Congruence theorems for triangles involving angle bisectors

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Abstract

Congruence theorems for triangles provide conditions that ensure that a triangle is determined up to congruence. In the classical statements, the conditions are the lengths of certain sides or the measure of certain interior angles. We consider congruence theorems for triangles that involve, respectively: the length of the angle bisectors (from a triangle vertex to the opposite side); the length of the segments on the angle bisectors from the incenter to the triangle vertices. In both cases there exists – up to congruence – precisely one triangle with prescribed arbitrary positive lenghts for those three segments.

1 Introduction

A *Congruence Theorem* for triangles usually contains a list of properties and it is phrased in one of the two following ways: one triangle that satisfies those properties is determined up to congruence; any two triangles that satisfy those properties are congruent. One classical Congruence Theorem, abbreviated as SSS, states that *the lengths of the three triangle sides determine a triangle up to congruence*. In other words, if two triangles have the same side lengths, then they are congruent. Certain Congruence Theorems for triangles were known at the time of Euclid and involve side lenghts and measures of interior angles.

Multiplying any side length with the length of the corresponding height always gives the same number, namely twice the triangle area, so we can deduce another Congruence Theorem: *a triangle is determined up to congruence by the lengths of its three heights.* (Hint: The ratios between the side lengths are determined hence the triangle is determined up to similarity.)

Another congruence theorem that could be presented as an exercise is the following:

Theorem 1. *A triangle is determined up to congruence by the lengths of its three medians.*

Proof. Let A, B, and C be the triangle vertices and (up to a translation) place the origin at the barycenter O. The lengths of $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are known because, for example, $\overrightarrow{OA}$ is $2/3$ of the length of the median at A. Recall from the definition of barycenter (as the center of mass of the vertices) that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{0}$. We deduce the angle measure $A\hat{O}B$ from the relation

$$||\overrightarrow{OC}||^2 = ||\overrightarrow{OA}+\overrightarrow{OB}||^2 = ||\overrightarrow{OA}||^2 + ||\overrightarrow{OB}||^2 + 2||\overrightarrow{OA}|| \cdot ||\overrightarrow{OB}|| \cdot \cos(A\hat{O}B)\,.$$

Up to a rotation, knowing $||\overrightarrow{OA}||$ determines A. Then, knowing $A\hat{O}B$ and $\overrightarrow{OA}$ determines B (up to a line symmetry). Finally, knowing $\overrightarrow{OC}=-(\overrightarrow{OA}+\overrightarrow{OB})$ determines C. Up to congruence, we have then determined the position of the vertices A, B, and C. □

In this paper, we prove Congruence Theorems that involve the lengths of angle bisectors or the lengths of the segments that are cut on an angle bisector by the incenter. Our main result is the following, for which we don't know of an easy proof as the one mentioned above:

Theorem 2. *A triangle is determined up to congruence by the length of the three bisectors. Moreover, for every triple of positive real numbers there exists a triangle such that those numbers are the length of the three angle bisectors.*

This result appeared previously in the work of Mironescu and Panaitopol [MP94] (see also related recent work [cMP22]), who provided an elegant argument by reducing it to an application of the Brouwer fixed-point theorem. Although their approach avoids many explicit computations, it does not offer a direct construction of the triangle, highlighting a central theme of this problem: recovering a triangle from its bisector lengths cannot be accomplished solely by straightedge-and-compass methods. By Galois theory, the algebraic constraints arising from the bisector-length equations are not solvable by radicals, thus ruling out purely classical constructions. Despite this inherent algebraic complexity, we believe a more hands-on and constructive geometric treatment is both instructive and worthwhile. In this paper, we present such a geometric proof. We combine algebraic and trigonometric manipulations with geometric insights, resulting in a self-contained approach that may appeal to those seeking a more elementary viewpoint.

As a variant of Theorem 2 we also prove:

Theorem 3. *A triangle is determined up to congruence by the length of the three segments connecting the incenter to the vertices. Moreover, for every triple of positive real numbers there exists a triangle such that those numbers are the lengths of said segments.*

Notice that we don't have a Congruence Theorem if we consider instead the length of the three segments connecting the incenter to the endpoints of the angle bisectors that are not vertices, see Remark 15.

We may also relate the angles to the lengths of the angle bisectors as follows:

Theorem 4. *In a triangle, the largest (respectively, smallest) angle corresponds to the shortest (respectively, longest) angle bisector. In particular, two angles are the same if and only if the corresponding angle bisectors have the same length.*

The above result clearly holds replacing the angle bisectors by the three segments connecting the incenter to the vertices (because the triangle sides are tangents to the incircle). However, the above result does not hold for a triangle ABC replacing the length of the angle bisectors by the length l'_A, l'_B, l'_C of the segments from the incenter to the endpoints of the angle bisectors (if the angle measures are $2\alpha > 2\beta > 2\gamma$ and $2\beta = \frac{\pi}{3}$ we have $l'_A = l'_C \neq l'_B$ by Theorem 10).

