ON A CLASS OF STOCHASTIC FRACTIONAL HEAT EQUATIONS

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Abstract. Considering the stochastic fractional heat equation driven by Gaussian noise with the covariance function defined by the heat kernel, we establish Feynman-Kac formulae for both Stratonovich and Skorohod solutions, along with their respective moments. One motivation lies in the occurrence of this equation in the study of a random walk in random environment which is generated by a field of independent random walks starting from a Poisson field.

Keywords. Stochastic fractional heat equation; Stratonovich solution; Skorohod solution; Feynman-Kac formula; directed polymer model.

AMS 2020 Subject Classifications. 60H15, 35R60, 60G60

1. Introduction

Consider the following stochastic fractional heat equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x) + u(t,x)\dot{W}(t,x),\ t\geq 0, x\in\mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $-(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian with $\alpha\in(0,2]$, the initial value u_0 is a bounded continuous function, the noise $\dot{W}$ is a generalized space-time Gaussian field with covariance function given by

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = p_{|t-s|}(x-y), \tag{1.2}$$

where $p_t(x) = (2\pi t)^{-\frac{d}{2}}\exp(-\frac{|x|^2}{2t})$ for $t>0$ and $x\in\mathbb{R}^d$ is the heat kernel, and the product in $u\dot{W}$ is either an ordinary product or a Wick product.

In this note, we shall prove Feynman-Kac formula for the Stratonovich solution of (1.1) (corresponding to the ordinary product in $u\dot{W}$) which has been done for the case $\alpha=2, d=1$ in [18], and prove the existence and uniqueness of the Skorohod solution (corresponding to the Wick product in $u\dot{W}$) and derive Feynman-Kac formulae for the solution and its moments. We obtain the Feynman-Kac formulae for the Stratonovich solution of (1.1) and its moments for $d<2$ (i.e. $d=1$ and $\alpha\in(0,2]$, see Theorem 3.3), prove that the Skorohod solution exists uniquely for $d<2+\alpha$ (see Theorem 4.1), and obtain the Feynman-Kac formulae for the Skorohod solution and its moments for $d<2$ and for $d<2+\alpha$ respectively (see Theorem 4.2 and Theorem 4.3).

We present a selection of related works within the SPDEs literature, recognizing that this list is not exhaustive. It was shown in [10] that (1.1) with space-time white noise $\dot{W}$ admits a unique square integrable Skorohod solution only when $d=1$. When the noise $\dot{W}$ is white in space and colored in time, the condition for the existence of a unique Skorohod solution of (1.1) was identified in [12]. In [13] the Feynman-Kac formulae for the Stratonovich and Skorohod solutions as well as for the moments were obtained, where $\dot{W}$ is the partial derivative of fractional Brownian sheet with the Hurst parameters within $(\frac{1}{2},1)$. The case that $\dot{W}$ is a general Gaussian noise was studied in [11]. All the above-mentioned works considered the case $\alpha=2$. The result in [13] was extended in [6] to the case $\alpha\in(0,2]$, and to SPDEs with the differential operator associated with a symmetric

Lévy process in [19]. We also refer to [21, 5, 9, 14, 2, 7, 3, 20] and the references therein for more related results.

The stochastic heat equation with multiplicative Gaussian noise in the form of (1.1) is intimately connected with the directed polymer model in random environment, of which the study was initiated in [1] and further developed in [4, 16, 18, 8, 17], etc. It was proved in [1] that for a simple symmetric random walk in i.i.d random environment on $\mathbb{N}\times\mathbb{Z}$, the rescaled partition function converges weakly to the Itô (i.e. Skorohod) solution of (1.1) with $d=1, \alpha=2$ and $\dot W$ being space-time white noise; this result was extended from a simple random walk to a long-range random walk in [4] and correspondingly α belongs to $(1,2]$. When the environment is correlated in space but still independent in time, the case of simple random walk (resp. long-range random walk) was considered in [16] (resp. [8]), in which they proved that the rescaled partition function converges weakly to the Itô solution of (1.1) with $\alpha=2$ (resp. $\alpha\in(1,2]$) and the noise being white in time and colored in space. When the environment is correlated in time but independent in space and the random walk is possibly long-range, it was shown in [17] that there are two types of rescaled partition functions which converge weakly to the Skorohod solution and the Stratonovich solution respectively of (1.1) with $d=1, \alpha\in(1,2]$ and the Gaussian noise $\dot W$ being colored in time and white in space. For a simple symmetric random walk in time-space correlated random environment generated by a Poisson field of independent random walks, the rescaled partition function was proved in [18] to converge weakly to the Stratonovich solution of (1.1) with $d=1, \alpha=2$ and the covariance of $\dot W$ being given by (1.2).

