_{k,i+1}, \ldots, a_{k,n}, \underline{a}_0, a_{k,1}, \ldots, a_{k,i-1})(a_{k,i})\| \\ &\leq \frac{1}{n+1} \cdot \sum_{i=1}^n \|\rho_{n-i,i-1}\| \cdot \sum_k \|a_{k,1}\| \cdots \|a_{k,n}\| \\ &< C^n \cdot \sum_k \|a_{k,1}\| \cdots \|a_{k,n}\|.\end{aligned}$$

Taking the infimum over all decompositions of a, it follows that $\|(h\rho)_n(a)\| \leq C^n \|a\|$, which shows that $h\rho$ is analytic. Hence h defines a map $\Omega^{2,\mathrm{cl},\mathrm{an}}(A)[1] \to \mathsf{C}^\bullet(A, A'_\Delta)$ which satisfies $S \circ h = \mathrm{id}$, showing that S is surjective.

Finally, if $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(A, B)$ is an analytic morphism, it follows from Lemma 3.10 and the subsequent discussion that the pullback morphisms f^* preserve analytic elements, so naturality follows. □

Corollary 4.18. *The map S induces an isomorphism $\mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta)/\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A) \xrightarrow{\sim} \Omega^{2,\mathrm{cl},\mathrm{an}}(A)[1]$.*

For the analytic analogue of Proposition 4.13 we note that $(\mathsf{B}A/\ell)^\vee \cong \mathcal{M}(A, A^\vee)[-2]$, and define

$$(\mathsf{B}A/\ell)^{\vee,\mathrm{an}} := (\mathsf{B}A/\ell)^\vee \cap \mathcal{A}(A, A')[-2],$$

as the subspace obtained from $\mathcal{A}(A, A')[-2]$ via this identification. The following now holds.

Lemma 4.19. *Let A be an analytic A_∞-algebra, then there is a short exact sequence*

$$0 \longrightarrow \mathsf{C}^{\bullet,\mathrm{an}}(A, A')/\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A) \xrightarrow{\mathrm{id}-t} (\mathsf{B}A/\ell)^{\mathrm{an}} \xrightarrow{N} \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A) \longrightarrow 0. \tag{11}$$

which is natural with respect to analytic A_∞-morphisms.

Proof. The operator $\mathrm{id} - t$ restricts to $\mathcal{A}(A, A')[-1] \subset \mathcal{M}(A, A^\vee)[-1]$ and clearly has the set of cyclic analytic elements as its kernel. Hence, it immediately follows that it defines an injective chain map

$$\mathsf{C}^{\bullet,\mathrm{an}}(A, A')/\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A) \xrightarrow{\mathrm{id}-t} (\mathsf{B}A/\ell)^\vee \cap \mathcal{A}(A, A')[-1] = (\mathsf{B}A/\ell)^{\vee,\mathrm{an}}.$$

To see that the norm operator likewise preserves $\mathcal{A}(A, A')$, note that if ϕ satisfies $\|\phi_n\| < C^n$ then

$$\|(N\phi)_n\| \leq \|\phi_n\| + \cdots + \|(t^n\phi)_n\| = (1+n)\|\phi_n\| < (1+n)C^n < (3C)^n,$$

so that $N\phi$ is again analytic. Therefore it follows that N defines a chain map

$$(\mathsf{B}A/\ell)^{\vee,\mathrm{an}} \xrightarrow{N} \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A).$$

It is moreover surjective, as every $\phi \in \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A)$ is the image of the sequence $(\frac{1}{n+1}\phi_n)_{n\in\mathbb{N}} \in \mathcal{A}(A, A')[-1]$. To see that the sequence (11) is exact at the middle term, one can consider the inverse operator

$$f \colon \ker N \to \mathsf{C}^{\bullet,\mathrm{an}}(A, A'_\Delta)/\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A), \quad (f\xi)_n = \frac{1}{n+1}(t + 2t^2 + \cdots + nt^n)\xi_n,$$

which maps an analytic element with $\|\xi_n\| < C^n$ for $n \geq 1$ to an element $f\xi$ satisfying $\|(f\xi)_n\| \leq \frac{n^2}{n+1}\|\xi_n\| < (2C)^n$, which is therefore again analytic. It follows that (11) is a short exact sequence as required. Naturality again follows from the naturality in (11) □

To construct the map $\Omega^{2,\mathrm{cl},\mathrm{an}}(A)[2] \to \mathsf{C}^\bullet_\lambda(A)$ it now suffices to find a contracting homotopy for $(\mathsf{B}A/\ell)^\vee$ which restricts to a contracting homotopy for $(\mathsf{B}A/\ell)^{\vee,\mathrm{an}}$. For this one can take e.g. the dual of the degeneracy map in [Lod92, Proposition-definition 1.1.12], which is given by the formula

$$s\colon (\mathsf{B}A/\ell)^\vee \to (\mathsf{B}A/\ell)^\vee[-1], \quad (sg)_n(a_1,\dots,a_n) = g_{n+1}(1,a_1,\dots,a_n),$$

where 1 is any choice of unit in A. The following lemma shows that this restricts to analytic elements.

Lemma 4.20. *Let A be a unital analytic A_∞-algebra. Then the degeneracy map restricts to a contracting homotopy $s\colon (\mathsf{B}A/\ell)^{\vee,\mathrm{an}} \to (\mathsf{B}A/\ell)^{\vee,\mathrm{an}}[-1]$.*

Proof. Let $f = (f_n)_{n\in\mathbb{N}} \in \mathcal{A}(A,A')$ be an analytic sequence, and pick $C > 0$ such that $\|f_n\| < C^n$ for all $n \geq n$. Then the corresponding element $g \in (\mathsf{B}A/\ell)^\vee$ is the map defined by $g_n(a_1,\dots,a_n) = f_{n-1}(a_1,\dots,a_{n-1})(a_n)$, hence

$$(sg)_n(a_1,\dots,a_n) = g_{n+1}(1,a_0,\dots,a_n) = f_n(1,a_1,\dots,a_{n-1})(a_n),$$

and the element $sf \in \mathcal{A}(A,A')$ corresponding to sg is given by $(sf)_n(a) = f_{n+1}(1\otimes a)$. Clearly this now satisfies the bound

$$\|(sf)_n(a)\| = \|f_{n+1}(1\otimes a)\| \leq \|f_{n+1}\|\|1\|\|a\| < C^{n+1}\cdot\|a\| < (C^2)^{n+1}\cdot\|a\|,$$

hence sf is analytic as claimed. □

Corollary 4.21. *For a unital analytic A_∞-algebra A there is a natural quasi-isomorphism*

$$\Omega^{2,\mathrm{cl},\mathrm{an}}(A)[2] \cong (\mathsf{C}^{\bullet,\mathrm{an}}(A,A'_\Delta)/\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A))[-1] \xrightarrow{N\circ s\circ(\mathrm{id}-t)} \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(A),$$

which maps analytic SHIPs to analytic right CY structures.

Example 4.22. Let (A,μ) be the analytic A_∞-algebra obtained from a normed DG algebra with ϕ an analytic right CY structure obtained from a bounded linear functional $\lambda\colon A^n \to \mathbb{C}$ as in Example 4.16. Then ϕ is the image of the analytic SHIP $\rho \in \Omega^{2,\mathrm{cl},\mathrm{an}}(A)$ defined by $\rho_0(a)(b) = \lambda(\mu_2(a,b))$ and $\rho_n = 0$ for $n > 0$: one has $\rho = S(\xi)$ for the analytic cocycle $\xi \in \mathsf{C}^{\bullet,\mathrm{an}}(A,A'_\Delta)$ with $\xi_1(a)(b) = \frac{1}{2}\lambda(a\cdot b)$ and $\xi_n = 0$ otherwise, and this cocycle maps to the Connes cocycle $(N\circ s\circ(\mathrm{id}-t)\xi)$ with

$$(N\circ s\circ(\mathrm{id}-t)\xi)_0(a) = \xi_1(1)(a) - (-1)^{|a|}\xi_1(a)(1) = \tfrac{1}{2}\lambda(\mu_2(1,a) - (-1)^{|a|}\mu_2(a,1)) = \lambda(a)$$

and vanishing higher terms. This example appears in the non-analytic case in [AT22, Example 2.25].

As result of the corollary, we find that every analytic SHIP is determined up to homotopy by an analytic right Calabi–Yau structure. Using the corollary above, we can restate the analytic Darboux lemma in terms of Calabi–Yau structures, which yields our main theorem.

Theorem 4.23. *Suppose $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$ is compact and admits a unital analytic minimal model. Then:*

1. *every analytic right d-CY structure ϕ, determines a d-cyclic analytic minimal model $(\mathrm{H}(A)^\phi, \sigma^\phi)$,*
2. *if ϕ,ψ are analytic right CY structures such that $[\psi]-[\phi]$ maps to 0 along the map $\mathrm{HC}^{-d,\mathrm{an}}_\lambda(A) \to \mathrm{HC}^{-d}_\lambda(A)$, then there is a cyclic analytic A_∞-isomorphism*

$$(\mathrm{H}(A)^\phi, \sigma^\phi) \cong_{\mathrm{an,cyc}} (\mathrm{H}(A)^\psi, \sigma^\psi)$$

3. *if $B \in \mathsf{Alg}^{\infty,\mathrm{an}}$ is also compact with a unital analytic minimal model, and $g\colon B \to A$ is an analytic quasi-isomorphism, then there is an induced a cyclic analytic A_∞-isomorphism*

$$(\mathrm{H}(B)^{g^*\phi}, \sigma^{g^*\phi}) \cong_{\mathrm{an,cyc}} (\mathrm{H}(A)^\phi, \sigma^\phi).$$

Effectively, there is a canonical cyclic analytic minimal model for every nondegenerate class in $\mathrm{HC}^{-d}_\lambda(A)$ which admits an analytic lift.

Proof. (1) Let $\mathrm{H}(A) \in \mathsf{Alg}^{\infty,\mathrm{an}}$ be a finite dimensional analytic minimal model of A with diagram

$$\mathrm{H}(A) \underset{P}{\overset{I}{\rightleftarrows}} A,$$

for analytic morphisms I and P. Given an analytic right CY structure $\phi \in \mathsf{C}_\lambda^{-d,\mathrm{an}}(A)$ it is immediate that the induced map $A \to A^\vee[-d]$ induces an isomorphism $\mathrm{H}(A) \to \mathrm{H}(A)^\vee[-d]$, which coincides with the morphism induced by $I^*\phi \in \mathsf{C}_\lambda^{\bullet,\mathrm{an}}(\mathrm{H}(A))$. Hence, $I^*\phi$ is an analytic right CY structure on $\mathrm{H}(A)$.

Because $\mathrm{H}(A)$ is unital, it follows by Corollary 4.21 that there exists an analytic strong homotopy inner product $\rho^\phi \in Z^{2-d}\Omega^{2,\mathrm{cl},\mathrm{an}}(A)$ with image in $\mathsf{C}_\lambda^{-d,\mathrm{an}}(A)$ homotopic to $I^*\phi$. Therefore Lemma 4.10 determines a cyclic A_∞-algebra $(\mathrm{H}(A)^\phi, \sigma^\phi) = (\mathrm{H}(A)^{\rho^\phi}, \sigma^{\rho^\phi})$ and an analytic A_∞-isomorphism $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathrm{H}(A)^\phi, \mathrm{H}(A))$ such that $f^*\rho^\phi = \sigma^\phi$. Because $\mathrm{H}(A)$ is finite dimensional, it follows by Lemma 2.14 the map f is invertible in $\mathsf{Alg}^{\infty,\mathrm{an}}$, and we obtain a diagram

$$\mathrm{H}(A)^\phi \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \mathrm{H}(A) \underset{P}{\overset{I}{\rightleftarrows}} A,$$

which exhibits $(\mathrm{H}(A)^\phi, \sigma^\phi)$ as a cyclic minimal model of A.

