A new R package for Finite Mixture Models with an application to clustering countries with respect to COVID data

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joint work with

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General description of Finite Mixture models

We have a collection of individual trajectories.

We try to divide the population into a number of homogenous sub-populations and to estimate, at the same time, a typical trajectory for each sub-population. (Nagin 2005, Schiltz 2015)

This model can be interpreted as functional fuzzy cluster analysis.

The basic model (Nagin 2005)

Consider a population of size N and a variable of interest Y.
Let $Y_i = y_{i_1}, y_{i_2}, ..., y_{i_T}$ be T measures of the variable, taken at times $t_1, ...t_T$ for subject number i and π_k the probability of a given subject to belong to group number k

For a given group G_k, we suppose conditional independence for the sequential realizations of the elements y_{i_t} over the T periods of measurements.

The density f of Y is given by

$$f(y_i; \psi) = \sum_{k=1}^{K} \pi_k g^k(y_i; \Theta_k), \tag{1}$$

where $g^k(\cdot)$ denotes the distribution of y_{it} conditional on membership in group k and the role of the parameters Θ_k is to describe the shape of the trajectories in group k.

Possible data distributions

- Poisson distribution
- Binary logit distribution
- (Censored) normal distribution
- Beta distribution (Noel & S. 2023)

Predictors of trajectory group membership

X : vector of variables potentially associated with group membership (measured before t_1).

Multinomial logit model:

$$\pi_k(x_i) = \frac{e^{x_i\theta_k}}{\sum_{k=1}^{K} e^{x_i\theta_k}}, \tag{2}$$

where θ_k denotes the effect of x_i on the probability of group membership for group k.

$$L = \prod_{i=1}^{N} \sum_{k=1}^{K} \frac{e^{x_i\theta_k}}{\sum_{k=1}^{K} e^{x_i\theta_k}} \prod_{t=1}^{T} p^k(y_{it}), \tag{3}$$

where $p^k(\cdot)$ denotes the distribution of y_{it} conditional on membership in group k.

Adding covariates to the trajectories

Let W be a vector of covariates potentially influencing Y.

The likelihood then becomes

$$L = \prod_{i=1}^{N} \sum_{k=1}^{K} \frac{e^{x_i \theta_k}}{\sum_{k=1}^{K} e^{x_i \theta_k}} \prod_{t=1}^{T} p^k(y_{it} | A_i, W_i, \Theta_k).$$