The study of the equation (1.1) with the covariance of the noise given by (1.2) is inspired by the above-mentioned works, in particular by [18] and [17]. In light of [18] and [17], our result suggests that for a long-range random walk in a Poisson field of independent of random walks, there should exist two types of partition functions converging weakly to the Stratonovich solution for $d=1$ and to the Skorohod solution for $d<2+\alpha$ respectively, as done in [17] (see Theorem 1.2 and Theorem 1.3 therein).

The rest of the paper is organized as follows. In section 2, some preliminary knowledge is provided. In Sections 3 and 4, the Stratonovich solution and the Skorohod solution are studied respectively.

2. Preliminaries

In this section, we provide some facts that will be used in this note on α-stale process and Gaussian analysis.

Let $X=\{X_t, t\ge 0\}$ be a d-dimensional α-stable process independent of $\dot W$ with the density function $g_\alpha(\cdot,\cdot)$. Note that when $\alpha=2$, $g_\alpha(t,x)=p_t(x)$. The Fourier transform of $g_\alpha(t,\cdot)$ with respect to the spatial variables is given by

$$\mathcal{F}g_\alpha(t,\xi)=\exp\left(-c_\alpha t|\xi|^\alpha\right) \tag{2.3}$$

for some $c_\alpha>0$. Moreover, for $p>0$,

$$\int_{\mathbb{R}^d}|\mathcal{F}g_\alpha(t,\xi)|^p\,d\xi=(c_\alpha tp)^{-\frac{d}{\alpha}}\int_{\mathbb{R}^d}e^{-|\xi|^\alpha}d\xi=Cp^{-\frac{d}{\alpha}}t^{-\frac{d}{\alpha}} \tag{2.4}$$

for some constant C that depends on c_α, α and d.

Let $\mathcal{H}$ be the completion of the space of $C_c^\infty([0,\infty)\times\mathbb{R}^d)$ of smooth functions with compact support equipped with the inner product

$$\langle f,g\rangle_{\mathcal{H}}=\int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}}f(s,x)g(t,y)p_{|t-s|}(x-y)dxdydsdt$$

$$= \int_{\mathbb{R}_+^2 \times \mathbb{R}^d} \mathcal{F}f(s,\cdot)(\xi)\overline{\mathcal{F}g(t,\cdot)(\xi)}\mathcal{F}p_{|t-s|}(\xi)d\xi dsdt. \tag{2.5}$$

We denote by $\|\cdot\|_{\mathcal{H}}$ the norm induced by the inner product.

In a complete probability space $(\Omega, \mathcal{F}, P)$, define an isonormal Gaussian process $\{W(g), g \in \mathcal{H}\}$ with covariance $\mathbb{E}[W(g)W(h)] = \langle g, h\rangle_{\mathcal{H}}$ for all $g, h \in \mathcal{H}$. In this paper, we also denote

$$W(g) := \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} g(t,x)W(dt,dx).$$

In particular, if $g(s,y) = \mathbf{1}_A(s,y)$ where A is of the form $[0,t]\times\prod_{j=1}^d[0,x_j]$, we also write $W(t,x) = W(I_A)$. The Gaussian noise $\dot{W}(t,x)$ can be identified as the generalized derivative $\frac{\partial^{1+d}}{\partial t\partial x_1\cdots\partial x_d}W(t,x)$.

Let $C_p^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ be the set of all infinitely continuously differentiable functions with all partial derivatives being polynomial growth. We define the set

$$\mathcal{S} = \left\{ f\left(W(g_1),\dots,W(g_n)\right) : n\in\mathbb{N}_+, f\in C_p^\infty(\mathbb{R}_+\times\mathbb{R}^d), g_1,\dots,g_n\in\mathcal{H}\right\}.$$

Then we can define the Malliavin derivative D on $\mathcal{S}$ by

$$D\left(f\left(W(g_1),\dots,W(g_n)\right)\right) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\left(W(g_1),\dots,W(g_n)\right)g_j.$$

The operator D is closable from $L^2(\Omega)$ to $L^2(\Omega;\mathcal{H})$ and we denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ under the norm $\|F\|_{1,2} = \left(\mathbb{E}\left[|F|^2\right] + \mathbb{E}\left[\,\|DF\|_{\mathcal{H}}^2\,\right]\right)^{\frac{1}{2}}$. Define

$$\mathrm{Dom}(\delta) := \left\{u : |\mathbb{E}\left[\langle DF, u\rangle_{\mathcal{H}}\right]| \le c_u\|F\|_2, \text{ for all } F\in\mathbb{D}^{1,2}\right\}.$$

For $u\in\mathrm{Dom}(\delta)$, the divergence operator (also called Skorohod integral) $\delta(u)$ defined by the following duality formula $\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}]$. We also write $\delta(u) := \int_0^\infty\int_{\mathbb{R}^d} u(s,x)W(ds,dx)$. For more details, we refer to [15].

Throughout this note, we use C to denote a generic constant which may vary in different contexts.

3. Stratonovich solution

In this section, we will study the solution of (1.1) in the Stratonovich sense, mainly following the approach used in [18, Section 2].

