THE TERNARY CYCLOTOMIC POLYNOMIALS Φ_{3pq}

Abstract. Cyclotomic polynomials are a classical and fundamental topic in number theory, and still an active field of research. The aim of this work is providing a formula for the family of ternary cyclotomic polynomials Φ_{3pq}, where $p < q$ are prime numbers greater than 3 such that $q \equiv \pm 1, \pm 2 \bmod 3p$ and $q > 3p$. We can derive various properties from our formula. In particular, we prove a conjecture of Zhang on the number of maximum gaps for the coefficients.

1. Introduction

Cyclotomic polynomials play an important role in several areas of mathematics and especially in number theory: they are a classical and fundamental topic, and still an active field of research. For the basic properties of cyclotomic polynomials, we refer the reader to [11, 12]. We denote by Φ_n the n-th cyclotomic polynomial. If n is a prime number, then $\Phi_n(X) = \sum_{i=0}^{n-1} X^i$. If n is the product of two distinct odd prime numbers, then Φ_n is called *binary* cyclotomic polynomial: these polynomials are flat (i.e. the non-zero coefficients are either 1 or -1) and their structure is known by Lam and Leung [8].

In this work we consider *ternary* cyclotomic polynomials, namely those polynomials Φ_n where $n = p_1 p_2 p_3$ for some odd prime numbers $p_1 < p_2 < p_3$. Such polynomials are not necessarily flat (for example, $\Phi_{3\cdot 5\cdot 7}$ has two coefficients equal to -2). Bounds for the coefficients and structural properties of ternary cyclotomic polynomials have been investigated however there are many open questions, see [11] for instance.

From now on, we suppose that $p_3 > p_1 p_2$, and decompose Φ_n into blocks, following Al-Kateeb [1, 2]. By grouping terms of exponents between two consecutive multiples of p_3, we obtain

$$\Phi_{p_1 p_2 p_3}(X) = \sum_{i=0}^{\varphi(p_1 p_2)-1} f_i(X) X^{i p_3} \tag{1}$$

where $f_i(X)$ is a polynomial of degree smaller than p_3 that is called a p_3-*block*. There are $\varphi(p_1 p_2)$ such blocks.

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Denote by q (respectively, r) the quotient (respectively, remainder) of p_3 after division by p_1p_2. We decompose each p_3-block as

$$f_i(X) = \sum_{j=0}^{q} f_{i,j}(X) X^{jp_1p_2} \tag{2}$$

where the polynomials $f_{i,j}$ have degree smaller than p_1p_2: they are called p_1p_2-*blocks* for $j < q$, while $f_{i,q}$ has degree smaller than r and it is called r-*block*. For example, $\Phi_{3\cdot 5\cdot 37}$ can be decomposed into eight 37-blocks, that in turn are composed of two 15-blocks and one 7-block.

There are iterative formulas for the construction of the p_1p_2-blocks and the r-blocks, and these blocks only depend on p_3 through $(p_3 \bmod p_1p_2)$, see [1] by Al-Kateeb. We decompose the p_1p_2-blocks into four slices (see Definition 2.6 and Proposition 2.5). Our main contribution is the following:

Theorem 1.1. *Setting $p_1 = 3$ and $p_3 \equiv \pm 1, \pm 2 \bmod 3p_2$, there are explicit formulas for $\Phi_{3p_2p_3}$. The expressions for the slices of the $3p_2$-blocks are given in Tables 1, 2 and 3 for $p_3 \equiv \pm 1 \bmod 3p_2$, and in Tables 5, 6, 7 and 8 for $p_3 \equiv \pm 2 \bmod 3p_2$.*

The proof of this result for the case $r \equiv \pm 1 \bmod 3p_2$ will be given in Section 3, and in Section 5 for $r \equiv \pm 2 \bmod 3p_2$. Our explicit formulas lead to the following result:

Proposition 1.2. *Under the assumptions $p_3 > 3p_2$ and $p_3 \equiv \pm 1, 2 \bmod 3p_2$, there are explicit formulas for the number of coefficients in $\Phi_{3p_2p_3}$ equal to a given value, see Tables 4 and 9.*

We also consider maximum gaps, which are defined as follows.

Definition 1.3. *Write $\Phi_{p_1p_2p_3}(X) = \sum_{k=0}^{\varphi(p_1p_2p_3)} a_k X^k$ and call k_i the finite growing sequence of the exponents of the non-zero coefficients. We call* gaps *the positive integers $g_i := k_{i+1} - k_i$ and we call $g := \max g_i$ the* maximum gap *of $\Phi_{p_1p_2p_3}$.*

The maximum gap has been determined by Al-Kateeb et al. in [2].

Theorem 1.4. *If $p_3 > p_1p_2$, the maximum gap of $\Phi_{p_1p_2p_3}$ is $(p_1 - 1)(p_2 - 1)$.*

The number of maximum gaps is the number of indices i such that $g_i = g$. In [13], Zhang stated the following conjecture and proved it for $p_1p_2 = 15$.

Conjecture 1.5 (Zhang, 2019)**.** *If $p_3 > p_1p_2$, the number of maximum gaps is $2q$.*

Thanks to our explicit formula we can prove the following:

Proposition 1.6. *Under the assumptions $p_3 > 3p_2$ and $p_3 \equiv \pm 1, 2 \bmod 3p_2$, Zhang's conjecture holds true.*

We prove Propositions 1.2 and 1.6 in Section 4 for $p_3 \equiv \pm 1 \bmod 3p_2$ and in Section 6 for $p_3 \equiv \pm 2 \bmod 3p_2$.

Finally, as an example, we study the family of cyclotomic polynomials Φ_{15p} for every odd prime number $p \neq 3, 5$ in Section 8.

2. Preliminaries

2.1. Binary cyclotomic polynomials.

We write

$$\Phi_{p_1p_2}(X) = \sum_{k=0}^{\varphi(p_1p_2)} b_k X^k .$$

We first specialise a result by Lam and Leung [8] to the case $p_1 = 3$.

Theorem 2.1. *Let $p_1 < p_2$ be odd prime numbers, and denote by u, v the unique non-negative integers such that $\varphi(p_1p_2) = up_1 + vp_2$. For every integer $0 \leq k \leq \varphi(p_1p_2)$, we have*

$$b_k = \begin{cases} 1 & \text{if } k = ip_1 + jp_2 \text{ with } 0 \leq i \leq u \text{ and } 0 \leq j \leq v \\ -1 & \text{if } k = ip_1 + jp_2 - p_1p_2 \text{ with } u+1 \leq i \leq p_2 - 1 \text{ and } v+1 \leq j \leq p_1 - 1 \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Corollary 2.2. *The coefficients b_k of the binary cyclotomic polynomial Φ_{3p_2} are periodically equal to*

$$\begin{cases} 1, -1, 0 & \text{for } 0 \leq k \leq p_2 - 1 \\ 1, 0, -1 & \text{for } p_2 - 1 \leq k \leq 2(p_2 - 1) \text{ and } p_2 \equiv 1 \bmod 3 \\ -1, 1, 0 & \text{for } p_2 - 1 \leq k \leq 2(p_2 - 1) \text{ and } p_2 \equiv 2 \bmod 3 . \end{cases}$$

Proof. We rely on Theorem 2.1. If $p_2 \equiv 1 \bmod 3$, we have $u = \frac{2(p_2-1)}{3}$ and $v = 0$. We deduce that $b_k = 1$ holds precisely for the indices $k = 3i$ with $0 \leq i \leq \frac{2(p_2-1)}{3}$. Similarly, $b_k = -1$ holds when $k = 3i - 2p_2$ or $k = 3i - p_2$ with $\frac{2(p_2-1)}{3} + 1 \leq i \leq p_2 - 1$. If $p_2 \equiv 2 \bmod 3$, we have $u = \frac{p_2-2}{3}$ and $v = 1$. We deduce that $b_k = 1$ holds precisely for the indices $k = 3i$ or $k = 3i + p_2$ with $0 \leq i \leq \frac{p_2-2}{3}$. Similarly, $b_k = -1$ holds when $k = 3i + 1$ with $0 \leq i \leq \frac{2p_2-4}{3}$. □

2.2. Operations on the blocks.

We recall the following operations from [2, Notation 5 and Section 2]:

Definition 2.3. *Let $f(X) = \sum_{k=0}^{p_1p_2-1} a_k X^k$ be a polynomial with degree smaller than p_1p_2. The truncation and rotation of f by a non-negative integer s are*

Truncation	$\mathcal{T}_s f(X) = \sum_{k=0}^{s-1} a_k X^k$
Rotation	$\mathcal{R}_s f(X) = \sum_{k=0}^{p_1p_2-1} a_k X^{\mathrm{rem}(k-s, p_1p_2)}$

where $\mathrm{rem}(k-s, p_1p_2)$ *is the remainder of* $k-s$ *after division by* p_1p_2*.*

Concretely, if we represent $f(X)$ with the list of coefficients $(a_0, ..., a_{p_1p_2-1})$, the effect of the rotation by s is a cyclical shift leftwards by s indices, leading to the coefficients

$$(a_s, ..., a_{p_1p_2-1}, a_0, ..., a_{s-1})\,.$$

Example 2.4. *For* $p_1p_2 = 15$ *and* $f(X) = 1 + X + X^2 - X^5 - X^6 - X^7$ *we have*

$$\mathcal{T}_2 f(X) = 1 + X,$$
$$\mathcal{R}_2 f(X) = 1 - X^3 - X^4 - X^5 + X^{13} + X^{14}.$$

Consider the inverse cyclotomic polynomial

$$\Psi_{p_1p_2}(X) = \frac{X^{p_1p_2}}{\Phi_{p_1p_2}(X)} = -1 - X - \ldots - X^{p_1-1} + X^{p_2} + \ldots + X^{p_2+p_1-1}\,.$$

We remark that adding or subtracting Ψ_{3p_2} to a polynomial only acts on the coefficients whose exponent is in $\{0, 1, 2, p_2, p_2+1, p_2+2\}$.