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The Beta distribution

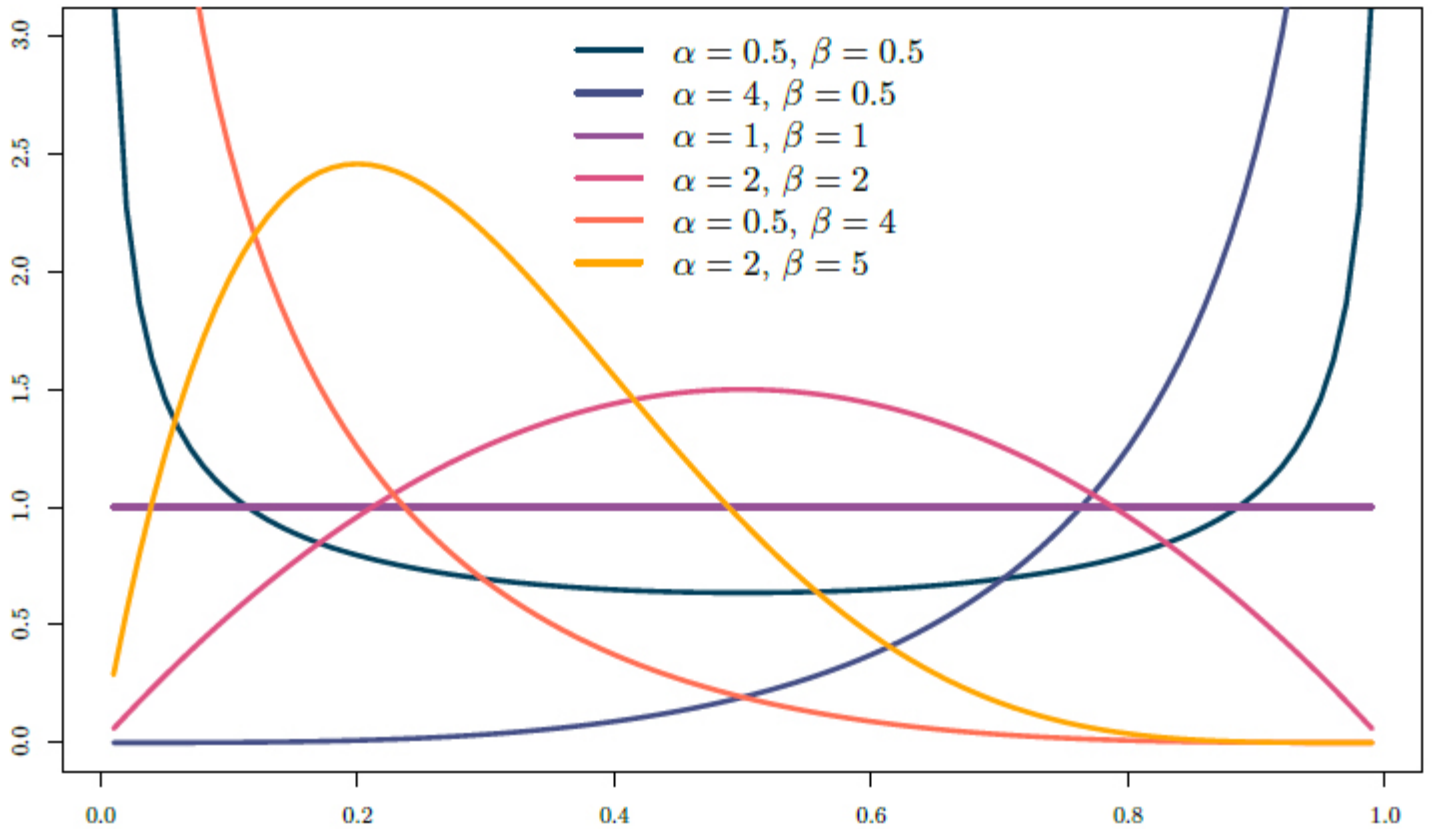

Figure 1 – *Example of different shapes of the Beta density for some parameters.*

Density of the Beta distribution

Let Y be a random variable following a Beta distribution with mean μ.

Consider the parameter ϕ defined by

$$var(Y) = \frac{\mu(1-\mu)}{1+\phi}.$$

ϕ can be interpreted as a precision parameter, in the sense that a large value of ϕ implies a small variance of Y.

The density f of Y can be written as

$$f(y;\mu;\phi) = \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1-\mu)\phi)} y^{\mu\phi-1}(1-y)^{(1-\mu)\phi-1},$$

where $0 < \mu < 1$ and $\phi > 0$.

Finite mixture models for an underlying Beta distribution

Density of y_{it} conditional to membership in group C_k :

$$g_k(y_{it}; \mu_{kit}, \phi_{kit}) = \frac{\Gamma(\phi_{kit})}{\Gamma(\mu_{kit}\phi_{kit})\Gamma((1-\mu_{kit})\phi_{kit})} y_{it}^{\mu_{kit}\phi_{kit}-1}(1-y_{it})^{(1-\mu_{kit})\phi_{kit}-1},$$

with

$$\mu_{kit} = \frac{e^{\beta_k A_{it}+\delta_k W_{it}}}{1+e^{\beta_k A_{it}+\delta_k W_{it}}} \text{ and } \phi_{kit} = \zeta_k A_{it}. \quad (4)$$

Likelihood of the data:

$$L = e^{\prod_{i=1}^{n}\left(\sum_{k=1}^{K}\pi_k \prod_{t=1}^{T} \frac{\Gamma(\phi_{kit})}{\Gamma(\mu_{kit}\phi_{kit})\Gamma((1-\mu_{kit})\phi_{kit})} y_{it}^{\mu_{kit}\phi_{kit}-1}(1-y_{it})^{(1-\mu_{kit})\phi_{kit}-1}\right)}. \quad (5)$$

