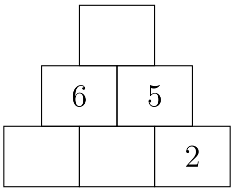
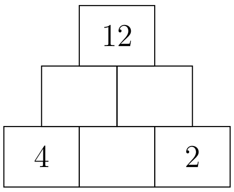
Addition walls

Addition walls provide mathematical amusement for all ages. Most notably, they are popular exercises for primary school to train addition and subtraction [1]. The aim is completing a number grid in the shape of a brick wall (alternatively, hexagonal cells are used). There must be one number in each brick. Every brick which is not in the bottom row lies on top of two other bricks and the *key rule* is the following: the number above is the sum of the two numbers below.

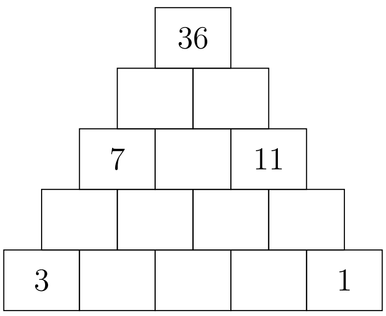
The easiest addition walls can be completed progressively going upwards by performing additions. Now try to solve this addition wall:



The number at the top is $6+5=11$, the second number in the bottom row is $5-2=3$ and then the first number in the bottom row is $6-3=3$. In general, if you have one brick on top of two other bricks and you know two out of the three numbers, then (because of the key rule) you can determine the unknown number with an addition or a subtraction. This observation allows you to solve the addition walls for primary school. Now try to solve this addition wall:



Call x the unknown number in the bottom row. The two numbers in the middle row are $4+x$ and $2+x$, so the number at the top is $12=(4+x)+(2+x)=6+2x$. Thus $x=3$ and the two numbers in the middle row are 7 and 5. In a more intuitive way, the number x counts twice for the top number 12 because it is used twice to generate the numbers in the second row. So we have $12=4+2+2x$ and again we can find $x=3$. We have discovered a strategy to solve any addition wall with a configuration of numbers like the one above. Now try to solve this addition wall:



The above part of the wall can be solved as in the previous example, the unknown number in the third row being 9. To solve the bottom part of the wall let x, y, and z be the unknown numbers in the bottom row. The numbers in the fourth row are then $3 + x$, $x + y$, $y + z$, $z + 1$. We know the numbers in the third row, and by the key rule they are sums of numbers in the fourth row. So we can write down the following equations:

$$\begin{aligned} 7 &= 3 + 2x + y \\ 9 &= x + 2y + z \\ 11 &= y + 2z + 1 \end{aligned}$$

We can solve this linear system and get $x = 1$, $y = 2$, $z = 4$. So we have the following number grid:

$$\begin{array}{c} 36 \\ 16 \quad 20 \\ 7 \quad 9 \quad 11 \\ 4 \quad 3 \quad 6 \quad 5 \\ 3 \quad 1 \quad 2 \quad 4 \quad 1 \end{array}$$

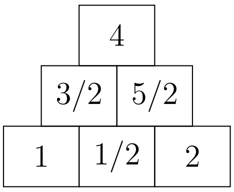
Had we known that all numbers were positive integers (and exploiting the fact that the known number 7 was very small) we could have solved the addition wall by trying out various possibilities. Indeed, the first two unknown numbers in the bottom row can only be 1 or 2 (because twice the first number plus the second number plus 3 gives 7). After determining the first two unknown numbers in the bottom row, by trial and error (or by a small computation) we can find the third unknown number in the bottom row. Then, as the bottom row is known, we can solve the addition wall progressively with additions.

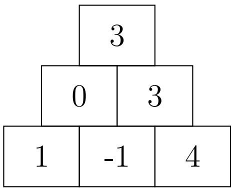
Solving any addition wall

If all numbers in an addition wall are positive integers, then with the help of a computer we can solve the problem by brute force (provided that the number of rows is not huge, and the known numbers are not huge). Indeed, we can fix larger and larger values for the numbers in the bottom row and complete the addition wall in the hope that we find back the known entries. This means searching over the possible addition walls and looking for a match with the constraints that are given.