The following result from [2, Lemma 6] describes relationships between the p_1p_2-blocks and the r-blocks of the ternary cyclotomic polynomial $\Phi_{p_1p_2p_3}$:

Proposition 2.5. *We have the following identities:*

$$f_{i,0} = f_{i,1} = \ldots = f_{i,q-1}, \tag{4}$$
$$f_{i,q} = \mathcal{T}_r f_{i,0}, \tag{5}$$
$$f_{i+1,0} = \mathcal{R}_r f_{i,0} - b_{i+1}\Psi_{p_1p_2}, \tag{6}$$
$$f_{0,0} = -\Psi_{p_1p_2}. \tag{7}$$

The equations (4) and (5) show that to determine $\Phi_{p_1p_2p_3}$ it is enough to compute the p_1p_2-blocks $f_{i,0}$ for $0 \le i < \varphi(p_1p_2)$. The other equations show an explicit expression for $f_{0,0}$ and how to compute $f_{i+1,0}$ from $f_{i,0}$. In this paper we compute all p_1p_2-blocks $f_{i,0}$ when $p_1 = 3$ and $p_3 \equiv \pm 1, \pm 2 \bmod 3p_2$.

Definition 2.6. *We decompose each block* $f_{i,0}(X) = \sum_{k=0}^{3p_2-1} a_{i,k}X^k$ *into four* slices

$$f_{i,0}(X) = s_{1,i}(X) + s_{2,i}(X) + s_{3,i}(X) + s_{4,i}(X)$$

by partitioning the exponents as follows (notice that $s_{1,i}$ *and* $s_{3,i}$ *have at most three non-zero coefficients):*

$$s_{1,i}(X) := \sum_{k=0}^{2} a_{i,k}X^k, \qquad s_{2,i}(X) := \sum_{k=3}^{p_2-1} a_{i,k}X^k,$$
$$s_{3,i}(X) := \sum_{k=p_2}^{p_2+2} a_{i,k}X^k, \qquad s_{4,i}(X) := \sum_{k=p_2+3}^{3p_2-1} a_{i,k}X^k\,.$$

3. Explicit block description for $\Phi_{3p_2p_3}$ with $p_3 \equiv \pm 1 \bmod 3p_2$

Consider the ternary cyclotomic polynomial $\Phi_{3p_2p_3}$ such that $p_3 > 3p_2$ and $p_3 \equiv \pm 1 \bmod p_1p_2$. We prove Theorem 1.1 in this case by computing explicitly the $3p_2$-blocks $f_{i,0}$ for $0 \le i \le 2(p_2-1)-1$. For $p_3 \equiv 1 \bmod 3p_2$ (respectively, $p_3 \equiv -1 \bmod 3p_2$) we gather the expressions of $s_{1,i}$ and $s_{3,i}$ in Table 1 (respectively, Table 2). For $p_3 \equiv \pm 1 \bmod 3p_2$ the expressions of $s_{2,i}$ and $s_{4,i}$ can be found in Table 3.

We will see that the expressions of the slices $s_{1,i}$ and $s_{3,i}$ of $f_{i,0}$ are periodic by varying i within certain intervals. The values of i for which the periodicity is affected will correspond to the values of i for which certain coefficients in $f_{i-1,0}$ are non-zero.

Suppose that $p_3 \equiv 1 \bmod 3p_2$. We speak about perturbations of the periodicity of $s_{1,i}$ that is caused by a non-zero coefficient of $s_{2,i-1}$ because rotating the polynomial $f_{i-1,0}$ by 1 the smallest possible exponent for $s_{2,i}$ becomes the largest possible exponent for $s_{1,i}$. Similarly, the periodicity for $s_{3,i}$ might be affected if the smallest exponent of $s_{4,i-1}$ is non-zero. For $0 \le i < 2(p_2-1)$ we then say that $s_{1,i}$ (respectively, $s_{3,i}$) is perturbed if the coefficient of X^3 (respectively, X^{p_2+3}) in $f_{i-1,0}$ is non-zero.

For $p_3 \equiv -1 \bmod 3p_2$, we consider instead the coefficient of X^{3p_2-1} (respectively, X^{p_2-1}) to say that $s_{1,i}$ (respectively, $s_{3,i}$) is perturbed.

3.1. The case $p_3 \equiv 1 \bmod 3p_2$.

We compute progressively $s_{k,i}$ for $k = 1, \dots, 4$ by considering small values of i first. We rely on those expressions to compute $s_{k,i}$ for further values of i.

The index $i = 0$. We have $f_{0,0} = -\Psi_{3p_2}$ by (7).

The indices $i \in \{1, 2, 3\}$. We computed the four slices by hand and we explicit here the computations for $i = 1$. Applying (6) with $b_1 = -1$, we find $f_{1,0} = \mathcal{R}_1 f_{0,0} + \Psi_{3p_2}$. Since

$$\mathcal{R}_1 f_{0,0} = 1 + X + -X^{p_2-1} - X^{p_2} - X^{p_2+1} + X^{3p_2-1}$$

we find

$$s_{1,1}(X) = -X^2 \qquad s_{2,1}(X) = -X^{p_2-1} \qquad s_{3,1}(X) = X^{p_2+2} \qquad s_{4,1}(X) = X^{3p_2-1}\,.$$

Then, $f_{2,0} = \mathcal{R}_1 f_{1,0}$, so

$$s_{1,2}(X) = -X \qquad s_{2,2}(X) = -X^{p_2-2} \qquad s_{3,2}(X) = X^{p_2+1} \qquad s_{4,2}(X) = X^{3p_2-2}.$$

Finally, $f_{3,0} = \mathcal{R}_1 f_{2,0} - \Psi_{3p_2}$ and hence

$$s_{1,3}(X) = X + X^2 \quad s_{2,3}(X) = -X^{p_2-3} \quad s_{3,3}(X) = -X^{p_2+1} - X^{p_2+2} \quad s_{4,3}(X) = X^{3p_2-3}.$$

The indices $3 < i \le p_2 - 3$. We can compute by hand that $s_{1,j} = s_{1,j+3}$ holds for $j = 1, 2, 3$. We will show that the expression of $s_{1,i}$ is 3-periodic for $1 \le i \le p_2 - 3$.

Since in (6) we only perform rotations by 1, the computation of $s_{1,i}$ from $f_{i-1,0}$ just requires b_i and the coefficients of X, X^2, X^3 in $f_{i-1,0}$ (because $s_{1,i}$ is a polynomial of degree at most 2). Similarly, computing the coefficients of X, X^2, X^3 in $f_{i-1,0}$ just requires b_{i-1} and the coefficients of X^2, X^3, X^4 in $f_{i-2,0}$, which are computable from b_{i-2} and the coefficients of X^3, X^4, X^5 in $f_{i-3,0}$. Thus, the expression of $s_{1,i}$ only depends on the coefficients of X^3, X^4 and X^5 in $f_{i-3,0}$ and on b_{i-2}, b_{i-1} and b_i. By Corollary 2.2, the coefficients b_i are periodic up to $i = p_2 - 1$. The only exponent for a non-zero coefficient in $s_{2,3}$ is $p_2 - 3$ (the coefficient is -1), so after applying $i-3$ rotations, we find that the smallest exponent corresponding to a non-zero coefficient in $s_{2,i}$ is $p_2 - i$ (also with coefficient -1). Hence, the coefficients of X^3, X^4 and X^5 in $f_{i-3,0}$ are zero if $i \leq p_2 - 3$. Thus, the expression of $s_{1,i}$ is 3-periodic for $3 < i \leq p_2 - 3$.

To study $s_{3,i}$ we mimic the reasoning for $s_{1,i}$. We compute by hand that $s_{3,j} = s_{3,j+3}$ for $j = 1, 2, 3$ and we prove that the expression for $s_{3,i}$ is 3-periodic for $1 \leq i \leq p_2 - 3$. The expression of $s_{3,i}$ only depends on the coefficients of $X^{p_2+3}, X^{p_2+4}, X^{p_2+5}$ in $f_{i-3,0}$ and on b_{i-2}, b_{i-1} and b_i. The only exponent for a non-zero coefficient in $s_{4,3}$ is $3p_2 - 3$ (with coefficient 1), so after applying $i - 3$ rotations, the smallest exponent for a non-zero coefficient in $s_{4,i}$ is $3p_2 - i$ (again with coefficient 1). Thus the coefficients of X^{p_2+3}, X^{p_2+4}, X^{p_2+5} in $f_{i-3,0}$ are zero for $3 < i \leq p_2 - 3$.

In our 3-periodic expressions of $s_{1,i}$ and $s_{3,i}$ for $1 \leq i < p_2 - 1$, the coefficients of X^0 and X^{p_2} are zero. Thus the rotation $\mathcal{R}_1 f_{i,0}$ doesn't introduce additional non-zero coefficients in $s_{2,i}$ (respectively, $s_{4,i}$) compared to $s_{2,3}$ (respectively, $s_{4,3}$). We deduce that $s_{2,i}(X) = -X^{p_2-i}$ and $s_{4,i}(X) = X^{3p_2-i}$.

The indices $i \in \{p_2 - 2, p_2 - 1, p_2\}$. The coefficient of X^3 in $f_{i,0}$ is equal to -1 for $i = p_2 - 3$, which affects the periodicity for the expression of $s_{1,i}$ (this is our first example of perturbation for the first slice). We could compute $s_{1,i}$ by hand for these values of i. Moreover, also by hand, $s_{2,i} = 0$ for $i \in \{p_2 - 2, p_2 - 1, p_2\}$ because s_{2,p_2-3} has only the monomial at exponent 3 while the smallest non-zero coefficient of $s_{3,i}$ has exponent larger than p_2 for $i \in \{p_2 - 3, p_2 - 2, p_2 - 1\}$. We may compute by hand that $s_{4,i}(X) = X^{3p_2-i}$ holds for $i \in \{p_2 - 2, p_2 - 1, p_2\}$.