Denote

$$A_{t,x}^{\varepsilon,\delta}(r,y) = \int_0^t \psi_\delta(t-s-r)p_\varepsilon(X_s^x - y)ds, \tag{3.6}$$

where $\psi_\delta(t) = \frac{1}{\delta}I_{[0,\delta]}(t)$ for $t\ge 0$, and $X_s^x = X_s + x$. Hence $A_{t,x}^{\varepsilon,\delta}(r,y)$ is an approximation of $\delta(X_{t-r}^x - y)$ when ε and δ are small. It was shown in [18, Proposition 2.3] that $A_{t,x}^{\varepsilon,\delta}$ belongs to $\mathcal{H}$ almost surely for all $\varepsilon,\delta>0$ and $I_{t,x}^{\varepsilon,\delta} = \int_0^t\int_{\mathbb{R}^d} A_{t,x}^{\varepsilon,\delta}(r,y)W(dr,dy) = W(A_{t,x}^{\varepsilon,\delta})$ is well-defined. By the same argument used in the proof of [18, Proposition 2.3], we can get the following result.

Theorem 3.1. *Assume $d=1$. Then we have $I_{t,x}^{\varepsilon,\delta}$ converges in L^2 as $(\varepsilon,\delta)\to 0$ to a limit $I_{t,x}$ denoted by*

$$I_{t,x} =: \int_0^t\int_{\mathbb{R}^d}\delta(X_{t-r}^x - y)W(dr,dy) = W(\delta(X_{t-\cdot}^x - \cdot)). \tag{3.7}$$

Conditional on X, $I_{t,x}$ is a Gaussian random variable with mean 0 and variance

$$Var[I_{t,x}|X] = \int_0^t \int_0^t p_{|s-r|}(X_s - X_r) drds. \tag{3.8}$$

Theorem 3.2. *The following estimate holds if and only if $d = 1$:*

$$\mathbb{E}\left[\exp\left(\int_0^t \int_0^t p_{|s-r|}(X_s - X_r) drds\right)\right] < \infty. \tag{3.9}$$

Proof. When $d = 1$, clearly (3.9) follows from

$$\int_0^t \int_0^t p_{|s-r|}(X_s - X_r) drds \le \int_0^t \int_0^t (2\pi|s-r|)^{-\frac{1}{2}} drds < \infty.$$

Now we show the necessity of the condition $d = 1$, for which we show that $d = 1$ is a necessary condition for $\mathbb{E}\left[\int_0^t \int_0^t p_{|r-s|}(X_s - X_r) drds\right] < \infty$. Indeed, we have

$$\mathbb{E}\left[\int_0^t \int_0^t p_{|s-r|}(X_s - X_r) drds\right] = 2\int_0^t \int_0^s \int_{\mathbb{R}^d} p_{s-r}(y) g_\alpha(s-r, y) dydrds$$
$$= 2\int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-(s-r)(|\xi|^2 + c_\alpha|\xi|^\alpha)} d\xi drds \ge 2\int_0^t \int_0^s \int_{|\xi|\ge 1} e^{-C(s-r)(|\xi|^2+|\xi|^\alpha)} d\xi drds$$
$$\ge 2\int_0^t \int_0^s \int_{|\xi|\ge 1} e^{-2C(s-r)|\xi|^2} d\xi drds.$$

Noting that

$$2\int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-2C(s-r)|\xi|^2} d\xi drds = C\int_0^t \int_0^s (s-r)^{-\frac{d}{2}} drds$$

is finite iff $d = 1$ and that $2\int_0^t \int_0^s \int_{|\xi|<1} e^{-2C(s-r)|\xi|^2} d\xi drds$ is finite for all d, we deduce that $d = 1$ is a necessary condition for $\mathbb{E}\left[\int_0^t \int_0^t p_{|s-r|}(X_s - X_r) drds\right] < \infty$ and hence for (3.9). □

Consider the following approximation of $\dot{W}(t, x)$:

$$\dot{W}^{\varepsilon,\delta}(t, x) = \int_{[0,t]\times\mathbb{R}} \psi_\delta(t-s) p_\varepsilon(x-y) W(ds, dy),$$

The following definition of Stratonovich integral is borrowed from [13, Definition 4.1].

Definition 3.1. *Given a random field $v = \{v(t, x), t \ge 0, x \in \mathbb{R}^d\}$ such that $\int_0^T \int_{\mathbb{R}^d} |v(t, x)| dxdt < \infty$ almost surely for all $T > 0$, the Stratonovich integral $\int_0^T \int_{\mathbb{R}^d} v(t, x) W(dt, dx)$ is defined as the following limit in probability,*

$$\lim_{\varepsilon,\delta\to 0} \int_0^T \int_{\mathbb{R}^d} v(t, x) \dot{W}^{\varepsilon,\delta}(t, x) dxdt.$$

Let $\mathcal{F}_t^W$ be the σ-algebra generated by $\{W(s, x), s \in [0, t], x \in \mathbb{R}^d\}$.