Fortunately, linear algebra [2] allows us to solve any addition wall in a smarter way. Let n be the number of rows, and consider the entries in each brick as variables. So there are $1 + 2 + \cdots + n = n(n + 1)/2$ variables. For every brick outside the bottom row, the key rule gives a linear relation between three variables (corresponding to the brick and the two bricks below it). So we have $n(n - 1)/2$ linear equations because of the key law. Each known entry provides one further linear equation. Collecting all these equations we get a linear system and solving it means solving the addition wall.

Provided that n is not huge, it is possible to solve this linear system and there is either no solution or precisely one solution or there are infinitely many solutions. It is customary that the addition walls given as exercises have precisely one solution. Moreover, the solution usually consists of positive integers. Warning: Even if the known entries are positive integers, we are solving a linear system over the rational numbers (because we do linear algebra over a field) so the entries are rational numbers. They are not necessarily integers, and not necessarily positive. For example consider the following addition walls (where the known entries are the number at the top and the first and third number in the bottom row):



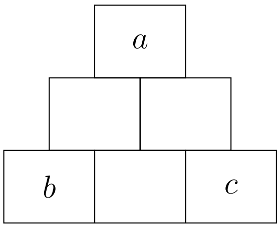


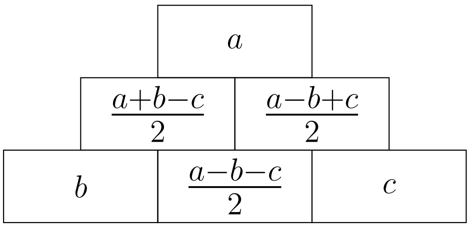
Now consider the n variables in the bottom row. Fixing any value for them results in precisely one addition wall. So, without knowing any entry, there are precisely n free variables (or degrees of freedom) for the addition wall.

To ensure that the solution is unique we then need to know at least n entries. More precisely, we need to know n or more "independent" entries. For example, three entries related by the key rule are not independent, while the known entries in all above examples of addition walls were independent. (The precise definition of independent entries is that the corresponding variables can be taken as free variables for the linear system provided by the key rule.)

Given some known entries which are not independent, we cannot be sure that a solution exists because the key rule gives compatibility relations that must be satisfied. For example, if we know the number in one brick, then we cannot arbitrarily prescribe the numbers in the two bricks below it.

Notice that linear algebra allows us to solve all addition walls with parametric linear systems. For example, we can solve the following addition wall for all values of a, b, c:





In this example we can make the following considerations: if a, b, c are strictly positive, then all entries are strictly positive if and only if $a > b + c$; if a, b, c are integers, then all entries are integers if and only if a and $b + c$ have the same parity.

Determining the top entry

Consider an addition wall with n rows and name x_1 to x_n the numbers in the bottom row. Progressively with additions we can determine all other entries. Each entry is then a sum of the numbers x_1 to x_n, possibly taken multiple times. In particular, we can determine the top entry. Its expression is as follows:

$$\begin{array}{ll} x_1 & \text{for } n = 1 \\ x_1 + x_2 & \text{for } n = 2 \\ x_1 + 2x_2 + x_3 & \text{for } n = 3 \\ x_1 + 3x_2 + 3x_3 + x_4 & \text{for } n = 4 \\ x_1 + 4x_2 + 6x_3 + 4x_4 + x_5 & \text{for } n = 5\,. \end{array}$$

These expressions are symmetric (namely, we can replace x_k by x_{n-k+1}) because the addition wall and the key rule are symmetric by swapping left and right. We may guess that for general n the expression of the top entry involves binomial coefficients and it is

$$x_1 + (n-1)x_2 + \binom{n-1}{2}x_3 + \cdots = \sum_{k=1}^{n} \binom{n-1}{k-1} x_k\,.$$

We can prove this formula by induction. The formula is clearly correct for the base case $n = 1$. To prove the induction step, we suppose that the formula holds for n (fixing

