STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS ASSOCIATED WITH FELLER PROCESSES

JIAN SONG, MENG WANG, AND WANGJUN YUAN

ABSTRACT. For the stochastic partial differential equation $\frac{\partial u}{\partial t} = \mathcal{L}u + u\dot{W}$ where $\dot{W}$ is Gaussian noise colored in time and $\mathcal{L}$ is the infinitesimal generator of a Feller process X, we obtain Feynman-Kac type of representations for the Stratonovich and Skorohod solutions as well as for their moments. The regularity of the law and the Hölder continuity of the solutions are also studied.

CONTENTS

1. INTRODUCTION

Consider the following SPDE in $\mathbb{R}^d$,

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \mathcal{L}u(t,x) + u(t,x)\dot{W}(t,x), & t \geq 0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $u_0(x)$ is a bounded measurable function, $\dot{W}(t,x)$ is Gaussian noise with covariance given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = |t-s|^{-\beta_0}\gamma(x-y), \tag{1.2}$$

with $\beta_0 \in [0,1)$ and $\gamma(x)$ being a non-negative and non-negative definite (generalized) function, and $\mathcal{L}$ is the infinitesimal generator of a time-homogeneous Markov process $X = \{X_t, t \geq 0\}$ which is independent of the noise $\dot{W}$. Let $\mu(d\xi) = \hat{\gamma}(\xi)d\xi$ be the spectral measure of the noise, i.e.,

$\int_{\mathbb{R}^d} \varphi(x)\gamma(x)dx = \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\mu(d\xi)$ for any Schwartz function $\varphi(x)$. Recall that μ is a positive and tempered measure, i.e., $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-p}\mu(d\xi) < \infty$ for some $p > 0$.

A typical example of $\dot{W}$ is given by the partial derivative of fractional Brownian sheet W^H with Hurst parameter H_0 in time and $(H_1, \cdots, H_d)$ in space satisfying $H = (H_0, H_1, \dots, H_d) \in (\frac{1}{2}, 1)^{1+d}$: the Gaussian noise $\dot{W}^H(t,x) = \frac{\partial^{d+1} W^H}{\partial t \partial x_1 \cdots \partial x_d}(t,x)$ is a distribution-valued Gaussian random variable with the covariance function (up to a multiplicative constant)

$$\mathbb{E}[\dot{W}^H(t,x)\dot{W}^H(s,y)] = |t-s|^{2H_0-2}\prod_{i=1}^d |x_i - y_i|^{2H_i-2}. \tag{1.3}$$

In this case, $\gamma(x) = \prod_{i=1}^d |x_i|^{2H_i-2}$ with $\hat{\gamma}(\xi) = \prod_{i=1}^d |\xi_i|^{1-2H_i}$. Another typical example of $\gamma(x)$ is the Dirac delta function $\delta(x)$ whose Fourier transform is the constant 1. Some other interesting examples of the spatial covariance function are, for instance, the Riesz kernel $\gamma(x) = |x|^{-\alpha}, \alpha \in (0,d)$ with $\hat{\gamma}(\xi) = |\xi|^{\alpha - d}$, the Cauchy kernel $\gamma(x) = \prod_{i=1}^d (1+x^2)^{-1}$ with $\hat{\gamma}(\xi) = \exp\left(\sum_{i=1}^d |\xi_i|\right)$, the Poisson kernel $\gamma(x) = (1+|x|^2)^{-(d+1)/2}$ with $\hat{\gamma}(\xi) = \exp(-|\xi|)$, and the Ornstein-Uhlenbeck kernel $\gamma(x) = \exp(-|x|^\alpha), \alpha \in (0,2]$.

Let $p_{t-s}^{(x)}(y) = \mathbb{P}(X_t = y | X_s = x)$ be the probability transition density of the time-homogeneous Markov process X. We assume the following condition on X throughout the paper:

Assumption (H). *We assume $p_t^{(x)}(y) \le C P_t(y-x)$ for some non-negative function $P_t(x)$ satisfying*

$$0 \le \hat{P}_t(\xi) \le c_1 \exp(-c_2 t \Psi(\xi)), \tag{1.4}$$

where c_1, c_2, C are positive constants and $\Psi(\xi) = \Psi(|\xi|)$ is a non-negative measurable function satisfying $\lim_{|\xi|\to\infty} \Psi(\xi) = \infty$.

Clearly, symmetric Lévy processes X such as Brownian motion and α-stable process satisfy Assumption (H) with $\Psi(\xi)$ being the characteristic exponent of X. Moreover, the diffusion process X^x governed by the stochastic differential equation

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dB_s, \quad s > 0, \tag{1.5}$$

where $\{B_t, t \ge 0\}$ a d-dimensional Brownian motion, also satisfies Assumption (H), if we assume the uniform ellipticity condition: there exists a constant $c > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$y^*\sigma(x)\sigma^*(x)y \ge c|y|^2. \tag{1.6}$$

Indeed, by By Theorem 1.5 of Baudoin, Nualart, Ouyang and Tindel (AOP 2016), under the condition (1.6), we have for all $y \in \mathbb{R}^d$,

$$p_t^{(x)}(y) \le c_1 t^{-d/2} \exp\left(-\frac{|y-x|^2}{c_2 t}\right), \tag{1.7}$$

and hence Assumption (H) is satisfied. In this situation, the differential operator $\mathcal{L}$ is of the following form:

$$\mathcal{L}u = \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^{\mathrm{T}})_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}u + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}u.$$

The equation (1.1) can be understood in the Stratonovich sense and in the Skorohod sense, depending on the definition of the product $u\dot{W}$. We say that it is a Stratovich equation if the product is ordinary and a Skorohod equation if the product is a Wick product. Note that if the noise is white in time, Skorohod solution reduces to Itô solution.

To obtain the existence of the solutions, we propose the following Dalang's conditions (see [6]):

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+\Psi(\xi)}\right)^{1-\beta_0} \mu(d\xi) < \infty \tag{1.8}$$

and

$$\int_{\mathbb{R}^d} \frac{1}{1+\Psi(\xi)} \mu(d\xi) < \infty \tag{1.9}$$

for Stratonovich equation and Skorohod equation, respectively. Clearly, (1.8) implies (1.9), noting that $\beta_0 \in [0,1)$ and $\Psi(\xi) \to \infty$ as $|\xi| \to \infty$.

Denote by X_t^x the Feller process X which starts at x at time $t=0$. We shall prove that under the stronger condition (1.8), the Stratonovich solution u^{st} and the Skorohod solution u^{sk} can be represented by the following Feynman-Kac type of formulas respectively,

$$u^{\mathrm{st}}(t,x) = \mathbb{E}_X \left[u_0(X_t) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta(X_{t-s}^x - y) W(ds, dy) \right) \right] \tag{1.10}$$

and

$$u^{\mathrm{sk}}(t,x) = \mathbb{E}_X \left[u_0(X_t) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta(X_{t-s}^x - y) W(ds, dy) - \frac{1}{2} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r^x - X_s^x) dr ds \right) \right], \tag{1.11}$$

where $\delta(\cdot)$ is the Dirac delta function, $\mathbb{E}_X$ is the expectation in the probability space generated by X (similarly, we will use $\mathbb{E}_W$ to denote the expectation with respect to the noise $\dot{W}$).

Note that there exists a unique Skorohod solution under the weaker condition (1.9) while the Feynman-Kac formula (1.11) may not hold as the Itô-Stratonovich correction term $\frac{1}{2}\int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r^x - X_s^x) dr ds$ is infinite if the condition (1.8) is violated. As a contrast, the moments of the Stratonovich solution and the Skorohod solution can be represented by Feynman-Kac type formulas under (1.8) and under (1.9), respectively. With the help of Feynman-Kac formula, we also study the properties of the solutions such as the regularity of the probability law and the Hölder continuity.

To close the introduction, we provide a short survey and comments on the related works which by no means is complete.

For the heat equation on $\mathbb{R}$ driven by space-time white noise, it has a unique Itô solution which cannot be presented by a formula in the form of (1.11) since the Itô-Stratonovich correction term is infinite (see [26]). For heat equations driven by Gaussian noise induced by fractional Brownian sheet, the existence of Feynman-Kac formulae was conjectured in [18] and was later on established in [14]. The result in [14] was extended to general Gaussian noise in [10] and to the equation $\frac{\partial u}{\partial t} = Lu + u\dot{W}$ where L is the infinitesimal generator of a symmetric Lévy process. One key ingredient in the validation of Feynman-Kac formulae is to establish the exponential integrability of the Hamiltonian $\int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r^x - X_s^x) dr ds$ which guarantees the well-definedness of the Feynman-Kac representations (1.10) and (1.11). In [14, 10] where X is Brownian motion, the exponential integrability was proved by using the technique developed in [16] which employs some unique properties of Brownian motion. For a general Feller process X, Le Gall's technique is not convenient. Our method is inspired by [24] and [23] which only involves the Markov property of X and some Fourier analysis techniques (see Section 3.1). Another key ingredient of validating Feynman–Kac formula (1.10) is to justify that it does solve (1.1) in some sense, and we shall follow the approach in [14, 10, 24] in which Malliavin calculus was used and prove that (1.10) is a mild Stratonovich solution to (1.1) (see Section 3.2).

With the help of Feynman-Kac formulae, it is more convenient to study the properties of the solutions such as the regularity of the distribution law which will be studied in Sections 3.3 and 4.2. For one-dimensional heat equation with space-time white noise, by using Malliavin calculus, the absolute continuity of the law of the solution was obtained in [22], and the smoothness of the probability density was proved in [2, 19] under proper conditions. For SPDEs of the form $Lu = b(u)+\sigma(u)\dot{W}$, the absolute continuity of the law was deduced in [25] when L is a psudodiffererential operator and $\dot{W}$ is space-time white noise, and the smoothness was obtained in [21] when L is a parabolic/hyperbolic operator and $\dot{W}$ is Gaussian noise white in time and homogeneous in space. The smoothness of the density for stochastic heat equations was obtained by using Malliavin calculus together with Feynman-Kac formula in [14, 15].

Another application of Feynman-Kac formula is to study the Hölder continuity for the Stratonovich solution (see Section 3.4); the Hölder continuity of the Skorohod solution is analysed by its Wiener chaos expansion (see Section 4.3), as in general it does not have the Feynman-Kac representation (1.11) under the condition (1.9). The Hölder continuity of the solutions to (fractional) heat equations has been studied in, for instance, [14, 10, 24, 1, 11]. We also refer to a survey [9] and the references therein.

This paper is organized as follows. In Section 2, some preliminaries on the Wiener space associated with the noise $\dot{W}$, in particular some fundamental ingredients in Malliavin calculus, are collected. In Section 3 and Section 4, the Stratonovich solution and the Skorohod solution are studied, respectively. In Appendix A, the Feynman-Kac formula for PDE $\partial_t u = \mathcal{L}u + f(t,x)u$ where $\mathcal{L}$ is the infinitesimal generator of a Feller process is provided.

2. Preliminaries

In this section, we provide some preliminaries on the Wiener space associated with the Gaussian noise $\dot{W}$ of which the covariance is given by (1.2). In particular, some basic elements in Malliavin calculus is recalled. We refer to [20, 8] for more details.

Let $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ be the space of smooth functions on $\mathbb{R}_+ \times \mathbb{R}^d$ with compact support. Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ with respect to the inner product given below,

$$\begin{aligned}\langle f, g\rangle_{\mathcal{H}} :&= \int_{\mathbb{R}_+^2}\int_{\mathbb{R}^{2d}} f(t,x)g(s,y)|t-s|^{-\beta_0}\gamma(x-y)dxdydsdt \\ &= \int_{\mathbb{R}_+^2}\int_{\mathbb{R}^{d}} \hat{f}(t,\xi)\overline{\hat{g}(s,\xi)}|t-s|^{-\beta_0}\mu(d\xi)dsdt.\end{aligned} \tag{2.12}$$

Here $\hat{f}$ means the Fourier transform in the space variable. In particular, for $f \in L^1(\mathbb{R}^d)$, its Fourier transform can be defined by the integral $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-\iota x\cdot\xi} f(x)dx$, where ι is the imaginary unit.

Let $W = \{W(f), f \in \mathcal{H}\}$ be a centered Gaussian family (also called an isonormal Gaussian process) with covariance

$$\mathbb{E}[W(f)W(g)] = \langle f, g\rangle_{\mathcal{H}}, \text{ for } f, g \in \mathcal{H}.$$

We call $W(f)$ a Wiener integral which is also denoted by $W(f) := \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} f(s,x)W(ds,dx)$. Denoting $W(t,x) := W(I_{[0,t]\times[0,x]})$, then $\{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a random field and we have $\dot{W}(t,x) = \frac{\partial^{1+d}}{\partial t \partial x_1 \cdots \partial x_d} W(t,x)$ in the sense of distribution. It is of interest to consider the following case as a toy model: when $\beta_0 = 2 - 2H_0$ and $\gamma(x-y) = \prod_{i=1}^d |x_i - y_i|^{2H_i - 2}$ with $H_i \in (0,1)$ for $i = 0, 1, \ldots, d$, the random field $\{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is the so-called fractional Brownian sheet (up to some multiplicative constant) with Hurst parameters H_0 in time and $(H_1, \ldots, H_d)$ in space.

Denote the mth Hermite polynomial by $H_m(x) := (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}$ for $m \in \mathbb{N} \cup \{0\}$. For $g \in \mathcal{H}$ with $\|g\|_{\mathcal{H}} = 1$, the mth multiple Wiener integral of $g^{\otimes m} \in \mathcal{H}^{\otimes m}$ is defined by $I_m(g^{\otimes m}) := H_m(W(g))$. In particular, we have $W(g) = I_1(g)$. For $f \in \mathcal{H}^{\otimes m}$, denote the symmetrization of f by

$$\tilde{f}(t_1, x_1, \dots, t_m, x_m) := \frac{1}{m!} \sum_{\sigma \in S_m} f(t_{\sigma(1)}, x_{\sigma(1)}, \dots, t_{\sigma(m)}, x_{\sigma(m)}),$$

where S_m is the set of all permutations of $\{1, 2, \dots, m\}$. Let $\mathcal{H}^{\tilde{\otimes} m}$ be the symmetrization of $\mathcal{H}^{\otimes m}$. Then for $f \in \mathcal{H}^{\tilde{\otimes} m}$, one can define the mth multiple Wiener integral $I_m(f)$ by a limiting argument. Moreover, for $f \in \mathcal{H}^{\tilde{\otimes} m}$ and $g \in \mathcal{H}^{\tilde{\otimes} n}$, we have

$$\mathbb{E}[I_m(f) I_n(g)] = m! \langle f, g \rangle_{\mathcal{H}^{\otimes m}} \delta_{mn}, \tag{2.13}$$

where δ_{mn} is the Kronecker delta function. For $f \in \mathcal{H}^{\otimes m}$ which is not necessarily symmetric, we simply define $I_m(f) := I_m(\tilde{f})$. For a square integrable random variable F which is measurable with respect to the σ-algebra generated by W, it has the following expansion (Wiener chaos expansion): $F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n)$ with $f_n \in \mathcal{H}^{\tilde{\otimes} n}$ for $n \in \mathbb{N}$ which is unique.

Now, let us collect some knowledge on Malliavin calculus that will be used in this paper. Let F be a smooth cylindrical random variable, i.e., F is of the form $F = f(W(\phi_1), \cdots, W(\phi_n))$, where $\phi_i \in \mathcal{H}$ and f is a smooth function of which all the derivatives are of polynomial growth. Then the Malliavin derivative DF of F is defined by

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} (W(\phi_1), \cdots, W(\phi_n)) \phi_j.$$

Noting that the operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$, we can define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space $\mathfrak{S}$ of all smooth cylindrical random variables under the norm

$$\|F\|_{1,2} = \left(\mathbb{E}[F^2] + \mathbb{E}[\|DF\|_{\mathcal{H}}^2] \right)^{1/2}.$$

Similarly, one can define the kth Malliavin derivative $D^k F$ as an $\mathcal{H}^{\otimes k}$-valued variable for $k \geq 2$, and for any $p > 1$ let $\mathbb{D}^{k,p}$ be the completion of $\mathfrak{S}$ under the norm

$$\|F\|_{k,p} = \left(\mathbb{E}[|F|^p] + \sum_{j=1}^{k} \mathbb{E}[\|D^j F\|_{\mathcal{H}^{\otimes j}}^p] \right)^{1/p}.$$

Then we define $\mathbb{D}^{\infty} = \cap_{k,p=1}^{\infty} \mathbb{D}^{k,p}$.

The divergence operator $\boldsymbol{\delta}$ (also called Skorohod integral) is defined by the duality

$$\mathbb{E}[\boldsymbol{\delta}(u) F] = \mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}]$$

for $F \in \mathbb{D}^{1,2}$ and $u \in \text{Dom}\, \boldsymbol{\delta}$, where $\text{Dom}\, \boldsymbol{\delta}$ is the domain of the divergence operator which is the set of $u \in L^2(\Omega, \mathcal{H})$ such that $|\mathbb{E}[\langle DF, u]\rangle_{\mathcal{H}}| \leq c_u \|F\|_2$ for all $F \in \mathbb{D}^{1,2}$. The second moment of $\boldsymbol{\delta}(u)$ has the following upper bound:

$$\mathbb{E}[|\boldsymbol{\delta}(u)|^2] \leq \mathbb{E}[\|u\|_{\mathcal{H}}^2] + \mathbb{E}[\|Du\|_{\mathcal{H}^{\otimes 2}}^2]. \tag{2.14}$$

The following formula will be used in the proof

$$FW(\phi) = \boldsymbol{\delta}(F\phi) + \langle DF, \phi \rangle_{\mathcal{H}}, \tag{2.15}$$

for $\phi \in \mathcal{H}$ and $F \in \mathbb{D}^{1,2}$.