(2) Let ψ is a second right CY structure and pick a lift $\rho^\psi \in \Omega^{2,\mathrm{cl},\mathrm{an}}(\mathrm{H}(A))$ for $I^*\psi$, yielding a cyclic minimal model $(\mathrm{H}(A)^\psi, \sigma^\psi)$ as above. Suppose $[\phi] - [\psi]$ maps to 0 in $\mathrm{HC}_\lambda^{-d}(A)$, then it follows by naturality that $[I^*\phi] - [I^*\psi]$ maps to 0 in $\mathrm{HC}_\lambda^{-d}(\mathrm{H}(A))$ and considering the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^{2-d}\Omega^{2,\mathrm{cl},\mathrm{an}}(\mathrm{H}(A)) & \longrightarrow & \mathrm{HC}_\lambda^{-d,\mathrm{an}}(\mathrm{H}(A)) \\
\downarrow & & \downarrow \\
\mathrm{H}^{2-d}\Omega^{2,\mathrm{cl}}(\mathrm{H}(A)) & \longrightarrow & \mathrm{HC}_\lambda^{-d}(\mathrm{H}(A))
\end{array}$$

it is also clear that $[\rho^\phi] - [\rho^\psi]$ maps to 0 in $\Omega^{2,\mathrm{cl}}(\mathrm{H}(A))$. Hence it follows from Lemma 4.10 that there is a cyclic analytic A_∞-isomorphism

$$(\mathrm{H}(A)^\psi, \sigma^\psi) \cong_{\mathrm{an,cyc}} (\mathrm{H}(A)^\phi, \sigma^\phi)$$

between the minimal models as claimed.

(3) Let $B \in \mathsf{Alg}^{\infty,\mathrm{an}}$ admit a finite dimensional analytic unital minimal model $\mathrm{H}(B)$ and suppose $g\colon B \to A$ is an analytic quasi-isomorphism, then $g^*\phi \in \mathsf{C}_\lambda^{\bullet,\mathrm{an}}(B)$ is again right CY and induces a cyclic minimal model $\mathrm{H}(B)^{g^*\phi}$. The map g induces an analytic A_∞-isomorphism $\mathrm{H}(g)$ fitting into a commutative diagram

$$\begin{array}{ccccc}
B & \xrightarrow{\quad g \quad} & A & & \\
I\uparrow & & I\uparrow & & \\
\mathrm{H}(B) & \xrightarrow{\mathrm{H}(g)} & \mathrm{H}(A) & \underset{f}{\overset{f^{-1}}{\rightleftarrows}} & \mathrm{H}(A)^\phi
\end{array}$$

This implies that the equality $[I^*(g^*\phi)] = [\mathrm{H}(g)^*I^*\phi]$ holds in cohomology, and therefore $[\mathrm{H}(g)^*\rho^\phi] = [\rho^{g^*\phi}]$ for the associated closed 2-forms. It therefore follows from Lemma 4.10 that the cyclic minimal model $(H^{g^*\phi}, \sigma^{g^*\phi})$ is cyclic-analytic A_∞-isomorphic to the cyclic minimal model corresponding to $\mathrm{H}(g)^*\rho^\phi$. The latter is given by $(\mathrm{H}(A)^\phi, \sigma^\phi)$, as the composition $\mathrm{H}(g)^{-1} \diamond f$ satisfies

$$(\mathrm{H}(g)^{-1} \diamond f)^*(\mathrm{H}(g)^*\rho^\phi) = f^*\rho^\phi = \sigma^\phi. \qquad \square$$

Corollary 4.24. *If $A \in \mathsf{Alg}^{\infty,\mathrm{an}}$ is compact with unital analytic minimal model with $\mathrm{H}^0(A) \cong \ell$, then any analytic 3-CY structure ϕ determines an analytic potential*

$$W_A^\phi \in \widetilde{\mathsf{T}}_\ell V$$

in the analytic tensor algebra over $V = \mathrm{H}^1(A)^*$*, which only depends on* $[\phi] \in \mathrm{HC}_\lambda^{-3}(A)$ *up to an change of coordinates* $\widetilde{\mathsf{T}}_\ell V \to \widetilde{\mathsf{T}}_\ell V$*. Moreover, if* $g\colon B \to A$ *is an analytic quasi-isomorphism for* B *another such analytic* A_∞*-algebra, then there is an isomorphism* $h\colon \widetilde{\mathsf{T}}_\ell W \to \widetilde{\mathsf{T}}_\ell V$ *from the tensor algebra over* $W = \mathrm{H}^1(B)$ *such that*

$$h^*(W_B^{g^*\phi}) = W_A^{\phi},$$

Proof. Letting $(\mathrm{H}(A)^\phi, \sigma^\phi)$ be the cyclic minimal model from Theorem 4.23(1) associated to a 3-CY structure ϕ, it follows from Proposition 4.2 that there is an associated analytic potential $W_A^\phi \in \widetilde{\mathsf{T}}_\ell V$ for $V = ((\mathrm{H}(A))^1)^* = \mathrm{H}^1(A)^*$. If ψ is another 3-CY structure with $[\psi] = [\phi]$ then it follows by Theorem 4.23(2) that there is an isomorphism $f \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathrm{H}(A)^\phi, \mathrm{H}(A)^\psi)$ which is cyclic. There is an induced algebra isomorphism $f^*\colon \widehat{\mathsf{T}}_\ell V \to \widehat{\mathsf{T}}_\ell V$, and it follows by [Kaj07, Proposition 4.16] that

$$f^*(W_A^\psi) = W_A^\phi.$$

Hence it remains to verify that f^* restricts to a map $\widetilde{\mathsf{T}}_\ell V \to \widetilde{\mathsf{T}}_\ell V$ of analytic tensor algebras. As in [HK19, Lemma 3.12] it suffices to show that $f^*(v) \in \widetilde{\mathsf{T}}_\ell V$ for each $v \in V$. Picking a basis $v_1, \ldots, v_m \in V$ and letting $a_1, \ldots, a_m \in \mathrm{H}^1(A)$ denote the dual basis, the element $f^*(v)$ can be written as the power series

$$f^*(v) = \sum_{n\geq 0} \sum_{i_1,\ldots,i_n} v(f_n(a_{i_1}, \ldots, a_{i_n})) \cdot v_{i_1} \otimes \cdots \otimes v_{i_n} \in \widehat{\mathsf{T}}_\ell V.$$

Because f is analytic, it follows that there exists $C_0 > 0$ such that $\|f_n\| < C_0^n$ for all $n \in \mathbb{N}$, $\|v\| < C_0$, and $\|a_i\| < C_0$ for each a_i. Then for each $i_1, \ldots, i_n$ the coefficient satisfies:

$$|v(f_n(a_{i_1}, \ldots, a_{i_n}))| \leq \|v\|\|f_n\|\|a_{i_1}\| \cdots \|a_{i_n}\| < C_0^{2n+1} \leq (C_0^3)^n,$$

which shows that $f^*(v) \in \widetilde{\mathsf{T}}_\ell V$. It follows that W_A^ϕ and W_A^ψ are related by an analytic change of coordinates. A similar argument now show the second statement. □

5. Analytic Calabi–Yau structures in geometry

We will now apply our main theorem to a geometric setting: we will consider A_∞-algebras which govern the deformations of sheaves on a smooth projective variety. To set notation, we will write X for a smooth projective variety of dimension n, which we interpret as an analytic space, and we will write Y for arbitrary complex manifold.

The A_∞-algebras arising in this setting are obtained from normed DG algebras, and for this reason it is more convenient to work in an *unshifted* setup involving sequences of maps $f_n^{\mathrm{unsh}}\colon A^{\otimes n} \to A$ of various degrees, rather than the maps $f_n\colon A[1]^{\otimes n} \to A[1]$ in the shifted setup. We remark that one can move between the two setup by taking the composition $f_n = s \circ f_n^{\mathrm{unsh}} \circ (s^{-1})^{\otimes n}$ with the canonical shift maps $s\colon A \to A[1]$. We will leave this implicit in what follows.

5.1. **The Dolbeault construction.** Recall that the sheaf of holomorphic functions $\mathcal{O}_Y$ on a complex manifold Y admits a fine resolution via the *Dolbeault complex*

$$\mathcal{O}_Y \hookrightarrow \mathscr{A}_Y^{0,0} \xrightarrow{\overline{\partial}} \mathscr{A}_Y^{0,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \mathscr{A}_Y^{0,n} \longrightarrow 0,$$

where $\mathscr{A}_X^{p,q}$ is the sheaf of smooth (p,q)-forms on Y. The sheaves $\mathscr{A}_X^{p,q}$ are acyclic, hence can be used to compute the sheaf cohomology of $\mathcal{O}_Y$ via $\mathrm{H}^i(U, \mathcal{O}_Y) \cong \mathrm{H}^i(\Gamma(U, \mathscr{A}_Y^{0,\bullet}), \overline{\partial})$ on any open $U \subset Y$. More generally, one can take any a bounded complex of locally free $\mathcal{O}_Y$-modules $\mathcal{E} = (\mathcal{E}^\bullet, \delta)$ on Y, i.e. perfect complex, and obtain an acyclic resolution of $\mathcal{E}$

$$\mathscr{A}_Y^{0,\bullet}(\mathcal{E}) := \quad \ldots \xrightarrow{D} \bigoplus_{q\in\mathbb{Z}} \mathcal{E}^{-q} \otimes_Y \mathscr{A}_Y^{0,q} \xrightarrow{D} \bigoplus_{q\in\mathbb{Z}} \mathcal{E}^{1-q} \otimes_Y \mathscr{A}_Y^{0,q} \xrightarrow{D} \ldots,$$

where the differential is given by $D = \overline{\partial} + \delta$. This complex again computes the hypercohomology $\mathbf{R}\Gamma^i(U, \mathcal{E}) \cong \mathrm{H}^i\Gamma(U, \mathscr{A}_Y^{0,\bullet}(\mathcal{E}))$ on any open $U \subset Y$.

The Dolbeault construction can be used to construct a DG enhancement of the derived category $\mathsf{D}^{\mathrm{perf}}(Y)$: for any perfect complexes $\mathcal{E}, \mathcal{F}$ there is a natural isomorphism

$$\mathrm{H}^i\Gamma(Y, \mathscr{A}_Y^{0,\bullet}(\mathcal{H}om(\mathcal{E},\mathcal{F}))) \cong \mathbf{R}^i\Gamma(Y, \mathcal{H}om_Y(\mathcal{E},\mathcal{F})) \cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{perf}}(Y)}(\mathcal{E}, \mathcal{F}[i]),$$

so the complex of global sections for $\mathscr{A}_Y^{0,\bullet}(\mathcal{H}om(\mathcal{E},\mathcal{F}))$ is a morphism complex between $\mathcal{E}$ and $\mathcal{F}$. The natural composition on $\mathsf{D}^{\mathrm{perf}}(Y)$ is given by a wedge product $\wedge$, which can be defined in terms of local sections $f \otimes \xi \in (\mathcal{H}om^{i_1}(\mathcal{F},\mathcal{G}) \otimes \mathscr{A}_Y^{0,q})(U)$ and $g \otimes \zeta \in (\mathcal{H}om^{i_1}(\mathcal{E},\mathcal{F}) \otimes \mathscr{A}_Y^{0,q_2})(U)$ as

$$(f \otimes \xi) \wedge (g \otimes \zeta) := (-1)^{i_2 q_1}(f \circ g) \otimes (\xi \wedge \zeta) \in (\mathcal{H}om^{i_1+i_2}(\mathcal{E},\mathcal{G}) \otimes \mathscr{A}_Y^{0,q_1+q_2})(U).$$

This product satisfies the Leibniz rule with respect to D, and therefore defines a DG enhancement of $\mathsf{D}^{\mathrm{perf}}(Y)$ called the Dolbeault enhancement, see e.g. [Toë11, §2.3 Example 9].