Acknowledgements

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2 Preliminaries on angle bisectors

Let ABC be a triangle with vertices A, B, and C. For convenience, we denote the measures of the interior angles at A, B and C by 2α, 2β and 2γ respectively (noticing that $\alpha+\beta+\gamma = \frac{\pi}{2}$). Call O the incenter (which is the intersection point of the three angle bisectors, and it is also the center of the incircle) and R the radius of the incircle. We write a for the length of the side opposite to A, b_A for the length of the angle bisector at A, we call A' the endpoint of said angle bisector, and we set $l_A := \overline{OA}$ and $l'_A := b_A - l_A$ (and we similarly define $b, c, B', C', b_B, b_C, l_B, l_C, l'_B, l'_C$). Without loss of generality $\alpha + \beta$ is the smaller angle made by the bisectors at A and B, see Figure 1.

Remark 5. *Consider the projection P of O on the side AB. Working in the right triangle APO and $C'PO$ respectively, we can see that $R = l_A \sin\alpha$ and $R = l'_C \cos(\alpha - \beta)$. Since R only depends on the triangle ABC (and not on the choice of a vertex), we also obtain*

$$R = l_A \sin\alpha = l'_A \cos(\beta - \gamma)\,. \tag{1}$$

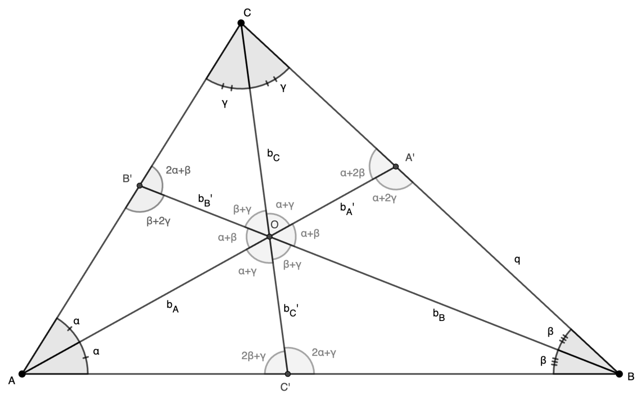


Figure 1: The angle bisectors in the triangle ABC, with related lengths and angle measures

Lemma 6. *We can express $\frac{l_A}{b_A}$, $\frac{l_A}{l'_A}$ (and hence $\frac{l'_A}{b_A}$) in terms of α, β, γ as follows:*

$$\frac{l_A}{b_A} = \frac{\sin(\alpha + 2\gamma)}{2\sin(\alpha+\gamma)\cos\gamma} \qquad \frac{l_A}{l'_A} = \frac{\sin(\alpha+2\gamma)}{\sin\alpha}.$$

Proof. By the previous remark, we have $l'_A = \frac{l_A \sin\alpha}{\sin(\alpha+2\gamma)}$. We deduce that

$$\frac{l_A}{b_A} = \frac{l_A}{l_A + l'_A} = \frac{l_A}{l_A + \frac{l_A \sin\alpha}{\sin(\alpha+2\gamma)}} = \frac{\sin(\alpha+2\gamma)}{\sin(\alpha+2\gamma)+\sin\alpha}.$$

We conclude because, as it can be seen by expanding the terms, we have $2\sin(\alpha+\gamma)\cos\gamma = \sin(\alpha+2\gamma)+\sin\alpha$. □

Remark 7. *The length of the angle bisector at A is*

$$b_A = \frac{2bc}{b+c}\cos\alpha = \sqrt{bc\Big(1 - \Big(\frac{a}{b+c}\Big)^2\Big)}$$

and analogous formulas hold for b_B and b_C. This is a well-known consequence of the Angle Bisector Theorem, see for example [pro]and [Wik25].

Example 8. *In a right-angled triangle ABC where $2\gamma = \frac{\pi}{2}$ and where, up to a rescaling, $R = 1$ we can express b_A, b_B, b_C in terms of α as follows (this easy computation is left to the reader):*

$$b_A(\alpha) = \frac{1}{\sin\alpha} + \frac{1}{\cos\alpha}$$

$$b_B(\alpha) = \frac{1}{\sin\left(\frac{\pi}{4} - \alpha\right)} + \frac{1}{\cos\left(\frac{\pi}{4}-\alpha\right)}$$

$$b_C(\alpha) = \frac{1}{\sin\left(2\alpha + \frac{\pi}{4}\right)} + \sqrt{2}$$

3 Isosceles triangles

The following result implies this observation on equilateral triangles:

Remark 9. *A triangle is equilateral if and only if $b_A = b_B = b_C$ if and only if $l_A = l_B = l_C$ if and only if $l'_A = l'_B = l'_C$.*

Theorem 10. *We have $2\alpha = 2\beta$ if and only if $b_A = b_B$ if and only if $l_A = l_B$. The property $l'_A = l'_B$ is equivalent to $2\alpha = 2\beta$ or $2\gamma = \frac{\pi}{3}$.*

Proof. If the triangle is isosceles with apex C, swapping A and B results in the same triangle so we deduce $b_A = b_B$ and $l_A = l_B$ and $l'_A = l'_B$. The property $l_A = l_B$ implies that $\alpha = \beta$ by Remark 5. Similarly, by Remark 5 the property $l'_A = l'_B$ is equivalent to $\cos(\alpha - \gamma) = \cos(\beta - \gamma)$ and hence $\alpha - \gamma = \pm(\beta - \gamma)$, which is equivalent to $\alpha = \beta$ or $\gamma = \frac{\pi}{6}$.