3. Stratonovich solution

In this section, we will derive the Feynman-Kac formula (1.10) for the Stratonovich equation (1.1).

3.1. **Exponential integrability of Hamiltonian.**

In this subsection, we shall define the Hamiltonian

$$\int_0^t \int_{\mathbb{R}^d} \delta(X^x_{t-s} - y) W(ds, dy)$$

appearing in Feynman-Kac formulas (1.10) via approximation, and then prove that it is exponentially integrable. We shall follow the approach used in [14, 24].

We use $\varphi_\delta(t)$ and $q_\varepsilon(x)$ to approximate the Dirac delta functions in space and in time, respectively, where $\varphi_\delta(t) = \frac{1}{\delta} I_{[0,\delta]}(t)$ for $t \geq 0$ and $q_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} e^{\frac{-|x|^2}{2\varepsilon}}$ for $x \in \mathbb{R}^d$. Thus,

$$A^{\varepsilon,\delta}_{t,x}(r, y) = \int_0^t \varphi_\delta(t - s - r) q_\varepsilon(X^x_s - y) ds, \tag{3.16}$$

is an approximation of $\delta(X^x_{t-r} - y)$ when ε and δ are small.

In the theorem below, we will show that $A^{\varepsilon,\delta}_{t,x} \in \mathcal{H}$ almost surely for $\varepsilon, \delta > 0$, indicating that

$$V^{\varepsilon,\delta}_{t,x} = \int_0^t \int_{\mathbb{R}^d} A^{\varepsilon,\delta}_{t,x}(r, y) W(dr, dy) = W(A^{\varepsilon,\delta}_{t,x}) \tag{3.17}$$

is a well-defined Wiener integral (conditional on X), and we shall show the L^2-convergence of $V^{\varepsilon,\delta}_{t,x}$ as $(\varepsilon, \delta) \to 0$ which defines the Hamiltonian.

Theorem 3.1. *Assume Assumption (H) and condition* (1.8)*. Then, $A^{\varepsilon,\delta}_{t,x}$ belongs to $\mathcal{H}$ almost surely for all $\varepsilon, \delta > 0$. Furthermore, $V^{\varepsilon,\delta}_{t,x}$ converges in L^2 as $(\varepsilon, \delta) \to 0$, the limit $V_{t,x}$ being denoted by*

$$V_{t,x} =: \int_0^t \int_{\mathbb{R}^d} \delta(X^x_{t-r} - y) W(dr, dy) = W(\delta(X^x_{t-\cdot} - \cdot)). \tag{3.18}$$

Conditional on X, $V_{t,x}$ is a Gaussian random variable with mean 0 *and variance*

$$Var[V_{t,x}] = \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X^x_r - X^x_s) dr ds. \tag{3.19}$$

Proof. The proof essentially follows from that of [24, Theorem 4.1]. Firstly, in order to show $A^{\varepsilon,\delta}_{t,x} \in \mathcal{H}$ for $\varepsilon, \delta > 0$, we compute

$$\begin{aligned}\left\langle A^{\varepsilon,\delta}_{t,x}, A^{\varepsilon',\delta'}_{t,x} \right\rangle_{\mathcal{H}} = \int_{[0,t]^4} \int_{\mathbb{R}^{2d}} &\varphi_\delta(t - s - u) q_\varepsilon(X^x_s - y) |u - v|^{-\beta_0} \\ &\times \gamma(y - z) \varphi_{\delta'}(t - r - v) q_{\varepsilon'}(X^x_r - z) dy dz du dv ds dr.\end{aligned} \tag{3.20}$$

By Lemma 3.1, we have, for $\varepsilon, \delta, \varepsilon', \delta' > 0$,

$$\left\langle A^{\varepsilon,\delta}_{t,x}, A^{\varepsilon',\delta'}_{t,x} \right\rangle_{\mathcal{H}} \leq C \int_{[0,t]^2} \int_{\mathbb{R}^{2d}} q_\varepsilon(X^x_s - y) q_{\varepsilon'}(X^x_r - z) |s - r|^{-\beta_0} \gamma(y - z) dy dz ds dr, \tag{3.21}$$

and then the Parseval–Plancherel identity implies that

$$\left\langle A^{\varepsilon,\delta}_{t,x}, A^{\varepsilon',\delta'}_{t,x} \right\rangle_{\mathcal{H}} \leq C \int_{[0,t]^2} |s - r|^{-\beta_0} ds dr \times \int_{\mathbb{R}^d} \hat{q}_\varepsilon(\xi) \hat{q}_{\varepsilon'}(\xi) \mu(d\xi).$$

The product of the integrals on the right-hand side is finite due to the condition $\beta_0 < 1$ and the fact

$$\int_{\mathbb{R}^d} \hat{q}_\varepsilon(\xi)\hat{q}_{\varepsilon'}(\xi)\mu(d\xi) \le \int_{\mathbb{R}^d} \hat{q}_\varepsilon(\xi)\mu(d\xi) < \infty$$

where the second step follows from the fact that $\mu(d\xi)$ is a tempered distribution while $\hat{q}_\varepsilon(\xi)$ decays exponentially fast as $|\xi| \to \infty$. Thus, we have $A^{\varepsilon,\delta}_{t,x} \in \mathcal{H}$ a.s.

Now we show that $\{A^{\varepsilon,\delta}_{t,x}\}_{\varepsilon,\delta>0}$ is a Cauchy sequence in $L^2(\Omega;\mathcal{H})$ as $(\varepsilon,\delta) \to 0$. Taking expectation for (3.20), we have

$$\begin{aligned}\mathbb{E}\left\langle A^{\varepsilon,\delta}_{t,x}, A^{\varepsilon',\delta'}_{t,x}\right\rangle_{\mathcal{H}} = \int_{[0,t]^4}\int_{\mathbb{R}^{2d}} &\varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{-\beta_0} \\ &\times q_\varepsilon(y)q_{\varepsilon'}(z)\mathbb{E}[\gamma(X^x_s - X^x_r - (y-z))]dydzdudvdsdr.\end{aligned}$$

Without loss of generality, we assume $r < s$. Then, by the Markov property of X and Assumption (H), we have

$$\begin{aligned}&\mathbb{E}[\gamma(X^x_s - X^x_r - (y-z))] = \mathbb{E}\left[\mathbb{E}\left[\gamma(X^x_s - X^x_r - (y-z))|X^x_r\right]\right] \\ &= \mathbb{E}\int_{\mathbb{R}^d} \gamma(u - X^x_r - (y-z))p^{(X^x_r)}_{s-r}(u)du \\ &\le C\mathbb{E}\int_{\mathbb{R}^d} \gamma(u - X^x_r - (y-z))P_{s-r}(u - X^x_r)du \\ &\le C\int_{\mathbb{R}^d} e^{-c_2(s-r)\Psi(\xi)}\mu(d\xi),\end{aligned}$$

and thus

$$\int_{\mathbb{R}^{2d}} q_\varepsilon(y)q_{\varepsilon'}(z)\mathbb{E}[\gamma(X^x_s - X^x_r - (y-z))]dydz \le C\int_{\mathbb{R}^d} e^{-c_2(s-r)\Psi(\xi)}\mu(d\xi).$$

Combining this fact with Lemma 3.1, by dominated convergence theorem we can get, as $\varepsilon, \delta, \varepsilon', \delta'$ go to zero,

$$\mathbb{E}\left\langle A^{\varepsilon,\delta}_{t,x}, A^{\varepsilon',\delta'}_{t,x}\right\rangle_{\mathcal{H}} \to \mathbb{E}\int_{[0,t]^2} |s-r|^{-\beta_0}\gamma(X^x_s - X^x_r)dsdr,$$

the right-hand side of which is finite due to [24, Proposition 3.2]. This implies that $\{V^{\varepsilon,\delta}_{t,x} = W(A^{\varepsilon,\delta}_{t,x})\}_{\varepsilon,\delta>0}$ is a Cauchy sequence in $L^2(\Omega)$ as $(\varepsilon,\delta) \to 0$.

Finally, the formula (3.19) holds true because

$$\lim_{\varepsilon,\delta\to 0}\left\langle A^{\varepsilon,\delta}_{t,x}, A^{\varepsilon,\delta}_{t,x}\right\rangle_{\mathcal{H}} = \int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X^x_r - X^x_s)drds$$

by Scheffé's Lemma, where the limit is taken in $L^1(\Omega)$. □

Theorem 3.2. *Assume Assumption (H) and condition* (1.8)*. Then, for any* $\beta \in \mathbb{R}$*, we have*

$$\mathbb{E}_W\mathbb{E}_X\left[\exp\left(\beta\int_0^t\int_{\mathbb{R}^d} \delta(X^x_{t-r} - y)W(dr,dy)\right)\right] < \infty.$$

Proof. Recalling (3.18), we have

$$\begin{aligned}&\mathbb{E}_W\mathbb{E}_X\left[\exp\left(\beta V_{t,x}\right)\right] = \mathbb{E}_X\left[\exp\left(\tfrac{\beta^2}{2}Var[V_{t,x}]\right)\right] \\ &= \mathbb{E}_X\left[\exp\left(\tfrac{\beta^2}{2}\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X^x_r - X^x_s)drds\right)\right].\end{aligned}$$

Then the desired result follows from Theorem 3.3. □

Theorem 3.3. *Assume Assumption (H) and condition* (1.8)*. Then, we have for all* $\beta \in \mathbb{R}$*,*

$$\mathbb{E}\left[\exp\left(\beta\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^x - X_s^x)drds\right)\right] < \infty.$$

Proof. In the proof we shall omit the superscript x in X_t^x. By Taylor's expansion, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\beta\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr\right)\right]\\
&=\sum_{m=0}^\infty \frac{\beta^m}{m!}\int_{[0,t]^{2m}}\prod_{i=1}^m |s_{2i}-s_{2i-1}|^{-\beta_0}\mathbb{E}\left[\prod_{i=1}^m \gamma(X_{s_{2i}} - X_{s_{2i-1}})\right]d\mathbf{s}\\
&=\sum_{m=0}^\infty \frac{\beta^m}{m!}\sum_{\tau\in S_{2m}}\int_{[0,t]^{2m}_<}\prod_{i=1}^m |s_{\tau(2i)}-s_{\tau(2i-1)}|^{-\beta_0}\mathbb{E}\left[\prod_{i=1}^m \gamma(X_{s_{\tau(2i)}} - X_{s_{\tau(2i-1)}})\right]d\mathbf{s},
\end{aligned}$$

where $[0,t]^{2m}_< = \{0 < s_1 < \ldots < s_{2m} < t\}$, $d\mathbf{s} = ds_1 \ldots ds_{2m}$, and S_{2m} is the set of all permutations on $\{1, 2, \ldots, 2m\}$.

For each fixed $\tau \in S_{2m}$, we denote by $t_i^+ = \max\{s_{\tau(2i)}, s_{\tau(2i-1)}\}$ and $t_i^- = \min\{s_{\tau(2i)}, s_{\tau(2i-1)}\}$. We have,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\beta\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr\right)\right]\\
&=\sum_{m=0}^\infty \frac{\beta^m}{m!}\sum_{\tau\in S_{2m}}\int_{[0,t]^{2m}_<}\prod_{i=1}^m |t_i^+ - t_i^-|^{-\beta_0}\mathbb{E}\left[\prod_{i=1}^m \gamma(X_{t_i^+} - X_{t_i^-})\right]d\mathbf{s},
\end{aligned}\tag{3.22}$$

Let t_i^* be the unique s_i that is closest to t_i^+ from the left. Then we always have $t_i^+ \geq t_i^* \geq t_i^-$ for $1 \leq i \leq m$. Let k be such that $t_k^+ = s_{2m}$. Then,

$$\mathbb{E}\left[\prod_{i=1}^m \gamma(X_{t_i^+} - X_{t_i^-})\right] = \mathbb{E}\left[\prod_{i\neq k}\gamma(X_{t_i^+} - X_{t_i^-})\mathbb{E}\left[\gamma(X_{t_k^+} - X_{t_k^-})\Big|\mathcal{F}_{t_k^*}\right]\right]$$

By Assumption (H), we have

$$\begin{aligned}
&\mathbb{E}\left[\gamma(X_{t_k^+} - X_{t_k^-})\Big|\mathcal{F}_{t_k^*}\right] = \int_{\mathbb{R}^d} p_{t_k^+ - t_k^*}^{(X_{t_k^*})}(y)\gamma(y - X_{t_k^-})dy\\
\leq& C\int_{\mathbb{R}^d} P_{t_k^+ - t_k^*}(y)\gamma(y + X_{t_k^*} - X_{t_k^-})dy\\
=& C\int_{\mathbb{R}^d}\exp\left(-c_2(t_k^+ - t_k^*)\Psi(\xi)\right)\hat{\gamma}(\xi)\exp\left(-\iota\xi\cdot(X_{t_k^*} - X_{t_k^-})\right)d\xi\\
\leq& C\int_{\mathbb{R}^d}\exp\left(-c_2(t_k^+ - t_k^*)\Psi(\xi)\right)\mu(d\xi),
\end{aligned}$$

where the second equality follows from the Parseval-Plancherel identity. Hence, we have

$$\mathbb{E}\left[\prod_{i=1}^m \gamma(X_{t_i^+} - X_{t_i^-})\right] \leq C\int_{\mathbb{R}^d}\exp\left(-c_2(t_k^+ - t_k^*)\Psi(\xi)\right)\mu(d\xi) \times \mathbb{E}\left[\prod_{i\neq k}\gamma(X_{t_i^+} - X_{t_i^-})\right],$$

and by repeating this procedure, we have

$$\mathbb{E}\left[\prod_{i=1}^m \gamma(X_{t_i^+} - X_{t_i^-})\right] \leq C^m\int_{\mathbb{R}^{md}}\prod_{i=1}^m\exp\left(-c_2(t_i^+ - t_i^*)\Psi(\xi_i)\right)\mu(d\xi_i)\tag{3.23}$$

Thus, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\beta\int_{[0,t]^2}|r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\right]\\
&\le\sum_{m=0}^{\infty}C^m\frac{\beta^m}{m!}\sum_{\tau\in S_{2m}}\int_{[0,t]_<^{2m}}d\mathbf{s}\int_{\mathbb{R}^{md}}\boldsymbol{\mu}(d\boldsymbol{\xi})\exp\left(-c_2\sum_{i=1}^m(t_i^+-t_i^*)\Psi(\xi_i)\right)\prod_{i=1}^m|t_i^+-t_i^-|^{-\beta_0}.
\end{aligned}$$

To deal with the above integrals, we apply the change of variables $r_i=s_i-s_{i-1}$ for $i=1,\dots,2m$ with the convention $s_0=0$ and then we have

$$[0,t]_<^{2m}=[0<s_1<\cdots<s_{2m}<t]=[0<r_1+\cdots+r_{2m}<t]\cap\mathbb{R}_+^{2m}:=\Sigma_t^{2m}.$$

By the symmetry of the integrals, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\beta\int_{[0,t]^2}|r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\right]\\
&\le\sum_{m=0}^{\infty}(C\beta)^m\frac{(2m)!}{m!}\int_{\Sigma_t^{2m}}d\mathbf{r}\int_{\mathbb{R}^{md}}\boldsymbol{\mu}(d\boldsymbol{\xi})\exp\left(-c_2\sum_{i=1}^m r_i\Psi(\xi_i)\right)\prod_{i=1}^m|r_i|^{-\beta_0}\\
&=\sum_{m=0}^{\infty}(C\beta)^m\frac{(2m)!}{m!}\int_{[0,t]_<^{2m}}d\mathbf{s}\int_{\mathbb{R}^{md}}\boldsymbol{\mu}(d\boldsymbol{\xi})\exp\left(-c_2\sum_{i=1}^m(s_i-s_{i-1})\Psi(\xi_i)\right)\prod_{i=1}^m|s_i-s_{i-1}|^{-\beta_0}\\
&\le\sum_{m=0}^{\infty}(C\beta)^m\frac{(2m)!}{m!}\frac{t^m}{m!}\int_{[0,t]_<^{m}}d\mathbf{s}\int_{\mathbb{R}^{md}}\boldsymbol{\mu}(d\boldsymbol{\xi})\exp\left(-c_2\sum_{i=1}^m(s_i-s_{i-1})\Psi(\xi_i)\right)\prod_{i=1}^m|s_i-s_{i-1}|^{-\beta_0}\\
&=\sum_{m=0}^{\infty}(C\beta)^m\frac{(2m)!}{m!}\frac{t^m}{m!}\int_{\Sigma_t^{m}}d\mathbf{r}\int_{\mathbb{R}^{md}}\boldsymbol{\mu}(d\boldsymbol{\xi})\exp\left(-c_2\sum_{i=1}^m r_i\Psi(\xi_i)\right)\prod_{i=1}^m|r_i|^{-\beta_0}\\
&\le e^{c_2Mt}\sum_{m=0}^{\infty}(C\beta)^m\frac{(2m)!}{m!}\frac{t^m}{m!}\int_{\Sigma_t^{m}}d\mathbf{r}\int_{\mathbb{R}^{md}}\boldsymbol{\mu}(d\boldsymbol{\xi})\exp\left(-c_2\sum_{i=1}^m r_i(M+\Psi(\xi_i))\right)\prod_{i=1}^m|r_i|^{-\beta_0}\\
&\le e^{c_2Mt}\sum_{m=0}^{\infty}(C\beta)^m\frac{(2m)!}{m!}\frac{t^m}{m!}\left(\int_0^t dr\int_{\mathbb{R}^d}\mu(d\xi)\exp\left(-c_2r(M+\Psi(\xi))\right)|r|^{-\beta_0}\right)^m\\
&\le e^{c_2Mt}\sum_{m=0}^{\infty}(C\beta)^m\frac{(2m)!}{m!}\frac{t^m}{m!}\left(\int_{\mathbb{R}^d}\left(\frac{1}{M+\Psi(\xi)}\right)^{1-\beta_0}\mu(d\xi)\right)^m, \qquad (3.24)
\end{aligned}$$

of which the right-hand side is finite if we choose M sufficiently large, by Stirling's formula and Dalang's condition (1.8). □