The indices $p_2 < i < 2(p_2 - 1)$. We apply the same procedure as above (computing $f_{i,0}$ by hand for the first four values of i greater than p_2 and reasoning by periodicity). The coefficients b_i are still periodic, but with a different order than in the case $i < p_2$ (see Corollary 2.2). Moreover, the cases $p_2 \equiv 1, 2 \bmod 3$ have to be distinguished because we have these two cases in Corollary 2.2. Notice that the coefficient of X^{p_2} in s_{3,p_2} is -1 hence, by rotation, $s_{2,p_2+1}(X) = -X^{p_2-1}$. Since the coefficient of X^{p_2} in $s_{3,i}$ is zero for $p_2 < i < 2(p_2 - 1)$, the rotations don't introduce new non-zero coefficients in $s_{2,i}$ for $p_2 + 1 < i < 2(p_2 - 1)$. Thus we deduce $s_{2,i}(X) = -X^{2p_2-i}$ from the expression of s_{2,p_2+1}. The coefficient of X^3 in $f_{i,0}$ is non-zero only if $i = 2p_2 - 3$, which is the last possible value of i, so there is no new perturbation in $s_{1,i}$ coming from the non-zero coefficient of $s_{2,i}$. Finally (looking at the explicit expressions that are obtained by 3-periodicity) the coefficient of X^0 of $s_{1,i}$ is equal to 0 for $p_2 < i < 2(p_2 - 1)$. Thus the rotation $\mathcal{R}_1 f_{i,0}$ doesn't introduce

i	$p_2 \bmod 3$	$s_{1,i}(X)$	$s_{3,i}(X)$
$i = 0$	1,2	$1 + X + X^2$	$-X^{p_2} - X^{p_2+1} - X^{p_2+2}$
$i \equiv 1 \bmod 3,\ i \le p_2 - 3$	1,2	$-X^2$	X^{p_2+2}
$i \equiv 2 \bmod 3,\ i \le p_2 - 3$	1,2	$-X$	X^{p_2+1}
$i \equiv 0 \bmod 3,\ 0 < i \le p_2 - 3$	1,2	$X + X^2$	$-X^{p_2+1} - X^{p_2+2}$
$i = p_2 - 2$	1	$-X - X^2$	X^{p_2+1}
	2	X	$-X^{p_2+1} - X^{p_2+2}$
$i = p_2 - 1$	1	X^2	$-X^{p_2+1} - X^{p_2+2}$
	2	$-X - X^2$	X^{p_2+2}
$i = p_2$	1	X	$-X^{p_2} - X^{p_2+1}$
	2	X^2	$-X^{p_2} - X^{p_2+2}$
$i \equiv 0 \bmod 3,\ p_2 < i < 2(p_2 - 1)$	1	X^2	$-X^{p_2+2}$
	2	X	$-X^{p_2+1}$
$i \equiv 1 \bmod 3,\ p_2 < i < 2(p_2 - 1)$	1	X	$-X^{p_2+1}$
	2	$-X - X^2$	$X^{p_2+1} + X^{p_2+2}$
$i \equiv 2 \bmod 3,\ p_2 < i < 2(p_2 - 1)$	1	$-X - X^2$	$X^{p_2+1} + X^{p_2+2}$
	2	X^2	$-X^{p_2+2}$

TABLE 1. Expressions of $s_{1,i}$ and $s_{3,i}$ in the case $p_3 \equiv 1 \bmod 3p_2$.

additional non-zero coefficients in $s_{4,i}$. We deduce that the expression $s_{4,i}(X) = X^{3p_2-i}$ (which holds for $i = p_2$) is still valid for $p_2 < i < 2(p_2 - 1)$.

3.2. **The case** $p_3 \equiv -1 \bmod 3p_2$**.** As this case is completely analogous to the case $p_3 \equiv 1 \bmod 3p_2$, we only sketch it. The main difference is that now the rotations are rightwards instead of leftwards (because shifting cyclically the coefficients by $3p_2 - 1$ indices leftwards consists in shifting the coefficients by 1 index rightwards). In particular, $s_{2,i}(X) = X^{2+i}$ for $1 \le i \le p_2 - 3$ and the coefficient of X^{p_2-1} of $f_{p_2-3,0}$ is equal to 1. Thus, the perturbation at index $i = p_2 - 2$ will affect $s_{3,i}$ instead of $s_{1,i}$.

4. PROPERTIES OF $\Phi_{3p_2p_3}$ FOR $p_3 \equiv \pm 1 \bmod 3p_2$

Recall that cyclotomic polynomials are symmetric with respect to the middle coefficient. For $\Phi_{3p_1p_2}$, the coefficient at exponent k is the same as the coefficient at exponent $2(p_2 - 1)(p_3 - 1) - k$, the middle coefficient being at exponent $(p_2 - 1)(p_3 - 1)$.

Notice (inspecting Tables 1 and 2) that each p_1p_2-block $f_{i,0}$ contains at least one positive coefficient and one negative coefficient. We now prove Proposition 1.6 for $p_3 \equiv \pm 1 \bmod 3p_2$, studying the gaps inside blocks and between blocks.

Let $0 \le i < 2(p_2 - 1)$. We write

$$F_i(X) := X^{p_3 i} f_i(X)$$

i	p_2 mod 3	$s_{1,i}(X)$	$s_{3,i}(X)$
$i=0$	1,2	$1+X+X^2$	$-X^{p_2}-X^{p_2+1}-X^{p_2+2}$
$i\equiv 1 \bmod 3,\ i\le p_2-3$	1,2	-1	X^{p_2}
$i\equiv 2 \bmod 3,\ i\le p_2-3$	1,2	$-X$	X^{p_2+1}
$i\equiv 0 \bmod 3,\ 0<i\le p_2-3$	1,2	$1+X$	$-X^{p_2}-X^{p_2+1}$
$i=p_2-2$	1	$-X$	$X^{p_2}+X^{p_2+1}$
	2	$1+X$	$-X^{p_2+1}$
$i=p_2-1$	1	$1+X$	$-X^{p_2}$
	2	-1	$X^{p_2}+X^{p_2+1}$
$i=p_2$	1	$X+X^2$	$-X^{p_2+1}$
	2	$1+X^2$	$-X^{p_2}$
$i\equiv 0 \bmod 3,\ p_2<i<2(p_2-1)$	1	1	$-X^{p_2}$
	2	X	$-X^{p_2+1}$
$i\equiv 1 \bmod 3,\ p_2<i<2(p_2-1)$	1	X	$-X^{p_2+1}$
	2	$-1-X$	$X^{p_2}+X^{p_2+1}$
$i\equiv 2 \bmod 3,\ p_2<i<2(p_2-1)$	1	$-1-X$	$X^{p_2}+X^{p_2+1}$
	2	1	$-X^{p_2}$

TABLE 2. Expressions of $s_{1,i}$ and $s_{3,i}$ in the case $p_3\equiv -1 \bmod 3p_2$.

i	p_3 mod $3p_2$	$s_{2,i}(X)$	$s_{4,i}(X)$
$i=0$	$1,-1$	0	0
$1\le i\le p_2-3$	1	$-X^{p_2-i}$	X^{3p_2-i}
	-1	X^{2+i}	$-X^{p_2+2+i}$
$p_2-2\le i\le p_2$	1	0	X^{3p_2-i}
	-1	0	$-X^{p_2+2+i}$
$p_2<i<2(p_2-1)$	1	$-X^{2p_2-i}$	X^{3p_2-i}
	-1	X^{i-p_2+2}	$-X^{p_2+2+i}$

TABLE 3. Expressions of $s_{2,i}$ and $s_{3,i}$ in the case $p_3\equiv \pm 1 \bmod 3p_2$.

for the block of $\Phi_{3p_2p_3}$ consisting of the terms with exponent from $p_3\cdot i$ to $p_3\cdot(i+1)-1$. We similarly write

$$F_{i,j}(X) := X^{p_3 i+3p_2 j} f_{i,j}(X) \qquad \text{where } 0\le j\le q$$

for the block of $\Phi_{3p_2p_3}$ consisting of the terms with exponent in the interval $[p_3 i+3p_2 j, p_3 i+3p_2(j+1)-1]$ for $j<q$ and $[p_3 i+3p_2 q, p_3(i+1)-1]$ for $j=q$.

We introduce the following notation, considering the exponents with non-zero coefficients. This will allow us to locate all maximum gaps.

- $g_{1,i}$: maximum gap of $F_{i,0}$ (for $0\le i<2(p_2-1)$ this is a positive integer)
- $g_{2,i}$: difference between the smallest exponent of $F_{i,1}$ and the largest exponent of $F_{i,0}$ (setting $g_{2,i}=0$ if $F_{i,1}=0$)

- $g_{3,i}$: difference between the smallest exponent of $F_{i,q}$ and the largest exponent of $F_{i,q-1}$ (setting $g_{2,i} = 0$ if $F_{i,q} = 0$; recall that $q \geq 1$)
- $g_{4,i}$: difference between the smallest exponent of F_{i+1} and the largest exponent of F_i (for $0 \leq i < 2(p_2 - 1)$, this is a positive integer).

By (4), there are q gaps equal to $g_{1,i}$ and $(q-1)$ gaps equal to $g_{2,i}$ in $\Phi_{3p_2p_3}$. Our expressions for $f_{i,0}$ show that $F_{i,q-1} = X^{p_3 i+3(q-1)p_2} f_{i,q-1} = X^{p_3 i+3(q-1)p_2} f_{i,0}$ is non-zero. Moreover, recall from (5) that $f_{i,q} = \mathcal{T}_r f_{i,0}$. We have $F_{i,q} = f_{i,q} = 0$ if and only if $p_3 \equiv 1 \bmod 3p_2$ and $i \geq 1$. In that case, one may directly study $g_{4,i}$ instead of $g_{3,i}$.

Proof of Proposition 1.6 for $p_3 \equiv \pm 1 \bmod 3p_2$. By Theorem 1.4 the maximum gap of $\Phi_{3p_2p_3}$ is $2(p_2 - 1)$ and we have to prove that the number of maximum gaps is $2q$. For $q \neq 1$, we have

$$f_{0,0}(X) = f_{0,1}(X) = 1 + X + X^2 - X^{p_2} - X^{p_2+1} - X^{p_2+2}$$

hence

$$F_{0,1}(X) = X^{3p_2} + X^{3p_2+1} + X^{3p_2+2} - X^{4p_2} - X^{4p_2+1} - X^{4p_2+2}.$$

So we have $g_{2,0} = 2(p_2 - 1)$, and there are $(q - 1)$ gaps equal to $g_{2,0}$ in $\Phi_{3p_2p_3}$. Moreover, the constant coefficient of $f_{0,q} = \mathcal{T}_1 f_{0,0}$ is 1, so $F_{0,q}(X) = X^{3p_2q}$ and $g_{3,0} = 2(p_2 - 1)$. Hence, we have located q gaps equal to the maximum gap. This also holds for $q = 1$, where $F_{0,1}(X) = X^{3p_2}$. Since F_0 corresponds to the terms of $\Phi_{3p_2p_3}$ with exponent in the interval $[0, p_3 - 1]$, these gaps occur before the middle coefficient of $\Phi_{3p_2p_3}$. By symmetry of the coefficients, there are also q maximum gaps after the middle coefficient. To conclude the proof, we have to show that all of the other gaps are smaller than $2(p_2 - 1)$, so we have to investigate $g_{1,i}$ and $g_{4,i}$ for $0 \leq i \leq p_2 - 2$ and $g_{2,i}$ and $g_{3,i}$ for $1 \leq i \leq p_2 - 2$.