Finally, suppose that $b_A = b_B$. By Remark 5 we have $l_B = \frac{l_A \sin\alpha}{\sin\beta}$ and by Lemma 6 we can express l'_A in terms of l_A and the triangle angles. The known equality $l_A + l'_A = l_B + l'_B$, expressing these lengths all in terms of l_A and the triangle angles, then gives

$$l_A + \frac{l_A \sin\alpha}{\sin(\alpha + 2\gamma)} = \frac{l_A \sin\alpha}{\sin\beta} + \frac{l_A \sin\alpha}{\sin(\beta + 2\gamma)} . \tag{2}$$

Calling $\omega = \alpha + \beta$ this equation is equivalent to

$$\frac{1}{\sin(\omega - \beta)} + \frac{1}{\sin(\omega + \beta)} = \frac{1}{\sin(\omega - \alpha)} + \frac{1}{\sin(\omega + \alpha)} . \tag{3}$$

Computing the left hand side (a similar computation holds for the right hand side), we get

$$\begin{aligned}\frac{1}{\sin(\omega - \beta)} + \frac{1}{\sin(\omega + \beta)} &= \frac{\sin(\omega + \beta) + \sin(\omega - \beta)}{\sin(\omega + \beta)\sin(\omega - \beta)} \\ &= \frac{2\sin\omega\cos\beta}{\sin^2\omega\cos^2\beta - \sin^2\beta\cos^2\omega} \\ &= \frac{2\sin\omega\cos\beta}{(1 - \cos^2\omega)\cos^2\beta - (1 - \cos^2\beta)\cos^2\omega} \\ &= \frac{2\sin\omega\cos\beta}{\cos^2\beta - \cos^2\omega} .\end{aligned} \tag{4}$$

If $\alpha \neq \beta$, then we have $\sin\omega \neq 0$ and we deduce

$$\frac{\cos\beta}{\cos^2\beta - \cos^2\omega} = \frac{\cos\alpha}{\cos^2\alpha - \cos^2\omega}$$

This is impossible because $f : x \mapsto \frac{x}{x^2 - k^2}$ is an injective function over $(k, 1)$. Indeed, we can compute that $f'(x) = \frac{-(x^2+k^2)}{(x^2-k^2)^2} < 0$. □

The following result is our main theorem in the case of isosceles triangles:

Theorem 11. *Given two positive real numbers h and b there exists a unique, up to congruence, isosceles triangle with height h and such that b is the length of the angle bisector at a basis angle.*

Proof. Consider an isosceles triangle ABC with apex C (thus $h = b_C$ and $b = b_A = b_B$). Fixing b (remark that, up to rescaling the triangle, we can obtain any value for b) and varying the basis angle $2\alpha \in (0, \frac{\pi}{2})$, we can see the height h as a function of α. It suffices to prove that this function is injective with range $(0, +\infty)$. Without loss of generality, fix $b = 1$ and consider the function $h_C(\alpha)$ for $\alpha \in (0, \frac{\pi}{4})$.

Call F the projection of A' on the side AB. In particular, $AA'F$ and $BA'F$ are right triangles. Recalling that $b_A = 1$, we have

$$\overline{AF} = \cos\alpha \qquad \overline{A'F} = \sin\alpha \qquad \overline{FB} = \frac{\overline{A'F}}{\tan 2\alpha} .$$

We deduce that

$$\begin{aligned} h_C(\alpha) &= \frac{1}{2}\overline{AB}\cdot\tan 2\alpha \\ &= \frac{1}{2}\sin(\alpha)+\frac{1}{2}\cos\alpha\cdot\tan 2\alpha \\ &= \frac{1}{2}\sin\alpha\cdot\Big(1+\frac{2\cos^2\alpha}{2\cos^2\alpha-1}\Big) \\ &= \frac{1}{2}\sin\alpha\cdot\Big(\frac{3-4\sin^2\alpha}{1-2\sin^2\alpha}\Big) \end{aligned} \tag{5}$$