Lemma 3.1. *Suppose that $\alpha\in(0,1)$. There exists a constant $C>0$ such that*

$$\sup_{\delta,\delta'}\int_0^t\int_0^t\varphi_\delta(t-s-r)\varphi_{\delta'}(t-s'-r')|r-r'|^{-\alpha}drdr'\le C|s-s'|^{-\alpha}.$$

Lemma 3.2. *Assume Assumption (H) and condition (1.8). Then, we have*

$$\sup_{x,y\in\mathbb{R}^d}\mathbb{E}\left(\int_0^t s^{-\beta_0}\gamma(X_s^x-y)ds\right)^2<\infty.$$

Proof. Note that

$$\mathbb{E}\left(\int_0^t s^{-\beta_0}\gamma(X_s^x-y)ds\right)^2$$

$$= 2\mathbb{E}\int_0^t\int_0^s (sr)^{-\beta_0}\gamma(X_r^x - y)\gamma(X_s^x - y)drds.$$

By Assumption (H) and the Markov property of X, we have for $0 < r < s < t$,

$$\begin{aligned}\mathbb{E}\left[\gamma(X_r^x - y)\gamma(X_s^x - y)\right] &= \mathbb{E}\left[\gamma(X_r^x - y)\int_{\mathbb{R}^d} p_{s-r}^{(X_r^x)}(z)\gamma(z-y)dz\right]\\ &\le C\mathbb{E}\left[\gamma(X_r^x - y)\int_{\mathbb{R}^d}\exp\left(-c_2(s-r)\Psi(\xi)\right)\mu(d\xi)\right]\\ &\le C^2\int_{\mathbb{R}^d}\exp\left(-c_2 r\Psi(\xi)\right)\mu(d\xi)\int_{\mathbb{R}^d}\exp\left(-c_2(s-r)\Psi(\xi)\right)\mu(d\xi)\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{E}\int_0^t\int_0^s (sr)^{-\beta_0}\gamma(X_r^x - y)\gamma(X_s^x - y)drds\\ &\le C\int_0^t\int_0^s\int_{\mathbb{R}^{2d}}(sr)^{-\beta_0}\exp\left(-c_2 r\Psi(\xi_1)\right)\int_{\mathbb{R}^d}\exp\left(-c_2(s-r)\Psi(\xi_2)\right)\mu(d\xi_1)\mu(d\xi_2)drds\\ &= C\int_{\left\{\substack{u_1,u_2>0\\ 0<u_1+u_2<t}\right\}}\int_{\mathbb{R}^{2d}}(u_1(u_1+u_2))^{-\beta_0}\exp\left(-c_2u_1\Psi(\xi_1)\right)\int_{\mathbb{R}^d}\exp\left(-c_2u_2\Psi(\xi_2)\right)\mu(d\xi_1)\mu(d\xi_2)du_1du_2\\ &\le C\left(\int_0^t\int_{\mathbb{R}^d}s^{-\beta_0}\exp\left(-c_2 s\Psi(\xi)\right)\mu(d\xi)ds\right)^2\end{aligned}$$

which is finite due to [24, Lemma 3.7] and condition (1.8). □

3.2. **Feynman-Kac formula.** In this subsection, we prove that (1.10) is a mild Stratonovich solution to (1.1). Let

$$\dot W^{\varepsilon,\delta}(t,x) = \int_0^t\int_{\mathbb{R}^d}\varphi_\delta(t-s)q_\varepsilon(x-y)W(ds,dy) = W(\varphi_\delta(t-\cdot)q_\varepsilon(x-\cdot)) \tag{3.25}$$

be an approximation of $\dot W(t,x)$. As in [14], we define Stratonovich integral as follows.

Definition 3.1. *Given a random field $v = \{v(t,x), t\ge 0, x\in\mathbb{R}^d\}$ such that*

$$\int_0^T\int_{\mathbb{R}^d}|v(t,x)|dxdt < \infty,$$

almost surely for all $T > 0$, the Stratonovich integral $\int_0^T\int_{\mathbb{R}^d} v(t,x)W(dt,dx)$ is defined as the following limit in probability,

$$\lim_{(\varepsilon,\delta)\to 0}\int_0^T\int_{\mathbb{R}^d} v(t,x)\dot W^{\varepsilon,\delta}(t,x)dxdt.$$

Let $\mathcal{F}_t^W$ be the σ-algebra generated by $\{W(s,x), 0\le s\le t, x\in\mathbb{R}^d\}$. A mild solution to (1.1) is defined below.

Definition 3.2. *A random field $\{u(t,x), t\ge 0, x\in\mathbb{R}^d\}$ is a mild Stratonovich solution to* (1.1) *if for all $t\ge 0$ and $x\in\mathbb{R}^d$, $u(t,x)$ is $\mathcal{F}_t^W$-measurable and the following integral equation holds*

$$u(t,x) = \int_{\mathbb{R}^d} p_t^{(x)}(y)u_0(y)dy + \int_0^t\int_{\mathbb{R}^d} p_{t-s}^{(x)}(y)u(s,y)W(ds,dy), \tag{3.26}$$

where the stochastic integral is in the Stratonivich sense of Definition (3.1).

Theorem 3.4. *Assume Assumption (H) and condition* (1.8)*, and suppose that* $u_0(x)$ *is a bounded measurable function. Then,*

$$u(t,x) = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta(X_{t-r}^x - y)W(dr,dy)\right)\right] \tag{3.27}$$

is a mild Stratonovich solution of (1.1)*.*

Proof. Consider the approximation of (1.1)

$$\begin{cases} \dfrac{\partial u^{\varepsilon,\delta}(t,x)}{\partial t} = \mathcal{L}u^{\varepsilon,\delta}(t,x) + u^{\varepsilon,\delta}(t,x)\dot{W}^{\varepsilon,\delta}(t,x), \quad t\geq 0,\, x\in\mathbb{R}^d, \\ u^{\varepsilon,\delta}(0,x) = u_0(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{3.28}$$

where $\dot{W}^{\varepsilon,\delta}$ is given in (3.25). Recall the definition of $V_{t,x}^{\varepsilon,\delta} = W(A_{t,x}^{\varepsilon,\delta})$ by (3.17) and note that

$$\mathbb{E}_W\mathbb{E}_X\left[\exp\left(\left|\int_0^t \dot{W}^{\varepsilon,\delta}(t-s,X_s^x)ds\right|\right)\right] = \mathbb{E}_W\mathbb{E}_X\left[\exp\left(\left|V_{t,x}^{\varepsilon,\delta}\right|\right)\right]. \tag{3.29}$$

We now show that for any $p>0$,

$$\sup_{\varepsilon,\delta>0}\mathbb{E}\left[\exp\left(pV_{t,x}^{\varepsilon,\delta}\right)\right] < \infty. \tag{3.30}$$

Indeed, by (3.21) and Taylor's expansion, we have the following estimation parallel with (3.22)

$$\begin{aligned} &\mathbb{E}\left[\exp\left(pV_{t,x}^{\varepsilon,\delta}\right)\right] = 2\mathbb{E}\left[\exp\left(\frac{p^2}{2}\|A_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2\right)\right] \\ &\leq \sum_{m=0}^\infty C^m\frac{p^{2m}}{m!}\int_{[0,t]^{2m}}\int_{\mathbb{R}^{2md}}\prod_{i=1}^m q_\varepsilon(y_i)q_\varepsilon(z_i)\prod_{i=1}^m|s_{2i}-s_{2i-1}|^{-\beta_0}\mathbb{E}\left[\prod_{i=1}^m\gamma(X_{s_{2i}}-X_{s_{2i-1}}-(y_i-z_i))\right]d\mathbf{y}d\mathbf{z}d\mathbf{s} \\ &= \sum_{m=0}^\infty C^m\frac{p^{2m}}{m!}\sum_{\tau\in S_{2m}}\int_{[0,t]^{2m}_<}\int_{\mathbb{R}^{2md}}\prod_{i=1}^m q_\varepsilon(y_i)q_\varepsilon(z_i)\prod_{i=1}^m|t_i^+-t_i^-|^{-\beta_0}\mathbb{E}\left[\prod_{i=1}^m\gamma(X_{t_i^+}-X_{t_i^-}-(y_i-z_i))\right]d\mathbf{y}d\mathbf{z}d\mathbf{s}, \end{aligned}$$

where the symbols $t_i^\pm$ are defined in the proof of Theorem 3.3. By the same argument that yields (3.23), we can show

$$\mathbb{E}\left[\prod_{i=1}^m\gamma(X_{t_i^+}-X_{t_i^-}-(y_i-z_i))\right] \leq C^m\int_{\mathbb{R}^{md}}\prod_{i=1}^m\exp\left(-c_2(t_i^+-t_i^*)\Psi(\xi_i)\right)\mu(d\xi_i),$$

the right-hand side of which is independent of $(y_i,z_i)'s$. Thus, use the same proof as in the rest part of the proof of Theorem 3.3, we can prove (3.30). Therefore, we have

$$\mathbb{E}\left[\exp\left(p|V_{t,x}^{\varepsilon,\delta}|\right)\right] \leq 2\mathbb{E}\left[\exp\left(pV_{t,x}^{\varepsilon,\delta}\right)\right] < \infty. \tag{3.31}$$

Noting (3.29) and (3.31), we can apply the classical Feynman-Kac formula (see Appendix A) and get that

$$u^{\varepsilon,\delta}(t,x) = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(\int_0^t\dot{W}^{\varepsilon,\delta}(t-s,X_s^x)ds\right)\right] = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(W(A_{t,x}^{\varepsilon,\delta})\right)\right], \tag{3.32}$$

is a mild solution to (3.28), that is

$$u^{\varepsilon,\delta}(t,x) = \int_{\mathbb{R}^d}p_t^{(x)}(y)u_0(y)dy + \int_0^t\int_{\mathbb{R}^d}p_{t-s}^{(x)}(y)u^{\varepsilon,\delta}(s,y)\dot{W}^{\varepsilon,\delta}(s,y)dyds. \tag{3.33}$$

To prove $u(t,x)$ given in (3.27) is a mild solution to (1.1), it suffices to show that both sides of (3.33) converge to those of (3.26) in probability as $(\varepsilon,\delta)\to 0$, respectively .

Step 1. Firstly, we shall prove that for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ and all $p\geq 1$,

$$\lim_{(\varepsilon,\delta)\to 0}\mathbb{E}[|u^{\varepsilon,\delta}(t,x)-u(t,x)|^p]=0.$$

By Theorem 3.1, $V^{\varepsilon,\delta}_{t,x}$ converges to $V_{t,x}$ in probability as $(\varepsilon,\delta)\to 0$. Hence, to get the L^p-convergence of $u^{\varepsilon,\delta}(t,x)$ to $u(t,x)$, noting that $u_0(x)$ bounded, we only need to show

$$\sup_{x\in\mathbb{R}^d}\sup_{(\varepsilon,\delta)>0}\mathbb{E}[|u^{\varepsilon,\delta}(t,x)|^p]<\infty, \tag{3.34}$$

which is a direct consequence of (3.30). Moreover, we can show $u^{\varepsilon,\delta}(t,x)\to u(t,x)$ in $\mathbb{D}^{1,2}$, for which it suffices to show noting that the Malliavin derivative D is closable,

$$\lim_{(\varepsilon,\delta),(\varepsilon',\delta')\to 0}\mathbb{E}[\|Du^{\varepsilon,\delta}(t,x)-Du^{\varepsilon',\delta'}(t,x)\|^2_{\mathcal{H}}]=0, \tag{3.35}$$

where the Malliavin derivative is taken with respect to $\dot W$. For simplicity of expressions, we assume $u_0(x)\equiv 1$ throughout the rest of the proof. Noting that

$$Du^{\varepsilon,\delta}(t,x)=\mathbb{E}_X\left[\exp\left(W(A^{\varepsilon,\delta}_{t,x})\right)A^{\varepsilon,\delta}_{t,x}\right], \tag{3.36}$$

we have

$$\mathbb{E}\left\langle Du^{\varepsilon,\delta}(t,x),Du^{\varepsilon',\delta'}(t,x)\right\rangle_{\mathcal{H}}=\mathbb{E}\left[\exp\left(W(A^{\varepsilon,\delta}_{t,x}+\tilde A^{\varepsilon',\delta'}_{t,x})\right)\left\langle A^{\varepsilon,\delta}_{t,x},\tilde A^{\varepsilon',\delta'}_{t,x}\right\rangle_{\mathcal{H}}\right],$$

where we recall that $A^{\varepsilon,\delta}_{t,x}(r,y)$ given in (3.16) is an approximation of $\delta(X^x_{t-r}-y)$ and $\tilde A^{\varepsilon',\delta'}_{t,x}$ is obtained from replacing X by its independent copy $\tilde X$ in $A^{\varepsilon',\delta'}_{t,x}$. Thus,

$$\begin{aligned}
&\mathbb{E}\left\langle Du^{\varepsilon,\delta}(t,x),Du^{\varepsilon',\delta'}(t,x)\right\rangle_{\mathcal{H}}\\
&=\mathbb{E}\left[\exp\left(W(A^{\varepsilon,\delta}_{t,x}+\tilde A^{\varepsilon',\delta'}_{t,x})\right)\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}A^{\varepsilon,\delta}_{t,x}(r_1,y_1)\tilde A^{\varepsilon',\delta'}_{t,x}(r_2,y_2)|r_1-r_2|^{-\beta_0}\gamma(y_1-y_2)d\mathbf{y}d\mathbf{r}\right]\\
&=\mathbb{E}\Bigg[\exp\left(W(A^{\varepsilon,\delta}_{t,x}+\tilde A^{\varepsilon',\delta'}_{t,x})\right)\int_{[0,t]^4}\int_{\mathbb{R}^{2d}}\varphi_\delta(t-s_1-r_1)\varphi_(\delta')(t-s_2-r_2)\\
&\qquad\times q_\varepsilon(X^x_{s_1}-y_1)q_{\varepsilon'}(\tilde X^x_{s_2}-y_2)|r_1-r_2|^{-\beta_0}\gamma(y_1-y_2)d\mathbf{y}d\mathbf{r}d\mathbf{s}\Bigg].
\end{aligned}$$

By a similar argument used in the proof of Theorem 3.2, one can show that as $\varepsilon,\delta,\varepsilon',\delta'$ go to zero, the above term converges to

$$\begin{aligned}
&\mathbb{E}\Bigg[\exp\left(\frac12\int_0^t\int_0^t|s-r|^{-\beta_0}\left(\gamma(X^x_s-X^x_r)+\gamma(\tilde X^x_s-\tilde X^x_r)+2\gamma(X^x_s-\tilde X^x_r)\right)dsdr\right)\\
&\qquad\times\int_0^t\int_0^t|s-r|^{-\beta_0}\gamma(X^x_s-\tilde X^x_r)dsdr\Bigg]<\infty.
\end{aligned}$$

This proves (3.35), and consequently we have

$$Du(t,x)=\mathbb{E}_X\left[\exp\left(W(\delta(X^x_{t-\cdot}-\cdot))\right)\delta(X^x_{t-\cdot}-\cdot)\right]. \tag{3.37}$$

Step 2. In this step, we prove the convergence of the right-hand side. Noting Definition 3.1, it suffices to prove

$$I^{\varepsilon,\delta}:=\int_0^t\int_{\mathbb{R}^d}p^{(x)}_{t-s}(y)v^{\varepsilon,\delta}_{s,y}\dot W^{\varepsilon,\delta}(s,y)dyds$$

converges to 0 in $L^1(\Omega)$ as (ε,δ) tends to zero, where $v^{\varepsilon,\delta}_{s,y}=u^{\varepsilon,\delta}(s,y)-u(s,y)$. Applying (2.15) to $v^{\varepsilon,\delta}_{s,y}\dot W^{\varepsilon,\delta}(s,y)=v^{\varepsilon,\delta}_{s,y}W(\phi^{\varepsilon,\delta}_{s,y})$ where

$$\phi^{\varepsilon,\delta}_{s,y}(r,z)=\varphi_\delta(s-r)q_\varepsilon(y-z),$$

we have

$$\begin{aligned}I^{\varepsilon,\delta}&=\int_0^t\int_{\mathbb{R}^d}p^{(x)}_{t-s}(y)\left[\boldsymbol{\delta}(v^{\varepsilon,\delta}_{s,y}\phi^{\varepsilon,\delta}_{s,y}(\cdot,\cdot))+\left\langle Dv^{\varepsilon,\delta}_{s,y},\phi^{\varepsilon,\delta}_{s,y}\right\rangle_{\mathcal{H}}\right]dyds\\&=\boldsymbol{\delta}\left(\int_0^t\int_{\mathbb{R}^d}p^{(x)}_{t-s}(y)v^{\varepsilon,\delta}_{s,y}\phi^{\varepsilon,\delta}_{s,y}(\cdot,\cdot)dyds\right)+\int_0^t\int_{\mathbb{R}^d}p^{(x)}_{t-s}(y)\left\langle Dv^{\varepsilon,\delta}_{s,y},\phi^{\varepsilon,\delta}_{s,y}\right\rangle_{\mathcal{H}}dyds\\&=:I^{\varepsilon,\delta}_1+I^{\varepsilon,\delta}_2.\end{aligned}\tag{3.38}$$