To each perfect complex $\mathcal{E} \in \mathsf{D}^{\mathrm{perf}}(Y)$ is associated an endomorphism DGA: the *Dolbeault DG algebra*

$$\mathfrak{g}_{\mathcal{E}} := \left(\Gamma(X, \mathscr{A}_Y^{0,q}(\mathcal{E}nd(\mathcal{E}))), D, \wedge\right),$$

with cohomology $\mathrm{Ext}^\bullet(\mathcal{E},\mathcal{E})$. If $\mathcal{E}$ admits a direct sum decomposition $\mathcal{E} = \mathcal{E}_1^{\oplus m_1} \oplus \ldots \oplus \mathcal{E}_k^{\oplus m_k}$ then the Dolbeault DG algebra is naturally defined over the base ring

$$\ell = \mathrm{Mat}_{\mathbb{C}}(m_1) \times \cdots \times \mathrm{Mat}_{\mathbb{C}}(m_k) \subset \mathfrak{g}_{\mathcal{E}}^0.$$

In what follows we will always assume such a splitting is given and work over the fixed base ℓ.

5.2. **Analytic minimal models in the compact setting.** Now let X be (the analytic space associated to) a smooth projective variety and fix a hermitian metric $\langle -,- \rangle_X \colon \mathscr{A}_X^{0,q} \otimes \mathscr{A}_X^{0,q} \to \mathscr{A}^{0,0}$. For a perfect complex $\mathcal{E} = (\mathcal{E}^\bullet, \delta)$ one can choose a compatible hermitian metric

$$\langle -,- \rangle_{\mathcal{E}} \colon \mathscr{A}_X^{0,0}(\mathcal{E}^i) \otimes \mathscr{A}_X^{0,0}(\mathcal{E}^i) \to \mathscr{A}_X^{0,0}$$

for each locally free sheaf $\mathcal{E}^i$; we will refer to $\mathcal{E}$ endowed which such a metric as a *hermitian perfect complex*. Recall (see e.g. [GH78]) that this data determines an L^2-norm $\|\cdot\|_{L^2} \colon \Gamma(X, \mathscr{A}_X^{0,\bullet}(\mathcal{E})) \to \mathbb{R}$, which is defined on pure tensors $\xi = f \otimes \omega$ with $f \in \Gamma(X, \mathscr{A}_X^{0,0}(\mathcal{E}^i))$ and $\omega \in \Gamma(X, \mathscr{A}^{0,q})$ by the formula

$$\|\xi\|_{L^2}^2 := \int_X \langle f, f \rangle_{\mathcal{E}} \cdot \langle \omega, \omega \rangle_X \, \mathrm{d}vol_X,$$

where $\mathrm{d}vol_X \in \Gamma(X, \mathscr{A}_X^{0,n}(\Omega_X^n))$ is the canonical real volume form on X. The *Sobolev* $(l,2)$*-norm* for $l \in \mathbb{N}$ is defined in terms of the L^2-norm as $\|\xi\|_{l,2} := \sum_{k \le l} \left\|\nabla^k \xi\right\|_{L^2}$, where the operator ∇ acts on each locally free sheaf $\mathcal{E}^i$ as the canonical metric connection.

Given hermitian perfect complexes $\mathcal{E}$ and $\mathcal{F}$, there is an induced hermitian metric on $\mathcal{H}om(\mathcal{E},\mathcal{F})$ and hence also induced norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{l,2}$ on the global sections of its Dolbeault construction. As remarked by Fukaya [Fuk01], the operators D and $\wedge$ satisfy a bound

$$\|D\|_{l,2} < C, \quad \|\wedge\|_{l,2} < C^2, \quad (C > 0)$$

if one chooses a sufficiently large $l > 2\dim_{\mathbb{C}} X$. In particular, the Dolbeault DG algebra $\mathfrak{g}_{\mathcal{E}}$ of a hermitian perfect complex is naturally an analytic A_∞-algebra. It was shown by Tu [Tu14] and Toda [Tod18] that this admits a strong analytic minimal model.

Theorem 5.1 ([Tu14, Appendix A][Tod18, Lemma 4.1])**.** *The DG algebra $\mathfrak{g}_{\mathcal{E}}$ admits a minimal model with diagram*

$$\mathcal{H}_{\mathcal{E}} \underset{P}{\overset{I}{\rightleftarrows}} \mathfrak{g}_{\mathcal{E}} \circlearrowleft Q$$

such that the A_∞-structure μ on $\mathcal{H}_{\mathcal{E}}$ and the maps I, P, and Q satisfy the bounds

$$\|\mu_k\|_{l,2} < C^k, \quad \|I_k\|_{l,2} < C^k, \quad \|P_k\|_{l,2} < C^k, \quad \|Q_k\|_{l,2} < C^k$$

for all $k \ge 1$ and a fixed constant $C > 0$ independent of k.

At first order, the minimal model is given by Hodge theory: $\mathcal{H}_\mathcal{E}$ is the space of harmonic forms in $\mathfrak{g}_\mathcal{E}$, with I_1 and P_1 being the inclusion and projection, and Q_1 is defined via the Green operator associated to D. The higher compositions are all defined via the standard homotopy transfer formula [Kad80] given via sums of trees.

If X is a Calabi–Yau variety, then Polishchuk [Pol01] showed that the Serre duality pairing pulls back to a cyclic structure on this analytic minimal model. Using the terminology employed in this paper, this can be phrased as follows.

Theorem 5.2 ([Pol01, Theorem 1.1])**.** *Suppose X is a projective Calabi–Yau variety with holomorphic volume form $\nu \in \mathrm{H}^0(X, \Omega_X^n)$, then the Serre pairing*

$$\rho(-)(-) = \int_X \nu \wedge \mathrm{tr}(- \wedge -)$$

pulls back to a cyclic structure $\sigma = I^*\rho$ on $\mathcal{H}_\mathcal{E}$.

The Serre pairing used in this theorem can be viewed as an analytic strong homotopy inner product $\rho \in \Omega^{2,\mathrm{cl},\mathrm{an}}(\mathfrak{g}_\mathcal{E})$, which corresponds to the right Calabi–Yau structure associated to the bounded linear functional

$$\tau_\nu = \int_X \nu \wedge \mathrm{tr}(-) \in \mathfrak{g}_\mathcal{E}' \in \mathbf{C}_\lambda^{-n}(\mathfrak{g}_\mathcal{E}),$$

as in Example 4.22. In view of the analytic Darboux theorem, one might suspect that there exists other cyclic analytic minimal models for other choices of analytic right Calabi–Yau structure.

In what follows we show how such CY structures can be obtained from a choice of holomorphic volume form on open subsets $U \subset X$, using a compactly supported version of the Dolbeault construction.

5.3. **The compactly supported setting.** Let $U \subset X$ be an open analytic subvariety of a smooth projective variety X, viewed as an analytic manifold. Given a bounded complex of locally free $\mathcal{O}_U$-modules $\mathcal{F}$ we let

$$\mathfrak{g}_{\mathcal{F},c} := \Big(\Gamma_c(U, \mathscr{A}_U^{0,\bullet}(\mathcal{E}nd(\mathcal{F}))),\ D,\ \wedge\Big)$$

denote the nonunital DG subalgebra of $\mathfrak{g}_\mathcal{F}$ of compactly supported sections. Henceforth, we will assume that the cohomology of $\mathcal{F}$ is supported on a compact subset $Z \subset U$, which implies that

$$\mathrm{H}^\bullet \mathfrak{g}_{\mathcal{F},c} = \mathbf{R}^i\Gamma_c(U, \mathscr{A}_U^{0,\bullet}(\mathcal{E}nd(\mathcal{F}))) \cong \mathbf{R}^i\Gamma(U, \mathscr{A}_U^{0,\bullet}(\mathcal{E}nd(\mathcal{F}))) = \mathrm{H}^\bullet \mathfrak{g}_\mathcal{F},$$

so that the inclusion $\mathfrak{g}_{\mathcal{F},c} \hookrightarrow \mathfrak{g}_\mathcal{F}$ is a quasi-isomorphism. We also fix a hermitian metric on $\mathcal{F}$, which induces a hermitian pairing

$$\langle -,-\rangle_{\mathcal{E}nd(\mathcal{F})} \colon \mathscr{A}_U^{0,0}(\mathcal{E}nd(\mathcal{F})) \otimes \mathscr{A}_U^{0,0}(\mathcal{E}nd(\mathcal{F})) \to \mathscr{A}_U^{0,0}$$

The metric data again determines a well-defined L^2-norm $\|\cdot\|_{L^2} \colon \mathfrak{g}_{\mathcal{F},c} \to \mathbb{R}$ which is defined on pure tensors $\xi = f \otimes \omega$ with $f \in \Gamma_c(U, \mathscr{A}^{0,0}(\mathcal{E}nd(\mathcal{F})))$ and $\omega \in \Gamma_c(U, \mathscr{A}^{0,q})$ as

$$\|\xi\|_{L^2} = \left(\int_X \langle f, f\rangle_{\mathcal{E}nd(\mathcal{F})} \cdot \langle \omega, \omega\rangle_X \mathrm{d}vol_X\right)^{1/2},$$

where we note that the compactly supported function $\langle f, f\rangle_{\mathcal{E}nd(\mathcal{F})}$ can be viewed as a function on X via extension by 0, and similarly ω extends to a smooth $(0,q)$-form on X. The metric connection then also determines a Sobolev norm $\|\cdot\|_{l,2} = \sum_{k\le l} \left\|\nabla^k -\right\|_{L^2}$ as in the compact setting. A priori, the operators D and $\wedge$ are *not* guaranteed to be bounded, however the reader can check that such a bound exists if U is chosen to be a sufficiently small neighbourhood of the compact set Z.

Now suppose that U admits a nowhere-vanishing holomorphic volume form $\nu \in \Gamma(U, \Omega_U^n)$. Then writing again $\mathrm{tr} \colon \mathscr{A}_U^{0,n}(\mathcal{E}nd(\mathcal{E})) \to \mathscr{A}_U^{0,n}$ for the point-wise trace, we obtain a linear functional

$$\tau_\nu \colon \mathfrak{g}_{\mathcal{F},c}^n \to \mathbb{C}, \quad \zeta \mapsto \int_U \nu \wedge \mathrm{tr}(\zeta).$$

We will view τ_ν as an element of $\mathfrak{g}_{\mathcal{F},c}^\vee \subset \mathcal{M}(\mathfrak{g}_{\mathcal{F},c}, \mathfrak{g}_{\mathcal{F},c}^\vee)[-1]$ in the obvious way, and claim that this is a negative cyclic cocycle.

Lemma 5.3. *The map τ_ν defines a right Calabi–Yau structure of degree $-n$.*

Proof. Unraveling the definition of the differential on $\mathbf{C}^\bullet(\mathfrak{g}_{\mathcal{F},c}, \mathfrak{g}_{\mathcal{F},c}^\vee) = (\mathcal{M}(\mathfrak{g}_{\mathcal{F},c}, \mathfrak{g}_{\mathcal{F},c}^\vee)[-1], b)$, we see that $b(\tau_\nu) = 0$ if and only if the following two equations hold for all homogeneous $\zeta, \xi \in \mathfrak{g}_{\mathcal{F},c}$:

$$\tau_\nu(D\zeta) = 0,$$
$$\tau_\nu(\zeta_1 \wedge \zeta_2) = (-1)^{|\zeta_1||\zeta_2|}\tau_v(\zeta_2 \wedge \zeta_1).$$

To show the first equation, we note that for a local form $\zeta = f \otimes \omega$ with f a local section of $\mathcal{E}nd^i(\mathcal{F})$ and ω a local $(0, n-i-1)$-form, we have the local identity

$$\mathrm{tr}(D(f\otimes\omega)) = \mathrm{tr}(f)\overline{\partial}\omega + \mathrm{tr}([\delta, f])\omega = \mathrm{tr}(f)\overline{\partial}\omega = \overline{\partial}\,\mathrm{tr}(f\otimes\omega).$$

Hence, the same identity also hold for a general compactly supported section ζ over U. Because ν is holomorphic it then follows by Stokes' theorem for compactly supported forms that

$$\tau_\nu(D\zeta) = \int_U \overline{\partial}(\nu \wedge \mathrm{tr}(\zeta)) = \int_U (\partial + \overline{\partial})(\nu\wedge\mathrm{tr}(\zeta)) = 0,$$

where we note that $\partial(\nu \wedge \mathrm{tr}(\zeta)) = 0$ for degree reasons. For the second equation, we can likewise consider local forms $\zeta_1 = f_1 \otimes \omega_1$ and $\zeta_2 = f_2 \otimes \xi_2$ with f_k a local section of $\mathcal{E}nd^{i_k}(\mathcal{F})$ and ω_k a local section of $\mathscr{A}_U^{0,q_k}$, there is an identity

$$\mathrm{tr}(\zeta_1\wedge\zeta_2) = (-1)^{q_1 i_2}\,\mathrm{tr}(f_1\circ f_2)\cdot\omega_1\wedge\omega_2 = (-1)^{q_1(q_2+i_2)+i_1 i_2}\,\mathrm{tr}(f_2\circ f_1)\cdot\omega_2\wedge\omega_1 = (-1)^{|\zeta_1||\zeta_2|}\,\mathrm{tr}(\zeta_2\wedge\zeta_1).$$

This identity then also holds for any compactly supported sections on U, so integrating against ν yields the section equality. It follows that τ_ν is a cocycle in $\mathsf{C}^\bullet(\mathfrak{g}_{\mathcal{F},c}, \mathfrak{g}_{\mathcal{F},c}^\vee)$, and it is immediate that it lies in $\mathsf{C}_\lambda^\bullet(\mathfrak{g}_{\mathcal{F},c})$ because the cyclic action is trivial on the summand $\mathfrak{g}_{\mathcal{F},c}^\vee \subset \mathsf{C}_\lambda^\bullet(\mathfrak{g}_{\mathcal{F},c})$.