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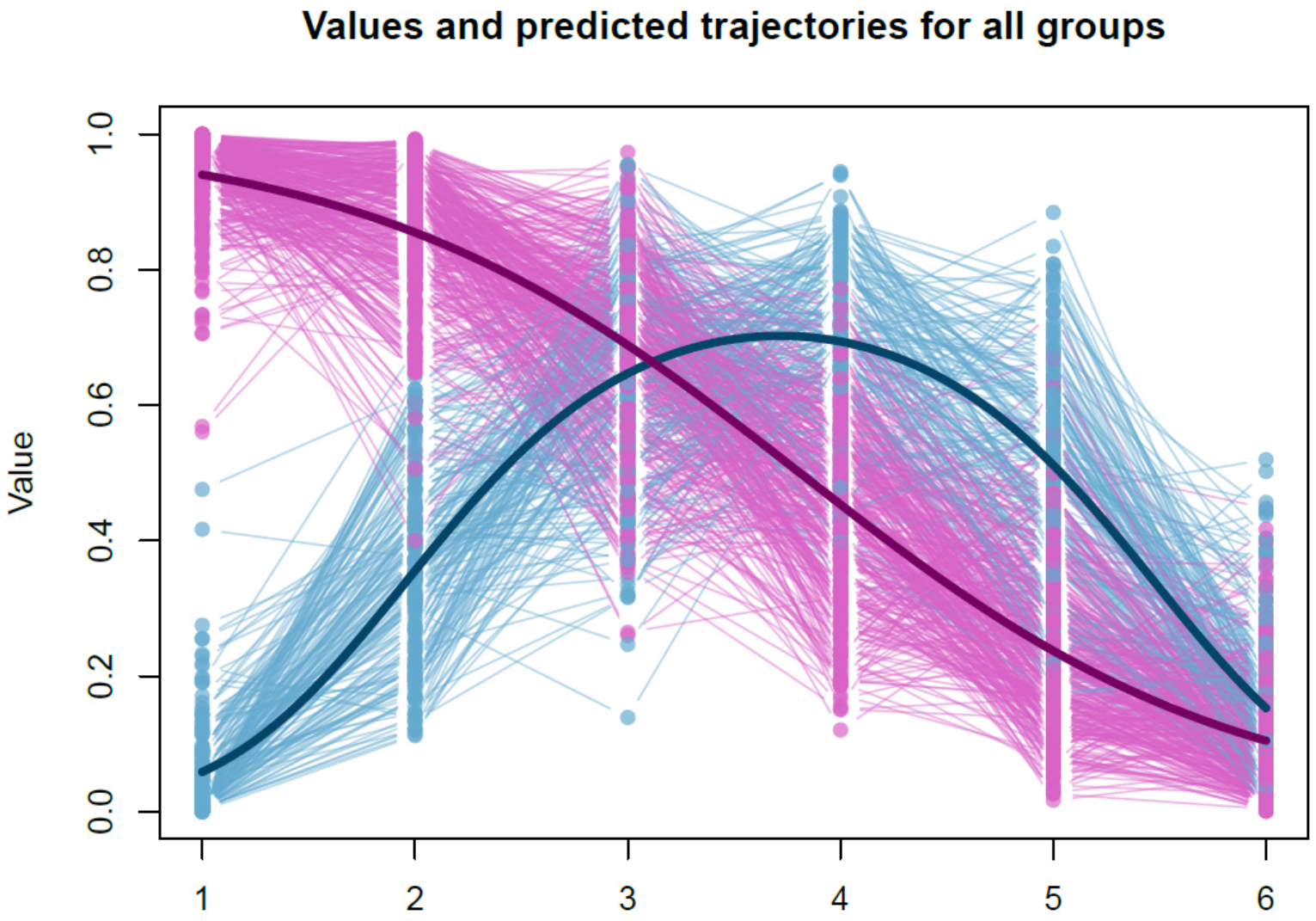
Function signature

```
R> trajeR(Y, A, Risk = NULL, TCOV = NULL, degre, degre.phi = 0,
+         Model, Method = "L",
+         ssigma = FALSE, ymax = max(Y) + 1, ymin = min(Y) - 1,
+         hessian = TRUE, itermax = 100, paraminit = NULL,
+         ProbIRLS = TRUE, refgr = 1, + fct = NULL, diffct = NULL,nbvar = NULL,
```

Output of result

```
## Model : Beta
## Method : Likelihood
##
##   group   Parameter   Estimate   Std. Error   T for H0:   Prob>|T|
##                                               param.=0
## ------------------------------------------------------------------
##    mean
##       1   Intercept   -5.95316      0.1281    -46.4734          0
##              Linear    3.66558     0.07649     47.92297         0
##           Quadratic   -0.49316     0.01027    -48.04232         0
##    zeta
##       1   Intercept    2.26533      0.0993     22.81197         0
##              Linear   -0.00558     0.02466     -0.22636   0.82094
##
##    mean
##       2   Intercept    3.73504     0.04525     82.53444         0
##              Linear   -0.98061     0.01144    -85.70519         0
##    zeta
##       2   Intercept    2.35458     0.07128     33.03302         0
##              Linear   -0.00144     0.01771     -0.08113   0.93534
## ------------------------------------------------------------------
##       1         pi1       0.344     0.02069            0         0
##       2         pi2       0.656     0.02069     31.19708         0
## ------------------------------------------------------------------
## Likelihood : 2516.737
```

Graphical illustration of result



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Data

Data from 190 countries from "Our World In Data".

Main variable of interest: **contamination rate**. We create a panel with monthly data from January 2020 till April 2021.

Covariates: new cases, population size (in million inhabitants), total cases per million people, median age of the population, population density, number of inhabitants over 65 (in million inhabitants), government response stringency index, GDP per capita, extreme poverty index, cardiovascular death rate, diabetes prevalence rate, index of handwashing facilities, rate of hospital beds per thousand inhabitants, life expectancy, index of human development and stringency index.

The nine metrics used to calculate the **stringency index** are: school closures; workplace closures; cancellation of public events; restrictions on public gatherings; closures of public transport; stay-at-home requirements; public information campaigns; restrictions on internal movements; and international travel controls.