Denote

$$\Phi^{\varepsilon,\delta}_{t,x}(r,z):=\int_0^t\int_{\mathbb{R}^d}p^{(x)}_{t-s}(y)v^{\varepsilon,\delta}_{s,y}\phi^{\varepsilon,\delta}_{s,y}(r,z)dyds.$$

To deal with the first term $I^{\varepsilon,\delta}_1=\boldsymbol{\delta}(\Phi^{\varepsilon,\delta}_{t,x})$ of $I^{\varepsilon,\delta}$, by (2.14) it suffices to show

$$\lim_{(\varepsilon,\delta)\to0}\mathbb{E}[\|\Phi^{\varepsilon,\delta}_{t,x}\|^2_{\mathcal{H}}]+\mathbb{E}[\|D\Phi^{\varepsilon,\delta}_{t,x}\|^2_{\mathcal{H}^{\otimes2}}]=0.$$

Note that

$$\mathbb{E}[\|\Phi^{\varepsilon,\delta}_{t,x}\|^2_{\mathcal{H}}]=\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}p^{(x)}_{t-s_1}(y_1)p^{(x)}_{t-s_2}(y_2)\mathbb{E}\left[v^{\varepsilon,\delta}_{s_1,y_1}v^{\varepsilon,\delta}_{s_2,y_2}\right]\left\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\phi^{\varepsilon,\delta}_{s_2,y_2}\right\rangle_{\mathcal{H}}d\mathbf{y}d\mathbf{s}\tag{3.39}$$

By Step 1, we know that $\mathbb{E}\left[v^{\varepsilon,\delta}_{s_1,y_1}v^{\varepsilon,\delta}_{s_2,y_2}\right]$ is uniformly bounded and converges to 0 as ε,δ go to zero. For the integral without $\mathbb{E}\left[v^{\varepsilon,\delta}_{s_1,y_1}v^{\varepsilon,\delta}_{s_2,y_2}\right]$, noting that

$$\begin{aligned}\left\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\phi^{\varepsilon,\delta}_{s_2,y_2}\right\rangle_{\mathcal{H}}&=\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}\phi^{\varepsilon,\delta}_{s_1,y_1}(r_1,z_1)\phi^{\varepsilon,\delta}_{s_2,y_2}(r_2,z_2)|r_1-r_2|^{-\beta_0}\gamma(z_1-z_2)d\mathbf{z}d\mathbf{r}\\&=\int_{[0,t]^2}\int_{\mathbb{R}^d}\varphi_\delta(s_1-r_1)\varphi_\delta(s_2-r_2)|\hat q_\varepsilon(\eta)|^2e^{\iota\eta\cdot(y_1-y_2)}|r_1-r_2|^{-\beta_0}\mu(d\eta)d\mathbf{r},\end{aligned}$$

we have

$$\begin{aligned}&\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}p^{(x)}_{t-s_1}(y_1)p^{(x)}_{t-s_2}(y_2)\left\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\phi^{\varepsilon,\delta}_{s_2,y_2}\right\rangle_{\mathcal{H}}d\mathbf{y}d\mathbf{s}\\&\le C\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}P_{t-s_1}(y_1-x)P_{t-s_2}(y_2-x)\left\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\phi^{\varepsilon,\delta}_{s_2,y_2}\right\rangle_{\mathcal{H}}d\mathbf{y}d\mathbf{s}\\&\le C\int_{[0,t]^4}\int_{\mathbb{R}^d}\varphi_\delta(s_1-r_1)\varphi_\delta(s_2-r_2)|\hat q_\varepsilon(\eta)|^2e^{-c_2[(t-s_1)+(t-s_2)]\Psi(\eta)}|r_1-r_2|^{-\beta_0}\mu(d\eta)d\mathbf{r}d\mathbf{s}\\&\le C\int_{[0,t]^2}\int_{\mathbb{R}^d}|s_1-s_2|^{-\beta_0}e^{-c_2(s_1+s_2)\Psi(\eta)}\mu(d\eta)d\mathbf{s}\end{aligned}$$

which is finite due to [24, Proposition 3.2]. Then, by dominated convergence theorem, we have $\lim_{\varepsilon,\delta\to0}\mathbb{E}[\|\Phi^{\varepsilon,\delta}_{t,x}\|^2_{\mathcal{H}}]=0$. Similar to (3.39), we have

$$\mathbb{E}[\|D\Phi^{\varepsilon,\delta}_{t,x}\|^2_{\mathcal{H}^{\otimes2}}]=\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}p^{(x)}_{t-s_1}(y_1)p^{(x)}_{t-s_2}(y_2)\mathbb{E}\left[\left\langle Dv^{\varepsilon,\delta}_{s_1,y_1},Dv^{\varepsilon,\delta}_{s_2,y_2}\right\rangle_{\mathcal{H}}\right]\left\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\phi^{\varepsilon,\delta}_{s_2,y_2}\right\rangle_{\mathcal{H}}d\mathbf{y}d\mathbf{s},$$

which converges to 0 as ε,δ tend to zero, noting that $\mathbb{E}\left[\left\langle Dv^{\varepsilon,\delta}_{s_1,y_1},Dv^{\varepsilon,\delta}_{s_2,y_2}\right\rangle_{\mathcal{H}}\right]$ is uniformly bounded and tends to 0 as $(\varepsilon,\delta)\to0$ by Step 1. This implies that the first term $I^{\varepsilon,\delta}_1$ in (3.38) converges to 0 in L^2.

To prove the second term $I_2^{\varepsilon,\delta}$ in (3.38) converges to 0 in L^1, it suffices to show that both $\int_0^t \int_{\mathbb{R}^d} p_{t-s}^{(x)}(y) \left\langle Du^{\varepsilon,\delta}(s,y), \phi_{s,y}^{\varepsilon,\delta} \right\rangle_{\mathcal{H}} dyds$ and $\int_0^t \int_{\mathbb{R}^d} p_{t-s}^{(x)}(y) \left\langle Du(s,y), \phi_{s,y}^{\varepsilon,\delta} \right\rangle_{\mathcal{H}} dyds$ converge to the same limit in L^1 (note that $\|\phi_{s,y}^{\varepsilon,\delta}\| \to \infty$ as $(\varepsilon,\delta) \to 0$). Note that by (3.36) we have

$$\begin{aligned}
&\left\langle Du^{\varepsilon,\delta}(s,y), \phi_{s,y}^{\varepsilon,\delta} \right\rangle_{\mathcal{H}} \\
&= \int_{[0,s]^3} \int_{\mathbb{R}^{2d}} \varphi_\delta(s-\tau-r_1)\mathbb{E}_X \left[\exp\left(A_{s,y}^{\varepsilon,\delta}\right) q_\varepsilon(X_\tau^x - z_1)\right] \\
&\qquad \times \varphi_\delta(s-r_2) q_\varepsilon(y-z_2)|r_1-r_2|^{-\beta_0}\gamma(z_1-z_2) d\mathbf{z} d\tau d\mathbf{r}.
\end{aligned}$$

By a similar argument used in the proof of Theorem 3.2, one can show that the above term converges in L^1 to

$$\int_0^s \tau^{-\beta_0} \mathbb{E}_X \left[\exp\left(\delta(X_{s-\cdot}^y - \cdot)\right) \gamma(X_\tau^x - y)\right] d\tau,$$

of which the L^1-norm is bounded in (s,y) noting Lemma 3.2 and $\sup_{y\in\mathbb{R}^d} \mathbb{E}[|u(s,y)|^p] < \infty$ for all $p \ge 1$ by (3.34). This shows that $\int_0^t \int_{\mathbb{R}^d} p_{t-s}^{(x)}(y) \left\langle Du^{\varepsilon,\delta}(s,y), \phi_{s,y}^{\varepsilon,\delta} \right\rangle_{\mathcal{H}} dyds$ converges in L^1 to

$$\mathbb{E}_X \int_0^t \int_{\mathbb{R}^d} p_{t-s}^{(x)}(y) \exp\left(\delta(X_{s-\cdot}^y - \cdot)\right) \int_0^s \tau^{-\beta_0} \gamma(X_\tau^y - y) d\tau dyds.$$

The convergence of $\int_0^t \int_{\mathbb{R}^d} p_{t-s}^{(x)}(y) \left\langle Du(s,y), \phi_{s,y}^{\varepsilon,\delta} \right\rangle_{\mathcal{H}} dyds$ to the same limit can be proven in a similar way. This justifies the L^1-convergence of $I_2^{\varepsilon,\delta}$ to zero.

Combining the convergences of $I_1^{\varepsilon,\delta}$ and $I_2^{\varepsilon,\delta}$, we get that $I^{\varepsilon,\delta}$ converges to 0 in L^1. The proof is concluded. □

3.3. **Regularity of the law.** In this subsection, we shall prove the existence of the probability density of the Stratonovich solution $u(t,x)$ of (1.1) given by (1.10), and show that the density is smooth under proper conditions.

Our tool of studying the probability law is Malliavin calculus. By Bouleau–Hirsch's criterion (see [3] or [20, Theorem 2.1.2]), for a random variable $F \in \mathbb{D}^{1,2}$, if $\|DF\|_{\mathcal{H}} > 0$ a.s., the law of F is absolutely continuous with respect to the Lebesgue measure, i.e., the law of F has a probability density. Moreover, if $F \in \mathbb{D}^\infty$ and $\mathbb{E}\|DF\|_{\mathcal{H}}^{-p} < \infty$ for all $p > 0$, then the law of F has an infinitely differentiable density (see [20, Theorem 2.1.4]).

Theorem 3.5. *Assume Assumption (H) and condition* (1.8). *Furthermore, suppose that* $u_0(x) > 0$ *almost everywhere and*

$$\mathbb{E}[\gamma(X_r^x - \tilde{X}_s^x)] > 0, \tag{3.40}$$

where $\tilde{X}$ *is an independent copy of* X. *Then, the law of* $u(t,x)$ *given by* (1.10) *has a probability density.*

Remark 3.1. *The condition* (3.40) *is satisfied, if we assume* $\gamma(x)$ *is a strictly positive measurable function, or if we assume* $\gamma(x) = \delta(x)$ *and the density function* $p_t^{(x)}(y) > 0$ *for all* $x, y \in \mathbb{R}$.

Proof. By Theorem 3.4, we have $u(t,x) = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(V_{t,x}\right)\right]$, where $V_{t,x} = W(\delta(X_{t-\cdot}^x - \cdot))$ is given by (3.18). By (3.37), The Malliavin derivative of $u(t,x)$ is,

$$D_{s,y}u(t,x) = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(V_{t,x}\right)\delta(X_{t-s}^x - y)\right], \tag{3.41}$$

and hence, denoting $\tilde{V}_{t,x} = \delta(\tilde{X}_{t-\cdot} - \cdot)$ where $\tilde{X}$ is an independent copy of X,

$$\begin{aligned}\|Du(t,x)\|_{\mathcal{H}}^2 &= \mathbb{E}_{X,\tilde{X}}\left[u_0(X_t^x)u_0(\tilde{X}_t^x)\exp\left(V_{t,x}+\tilde{V}_{t,x}\right)\left\langle \delta(X_{t-\cdot}^x-\cdot),\delta(\tilde{X}_{t-\cdot}^x-\cdot)\right\rangle_{\mathcal{H}}\right]\\ &= \mathbb{E}_{X,\tilde{X}}\left[u_0(X_t^x)u_0(\tilde{X}_t^x)\exp\left(V_{t,x}+\tilde{V}_{t,x}\right)\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^x-\tilde{X}_s^x)drds\right].\end{aligned} \tag{3.42}$$

Noting that $u_0(X_t^x)u_0(\tilde{X}_t^x)\exp\left(V_{t,x}+\tilde{V}_{t,x}\right) > 0$, by (3.40) and (3.42), we have $\|Du(t,x)\|_{\mathcal{H}}^2$ is positive a.s. The proof is concluded. □

Theorem 3.6. *Assume Assumption (H) and condition* (1.8). *Furthermore, suppose that for all* $p > 0$, *we have* $\mathbb{E}|u_0(X_t^x)|^{-p} < \infty$ *and*

$$\mathbb{E}\left(\int_0^t\int_0^t \gamma(X_r^x-\tilde{X}_s^x)drds\right)^{-p} < \infty, \tag{3.43}$$

where $\tilde{X}$ *is an independent copy of* X. *Then, the law of* $u(t,x)$ *given by* (1.10) *has a smooth probability density.*

Proof. Using a similar argument leading to (3.37), we can show $u(t,x) \in \mathbb{D}^\infty$. Then, to prove that $u(t,x)$ has a smooth density, it suffices to show (see [20, Theorem 2.1.4]) $\mathbb{E}[\|Du(t,x)\|_{\mathcal{H}}^{-2p}] < \infty$. Applying Jensen's inequality to (3.42) yields

$$\|Du(t,x)\|_{\mathcal{H}}^{-2p} \le \mathbb{E}_X\left[\left|u_0(X_t^x)u_0(\tilde{X}_t^x)\right|^{-p}\exp\left(-p[V_{t,x}+\tilde{V}_{t,x}]\right)\left|\left\langle \delta(X_{t-\cdot}^x-\cdot),\delta(\tilde{X}_{t-\cdot}^x-\cdot)\right\rangle_{\mathcal{H}}\right|^{-p}\right],$$

and then Hölder's inequality implies

$$\mathbb{E}\|Du(t,x)\|_{\mathcal{H}}^{-2p} \le I_1I_2I_3 \tag{3.44}$$

where $I_1 = \left(\mathbb{E}\left|u_0(X_t^x)u_0(\tilde{X}_t^x)\right|^{-pp_1}\right)^{1/p_1}$, $I_2 = \left(\mathbb{E}\exp\left(-pp_2[V_{t,x}+\tilde{V}_{t,x}]\right)\right)^{1/p_2}$ and

$$I_3 = \left(\mathbb{E}\left|\left\langle \delta(X_{t-\cdot}^x-\cdot),\delta(\tilde{X}_{t-\cdot}^x-\cdot)\right\rangle_{\mathcal{H}}\right|^{-pp_3}\right)^{1/p_3},$$

with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. By the assumption on $u_0(x)$ and Theorem 3.2, it is clear that both I_1 and I_2 are finite. For the term I_3, by (2.12) we have

$$I_3^{p_3} = \mathbb{E}\left(\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^x-\tilde{X}_s^x)drds\right)^{-pp_3} \le t^{\beta_0pp_3}\mathbb{E}\left(\int_0^t\int_0^t \gamma(X_r^x-\tilde{X}_s^x)drds\right)^{-pp_3},$$

where we use the fact that $|r-s| \le t$. The right-hand side is finite due to the assumption (3.43). □

Condition (3.43) relates to the small ball probability of $\int_0^t\int_0^t\gamma(X_r^x-\tilde{X}_s^x)drds$, i.e. the asymptotic behavior of $P(\int_0^t\int_0^t\gamma(X_r^x-\tilde{X}_s^x)drds < \varepsilon)$ as ε goes to zero. Usually it is not an easy task to estimate small ball probabilities. In particular, when the spatial covariance function $\gamma(x)$ is the Dirac delta function $\delta(x)$, Condition (3.43) concerns the small ball probabilities of mutual intersection local time of X, which is still open even for Brownian motion.

If $\gamma(x)$ is a classical measurable function rather than a distribution, by Jensen's inequality, a sufficient condition for (3.43) which is easier to justify is the following

$$\sup_{r,s\in[0,t]}\mathbb{E}\left[|\gamma(X_r^x-\tilde{X}_s^x)|^{-p}\right] < \infty. \tag{3.45}$$

Corollary 3.1. *Assume Assumption (H) and condition* (1.8). *Suppose that* $\mathbb{E}|u_0(X_t^x)|^{-p} < \infty$ *and that the covariance function* γ *is a measurable function such that* (3.45) *is satisfied for all* $p > 0$. *Then, the law of* $u(t,x)$ *given by* (1.10) *has a smooth probability density.*

Remark 3.2. *By Assumption (H),*

$$\mathbb{E}\left[|\gamma(X_r^x - \tilde{X}_s^x)|^{-p}\right] = \mathbb{E}\int_{\mathbb{R}^d} p_r^{(\tilde{X}_s^x)}(y)|\gamma(y-\tilde{X}_r^x)|^{-p}dy \le C\int_{\mathbb{R}^d} P_r(y)|\gamma(y)|^{-p}dy. \tag{3.46}$$

The right-hand side of (3.46) *is uniformly bounded for* $r,s\in[0,t]$, *if we assume, for instance, the Feller process* X *is a diffusion process given by* (1.5) *whose density function satisfies* (1.7) *and*

$$(\gamma(x))^{-1} \le Ce^{C|x|^\alpha} \text{ for some } \alpha\in(0,2). \tag{3.47}$$

The spatial covariance satisfying (3.47) *includes the kernels mentioned in the Introduction such as the Riesz kernel, Poisson kernel, Cauchy Kernel, and Ornstein-Uhlenbeck kernel with* $\alpha\in(0,2)$. *This result is stated as a corollary below.*

Another typical situation in which (3.46) *is uniformly bounded for all* $p>0$ *is the following:* $P_r(y)$ *is a rapidly decreasing function and* $(\gamma(x))^{-1}$ *is at polynomial growth.*

3.4. **Hölder continuity of Stratonovich solution.** In this subsection, we will obtain Hölder continuity in space for the Stratonovich solution $u(t,x)$ of (1.1).