It remains to show that the map $\zeta \mapsto \tau_\nu(\zeta \wedge -)$ is a quasi-isomorphism, or equivalently that

$$(\zeta, \xi) \mapsto \tau_\nu(\zeta\wedge\xi) = \int_U \nu \wedge \mathrm{tr}(\zeta\wedge\xi) \tag{12}$$

induces a nondegenerate pairing on cohomology. For this, we note that a holomorphic volume form $\nu \in \Gamma(U, \Omega_U^n)$ represents an isomorphism $[\nu] \in \mathrm{Hom}_{\mathsf{D}(U)}(\mathcal{O}_U, \Omega_U^n)$ in the derived category. Therefore, the wedge product

$$\nu \wedge -\colon \Gamma_c(U, \mathscr{A}_U^{0,\bullet}(\mathcal{E}nd(\mathcal{F}))) \to \Gamma_c(U, \mathscr{A}_U^{0,\bullet}(\mathcal{E}nd(\mathcal{F}) \otimes \Omega_U^n)), \tag{13}$$

induces the isomorphisms $\mathrm{Hom}_{\mathsf{D}(U)}(\mathcal{E}, \mathcal{E}[i]) \to \mathrm{Hom}_{\mathsf{D}(U)}(\mathcal{E}, \mathcal{E}\otimes\Omega_U^n[i])$ in the derived category. By inspection, (12) is the composition of (13) with the Serre duality pairing

$$\Gamma_c(U, \mathscr{A}_U^{0,n-i}(\mathcal{E}nd(\mathcal{E}))) \otimes \Gamma_c(U, \mathscr{A}_U^{0,i}(\mathcal{E}nd(\mathcal{E})\otimes\Omega_U^n)) \xrightarrow{\int_U \mathrm{tr}(-\wedge-)} \mathbb{C},$$

which induces a nondegenerate pairing $\mathrm{Hom}_{\mathsf{D}(U)}(\mathcal{E}, \mathcal{E}[n-i]) \otimes \mathrm{Hom}_{\mathsf{D}(U)}(\mathcal{E}, \mathcal{E}\otimes\Omega^n[i]) \to \mathbb{C}$ on cohomology. Therefore the pairing induced by (12) on cohomology is also nondegenerate, and it follows that τ_ν is a right Calabi–Yau structure. □

We now want to show that the right Calabi–Yau structure τ_ν is analytic if $\mathfrak{g}_{\mathcal{F},c}$ is an analytic A_∞-algebra. For this it suffices to show that τ_ν is bounded with respect to the Sobolev norm.

Lemma 5.4. *If $\nu \in \Gamma(U, \Omega_U^n)$ is an L^2-integrable holomorphic volume form, then τ_ν is bounded.*

Proof. For any $\zeta \in \Gamma_c(U, \mathscr{A}^{0,\bullet}(\mathcal{E}nd(\mathcal{F}))$ there is a Cauchy-Schwarz type inequality

$$|\tau_\nu(\zeta)| = \left|\int_U \nu\wedge\mathrm{tr}(\zeta)\right| \le \|\nu\|_{L^2}\|\mathrm{tr}(\zeta)\|_{L^2},$$

with respect to the L^2-inner product on smooth forms determined by the hermitian metric. It is then easy to check that $\|\mathrm{tr}(\zeta)\|_{L^2} \le \|\zeta\|_{L^2} \le \|\zeta\|_{l,2}$, so that $\|\tau_\nu\|_{l,2}$ is bounded by $\|\nu\|_{L^2}$. □

We again remark that one can ensure that any holomorphic volume form $\nu \in \Gamma(U, \Omega_U^n)$ is bounded in the L^2-norm by replacing U with a sufficiently small neighbourhood of Z. After making this slight modification, we find an analytic right Calabi–Yau structure corresponding to the volume form.

5.4. **Comparing the compact and noncompact settings.** Let $\mathcal{E}$ be a hermitian perfect complex on a projective variety X, with cohomology supported on a subset Z, and choose an open neighbourhood $U \supset Z$ with inclusion map $i\colon U \hookrightarrow X$. Then the extension by zero yields an injective map

$$i_!\colon \mathfrak{g}_{\mathcal{E}|_U,c} = \Gamma_c(U, \mathscr{A}_U^{0,\bullet}(\mathcal{E}nd(\mathcal{E}|_U))) \longrightarrow \Gamma(X, \mathscr{A}_X^{0,\bullet}(\mathcal{E}nd(\mathcal{E}))) = \mathfrak{g}_{\mathcal{E}},$$

which exhibits $\mathfrak{g}_{\mathcal{E}|_U,c}$ as a DG ideal of $\mathfrak{g}_{\mathcal{E}}$. Equipping $\mathcal{E}|_U$ with the induced metric, this is an isometry with respect to the Sobolev norm, making $\mathfrak{g}_{\mathcal{E}|_U,c}$ into an analytic A_∞-subalgebra of $\mathfrak{g}_{\mathcal{E}}$. The assumption on the support moreover guarantees that $i_!$ is a quasi-isomorphism.

In order to pull back the analytic right CY structures found above from $\mathfrak{g}_{\mathcal{E}|_U,c}$ to $\mathfrak{g}_{\mathcal{E}}$ we would like to construct a quasi-inverse for $i_!$ using Lemma 2.18. This requires us to first construct a quasi-isomorphism $\mathfrak{g}_{\mathcal{E}} \to \mathfrak{g}_{\mathcal{E}|_U,c}$, for which we use a choice of weak unit.

Lemma 5.5. *There exists a pair* $(u, h) \in \mathfrak{g}^0_{\mathcal{E}|_U,c} \times \mathfrak{g}^{-1}_{\mathcal{E}}$ *such that* $Du = 0$ *and*

$$Dh = i_!u - \mathrm{id}_{\mathcal{E}} \in \mathfrak{g}^0_{\mathcal{E}}.$$

In particular, the map $\mathbb{1}\colon l \mapsto l \cdot u$ *is a weak unit for the analytic* A_∞*-algebra* $\mathfrak{g}_{\mathcal{E}|_U,c}$.

Proof. Because $i_!\colon \mathfrak{g}_{\mathcal{E}|_U,c} \to \mathfrak{g}_{\mathcal{E}}$ is a quasi-isomorphism, it follows that there exists a cocycle u with $[i_!u] = [\mathrm{id}_{\mathcal{E}}]$, and a homotopy h for the difference $i_!u - \mathrm{id}_{\mathcal{E}}$. If $\xi \in \mathfrak{g}_{\mathcal{E}|_U,c}$ is D-closed, then

$$[i_!(u \wedge \xi)] = [\mathrm{id}_{\mathcal{E}} \wedge i_!\xi] + [Dh \wedge i_!\xi] = [i_!\xi] \in \mathrm{H}^\bullet \mathfrak{g}_{\mathcal{E}}.$$

It follows that $[u \wedge \xi] = [\xi]$ and likewise $[\xi \wedge u] = [\xi]$ hold in $\mathrm{H}^\bullet \mathfrak{g}_{\mathcal{E}|_U,c}$. The map $\mathbb{1}\colon \ell \to \mathfrak{g}_{\mathcal{E}|_U,c}$ then satisfies $[\mathbb{1}(l) \wedge \xi] = [l \cdot u \wedge \xi] = [l \cdot \xi]$ and $[\xi \wedge \mathbb{1}(l)] = [\xi \cdot l \wedge u] = [\xi \cdot l]$, making it a weak unit. □

For a unit/homotopy pair (u, h) as above, the wedge product $i_!u \wedge \xi$ with any section $\xi \in \mathfrak{g}_{\mathcal{E}}$ has compact support contained in U. Hence, we obtain a well-defined retraction $i^*(i_!u \wedge -)\colon \mathfrak{g}_{\mathcal{E}} \to \mathfrak{g}_{\mathcal{E}|_U,c}$ which is quasi-inverse to $i_!$ as a map of chain complexes. The following lemma shows that this map extends to a full analytic A_∞-morphism, hence a quasi-isomorphism.

Lemma 5.6. *Let* (u, h) *be a unit/homotopy pair as above, then the sequence of maps*

$$K_n(\xi_1, \dots, \xi_n) = (-1)^{(n-1)|\xi_1| + (n-2)|\xi_2| + \dots + |\xi_{n-1}|} \cdot i^*(i_!u \wedge \xi_1 \wedge h \wedge \dots \wedge h \wedge \xi_n)$$

defines an analytic A_∞*-quasi-isomorphism* $K \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E}|_U,c})$.

Proof. Because the A_∞-algebra structures on $\mathfrak{g}_{\mathcal{E}}$ and $\mathfrak{g}_{\mathcal{E}|_U,c}$ have only the first and second order multiplication, the system of maps $K_n\colon \mathfrak{g}_{\mathcal{E}}^{\otimes n} \to \mathfrak{g}_{\mathcal{E}|_U,c}$ defines an A_∞-morphism in the unshifted convention, if and only if the following identity holds for all $\xi_1, \dots, \xi_n \in \mathfrak{g}_{\mathcal{E}}$:

$$\begin{aligned} DK_n(\xi_1, \dots, \xi_n) = {} & \sum_{i=0}^{n-1} (-1)^{n + |\xi_1| + \dots + |\xi_{i-1}|} K_n(\xi_1, \dots, \xi_{i-1}, D\xi_i, \xi_{i+1}, \dots, \xi_n) \\ & + \sum_{i=1}^{n-1} (-1)^{i(|\xi_1| + \dots + |\xi_i|)} K_i(\xi_1, \dots, \xi_i) \wedge K_{n-i}(\xi_{i+1}, \dots, \xi_n) \\ & + \sum_{i=0}^{n-2} (-1)^i K_{n-1}(\xi_1, \dots, \xi_{i-1}, \xi_i \wedge \xi_{i+1}, \xi_{i+2}, \dots, \xi_n). \end{aligned}$$