Definition 3.2. *A random field $\{u(t, x), t \ge 0, x \in \mathbb{R}^d\}$ is a mild Stratonovich solution to (1.1) if for all $t \ge 0$ and $x \in \mathbb{R}^d$, $u(t, x)$ is $\mathcal{F}_t^W$-measurable and the following integral equation holds*

$$u(t, x) = (g_\alpha(t, \cdot) * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} g_\alpha(t-s, x-y) u(s, y) W(ds, dy), \tag{3.10}$$

where $(g_\alpha(t, \cdot) * u_0)(x) = \int_{\mathbb{R}^d} g_\alpha(t, x-y) u_0(y) dy$ and the stochastic integral is in the Stratonivich sense of Definition (3.1).

Below is the main result for mild Stratonovich solution.

Theorem 3.3. *Assume $d = 1$. Then,*

$$u(t,x) = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta(X_{t-r}^x - y)W(dr,dy)\right)\right] \tag{3.11}$$

is a mild Stratonovich solution of (1.1). Furthermore, for any positive integer p,

$$\mathbb{E}\left[u(t,x)^p\right] = \mathbb{E}\Big[\prod_{j=1}^p u_0(X_t^{(j)} + x)\exp\left(\frac{1}{2}\sum_{j,k=1}^p\int_{[0,t]^2} p_{|s-r|}\left(X_s^{(j)} - X_r^{(k)}\right)dsdr\right)\Big], \tag{3.12}$$

where $X^{(1)}, \cdots, X^{(p)}$ are independent copies of X.

Proof. Noting Theorem 3.2, the proof of (3.11) follows exactly the same argument used in [19, Theorem 4.6] (see also the proof of [18, Proposition 1.7]). The formula (3.12) is a direct consequence of (3.11). □

4. Skorohod solution

In this section, we will prove the existence and uniqueness of the Skorohod solution of (1.1), and present Feynman-Kac formulae for the solution and its moments. The method is inspired, for instance, by [19].

4.1. **The existence and uniqueness of the Skorohod solution.**

Definition 4.3. *A random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a mild Skorohod solution of (1.1) if for all $t \geq 0$ and $x \in \mathbb{R}^d$, $u(t,x)$ is $\mathcal{F}_t^W$-measurable, square-integrable, and satisfies the following integral equation:*

$$u(t,x) = (g_\alpha(t,\cdot) * u_0)(x) + \int_0^t\int_{\mathbb{R}^d} g_\alpha(t-s, x-y)u(s,y)W(ds,dy), \tag{4.13}$$

where $(g_\alpha(t,\cdot) * u_0)(x) = \int_{\mathbb{R}^d} g_\alpha(t, x-y)u_0(y)dy$ and the stochastic integral is in the Skorohod sense.

If (1.1) admits a solution u, then by iteration, one can formally write

$$u(t,x) = (g_\alpha(t,\cdot) * u_0)(x) + \sum_{n=1}^\infty I_n(\tilde{f}_n), \tag{4.14}$$

where the function f_n is given by

$$f_n(x_1, s_1, \ldots, x_n, s_n, x, t) = \prod_{j=1}^n g_\alpha(s_{j+1} - s_j, x_{j+1} - x_j)u_0(x_1)\mathbf{1}_{\{0<s_1<\ldots<s_n<t\}},$$

with convention $s_{n+1} = t$ and $x_{n+1} = x$, $\tilde{f}_n$ is the symmetrization of f_n, i.e.

$$\tilde{f}_n(x_1, s_1, \ldots, x_n, s_n, x, t) = \frac{1}{n!}f_n(x_{\tau(1)}, s_{\tau(1)}, \ldots, x_{\tau(n)}, s_{\tau(n)}, x, t),$$

where τ is the permutation such that $0 < s_{\tau(1)} < \ldots < s_{\tau(n)} < t$ and $I_n(\cdot)$ is the multiple Wiener integral given by

$$I_n(f_n) = \int_{\mathbb{R}_+^n}\int_{\mathbb{R}^{nd}} f_n(x_1, s_1, \ldots, x_n, s_n, x, t)W(ds_1, dx_1)\ldots W(ds_n, dx_n).$$

Noting the expansion (4.14) and the uniqueness of the Wiener chaos expansion, we have that the existence of a unique mild Skorohod solution to (1.1) is equivalent to

$$\mathbb{E}\left[\left|(g_\alpha(t,\cdot) * u_0)(x)\right|^2\right] + \sum_{n=1}^\infty n!\left\|\tilde{f}_n(\cdot, t, x)\right\|_{\mathcal{H}^{\otimes n}}^2 < \infty, \tag{4.15}$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$. The following theorem is the main theorem of this paper.