Assumption (H1). *Assume Assumption (H) and* $p_t^{(x)}(y)=p_t(y-x)$. *Furthermore, we assume there exists* $\theta\in(0,1]$ *and* $C>0$ *such that,*

$$\int_0^T\int_0^T\int_{\mathbb{R}^d}|r-s|^{-\beta_0}p_{|r-s|}(y)\Big[\gamma(y)-\gamma(y+z)\Big]dydrds \le C|z|^{2\theta}. \tag{3.48}$$

Note that if we assume $\hat{p}_t(\xi)\sim e^{-t\Psi(\xi)}$, then condition (3.48) is equivalent to the condition in [24, Hypothesis S1]:

$$\int_0^T\int_0^T\int_{\mathbb{R}^d}|r-s|^{-\beta_0}\exp\left(-|r-s|\Psi(\xi)\right)(1-\cos(\xi\cdot z))\mu(d\xi)drds \le C|z|^{2\theta},$$

for which a sufficient condition is (see[24, Remark 4.10])

$$\int_{\mathbb{R}^d}\frac{|\xi|^{2\theta}}{1+(\Psi(\xi))^{1-\beta_0}}\mu(d\xi)<\infty.$$

Theorem 3.7. *Assume Assumption (H1). Then the solution* $u(t,x)$ *given by the Feynman-Kac formula* (1.10) *is* κ*-Hölder continuous in* x *with* $\kappa\in(0,\theta)$ *on any compact set of* $[0,\infty)\times\mathbb{R}^d$.

Remark 3.3. *Note that if we assume* $\mathbb{E}e^{\iota\xi\cdot X_t}=e^{-t\Psi(\xi)}$, *then* [24, Theorem 4.11] *also holds in our setting.*

Proof. The proof essentially follows the proofs of [14, Theorem 5.1] and [24, Theorem 4.11]. Recalling (3.18): $V_{t,x}=\int_0^t\int_{\mathbb{R}^d}\delta(X_{t-s}^x-y)W(ds,dy)$ and using the inequality $|e^a-e^b|\le(e^a+e^b)|a-b|$, we have for all $p\ge 2$,

$$\begin{aligned}
&\mathbb{E}[|u(t,x_1)-u(t,x_2)|^p] = \mathbb{E}_W\left[|\mathbb{E}_X\left[\exp(V_{t,x_1})-\exp\left(V_{t,x_2}\right)\right]|^p\right]\\
&\le \mathbb{E}_W\left[(\mathbb{E}_X\left[\exp(2V_{t,x_1})+\exp(2V_{t,x_2})\right])^{p/2}\left(\mathbb{E}_X|V_{t,x_1}-V_{t,x_2}|^2\right)^{p/2}\right]\\
&\le C\mathbb{E}\left[\exp(pV_{t,x_1})+\exp(pV_{t,x_2})\right]\left(\mathbb{E}_W\left[\left(\mathbb{E}_X|V_{t,x_1}-V_{t,x_2}|^2\right)^p\right]\right)^{1/2}\\
&\le C\left(\mathbb{E}_W\left[\left(\mathbb{E}_X|V_{t,x_1}-V_{t,x_2}|^2\right)^p\right]\right)^{1/2}
\end{aligned}$$

where the last step follows from Theorem 3.2. Applying Minkowski's inequality and noting the equivalence between the L^p-norm and L^2-norm of Gaussian random variables, we have

$$\left(\mathbb{E}_W\left[\left(\mathbb{E}_X|V_{t,x_1}-V_{t,x_2}|^2\right)^p\right]\right)^{1/2} \le \left(\mathbb{E}_X\left[\left(\mathbb{E}_W|V_{t,x_1}-V_{t,x_2}|^{2p}\right)^{1/p}\right]\right)^{p/2}$$
$$\le C_p\left(\mathbb{E}|V_{t,x_1}-V_{t,x_2}|^2\right)^{p/2}.$$

Thus, we have

$$\mathbb{E}[|u(t,x_1)-u(t,x_2)|^p] \le C\left(\mathbb{E}|V_{t,x_1}-V_{t,x_2}|^2\right)^{p/2}. \tag{3.49}$$

Note that

$$\begin{aligned}\mathbb{E}|V_{t,x_1}-V_{t,x_2}|^2 &= \int_0^t\int_0^t |r-s|^{-\beta_0}\mathbb{E}_X\left[\gamma(X_r^{x_1}-X_s^{x_1})+\gamma(X_r^{x_2}-X_s^{x_2})-2\gamma(X_r^{x_1}-X_s^{x_2})\right]drds\\ &= 2\int_0^t\int_0^t |r-s|^{-\beta_0}\mathbb{E}_X\left[\gamma(X_r-X_s)-\gamma(X_r-X_s+x_1-x_2)\right]drds,\end{aligned}$$

where the second equality holds since $(X_r^x)_{r\ge0}$ has the same distribution of $(X_r+x=X_r^0+x)_{r\ge0}$ due to the condition $p_t^{(x)}(y)=p_t(y-x)$. Hence,

$$\mathbb{E}|V_{t,x_1}-V_{t,x_2}|^2 = 2\int_0^t\int_0^t\int_{\mathbb{R}^d}|r-s|^{-\beta_0}p_{|r-s|}(y)\Big[\gamma(y)-\gamma(y+x_1-x_2)\Big]dydrds. \tag{3.50}$$

Thus, by (3.49), (3.50) and (3.48), we have

$$\mathbb{E}|u(t,x_1)-u(t,x_2)|^p \le C|x_1-x_2|^{\theta p}$$

for all $p>0$. The desired result then follows from Kolmogorov's continuity criterion. □

4. Skorohod solution

In this section, we consider the equation (1.1) in the Skorohod sense.

Definition 4.3. *A random field $u=\{u(t,x): t\ge0, x\in\mathbb{R}^d\}$ is a mild Skorohod solution to* (1.1) *if for all $t\ge0$ and $x\in\mathbb{R}^d$, $\mathbb{E}|u(t,x)|^2<\infty$, $u(t,x)$ is $\mathcal{F}_t^W$-measurable, and the following integral equation holds*

$$u(t,x)=p_t^{(0)}*u_0(x)+\int_0^t\int_{\mathbb{R}^d}p_{t-s}^{(0)}(x-y)u(s,y)W(ds,dy), \tag{4.51}$$

where the stochastic integral is in the Skorohod sense.

Suppose u is a mild Skorohod solution to (1.1). Then repeating (4.51), we have the following Wiener chaos expansion of $u(t,x)$:

$$u(t,x)=\sum_{n=0}^\infty I_n(\tilde f_n(\cdot,t,x)), \tag{4.52}$$

where

$$f_n(t_1,x_1,\dots,t_n,x_n,t,x)=p_{t-t_n}^{(0)}(x-x_n)\dots p_{t_2-t_1}^{(0)}(x_2-x_1)p_{t_1}^{(0)}*u_0(x_1)\mathbf{1}_{\{0<t_1<\dots<t_n<t\}} \tag{4.53}$$

and $\tilde f_n$ is the symmetrization of f_n, i.e.,

$$\begin{aligned}\tilde f_n(t_1,x_1,\dots,t_n,x_n,t,x) =& \frac{1}{n!}\sum_{\sigma\in S_n}f_n(t_{\sigma(1)},x_{\sigma(1)},\dots,t_{\sigma(n)},x_{\sigma(n)},t,x)\\ =& \frac{1}{n!}f_n(t_{\tau(1)},x_{\tau(1)},\dots,t_{\tau(n)},x_{\tau(n)},t,x),\end{aligned}$$

where S_n is the set of all permutations of $\{1,2,\ldots,n\}$ and $\tau \in S_n$ is the permutation such that $0 < t_{\tau(1)} < \ldots < t_{\tau(n)} < t$. In view of the expansion (4.52), there exists a unique mild Skorohod solution to (1.1) if and only if

$$\sum_{n=0}^{\infty} n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} < \infty, \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R}^d. \tag{4.54}$$

4.1. **Existence and uniqueness of Skorohod solution.**

Theorem 4.1. *Assume assumption (H) and the Dalang's condition* (1.9)*. Suppose u_0 is a bounded function. Then,* (4.54) *holds, and hence $u(t,x)$ given by* (4.52) *is the unique mild Skorohod solution to* (1.1)*.*

Proof. Without loss of generality, we assume that $u_0 \equiv 1$. We define the following function:

$$F_n(t_1,x_1,\ldots,t_n,x_n,t,x) = P_{t-t_n}(x_n - x)\ldots P_{t_2-t_1}(x_1-x_2)\mathbf{1}_{\{0<t_1<\ldots<t_n<t\}} \tag{4.55}$$

Then by assumption (H), we have

$$f_n(t_1,x_1,\ldots,t_n,x_n,t,x) \le C^n F_n(t_1,x_1,\ldots,t_n,x_n,t,x). \tag{4.56}$$

Note that

$$\mathcal{F}F_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n) = e^{-ix\cdot(\xi_1+\ldots+\xi_n)}\prod_{j=1}^n \mathcal{F}P_{t_{j+1}-t_j}(\xi_1+\ldots+\xi_j)\mathbf{1}_{\{0<t_1<\ldots<t_n<t\}}. \tag{4.57}$$

with the convention that $t_{n+1} = t$. By the definition (2.12) of $\mathcal{H}$-norm together with (4.56), we have

$$\begin{aligned}
& n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \le n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \\
=& n!\int_{\mathbb{R}_+^{2n}}\int_{\mathbb{R}^{2nd}} f_n(t_1,x_1,\ldots,t_n,x_n,t,x)f_n(s_1,y_1,\ldots,s_n,y_n,t,x)\prod_{j=1}^n |t_j-s_j|^{-\beta_0}\gamma(x_j-y_j)d\mathbf{x}d\mathbf{t}d\mathbf{s} \\
\le& C^{2n}n!\int_{\mathbb{R}_+^{2n}}\int_{\mathbb{R}^{2nd}} F_n(t_1,x_1,\ldots,t_n,x_n,t,x)F_n(s_1,y_1,\ldots,s_n,y_n,t,x)\prod_{j=1}^n |t_j-s_j|^{-\beta_0}\gamma(x_j-y_j)d\mathbf{x}d\mathbf{t}d\mathbf{s}.
\end{aligned}$$

Then, by (1.4) and (4.57) we have

$$\begin{aligned}
& n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \\
=& C^{2n}n!\int_{\mathbb{R}_+^{2n}} d\mathbf{t}d\mathbf{s}\prod_{j=1}^d |t_j-s_j|^{-\beta_0}\int_{\mathbb{R}^{nd}}\mu(d\boldsymbol{\xi})\mathcal{F}F_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n) \\
& \qquad \times\overline{\mathcal{F}F_n(s_1,\cdot,\ldots,s_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)} \\
=& C^{2n}n!\int_{([0,t]^n_<])^2} d\mathbf{t}d\mathbf{s}\prod_{j=1}^n |t_j-s_j|^{-\beta_0}\int_{\mathbb{R}^{nd}}\mu(d\boldsymbol{\xi})\prod_{j=1}^n \mathcal{F}P_{t_{j+1}-t_j}(\xi_1+\ldots+\xi_j)\mathcal{F}P_{s_{j+1}-s_j}(\xi_1+\ldots+\xi_j) \\
\le& C^{2n}n!\int_{([0,t]^n_<])^2} d\mathbf{t}d\mathbf{s}\prod_{j=1}^n |t_j-s_j|^{-\beta_0}\int_{\mathbb{R}^{nd}}\mu(d\boldsymbol{\xi})\prod_{j=1}^n \exp\left(-c_2(t_{j+1}-t_j+s_{j+1}-s_j)\Psi(\xi_1+\ldots+\xi_j)\right),
\end{aligned}$$

where the right-hand side has been shown to be summable in the proof of [24, Theorem 5.3]. Thus, (4.54) follows and the proof is concluded. □

Using the same argument as in the proof of [14, Theorem 7.2], we can show that the Feynman-Kac representation (1.11) is a mild Skorohod solution under the condition (1.8).

Theorem 4.2. *Assume the Dalang's condition* (1.8). *Then the unique Skorohod solution to* (1.1) *can be represented by*

$$u(t,x)=\mathbb{E}_X\left[u_0(X_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta(X_{t-r}^x-y)W(dr,dy)-\frac{1}{2}\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r^x-X_s^x)drds\right)\right].$$

Theorem 4.1 indicates that under the condition (1.9) (which is weaker than (1.8)), there is a unique Skorohod solution. On the other hand, as shown in [14, Proposition 3.2], if the condition (1.8) is violated, the sequence $V_{t,x}^{\varepsilon,\delta}$ may diverge and hence the term $\int_0^t\int_{\mathbb{R}^d}\delta(X_{t-r}^x-y)W(dr,dy)$ given in (3.18) may not be well-defined. In this situation, Feynman-Kac representation of the form (1.11) for the Skorohod solution is not available. However, we do have Feynman-Kac representation for the moments of the Skorohod solution under (1.9).

Theorem 4.3. *Assume assumption (H) and the Dalang's condition* (1.9). *Suppose u_0 is a bounded function. Then for any $p\in\mathbb{N}_+$, we have the following representation of the unique Skorohod solution to* (1.1)*:*

$$\mathbb{E}\left[u(t,x)^p\right]=\mathbb{E}\left[\prod_{j=1}^p u_0(X_t^{(j)}+x)\exp\left(\sum_{1\le j<k\le p}\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma\left(X_r^{(j)}-X_s^{(k)}\right)drds\right)\right].\qquad(4.58)$$

Here, $X^{(1)},\dots,X^{(p)}$ are i.i.d. copies of X.

Proof. Without loss of generality, we assume $u_0(x)\equiv 1$. Similar to (3.28), we consider the following approximation of (1.1):

$$\begin{cases}\dfrac{\partial u^{\varepsilon,\delta}(t,x)}{\partial t}=\mathcal{L}u^{\varepsilon,\delta}(t,x)+u^{\varepsilon,\delta}(t,x)\diamond\dot{W}^{\varepsilon,\delta}(t,x), & t\ge 0,\ x\in\mathbb{R}^d,\\ u^{\varepsilon,\delta}(0,x)=u_0(x), & x\in\mathbb{R}^d,\end{cases}\qquad(4.59)$$

where the symbol $\diamond$ means Wick product and $\dot{W}^{\varepsilon,\delta}$ is given in (3.25). The mid Skorohod solution $u^{\varepsilon,\delta}(t,x)$ is given by the following integral equation

$$u^{\varepsilon,\delta}(t,x)=1+\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}p_{t-s}^{(0)}(x-y)\varphi_\delta(s-r)q_\varepsilon(y-z)u^{\varepsilon,\delta}(s,y)\diamond W(dr,dz)dsdy,$$

where the integral on the right-hand side is in the Skorhod sense. Recall the notation $A_{t,x}^{\varepsilon,\delta}$ given by (3.16). By a similar argument used in Step 1 in the proof of [24, Theorem 5.6] (see also [12, Proposition 5.2]), one can show that $W(A_{t,x}^{\varepsilon,\delta})$ is well-defined and that the following Feynman-Kac formula holds:

$$u^{\varepsilon,\delta}(t,x)=\mathbb{E}_X\left[\exp\left(W\left(A_{t,x}^{\varepsilon,\delta}\right)-\frac{1}{2}\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2\right)\right].\qquad(4.60)$$

As in Step 3 in the proof of [24, Theorem 5.6], we can prove $u^{\varepsilon,\delta}(t,x)$ converges to $u(t,x)$ in L^p for all $p>0$ as $(\varepsilon,\delta)\to 0$. By (4.60), the pth moment of $u^{\varepsilon,\delta}(t,x)$ can be calculated explicitly:

$$\begin{aligned}\mathbb{E}\left[\left(u^{\varepsilon,\delta}(t,x)\right)^p\right]&=\mathbb{E}\left[\exp\left(W\left(\sum_{j=1}^p A_{t,x}^{(\varepsilon,\delta,j)}\right)-\frac{1}{2}\sum_{j=1}^p\left\|A_{t,x}^{(\varepsilon,\delta,j)}\right\|_{\mathcal{H}}^2\right)\right]\\&=\mathbb{E}_X\left[\exp\left(\frac{1}{2}\left\|\sum_{j=1}^p A_{t,x}^{(\varepsilon,\delta,j)}\right\|_{\mathcal{H}}^2-\frac{1}{2}\sum_{j=1}^p\left\|A_{t,x}^{(\varepsilon,\delta,j)}\right\|_{\mathcal{H}}^2\right)\right]\\&=\mathbb{E}_X\left[\exp\left(\sum_{1\le j<k\le p}\left\langle A_{t,x}^{(\varepsilon,\delta,j)},A_{t,x}^{(\varepsilon,\delta,k)}\right\rangle_{\mathcal{H}}\right)\right],\end{aligned}\qquad(4.61)$$

where $A_{t,x}^{(\varepsilon,\delta,j)}$ is given by (3.16) with X^x being replaced by $X^{(j)}$, i.i.d. copies of X^x, and $\mathbb{E}_X$ is the expectation with respect to the randomness of $X^{(1)},\dots,X^{(p)}$. Thus, by the same method used in the proof of Theorem 3.1, we can prove the following L^1-convergence:

$$\lim_{\varepsilon,\delta\to 0+}\left\langle A_{t,x}^{(\varepsilon,\delta,j)}, A_{t,x}^{(\varepsilon,\delta,k)}\right\rangle_{\mathcal{H}} = \int_{[0,t]^2}|r_1-r_2|^{-\beta_0}\gamma\left(X_{r_2}-X_{r_1}\right)dr_1dr_2. \tag{4.62}$$

We claim that for any $\lambda>0$,

$$\sup_{\epsilon,\delta>0}\mathbb{E}\left[\exp\left(\lambda\left\langle A_{t,x}^{(\varepsilon,\delta,j)}, A_{t,x}^{(\varepsilon,\delta,k)}\right\rangle_{\mathcal{H}}\right)\right]<\infty. \tag{4.63}$$

Then the desired result (4.58) follows from (4.61), (4.62) and (4.63).