This identity is easily verified via the graded Leibniz rule for D and $\wedge$: because u is D-closed of degree 0 and h is an element of degree -1 with $Dh = i_!u - \mathrm{id}_{\mathcal{E}|_U}$ it follows that

$$\begin{aligned} D(u \wedge \xi_1 \wedge h \wedge \dots \wedge \xi_n) = {} & \sum_{i=0}^{n-1} (-1)^{|\xi_1| + \dots + |\xi_{i-1}| + (i-1)} i^*(i_!u \wedge \dots \wedge h \wedge D\xi_i \wedge h \wedge \dots \wedge \xi_n) \\ & + \sum_{i=0}^{n-2} (-1)^{|\xi_1| + \dots + |\xi_i| + (i-1)} i^*(i_!u \wedge \dots \wedge h \wedge \xi_i \wedge u \wedge \xi_{i+1} \wedge h \wedge \dots \wedge \xi_n). \\ & - \sum_{i=0}^{n-2} (-1)^{|\xi_1| + \dots + |\xi_i| + (i-1)} i^*(i_!u \wedge \dots \wedge h \wedge \xi_i \wedge \xi_{i+1} \wedge h \wedge \dots \wedge \xi_n), \end{aligned}$$

and the required identity drops out after adding the appropriate signs in the definition of the maps K_n. Hence, the system of maps K defines an A_∞-morphism. To see that K is analytic, let $C \geq 1$ be any constant so that $\|\wedge\|_{l,2} \leq C$, $\|u\|_{l,2} < C$ and $\|h\|_{l,2} < C$. Then

$$\|K_n(\xi_1, \dots, \xi_n)\|_{l,2} \leq C^{3n-1} \cdot \|\xi_1\|_{l,2} \cdots \|\xi_n\|_{l,2},$$

holds for any sections $\xi_1, \dots, \xi_n$, which implies that $\|K_n\| < C^{3n-1} < (C^3)^n$. □

Using the maps K and $i_!$, Lemma 2.18 now shows that there exists a quasi-automorphism $T \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_\mathcal{E}, \mathfrak{g}_\mathcal{E})$ such that $T \diamond i_!$ is left quasi-inverse to K and $K \diamond T$ is right quasi-inverse to $i_!$. Moreover, either of the diagrams

$$\mathcal{H}_\mathcal{E} \underset{P}{\overset{I}{\rightleftarrows}} \mathfrak{g}_\mathcal{E} \underset{i_!}{\overset{K\diamond T}{\rightleftarrows}} \mathfrak{g}_{\mathcal{E}|_U,c}, \qquad \mathcal{H}_\mathcal{E} \underset{P}{\overset{I}{\rightleftarrows}} \mathfrak{g}_\mathcal{E} \underset{T\diamond i_!}{\overset{K}{\rightleftarrows}} \mathfrak{g}_{\mathcal{E}|_U,c}. \tag{14}$$

exhibit $\mathcal{H}_\mathcal{E}$ as a minimal model for $\mathfrak{g}_{\mathcal{E}|_U,c}$. These maps allow one to compare invariants of $\mathfrak{g}_{\mathcal{E}|_U,c}$, $\mathfrak{g}_\mathcal{E}$ and $\mathcal{H}_\mathcal{E}$, in particular right CY structures.

Finally, we wish to compare the situation where $\mathcal{E}|_U$ is replaced by an arbitrary hermitian perfect complex on U. It turns out that we can again compare the corresponding Dolbeault DG algebras via a quasi-isomorphism, as the following lemma shows.

Lemma 5.7. *Suppose $\mathcal{F}$ is a hermitian perfect complex quasi-isomorphic to $\mathcal{E}|_U$, such that $\mathfrak{g}_{\mathcal{F},c}$ is analytic with respect to the Sobolev norm. Then there exists a triple (r, t, h_{rt}) with*

$$r \in \Gamma_c(U, \mathscr{A}^{0,\bullet}_X(\mathcal{H}om(\mathcal{F}, \mathcal{E}|_U))), \quad t \in \Gamma_c(U, \mathscr{A}^{0,\bullet}_X(\mathcal{H}om(\mathcal{E}|_U, \mathcal{F})))$$

degree 0 cocycles and $h_{rt} \in \mathfrak{g}^{-1}_\mathcal{E}$ satisfying $Dh_{rt} = i_!(r \wedge t) - \mathrm{id}_\mathcal{E}$. Such that the maps

$$F_n(\xi_1, \dots, \xi_n) := (-1)^{(n-1)|\xi_1|+(n-2)|\xi_2|+\cdots+|\xi_{n-1}|} \cdot t \wedge \xi_1 \wedge h_{rt} \wedge \cdots \wedge h_{rt} \wedge \xi_n \wedge r$$

determine an analytic A_∞-quasi-isomorphism $F \in \mathrm{Hom}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}|_U,c}, \mathfrak{g}_{\mathcal{F},c})$.

Proof. Because the cohomologies of $\mathcal{F}$ and $\mathcal{E}|_U$ are compactly supported on U, the morphism spaces between $\mathcal{E}|_U$ and $\mathcal{F}$ in the derived category can be computed as

$$\mathrm{Hom}_{\mathsf{D}(U)}(\mathcal{E}|_U, \mathcal{F}) \cong \mathbf{R}^0\Gamma_c(U, \mathcal{H}om(\mathcal{E}|_U, \mathcal{F})) \cong \mathrm{H}^0(\Gamma_c(U, \mathscr{A}^{0,\bullet}_U(\mathcal{H}om(\mathcal{E}|_U, \mathcal{F}))), D),$$

and similarly $\mathrm{Hom}_{\mathsf{D}(U)}(\mathcal{F}, \mathcal{E}|_U) \cong \mathrm{H}^0(\Gamma_c(U, \mathscr{A}^{0,\bullet}_U(\mathcal{H}om(\mathcal{E}|_U, \mathcal{F})), D)$. Hence, there exists degree 0 cocycles $t \in \Gamma_c(U, \mathscr{A}^{0,\bullet}_U(\mathcal{H}om(\mathcal{E}|_U, \mathcal{F})))$ and $r \in \Gamma_c(U, \mathscr{A}^{0,\bullet}_U(\mathcal{H}om(\mathcal{E}|_U, \mathcal{F})))$ which induces the isomorphism between $\mathcal{F}$ and $\mathcal{E}|_U$ in the derived category. The composition $r \wedge t$ induces the identity on $\mathcal{E}|_U$ in the derived category, which implies that

$$i_!(r \wedge t) = \mathrm{id}_\mathcal{E} + Dh_{rt},$$

for some $h_{rt} \in \mathfrak{g}^{-1}_\mathcal{E}$. The A_∞-morphism axioms for the map F can be checked in an analogous fashion to the proof of Lemma 5.6, so it remains to show that the first component $F_1 = t \wedge - \wedge r$ induces an isomorphism in cohomology. This follows by the existence of a quasi-inverse $F_1^{-1} := r \wedge - \wedge t$ which satisfies

$$[i_!(F_1^{-1}(F(\xi)))] = [i_!(r \wedge t) \wedge i_!\xi \wedge i_!(r \wedge t)] = [(\mathrm{id}_\mathcal{E} + Dh_{rt}) \wedge i_!\xi \wedge (\mathrm{id}_\mathcal{E} + Dh_{rt})] = [i_!\xi] \in \mathrm{H}^0\mathfrak{g}_\mathcal{E}.$$

Because $i_! \colon \mathfrak{g}_{\mathcal{E}|_U,c} \to \mathfrak{g}_\mathcal{E}$ is a quasi-isomorphism, it follows that $F_1^{-1} \circ F_1$ induces the identity on $\mathrm{H}^\bullet\mathfrak{g}_{\mathcal{E}|_U,c}$. Finally, to check that F is analytic it suffices to choose $C > \max\{\|t\|_{l,2}, \|r\|_{l,2}, \|h\|_{l,2}, \|\wedge\|_{l,2}\}$, so that

$$\|F\|_{l,2} \leq \|\wedge\|^{2n}_{l,2} \cdot \|r\|_{l,2} \cdot \|t\|_{l,2} \cdot \|h_{rt}\|^{n-1}_{l,2} \leq C^{3n+1} \leq C^{4n},$$

as in the proof of Lemma 5.6. It follows that F is analytic when $\mathfrak{g}_{\mathcal{F},c}$ is an analytic A_∞-algebra. □

If $\mathcal{F}$ is an arbitrary perfect complex on U with compactly supported cohomology then $\mathcal{F}$ is quasi-isomorphic to $i^*\mathcal{E} = \mathcal{E}|_U$, where $\mathcal{E} \to \mathbf{R}i_!\mathcal{F}$ is any resolution of the exceptional direct image. With the above lemma, this then yields a sequence of quasi-isomorphisms

$$\mathcal{H}_\mathcal{E} \xrightarrow{I} \mathfrak{g}_\mathcal{E} \xrightarrow{K} \mathfrak{g}_{\mathcal{E}|_U,c} \xrightarrow{F} \mathfrak{g}_{\mathcal{F},c},$$

along which right Calabi–Yau structures can be pulled back. With these comparison maps, we are now ready to prove the main theorems of the paper.

5.5. **Cyclic minimal models from local volumes.** We now combine the results of the previous subsections with the main theorem of the previous section to obtain new cyclic minimal models associated to local holomorphic volume forms. It turns out that the cyclic minimal model one obtains does not depend on the neighbourhood chosen, so we can take an agnostic approach: given a closed subset Z we consider the space of germs along Z

$$(\Omega^n_X)_Z := \{\nu \in \Gamma(V, \Omega^n_X) \mid V \supset Z \text{ open in } X\}/\sim$$

where $\nu \sim \nu'$ for two sections $\nu \in \Gamma(V, \Omega^n_X)$ and $\nu' \in \Gamma(V', \Omega^n_X)$ if there exists $V'' \subset V \cap V'$ such that $Z \subset V''$ and $\nu|_{V''} = \nu'|_{V''}$. We consider germs which locally act as a volume.

Definition 5.8. An element $\nu \in (\Omega^n_X)_Z$ is a *volume germ* along Z if there is some open neighbourhood $V \supset Z$ such that $\nu|_V$ is nonzero for all $p \in V$.

The main result of this section is the following theorem.

Theorem 5.9. *Let X be a smooth projective variety of dimension n, and $\mathcal{E}$ a perfect complex on X with cohomology supported on $Z \subset X$. Then any volume germ $\nu \in (\Omega^n_X)_Z$ determines a canonical analytic right Calabi–Yau structure ϕ^ν, which is well-defined up to non-analytic homotopy, and this determines a corresponding* analytic cyclic minimal model

$$(\mathcal{H}^\nu_{\mathcal{E}}, \sigma^\nu) = (\mathrm{Ext}^\bullet_X(\mathcal{E}, \mathcal{E}), \mu^\nu, \sigma^\nu),$$

for $\mathfrak{g}_{\mathcal{E}}$, which is well-defined up to cyclic analytic A_∞-isomorphism.

Proof. We fix a hermitian structure on X and $\mathcal{E}$ as before, so that $\mathfrak{g}_{\mathcal{E}}$ is an analytic DG algebra which admits a strong analytic minimal model by Theorem 5.1 with underlying space

$$\mathcal{H}_{\mathcal{E}} \cong \mathrm{H}^\bullet \mathfrak{g}_{\mathcal{E}} \cong \mathrm{Ext}^\bullet_X(\mathcal{E}, \mathcal{E}).$$

We claim that every volume germ along Z has a representative $\nu \in \Gamma(U, \Omega^n_X)$ such that ν is L^2-integrable on U. To construct this, simply pick any representative $\nu^0 \in \Gamma(V, \Omega^n_X)$, and note that the pointwise squared-norm is a continuous function $p \mapsto |\nu^0(p)|^2_X$ on V. Because Z is compact, this function is bounded on Z, so picking any $\epsilon > 0$ we obtain an open neighbourhood

$$U = \left\{ p \in Y \;\middle|\; |\nu^0_p|^2_X < \epsilon + \max_{q \in Z} |\nu^0_X(q)|^2 \right\},$$

on which $|\nu^0_p|^2$ is bounded, hence square-integrable. Setting $\nu = \nu^0|_U$ then yields the required representative.

