Mathematics of the Vichten Mosaic

Abstract

We explore mathematically the Vichten Roman Mosaic. We have invented a Sangaku out of the central part of the mosaic, which is a tessellation involving octagons, half-squares, and symmetric pentagons. We then investigate the geometric figures and the braids in the mosaic. Finally, we consider the “whirls” pattern and the related optical illusion. The activities only require basic school geometry and show a nice interplay of mathematics and art.

The country of Luxembourg was part of the Roman Empire. In 1995, a significant archaeological discovery was made in the town of Vichten, namely a 60 square meter floor mosaic that adorned the reception area of a large Gallo-Roman villa. Today, the *Vichten mosaic* is on display at the MNAHA Museum in Luxembourg City.

Figure 1: The central part of the Vichten Mosaic.

We ignore the artistic decorations and focus on the mosaic structure. We will analyze the mosaic in a mathematically way, by investigating the various geometric figures. The central part of the Vichten mosaic is the tessellation of a square. Such tessellation can be considered to be a *Sangaku*. Sangakus (in Japanese, this means “mathematical tablet”) are special kind of geometry exercises that are often presented as pictures without text.

Inside and outside the central square of the Vichten mosaic, there is an interesting pattern with braids. Additionally, above and below the central square, we find two stripes featuring a fascinating "whirl" motif (there is even an optical illusion). Our investigation will also lead us to study octagons, eight-pointed stars, and symmetric pentagons.

One giant Roman Sangaku

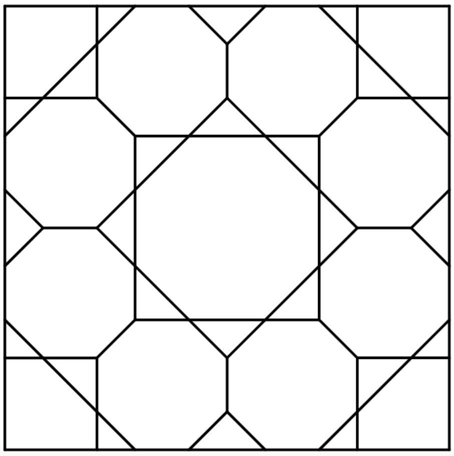

Figure 2 : The tessellation of the central square of the Vichten mosaic.

Let us examine the central square of the Vichten mosaic, see Figure 2. The unspoken convention in Sangakus is the following: “what looks regular, is regular”. In other words, when faced with geometric shapes, one does not encounter an almost regular octagon or an angle of 91°. Moreover, it is customary that one may determine the ratio between lengths or areas of the given geometric forms without any measurement.

- ***Describe the geometrical shapes in the central square of the Vichten mosaic.***
  Considering only the disjoint pieces of the tessellation of the central square frame, we can identify several convex polygons, namely: regular octagons (that appear in two different sizes), half-square triangles (in two different sizes) and a symmetric pentagon that is “a square with a broken corner”.
  We also mention the eight-pointed star in the middle of the square, and the four squares at the corner of the square frame.
- ***Assign sizes to the geometrical shapes of the tessellation (*****S*****=small,*** **M*****=medium,*** **L*****=Large,*** **XL*****=Extra Large).***

The square frame is as large as possible, so it is XL. Its inscribed octagon is thus also XL. The second largest octagon at the center is of size L. Moreover, there are eight octagons of size M surrounding it.

The two central squares are of size L because they circumscribe the octagon of size L. There is no square of size M (because such a square circumscribes the octagon of size M). The four squares located at the corners of the square frame have size S.

The four triangles at the corner of the square frame are of size XL (because they relate the octagon and the square of size XL). Similarly, the eight triangles forming the eight-pointed star are of size L. The triangles positioned at the middle of the sides of the square frame are also of size L, while the remaining eight triangles are of size M.

- ***Describe the ratio between figures of different sizes.***

  Answer: Using the notation S,M,L,XL for the side of a square with the corresponding size we are going to prove $\frac{XL}{L} = 1 + \sqrt{2}$ and $\frac{L}{M} = \frac{M}{S} = \sqrt{2}$.

  Using these ratios we can determine all ratios between S,M,L,XL, and in particular we have $\frac{L}{S} = 2$.

  Notice that to compare the areas instead we only need to take the square of the ratios above, namely $\frac{XL^2}{L^2} = 3 + 2\sqrt{2}$ and $\frac{L^2}{M^2} = \frac{M^2}{S^2} = 2$.