To conclude the proof, we prove the claim (4.63). By (3.16) and Lemma 3.1, we have

$$\begin{aligned}\left\langle A_{t,x}^{(\varepsilon,\delta,j)}, A_{t,x}^{(\varepsilon,\delta,k)}\right\rangle_{\mathcal{H}} \le& C\int_{[0,t]^2}|s-r|^{-\beta_0}dsdr\int_{\mathbb{R}^{2d}}q_\varepsilon(X_s^{(j)}-y)q_\varepsilon(X_r^{(k)}-z)\gamma(y-z)dydz\\ =&C\int_{[0,t]^2}|s-r|^{-\beta_0}dsdr\int_{\mathbb{R}^{2d}}q_\varepsilon(y)q_\varepsilon(z)\gamma(X_s^{(j)}-X_r^{(k)}-y+z)dydz.\end{aligned}$$

By Taylor expansion, we get

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\lambda\left\langle A_{t,x}^{(\varepsilon,\delta,j)}, A_{t,x}^{(\varepsilon,\delta,k)}\right\rangle_{\mathcal{H}}\right)\right]\\ \le& 1+\sum_{m=1}^\infty\frac{\lambda^mC^m}{m!}\int_{[0,t]^{2m}}d\mathbf{s}d\mathbf{r}\prod_{l=1}^m|s_l-r_l|^{-\beta_0}\\ &\times\int_{\mathbb{R}^{2md}}\prod_{l=1}^m q_\varepsilon(y_l)q_\varepsilon(z_l)\mathbb{E}\left[\prod_{l=1}^m\gamma(X_{s_l}^{(j)}-X_{r_l}^{(k)}-y_l+z_l)\right]d\mathbf{y}d\mathbf{z}.\end{aligned}\tag{4.64}$$

To compute the expectation in (4.64), we apply Lemma 4.3 below to the conditional expectation given $X^{(k)}$ and get, for $0=s_0<s_1<\cdots,s_m<t$,

$$\mathbb{E}\left[\prod_{l=1}^m\gamma(X_{s_l}^{(j)}-X_{r_l}^{(k)}-y_l+z_l)\right]\le C^mc_1^m\int_{\mathbb{R}^{md}}\prod_{l=1}^m\exp\left(-c_2(s_l-s_{l-1})\Psi(\xi_l)\right)\mu(d\xi_1)\dots\mu(d\xi_m). \tag{4.65}$$

Note that (4.65) is independent of $y_1,\dots,y_m,z_1,\dots,z_m$ and $r_1,\dots,r_m$. and hence

Thus, to estimate (4.64), we can integrate with respet to $dy_1\dots dy_mdz_1\dots dz_m$ first, and then to $dr_1\dots dr_m$ using the fact

$$\int_{[0,t]^n}\prod_{j=1}^n|s_j-r_j|^{-\beta_0}d\mathbf{r}\le D_t^n,\ \text{ with } D_t=2\int_0^t|s|^{-\beta_0}ds<\infty, \tag{4.66}$$

to obtain

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\lambda\left\langle A_{t,x}^{(\varepsilon,\delta,j)}, A_{t,x}^{(\varepsilon,\delta,k)}\right\rangle_{\mathcal{H}}\right)\right]\\ \le& 1+\sum_{m=1}^\infty\frac{\lambda^mC^mc_1^m}{m!}\int_{[0,t]^{2m}}d\mathbf{s}d\mathbf{r}\prod_{l=1}^m|s_l-r_l|^{-\beta_0}\int_{\mathbb{R}^{md}}\prod_{l=1}^m\exp\left(-c_2(s_l-s_{l-1})\Psi(\xi_l)\right)\mu(d\boldsymbol{\xi})\\ \le& 1+\sum_{m=1}^\infty D_t^m\lambda^mC^mc_1^m\int_{[0,t]_<^m}d\mathbf{s}\int_{\mathbb{R}^{md}}\prod_{l=1}^m\exp\left(-c_2(s_l-s_{l-1})\Psi(\xi_l)\right)\mu(d\boldsymbol{\xi}).\end{aligned}$$

Then we can use the same technique leading to (3.24) to prove the convergence of the series. Applying the change of variable $\tau_l=s_l-s_{l-1}$ for $l=1,\dots,m$ and recalling the notation $\Sigma_t^m=[0<\tau_1+\cdots+\tau_m<t]\cap\mathbb{R}_+^m$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(\lambda\left\langle A_{t,x}^{(\varepsilon,\delta,j)},A_{t,x}^{(\varepsilon,\delta,k)}\right\rangle_{\mathcal{H}}\right)\right]\\
=&1+\sum_{m=1}^{\infty}D_t^m\lambda^mC^mc_1^m\int_{\Sigma_t^m}d\boldsymbol{\tau}\int_{\mathbb{R}^{md}}\boldsymbol{\mu}(d\boldsymbol{\xi})\exp\left(-c_2\sum_{i=1}^m r_i\Psi(\xi_i)\right)\\
\le&1+e^{c_2Mt}\sum_{m=0}^{\infty}D_t^m\lambda^mC^mc_1^m\int_{\Sigma_t^m}d\boldsymbol{\tau}\int_{\mathbb{R}^{md}}\boldsymbol{\mu}(d\boldsymbol{\xi})\exp\left(-c_2\sum_{i=1}^m\tau_i(M+\Psi(\xi_i))\right)\\
\le&1+e^{c_2Mt}\sum_{m=1}^{\infty}D_t^m\lambda^mC^mc_1^m\left(\int_0^t d\tau\int_{\mathbb{R}^d}\mu(d\xi)\exp\left(-c_2\tau(M+\Psi(\xi))\right)\right)^m\\
\le&1+e^{c_2Mt}\sum_{m=1}^{\infty}D_t^m\lambda^mC^mc_1^m\left(\int_{\mathbb{R}^d}\frac{1}{M+\Psi(\xi)}\mu(d\xi)\right)^m.
\end{aligned}
$$

Thus, by the Dalang's condition (1.9), we can find M sufficiently large such that the above series converges. The proof is concluded. □

The following lemma has been used in the proof of Theorem 4.3.

Lemma 4.3. *For $0=r_0<r_1<\cdots<r_m<\infty$, we have*

$$
\sup_{(a_1,\dots,a_m)\in\mathbb{R}^m}\mathbb{E}\left[\prod_{i=1}^m\gamma\left(X_{r_i}-a_i\right)\right]\le C^mc_1^m\int_{\mathbb{R}^{md}}\prod_{j=1}^m\exp\Big(-c_2(r_j-r_{j-1})\Psi(\xi_j)\Big)\mu(d\boldsymbol{\xi}).
$$

Proof. The proof essentially follows the argument leading to (3.23). By Markov property of Xwe have

$$
\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^m\gamma\left(X_{r_i}-a_i\right)\right]=&\mathbb{E}\left[\prod_{i=1}^{m-1}\gamma\left(X_{r_i}-a_i\right)\mathbb{E}\left[\gamma\left(X_{r_m}-a_m\right)|\mathcal{F}_{r_{m-1}}\right]\right]\\
=&\mathbb{E}\left[\prod_{i=1}^{m-1}\gamma\left(X_{r_i}-a_i\right)\mathbb{E}\left[\gamma\left(X_{r_m}-a_m\right)|X_{r_{m-1}}\right]\right]\\
=&\mathbb{E}\left[\prod_{i=1}^{m-1}\gamma\left(X_{r_i}-a_i\right)\int_{\mathbb{R}^d}\gamma(y_m-a_m)p_{r_m-r_{m-1}}^{(X_{r_{m-1}})}(y_m)dy_m\right]
\end{aligned}
$$

Applying assumption (H) and Parseval-Plancherel indentity to the integral, we have

$$
\begin{aligned}
\int_{\mathbb{R}^d}\gamma(y_m-a_m)p_{r_m-r_{m-1}}^{(X_{r_{m-1}})}(y_m)dy_m\le&C\int_{\mathbb{R}^d}\gamma(y_m-a_m)P_{r_m-r_{m-1}}(X_{r_{m-1}}-y_m)dy_m\\
=&C\int_{\mathbb{R}^d}\exp(-i(X_{r_{m-1}}-a_m)\cdot\xi_m)\mathcal{F}P_{r_m-r_{m-1}}(-\xi_m)\mu(d\xi_m)\\
\le&Cc_1\int_{\mathbb{R}^d}\exp\big(-c_2(r_m-r_{m-1})\Psi(\xi_m)\big)\mu(d\xi_m).
\end{aligned}
$$

Combining the above calculations, we get

$$
\mathbb{E}\left[\prod_{i=1}^m\gamma\left(X_{r_i}-a_i\right)\right]\le Cc_1\mathbb{E}\left[\prod_{i=1}^{m-1}\gamma\left(X_{r_i}-a_i\right)\right]\int_{\mathbb{R}^d}\exp\big(-c_2(r_m-r_{m-1})\Psi(\xi_m)\big)\mu(d\xi_m).
$$

The proof is then concluded by repeating this procedure. □

4.2. **Regularity of the law.** In this subsection, we study the regularity of the law of the Skorohod solution to (1.1).

In light of Theorem 4.2, using a similar argument in the proof of Theorem 3.5 we can prove the existence of probability density under the stronger condition (1.8).

Theorem 4.4. *Assume Assumption (H) and condition* (1.8). *Furthermore, suppose that* $u_0(x) > 0$ *almost everywhere and* (3.40) *is satisfied. Then, the law of* $u(t,x)$ *given by* (1.10) *has a probability density.*

Still assuming (1.8), we can prove the smoothness of the density with the help of Feynman-Kac formula given in Theorem 4.2 under some proper condition.

Theorem 4.5. *Assume Assumption (H) and condition* (1.8). *Suppose that for all* $p > 0$, *we have*

$$\mathbb{E}\left[|u_0(X_t^x)|^{-p}\right] + \mathbb{E}\left[\left|\int_{[0,t]^2} \gamma(X_r^x - \tilde{X}_s^x)drds\right|^{-p}\right] < \infty. \tag{4.67}$$

Then the probability density of the mild Skorohod solution $u(t,x)$ *to* (1.1) *exists and is smooth.*

Proof. The proof is similar to the proof of Theorem 3.6 and is sketched below. By Theorem 4.2, we have

$$u(t,x) = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(V_{t,x} - U_{t,x}\right)\right],$$

where $V_{t,x} = W(\delta(X_{t-\cdot}^x - \cdot))$ is given in (3.18) and we denote

$$U_{t,x} = \frac{1}{2}\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^x - X_s^x)drds.$$

Then the Malliaven derivative is given by

$$D_{s,y}u(t,x) = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(V_{t,x} - U_{t,x}\right)D_{s,y}V_{t,x}\right] = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(V_{t,x} - U_{t,x}\right)\delta(X_{t-s}^x - y)\right],$$

and thus

$$\|Du(t,x)\|_{\mathcal{H}}^2 = \mathbb{E}\left[u_0(X_t^x)u_0(\tilde{X}_t^x)\exp\left(V_{t,x} - U_{t,x} + \tilde{V}_{t,x} - \tilde{U}_{t,x}\right)\left\langle \delta(X_{t-\cdot}^x - \cdot), \delta(\tilde{X}_{t-\cdot}^x - \cdot)\right\rangle_{\mathcal{H}}\right],$$

where $\tilde{X}$ is an i.i.d. copy of X, and $\tilde{V}_{t,x}, \tilde{U}_{t,x}$ are the corresponding quantities of $V_{t,x}, U_{t,x}$ with X replaced by $\tilde{X}$. Then by Jensen's inequality and Hölder inequality, we have

$$\begin{aligned}
&\|Du(t,x)\|_{\mathcal{H}}^{-2p} \\
\le&\mathbb{E}\left[\left(u_0(X_t^x)u_0(\tilde{X}_t^x)\right)^{-p}\exp\left(-p\left(V_{t,x} - U_{t,x} + \tilde{V}_{t,x} - \tilde{U}_{t,x}\right)\right)\left|\left\langle \delta(X_{t-\cdot}^x - \cdot), \delta(\tilde{X}_{t-\cdot}^x - \cdot)\right\rangle_{\mathcal{H}}\right|^{-p}\right] \\
\le&\left(\mathbb{E}\left[\left(u_0(X_t^x)u_0(\tilde{X}_t^x)\right)^{-pp_1}\right]\right)^{1/p_1}\left(\mathbb{E}\left[\exp\left(-pp_2\left(V_{t,x} + \tilde{V}_{t,x}\right)\right)\right]\right)^{1/p_2} \\
&\times\left(\mathbb{E}\left[\exp\left(pp_3\left(U_{t,x} + \tilde{U}_{t,x}\right)\right)\right]\right)^{1/p_3}\left(\mathbb{E}\left[\left|\left\langle \delta(X_{t-\cdot}^x - \cdot), \delta(\tilde{X}_{t-\cdot}^x - \cdot)\right\rangle_{\mathcal{H}}\right|^{-pp_4}\right]\right)^{1/p_4},
\end{aligned} \tag{4.68}$$

for any positive numbers p_1, p_2, p_3, p_4 satisfying $p_1^{-1} + p_2^{-1} + p_3^{-1} + p_4^{-1} = 1$.

By (4.67) and independence, we have

$$\left(\mathbb{E}\left[\left(u_0(X_t^x)u_0(\tilde{X}_t^x)\right)^{-pp_1}\right]\right)^{1/p_1} = \left(\mathbb{E}\left[(u_0(X_t^x))^{-pp_1}\right]\right)^{2/p_1} < \infty. \tag{4.69}$$

Since X and $\tilde{X}$ has identical distribution, by Cauchy-Schwarz inequality and Theorem 3.2, we have

$$\left(\mathbb{E}\left[\exp\left(-pp_2\left(V_{t,x} + \tilde{V}_{t,x}\right)\right)\right]\right)^{1/p_2} \le \left(\mathbb{E}\left[\exp\left(-2pp_2V_{t,x}\right)\right]\right)^{1/p_2} < \infty. \tag{4.70}$$

By independence and Theorem 3.3, we have

$$\left(\mathbb{E}\left[\exp\left(pp_3\left(U_{t,x}+\tilde{U}_{t,x}\right)\right)\right]\right)^{1/p_3}=\left(\mathbb{E}\left[\exp\left(pp_3U_{t,x}\right)\right]\right)^{2/p_3}<\infty. \tag{4.71}$$

By (4.67) together with the fact that $|r-s|^{-\beta_0}\geq t^{-\beta_0}$, we have

$$\begin{aligned}&\left(\mathbb{E}\left[\left|\left\langle \delta(X^x_{t-\cdot}-\cdot),\delta(\tilde{X}^x_{t-\cdot}-\cdot)\right\rangle_{\mathcal{H}}\right|^{-pp_4}\right]\right)^{1/p_4}\\ =&\left(\mathbb{E}\left[\left|\int_{[0,t]^2}|r-s|^{-\beta_0}\gamma(X^x_r-\tilde{X}^x_s)drds\right|^{-pp_4}\right]\right)^{1/p_4}\\ \leq& t^{-\beta_0 p}\left(\mathbb{E}\left[\left|\int_{[0,t]^2}\gamma(X^x_r-\tilde{X}^x_s)drds\right|^{-pp_4}\right]\right)^{1/p_4}<\infty.\end{aligned} \tag{4.72}$$

Substituting (4.69), (4.70), (4.71) and (4.72) to (4.68), we have for all $p>0$,

$$\|D_{s,y}u(t,x)\|_{\mathcal{H}}^{-2p}<\infty.$$

The proof is concluded. ◻

The proof of Theorem 4.5 involves the Feymann-Kac formula, which requires the Dalang's condition (1.8). However, the Skorohod solution exists under the weaker Dalang's condition (1.9). The following theorem study regularity of the density of Skorohod solution under weaker conditions.