Theorem 4.1. *Assume $d<2+\alpha$. Then the condition (4.15) is satisfied and hence equation (1.1) admits a unique mild Skorohod solution.*

Proof. Without loss of generality, we assume that $u_0\equiv 1$. For $n\geq 1$, by (2.5), we have

$$\begin{aligned}
&n!\left\|\tilde f_n(\cdot,t,x)\right\|^2_{\mathcal{H}^{\otimes n}}\\
=&\frac{1}{n!}\int_{[0,t]^{2n}}\int_{\mathbb{R}^{2nd}} f_n(x_{\tau(1)},s_{\tau(1)},\dots,x_{\tau(n)},s_{\tau(n)},x,t)\prod_{j=1}^n p_{|s_j-r_j|}(x_j-y_j)\\
&\qquad\qquad\times f_n(y_{\sigma(1)},r_{\sigma(1)},\dots,y_{\sigma(n)},r_{\sigma(n)},x,t)d\mathbf{x}d\mathbf{y}d\mathbf{r}d\mathbf{s}\\
=&\frac{1}{n!}\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}} \mathcal{F}f_n(\cdot,s_{\tau(1)},\dots,\cdot,s_{\tau(n)},x,t)(\xi_1,\dots,\xi_n)\prod_{j=1}^n \mathcal{F}p_{|s_j-r_j|}(\xi_j)\\
&\qquad\qquad\times \overline{\mathcal{F}f_n(\cdot,r_{\sigma(1)},\dots,\cdot,r_{\sigma(n)},x,t)(\xi_1,\dots,\xi_n)}d\boldsymbol{\xi}d\mathbf{r}d\mathbf{s}\\
\leq&\frac{1}{n!}\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\exp\left(-c_\alpha\left[(s_{\tau(j+1)}-s_{\tau(j)})+(r_{\sigma(j+1)}-r_{\sigma(j)})\right]|\xi_j|^\alpha\right)\exp\left(-|s_j-r_j||\xi_j|^2\right)d\boldsymbol{\xi}d\mathbf{r}d\mathbf{s},
\end{aligned}\tag{4.16}$$

with $0<s_{\tau(1)}<\dots<s_{\tau(n)}<t$ and $0<r_{\sigma(1)}<\dots<r_{\sigma(n)}<t$. The last inequality in (4.16) follows from (2.3), maximum principle in [19, Lemma 4.9] and the fact that

$$\mathcal{F}f_n(\cdot,s_{\tau(1)},\dots,\cdot,s_{\tau(n)},x,t)(\xi_1,\dots,\xi_n)=\exp\left(-ix\cdot\sum_{j=1}^n\xi_j\right)\prod_{j=1}^n\mathcal{F}g_\alpha(s_{\tau(j+1)}-s_{\tau(j)},\xi_1+\dots+\xi_j).$$

Let $p,q>0$ such that $\frac{2}{p}+\frac{1}{q}$=1. Then by utilizing Hölder inequality and (2.4), we have that for some constant C depending on p,q,α,d,

$$\begin{aligned}
&n!\left\|\tilde f_n(\cdot,t,x)\right\|^2_{\mathcal{H}^{\otimes n}}\\
&\leq\frac{1}{n!}\int_{[0,t]^{2n}}\left(\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\exp\left(-c_\alpha p(s_{\tau(j+1)}-s_{\tau(j)})|\xi_j|^\alpha\right)d\xi\right)^{\frac1p}\left(\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\exp\left(-q(s_j-r_j)|\xi_j|^2\right)d\xi\right)^{\frac1q}\\
&\qquad\times\left(\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\exp\left(-c_\alpha p(r_{\sigma(j+1)}-r_{\sigma(j)})|\xi_j|^\alpha\right)d\xi\right)^{\frac1p}d\mathbf{r}d\mathbf{s}\\
&=\frac{1}{n!}C^n\int_{[0,t]^{2n}}\prod_{j=1}^n(s_{\tau(j+1)}-s_{\tau(j)})^{-\frac{d}{p\alpha}}\prod_{j=1}^n(r_{\sigma(j+1)}-r_{\sigma(j)})^{-\frac{d}{p\alpha}}\prod_{j=1}^n(s_j-r_j)^{-\frac{d}{2q}}d\mathbf{r}d\mathbf{s} &(4.17)\\
&\leq\frac{1}{n!}C^n\left(\int_{[0,t]^n}\prod_{j=1}^n\left(s_{\tau(j+1)}-s_{\tau(j)}\right)^{-\frac{d}{p\alpha}\times\frac{2}{(2-\frac{d}{2q})}}d\mathbf{s}\right)^{2-\frac{d}{2q}} &(4.18)\\
&=\frac{1}{n!}C^n(n!)^{2-\frac{d}{2q}}\left(\int_{\{0<s_1<\dots<s_n<t\}}\prod_{j=1}^n(s_{j+1}-s_j)^{-\frac{d}{p\alpha}\times\frac{2}{(2-\frac{d}{2q})}}d\mathbf{s}\right)^{2-\frac{d}{2q}} &(4.19)\\
&=C^n(n!)^{1-\frac{d}{2q}}\frac{\left(\Gamma(1-\frac{2d}{p\alpha(2-\frac{d}{2q})})\right)^{n(2-\frac{d}{2q})}}{\left(\Gamma(n-\frac{2nd}{p\alpha(2-\frac{d}{2q})}+1)\right)^{2-\frac{d}{2q}}}. &(4.20)
\end{aligned}$$