The block F_{p_2-2} corresponds to the terms of $\Phi_{3p_2p_3}$ with exponent in the interval $[(p_2 - 2)p_3, p_2p_3 - p_3 - 1]$, the blocks F_i for $0 \leq i \leq p_2 - 2$ cover all the coefficients of $\Phi_{3p_2p_3}$ up to the middle coefficient. By the symmetry of the coefficients, we will only need to consider the values of i in the interval $[0, p_2 - 2]$. We first suppose that $p_3 \equiv 1 \bmod 3p_2$.

For $0 \leq i \leq p_2 - 2$, the quantity $g_{1,i}$ is also the maximum gap of $f_{i,0}$. Differences between exponents in $s_{3,i}$ and $s_{1,i}$ are smaller than $p_2 + 3$ (this can be read from the explicit expressions). As $s_{2,0}(X) = s_{4,0}(X) = 0$, we have $g_{1,0} < p_2 + 3 < 2(p_2 - 1)$. For $1 \leq i \leq p_2 - 2$, in order to show that $g_{1,i} < 2(p_2 - 1)$, it is sufficient to show that the difference between the smallest exponent of $s_{4,i}$ and the largest exponent of $s_{3,i}$ is smaller than $2(p_2 - 1)$. For these values of i, we have $s_{4,i}(X) = X^{3p_2-i}$, so we may check that $g_{1,0} = p_2 - 2$, $g_{1,1} = g_{1,2} = 2p_2 - 3$, and $g_{1,i} \leq 2p_2 - i$ for $i \geq 3$.

We have already studied $g_{2,0}$ and $g_{3,0}$. We now assume $1 \leq i \leq p_2 - 2$. If $q = 1$, since the coefficient of X^0 in $f_{i,0}$ is 0, then $F_{i,1} = X^{p_3 i+3p_2} \cdot \mathcal{T}_1 f_{i,0} = 0$ and hence $g_{2,i} = 0$ according to our convention. Else, $g_{2,i}$ is by definition the difference between the smallest exponent of

$$F_{i,1} = X^{p_3 i+3p_2} f_{i,1} = X^{p_3 i+3p_2} f_{i,0}$$

and the largest exponent of $F_{i,0} = X^{p_3 i} f_{i,0}$. Then $g_{2,i}$ is the difference between the smallest exponent in $X^{3p_2} f_{i,0}$ and the largest exponent in $f_{i,0}$. Since $s_{4,i}(X) = X^{3p_2-i}$, the largest exponent of $f_{i,0}$ is $3p_2 - i$. Because $s_{1,i}(X) \neq 0$, the smallest exponent in $X^{3p_2} f_{i,0}$ is at most $3p_2 + 2$, so $g_{2,i} \leq 2 + i < 2(p_2 - 1)$. To study $g_{3,i}$, we may reason as for $g_{2,i}$ because $F_{i,q}$ is a truncation of $F_{i,0}$.

We now study $g_{4,i}$ for $0 \leq i \leq p_2 - 2$. We have $F_{0,q} = X^{3p_2 q} \cdot \mathcal{T}_1 f_{0,0} \neq 0$. Moreover, we have $s_{1,1} \neq 0$, so $g_{4,0} \leq 3$. For $0 < i \leq p_2 - 2$ we have $F_{i,q} = 0$, so $g_{4,i}$ is the difference between the smallest exponent of $X^{p_3(i+1)} f_{i+1,0}$ and the largest exponent of $X^{p_3 i + 3p_2(q-1)} f_{i,0}$. Since $s_{4,i}(X) = X^{3p_2 - i}$, we get that $g_{4,i} \leq 3 + i < 2(p_2 - 1)$.

The case $p_3 \equiv -1 \bmod 3p_2$ is analogous. Now we know that $f_{i,q} \neq 0$ (because $s_{1,i}$ has non-zero coefficients). Our explicit expressions for the blocks $f_{i,0}$ make it possible to verify that $g_{1,i}, g_{2,i}, g_{3,i} g_{4,i}$ are strictly less than $2(p_2 - 1)$ for $0 \leq i \leq p_2 - 2$ with the exception of $g_{2,0}$ and $g_{3,0}$. □

For the ternary cyclotomic polynomials $\Phi_{p_1 p_2 p_3}$ such that $p_2 \equiv \pm 1 \bmod p_1$ and $p_3 \equiv \pm 1 \bmod p_1 p_2$, Al-Kateeb has given in [1, Chapter 7] formulas for the number of non-zero coefficients.

Remark 4.1. *We consider the ternary cyclotomic polynomial $\Phi_{3p_2p_3}$ where $p_3 > 3p_2$ and $p_3 \equiv \pm 1 \bmod 3p_2$. We call N_c the number of coefficients equal to c in $\Phi_{3p_2p_3}$. The expressions in Tables 1, 2 and 4 show that $N_c = 0$ for all $c \neq 0, 1, -1$ (so the polynomial is flat, as known from* [7, Theorem 1]*). Consequently, we have*

$$N_0 = 2(p_2 - 1)(p_3 - 1) + 1 - N_1 - N_{-1}\,.$$

Proof of Proposition 1.2 for $p_3 \equiv \pm 1 \bmod 3p_2$. Calling $A_{c,i}$ (respectively, $B_{c,i}$) the number of coefficients equal to c in the block $f_{i,j} = f_{i,0}$ for any $0 \leq j < q$ (respectively, in the block $f_{i,q} = \mathcal{T}_r f_{i,0}$), we have

$$N_c = \sum_{i=0}^{2(p_2-1)-1} (A_{c,i} q + B_{c,i})\,.$$

For $c = \pm 1$, the numbers $A_{c,i}$ and $B_{c,i}$ can be determined with our explicit formulas for the blocks in Tables 1, 2 and 3. We gather the expressions for the number of coefficients in Table 4 (notice that $N_1 + N_{-1}$ agrees with Al-Kateeb's result on the number on non-zero coefficients). □

Remark 4.2. *Consider $\Phi_{3p_2p_3}$ where $p_3 \equiv 1 \bmod 3p_2$ and $p_2 \equiv 1 \bmod 3$. According to Table 4, the quantity N_1 divided by the total number of coefficients is*

$$\frac{N_1}{2(p_2-1)(p_3-1)+1} = \frac{14qq_2+1}{2(p_2-1)(p_3-1)+1} \approx \frac{14\frac{p_3}{3p_2}\frac{p_2}{3}}{2p_2p_3} \approx \frac{7}{9p_2}\,.$$

	$p_3 \bmod 3p_2$	$p_2 \equiv 1 \bmod 3$	$p_2 \equiv 2 \bmod 3$
N_1	1	$14qq_2+1$	$14qq_2+4q+1$
	-1	$14qq_2+14q_2$	$14qq_2+14q_2+4q+4$
N_{-1}	1	$14qq_2$	$14qq_2+4q$
	-1	$14qq_2+14q_2-1$	$14qq_2+14q_2+4q+3$
N_0	1	$54qq_2^2-10qq_2$	$54qq_2^2+26qq_2+4q$
	-1	$54qq_2^2-10qq_2+54q_2^2-22q_2+2$	$54qq_2^2+54q_2^2+26qq_2+14q_2+4q+2$

TABLE 4. Numbers of coefficients for $\Phi_{3p_2p_3}$, where $q_2 = \lfloor \frac{p_2}{3} \rfloor$.

In particular, this ratio goes to 0 *when* p_2 *(hence also* p_3*) goes to infinity. In the same way, one may consider* N_{-1} *or* N_0*, or* $p_2 \equiv 2 \bmod 3$*, or* $p_3 \equiv -1 \bmod 3p_2$*. In the following table we indicate approximate values when* p_2 *goes to infinity.*

	$p_3 \bmod 3p_2$	$p_2 \equiv 1 \bmod 3$	$p_2 \equiv 2 \bmod 3$
$\frac{N_1}{2(p_2-1)(p_3-1)+1}$	1	$\frac{7}{9p_2}$	$\frac{7}{9p_2}+\frac{2}{3p_2^2}$
	-1	$\frac{7}{9p_2}+\frac{7}{3p_3}$	$\frac{7}{9p_2}+\frac{7}{3p_3}+\frac{2}{3p_2^2}$
$\frac{N_{-1}}{2(p_2-1)(p_3-1)+1}$	1	$\frac{7}{9p_2}$	$\frac{7}{9p_2}+\frac{2}{3p_2^2}$
	-1	$\frac{7}{9p_2}+\frac{7}{3p_3}$	$\frac{7}{9p_2}+\frac{7}{3p_3}+\frac{2}{3p_2^2}$
$\frac{N_0}{2(p_2-1)(p_3-1)+1}$	1	$1-\frac{14}{9p_2}$	$1-\frac{14}{9p_2}-\frac{4}{3p_2^2}$
	-1	$1-\frac{14}{9p_2}-\frac{14}{3p_3}$	$1-\frac{14}{9p_2}-\frac{14}{3p_3}-\frac{4}{3p_2^2}$

5. EXPLICIT BLOCK DESCRIPTION FOR $\Phi_{3p_2p_3}$ WITH $p_3 \equiv \pm 2 \bmod 3p_2$

In this section we prove Theorem 1.1 for $p_3 \equiv \pm 2 \bmod 3p_2$. Since we may reason as for the case $p_3 \equiv \pm 1 \bmod 3p_2$, we only illustrate the differences that occur. As we work with rotations by 2 instead of by 1, we need to adapt our definition of perturbation. Namely, we say that $s_{1,i}$ (respectively, $s_{3,i}$) is perturbed if the coefficient of X^3 or X^4 (respectively, X^{p_2+3} or X^{p_2+4}) in $f_{i-1,0}$ is non-zero.