  Explanation: We have $\frac{L}{M} = \sqrt{2}$ because we can compare octagons of these sizes by looking at the triangle (half a square) between them.

  The ratio $\frac{XL}{L} = 1 + \sqrt{2}$ can then be deduced from the equality $XL = L + 2M$, which holds because the height of the central L-sized octagon and those of two M-sized octagons cover the XL-sized square side (recall that the height of the regular octagon equals the side of the circumscribed square).

  The ratio $\frac{M}{S} = \sqrt{2}$ can then be deduced from $\frac{L}{S} = 2$. The latter ratio can be obtained by combining the equalities $XL = 2M + 2S$ and $XL = 2M + L$. These equalities amount respectively to decomposing the side of the XL-sized square as two sides of M-sized squares (look at the M-sized octagons) together with two sides of S-sized squares and one side of an L-sized square respectively.

Squares with broken corners

The regular octagon has a circumscribed square, and hence it can be seen as "a square with four broken corners", see Figure 4. There are other octagons that show the same feature, see for example Figure 3: there we draw the “almost regular octagon” that has vertices on a 3x3 square grid (thus it is very easy to draw on squared paper). The almost regular octagon has four symmetry axes, and it has 90° rotational symmetry; there are two side lengths, whose ratio is $\sqrt{2}$; the shorter and longer sides alternate; the height is three times the shorter side length.

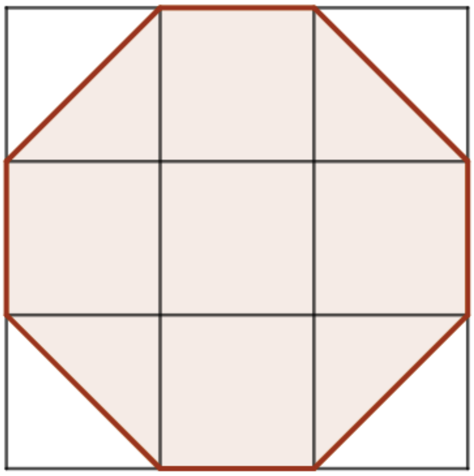

Figure 5: Octagon at the 3x3 square grid.

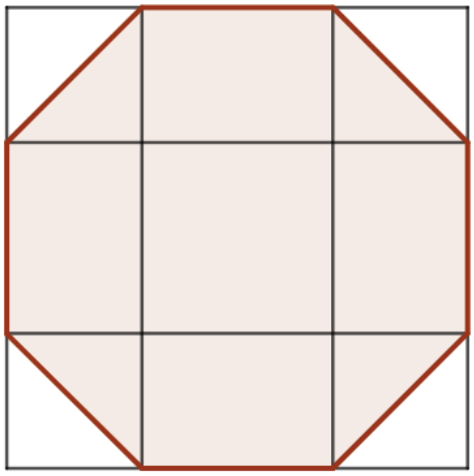

Figure 4: The regular octagon.

Now let us focus on the tessellation piece at the corner of the square frame, namely the pentagon which is “a square with a broken corner”. We will also consider the analogous pentagon stemming from the 2x2 square grid.

- ***Describe the pentagon stemming from the 2x2 grid and the pentagon in the Vichten mosaic.***

The two pentagons are irregular, but they are convex and symmetric; they both have three 90° angles and two 135° angles, but they are not similar, see Figures 5 and 6. The former has two side lengths with ratio $\sqrt{2}$, while the latter has three side lengths that are proportional to $1, \sqrt{2}, \ 1+\sqrt{2}$.

By cutting a symmetric corner out from a square we obtain an infinite family of symmetric pentagons with the same angles and where the side lengths are proportional to t, $1-t$, 1 (the parameter t being strictly between 0 and 1).

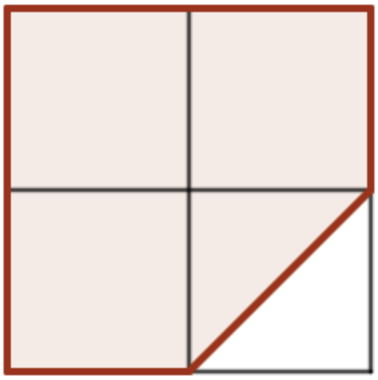

Figure 5: Pentagon at the 2x2 square grid.