Fixing such a choice of $\nu \in \Gamma(U, \Omega^n_X)$, it follows from Lemma 5.3 and Lemma 5.4 that $\tau_\nu \in \hom_{\ell^e}(\ell, \mathfrak{g}'_{\mathcal{E}|_U,c})$ is an analytic right CY structure on $\mathfrak{g}_{\mathcal{E}|_U,c}$. Writing $i\colon U \to X$ for the embedding of U, it follows by Lemma 5.5 that there is a unit/homotopy pair $(u, h) \in \mathfrak{g}^0_{\mathcal{E}|_U,c} \times \mathfrak{g}^{-1}_{\mathcal{E}}$, which induces an analytic quasi-isomorphism $K \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E}|_U,c})$ by Lemma 5.6. Because $\mathfrak{g}_{\mathcal{E}}$ admits a strong minimal model, it follows from Lemma 2.18 that there exists a quasi-automorphism $T \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E}})$ such that $K \diamond T$ is right quasi-inverse to $i_!$ as in (14). We define

$$\phi^\nu := (K \diamond T)^* \tau_\nu \in \mathsf{C}^{\mathrm{an},\bullet}(\mathfrak{g}_{\mathcal{E}}),$$

and note that it is again an analytic right CY structure on $\mathfrak{g}_{\mathcal{E}}$. Because the minimal model $\mathcal{H}_{\mathcal{E}}$ is unital, it follows from Theorem 4.23(1) that ϕ^ν determines an analytic cyclic minimal model

$$(\mathcal{H}^\nu_{\mathcal{E}}, \sigma^\nu) := (\mathcal{H}^{\phi^\nu}_{\mathcal{E}}, \sigma^{\phi^\nu}),$$

so that ϕ^ν pulls back to a class in $\mathsf{C}^{\bullet,\mathrm{an}}_\lambda(\mathcal{H}^\nu_{\mathcal{E}})$ which is homotopic to the image of σ^ν along the map in Corollary 4.21. It now remains to show that the right CY structure and the cyclic minimal model are independent of the choice of neighbourhood U or the unit/homotopy pair (u, h).

Suppose $V \subset U$ is another open subset with inclusion $j\colon V \hookrightarrow U$, and let (u_V, h_V) be a choice of unit/homotopy pair such that $(i \circ j)_! u_V = \mathrm{id}_{\mathcal{E}} + Dh_V$, and let $K_V \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E}|_V,c})$ and $T_V \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E}})$ be the induced quasi-isomorphisms such that $K_V \diamond T_V$ is a right quasi-inverse for $(i \circ j)_!$. Then $\nu|_V \sim \nu$ is another representative of the germ and induces the analytic right CY structure $(K_V \diamond T_V)^*\tau_{\nu|_V}$ as before. We claim that this induces the same class in the non-analytic cohomology $\mathrm{HC}_\lambda^{-d}(\mathfrak{g}_\mathcal{E})$. To see this, firstly note that by the identity

$$(j_!)^*\tau_\nu(\xi) = \int_U \nu \wedge \mathrm{tr}(j_!\xi) = \int_V \nu|_V \wedge \mathrm{tr}(\xi) = \tau_{\nu|_V}(\xi),$$

the strict morphism $j_!$ identifies the cocycles τ_ν and $\tau_{\nu|_V}$, and hence yields the equality

$$(K_V \diamond T_V)^*\tau_{\nu|_V} = (K_V \diamond T_V)^*(j_!)^*\tau_\nu = (j_! \diamond K_V \diamond T_V)^*\tau_\nu \in \mathsf{C}_\lambda^{\bullet,\mathrm{an}}(\mathfrak{g}_\mathcal{E}).$$

Because $K \diamond T$ is a right quasi-inverse to $i_!$ in $\mathsf{Alg}^{\infty,\mathrm{an}}$ it is in particular a right quasi-inverse in the nonanalytic category Alg^∞. But then $K \diamond T$ is also a left quasi-inverse in Alg^∞, because any quasi-inverse in Alg^∞ is two-sided. Hence there is a non-analytic homotopy $K \diamond T \diamond i_! \sim \mathrm{id}_{\mathfrak{g}_{\mathcal{E}|_U,c}}$, and hence an equality

$$\begin{aligned}[(j_! \diamond K_V \diamond T_V)^*\tau_\nu] &= [(j_! \diamond K_V \diamond T_V)^*(K \diamond T \diamond i_!)^*\tau_\nu] \\ &= [((i_! \circ j_!) \diamond (K_V \diamond T_V))^*(K \diamond T)^*\tau_\nu] \\ &= [(K \diamond T)^*\tau_\nu] = \phi^\nu.\end{aligned}$$

in the non-analytic Hochschild cohomology $\mathrm{HC}_\lambda^\bullet(\mathfrak{g}_\mathcal{E})$. Hence, $(K_V \diamond T_V)^*\tau_{\nu|_V}$ differs by ϕ^ν by a non-analytic homotopy as claimed. In particular, it follows by Theorem 4.23(2) that the cyclic minimal model induced by $(K_V \diamond T_V)^*\tau_{\nu|_V}$ is cyclic analytic A_∞-isomorphic to $\mathcal{H}_\mathcal{E}^\nu$. □

In the 3-Calabi–Yau case, a potential can be constructed using Corollary 4.24.

Corollary 5.10. *Suppose X is a threefold, then any volume germ ν induces an analytic potential*

$$W^\nu \in \widetilde{\mathsf{T}}_\ell \mathrm{Ext}^1(\mathcal{E},\mathcal{E})^\vee,$$

which is well-defined up to an analytic change of coordinates $\widetilde{\mathsf{T}}_\ell \mathrm{Ext}^1(\mathcal{E},\mathcal{E})^\vee \xrightarrow{\sim} \widetilde{\mathsf{T}}_\ell \mathrm{Ext}^1(\mathcal{E},\mathcal{E})^\vee$.

Remark 5.11. If X is itself Calabi–Yau then the global volume form $\nu \in \Gamma(X, \Omega_X^n)$ is L^2-integrable and $(u,h) = (\mathrm{id}_\mathcal{E}, 0)$ is a valid unit/homotopy inducing the identity maps $K = T = \mathrm{id}$ on $\mathfrak{g}_{\mathcal{E},c} = \mathfrak{g}_\mathcal{E}$. The trace $(K \diamond T)^*\tau_\nu = \tau_\nu$ induces the Serre pairing, and $(\mathcal{H}^\nu, \sigma^\nu)$ is simply the cyclic analytic minimal model of [Pol01; Tu14; Tod18] as given in Theorem 5.2.

The cyclic minimal models obtained from the above theorem express the local geometry on an open subset, but are always defined with respect to a global choice of projective space. We claim however, that the cyclic minimal model really only depends on the local geometry. To substantiate this claim, we will consider diagrams of the form

$$X \xleftarrow{\;i\;} Y \xrightarrow{\;f\;} X' \tag{15}$$

where Y is an open analytic subvariety in a smooth projective varieties X, and f is an open embedding into a second smooth projective variety X'. We claim that any cyclic minimal models for a perfect complexes with support on Y can be computed equivalently on X' or X.

Theorem 5.12. *In the situation of Equation (15), let $\mathcal{E}' \in \mathsf{D}^{\mathrm{perf}}(X')$ have support $f(Z) \subset f(Y)$ for some compact $Z \subset Y$, let $\nu \in (\Omega_{X'}^n)_Z$ be a volume form germ, and let $\mathcal{E} \in \mathsf{D}^{\mathrm{perf}}(X)$ be such that $\mathcal{E}|_Y \simeq f^*\mathcal{E}'$. Then there exists an analytic quasi-isomorphism*

$$\mathfrak{g}_\mathcal{E} \simeq_{\mathrm{an}} \mathfrak{g}_{\mathcal{E}'}$$

along which the analytic right CY structure ϕ^ν pulls back to a an analytic right CY structure which is (non-analytically) homotopic to $\phi^{f^*\phi}$. In particular, there is an analytic cyclic A_∞-isomorphism

$$(\mathcal{H}_\mathcal{E}^{f^*\nu}, \sigma^{f^*\nu}) \cong_{\mathrm{an,cyc}} (\mathcal{H}_{\mathcal{E}'}^\nu, \sigma^\nu),$$

between the cyclic analytic minimal models.

Proof. Given a germ in $(\Omega^n_{X'})_{f(Z)}$ we can find a representative $\nu \in \Gamma(f(U), \Omega^n_{X'})$ defined on an the image of some open neighbourhood $U \supset Z$. As before, this neighbourhood can be chosen so that ν and $f^*\nu$ are L^2-integrable forms on $f(U)$ and U respectively, yielding $\tau_\nu \in \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(\mathfrak{g}_{\mathcal{E}|_{f(U)},c})$ and $\tau_{f^*\nu} \in \mathsf{C}^{\bullet,\mathrm{an}}_\lambda(\mathfrak{g}_{\mathcal{E}|_U,c})$. Now for any choice of complexes $\mathcal{E}$ and $\mathcal{E}'$ as above, we can consider the chain of quasi-isomorphisms

$$\mathfrak{g}_{\mathcal{E}} \xrightarrow{K} \mathfrak{g}_{\mathcal{E}|_U,c} \xrightarrow{F} \mathfrak{g}_{f^*\mathcal{E}'|_U,c} \xrightarrow{(f^*)^{-1}} \mathfrak{g}_{\mathcal{E}'|_{f(U)},c}, \tag{16}$$

where $F\colon \mathfrak{g}_{\mathcal{E}|_U,c} \to \mathfrak{g}_{f^*\mathcal{E}'|_U,c}$ is the map in Lemma 5.7 associated to a triple (r, t, h_{rt}) of mutually inverse quasi-isomorphisms $t\colon \mathcal{E}|_U \to f^*\mathcal{E}'|_U$ and $t\colon \mathcal{E}|_U \to f^*\mathcal{E}'|_U$ with homotopy h_{rt} for the composition rt, and $(f^*)^{-1}$ denotes the DG algebra morphism given by the inverse of the pullback map

$$f^*\colon \Gamma(f(U), \mathscr{A}^{0,\bullet}_X(\mathcal{E}nd(\mathcal{E}'))) \to \Gamma(U, \mathscr{A}^{0,\bullet}_Y(\mathcal{E}nd(f^*\mathcal{E}'))),$$

which is a DG algebra isomorphism because f is a diffeomorphism onto its image. Choosing hermitian metrics on $\mathcal{E}$ and $\mathcal{E}'$, there is an induced hermitian metric on $f^*\mathcal{E}'$, and the maps K and F are analytic. The strict map f^* is again bounded provided that U is chosen sufficiently small, as the ratio between the metric on U and the metric pulled back from $f(U)$ is again bounded on a neighbourhood of the compact subset Z. Now we observe that along the quasi-isomorphism $(f^*)^{-1} \diamond F$ the functional $\tau_{f^*\nu}$ pulls back to an analytic negative cyclic cocycle of the form

$$\begin{aligned}
(((f^*)^{-1} \diamond F)^*\tau_\nu)_n(\xi_1, \dots, \xi_n) &= \int_{f(U)} \nu \wedge \mathrm{tr}((f^*)^{-1}(t \wedge \xi_1 \wedge h_{rt} \wedge \cdots \wedge h_{rt} \wedge \xi_n \wedge r)) \\
&= \int_U f^*\nu \wedge \mathrm{tr}(t \wedge \xi_1 \wedge h_{rt} \wedge \cdots \wedge h_{rt} \wedge \xi_n \wedge r) \\
&= \int_U \nu \wedge \mathrm{tr}(r \wedge t \wedge \xi_1 \wedge h_{rt} \wedge \cdots \wedge h_{rt} \wedge \xi_n) \\
&= ((K \diamond i_!)^*\tau_{f^*\nu})_n(\xi_1, \dots, \xi_n),
\end{aligned}$$

where K is the quasi-isomorphism associated to the unit/homotopy pair $(u, h) = (r \wedge t, h_{rt})$ constructed in Lemma 5.6. Taking again the quasi-isomorphism $T \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E}})$ such that $K \diamond T$ is right quasi-inverse to $i_!$ as before, then we find the relation:

$$\begin{aligned}
((f^*)^{-1} \diamond F \diamond K \diamond T^2)^*\tau_\nu &= (K \diamond T^2)^*((f^*)^{-1} \diamond F)^*\tau_\nu \\
&= (K \diamond T^2)^*(K \diamond i_!)^*\tau_{f^*\nu} \\
&= (K \diamond (i_! \diamond K \diamond T) \diamond T)^*\tau_{f^*\nu} \\
&\sim_{\mathrm{an}} (K \diamond T)^*\tau_{f^*\nu},
\end{aligned}$$

where $\sim_{\mathrm{an}}$ denotes the homotopy induced by the analytic homotopy $i_! \diamond K \diamond T \sim_{\mathrm{an}} \mathrm{id}$. The cocycle $(K \diamond T)^*\tau_{f^*\nu}$ is (up to nonanalytic homotopy) equal to the right CY structure $\phi^{f^*\nu}$ in Theorem 5.9. Now let $j\colon f(U) \hookrightarrow X'$ denote the inclusion of $f(U)$, and let K', T' be the analytic quasi-isomorphisms defining the analytic right CY structure ϕ^ν. Then

$$j_! \diamond (f^*)^{-1} \diamond F \diamond K \diamond T^2 \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^{\infty,\mathrm{an}}}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E}'})$$

is an analytic quasi-isomorphism for which the pullback of the cocycle ϕ^ν satisfies

$$\begin{aligned}
[(j_! \diamond (f^*)^{-1} \diamond F \diamond K \diamond T^2)\phi^\nu] &= [(K' \diamond T' \diamond j_! \diamond (f^*)^{-1} \diamond F \diamond K \diamond T^2)^*\tau_\nu] \\
&= [((f^*)^{-1} \diamond F \diamond K \diamond T^2)^*\tau_\nu] \\
&= [\phi^{f^*\nu}] \in \mathrm{HC}^\bullet_\lambda(\mathfrak{g}_{\mathcal{E}}),
\end{aligned}$$

where we used the fact that $K' \diamond T'$ is also left quasi-inverse to $j_!$ in Alg^∞. It now follows by Theorem 4.23(3) that there is a cyclic analytic A_∞-isomorphism between the cyclic analytic minimal models, which finishes the proof. □

In the threefold case, the second part of Corollary 4.24 now directly implies the following.