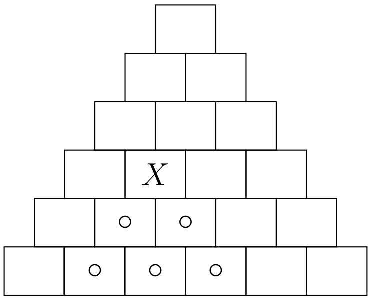
$n \geq 1$) and we prove it for $n+1$. Naming y_1 to y_n the entries in the n-th row, we know by assumption that the top entry is $T = \sum_{k=1}^{n} \binom{n-1}{k-1} y_k$ and we want to prove that $T = \sum_{k=1}^{n+1} \binom{n}{k-1} x_k$. By the key rule we also know that $y_k = x_k + x_{k+1}$, so we can write

$$T = \sum_{k=1}^{n} \binom{n-1}{k-1} y_k = \sum_{k=1}^{n} \binom{n-1}{k-1} x_k + \sum_{k=1}^{n} \binom{n-1}{k-1} x_{k+1}$$

$$= x_1 + x_{n+1} + \sum_{k=2}^{n} \binom{n-1}{k-1} x_k + \sum_{k=2}^{n} \binom{n-1}{k-2} x_k = \sum_{k=1}^{n+1} \binom{n}{k-1} x_k$$

where for the last step we applied the recurrence relation for binomial coefficients.

Now we wish to similarly determine an entry which is not the top entry. Luckily, we may resolve to a trick. Namely, we can see the entry X as the top entry of a smaller addition wall.



So we get a similar formula for X that involves only some of the numbers from the bottom row. Concretely, suppose that we want to know the i-th entry of the j-th row (in the above picture we have $i = 2$ and $j = 4$). The smaller addition wall has $n - j + 1$ rows and its bottom entries y_1 to y_{n-j+1} are the entries x_k with k varing from i to $i + n - j$. The formula for the top entry applied to the smaller addition wall then gives

$$X = \sum_{h=1}^{n-j+1} \binom{n-j}{h-1} y_h = \sum_{k=i}^{n-j+i} \binom{n-j}{k-i} x_k \,.$$

In the example from the above picture we simply get $X = x_2 + 2x_3 + x_4$.

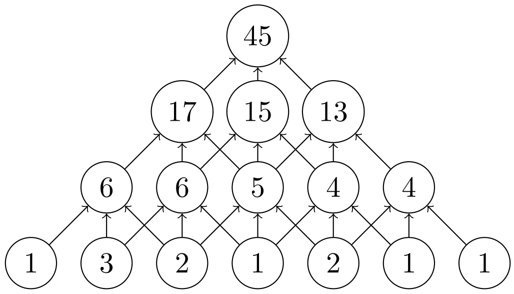
Generalizations of addition walls

Firstly, addition walls can be generalized by changing the key rule. For example, the number in a brick could be the difference of the numbers in the two bricks below it. More generally, the number in a brick could be a linear expression of numbers in other bricks. In this case, we can still write down a linear system and use linear algebra to solve the problem.

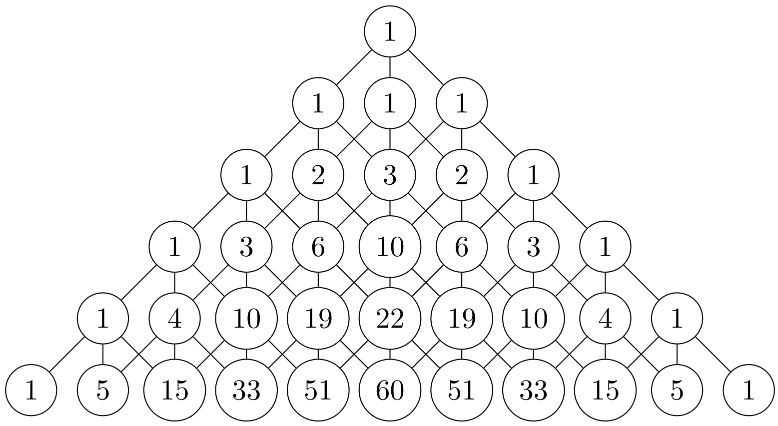
Secondly, we could involve operations other than addition in the key rule. This could get complicated because we may loose the possibility to work with linear algebra. Still within reach are the *multiplication walls*, where the number in a brick is the product of the numbers in the two bricks below it. In fact, by taking logarithms we have a corresponding addition wall that we can solve with linear algebra and then, by exponentiating back the numbers, we find the solution to the multiplication wall.

Thirdly, we could vary the problem a bit by having other kind of information on the entries of the addition wall. For example, we can be told that the unknown entries in the bottom row belong to a certain set and we only need to correctly assign them with a small reasoning. These variants are particularly meaningful when they allow to determine the entries of the addition wall progressively.