In particular, the function $h_C(\alpha)$ is continuous. Let $x = \sin\alpha \in (0, \frac{\sqrt{2}}{2})$, so $\sin^2\alpha \in (0, \frac{1}{2})$. The range of $h_C(\alpha)$ is the range of the function

$$f(x) = x\cdot\frac{3-4x^2}{1-2x^2}$$

where $x \in (0, \frac{\sqrt{2}}{2})$. Moreover, as x is an injective function of α, we deduce that $h_C(\alpha)$ is injective if $f(x)$ is injective. The function f is continuous, its right limit for $x \to 0$ equals 0 and its left limit for $x \to \frac{\sqrt{2}}{2}$ equals $+\infty$. We conclude by proving that f is strictly increasing. We compute

$$f'(x) = \frac{8x^4-6x^2+3}{(2x-1)^2}\,.$$

The numerator (seen as a quadratic expression in x^2) has negative discriminant and hence it is strictly positive. □

Remark 12. *The above proof also shows (starting with an equilateral triangle) that, for an isosceles triangle, we have $h > b$ if and only if the basis is smaller than the other two sides if and only if the basis angles are larger than the angle at the apex.*

4 The proof of Theorem 3

To prove Theorem 3, we separate it into two results:

Theorem 13. *A triangle is determined up to congruence by the length of $\overline{OA}$, $\overline{OB}$, $\overline{OC}$.*

Proof. It suffices to prove that the ratios l_A/l_B and l_A/l_C determine the triangle up to similarity. By Remark 5, the numbers $c = \frac{\sin\beta}{\sin\alpha}$ and $d = \frac{\sin\gamma}{\sin\alpha}$ are known, and it suffices to prove that they determine $\sin\alpha$ (then $\sin\beta$ and $\sin\gamma$ are also determined, and hence also the acute angles α, β, and γ). We have

$$\sin\gamma = \cos(\alpha+\beta) = \sqrt{1-\sin^2\alpha}\cdot\sqrt{1-c^2\sin^2\alpha} - c\sin^2\alpha$$

and hence

$$d\sin\alpha + c\sin^2\alpha = \sqrt{1-\sin^2\alpha}\cdot\sqrt{1-c^2\sin^2\alpha}\,.$$

Setting $S = \sin\alpha$ and squaring gives

$$2cdS^3 + (1+c^2+d^2)S^2 - 1 = 0\,.$$

In the variable $t = \frac{1}{\sin\alpha}$ we obtain the depressed cubic

$$t^3 + pt + q = 0 \qquad p = -(1+c^2+d^2) \qquad q = -2cd$$

We conclude by proving that there is at most one real root of the cubic which is strictly positive. Call Δ the discriminant of the cubic. Notice that 0 is not a root (as $q \neq 0$) and the sum of the roots is zero (as the coefficient at t^2 is zero). We then exclude $\Delta < 0$ (because there cannot be precisely one root). If $\Delta = 0$, there are two values for the roots, so at most one of them is a strictly positive real number. Now suppose that $\Delta > 0$: there are three distinct real roots, their sum is 0 and their product is positive (as $q < 0$), so there is precisely one strictly positive real root. □

Theorem 14. *For any triple of positive real numbers, there is a triangle such that these are the lengths of $\overline{OA}$, $\overline{OB}$, $\overline{OC}$.*

Proof. Up to a rescaling we may fix $R = 1$ and it suffices to prove that there is a triangle with any given ratios $l_C/l_A = \frac{\sin(\alpha)}{\sin(\gamma)} = \frac{\sin(\alpha)}{\cos(\alpha+\beta)}$ and $l_C/l_B = \frac{\sin(\beta)}{\sin(\gamma)} = \frac{\cos(\alpha+\gamma)}{\sin(\gamma)} = \frac{\sin(\beta)}{\cos(\alpha+\beta)}$. So call

$$f(\alpha, \omega) = \Big(\frac{\sin(\alpha)}{\cos(\omega)}, \frac{\sin(\omega - \alpha)}{\cos(\omega)}\Big),$$

where ω represents $\alpha + \beta$. The domain of this function is (α, ω) such that $0 < \alpha < \omega < \frac{\pi}{2}$. It suffices to show that f is onto with codomain $(0, \infty)^2$.

We let $(x, y) \in (0, \infty)^2$ be arbitrary and aim to find $0 < \alpha < \omega < \frac{\pi}{2}$ such that

$$\frac{\sin(\alpha)}{\cos(\omega)} = x, \qquad \frac{\sin(\omega - \alpha)}{\cos(\omega)} = y.$$

For $\omega \in (\arctan x, \frac{\pi}{2})$ we have $\tan\omega > x$, hence $\sin\omega > x\cos\omega$, which gives

$$0 < x\cos\omega < 1,$$

so $\alpha(\omega) = \arcsin(x\cos\omega) \in (0, \frac{\pi}{2})$, and moreover $\alpha(\omega) < \omega$. Thus, for fixed $x > 0$, the equation

$$x = \frac{\sin\alpha}{\cos\omega}$$

forces α to depend on ω, giving a one-parameter family $(\alpha(\omega), \omega)$. The second coordinate y is then given by the function

$$g(\omega) := \frac{\sin(\omega - \alpha(\omega))}{\cos\omega}, \qquad \omega \in (\arctan x, \tfrac{\pi}{2}).$$