In the derivation above, we assume α, d, p, q with $\frac{2}{p}+\frac{1}{q}=1$ satisfy the following three conditions:

$$\begin{aligned}
&-\frac{d}{2q}\in(-1,0)\iff d<2q,\\
&\frac{2d}{p\alpha(2-\frac{d}{2q})}<1\iff d<\frac{4pq\alpha}{4q+p\alpha},\\
&\Big(1-\frac{2d}{p\alpha(2-\frac{d}{2q})}\Big)(2-\frac{d}{2q})>1-\frac{d}{2q}\iff d<\frac{p\alpha}{2}.
\end{aligned}$$

where the first condition is due to Lemma 5.1.1 in [15] by which we deduce (4.18), the second one ensures the integral in (4.19) is finite, and the third one is used to obtain the convergence of the series. It turns out that if we choose $p=(4+2\alpha)/\alpha$ and $q=1+\frac{\alpha}{2}$, those three conditions are satisfied, and we get the optimal condition $d<2+\alpha$ which is assumed in this theorem. The last equality (4.20) follows from Lemma C.3 of [17], by which we deduce (4.15) from Stirling's formula and the condition $d<2+\alpha$. The proof is complete. □

4.2. **Feynman-Kac formulae.** In this subsection, we present Feynman-Kac type representations for the Skorohod solution and its moments.

Theorem 4.2. *When $d=1$, the process*

$$u(t,x)=\mathbb{E}_X\left[u_0(X_t^x)\exp\left(\int_{[0,t]^2}\delta(X_{t-r}^x-y)W(dr,dy)-\frac{1}{2}\int_{[0,t]^2}p_{|s-r|}(X_s-X_r)dsdr\right)\right] \tag{4.21}$$

is the unique mild Skorohod solution to (1.1).

Proof. Note that when $d=1$, Theorem 3.3 holds. Then the proof follows directly from the argument used in the proof of [13, Theorem 7.2]. □

Theorem 4.1 indicates that the existence and uniqueness of the Skorohod solution hold under the condition $d<2+\alpha$. However, the Feynman-Kac formula (4.21) for the Skorohod solution of equation (1.1) is valid only when $d=1$, as referenced in [13, Proposition 3.2]. Nevertheless, we can investigate the Feynman-Kac formula for the p-th order moments of the Skorohod solution in the case $d<2+\alpha$.

Theorem 4.3. *Assume $d<2+\alpha$ and $\{u(t,x),t\geq 0,x\in\mathbb{R}^d\}$ is a mild Skorohod solution of (1.1). Then for all positive integer p, we have*

$$\mathbb{E}\left[u(t,x)^p\right]=\mathbb{E}\Big[\prod_{j=1}^p u_0(X_t^{(j)}+x)\exp\Big(\sum_{1\leq j<k\leq p}\int_{[0,t]^2}p_{|s-r|}\left(X_s^{(j)}-X_r^{(k)}\right)dsdr\Big)\Big], \tag{4.22}$$

where $X^{(1)},\cdots,X^{(p)}$ are independent copies of X.

Proof. When $d=1$, the desired formula (4.22) follows from (4.21) directly. For $d\in(1,2+\alpha)$, we consider the following approximation of (1.1):

$$\begin{cases}\dfrac{\partial}{\partial t}u^{\varepsilon,\delta}(t,x)=-(-\Delta)^{\frac{\alpha}{2}}u^{\varepsilon,\delta}(t,x)+u^{\varepsilon,\delta}(t,x)W_{\varepsilon,\delta}(t,x),\\ \quad u^{\varepsilon,\delta}(0,x)=u_0(x).\end{cases} \tag{4.23}$$

In accordance with Definition 4.3, the mild solution of (4.23) satisfies

$$u^{\varepsilon,\delta}(t,x)=(g_\alpha(t,\cdot)*u_0)(x)+\int_0^t\int_{\mathbb{R}^d}g_\alpha(t-s,x-y)u^{\varepsilon,\delta}(s,y)W^{\varepsilon,\delta}(ds,dy). \tag{4.24}$$

Using a similar argument as in the proof of [19, Theorem 5.6], we have

$$u^{\varepsilon,\delta}(t,x) = \mathbb{E}_X\Big[u_0(X_t^x)\exp\Big(W(A_{t,x}^{\varepsilon,\delta}) - \|A_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2\Big)\Big]. \tag{4.25}$$

satisfies (4.24) and hence is a mild solution of equation (4.23).