Theorem 4.6. *Assume that assumption (H) and the Dalang's condition* (1.9) *hold. Suppose that for all* $p>0$*, we have*

$$\mathbb{E}\left[|u_0(X^x_t)|^{-p}\right]+\int_{[0,t]^2}\sup_{a\in\mathbb{R}^d}\mathbb{E}\left[\left|\gamma(X^x_s-\tilde{X}^x_r+a)\right|^{-p}\right]dsdr<\infty. \tag{4.73}$$

Then the probability density of the mild Skorohod solution $u(t,x)$ *to* (1.1) *exists and is smooth.*

Proof. We approximate the solution to (1.1) using (4.59) as in the proof of Theorem 3.4. Note that the solution $u^{\varepsilon,\delta}(t,x)$ to (4.59) is given by the Feynman-Kac formula (4.60). We will show that $Du^{\varepsilon,\delta}(t,x)$ converges in L^2 using the idea of Step 1 of the proof of Theorem 3.4. We have

$$\begin{aligned}D_{s,y}u^{\varepsilon,\delta}(t,x)=&\mathbb{E}_X\left[u_0(X^x_t)\exp\left(W(A^{\varepsilon,\delta}_{t,x})-\frac{1}{2}\left\|A^{\varepsilon,\delta}_{t,x}\right\|^2_{\mathcal{H}}\right)D_{s,y}\left(W(A^{\varepsilon,\delta}_{t,x})-\frac{1}{2}\left\|A^{\varepsilon,\delta}_{t,x}\right\|^2_{\mathcal{H}}\right)\right]\\ =&\mathbb{E}_X\left[u_0(X^x_t)\exp\left(W(A^{\varepsilon,\delta}_{t,x})-\frac{1}{2}\left\|A^{\varepsilon,\delta}_{t,x}\right\|^2_{\mathcal{H}}\right)A^{\varepsilon,\delta}_{t,x}\right].\end{aligned} \tag{4.74}$$

Thus,

$$\begin{aligned}&\mathbb{E}\left[\left\langle Du^{\varepsilon,\delta}(t,x),Du^{\epsilon',\delta'}(t,x)\right\rangle_{\mathcal{H}}\right]\\ =&\mathbb{E}\Bigg[u_0(X^x_t)u_0(\tilde{X}^x_t)\exp\left(W\left(A^{\varepsilon,\delta}_{t,x}+\tilde{A}^{\varepsilon,\delta}_{t,x}\right)-\frac{1}{2}\left\|A^{(\epsilon,\delta)}_{t,x}\right\|^2_{\mathcal{H}}-\frac{1}{2}\left\|\tilde{A}^{\varepsilon,\delta}_{t,x}\right\|^2_{\mathcal{H}}\right)\int_{[0,t]^4}\int_{\mathbb{R}^{2d}}\varphi_\delta(t-s_1-r_1)\\ &\times\varphi_{\delta'}(t-s_2-r_2)q_\varepsilon(X^x_{s_1}-y_1)q_{\varepsilon'}(\tilde{X}^x_{s_2}-y_2)|r_1-r_2|^{-\beta_0}\gamma(y_1-y_2)d\mathbf{y}d\mathbf{r}d\mathbf{s}\Bigg].\end{aligned}$$

Computing $\mathbb{E}_W$ first, we can show that

$$\mathbb{E}\left[\left\langle Du^{\varepsilon,\delta}(t,x),Du^{\epsilon',\delta'}(t,x)\right\rangle_{\mathcal{H}}\right]$$

$$\to \mathbb{E}\left[u_0(X_t^x)u_0(\tilde{X}_t^x)\exp\left(\int_0^t\int_0^t |s-r|^{-\beta_0}\gamma(X_s^x-\tilde{X}_r^x)dsdr\right)\int_0^t\int_0^t |s-r|^{-\beta_0}\gamma(X_s^x-\tilde{X}_r^x)dsdr\right].$$

as $\varepsilon,\delta,\varepsilon',\delta'\to 0$. The limit is finite due to the boundedness of u_0 and (4.63). Thus, we have

$$\lim_{\epsilon,\delta,\epsilon',\delta'\to 0+}\mathbb{E}\left[\left\|Du^{\varepsilon,\delta}(t,x)-Du^{\epsilon',\delta'}(t,x)\right\|_{\mathcal{H}}^2\right]=0.$$

Hence, $Du^{\varepsilon,\delta}(t,x)$ is a Cauchy sequence in L^2, so it is convergent in L^2. For any $p>2$, we have that both $u^{\varepsilon,\delta}$ and u belongs to L^p. Thus, By Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\left\|Du^{\varepsilon,\delta}(t,x)-Du^{\epsilon',\delta'}(t,x)\right\|_{\mathcal{H}}^p\right]\\&\le\left(\mathbb{E}\left[\left\|Du^{\varepsilon,\delta}(t,x)-Du^{\epsilon',\delta'}(t,x)\right\|_{\mathcal{H}}^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left\|Du^{\varepsilon,\delta}(t,x)-Du^{\epsilon',\delta'}(t,x)\right\|_{\mathcal{H}}^{2p-2}\right]\right)^{1/2},\end{aligned}$$

which converges to 0 as $\epsilon,\delta,\epsilon',\delta'\to 0$. Therefore, $Du^{\varepsilon,\delta}(t,x)$ converges in L^p for all $p\ge 2$.

Next, we show that

$$\sup_{\varepsilon,\delta}\mathbb{E}\left[\left\|Du^{\varepsilon,\delta}(t,x)\right\|_{\mathcal{H}}^{-p}\right]<\infty. \tag{4.75}$$

By Jensen's inequality, we have

$$\begin{aligned}&\left\|Du^{\varepsilon,\delta}(t,x)\right\|_{\mathcal{H}}^{-2p}\\&\le\mathbb{E}\left[\left(u_0(X_t^x)u_0(\tilde{X}_t^x)\right)^{-p}\exp\left(-p\left(W(A_{t,x}^{\varepsilon,\delta})-\frac{1}{2}\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2+W(\tilde{A}_{t,x}^{\varepsilon,\delta})-\frac{1}{2}\left\|\tilde{A}_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2\right)\right)\right.\\&\quad\left.\times\left|\left\langle A_{t,x}^{\epsilon,\delta},\tilde{A}_{t,x}^{\epsilon,\delta}\right\rangle_{\mathcal{H}}\right|^{-p}\right]\\&\le\left(\mathbb{E}\left[\left(u_0(X_t^x)u_0(\tilde{X}_t^x)\right)^{-pp_1}\right]\right)^{1/p_1}\left(\mathbb{E}\left[\exp\left(-pp_2\left(W(A_{t,x}^{\varepsilon,\delta})-\frac{1}{2}\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2+W(\tilde{A}_{t,x}^{\varepsilon,\delta})-\frac{1}{2}\left\|\tilde{A}_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2\right)\right)\right]\right)^{1/p_2}\\&\quad\times\left(\mathbb{E}\left[\left|\left\langle A_{t,x}^{\epsilon,\delta},\tilde{A}_{t,x}^{\epsilon,\delta}\right\rangle_{\mathcal{H}}\right|^{-pp_3}\right]\right)^{1/p_3},\end{aligned}$$

By (4.73) and (4.69), in order to show (4.75), we only need to prove for any $\lambda\in\mathbb{R}$:

$$\mathbb{E}\left[\exp\left(\lambda\left(W(A_{t,x}^{\varepsilon,\delta})-\frac{1}{2}\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2+W(\tilde{A}_{t,x}^{\varepsilon,\delta})-\frac{1}{2}\left\|\tilde{A}_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2\right)\right)\right]<\infty, \tag{4.76}$$

and for any $p>0$,

$$\mathbb{E}\left[\left|\left\langle A_{t,x}^{\epsilon,\delta},\tilde{A}_{t,x}^{\epsilon,\delta}\right\rangle_{\mathcal{H}}\right|^{-p}\right]<\infty. \tag{4.77}$$

For (4.76), compute the expectation with respect to W first, we have

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\lambda\left(W(A_{t,x}^{\varepsilon,\delta})-\frac{1}{2}\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2+W(\tilde{A}_{t,x}^{\varepsilon,\delta})-\frac{1}{2}\left\|\tilde{A}_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2\right)\right)\right]\\&=\mathbb{E}\left[\exp\left(\lambda\left(\frac{1}{2}\left\|A_{t,x}^{\varepsilon,\delta}+\tilde{A}_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2-\frac{1}{2}\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2-\frac{1}{2}\left\|\tilde{A}_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2\right)\right)\right]\\&=\mathbb{E}\left[\exp\left(\lambda\left\langle A_{t,x}^{\varepsilon,\delta},\tilde{A}_{t,x}^{\varepsilon,\delta}\right\rangle_{\mathcal{H}}\right)\right].\end{aligned}$$

This quantity is bounded uniformly in ϵ,δ because of (4.63). For (4.77), by (3.20) and Jensen's inequality with respect to the measure $\varphi_\delta(t-s-u)\varphi_\delta(t-r-v)q_\varepsilon(X_s^x-y)q_\varepsilon(\tilde{X}_r^x-z)dydzdudvdrds$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left|\left\langle A^{\epsilon,\delta}_{t,x}, \tilde{A}^{\epsilon,\delta}_{t,x}\right\rangle_{\mathcal{H}}\right|^{-p}\right]\\
\le& C(t,p)\mathbb{E}\left[\int_{[0,t]^4}\int_{\mathbb{R}^{2d}} |\gamma(y-z)|^{-p}\,|u-v|^{\beta_0 p}\varphi_\delta(t-s-u)\varphi_\delta(t-r-v)q_\varepsilon(X^x_s-y)q_\varepsilon(\tilde{X}^x_r-z)dydzdudvdsdr\right]\\
\le& C(t,p)t^{\beta_0 p}\mathbb{E}\left[\int_{[0,t]^4}\int_{\mathbb{R}^{2d}} |\gamma(y-z)|^{-p}\,\varphi_\delta(t-s-u)\varphi_\delta(t-r-v)q_\varepsilon(X^x_s-y)q_\varepsilon(\tilde{X}^x_r-z)dydzdudvdsdr\right]\\
=& C(t,p)t^{\beta_0 p}\mathbb{E}\left[\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} |\gamma(y-z)|^{-p}\,q_\varepsilon(X^x_s-y)q_\varepsilon(\tilde{X}^x_r-z)dydzdsdr\right]\\
=& C(t,p)t^{\beta_0 p}\mathbb{E}\left[\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} \left|\gamma(X^x_s-\tilde{X}^x_r-y+z)\right|^{-p}q_\varepsilon(y)q_\varepsilon(z)dydzdsdr\right]\\
\le& C(t,p)t^{\beta_0 p}\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}\sup_{a\in\mathbb{R}^d}\mathbb{E}\left[\left|\gamma(X^x_s-\tilde{X}^x_r+a)\right|^{-p}\right]q_\varepsilon(y)q_\varepsilon(z)dydzdsdr\\
=& C(t,p)t^{\beta_0 p}\int_{[0,t]^2}\sup_{a\in\mathbb{R}^d}\mathbb{E}\left[\left|\gamma(X^x_s-\tilde{X}^x_r+a)\right|^{-p}\right]dsdr.
\end{aligned}
$$

where we integrate $dudv$ in the fourth line, change of variables in the fifth line, and integrate $dydz$ in the last line. Here, $C(t,p)$ is a positive constant that depends on t,p only. Thus, (4.77) follows directly from (4.73).

Now we show that for any p,

$$
\mathbb{E}\left[\|Du(t,x)\|_{\mathcal{H}}^{-p}\right]<\infty. \tag{4.78}
$$

We choose $\varepsilon_n,\delta_n\to 0+$, then we have the convergence of $\|Du^{\epsilon_n,\delta_n}(t,x)\|_{\mathcal{H}}$ to $\|Du(t,x)\|_{\mathcal{H}}$ in $L^p(\Omega)$. The convergence is also in probability, and thus, there is a subsequence that converges almost surely. Without loss of generality, we assume that $\|Du^{\epsilon_n,\delta_n}(t,x)\|_{\mathcal{H}}$ converges to $\|Du(t,x)\|_{\mathcal{H}}$ almost surely. By Fatou's lemma and (4.75), we have

$$
\mathbb{E}\left[\|Du(t,x)\|_{\mathcal{H}}^{-p}\right]=\mathbb{E}\left[\lim_{n\to\infty}\left\|Du^{\epsilon_n,\delta_n}(t,x)\right\|_{\mathcal{H}}^{-p}\right]\le\liminf_{n\to\infty}\mathbb{E}\left[\left\|Du^{\epsilon_n,\delta_n}(t,x)\right\|_{\mathcal{H}}^{-p}\right]<\infty.
$$

The proof is concluded. □

4.3. **Hölder continuity of Skorohod solution.** Let $u(t,x)$ be the unique mild solution to (1.1). In this subsection, we study the Hölder continuity of $u(t,x)$. For simplicity, we assume that $u_0(x)\equiv 1$. In this case, the chaos decomposition (4.52) of $u(t,x)$ still holds with

$$
f_n(t_1,x_1,\dots,t_n,x_n,t,x)=p^{(0)}_{t-t_n}(x-x_n)\dots p^{(0)}_{t_2-t_1}(x_2-x_1)\mathbf{1}_{\{0<t_1<\ldots<t_n<t\}},
$$

and the Fourier transform on spatial variables is

$$
\mathcal{F}f_n(t_1,\cdot,\dots,t_n,\cdot,t,x)(\xi_1,\dots,\xi_n)=e^{-ix\cdot(\xi_1+\ldots+\xi_n)}\prod_{j=1}^n\mathcal{F}p^{(0)}_{t_{j+1}-t_j}(\xi_1+\ldots+\xi_j)\mathbf{1}_{\{0<t_1<\ldots<t_n<t\}}.
$$

We have the following theorem on the Hölder continuity of spatial variable.

Theorem 4.7. *Suppose that there exists $\alpha_1\in(0,1)$ and $C_T>0$, such that for all $x\in\mathbb{R}^d$, the following holds:*

$$
\sup_{\eta\in\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d}\left|1-e^{ix(\xi+\eta)}\right|^2\left|\mathcal{F}p^{(0)}_t(\xi+\eta)\right|^2\mu(d\xi)dt\le C_T|x|^{2\alpha_1}. \tag{4.79}
$$

Under the assumption (H), then $u(t,x)$ has a version that is α_1-Hölder continuous in x on any compact set of $[0,\infty)\times\mathbb{R}^d$.

Proof. By the chaos expansion (4.52) of $u(t,x)$, triangle inequality and hypercontractivity, we have

$$\begin{aligned}\|u(t,x)-u(t,y)\|_{L^p} &\le \sum_{n=1}^{\infty}\left\|I_n(\tilde{f}_n(\cdot,t,x))-I_n(\tilde{f}_n(\cdot,t,y))\right\|_{L^p}\\ &\le \sum_{n=1}^{\infty}(p-1)^{n/2}\left\|I_n(\tilde{f}_n(\cdot,t,x))-I_n(\tilde{f}_n(\cdot,t,y))\right\|_{L^2}\\ &= \sum_{n=1}^{\infty}(p-1)^{n/2}(n!)^{1/2}\left\|\tilde{f}_n(\cdot,t,x)-\tilde{f}_n(\cdot,t,y)\right\|_{\mathcal{H}^{\otimes n}} \qquad (4.80)\end{aligned}$$

for any $p\ge 2$.

Note that the Fourier transform on spatial variables of $\tilde{f}_n(\cdot,t,x)-\tilde{f}_n(\cdot,t,y)$ is

$$\begin{aligned}&\mathcal{F}\left(\tilde{f}_n(t_1,\cdot,\dots,t_n,\cdot,t,x)-\tilde{f}_n(t_1,\cdot,\dots,t_n,\cdot,t,y)\right)(\xi_1,\dots,\xi_n)\\ =&\frac{1}{n!}\left(e^{-ix\cdot(\xi_1+\dots+\xi_n)}-e^{-iy\cdot(\xi_1+\dots+\xi_n)}\right)\prod_{j=1}^{n}\mathcal{F}p^{(0)}_{t_{\tau(j+1)}-t_{\tau(j)}}\left(\sum_{l=1}^{j}\xi_{\tau(l)}\right),\end{aligned}$$

where $\tau\in S_n$ is the permutation such that $0<t_{\tau(1)}<\dots<t_{\tau(n)}<t$ and we set $t_{\tau(n+1)}=t$. By definition of $\mathcal{H}$-norm, we have

$$\begin{aligned}&n!\left\|\tilde{f}_n(\cdot,t,x)-\tilde{f}_n(\cdot,t,y)\right\|^2_{\mathcal{H}^{\otimes n}}\\ =&n!\int_{[0,t]^{2n}}\prod_{j=1}^{n}|s_j-t_j|^{-\beta_0}ds_1\dots ds_n dt_1\dots dt_n\int_{\mathbb{R}^{nd}}\mu(d\xi_1)\dots\mu(d\xi_n)\\ &\times\mathcal{F}\left(\tilde{f}_n(t_1,\cdot,\dots,t_n,\cdot,t,x)-\tilde{f}_n(t_1,\cdot,\dots,t_n,\cdot,t,y)\right)(\xi_1,\dots,\xi_n)\\ &\times\overline{\mathcal{F}\left(\tilde{f}_n(s_1,\cdot,\dots,s_n,\cdot,t,x)-\tilde{f}_n(s_1,\cdot,\dots,s_n,\cdot,t,y)\right)(\xi_1,\dots,\xi_n)}\\ =&\frac{1}{n!}\int_{[0,t]^{2n}}\prod_{j=1}^{n}|s_j-t_j|^{-\beta_0}ds_1\dots ds_n dt_1\dots dt_n\int_{\mathbb{R}^{nd}}\mu(d\xi_1)\dots\mu(d\xi_n)\\ &\times\left|e^{-ix\cdot(\xi_1+\dots+\xi_n)}-e^{-iy\cdot(\xi_1+\dots+\xi_n)}\right|^2\prod_{j=1}^{n}\mathcal{F}p^{(0)}_{t_{\tau(j+1)}-t_{\tau(j)}}\left(\sum_{l=1}^{j}\xi_{\tau(l)}\right)\prod_{j=1}^{n}\overline{\mathcal{F}p^{(0)}_{s_{\theta(j+1)}-s_{\theta(j)}}\left(\sum_{l=1}^{j}\xi_{\theta(l)}\right)},\end{aligned}$$

where $\theta\in S_n$ is the permutation such that $0<s_{\theta(1)}<\dots<s_{\theta(n)}<t$ and we use the convention $s_{\theta(n+1)}=t$. Using the inequality $2|ab|\le a^2+b^2$ and symmetry, we have

$$\begin{aligned}&n!\left\|\tilde{f}_n(\cdot,t,x)-\tilde{f}_n(\cdot,t,y)\right\|^2_{\mathcal{H}^{\otimes n}}\\ \le&\frac{1}{2n!}\int_{[0,t]^{2n}}\prod_{j=1}^{n}|s_j-t_j|^{-\beta_0}ds_1\dots ds_n dt_1\dots dt_n\int_{\mathbb{R}^{nd}}\mu(d\xi_1)\dots\mu(d\xi_n)\\ &\times\left|e^{-ix\cdot(\xi_1+\dots+\xi_n)}-e^{-iy\cdot(\xi_1+\dots+\xi_n)}\right|^2\left|\prod_{j=1}^{n}\mathcal{F}p^{(0)}_{t_{\tau(j+1)}-t_{\tau(j)}}\left(\sum_{l=1}^{j}\xi_{\tau(l)}\right)\right|^2\\ &+\frac{1}{2n!}\int_{[0,t]^{2n}}\prod_{j=1}^{n}|s_j-t_j|^{-\beta_0}ds_1\dots ds_n dt_1\dots dt_n\int_{\mathbb{R}^{nd}}\mu(d\xi_1)\dots\mu(d\xi_n)\end{aligned}$$

$$\times \left| e^{-ix\cdot(\xi_1+\ldots+\xi_n)} - e^{-iy\cdot(\xi_1+\ldots+\xi_n)} \right|^2 \left| \prod_{j=1}^n \mathcal{F}p^{(0)}_{s_{\theta(j+1)}-s_{\theta(j)}} \left(\sum_{l=1}^j \xi_{\theta(l)} \right) \right|^2 . \tag{4.81}$$