Individual trajectories

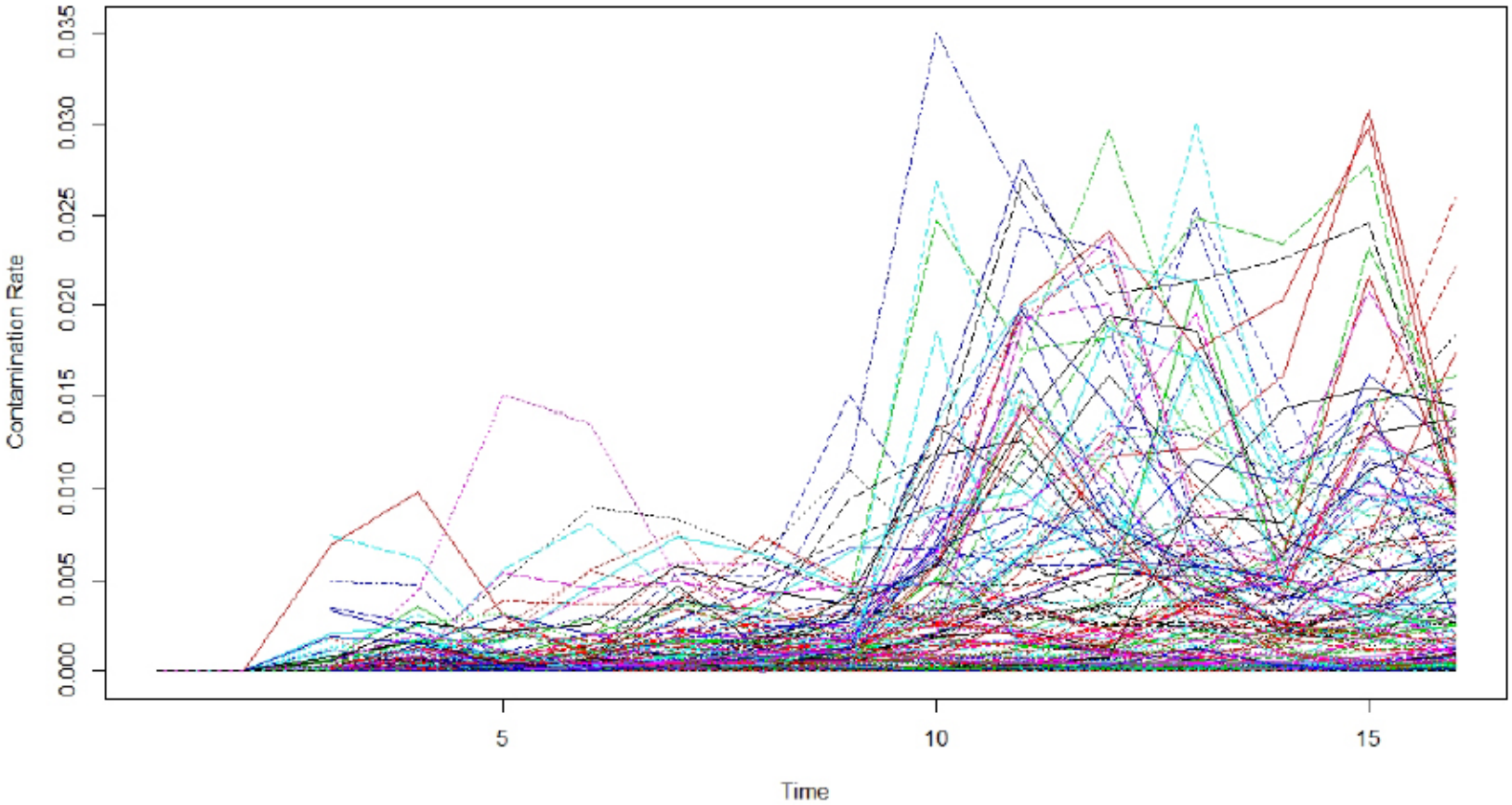


Figure 2 – *Contamination rates for all countries.*

Model selection

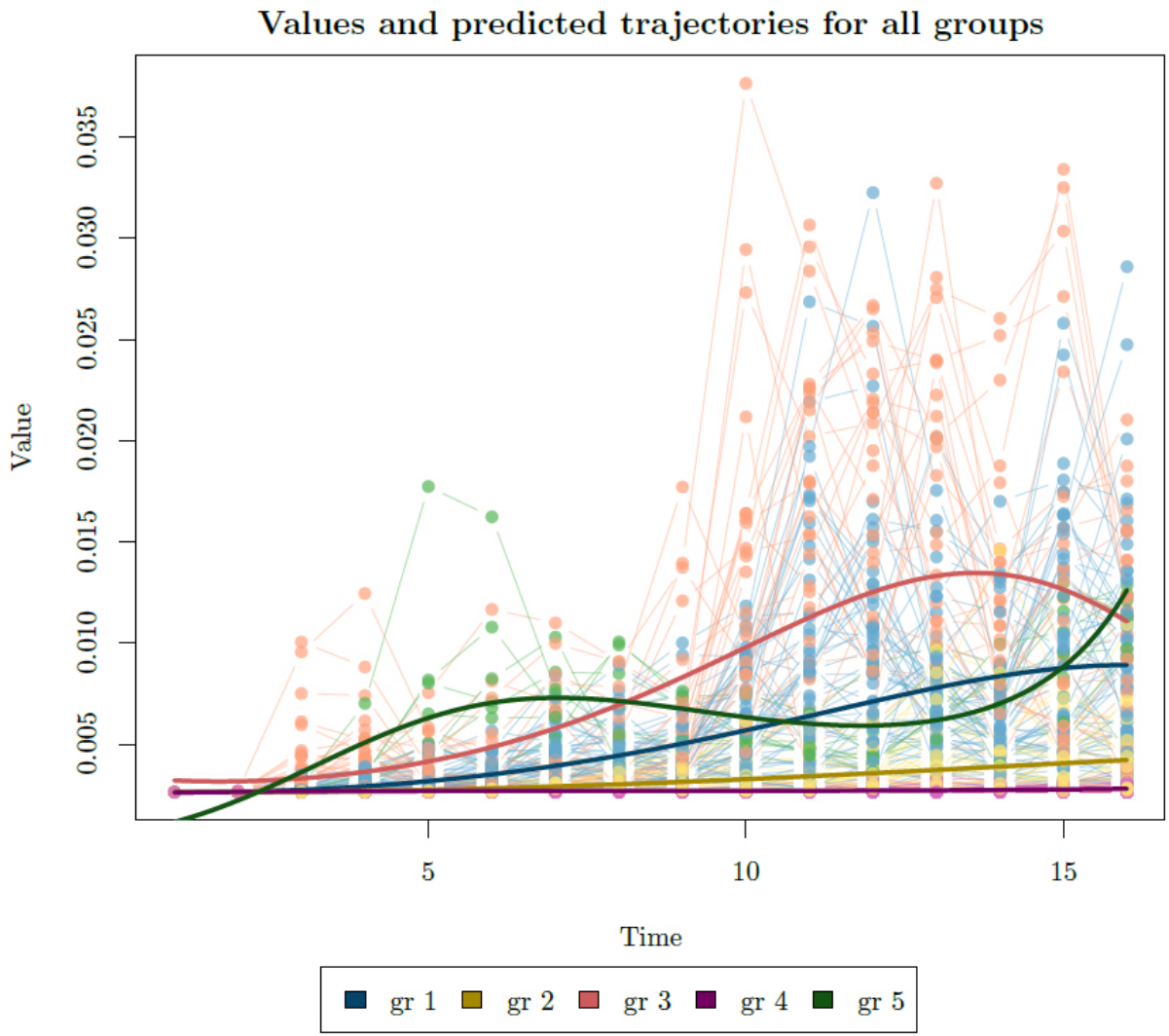
Kass and Wasserman's crierion: Let p_k be the probability that a model with k groups is the correct model. They show that p_k can be approximated by