Indeed, the function g is continuous, strictly increasing, and satisfies

$$\lim_{\omega \downarrow \arctan x} g(\omega) = 0, \qquad \lim_{\omega \uparrow \frac{\pi}{2}} g(\omega) = +\infty.$$

Thus g is a bijection $(\arctan x, \frac{\pi}{2}) \to (0, \infty)$. So, for any $y > 0$ there exists ω with $g(\omega) = y$. Setting $\alpha = \alpha(\omega)$ yields $f(\alpha, \omega) = (x, y)$. □

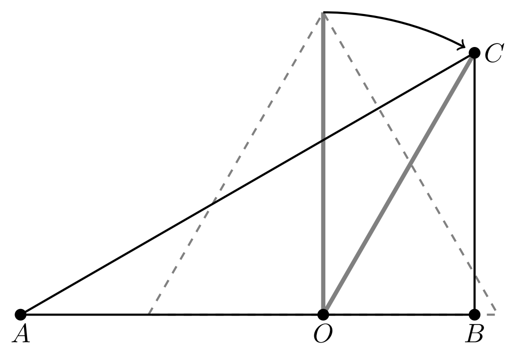
Remark 15. *Knowing the lengths of l'_A, l'_B, l'_C is not sufficient to determine the triangle up to congruence. Indeed, consider two triangles ABC and $A'B'C'$ and fix $\alpha = \beta = \frac{2\pi}{15}$ and $\alpha' = \beta' = \frac{\pi}{5}$. Notice that we have $\sin(3\alpha) = \sin(3\alpha')$. Moreover, we have $l'_A = l'_B$ and $l'_{A'} = l'_{B'}$. Up to rescaling one of the triangles, we may suppose that $l_C = l'_C$ and we conclude because*

$$l'_C/l'_A = \frac{\cos(\beta - \gamma)}{\cos(\beta - \alpha)} = \sin(3\alpha) = \sin(3\alpha') = l'_{C'}/l'_{A'}\,.$$

5 The proof of the main result

This section is devoted to proving Theorem 2. The following transformation is depicted in Figure 2.

Figure 2: Transformation I applied to the equilateral triangle



Lemma 16 (Transformation I)**.** *Consider a triangle ABC and suppose that $\beta \geqslant \alpha$. Modify the triangle by keeping b_C and γ and the line AB invariant (thus C rotates around the endpoint of b_C) and preserving $\beta \geqslant \alpha$. Increasing β (equivalently, decreasing α) we have that b_A increases and b_B decreases.*

Proof. We have

$$b_A = b_C \sin(2\gamma) \cdot \frac{\sin(2\alpha + \gamma)}{\sin(\alpha + 2\gamma)\sin(2\alpha)}$$

and the analogous expression for b_B (replacing α by β). We can see $\alpha(t)$ (respectively, $\beta(t)$) as a function of the time, such that $\alpha' = -1$ (respectively, $\beta' = 1$). To study the sign of b'_A and b'_B we may neglect the positive constant $b_C \sin(2\gamma)$. So we have to derive a quotient f/g and the sign of its derivative is the sign of $f'g - g'f$. We now show that $b'_A > 0$:

$$\begin{aligned}
&-2\cos(2\alpha+\gamma)\sin(\alpha+2\gamma)\sin(2\alpha) + \sin(2\alpha+\gamma)\cos(\alpha+2\gamma)\sin(2\alpha) \\
&\quad + 2\sin(2\alpha+\gamma)\sin(\alpha+2\gamma)\cos(2\alpha) \\
&= 2\sin(\gamma)\sin(\alpha+2\gamma) + \sin(2\alpha+\gamma)\cos(\alpha+2\gamma)\sin(2\alpha) \\
&= 2\sin(\gamma)\sin(\alpha+2\gamma) + \cos(\gamma)\cos(\alpha+2\gamma) - \cos(2\alpha+\gamma)\cos(2\alpha)\cos(\alpha+2\gamma) \\
&= \sin(\gamma)\sin(\alpha+2\gamma) + \cos(\alpha+\gamma) - \cos(2\alpha+\gamma)\cos(2\alpha)\cos(\alpha+2\gamma) \\
&> 0\,.
\end{aligned}$$

In the last inequality we used the following:

1. For the first summand: $\alpha + 2\gamma < \pi$, so that $\sin(\gamma)\sin(\alpha + 2\gamma) \geqslant 0$.
2. For the second summand: $\alpha + \gamma < \pi/2$, so that $\cos(\alpha + \gamma) > 0$.
3. For the third summand, we can lower bound it by $-\cos(2\alpha + \gamma)$, and since $2\alpha + \gamma \leqslant \frac{\pi}{2}$ (because $\alpha \leqslant \beta$), we have that $\cos(\alpha + \gamma) > \cos(2\alpha + \gamma)$.