Corollary 5.13. *Suppose X and X' are threefolds with complexes $\mathcal{E}$ and $\mathcal{E}'$ as above. Then for any volume germ $\nu \in (\Omega^3_{X'})_Z$ there is an algebra isomorphism*

$$g\colon \widetilde{\mathsf{T}}_\ell \mathrm{Ext}^1(\mathcal{E}',\mathcal{E}')^\vee \xrightarrow{\sim} \widetilde{\mathsf{T}}_\ell \mathrm{Ext}^1(\mathcal{E},\mathcal{E})^\vee.$$

which relates the two potentials via $g(W^\nu) = W^{f^*\nu}$

5.6. **The example of a point.** We will give an explicit computation of the cyclic minimal model associated to the point sheaf of the origin in $\mathbb{A}^n$, which is a quasi-projective Calabi–Yau with standard volume form $\omega = \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n$ in standard coordinates.

Let $\mathbb{P}^n \subset \mathbb{A}^n$ be the projective compactification with coordinates $z_0, z_1, \dots, z_n$ and write $o \in \mathbb{A}^n \subset \mathbb{P}^n$ for the origin. Writing $V = (T_o\mathbb{A}^n)^* \cong \mathbb{C}^n$, the point sheaf $\mathcal{O}_o$ is resolved by the Koszul complex

$$\mathcal{E} := \quad \wedge^n(V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n}(-1)) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \wedge^2(V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n}(-1)) \xrightarrow{\delta} V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}^n},$$

where the differential acts $v \otimes f \mapsto \sum_{i=1}^n v(\partial_{z_i}) z_i \cdot f$ on $\mathcal{E}^{-1} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n}(-1)$ and acts on the other terms by extension over the wedge product. We endow $\mathcal{E}$ with a hermitian structure from the Fubini-Study metric on $\mathbb{P}^n$, yielding a Dolbeault DG algebra $\mathfrak{g}_{\mathcal{E}}$ as a model for $\mathrm{Ext}^\bullet(\mathcal{O}_o, \mathcal{O}_o)$. It is known (see e.g. [RS21, Lemma 3.6] for the threefold case) that the algebra of polyvectors

$$\mathfrak{h} := \left(\bigoplus_{i=0}^n (T_o\mathbb{A}^n)^{\wedge i}, \wedge \right),$$

at o forms an A_∞-minimal model for $\mathfrak{g}_{\mathcal{E}}$. Here we include a proof for the analytic setting.

Lemma 5.14. *There is an analytic DG algebra quasi-isomorphism $I\colon \mathfrak{h} \xrightarrow{\sim} \mathfrak{g}_{\mathcal{E}}$, which makes $\mathfrak{h}$ into an analytic minimal model for $\mathfrak{g}_{\mathcal{E}}$.*

Proof. It suffices to define $I(\xi)$ for $\xi \in T_o\mathbb{A}^n$ and check that $I(\xi) \wedge I(\xi) = 0$ and $DI(\xi) = 0$ hold. For a vector $\xi \in T_o\mathbb{A}^n$ the element $I(\xi)$ is defined as the holomorphic endomorphism acting by $v \otimes f \mapsto v(\xi) \cdot f \in \mathcal{E}^0$ on $\mathcal{E}^{-1}$ and extended to maps $\mathcal{E}^{-k} \mapsto \mathcal{E}^{1-k}$ over the wedge product. A standard computation shows that $I(\xi) \wedge I(\xi) = I(\xi) \circ I(\xi)$ acts by

$$\begin{aligned}(I(\xi) \circ I(\xi))(v_1 \otimes f_1 \wedge \dots \wedge v_k \otimes f_k) &= \sum_{i=1}^n (-1)^i v_i(\xi) f_i \cdot I(\xi)(v_1 \otimes f_1 \wedge \dots \widehat{v_i \otimes f_i} \dots \wedge v_k \otimes f_k \\ &= \sum_{i<j} ((-1)^{i+j} v_i(\xi) v_j(\xi) f_i f_j + (-1)^{i+j-1} v_i(\xi) v_j(\xi) f_i f_j) \cdot \\ &\qquad (v_1 \otimes f_1 \wedge \dots \dots \widehat{v_i \otimes f_i} \dots \widehat{v_j \otimes f_j} \dots \wedge v_k \otimes f_k \\ &= 0,\end{aligned}$$

where the hats indicate omission. It follows that I is an algebra morphism, and a similar argument shows that $D(I(\xi)) = 0$, making I into an analytic DG algebra morphism. It is moreover a quasi-isomorphism because the dimension of $\mathfrak{h}$ is $\dim_{\mathbb{C}} \mathrm{Ext}^i(\mathcal{O}_o, \mathcal{O}_o) = \dim_{\mathbb{C}} (T_o\mathbb{A}^n)^{\wedge i}$. By Corollary 2.16 it follows that $\mathfrak{h}$ is equivalent to the analytic minimal model $\mathcal{H}_{\mathcal{E}}$ of $\mathfrak{g}_{\mathcal{E}}$. □

It follows easily from [VdB15, Lemma 11.2] that $\mathfrak{h}$ admits a cyclic structure: for every choice of trace $\lambda\colon \mathfrak{h}^n \to \mathbb{C}$ the composition

$$\sigma := \lambda \diamond (- \wedge -),$$

is a cyclic structure. Viewing λ as a cocycle in $\mathsf{C}_\lambda^{\bullet,\mathrm{an}}(\mathfrak{h})$, one checks that it corresponds to σ along the map in Corollary 4.21. The following lemma relates λ to an analytic right CY structure on $\mathfrak{g}_{\mathcal{E}|_U,c}$ induced by the volume ω.

Lemma 5.15. *There exists an open $U \subset \mathbb{A}^n$ and unit/homotopy pair (u, h) for $\mathfrak{g}_{\mathcal{E}|_U,c} \subset \mathfrak{g}_{\mathcal{E}}$ with associated quasi-isomorphism K, such that $(K \diamond I)^* \tau_{\omega|_U}$ is homotopic to a trace λ as above.*

Proof. Let $q\colon \mathbb{R} \to \mathbb{R}$ be any smooth real function such that $q(t) = 1$ for $|t| \geq 1$ and let $p\colon \mathbb{R} \to \mathbb{R}$ be the smooth compactly supported function

$$p(t) = \begin{cases} 1 - q(t)t & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| \geq 1 \end{cases}$$

Writing $\partial^*_{z_i} \in (T_o\mathbb{A}^n)^* = V$ for the dual of the standard basis vector ∂_{z_i}, we consider for each $i = 1, \dots, n$ the smooth sections $u_i \in \mathfrak{g}^0_{\mathcal{E}|_{\mathbb{A}^n}}$ and $h_i \in \mathfrak{g}^{-1}_{\mathcal{E}|_{\mathbb{A}^n}}$ defined by

$$u_i(z) = p(|z_i|^2) + p'(|z_i|^2)(\partial^*_{z_i} \wedge -) \cdot \mathrm{d}\overline{z}_i, \quad h_i(z) = q(|z_i|^2)\overline{z}_i(\partial^*_{z_i} \wedge -),$$

which clearly satisfy $u_i = \mathrm{id}_{\mathcal{E}|_{\mathbb{A}^n}} + Dh_i$. The element $u = u_1 \wedge \dots \wedge u_n$ has the closed polydisc $\{z \mid |z_i| \leq 1\ \forall i\}$ as its support and can therefore be interpreted as an element of $\mathfrak{g}_{\mathcal{E}|_U,c}$ for a bounded open neighbourhood U of said polydisc. The element u satisfies $u = \mathrm{id}_{\mathcal{E}|_{\mathbb{A}^n}} + D\widetilde{h}$ with respect to

$$\widetilde{h} = \sum_{k=1}^n h_k + \sum_{i<j} h_i \wedge Dh_j + \ldots + \sum_{j_1 < \dots < j_{n-1}} h_{j_1} \wedge Dh_{j_2} \wedge \dots \wedge Dh_{j_n} + h_1 \wedge Dh_2 \wedge \dots \wedge Dh_n,$$

so a valid unit/homotopy pair can be constructed by extending $\widetilde{h}$ to all of $\mathbb{P}^n$. For this, consider a smooth function t with compact support on $\mathbb{A}^n$ such that $t|_U = 1$. Then $t \cdot \widetilde{h}$ has compact support, and hence extends to a form $i_!(t \cdot \widetilde{h})$ on $\mathbb{P}^n$ which satisfies

$$D(i_!(t \cdot \widetilde{h})) = t(i_!u - \mathrm{id}_{\mathcal{E}}) + \overline{\partial}t \cdot \widetilde{h} = i_!u - \mathrm{id}_{\mathcal{E}} + ((1-t)\mathrm{id}_{\mathcal{E}} + i_!(\overline{\partial}t \cdot \widetilde{h})).$$

Now the element $(1-t)\mathrm{id}_{\mathcal{E}} + i_!(\overline{\partial}t \cdot \widetilde{h}) \in \mathfrak{g}^0_{\mathcal{E}}$ has support in $\mathbb{P}^n \setminus U$, and is therefore given by $D\beta$ for some $\beta \in \mathfrak{g}^{-1}_{\mathcal{E}}$ with compact support in $\mathbb{P}^n \setminus U$. Taking $h = i_!(t \cdot \widetilde{h}) - \beta$ then yields the required extension, making (u, h) into a homotopy pair with $h|_U = \widetilde{h}$. This yields a quasi-isomorphism $K\colon \mathfrak{g}_{\mathcal{E}} \to \mathfrak{g}_{\mathcal{E}|_U,c}$, and $I^*K^*\tau_{\omega|_U}$ is now an analytic negative cyclic cocycle for $\mathfrak{h}$. A computation in polar coordinates shows that the zeroth component maps the top polyvector $\xi = \partial_{z_1} \wedge \dots \wedge \partial_{z_n} \in \mathfrak{h}^n$ to the nonzero value

$$\begin{aligned}(I^*K^*\tau_{\omega|_U})_0(\xi) &= \int_U p'(|z_1|^2) \cdots p'(|z_n|^2) \cdot \omega|_U \wedge \mathrm{d}\overline{z}_1 \wedge \dots \wedge \mathrm{d}\overline{z}_n. \\ &= (-1)^n \int_0^{2\pi} \int_0^1 p'(r_1^2) r_1 \mathrm{d}r_1 \mathrm{d}\theta_1 \cdots \int_0^{2\pi} \int_0^1 p'(r_n)^2 r_n \mathrm{d}r_n \mathrm{d}\theta_n \\ &= (-1)^n (\pi p(1) - \pi \cdot p(0))^n = \pi^n.\end{aligned}$$