$$p_k \approx \frac{e^{BIC_k - BIC_{max}}}{\sum_k e^{BIC_k - BIC_{max}}}.$$

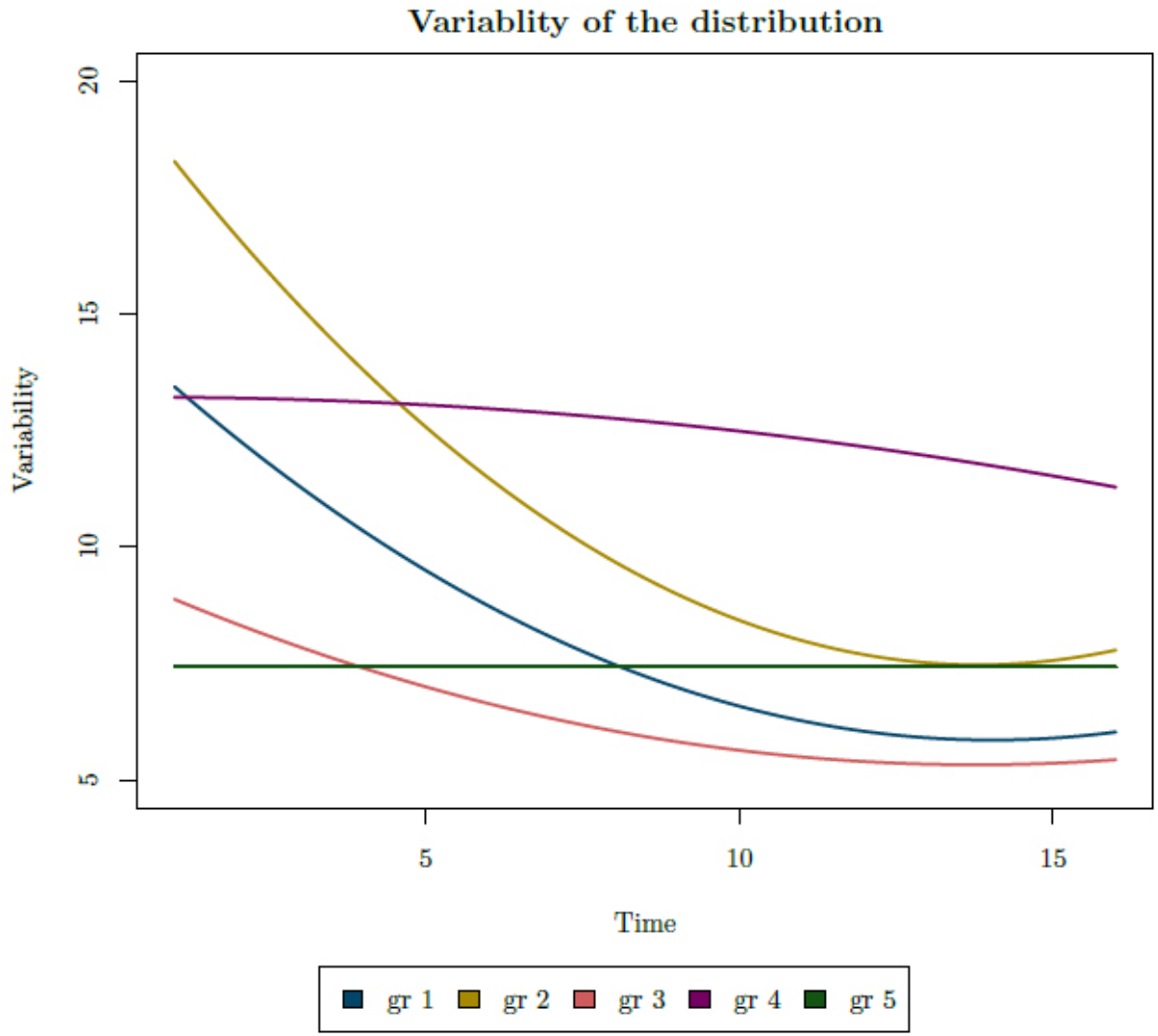
Number of groups	AIC	BIC	Prob
2	29851.99	14902.64	0.00000
3	30341.00	15142.28	0.00000
3	29945.96	14936.64	0.00000
3	30777.14	15352.23	0.00000
4	30839.69	15370.52	0.00000
4	31192.78	15547.06	0.00001
5	31241.46	15558.41	0.99999