For the derivative b'_B we cannot reduce directly to the previous argument, because the assumption $\beta \geqslant \alpha$ breaks the symmetry. In this case, we have to show the following expression is negative:

$$\begin{aligned}
&2\cos(2\beta+\gamma)\sin(\beta+2\gamma)\sin(2\beta) - \sin(2\beta+\gamma)\cos(\beta+2\gamma)\sin(2\beta) \\
&\quad - 2\sin(2\beta+\gamma)\sin(\beta+2\gamma)\cos(2\beta)\,.
\end{aligned}$$

The computational knowledge engine WolframAlpha rewrites this expression as $\frac{1}{4}$ times

$$\cos(3\beta - \gamma) - 5\cos(\beta + \gamma) + 3\cos(\beta + 3\gamma) + \cos(5\beta + 3\gamma)\,.$$

Write $x = \beta + \gamma$ and $y = 2\gamma$, so that $\frac{\pi}{6} < x < \frac{\pi}{2}$ and $\frac{\pi}{6} < y < \pi$. So the above expression becomes

$$\cos(3x - 2y) - 5\cos(x) + 3\cos(x + y) + \cos(5x - y)\,.$$

Writing $x = \frac{\pi}{2} - \delta$ and $y = \pi - \varepsilon$, and recalling

$$\cos\left(-\frac{\pi}{2} + z\right) = \sin(z), \qquad \cos\left(\frac{\pi}{2} + z\right) = -\sin(z),$$

the expression becomes

$$\sin(2\varepsilon - 3\delta) - 5\sin(\delta) - 3\sin(\varepsilon + \delta) + \sin(\varepsilon - 5\delta)\,.$$

We have $0 < \delta < \frac{\pi}{6}$ and $0 < \varepsilon < \frac{2\pi}{3}$. Suppose first that $\varepsilon < \frac{\pi}{4}$ and $\delta < \frac{\pi}{10}$. Then

$$\sin(2\varepsilon - 3\delta) < \sin(2\varepsilon) \quad \text{and} \quad \sin(\varepsilon - 5\delta) < \sin(\varepsilon).$$

An upper bound for the full expression is then

$$\sin(2\varepsilon) - 3\sin(\varepsilon) + \sin(\varepsilon) = 2\sin(\varepsilon)(\cos(\varepsilon) - 1) < 0\,.$$

It suffices to prove that in the remaining region we have

$$5\sin(\delta) + 3\sin(\varepsilon + \delta) > 2\,.$$

We can check by hand that this inequality holds for the extremal values (δ, ε) in the following list:

$$\left(0, \tfrac{\pi}{4}\right) \quad \left(\tfrac{\pi}{10}, \tfrac{\pi}{5}\right) \quad \left(0, \tfrac{2\pi}{3}\right) \quad \left(\tfrac{\pi}{3}, \tfrac{2\pi}{3}\right)\,.$$

The convex hull of the four points is a convex quadrilateral that contains the remaining region because $\varepsilon \geqslant 2\delta$ (as $\varepsilon = 2\alpha + 2\beta$ and $\delta = \alpha$). We then reason by convexity. Let $f(\epsilon, \delta) := 5\sin(\delta) + 3\sin(\epsilon + \delta)$. We claim f is concave on $\{(\epsilon, \delta) : 0 \leqslant \epsilon + \delta < \pi\}$. This follows because a $\mathcal{C}^2$ function is concave if and only if its Hessian is negative semi-definite, and f is negative semi-definite if and only if $15\sin(\delta)\sin(\epsilon + \delta) \geqslant 0$, which is true because $0 \leqslant \epsilon + \delta < \pi$. (As a side remark: by moving along lines it would be possible to prove and make use of concavity only for functions in one-variable.) We deduce that a lower bound for the values inside the region is the minimum of the four extremal values, which is strictly greater than 2 and we conclude. □

The following result implies Theorem 4:

Lemma 17. *If the angles in the triangle ABC satisfy $\alpha \leqslant \beta \leqslant \gamma$ then we have $b_A \geqslant b_B \geqslant b_C$.*