A similar computation shows that the higher components of $I^*K^*\tau_{\omega|_U}$ vanish: by inspection

$$\begin{aligned}(I^*K^*\tau_{\omega|_U})_k(\xi_1, \dots, \xi_k)(\xi_0) &= \int_{\mathbb{A}^n} \omega|_U \wedge \mathrm{tr}(u \wedge I(\xi_0) \wedge h \wedge \dots \wedge h \wedge I(\xi_k)) \\ &= \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \int_{\mathbb{A}^n} F_{\xi,\alpha}(|z_1|^2, \dots, |z_n|^2) \overline{z}_1^{\alpha_1} \cdots \overline{z}_n^{\alpha_n} \omega|_U \wedge \mathrm{d}\overline{z}_1 \wedge \dots \wedge \mathrm{d}\overline{z}_n \\ &= \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} (-1)^n \int_{[0,1]^n} F_{\xi,\alpha}(r_1^2, \dots, r_n^2) r^{\alpha+1} \mathrm{d}r \cdot \prod_{j=1}^n \int_0^{2\pi} e^{-\alpha_j \theta_j \cdot i} \mathrm{d}\theta_j.\end{aligned}$$

for some smooth functions $F_{\xi,\alpha}\colon \mathbb{R}^n \to \mathbb{R}$ supported on $[-1,1]^n$, which depend on ξ and a multi-index α with $|\alpha| = \alpha_1 + \ldots + \alpha_n = k$. For $k \geq 1$ there is at least one $\alpha_j \neq 0$, contributing a factor $\int_0^{2\pi} e^{-\alpha_j \theta_j \cdot i} \mathrm{d}\theta_j = 0$ which makes the term in the sum vanish. It follows that $(I^*K^*\tau_{\omega|_U})_k = 0$ for $k \geq 1$, and therefore $\lambda = I^*K^*\tau_{\omega|_U} = (I^*K^*\tau_{\omega|_U})_0$ is a linear functional. ☐

Corollary 5.16. *The pair* $(\mathfrak{h}, \sigma)$ *is equivalent to the cyclic analytic minimal model* $(\mathcal{H}^\omega_{\mathcal{E}}, \sigma^\omega)$.

Proof. Letting U and K be as in Lemma 5.15 above, the map $K \diamond I$ makes the $(\mathfrak{h}, \sigma)$ into the cyclic analytic minimal model of $\mathfrak{g}_{\mathcal{E}|_U,c}$ corresponding to the analytic right CY structure $\tau_{\omega|_U}$ as in

Theorem 4.23. Letting $T \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E}})$ it follows that $K \diamond T \in \mathrm{Hom}^{\mathrm{an}}_{\mathsf{Alg}^\infty}(\mathfrak{g}_{\mathcal{E}}, \mathfrak{g}_{\mathcal{E},c})$ is a quasi-isomorphism such that

$$[(K \diamond T)^* \tau_{\omega|_U}] = [\phi^\omega] \in \mathrm{HC}^\bullet_\lambda(\mathfrak{g}_{\mathcal{E}}),$$

is the canonical analytic right CY structure of Theorem 5.9. But then it follows from Theorem 4.23(3) that there is a cyclic analytic A_∞-isomorphism $(\mathfrak{h}, \sigma) \cong_{\mathrm{cyc,an}} (\mathcal{H}^\omega_{\mathcal{E}}, \sigma^\omega)$. □

Because any point of a smooth variety has an analytic neighbourhood biholomorphic to an open analytic subset of $\mathbb{A}^n \subset \mathbb{P}^n$, we now find a similar result for the cyclic analytic minimal models of arbitrary points.

Proposition 5.17. *Let $p \in X'$ be a closed point in a smooth projective variety X' of dimension n, and let $\mathcal{E}' \to \mathcal{O}_p$ be any resolution by a perfect complex. Then for* every *holomorphic volume germ $\nu \in (\Omega^d_X)_Z$ the cyclic analytic minimal model $\mathcal{H}^\nu_{\mathcal{E}'}$ is cyclic-analytic A_∞-isomorphic to $(\mathfrak{h}, \sigma)$.*

Proof. Let $Y \subset \mathbb{A}^n \subset \mathbb{P}^n$ be a sufficiently small neighbourhood of $o \in \mathbb{A}^n$ so that there is an open embedding $f\colon Y \to X'$ with $f(o) = p$. Then along this map $f^*\mathcal{O}_p \cong \mathcal{O}_o$ and the volume ν pulls back to a form

$$f^*\nu = g(z_1, \dots, z_n)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n,$$

for some nonvanishing holomorphic function g on Y. Let $G(z_1, \dots, z_n)$ be a holomorphic function such that $\frac{\partial G}{\partial z_1} = g$ and $G(0, \dots, 0) = 0$. Then the map $\varphi\colon Y \to \mathbb{A}^n$ given by

$$\varphi(z_1, \dots, z_n) = (G(z_1, \dots, z_n), z_2, \dots, z_n)$$

has a Jacobian $J(\varphi) = \det \frac{\partial G_i}{\partial z_j} = g$ which does not vanish on Y. Shrinking Y if necessary, we obtain an open embedding $\varphi\colon Y \to \mathbb{A}^n$ such that $\varphi^*\mathcal{O}_o \cong \mathcal{O}_o$ and

$$\varphi^*\omega = \mathrm{d}g(z_1, \dots, z_n) \wedge \cdots \wedge \mathrm{d}z_n = f(z_1, \dots, z_n)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = f^*\nu.$$

Applying Theorem 5.12 twice then yields cyclic-analytic A_∞-isomorphisms

$$(\mathfrak{h}, \sigma) = \mathcal{H}^\omega_{\mathcal{E}} \cong_{\mathrm{an,cyc}} \mathcal{H}^{f^*\nu}_{\mathcal{E}} \cong_{\mathrm{an,cyc}} \mathcal{H}^\nu_{\mathcal{E}'}$$ □

References

[AT22] L. Amorim and J. Tu. "On the Morita invariance of Categorical Enumerative Invariants". In: (Sept. 2022). arXiv: 2209.02744 [math.AG].

[BBS13] K. Behrend, J. Bryan, and B. Szendrői. "Motivic degree zero Donaldson-Thomas invariants". In: *Inventiones Mathematicae* 192.1 (2013), pp. 111–160.

[BD19] C. Brav and T. Dyckerhoff. "Relative Calabi-Yau structures". In: *Compositio Mathematica* 155.2 (2019), pp. 372–412.

[Beh09] K. Behrend. "Donaldson-Thomas type invariants via microlocal geometry". In: *Annals of Mathematics. Second Series* 170.3 (2009), pp. 1307–1338.

[Cho08] C.-H. Cho. "Strong homotopy inner product of an A_∞-algebra". English. In: *IMRN. International Mathematics Research Notices* 2008 (2008). Id/No rnn041, p. 35.

[CL11] C.-H. Cho and S. Lee. "Notes on Kontsevich-Soibelman's theorem about cyclic A_∞-algebras". In: *International Mathematics Research Notices. IMRN* 14 (2011), pp. 3095–3140.

[DT98] S. K. Donaldson and R. P. Thomas. "Gauge theory in higher dimensions". In: *The geometric universe (Oxford, 1996)*. Oxford Univ. Press, Oxford, 1998, pp. 31–47.

[Fuk01] K. Fukaya. "Deformation theory, homological algebra and mirror symmetry". In: *4th SIGRAV Graduate School on Contemporary Relativity and Gravitational Physics and 2001 School on Algebraic Geometry and Physics: Geometry and Physics of Branes (SAGP 2001)*. May 2001, pp. 121–209.

[Gan12] S. Ganatra. *Symplectic Cohomology and Duality for the Wrapped Fukaya Category*. Thesis (Ph.D.)–Massachusetts Institute of Technology. ProQuest LLC, Ann Arbor, MI, 2012, (no paging).

[GH78] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978, pp. xii+813.

[HK19] Z. Hua and B. Keller. "Quivers with analytic potentials". In: (Sept. 2019). arXiv: 1909.13517 [math.AG].

[Joy15] D. Joyce. "A classical model for derived critical loci". In: *Journal of Differential Geometry* 101.2 (2015), pp. 289–367.

[JS12] D. Joyce and Y. Song. "A theory of generalized Donaldson-Thomas invariants". In: *Memoirs of the American Mathematical Society* 217.1020 (2012), pp. iv+199.

[Kad80] T. Kadeishvili. "On the homology theory of fibre spaces". In: *Russian Mathematical Surveys* 35 (1980), pp. 231–238. arXiv: math/0504437.

[Kaj07] H. Kajiura. "Noncommutative homotopy algebras associated with open strings". In: *Reviews in Mathematical Physics. A Journal for Both Review and Original Research Papers in the Field of Mathematical Physics* 19.1 (2007), pp. 1–99.

[Kat08] S. Katz. "Genus zero Gopakumar-Vafa invariants of contractible curves". In: *Journal of Differential Geometry* 79.2 (2008), pp. 185–195.

[KS08] M. Kontsevich and Y. Soibelman. *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations.* Nov. 2008. arXiv: 0811.2435 [math.AG].

[KS09] M. Kontsevich and Y. Soibelman. "Notes on A_∞-algebras, A_∞-categories and non-commutative geometry". In: *Homological mirror symmetry.* Vol. 757. Lecture Notes in Phys. Springer, Berlin, 2009, pp. 153–219.

[Lod92] J.-L. Loday. *Cyclic homology.* Vol. 301. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Appendix E by Maria O. Ronco. Springer-Verlag, Berlin, 1992, pp. xviii+454.

[MMNS12] A. Morrison, S. Mozgovoy, K. Nagao, and B. Szendrői. "Motivic Donaldson-Thomas invariants of the conifold and the refined topological vertex". English. In: *Advances in Mathematics* 230.4-6 (2012), pp. 2065–2093.

[Pol01] A. Polishchuk. "Homological mirror symmetry with higher products". English. In: *Winter school on mirror symmetry, vector bundles and Lagrangian submanifolds. Proceedings of the winter school on mirror symmetry, Cambridge, MA, USA, January 1999.* Providence, RI: American Mathematical Society (AMS); Somerville, MA: International Press, 2001, pp. 247–259.

[RS21] A. T. Ricolfi and M. Savvas. "The d-critical structure on the Quot scheme of points of a Calabi-Yau 3-fold". In: (June 2021). arXiv:2106.16133: 2106.16133 (math.AG).

[Sze08] B. Szendrői. "Non-commutative Donaldson-Thomas invariants and the conifold". In: *Geometry & Topology* 12.2 (2008), pp. 1171–1202.

[Tho00] R. P. Thomas. "A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations". In: *Journal of Differential Geometry* 54.2 (2000), pp. 367–438.

[Tod18] Y. Toda. "Moduli stacks of semistable sheaves and representations of Ext-quivers". English. In: *Geometry & Topology* 22.5 (2018), pp. 3083–3144.

[Tod21] Y. Toda. *Recent progress on the Donaldson-Thomas theory—wall-crossing and refined invariants.* Vol. 43. SpringerBriefs in Mathematical Physics. Springer, Singapore, 2021, pp. viii+104.

[Toë11] B. Toën. "Lectures on dg-categories". In: *Topics in algebraic and topological K-theory.* Vol. 2008. Lecture Notes in Math. Springer, Berlin, 2011, pp. 243–302.

[Tra08] T. Tradler. "Infinity-inner-products on A-infinity-algebras." English. In: *Journal of Homotopy and Related Structures* 3.1 (2008), pp. 245–271.

[Tu14] J. Tu. "Homotopy L-infinity spaces". In: (Nov. 2014). arXiv: 1411.5115 [math.AG].

[VdB15] M. Van den Bergh. "Calabi-Yau algebras and superpotentials". In: *Selecta Mathematica. New Series* 21.2 (2015), pp. 555–603.