For the p-th moment of $u^{\varepsilon,\delta}(t,x)$, we have

$$\begin{aligned}
\mathbb{E}\Big[|u^{\varepsilon,\delta}(t,x)|^p\Big] =&\mathbb{E}\Big[\prod_{j=1}^p u_0(X_t^{(j)}+x)\exp\Big(\sum_{j=1}^p W(A_{t,x}^{\varepsilon,\delta,(j)}) - \frac12\sum_{j=1}^p\|A_{t,x}^{\varepsilon,\delta,(j)}\|_{\mathcal{H}}^2\Big)\Big]\\
=&\mathbb{E}\Big[\mathbb{E}_W\Big[\prod_{j=1}^p u_0(X_t^{(j)}+x)\exp\Big(W\Big(\sum_{j=1}^p A_{t,x}^{\varepsilon,\delta,(j)}\Big) - \frac12\sum_{j=1}^p\|A_{t,x}^{\varepsilon,\delta,(j)}\|_{\mathcal{H}}^2\Big)\Big]\Big]\\
=&\mathbb{E}\Big[\prod_{j=1}^p u_0(X_t^{(j)}+x)\exp\Big(\frac12\Big\|\sum_{j=1}^p A_{t,x}^{\varepsilon,\delta,(j)}\Big\|_{\mathcal{H}}^2 - \frac12\sum_{j=1}^p\|A_{t,x}^{\varepsilon,\delta,(j)}\|_{\mathcal{H}}^2\Big)\Big]\\
=&\mathbb{E}\Big[\prod_{j=1}^p u_0(X_t^{(j)}+x)\exp\Big(\sum_{1\le j<k\le p}\Big\langle A_{t,x}^{\varepsilon,\delta,(j)}, A_{t,x}^{\varepsilon,\delta,(k)}\Big\rangle_{\mathcal{H}}\Big)\Big],
\end{aligned} \tag{4.26}$$

where $A_{t,x}^{\varepsilon,\delta,(j)}$ is defined by (3.6) with X replaced by $X^{(j)}$.

By (3.6) and (2.5), we have

$$\begin{aligned}
\Big\langle A_{t,x}^{\varepsilon,\delta,(j)}, A_{t,x}^{\varepsilon,\delta,(k)}\Big\rangle_{\mathcal{H}} = \int_{[0,t]^4}\int_{\mathbb{R}^{2d}} &\psi_\delta(t-s-u)\psi_\delta(t-r-v)p_{|u-v|}(y-z)\\
&\times p_\varepsilon(X_s^{(j),x}-y)p_\varepsilon(X_r^{(k),x}-z)dydzdrdsdudv,
\end{aligned} \tag{4.27}$$

where $X_s^{(j),x} = X_s^{(j)}+x$. By the semigroup property of the heat kernel, we have

$$\int_{\mathbb{R}^{2d}} p_\varepsilon(X_s^{(j),x}-y)p_\varepsilon(X_r^{(k),x}-z)p_{|u-v|}(y-z)dydz = p_{|u-v|+2\varepsilon}(X_s^{(j)}-X_r^{(k)}),$$

and hence

$$\begin{aligned}
\Big\langle A_{t,x}^{\varepsilon,\delta,(j)}, A_{t,x}^{\varepsilon,\delta,(k)}\Big\rangle_{\mathcal{H}} &= \int_{[0,t]^4}\psi_\delta(t-s-u)\psi_\delta(t-r-v)p_{|u-v|+2\varepsilon}(X_s^{(j)}-X_r^{(k)})drdsdudv\\
&= \int_{[0,t]^4}\psi_\delta(u-s)\psi_\delta(v-r)p_{|u-v|+2\varepsilon}(X_s^{(j)}-X_r^{(k)})drdsdudv,
\end{aligned} \tag{4.28}$$

Therefore, noting

$$\lim_{\varepsilon,\delta\to0}\Big\langle A_{t,x}^{\varepsilon,\delta,(j)}, A_{t,x}^{\varepsilon,\delta,(k)}\Big\rangle_{\mathcal{H}} = \int_{[0,t]^2} p_{|s-r|}(X_s^{(j)}-X_r^{(k)})drds,$$

to prove the desired result (4.22) by the dominated convergence theorem, it suffices to show, for $j\neq k$,

$$\sup_{\varepsilon,\delta>0}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\lambda\Big\langle A_{t,x}^{\varepsilon,\delta,(j)}, A_{t,x}^{\varepsilon,\delta,(k)}\Big\rangle_{\mathcal{H}}\right)\right] < \infty, \quad \text{for all } \lambda>0. \tag{4.29}$$