Table 1 – *Model selection criteria*

Typical trajectories



Variability of the distribution in the different groups



Explanation of the groups

Table 3 – *Means and standard deviations of the descriptive variables for each group*

Variables	Group 1 mean (sd)	Group 2 mean (sd)	Group 3 mean (sd)	Group 4 mean (sd)	Group 5 mean (sd)
population size	19.97 (31.3)	76.41 (235.85)	23.38 (61.44)	48.49 (184.5)	39.64 (77.13)
median age	33.67 (7.76)	28.24 (6.88)	41.67 (5.29)	23.49 (7.51)	32.13 (2.19)
population density	122.67 (220.76)	207.31 (242.47)	1148.75 (3729.09)	126.64 (153.86)	286.19 (524.35)
aged 65 and older	10.54 (6.22)	6.75 (4.69)	17.08 (5.38)	4.83 (3.3)	5.27 (3.7)
aged 70 and older	6.87 (4.25)	4.16 (3.21)	10.76 (3.43)	2.81 (2.13)	3.23 (2.34)
gdp per capita	20583.62 (14511.5)	10267.22 (8325.13)	50138.17 (41050.34)	7655.52 (12468.09)	40673.95 (38570.57)
extreme poverty	3.63 (6.68)	12.96 (16.69)	0.81 (0.67)	31.28 (24.1)	2.73 (1.24)
cardiovascular death rate	259.68 (110.33)	285.53 (113.24)	172.99 (108)	306.11 (121.94)	161.7 (56.6)
diabetes prevalence	8.35 (3.3)	7.75 (3.63)	6.93 (2.84)	7.72 (5.59)	10.95 (4.08)
handwashing facilities	72.84 (22.95)	55.53 (28.91)	95.88 (2.6)	32.5 (27)	96.6 (1.13)
hospital beds	3.63 (2.18)	2.47 (2.5)	4.45 (2.67)	1.87 (2.1)	1.79 (0.38)
life expectancy	75.2 (4.56)	70.73 (6.42)	80.97 (3.16)	67.18 (7.41)	77.9 (1.95)
human development index	0.79 (0.09)	0.68 (0.1)	0.9 (0.05)	0.59 (0.14)	0.8 (0.04)
Stringency index	64.75 (16.98)	64.1 (15.74)	58.01 (18.17)	49.9 (16.22)	68.72 (22.02)