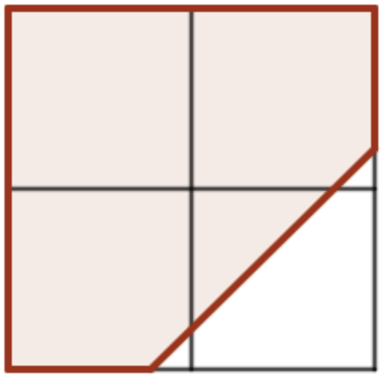

Figure 6: Pentagon from the Vichten mosaic.

Eight-pointed stars

In the center of the square frame of the Vichten mosaic, as shown in Figure 2, there is an eight-pointed star. There is a mathematical classification of regular stars, and there are two regular eight-pointed stars. The one in the mosaic is called 8/2 star (Achtort, in German), the other is the 8/3 star (octagram). The 8/2-star is formed by two regularly interlaced squares.

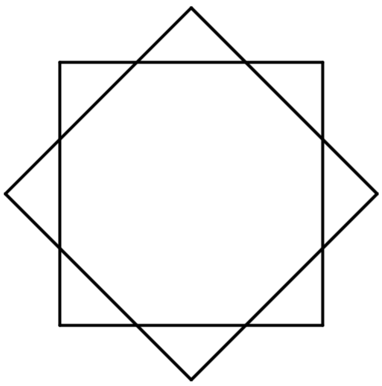

Figure 6: The 8/2 star.

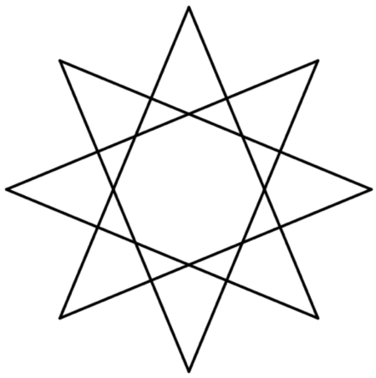

Figure 7: The 8/3 star.

- ***Describe the construction with ruler and compass of the two regular eight-pointed stars.***

  We can exploit the classical construction of the regular octagon because the two stars consist of selected diagonals of the regular octagon. The 8/2 stars is given by the eight smallest diagonals (among vertices with 2 sides in between), while the 8/3 stars is made with the eight intermediate diagonals (among vertices with 3 sides in between).

- ***Draw an 8/2-star starting from an 8/3-star and conversely.***

The 8/3 star shape consists of 8 lines: the segment in the interior of the star form an 8/2 star.

Conversely, prolonge the eight lines forming an 8/2 stars. Doing this creates eight additional intersection points which are the vertices of an 8/3 star.

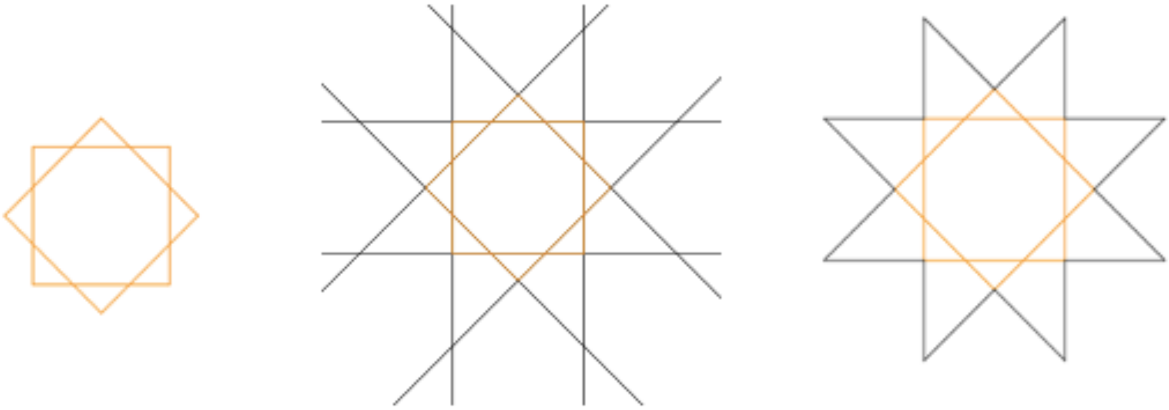

Figure 8: Constructing the 8/3 star out of the 8/2 star.