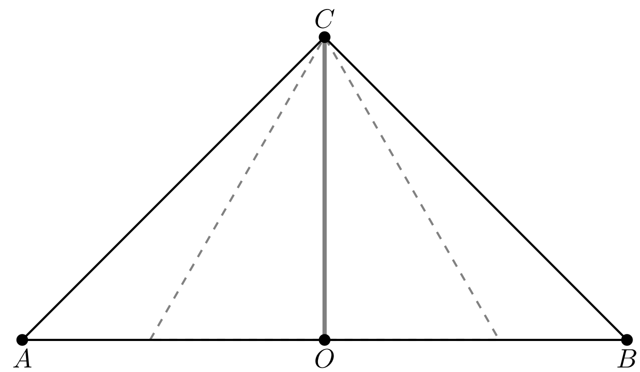
Proof. By Theorem 10 we have an equality of angles if and only if the corresponding bisectors are equal. So the case of an equilateral triangle is evident. Now suppose that $\alpha < \beta = \gamma$ or that $\alpha = \beta < \gamma$. From Remark 12 we deduce that $b_A > b_C$ and we conclude. Finally, suppose that $\alpha < \beta < \gamma$. We consider a continuous transformation that changes the angles while keeping the triangle scalene. As such a transformation never gives an isosceles triangle, by continuity the inequalities among the bisectors are preserved. We conclude by Lemma 16 (starting from an isosceles triangle with strict largest angle 2γ and considering Remark 12). □

Lemma 18 (Transformation II)**.** *Consider a triangle ABC such that 2γ is the largest angle. Fixing b_C and $\beta - \alpha$ and increasing γ (thus, α and β decrease by the same amount), both b_A and b_B increase.*

Proof. Seeing the angles as a function of the time we may suppose that $\gamma' > 0$ and $\beta', \alpha' < 0$. The role of A and B is interchangeable, so it suffices to prove that b_A increases. We have

$$b_A = b_C \sin(2\alpha + \gamma) \cdot \frac{\sin(2\gamma)}{\sin(\alpha + 2\gamma)\sin(2\alpha)}\,.$$

Figure 3: Transformation II applied to the equilateral triangle



Since $2\alpha + \gamma$ is constant, up to neglecting the positive constant $b_C \cdot \sin(2\alpha + \gamma)$ we get

$$\frac{db_A}{dt} = \frac{\alpha'}{[\sin(\alpha + 2\gamma)\sin(2\alpha)]^2} \cdot \Big[-4\cos(2\gamma)\sin(\alpha+2\gamma)\sin(2\alpha) + 3\sin(2\gamma)\cos(\alpha+2\gamma)\sin(2\alpha) - 2\sin(2\gamma)\sin(\alpha+2\gamma)\cos(2\alpha) \Big] \tag{6}$$

Thus, in order to prove that $\frac{db_A}{dt} > 0$, it suffices to show that the sign of the following expression is positive:

$$\begin{aligned} &4\cos(2\gamma)\sin(\alpha+2\gamma)\sin(2\alpha) - 3\sin(2\gamma)\cos(\alpha+2\gamma)\sin(2\alpha) + 2\sin(2\gamma)\sin(\alpha+2\gamma)\cos(2\alpha) \\ &= \cos(2\gamma)\sin(\alpha+2\gamma)\sin(2\alpha) + 3\sin(\alpha)\sin(2\alpha) + 2\sin(2\gamma)\sin(\alpha+2\gamma)\cos(2\alpha) \\ &= \sin(\alpha+2\gamma)\sin(2\alpha+2\gamma) + 3\sin(\alpha)\sin(2\alpha) + \sin(2\gamma)\sin(\alpha+2\gamma)\cos(2\alpha) > 0\,. \end{aligned}$$

In the last inequality, we used that since $2(\alpha + \beta + \gamma) = \pi$, we have

1. First summand is positive: since $\alpha + 2\gamma < \pi$, and $2\alpha + 2\gamma < \pi$.
2. Second summand is positive: since $2\alpha < \pi$.
3. Third summand is positive: since $2\gamma < \pi$, $\alpha + 2\gamma < \pi$ and $2\alpha < \pi/2$.

□

Proof of the unicity claim in the Main Theorem. By the result for isosceles triangles we may suppose that our triangle is scalene. Up to rescaling we may suppose without loss of generality that $b_C = 1$ is the shortest angle bisector and that $b_A > b_B$, so that $\gamma > \beta > \alpha$ by Lemma 17. Suppose that $A'B'C'$ is another triangle with $b_{C'} = 1$ and $b_{A'} = b_A$ and $b_{B'} = b_B$. So $\gamma' > \beta' > \alpha'$ and we may suppose without loss of generality that $\gamma' \geqslant \gamma$. Then we obtain $A'B'C'$ from ABC by combining the two transformations seen in the previous lemmas: keeping $b_C = 1$, opening γ to γ' (where b_A and b_B do not decrease, and strictly increase if $\gamma' > \gamma$). Then, keeping $b_C = 1$, increasing β and decreasing α (so that $\alpha + \beta$ is constant) or conversely. In the former transformation, b_A strictly increases and in the latter b_B strictly increases (unless we already are at the triangle $A'B'C'$). This shows that, unless ABC and $A'B'C'$ are similar (and hence congruent because $b_C = b_{C'} = 1$) then b_A or b_B increases while passing from ABC to $A'B'C'$, against our assumptions. □