Finally, we could vary the structure of addition walls. Now consider an addition wall as a graph. More precisely, it is a rooted tree and (with the terminology of rooted trees), the key rule states that the number of a parent is the sum of the numbers of its two children. One natural generalization would then be increasing the number of children, for example from two to three. Then the key rule is that the number of the parent (vertex above) is the sum of the numbers of its three children (the three vertices below). Here a solved example of such a *ternary addition wall*:



Luckily, we can still resolve to linear algebra to solve ternary addition walls. Moreover, we can still compute an expression for the top entry in terms of the bottom entries. The coefficients that we find in that expression are generalizations of the binomial coefficients. Namely, they are the entries of the *trinomial triangle*:



Such numbers, sometimes called *trinomial coefficients*, have an easy recurrence relation (three numbers must now be added to obtain a new number) and have further nice and deep properties [4].

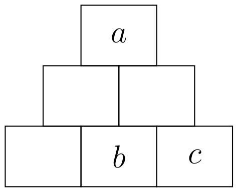
We could also conceive other addition walls leading to further generalizations of the binomial coefficients and Pascal's triangle [3]. There are also other kinds of number walls that do not show a triangular structure [5], and they are also extremely interesting.

This investigation comes to an end, and now it is your turn. You may invent addition walls for your friends (or for mathematical competitions); you may provide your college students with funny exercises of linear algebra and combinatorics; you may investigate generalizations of addition walls within projects of undergraduate research. Have fun!

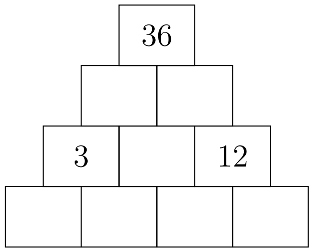
Exercises for the reader

1. Consider an addition wall with n rows. If all numbers in the bottom row equal some constant c, what is the number at the top?

2. Solve the following addition wall for all values of a, b, c and show that if a, b, c are integers, then all entries are integers.



3. Invent an addition wall with infinitely many rows.
4. Solve the following multiplication wall by direct inspection, knowing that all entries are positive integers.



Summary

Easy addition walls are popular exercises for primary school, while more complicated ones are nifty problems for the general public. College students can tackle them with linear algebra. Moreover, addition walls have a strong link with combinatorics and they can be generalized in fascinating ways. In short, addition walls are suitable for all ages.

References

[1] Cherri Moseley and Janet Rees, *Cambridge Primary Mathematics Workbook 1*, 2nd Edition, Cambridge University Press, 2021.

[2] Howard Anton, Chris Rorres and Anton Kaul, *Elementary Linear Algebra, Applications Version*, 12th Edition, Wiley, 2019.

[3] Gábor Kallós, *A generalization of Pascal's triangle using powers of base numbers*, Annales Mathématiques Blaise Pascal, Volume 13, no. 1 (2006), p. 1-15.

[4] Wikipedia contributors, *Trinomial triangle*, 2013, `https://en.wikipedia.org/wiki/Trinomial_triangle`, Online; accessed 11-August-2023.

[5] Burkard Polster (and collaborators), *Secrets of the lost number walls*, Mathologer YouTube Channel, `https://www.youtube.com/watch?v=NO1_-qptr6c`.

Solutions to the exercises for the reader

1. Suppose first that $c = 1$. In the n-th row there are n numbers equal to 1. In the $(n-1)$-th row there are $n-1$ numbers equal to 2. In general, the numbers in each row are all the same and the value for one row is twice the value of the row below (this can be seen by iteration using the key rule). So the top value for $c = 1$ is 2^n and for general c (by the distributivity law) it is $c \cdot 2^n$.

2. We can progressively determine the missing entries. The second entry in the second row is $b + c$. Then the first entry in the second row is $a - b - c$. Finally, the first entry in the third row is $a - 2b - c$. The missing entries are all sums of integers, so they are all integers.

3. One example is as follows. The top entry is 2. In all other rows, the first and the last entry are 1 while the middle entries are all 0. The key rule is respected: the key rule at the top entry states $2 = 1 + 1$; the key rule at the first (respectively, last) entry of a further row states $1 = 1 + 0$; the key rule at a middle entry states $0 = 0 + 0$.

4. We can progressively determine the second, third, and fourth row. We obtain

36			
3	12		
3	1	12	
3	1	1	12