Braid patterns

The braid patterns in the Vichten mosaic serve as a visual element to separate the different geometrical forms. The braid patterns give the tessellation of the central square of the mosaic, but also extend beyond the central square.

The braids are visually made by ropes that do not end (closed pieces of rope). Along line segments we see classical braids consisting of 3 ropes. Interestingly, such patterns merge at corners: two braids or even three braids merge.

- ***Invent a strategy to count the number of distinct pieces of ropes used in the braids (if you are given an image of the mosaic).***

  We begin with one rope, say in the upper-left corner, and follow its path until we reach its starting point. Visually, we may color the rope to mark that we have already consider it. We then select another rope and repeat the process until the entire pattern is covered.

Notice that a single rope is needed to construct the braid pattern which encircles the central octagon (the rope circles the octagon three times), see Figure 9.

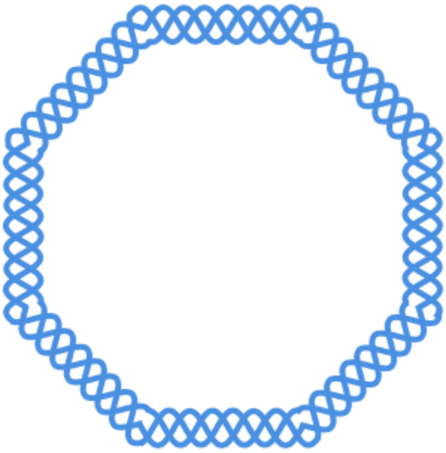

Figure 9: Braid pattern in the center of the mosaic.

It is not always easy to follow ropes in the braid patterns, mistakes can arise. What we found for the central square (excluding the octagonal braid pattern mentioned above) was braid patterns made by 5 ropes in a non-symmetric way, see Figure 10.

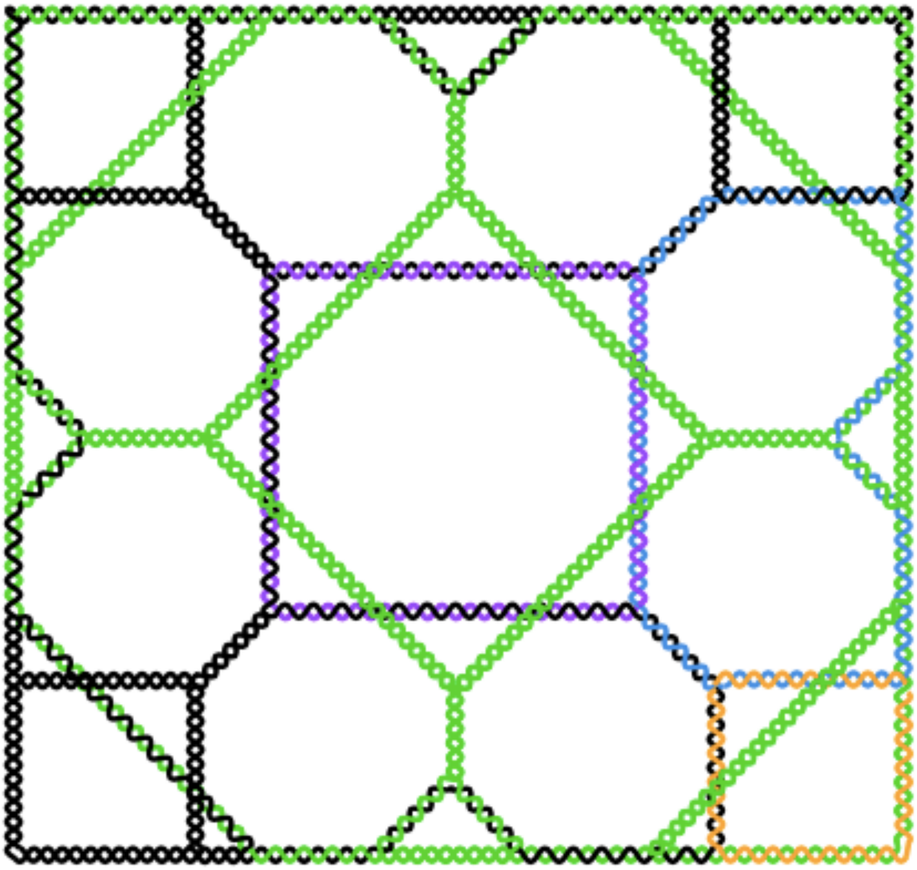

Figure 10: Braid patterns from the central square of the Vichten mosaic.