Proof of the existence claim in the Main Theorem. Transformations I and II proven, respectively, in Lemma 16 and Lemma 18 allow us to modify the angles of the triangles (keeping $b_C = 1$) so that b_A and b_B both increase and, respectively, b_A increases and b_B decreases. The idea of this proof is to show that, by combining the two transformations, we can settle for the desired values of b_A and b_B. Let $\mathcal{T}$ be the space of Euclidean triangle up to similarity, which we parametrize as follows. We assume the base of the triangle lies on the x-axis, and we scale the triangle so that $b_C = 1$ is the shortest angle bisector (consequently, 2γ is the largest angle). We may then parametrize the triangle by the angles (α, β), since $\gamma = \pi - \alpha - \beta$ is determined. Let $b_A(\alpha, \beta)$, $b_B(\alpha, \beta)$ denote the bisector lengths corresponding to the vertices A and B for a triangle $T = (\alpha, \beta)$.

Define the smooth map:

$$\Phi : \mathcal{T} \to \mathbb{R}^2, \quad \Phi(\alpha, \beta) = (b_A(\alpha, \beta), b_B(\alpha, \beta)).$$

Recall the Transformation I (Lemma 16) and Transformation II (Lemma 18), which we will tailor specifically for this setting. We start with a base triangle $T = (\alpha_0, \beta_0)$.

- **Transformation I:** Fixes γ. This transformation increases α (thus decreasing β by the same amount s). This increases b_A and decreases b_B (the signs reverse if we decrease α). Let $s > 1$. We consider the transformation where α_0 is replaced by

  $\alpha(s) = \alpha_0 - \log(s)$,and $\beta(s) = \beta_0 + \log(s)$.

  Let $X_1^s(\alpha, \beta)$ denote the triangle obtained by performing Transformation I of magnitude s.

- **Transformation II:** Fix $\beta - \alpha =$ constant (i.e., move along a line of slope 1 in angle space) i.e., decrease α and β by the same amount. This causes both b_A and b_B to increase. Let $t > 1$. We will replace γ_0 by $\gamma(t) = 2\log(t) + \gamma_0$, or equivalently, α and β by $\alpha(t) = \alpha - \log(t)$ and $\beta(t) = \beta_0 - \log(t)$ respectively. Let $X_2^t(\alpha, \beta)$ denote the triangle obtained by performing Transformation II of magnitude t.

Since the only assumptions in Lemma 16 and Lemma 18 are that $\gamma \geqslant \beta \geqslant \alpha$, these transformations are well-defined for all $s, t > 1$ such that we still have $\gamma \geqslant \beta$. By Theorem 11, we may suppose that we have $\gamma > \beta > \alpha$.

Given $T = (\alpha, \beta) \in \mathcal{T}$, consider the map $F \colon (1, \infty)^2 \to \mathbb{R}_+^2$, defined as

$$F = F_T(s, t) := X_2^t(X_1^s(\alpha, \beta)) = (b_A(s, t), b_B(s, t)).$$

Our goal is to show that the image of F is the set S consisting of the pairs (x, y) such that $x > y > 1$. Lemma 16 implies

$$\frac{\partial b_A}{\partial s} < 0 \quad \text{and} \quad \frac{\partial b_B}{\partial s} > 0.$$

Also, Lemma 18 implies that

$$\frac{\partial b_A}{\partial t} > 0 \quad \text{and} \quad \frac{\partial b_B}{\partial t} > 0.$$

Thus the matrix differential DF of F at the point $(s, t) = (0, 0)$ has rank 2. From the inverse function theorem it follows that F is a C^1-diffeomorphism in a neighborhood V of $(0, 0)$, i.e., it is a local diffeomorphism. It follows that F is an open map.

If $\vec{B} \in S$ corresponds to the triangle T, we let $A_{\vec{B}}$ denote the set of $\vec{B}' \in S$ so that $\vec{B}' = F_T(s, t)$ for some $s, t \in (1, \infty)^2$.

This corresponds to the pairs in S arising as a pair of bisectors obtained by traveling from triangle T by some time s along X_1 and some time t along X_2. We want to show $A_{\vec{B}} = S$.

Since F is an open map, $A_{\vec{B}}$ is open. We define the relation $\vec{B} \sim \vec{B}'$ whenever $\vec{B}' \in A_{\vec{B}}$. We claim this is an equivalence relation. Indeed, we clearly have $\vec{B} \in A_{\vec{B}}$; moreover, if $\vec{B}' \in A_{\vec{B}}$ then $\vec{B} \in A_{\vec{B}'}$, by running the opposite trajectory. Finally, if $\vec{B}_1 \in A_{\vec{B}_2}$ and $\vec{B}_2 \in A_{\vec{B}_3}$ implies $\vec{B}_1 \in A_{\vec{B}_3}$ by composition. Then, we have a partition into open sets

$$S = \bigcup_{\vec{B} \in S} A_{\vec{B}}.$$

But S is connected, hence there can only be one open set in this partition, so $S = A_{\vec{B}}$. This shows F is surjective onto S, and finishes the proof of the existence result. $\square$

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