5.1. **The case** $p_3 \equiv 2 \bmod 3p_2$**.** We work with rotations by 2 instead of 1, and it turns out that $s_{2,i}$ and $s_{4,i}$ contain more non-zero coefficients. These two slices are still computable in practice because their non-zero coefficients come from non-zero coefficients of $s_{1,k}$ and $s_{3,k}$ for the values of $k < i$. Notice that the expressions of $s_{1,i}$ and $s_{3,i}$ are still 3-periodic unless there is a perturbation.

We find that the coefficient of X^3 in $f_{i,0}$ is -1 for $i = \frac{p_2-3}{2}$, which leads to a perturbation of $s_{1,i}$ for $i = \frac{p_2-1}{2}$. From the index $i = \frac{p_2-1}{2}$ the coefficients of X^3 and X^4 are 3-periodic (this is a consequence of the periodicity of the coefficients of $s_{3,k}$ for $k < i$). It will follow from the expressions of the next $s_{3,k}$'s that this is the case up to index $i = \frac{3(p_2-1)-2}{2}$.

Hence we retrieve the 3-periodicity for $s_{1,i}$ (but the expressions are different than in the case $i < \frac{p_2-1}{2}$). Moreover, the coefficient of X^{p_2+4} in $f_{i,0}$ is 1 for $i = p_2 - 2$, which leads to a perturbation for $s_{3,i}$ for $i = p_2 - 1$. The coefficients of X^{p_2+3} and X^{p_2+4} in $f_{i,0}$ are periodic from $i = p_2 - 1$, so we also retrieve the 3-periodicity for $s_{3,i}$. As the expressions of $s_{1,i}$ have changed from $i = \frac{p_2-1}{2}$, the expressions of the coefficients of X^{p_2+3} and X^{p_2+4} in $f_{i,0}$ change from $i = \frac{3(p_2-1)-2}{2}$, and so do the expressions of $s_{3,i}$ from $i = \frac{3(p_2-1)}{2}$. Recall also that the coefficients b_i of Φ_{3p_2} change at $i = p_2 - 1$, so do the expressions of $s_{1,i}$.

We distinguish the cases $p_2 \equiv 1 \bmod 3$ and $p_2 \equiv 2 \bmod 3$. The expressions of the first and third slices are given in Table 5.

i	$p_2 \bmod 3$	$s_{1,i}(X)$	$s_{3,i}(X)$
$i \equiv 0 \bmod 3,\ i < \frac{p_2-1}{2}$	1,2	$1+X+X^2$	$-X^{p_2}-X^{p_2+1}-X^{p_2+2}$
$i \equiv 1 \bmod 3,\ i < \frac{p_2-1}{2}$	1,2	$-X-X^2$	$X^{p_2+1}+X^{p_2+2}$
$i \equiv 2 \bmod 3,\ i < \frac{p_2-1}{2}$	1,2	-1	X^{p_2}
$i \equiv 0 \bmod 3,\ \frac{p_2-1}{2} \le i < p_2-1$	1	1	$-X^{p_2}-X^{p_2+1}-X^{p_2+2}$
	2	$X+2X^2$	$-X^{p_2}-X^{p_2+1}-X^{p_2+2}$
$i \equiv 1 \bmod 3,\ \frac{p_2-1}{2} \le i < p_2-1$	1	$-1-X$	$X^{p_2+1}+X^{p_2+2}$
	2	$1-X^2$	$X^{p_2+1}+X^{p_2+2}$
$i \equiv 2 \bmod 3,\ \frac{p_2-1}{2} \le i < p_2-1$	1	X	X^{p_2}
	2	$-1-X-X^2$	X^{p_2}
$i = p_2-1$	1	1	$-X^{p_2}-X^{p_2+1}$
	2	$1-X^2$	$X^{p_2+1}+2X^{p_2+2}$
$i \equiv 0 \bmod 3,\ p_2-1 < i < \frac{3(p_2-1)}{2}$	1	0	$-X^{p_2}-X^{p_2+1}$
	2	X^2	$-X^{p_2}-X^{p_2+1}-X^{p_2+2}$
$i \equiv 1 \bmod 3,\ p_2-1 < i < \frac{3(p_2-1)}{2}$	1	X^2	X^{p_2+1}
	2	$-X^2$	$X^{p_2+1}+2X^{p_2+2}$
$i \equiv 2 \bmod 3,\ p_2-1 < i < \frac{3(p_2-1)}{2}$	1	$-X^2$	X^{p_2}
	2	0	$X^{p_2}-X^{p_2+2}$
$i = \frac{3(p_2-1)}{2}$	1	0	$-X^{p_2}-X^{p_2+1}$
	2	X^2	$-X^{p_2}-X^{p_2+1}-X^{p_2+2}$
$i \equiv 0 \bmod 3,\ \frac{3(p_2-1)}{2} < i < 2(p_2-1)$	1	0	0
	2	X^2	$-X^{p_2+2}$
$i \equiv 1 \bmod 3,\ \frac{3(p_2-1)}{2} < i < 2(p_2-1)$	1	X^2	$-X^{p_2+2}$
	2	$-X^2$	X^{p_2+2}
$i \equiv 2 \bmod 3,\ \frac{3(p_2-1)}{2} < i < 2(p_2-1)$	1	$-X^2$	X^{p_2+2}
	2	0	0

TABLE 5. Expressions of $s_{1,i}$ and $s_{3,i}$ in the case $p_3 \equiv 2 \bmod 3p_2$.

The second and fourth slices are given in Table 7, described by the list of their coefficients. For example, $\sum_{k=3}^{p_2-1} a_{k,i}X^k$ has coefficients $(a_3, ..., a_{p_2-1})$. If L and L' are two lists, then $L+L'$ denotes the concatenation of L and L' and $k \cdot L = \underbrace{L + \ldots + L}_{k \text{ times}}$ and $0 \cdot L$ is the empty list. We also set

$$A = (-1,-1,0,1,1,0), \qquad B = (1,1,0,-1,-1,0), \qquad C = (1,0,-1,-1,0,1),$$

$$D = (0,-1,-1,0,1,1), \qquad E = (0,1,1,0,-1,-1), \qquad F = (-1,0,1,1,0,-1)\,.$$

In Table 7, the list $(0\ldots0)$ consists of zero coefficients, and it is empty in case the list concatenated to it already contains all coefficients. This happens, for example, if $p_2=7$, because $s_{2,2}$ has 4 coefficients (at X^3, X^4, X^5, X^6) and we write

$$s_{2,2}(X) = (0...0) + 0\cdot A + (-1,-1,0,1)\,.$$

Finally, we make use of the notation

$$h_1(k) = -k + \frac{4p_2-10}{6}, \qquad h_2(k) = -k + \frac{4p_2-8}{6}, \qquad h_3(k) = -k + \frac{3p_2-9}{6}\,.$$

Disclaimer: For the special case $p_2 = 5$, the second slice has only two coefficients, and the fourth slice has seven coefficients, so we have to take the truncation by two for $s_{2,i}$ and the truncation by seven for $s_{4,i}$ in the expressions given in our tables.

5.2. **The case $p_3 \equiv -2 \bmod 3p_2$.** This case is analogous to the case $p_3 \equiv 2 \bmod 3p_2$. Notice that rotations now go rightwards. The results are gathered in Tables 6 and 8 (the description of the latter table is the same as the one given above for Table 7).

i	p_2 mod 3	$s_{1,i}(X)$	$s_{3,i}(X)$
$i \equiv 0 \mod 3,\ i < \frac{p_2-1}{2}$	1,2	$1+X+X^2$	$-X^{p_2} - X^{p_2+1} - X^{p_2+2}$
$i \equiv 1 \bmod 3,\ i < \frac{p_2-1}{2}$	1,2	$-1-X$	$X^{p_2} + X^{p_2+1}$
$i \equiv 2 \bmod 3,\ i < \frac{p_2-1}{2}$	1,2	$-X^2$	X^{p_2+2}
$i \equiv 0 \bmod 3,\ \frac{p_2-1}{2} \le i < p_2-1$	1	$1+X+X^2$	$-X^{p_2+2}$
	2	$1+X+X^2$	$-2X^{p_2} - X^{p_2+1}$
$i \equiv 1 \bmod 3,\ \frac{p_2-1}{2} \le i < p_2-1$	1	$-1-X$	$X^{p_2+1} + X^{p_2+2}$
	2	$-1-X$	$X^{p_2} - X^{p_2+2}$
$i \equiv 2 \bmod 3,\ \frac{p_2-1}{2} \le i < p_2-1$	1	$-X^2$	$-X^{p_2+1}$
	2	$-X^2$	$X^{p_2} + X^{p_2+1} + X^{p_2+2}$
$i = p_2-1$	1	$X+X^2$	$-X^{p_2+2}$
	2	$-2-X$	$X^{p_2} - X^{p_2+2}$
$i \equiv 0 \bmod 3,\ p_2-1 < i < \frac{3(p_2-1)}{2}$	1	$X+X^2$	0
	2	$1+X+X^2$	$-X^{p_2}$
$i \equiv 1 \bmod 3,\ p_2-1 < i < \frac{3(p_2-1)}{2}$	1	$-X$	$-X^{p_2}$
	2	$-2-X$	X^{p_2}
$i \equiv 2 \bmod 3,\ p_2-1 < i < \frac{3(p_2-1)}{2}$	1	$-X^2$	X^{p_2}
	2	$1-X^2$	0
$i = \frac{3(p_2-1)}{2}$	1	$X+X^2$	0
	2	$1+X+X^2$	$-X^{p_2}$
$i \equiv 0 \bmod 3,\ \frac{3(p_2-1)}{2} < i < 2(p_2-1)$	1	0	0
	2	1	$-X^{p_2}$
$i \equiv 1 \bmod 3,\ \frac{3(p_2-1)}{2} < i < 2(p_2-1)$	1	1	$-X^{p_2}$
	2	-1	X^{p_2}
$i \equiv 2 \bmod 3,\ \frac{3(p_2-1)}{2} < i < 2(p_2-1)$	1	-1	X^{p_2}
	2	0	0

TABLE 6. Expressions of $s_{1,i}$ and $s_{3,i}$ in the case $p_3 \equiv -2 \bmod 3p_2$.