World Map with the five clusters

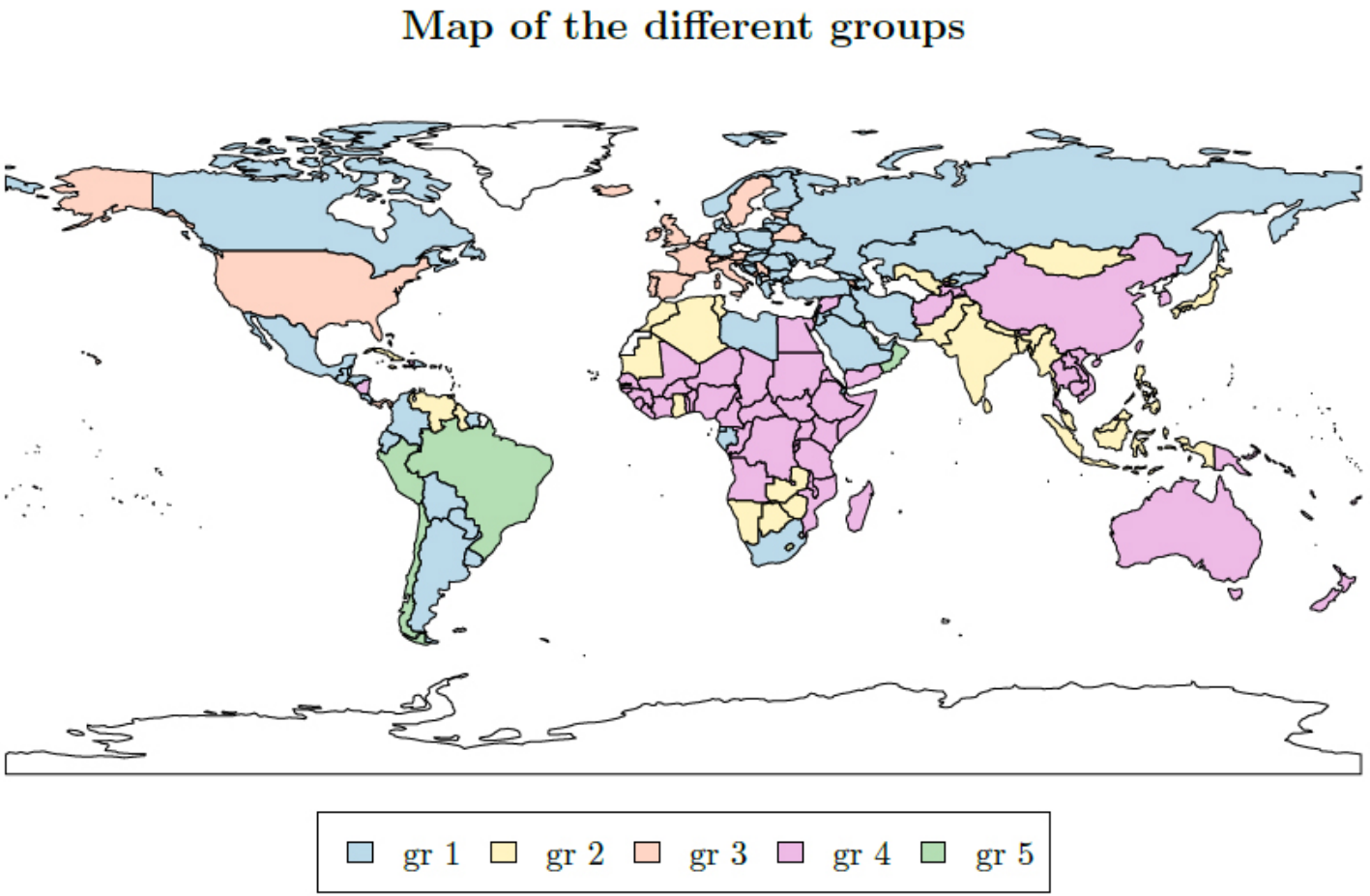


Figure 5 – *World map with the geographic distribution of the five groups*

Predictors of group membership

	Group 1			Group 2		
	Estimate	Std. Error	Prob>|T|	Estimate	Std. Error	Prob>|T|
intercept	-16.812	4.681	0	-4.805	3.422	0.16
median age	0.193	0.086	**0.024**	0.172	0.101	**0.088**
population density	-0.003	0.002	0.093	0.000	0.001	0.869
aged 65 older	-0.021	0.132	0.871	-0.060	0.126	0.631
life expectancy	0.073	0.080	0.364	-0.073	0.071	0.304
mean of stringency	0.112	0.023	**0**	0.092	0.023	**0**

	Group 3			Group 5		
	Estimate	Std. Error	Prob>|T|	Estimate	Std. Error	Prob>|T|
intercept	-67.733	19.400	0	-73.689	23.469	0.002
median age	0.129	0.158	0.412	0.418	0.205	**0.041**
population density	0.000	0.001	0.784	0.000	0.001	0.926
aged 65 older	0.109	0.178	0.542	-0.640	0.206	**0.002**
life expectancy	0.646	0.223	**0.004**	0.646	0.283	**0.023**
mean of stringency	0.185	0.054	**0.001**	0.228	0.075	**0.002**

