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Numeral coins as hands-on material, material for digital tools, and for the conceptual understanding of numeral systems

Place value numeral systems with a base different from 10 look abstruse, however the key principles of such representation can be visualized, grasped, and understood in an intuitive way by working with numeral coins whose values are the powers of the given base.

We have produced numeral coins in suitable shapes that allow to represent all numbers up to 100 in the bases 2,3,5 and 10. Moreover, with *Imaginary* we have developed an interactive website to play with numeral coins.

Fig. 1: Binary coins to express the numbers up to 100.

Alternative place value numeral systems

The binary numeral system is used by computers, the digits being the bits. The hexadecimal numeral system is used to represent html colors. The Babylonians used the base-60 numeral system. The Cantor set fractal can be understood with the ternary numeral system (it depicts the real numbers from 0 to 1 that do not make use of the digit 1). Some mathematical problems have elegant solutions that make use of different place value numeral systems.

Beyond their cultural value, the main advantage of numeral bases alternative to base-10 is disconnecting biology from arithmetic. Indeed, we are so accustomed to the decimal system that we can hardly believe that any operation, calculation, and reasoning would still have been possible have we had a different number of fingers (the etymology of the word *digit* is 'finger').

An indirect but important advantage is that, by comparison, alternative numeral systems allow for a better understanding of our familiar decimal system. In the same way that learning a new language sheds a new light on the native tongue and it gives some perspective.

A further advantage (surely not appreciated by most pupils) is that alternative numeral systems train and put to a test the pupils' understanding of place value. Indeed, children can learn the numbers up to 20 without a conceptual understanding of digits because they can rely on memorization. And, for larger integers written in the decimal systems, pupils may still transfer the intuition that they get from the familiar small numbers. However, intuition is gone for good while working with other numeral bases. What is the number written as 13 in base 7? The immediate grasp of a quantity is substituted by a reasoning that allows to estimate such quantity or a calculation that allows to compute it.

Even worse (even better for those who like challenges), the intuition coming from the decimal numeral system can be misleading for another numeral system like a false friend in linguistics. Some of what we usually take for granted still holds, for example "a natural number with more digits is larger". Other true and seemingly obvious facts in base-10 become utterly false in other bases, for example "a natural number ending with 0 is even" (e.g. 10 is odd if the base is odd).

And there we discover the close relatives of the number 10. Indeed, 2 and 5 play a special role in the decimal system because they are divisors of 10. And 9=10-1 (with its divisor 3) and 11=10+1 also play a special role, for example for divisibility criteria. It's an instructive exercise in mathematical generalization determining, if it exits, the analogue of some property that holds in base-10. For example, $9\times2=18$ and $9^2=81$ have the same digits also in other bases because $(b-1)\times2$ and $(b-1)^2$ are written 1(b-2) and (b-2)1 respectively.

Let's not forget that conversion algorithms between different place value numeral systems provide interesting examples of arithmetical algorithms. There are several algorithms to compute the same quantity and there can be quicker algorithms in special cases, for example converting a number from base-10 to base-100. So, we also have a playground for algorithmic thinking.

Some intriguing aspect of numeral systems is also that we don't have enough icons to represent the possible digits for a large base. Indeed, we could pick as base a number which is larger than the number of atoms in the universe. However, digits are numbers, and we can (in principle) represent any number with the decimal number system. For example, 13 is a digit in base 20. And, working in base 20, it is important to distinguish, for example, the two-digit number (13)0 from the three-digit number 130.

A final challenge that goes beyond school mathematics but that is it within reach of mathematical clubs for talented pupils: writing rational numbers in an alternative place value numeral system (by taking negative powers of the basis as well), and determining which rational numbers have a terminating digital representation.

How numeral coins promote the understanding of numeral systems

To teach numbers, educators rely on manipulatives (such as counting blocks or abacuses) to bridge the gap between the abstract world of symbols and the tangible understanding of quantity. We have developed numeral coins in suitable shapes (ogival, triangular, pentagonal, and circular) to introduce numeral systems. Our choice has been producing coins for the numeral systems with base 2,3,5, and 10. Our sets of coins allow to represent all natural numbers up to 100 included. We have also produced placeholder grids for the coins to enforce the understanding of place value (as customary, smaller values are on the right).

Fig. 2: The set of numeral coins for the base 2,3,5 and 10.

Key mathematical ideas conveyed by the numeral coins are the following:

- We are only allowed to use as values the powers of the base (indeed, we are only allowed to use the given numeral coins).
- We can represent all numbers as a sum of powers of the base (indeed, we can pay any amount with the given numeral coins).
- We only need few copies of each power of the base (indeed, we only need few coins with the same value).
- We use the largest possible powers of the base, and with this constraint the decomposition of the number is unique. We make coin exchanges as to have coins with the highest possible value.
- The digits of the numeral representation are simple "bookkeeping", counting how many coins of a certain value we use.
- Grouping and arranging the coins (in the canonical descending order) provides a practical understanding of place value.

We can implement with coins the (probably) easiest algorithm to represent a natural number with an alternative base. We subtract powers of the base as to make the number smaller (but still non-negative) and eventually we arrive at 0. This means that we replace the given number by a bunch of coins.

We then make coin exchanges as to use the largest possible values. Finally, we count coins with different values (bookkeeping) and write down their numbers as digits for the requested number representation.

For younger pupils, we keep in mind the following adaptations: we do not use the formal power notation (for example, we write 9 instead of 3^2); we do not make multiplications (say, we do not multiply a digit by the value that it represents) but we only do additions/subtractions and counting.

Learning material for numeral systems

While the idea of hands-on materials in education is often associated with early childhood learning, there is a compelling case for the necessity of hands-on materials even at the high school level, especially in mathematics. Additionally remark that manipulatives have been chosen as the most effective approach to engaging the public of all ages in mathematical museums and exhibitions.
To provide an alternative to coins manipulatives in an era shaped by technology, we have devised an interactive website to work and experiment with numeral coins. In particular, it is possible to practice the above-mentioned algorithm and make coin exchanges. The tool has a *Junior* setting and a *Senior* setting, the former having the above-mentioned adaptations and keeping symbolic abstraction to the bare minimum. Remark that the tool is language-free to promote inclusiveness for diverse set of learners.

We are very grateful for any feedback concerning the didactical concept and its practical implementation in class. All valuable contributions will be acknowledged on our project webpage.

References (freely available learning material)

The project webpage is https://math.uni.lu/numeralsystems.

The .stl files for the numeral coins (ready for 3D-printing) with printing instructions are free for download. We thank Dominique Breton for the files.

The online tool is linked from the project webpage. We thank Daniel Ramos from the *Imaginary gGmbH* for producing it.