i	$p_2 \bmod 3$	$s_{2,i}(X)$	$s_{4,i}(X)$
$i=3k,\ i<\frac{p_2-1}{2}$	1,2	$(0\dots0)+k\cdot A$	$(0\dots0)+k\cdot B$
$i=3k+1,\ i<\frac{p_2-1}{2}$	1,2	$(0\dots0)+k\cdot A+(-1,-1)$	$(0\dots0)+k\cdot B+(1,1)$
$i=3k+2,\ i<\frac{p_2-1}{2}$	1,2	$(0\dots0)+k\cdot A+(-1,-1,0,1)$	$(0\dots0)+k\cdot B+(1,1,0,-1)$
$i=3k,\ \frac{p_2-1}{2}\le i<p_2-1$	1	$(0,1,1,0)+\frac{p_2-7}{6}\cdot A$	$(0\dots0)+\frac{p_2-1}{6}\cdot B+(k-\frac{p_2-1}{6})\cdot C$
	2	$(1,0)+\frac{p_2-5}{6}\cdot A$	$(0\dots0)+\frac{p_2-5}{6}\cdot B+(1,1,0,-1)+(k-\frac{p_2+1}{6})\cdot A+(-1,-1)$
$i=3k+1,\ \frac{p_2-1}{2}\le i<p_2-1$	1	$(1,0)+\frac{p_2-7}{6}\cdot A+(-1,-1)$	$(0\dots0)+\frac{p_2-1}{6}\cdot B+(k-\frac{p_2-1}{6})\cdot C+(1,0)$
	2	$\frac{p_2-5}{6}\cdot A+(-1,-1)$	$(0\dots0)+\frac{p_2-5}{6}\cdot B+(1,1,0,-1)+(k-\frac{p_2+1}{6})\cdot A+(-1,-1,0,1)$
$i=3k+2,\ \frac{p_2-1}{2}\le i<p_2-1$	1	$\frac{p_2-7}{6}\cdot A+(-1,-1,0,1)$	$(0\dots0)+\frac{p_2-1}{6}\cdot B+(k-\frac{p_2-1}{6})\cdot C+(1,0,-1,-1)$
	2	$(0,1)+\frac{p_2-5}{6}\cdot C$	$(0\dots0)+\frac{p_2-5}{6}\cdot B+(1,1,0,-1)+(k-\frac{p_2-5}{6})\cdot A$
$i=p_2-1$	1	$(0,1,1,0)+\frac{p_2-7}{6}\cdot A$	$(1,0,-1,-1,0)+\frac{p_2-7}{6}\cdot B+\frac{p_2-1}{6}\cdot C$
	2	$\frac{p_2-5}{6}\cdot A+(-1,-1)$	$(1,0)+\frac{p_2-5}{6}\cdot A+(-1)+\frac{p_2-5}{6}\cdot A+(-1,-1,0,1,0\dots0)$
$i=3k,\ p_2-1<i<\frac{3(p_2-1)}{2}$	1	$(0,1,1,0)+\frac{p_2-7}{6}\cdot A$	$(1,0,-1,-1,0)+h_3(k)\cdot B+\frac{p_2-1}{6}\cdot C+(1,0\dots0)$
	2	$(1,0)+\frac{p_2-5}{6}\cdot A$	$(0,1,1,0)+h_3(k)\cdot A+(-1)+\frac{p_2-5}{6}\cdot A+(-1,-1,0,1,1,0\dots0)$
$i=3k+1,\ p_2-1<i<\frac{3(p_2-1)}{2}$	1	$(1,0)+\frac{p_2-7}{6}\cdot A+(-1,-1)$	$(-1,-1,0)+h_3(k)\cdot B+\frac{p_2-1}{6}\cdot C+(1,0\dots0)$
	2	$\frac{p_2-5}{6}\cdot A+(-1,-1)$	$(1,0)+h_3(k)\cdot A+(-1)+\frac{p_2-5}{6}\cdot A+(-1,-1,0,1,1,0\dots0)$
$i=3k+2,\ p_2-1<i<\frac{3(p_2-1)}{2}$	1	$\frac{p_2-7}{6}\cdot A+(-1,-1,0,1)$	$(0)+h_3(k)\cdot B+\frac{p_2-1}{6}\cdot C+(1,0\dots0)$
	2	$(0,1)+\frac{p_2-5}{6}\cdot C$	$h_3(k)\cdot A+(-1)+\frac{p_2-5}{6}\cdot A+(-1,-1,0,1,1,0\dots0)$
$i=\frac{3(p_2-1)}{2}$	1	$(0,1,1,0)+\frac{p_2-7}{6}\cdot A$	$(0,-1,-1,0,1)+\frac{p_2-7}{6}\cdot C+(1,0\dots0)$
	2	$(1,0)+\frac{p_2-5}{6}\cdot A$	$\frac{p_2-5}{6}\cdot F+(-1,0,1,1,0\dots0)$
$i=3k,\ \frac{3(p_2-1)}{2}<i$	1	$(0,1,1,0)+h_1(k)\cdot A+(-1,-1)+(0\dots0)$	$(0,-1,-1,0,1)+h_1(k)\cdot C+(1,0\dots0)$
	2	$(1,0)+h_2(k)\cdot A+(-1,-1)+(0\dots0)$	$h_2(k)\cdot F+(-1,0,1,1,0\dots0)$
$i=3k+1,\ \frac{3(p_2-1)}{2}\le i$	1	$(1,0)+h_1(k)\cdot A+(-1,-1)+(0\dots0)$	$(-1,0,1)+h_1(k)\cdot C+(1,0\dots0)$
	2	$h_2(k)\cdot A+(-1,-1)+(0\dots0)$	$h_2(k)\cdot B+(1,1,0\dots0)$
$i=3k+2,\ \frac{3(p_2-1)}{2}\le i$	1	$h_1(k)\cdot A+(-1,-1)+(0\dots0)$	$(1)+h_1(k)\cdot C+(1,0\dots0)$
	2	$h_2(k)\cdot E+(0\dots0)$	$h_2(k)\cdot D+(0\dots0)$

TABLE 7. Expressions of $s_{2,i}$ and $s_{4,i}$ in the case $p_3\equiv 2 \bmod 3p_2$.

i	p_2 mod 3	$s_{2,i}(X)$	$s_{4,i}(X)$
$i=3k,\ i<\frac{p_2-1}{2}$	1,2	$k\cdot D+(0\dots0)$	$k\cdot E+(0\dots0)$
$i=3k+1,\ i<\frac{p_2-1}{2}$	1,2	$(1,1)+k\cdot D+(0\dots0)$	$(-1,-1)+k\cdot E+(0\dots0)$
$i=3k+2,\ i<\frac{p_2-1}{2}$	1,2	$(-1,0,1,1)+k\cdot D+(0\dots0)$	$(1,0,-1,-1)+k\cdot E+(0\dots0)$
$i=3k,\ \frac{p_2-1}{2}\le i<p_2-1$	1	$\frac{p_2-7}{6}\cdot D+(0,-1,-1,0)$	$(k-\frac{p_2-1}{6})\cdot F+\frac{p_2-1}{6}\cdot E+(0\dots0)$
	2	$(0,-1)+\frac{p_2-5}{6}\cdot F$	$(1,1)+(k-\frac{p_2+1}{6})\cdot D+(1,0,-1,-1)+\frac{p_2-5}{6}\cdot E+(0\dots0)$
$i=3k+1,\ \frac{p_2-1}{2}\le i<p_2-1$	1	$(1,1)+\frac{p_2-7}{6}\cdot D+(0,-1)$	$(0,-1)+(k-\frac{p_2-1}{6})\cdot F+\frac{p_2-1}{6}\cdot E+(0\dots0)$
	2	$(1,1)+\frac{p_2-5}{6}\cdot D$	$(-1,0,1,1)+(k-\frac{p_2+1}{6})\cdot D+(1,0,-1,-1)+\frac{p_2-5}{6}\cdot E+(0\dots0)$
$i=3k+2,\ \frac{p_2-1}{2}\le i<p_2-1$	1	$(-1,0,1,1)+\frac{p_2-7}{6}\cdot D$	$(1,1,0,-1)+(k-\frac{p_2-1}{6})\cdot F+\frac{p_2-1}{6}\cdot E+(0\dots0)$
	2	$(-1,0)+\frac{p_2-5}{6}\cdot B$	$(k+1-\frac{p_2+1}{6})\cdot D+(1,0,-1,-1)+\frac{p_2-5}{6}\cdot E+(0\dots0)$
$i=p_2-1$	1	$\frac{p_2-7}{6}\cdot D+(0,-1,-1,0)$	$\frac{p_2-1}{6}\cdot F+\frac{p_2-7}{6}\cdot E+(0,1,1,0,-1)$
	2	$(1,1)+\frac{p_2-5}{6}\cdot D$	$(-1,0,1,1)+\frac{p_2-5}{6}\cdot D+(1)+\frac{p_2-5}{6}\cdot D+(0,-1)$
$i=3k,\ p_2-1<i<\frac{3(p_2-1)}{2}$	1	$\frac{p_2-7}{6}\cdot D+(0,-1,-1,0)$	$(0\dots0,-1)+\frac{p_2-1}{6}\cdot F+h_3(k)\cdot E+(0,1,1,0,-1)$
	2	$(0,-1)+\frac{p_2-5}{6}\cdot F$	$(0\dots0)+\frac{p_2+1}{6}\cdot D+(1)+h_3(k)\cdot D+(0,-1,-1,0)$
$i=3k+1,\ p_2-1<i<\frac{3(p_2-1)}{2}$	1	$(1,1)+\frac{p_2-7}{6}\cdot D+(0,-1)$	$(0\dots0,-1)+\frac{p_2-1}{6}\cdot F+h_3(k)\cdot E+(0,1,1)$
	2	$(1,1)+\frac{p_2-5}{6}\cdot D$	$(0\dots0)+\frac{p_2+1}{6}\cdot D+(1)+h_3(k)\cdot D+(0,-1)$
$i=3k+2,\ p_2-1<i<\frac{3(p_2-1)}{2}$	1	$(-1,0,1,1)+\frac{p_2-7}{6}\cdot D$	$(0\dots0,-1)+\frac{p_2-1}{6}\cdot F+h_3(k)\cdot E+(0)$
	2	$(-1,0)+\frac{p_2-5}{6}\cdot B$	$(0\dots0)+\frac{p_2+1}{6}\cdot D+(1)+h_3(k)\cdot D$
$i=\frac{3(p_2-1)}{2}$	1	$\frac{p_2-7}{6}\cdot D+(0,-1,-1,0)$	$(0\dots0,-1)+\frac{p_2-7}{6}\cdot F+(-1,0,1,1,0)$
	2	$(0,-1)+\frac{p_2-5}{6}\cdot F$	$(0\dots0)+\frac{p_2-5}{6}\cdot A+(-1,-1,0,1)$
$i=3k,\ \frac{3(p_2-1)}{2}<i$	1	$(0\dots0)+(1,1)+h_1(k)\cdot D+(0,-1,-1,0)$	$(0\dots0,-1)+h_1(k)\cdot F+(-1,0,1,1,0)$
	2	$(0\dots0)+(1,1)+h_2(k)\cdot D+(0,-1)$	$(0\dots0)+h_2(k)\cdot A+(-1,-1,0,1)$
$i=3k+1,\ \frac{3(p_2-1)}{2}\le i$	1	$(0\dots0)+(1,1)+h_1(k)\cdot D+(0,-1)$	$(0\dots0,-1)+h_1(k)\cdot F+(-1,0,1)$
	2	$(0\dots0)+(1,1)+h_2(k)\cdot D$	$(0\dots0)+h_2(k)\cdot A+(-1,-1)$
$i=3k+2,\ \frac{3(p_2-1)}{2}\le i$	1	$(0\dots0)+(1,1)+h_1(k)\cdot D$	$(0\dots0,-1)+h_1(k)\cdot F+(-1)$
	2	$(0\dots0)+(1,1)+(h_2(k)-1)\cdot D+(0,-1,-1,0)$	$(0\dots0)+h_2(k)\cdot A$