Table 4 – *Predictors of group membership.*

Distribution of the median age

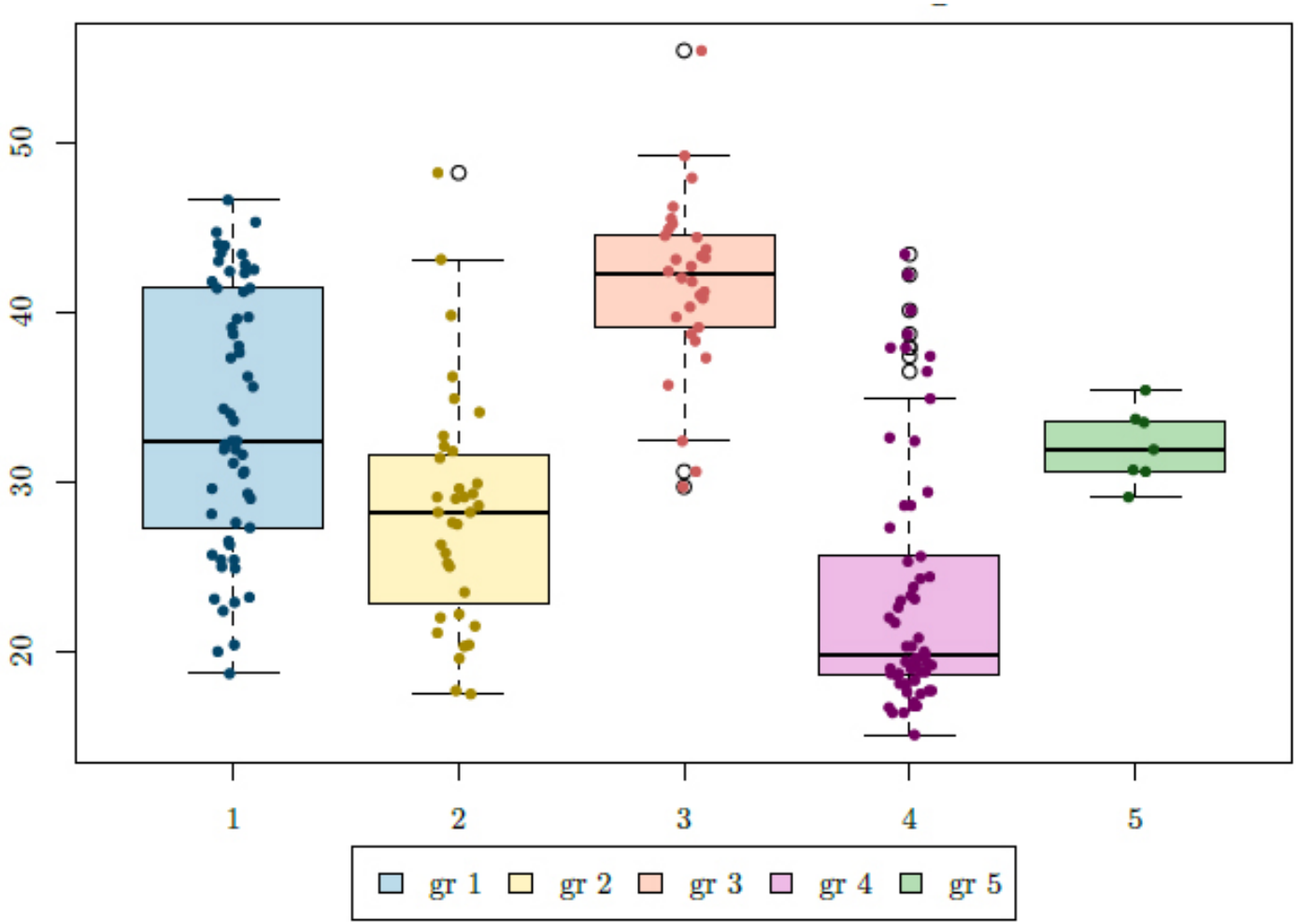


Figure 12 – *Boxplots of the median age for all 5 groups.*

Stringency index as time dependent covariate

Param.	sd	Test
Beta 1		
-5.843	0.026	0.000
-0.120	0.024	0.000
0.029	0.004	0.000
-0.001	0.000	0.000
Beta 2		
-5.927	0.003	0.000
-0.014	0.004	0.000
0.005	0.001	0.000
0.000	0.000	0.001
Beta 3		
-5.602	0.117	0.000
-0.421	0.070	0.000
0.076	0.009	0.000
-0.003	0.000	0.000
Beta 4		
-5.972	0.012	0.000
0.012	0.005	0.018
-0.001	0.001	0.043
0.000	0.000	0.027
Beta 5		
-7.304	0.366	0.000
0.701	0.147	0.000
-0.078	0.017	0.000
0.003	0.001	0.000

Param.	sd	Test
Phi 1		
14.337	0.317	0.000
-1.164	0.076	0.000
0.040	0.004	0.000
Phi 2		
19.866	0.570	0.000
-1.710	0.125	0.000
0.061	0.006	0.000
Phi 3		
9.624	0.369	0.000
-0.521	0.097	0.000
0.016	0.005	0.003
Phi 4		
12.887	0.372	0.000
0.148	0.085	0.082
-0.015	0.004	0.000
Phi 5		
7.384	0.137	0.000

Param.	sd	Test
Delta 1		
0.001	0.000	0.001
Delta 2		
0.000	0.000	0.955
Delta 3		
0.010	0.001	0.000
Delta 4		
0.000	0.000	0.000
Delta 5		
0.004	0.001	0.004

Param.	sd	Test
Prob. 1		
0.328	0.039	0.00
Prob. 2		
0.175	0.030	0.00
Prob. 3		
0.156	0.030	0.00
Prob. 4		
0.301	0.035	0.00
Prob. 5		
0.040	0.016	0.01

Table 5 – *parameters of the final model with time dependent covariates.*

Bibliography

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