By (4.28) and the independence between $X^{(j)}$ and $X^{(k)}$ for $j\neq k$, we have

$$\begin{aligned}
&\frac{1}{n!}\mathbb{E}\left[\left(\Big\langle A_{t,x}^{\varepsilon,\delta,(j)}, A_{t,x}^{\varepsilon,\delta,(k)}\Big\rangle_{\mathcal{H}}\right)^n\right]\\
=&\frac{1}{n!}\int_{[0,t]^{4n}}\int_{\mathbb{R}^{2nd}} f_n(x_{\tau(1)},s_{\tau(1)},\dots,x_{\tau(n)},s_{\tau(n)},x,t)\prod_{i=1}^n p_{|u_i-v_i|+2\varepsilon}(x_i-y_i)\psi_\delta(u_i-s_i)\psi_\delta(v_i-r_i)
\end{aligned}$$

$$\times f_n(y_{\sigma(1)}, t_{\sigma(1)}, \dots, y_{\sigma(n)}, t_{\sigma(n)}, x, t) d\mathbf{x} d\mathbf{y} d\mathbf{s} d\mathbf{r} d\mathbf{u} d\mathbf{v}$$

$$= \frac{1}{n!} \int_{[0,t]^{4n}} \prod_{i=1}^{n} \psi_\delta(u_i - s_{)} \psi_\delta(v_i - r_i) \int_{\mathbb{R}^{nd}} \mathcal{F} f_n(\cdot, r_{\tau(1)}, \dots, \cdot, r_{\tau(n)}, x, t)(\xi_1, \dots, \xi_n) \prod_{i=1}^{n} \mathcal{F} p_{|u_i - v_i| + 2\varepsilon}(\xi_i)$$
$$\times \overline{\mathcal{F} f_n(\cdot, s_{\sigma(1)}, \dots, \cdot, s_{\sigma(n)}, x, t)(\xi_1, \dots, \xi_n)} d\boldsymbol{\xi} d\mathbf{s} d\mathbf{r} d\mathbf{u} d\mathbf{v}$$

$$\le \frac{1}{n!} \int_{[0,t]^{4n}} \prod_{i=1}^{n} \psi_\delta(u_i - s_i) \psi_\delta(v_i - r_i)$$
$$\times \int_{\mathbb{R}^{nd}} \prod_{i=1}^{n} \exp\left(-c_\alpha \left[(s_{\tau(i+1)} - s_{\tau(i)}) + (r_{\sigma(i+1)} - r_{\sigma(i)})\right] |\xi_i|^\alpha\right) \exp\left(-|u_i - v_i| |\xi_i|^2\right) d\boldsymbol{\xi} d\mathbf{s} d\mathbf{r} d\mathbf{u} d\mathbf{v},$$

with $0 < s_{\tau(1)} < \dots < s_{\tau(n)} < t$ and $0 < r_{\sigma(1)} < \dots < r_{\sigma(n)} < t$. Then following the argument leading to (4.17) in the proof of Theorem 4.1, we get

$$\frac{1}{n!} \mathbb{E}\left[\left(\left\langle A_{t,x}^{\varepsilon,\delta,(j)}, A_{t,x}^{\varepsilon,\delta,(k)} \right\rangle_{\mathcal{H}}\right)^n\right]$$
$$\le \frac{1}{n!} C^n \int_{[0,t]^{4n}} \prod_{i=1}^{n} \psi_\delta(u_i - s_i) \psi_\delta(v_i - r_i)$$
$$\times \prod_{i=1}^{n} (r_{\sigma(i+1)} - r_{\sigma(i)})^{-\frac{d}{p\alpha}} \prod_{i=1}^{n} (s_{\tau(i+1)} - s_{\tau(i)})^{-\frac{d}{p\alpha}} \prod_{i=1}^{n} (u_i - v_i)^{-\frac{d}{2q}} d\mathbf{r} d\mathbf{s} d\mathbf{u} d\mathbf{v}$$
$$\le \frac{1}{n!} C^n \int_{[0,t]^{2n}} \prod_{i=1}^{n} (r_{\sigma(i+1)} - r_{\sigma(i)})^{-\frac{d}{p\alpha}} \prod_{i=1}^{n} (s_{\tau(i+1)} - s_{\tau(j)})^{-\frac{d}{p\alpha}} \prod_{j=1}^{n} (s_i - r_i)^{-\frac{d}{2q}} d\mathbf{r} d\mathbf{s}$$

where the last step follows from Lemma A.3 of [13]: noting $\frac{d}{2q} \in (0,1)$,

$$\int_{[0,t]^2} \psi_\delta(u - s) \psi_\delta(v - r) |u - v|^{-\frac{d}{2q}} du dv \le C |s - r|^{-\frac{d}{2q}}.$$

Then estimate (4.29) can be proved by using the same argument as in the proof of Theorem 4.1. The proof is concluded. □

Acknowledgement J. Song is partially supported by the National Natural Science Foundation of China (nos. 12071256 and 12226001), the Major Basic Research Program of the Natural Science Foundation of Shandong Province in China (nos. ZR2019ZD42 and ZR2020ZD24). W. Yuan is partially supported by ERC Consolidator Grant 815703 "STAMFORD: Statistical Methods for High Dimensional Diffusions".

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