TABLE 8. Expressions of $s_{2,i}$ and $s_{4,i}$ in the case $p_3\equiv-2 \bmod 3p_2$.

6. Properties of $\Phi_{3p_2p_3}$ for $p_3 \equiv \pm 2 \bmod 3p_2$

In this section we consider the case $p_3 \equiv \pm 2 \bmod 3p_2$, and we mimic our proofs from the case $p_3 \equiv \pm 1 \bmod 3p_2$. Our explicit formulas for $\Phi_{3p_2p_3}$ show that Zhang's conjecture holds true.

Proof of Proposition 1.6 for $p_3 \equiv \pm 2 \bmod 3p_2$. The maximum gap is $2(p_2 - 1)$. We have q maximum gaps located inside F_1, between the smallest exponent of $F_{0,i+1}$ and the largest exponent of $F_{0,i}$ for $0 \leq i < q$, and q other maximum gaps are given by symmetry at the middle coefficient. So we only have to check that the other gaps are smaller than $2(p_2 - 1)$. We only treat the case $p_3 \equiv 2 \bmod 3p_2$, the case $p_3 \equiv -2 \bmod 3p_2$ being completely analogous.

Since $g_{2,0}$ and $g_{3,0}$ give a maximum gap, by the symmetry of the coefficients of $\Phi_{3p_2p_3}$ we are left to study $g_{2,i}$ and $g_{3,i}$ for $1 \leq i < p_2 - 1$ and $g_{1,i}$ and $g_{4,i}$ for $0 \leq i < p_2 - 1$. We have $g_{1,0} = p_2 - 2$ and $g_{1,1} = 2p_2 - 4$, while for $2 \leq i < p_2 - 1$ we have $g_{1,i} < 2(p_2 - 1)$ because the difference between the smallest exponent of $s_{4,i}$ and the largest exponent of $s_{3,i}$ is at most $3p_2 - 2i - p_2$. We have $g_{2,i} \leq 1 + i < 2(p_2 - 1)$ because the coefficient of X in $s_{1,i}$ is non-zero (thus $f_{i,1} \neq 0$) and the coefficient of X^{3p_2-2i} in $s_{4,i}$ is non-zero. For $g_{3,i}$ we may reason as for $g_{2,i}$, while we have $g_{4,i} \leq 3$ because $f_{i,q} = \mathcal{T}_2 f_{i,0} \neq 0$ for $0 \leq i \leq p_2 - 2$.

□

Finally, we compute the number N_c of coefficients equal to c. The non-zero coefficients are in the set $\{\pm 1, \pm 2\}$ (see for instance [5, Theorem 1]). In particular we may deduce the expression of N_0 from N_1, N_{-1}, N_2, N_{-2}. From the explicit expressions in Table 9 we have $N_2 + N_{-2} > 0$ hence the polynomial $\Phi_{3p_2p_3}$ is not flat.

The computations of N_c are less evident, but only because $s_{2,i}$ and $s_{4,i}$ have more complex expressions. Notice that Al-Kateeb and Dagher recently found formulas for the number of non-zero coefficients (see [3]) that clearly match our expressions.

Remark 6.1. *The behavior of the number of coefficients is different w.r.t. Remark 4.2. Indeed, we have*

$$\frac{N_{\pm 2}}{2(p_2-1)(p_3-1)+1} \approx 0, \qquad \frac{N_0}{2(p_2-1)(p_3-1)+1} \approx \frac{22}{36},$$

$$\frac{N_{\pm 1}}{2(p_2-1)(p_3-1)+1} \approx \frac{\frac{7p_2^2}{6} \cdot q}{2(p_2-1)(p_3-1)} \approx \frac{\frac{7p_2^2}{6} \cdot \frac{p_3}{3p_2}}{2p_2p_3} \approx \frac{\frac{7p_2^2}{6} \cdot \frac{p_3}{3p_2}}{2p_2p_3} \approx \frac{7}{36}.$$

7. General algorithm for $p_3 > p_1p_2$ and $p_3 \equiv \pm 1, \pm 2 \bmod p_1p_2$

We have computed the blocks $f_{i,0}$ of $\Phi_{p_1p_2p_3}$ when $p_1 = 3$, $p_3 > 3p_2$, and $p_3 \equiv \pm 1, \pm 2 \bmod 3p_2$. In this section we argument that our method allows to compute these blocks for $\Phi_{p_1p_2p_3}$ when p_1 is fixed, $p_3 > p_1p_2$, and $p_3 \equiv \pm 1, \pm 2 \bmod p_1p_2$.

	$p_3 \bmod 3p_2$	$p_2 \equiv 1 \bmod 3$	$p_2 \equiv 2 \bmod 3$
N_2	2	0	$\frac{p_2+1}{3} \cdot q$
	-2	0	0
N_1	2	$\left(\frac{7p_2^2}{6} - \frac{7}{6}\right) \cdot q + \frac{2p_2+1}{3}$	$\left(\frac{7p_2^2}{6} + \frac{p_2}{3} - \frac{5}{6}\right) \cdot q + \frac{2p_2+2}{3}$
	-2	$\left(\frac{7p_2^2}{6} - \frac{7}{6}\right) \cdot (q+1) - \frac{2p_2-2}{3}$	$\left(\frac{7p_2^2}{6} + p_2 - \frac{1}{6}\right) \cdot (q+1) + \frac{1-2p_2}{3}$
N_{-1}	2	$\left(\frac{7p_2^2}{6} - \frac{7}{6}\right) \cdot q + \frac{2p_2-2}{3}$	$\left(\frac{7p_2^2}{6} + p_2 - \frac{1}{6}\right) \cdot q + \frac{2p_2-1}{3}$
	-2	$\left(\frac{7p_2^2}{6} - \frac{7}{6}\right) \cdot (q+1) - \frac{2p_2+1}{3}$	$\left(\frac{7p_2^2}{6} + \frac{p_2}{3} - \frac{5}{6}\right) \cdot (q+1) - \frac{2p_2+2}{3}$
N_{-2}	2	0	0
	-2	0	$\left(\frac{p_2+1}{3}\right) \cdot (q+1)$
N_0	2	$\left(\frac{11p_2^2}{3} - 6p_2\right) \cdot q + \frac{7q}{3} + \frac{2p_2-2}{3}$	$\frac{11p_2^2 q}{3} - \frac{23p_2 q}{3} + \frac{2p_2}{3} + \frac{2q}{3} - \frac{4}{3}$
	-2	$\frac{11p_2^2}{3} \cdot (q+1) - 6p_2 q - \frac{32p_2}{3} + \frac{7q}{3} + 9$	$\frac{11p_2^2}{3} \cdot (q+1) - \frac{23p_2 q}{3} - \frac{37p_2}{3} + \frac{2q}{3} + 8$

TABLE 9. Numbers of coefficients for Φ_{3p_3}.

We only need to investigate $p_3 \equiv 1, 2 \bmod p_1p_2$ because the cases $p_3 \equiv -1, -2 \bmod p_1p_2$ are analogous (with rotations going rightwards instead of leftwards). We denote by q_2 and r_2 the quotient and remainder of p_2 after division by p_1. We decompose

$$f_{i,0}(X) = \sum_{k=0}^{p_1p_2-1} a_{i,k}X^k = \sum_{j=1}^{4} s_{j,i}(X)$$

with the four slices

$$s_{1,i}(X) := \sum_{k=0}^{p_1-1} a_{i,k}X^k, \qquad s_{2,i}(X) := \sum_{k=p_1}^{p_2-1} a_{i,k}X^k,$$

$$s_{3,i}(X) := \sum_{k=p_2}^{p_2+p_1-1} a_{i,k}X^k, \qquad s_{4,i}(X) := \sum_{k=p_2+p_1}^{p_1p_2-1} a_{i,k}X^k\,.$$

The binary cyclotomic polynomial $\Phi_{p_1p_2}$ is more complex than Φ_{3p_2}, however Theorem 2.1 applies. We may decompose Φ_{3p_2} with p_2-blocks $\widetilde{f_i}$ (for $0 \le i \le p_1 - 1$) that in turn may be decomposed with p_1-blocks $\widetilde{f_{i,j}}$ for $0 \le j \le q_2 - 1$ and a final r_2-block $\widetilde{f_{i,q_2}}$, satisfying the same structure property as seen for the ternary case, namely

$$\widetilde{f_{i,0}} = \ldots = \widetilde{f_{i,q_2-1}} \qquad \text{and} \qquad \widetilde{f_{i,q_2}} = \mathcal{T}_{r_2}\widetilde{f_{i,0}}\,.$$

Inside a block $\widetilde{f_i}$ the coefficients of $\Phi_{p_1p_2}$ are p_1-periodic. Hence we will get a certain periodicity for the expressions of $s_{1,i}$ and $s_{3,i}$ (there are intervals of values of i for which these expressions are p_1-periodic).