We treat the two terms separately. For the first term, we integrate $ds_1 \ldots ds_n$ first. Substitute (4.66) to the first term of (4.81), we have

$$\begin{aligned}
&\frac{1}{2n!} \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{-\beta_0} ds_1 \ldots ds_n dt_1 \ldots dt_n \int_{\mathbb{R}^{nd}} \mu(d\xi_1) \ldots \mu(d\xi_n) \\
&\quad \times \left| e^{-ix\cdot(\xi_1+\ldots+\xi_n)} - e^{-iy\cdot(\xi_1+\ldots+\xi_n)} \right|^2 \left| \prod_{j=1}^n \mathcal{F}p^{(0)}_{t_{\tau(j+1)}-t_{\tau(j)}} \left(\sum_{l=1}^j \xi_{\tau(l)} \right) \right|^2 \\
\le & \frac{D_t^n}{2n!} \int_{[0,t]^n} dt_1 \ldots dt_n \int_{\mathbb{R}^{nd}} \mu(d\xi_1) \ldots \mu(d\xi_n) \\
&\quad \times \left| 1 - e^{i(x-y)\cdot(\xi_1+\ldots+\xi_n)} \right|^2 \left| \prod_{j=1}^n \mathcal{F}p^{(0)}_{t_{\tau(j+1)}-t_{\tau(j)}} \left(\sum_{l=1}^j \xi_{\tau(l)} \right) \right|^2 \\
= & \frac{D_t^n}{2} \int_{0<t_1<\ldots<t_n<t} dt_1 \ldots dt_n \int_{\mathbb{R}^{nd}} \mu(d\xi_1) \ldots \mu(d\xi_n) \\
&\quad \times \left| 1 - e^{i(x-y)\cdot(\xi_1+\ldots+\xi_n)} \right|^2 \left| \prod_{j=1}^n \mathcal{F}p^{(0)}_{t_{j+1}-t_j} \left(\sum_{l=1}^j \xi_l \right) \right|^2 .
\end{aligned} \tag{4.82}$$

For the second term of (4.81), we have similar result

$$\begin{aligned}
&\frac{1}{2n!} \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{-\beta_0} ds_1 \ldots ds_n dt_1 \ldots dt_n \int_{\mathbb{R}^{nd}} \mu(d\xi_1) \ldots \mu(d\xi_n) \\
&\quad \times \left| e^{-ix\cdot(\xi_1+\ldots+\xi_n)} - e^{-iy\cdot(\xi_1+\ldots+\xi_n)} \right|^2 \left| \prod_{j=1}^n \mathcal{F}p^{(0)}_{s_{\theta(j+1)}-s_{\theta(j)}} \left(\sum_{l=1}^j \xi_{\theta(l)} \right) \right|^2 \\
\le & \frac{D_t^n}{2} \int_{0<t_1<\ldots<t_n<t} dt_1 \ldots dt_n \int_{\mathbb{R}^{nd}} \mu(d\xi_1) \ldots \mu(d\xi_n) \\
&\quad \times \left| 1 - e^{i(x-y)\cdot(\xi_1+\ldots+\xi_n)} \right|^2 \left| \prod_{j=1}^n \mathcal{F}p^{(0)}_{t_{j+1}-t_j} \left(\sum_{l=1}^j \xi_l \right) \right|^2 .
\end{aligned} \tag{4.83}$$

Hence, substitute (4.82) and (4.83) to (4.81), we have

$$\begin{aligned}
& n! \left\| \tilde{f}_n(\cdot,t,x) - \tilde{f}_n(\cdot,t,y) \right\|^2_{\mathcal{H}^{\otimes n}} \\
\le & D_t^n \int_{0<t_1<\ldots<t_n<t} dt_1 \ldots dt_n \int_{\mathbb{R}^{nd}} \mu(d\xi_1) \ldots \mu(d\xi_n) \\
&\quad \times \left| 1 - e^{i(x-y)\cdot(\xi_1+\ldots+\xi_n)} \right|^2 \left| \prod_{j=1}^n \mathcal{F}p^{(0)}_{t_{j+1}-t_j} \left(\sum_{l=1}^j \xi_l \right) \right|^2 \\
\le & D_t^n \int_0^t \int_{\mathbb{R}^d} \left| 1 - e^{i(x-y)\cdot(\xi_1+\ldots+\xi_n)} \right|^2 \left| \mathcal{F}p^{(0)}_{t-t_n} \left(\sum_{l=1}^n \xi_l \right) \right|^2 \mu(d\xi_n) dt_n
\end{aligned}$$

$$\times \int_{0<t_1<\ldots<t_{n-1}<t} dt_1 \ldots dt_{n-1} \int_{\mathbb{R}^{(n-1)d}} \mu(d\xi_1)\ldots\mu(d\xi_{n-1}) \left|\prod_{j=1}^{n-1} \mathcal{F}p^{(0)}_{t_{j+1}-t_j}\left(\sum_{l=1}^{j}\xi_l\right)\right|^2$$

$$\leq D_t^n C_T|x-y|^{2\alpha_1} \int_{0<t_1<\ldots<t_{n-1}<t} dt_1 \ldots dt_{n-1} \int_{\mathbb{R}^{(n-1)d}} \mu(d\xi_1)\ldots\mu(d\xi_{n-1}) \left|\prod_{j=1}^{n-1} \mathcal{F}p^{(0)}_{t_{j+1}-t_j}\left(\sum_{l=1}^{j}\xi_l\right)\right|^2, \tag{4.84}$$

where we use the assumption (4.79) in the third inequality.

To compute the integral, one option is to integrate the spatial variables in the order $d\xi_{n-1}, \ldots, d\xi_1$ using the maximal principle (Lemma 4.4) and assumption (H). Then we will have

$$n!\left\|\tilde{f}_n(\cdot,t,x)-\tilde{f}_n(\cdot,t,y)\right\|^2_{\mathcal{H}^{\otimes n}}$$

$$\leq D_t^n C_T|x-y|^{2\alpha_1} \int_{0<t_1<\ldots<t_{n-1}<t} dt_1 \ldots dt_{n-1} \prod_{j=1}^{n-1} \int_{\mathbb{R}^d} \left|\mathcal{F}p^{(0)}_{t_{j+1}-t_j}(\xi_j)\right|^2 \mu(d\xi_j)$$

$$= D_t^n C_T|x-y|^{2\alpha_1} \int_{0<t_1<\ldots<t_{n-1}<t} dt_1 \ldots dt_{n-1} \prod_{j=1}^{n-1} \int_{\mathbb{R}^{2d}} p^{(0)}_{t_{j+1}-t_j}(x_j)p^{(0)}_{t_{j+1}-t_j}(y_j)\gamma(x_j-y_j)dx_jdy_j$$

$$\leq D_t^n C^{2n-2} C_T|x-y|^{2\alpha_1} \int_{0<t_1<\ldots<t_{n-1}<t} dt_1 \ldots dt_{n-1} \prod_{j=1}^{n-1} \int_{\mathbb{R}^{2d}} P_{t_{j+1}-t_j}(-x_j)P_{t_{j+1}-t_j}(-y_j)\gamma(x_j-y_j)dx_jdy_j$$

$$= D_t^n C^{2n-2} C_T|x-y|^{2\alpha_1} \int_{0<t_1<\ldots<t_{n-1}<t} dt_1 \ldots dt_{n-1} \prod_{j=1}^{n-1} \int_{\mathbb{R}^d} \left|\mathcal{F}P_{t_{j+1}-t_j}(-\xi_j)\right|^2 \mu(d\xi_j)$$

$$\leq D_t^n C^{2n-2} c_1^{2n-2} C_T|x-y|^{2\alpha_1} \int_{0<t_1<\ldots<t_{n-1}<t} dt_1 \ldots dt_{n-1} \prod_{j=1}^{n-1} \int_{\mathbb{R}^d} \exp\left(-2c_2(t_{j+1}-t_j)\Psi(-\xi_j)\right)\mu(d\xi_j),$$

where we use definition (2.12) of the inner product in the third line and fifth line. The integral coincides with the integral line 2 of page 66 of [24]. Thus, $n!\|\tilde{f}_n(\cdot,t,x)-\tilde{f}_n(\cdot,t,y)\|^2_{\mathcal{H}^{\otimes n}}/|x-y|^{2\alpha_1}$ is finite and sumable in n. Therefore, the Hölder continuity of $u(t,x)$ follows directly from (4.80) and the Hölder continuity of each $\tilde{f}_n(x_1,\ldots,x_n,t,\cdot)$. ◻

We have the following maximal principle.

Lemma 4.4. *For any $a\in\mathbb{R}^d$, $t\in\mathbb{R}_+$, we have*

$$\int_{\mathbb{R}^d}\left|\mathcal{F}p_t^{(0)}(\xi+a)\right|^2\mu(d\xi) \leq \int_{\mathbb{R}^d}\left|\mathcal{F}p_t^{(0)}(\xi)\right|^2\mu(d\xi).$$

Proof. By Parseval–Plancherel identity, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\mathcal{F}p_t^{(0)}(\xi+a)\right|^2\mu(d\xi) &= \int_{\mathbb{R}^{2d}} p_t^{(0)}(x)e^{-\iota x\cdot a}p_t^{(0)}(y)e^{-\iota y\cdot a}\gamma(x-y)dxdy \\ &\leq \int_{\mathbb{R}^{2d}} p_t^{(0)}(x)p_t^{(0)}(y)\gamma(x-y)dxdy \\ &= \int_{\mathbb{R}^d}\left|\mathcal{F}p_t^{(0)}(\xi)\right|^2\mu(d\xi).\end{aligned}$$

◻

Acknowledgement J. Song is partially supported by the National Natural Science Foundation of China (nos. 12071256 and 12226001), the Major Basic Re- search Program of the Natural Science

Foundation of Shandong Province in China (nos. ZR2019ZD42 and ZR2020ZD24). W. Yuan gratefully acknowledges the financial support of ERC Consolidator Grant 815703 "STAMFORD: Statistical Methods for High Dimensional Diffusions".

Appendix A. Feynman-Kac formula

Consider the following partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u(t,x) + f(t,x)u(t,x), & t \geq 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.85}$$

where $\mathcal{L}$ is the infinitesimal generator of a Feller process X, $u_0(x)$ is a bounded measurable function, and $f(t,x)$ is a measurable function. The Feynman-Kac formula for (1.85) can be found in [17, Theorem 3.47]. Nevertheless, we provide our version of Feynman-Kac formula which suits our purpose.

Assume

$$\mathbb{E}_X \left[\exp\left(\left| \int_0^t f(t-s, X_s^x) ds \right| \right) \right] < \infty, \text{ for all } x \in \mathbb{R}^d. \tag{1.86}$$

Then, the following Feynman-Kac representation

$$u(t,x) = \mathbb{E}_X \left[u_0(X_t^x) \exp\left(\int_0^t f(t-s, X_s^x) ds \right) \right] \tag{1.87}$$

is a Duhamel's solution to (1.85), i.e.,

$$u(t,x) = \int_{\mathbb{R}} p_t^{(x)}(y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} p_{t-s}^{(x)}(y) u(s,y) f(s,y) dy ds. \tag{1.88}$$

Proof. One can verify directly that $u(t,x)$ given by (1.87) satisfies (1.88). For simplicity, we assume $u_0(x) \equiv 1$. Plugging the expression (1.87) to (1.88), we have that the right-hand side of (1.88) is

$$\begin{aligned} & 1 + \int_0^t \mathbb{E}_X [u(s, X_{t-s}^x) f(s, X_{t-s}^x)] ds \\ =& 1 + \int_0^t \mathbb{E}_X [u(t-s, X_s^x) f(t-s, X_s^x)] ds \\ =& 1 + \int_0^t \mathbb{E}_X \left[\tilde{\mathbb{E}}_{\tilde{X}} \left[\exp\left(\int_0^{t-s} f(t-s-r, \tilde{X}_r^{X_s^x}) dr \right) \right] f(t-s, X_s^x) \right] ds, \end{aligned} \tag{1.89}$$

where $\tilde{X}$ is an independent copy of X and $\tilde{\mathbb{E}}$ means the expectation with respect to $\tilde{X}$. Applying Taylor's expansion to the function e^x and then taking expectation, one can show that the resulting series of the right-hand side of (1.89) is absolute convergent under the condition (1.86) and coincides with the series expansion of $u(t,x) = \mathbb{E}_X \left[\exp\left(\int_0^t f(t-s, X_s^x) ds \right) \right]$ on the left-hand side of (1.88). □

References

[1] Raluca M. Balan, Lluís Quer-Sardanyons, and Jian Song. Hölder continuity for the parabolic Anderson model with space-time homogeneous Gaussian noise. Acta Mathematica Scientia, 2019, 39B(3): 717–730.

[2] V. Bally and E. Pardoux: Malliavin calculus for white noise driven Parabolic SPDEs. Potential Analysis 9(1998) 27-64.

[3] N. Bouleau, F. Hirsch. Dirichlet forms and analysis on Wiener space. Walter de Gruyter, Berlin (1991).

[4] X. Chen. Random Walk Intersections: Large Deviations and Related Topics. Mathematical Surveys and Monographs 157. Amer. Math. Soc., Providence, RI, 2010

[5] X Chen, Y Hu, J Song, X Song (2018). Temporal asymptotics for fractional parabolic Anderson model. Electron. J. Probab. 23, no. 14, 1–39.

[6] R.C. Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s, Electron. J. Probab. 4 (6) (1999) 29.
[7] X Chen, Y Hu, J Song, F Xing (2015). Exponential asymptotics for time–space Hamiltonians Annales de l'IHP Probabilités et statistiques 51 (4), 1529-1561
[8] Yaozhong Hu (2016). Analysis on Gaussian spaces. World Scientific.
[9] . Y. Hu. Some recent progress on stochastic heat equations. Acta Mathematica Scientia, 2019, 39B(3): 874–914
[10] Yaozhong Hu, Jingyu Huang, David Nualart, Samy Tindel (2015). Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. Electron. J. Probab. **20**, no. 55, 1–50.
[11] Y. Hu, K. Lê. Joint Hölder continuity of parabolic Anderson model. Acta Mathematics Scientia, 2019, 39B(3): 764-780.
[12] Y. Hu, and D. Nualart. Stochastic heat equation driven by fractional noise and local time, Probab. Theory Related Fields 143 (1-2) (2009) 285-328.
[13] Huang, J., Lê, K., and Nualart, D. (2017). Large time asymptotics for the parabolic Anderson model driven by spatially correlated noise. In Annales de l'Institut Henri Poincaré, Probabilités et Statistiques (Vol. 53, No. 3, pp. 1305-1340). Institut Henri Poincaré.
[14] Yaozhong Hu, David Nualart, Jian Song. Feynman-Kac formula for heat equation driven by fractional white noise. The Annals of Probability 2011, Vol. 39, No. 1, 291-326.
[15] Yaozhong Hu, David Nualart, Jian Song. A nonlinear stochastic heat equation: Hölder continuity and smoothness of the density of the solution. Stochastic Processes and their Applications 123 (2013) 1083-1103.
[16] J. Le Gall (1994). Exponential moments for the renormalized self-intersection local time of planar Brownian motion. In Séminaire de Probabilités, XXVIII. Lecture Notes in Math. 1583 172-180. Springer, Berlin.
[17] T. M. Liggett. Continuous time Markov processes: an introduction. American Mathematical Society, Providence, Rhode Island.
[18] O. Mocioalca, F. Vienns (2005). Skorohod integration and stochastic calculus beyond the fractional Brownian scale. J. Funct. Anal. 222 385-434.
[19] C. Mueller and D. Nualart. Regularity of the density for the stochastic heat equation, Electron. J. Probab. 13 (74)(2008) 2248-2258.
[20] D. Nualart, The Malliavin Calculus and Related Topics, seconded.,in: Probability and its Applications (NewYork), Springer-Verlag, Berlin, 2006.
[21] Nualart David, Lluís Quer-Sardanyons. Existence and Smoothness of the Density for Spatially Homogeneous SPDEs. Potential Anal (2007) 27: 281-299.
[22] E. Pardoux and T. Zhang, Absolute continuity of the law of the solution of a parabolic SPDE, J. Funct. Anal. 112 (1993), 447-458.
[23] G. Rang, J. Song, and M. Wang. Scaling limit of a long-range random walk in time-correlated random environment. arXiv:2210.01009.
[24] Jian Song (2017). On a class of stochastic partial differential equations. Stochastic Processes and their Applications 127, 37-79.
[25] Samy Tindel. SPDEs with psedodifferential generators: the existence of a density. Appl. Math. 27(3) (2000)287-308.
[26] J. B. Walsh (1986). An introduction to stochastic partial differential equations. In École D'Éeté de Probabilités de Saint–Flour, XIV—1984. Lecture Notes in Math. 1180 265–439. Springer, Berlin.

Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, China

Email address: txjsong@sdu.edu.cn

School of Mathematics, Shandong University, Jinan, Shandong, China

Email address: wangmeng22@mail.sdu.edu.cn

University of Luxembourg, Department of Mathematics, Maison du Nombre 6, Avenue de la Fonte L-4364 Esch-sur-Alzette. Luxembourg.

Email address: ywangjun@connect.hku.hk