Recall from (7) that $f_{0,0} = -\Psi_{p_1p_2}$. Then to compute $f_{i,0}$ we start from $f_{0,0}$ with the iterative formula (6). As explained above, the p_1-periodicity for the coefficients b_{i+1} (inside a p_2-block of $\Phi_{p_1p_2}$) leads to a periodicity for $s_{1,i}$ and $s_{3,i}$, as seen in the case $p_1 = 3$, after at most p_1+1 steps. This reasoning leads to the expression of $f_{i,0}$ for the p_2 first values of i. When we take b_{i+1} in a new block for $\Phi_{p_1p_2}$ we have a new p_1-periodicity in the ternary cyclotomic polynomial (this change happens p_1-2 many times because the number of p_2-blocks in $\Phi_{p_1p_2}$ is p_1-1) hence we may have a perturbation of the periodicity for $s_{1,i}$ and $s_{3,i}$, which means that we have to start anew to apply the iterative formula. For $i \leq p_2-p_1$ there is no perturbation for the periodicity coming from changing p_2-block in $\Phi_{p_1p_2}$ and, as we will see below from the expressions of $s_{2,i}$ and $s_{4,i}$, there is no further perturbation coming from non-zero coefficients of $s_{2,i}$ and $s_{4,i}$ by rotation. For $i > p_2-p_1$ those further perturbations may occur.

Our definition of perturbation for $p_1=3$ adapts to this more general setting. We first consider the case $p_3 \equiv 1 \bmod p_1p_2$. We say that there is a perturbation for $s_{3,i}$ (respectively, $s_{1,i}$) if the coefficient of $X^{p_2+p_1}$ (resp. X^{p_1}) in $f_{i-1,0}$ is non-zero.

As in the case $p_1=3$, $s_{4,0}=0$ and for $i \geq 1$ the smallest exponent in $s_{4,i}$ is p_1p_2-i. So there is no perturbation of $s_{3,i}$ caused by non-zero coefficients of $s_{4,i}$, because the coefficient of $X^{p_2+p_1}$ in $f_{i,0}$ is non-zero only for $i=\varphi(p_1p_2)-1$, which is the last possible value of i.

As in the case $p_1=3$, $s_{2,0}=0$ and for $1 \leq i \leq p_2-1$ the coefficient of X^{p_2} in $s_{3,i}$ is equal to 0. In fact, by (3) we have $\widetilde{f_{0,0}}(X)=1-X$, so applying (6) successively yields

$$\begin{aligned} s_{3,i}(X) &= X^{p_1+p_2-i} \qquad \text{for} \qquad 1 \leq i < p_1 \\ s_{3,p_1}(X) &= X^{p_2+1}+\ldots+X^{p_2+p_1-1} \\ \text{and} \qquad s_{3,p_1+1}(X) &= X^{p_2+p_1-1} = s_{3,1}(X)\,. \end{aligned}$$

Then, by p_1-periodicity, the coefficient of X^{p_2} of $s_{3,i}$ is equal to 0 for other $i \leq p_2-1$. As for the case $p_1=3$, we have

$$s_{2,i}(X) = -X^{p_2-i} \qquad \text{for} \qquad 1 \leq i \leq p_2-3\,.$$

Therefore, the coefficient of X^{p_1} of $f_{i,0}$ is -1 for $i=p_2-p_1$, which leads to the first perturbation of $s_{1,i}$. Depending on the expression of the polynomials $s_{3,i}$, the polynomial $s_{2,i}$ could contain non-zero coefficients for larger values of i, and new perturbations for $s_{1,i}$ may occur. Notice that the coefficients in $s_{2,i}$ and $s_{4,i}$ are coefficients of $s_{1,j}$ and $s_{3,j}$ for some $j<i$.

In the case $p_2 \equiv 2 \bmod 3p_2$, we say that there is a perturbation of $s_{3,i}$ (respectively, $s_{1,i}$) if the coefficient of $X^{p_2+p_1}$ or $X^{p_2+p_1+1}$ (respectively, X^{p_1} or X^{p_1+1}) in $f_{i-1,0}$ is non-zero. Studying perturbations for $p_3 \equiv 2 \bmod p_1p_2$ is more complex but similar.

8. The family of ternary polynomials Φ_{15p_3}

Consider the family of ternary polynomials $\Phi_{p_1p_2p_3}$ for some fixed value of p_1p_2. We argument that is possible to prove our results for the polynomials in this family (without the assumptions $p_3 > p_1p_2$ or $p_3 \equiv \pm1, \pm2 \bmod p_1p_2$). Firstly, we may deal with the finitely many primes $p_3 \leq p_1p_2$ separately. Secondly, there are only finitely many remainders r for

p_3 after division by p_1p_2 (the possible remainders being those coprime to p_1p_2). Proposition 2.5 shows that the expression of the blocks $f_{i,0}$ only depend on r. Therefore, we may resolve to choosing some prime $P_r > p_1p_2$ that leaves remainder r after division by p_1p_2 and computing $\Phi_{p_1p_2P_r}$ (see [4] for algorithms to compute cyclotomic polynomials). To get the blocks $f_{i,0}$, it suffices to extract from $\Phi_{p_1p_2P_r}$ the appropriate slice of coefficients. The knowledge of these blocks leads straight-forwardly (see Section 4) to the determination the number of maximum gaps. Moreover, the coefficients of $\Phi_{p_1p_2p_3}$ are bounded in absolute value by $p_1 - 1$ (see [11, Section 3.1] for bounds on the coefficients of ternary cyclotomic polynomials) so we have to compute the number of coefficients N_c only for $|c| < p_1$ (and we have $N_0 = \varphi(p_1p_2p_3) + 1 - \sum_{1<|c|<p_1} N_c$). The computation of N_c can be done by counting the coefficients equal to c in each block, see Section 4.

We now fix $p_1p_2 = 15$ and illustrate the above method in this example. For $p_3 \leq 15$ we may compute Φ_{15p_3} explicitly. In particular, we get

p_3	N_0	N_1	N_{-1}	N_2	N_{-2}	maximum gap	# maximum gaps
7	16	18	13	0	2	3	6
11	24	14	33	10	0	4	4
13	38	31	26	0	2	6	4

We may now suppose that $p_3 > 15$. The maximum gap is 8 (by Theorem 1.4) and the number of maximum gaps is $2q$ (by the known result on Zhang's conjecture [13]).

For an invertible class ($r \bmod 15$), the knowledge of the blocks $f_{i,0}$ leads to formulas for N_c if $p_3 \equiv r \bmod 15$ with $|c| \leq 2$, as we did in Sections 4 and 6. We computed the blocks $f_{i,0}$ recursively by hand using equation (6). Another possibility would have been to compute Φ_{15p_3} numerically (for some fixed $p_3 > 15$ with remainder r) and to extract the slices of coefficients that correspond to the blocks. As an illustration, we detail the case $p_3 \equiv 4 \bmod 15$. Recall that we have

$$\Phi_{15}(X) = 1 - X + X^3 - X^4 + X^5 - X^7 + X^8 \,.$$

We then compute the 15-blocks $f_{i,0}$ where $i = 0, \ldots, 7$:

$$\begin{cases} f_{0,0}(X) &= -\Psi_{15}(X) = 1 + X + X^2 - X^5 - X^6 - X^7 \\ f_{1,0}(X) &= \mathcal{R}_4(f_{0,0}(X)) + \Psi_{15}(X) \\ &= -1 - 2X - 2X^2 - X^3 + X^5 + X^6 + X^7 + X^{11} + X^{12} + X^{13} \\ f_{2,0}(X) &= \mathcal{R}_4(f_{1,0}(X)) = X + X^2 + X^3 + X^7 + X^8 + X^9 - X^{11} - 2X^{12} - 2X^{13} - X^{14} \\ f_{3,0}(X) &= \mathcal{R}_4(f_{2,0}(X)) - \Psi_{15}(X) \\ &= 1 + X + X^2 + X^3 + X^4 - X^6 - 2X^7 - 2X^8 - 2X^9 - X^{10} + X^{12} + X^{13} + X^{14} \\ f_{4,0}(X) &= \mathcal{R}_4(f_{3,0}(X)) + \Psi_{15}(X) \\ &= -X - 2X^2 - 2X^3 - 2X^4 - X^5 + X^7 + X^8 + X^9 + X^{10} + X^{11} + X^{12} + X^{13} + X^{14} \\ f_{5,0}(X) &= \mathcal{R}_4(f_{4,0}(X)) - \Psi_{15}(X) \\ &= -1 + X^2 + X^3 + X^4 + X^8 + X^9 + X^{10} - X^{12} - 2X^{13} - 2X^{14} \\ f_{6,0}(X) &= \mathcal{R}_4(f_{5,0}(X)) = 1 + X^4 + X^5 + X^6 - X^8 - 2X^9 - 2X^{10} - X^{11} + X^{13} + X^{14} \\ f_{7,0}(X) &= \mathcal{R}_4(f_{6,0}(X)) + \Psi_{15}(X) \\ &= -X^4 - X^5 - X^6 + X^9 + X^{10} + X^{11}. \end{cases}$$

We deduce that, for $r = 4$, the numbers $A_{1,i}$ and $B_{1,i}$ introduced in Section 4 are

i	0	1	2	3	4	5	6	7
$A_{1,i}$	3	6	6	8	8	6	6	3
$B_{1,i}$	3	0	3	4	0	2	1	0

so we deduce that $N_1 = \sum_{i=0}^{7} (A_{1,i}q + B_{1,i}) = 46q + 13$. Analogous computations lead to the following values:

$p_3 \mod 15$	N_0	N_1	N_{-1}	N_2	N_{-2}
1	$84q$	$18q + 1$	$18q$	0	0
2	$54q + 2$	$30q + 4$	$34q + 3$	$2q$	0
4	$42q + 4$	$46q + 13$	$18q + 4$	0	$14q + 4$
7	$48q + 16$	$38q + 18$	$30q + 13$	0	$4q + 2$
8	$48q + 18$	$30q + 17$	$38q + 20$	$4q + 2$	0
11	$42q + 24$	$18q + 14$	$46q + 33$	$14q + 10$	0
13	$54q + 38$	$34q + 31$	$30q + 26$	0	$2q + 2$
14	$84q + 70$	$18q + 18$	$18q + 17$	0	0

Table 10. Number of coefficients for Φ_{15p_3}, where $p_3 > 15$.

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