The whirl pattern

Beyond the central part of the mosaic there are two rectangular stripes that contain several repetitions of a "whirl" shape, see Figure 11.

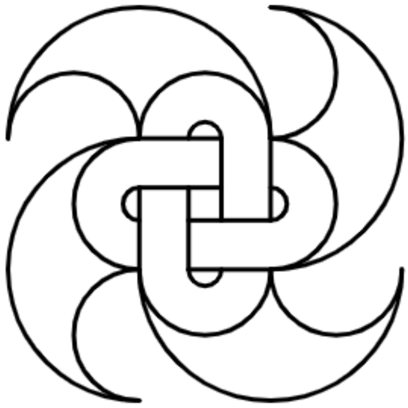

Figure 11: Whirl from the Vichten mosaic

- ***Describe the whirl shape.***

  The whirl shape is the union of four congruent figures that differ by rotations of 90°. Each figure, in turn, consists of two parts.

  The first part is a large half-circle without two small half-circles (the ratio between the diameter lengths being 2). The diameters of the two small half-circles cover the diameter of the large half-circle.

  The second part is a half-annulus whose larger arc is the arc of one of the small half-circles. Then there is a rectangle. Notice that some of the segments at the border of the given figures are not drawn in the picture, see Figure 11.

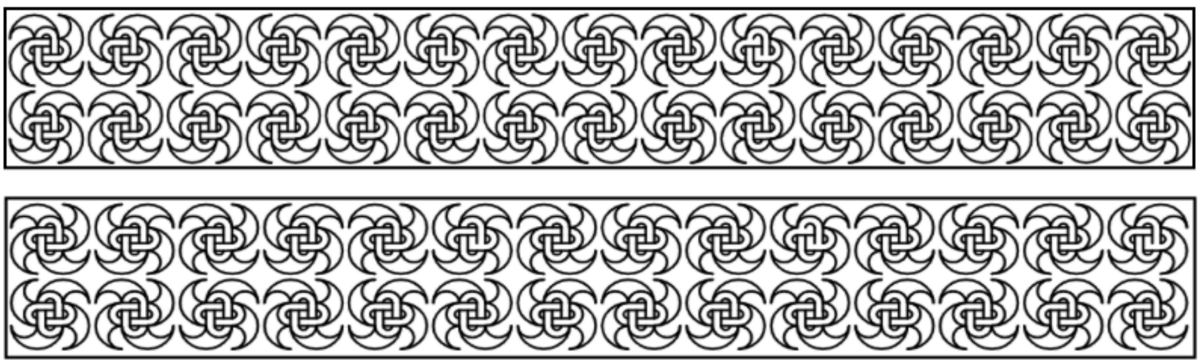

Figure 12: The two stripes with whirls in the Vichten mosaic.

The following exercise concerns Figure 12.

- ***Describe the whirls inside the two stripes (and an optical illusion concerning them).***

The two rectangular stripes in the Vichten mosaic (see Figure 13) have the same area. The different number of whirls contained in the two rectangles is noticeable by comparing the orientation of the first and the last whirls in a row. Indeed, the orientation of the whirls alternate so in the stripe above (respectively, below) there is an odd (respectively, even) number of whirls.
One may count 15 and 14 whirls respectively in each row. The optical inclusion arises from the fact that the whirls in the stripe above are slightly smaller.

Conclusions

We have studied some features of the Vichten mosaic. Looking at the artistic details would provide for more mathematical exercises because further geometrical shapes appear in the decoration. Moreover, there are exercises stemming from constructing the given geometrical figures with small mosaic tiles.
In general, Roman mosaics have a very rich geometry and each mosaic may provide with a unique set of mathematical challenges.
A classroom exploration of Roman mosaics is an interdisciplinary subject because it relates history, art, and mathematics. The mosaics provide visually engaging examples of geometric patterns and tessellations, that are beautiful because of their symmetry but also because of the richness of their structure.
The online programs [Polypad] and [GeoGebra] allow for playing with the geometrical shapes present in the mosaic.
The mathematical exploration that we provide only relies on basic school geometry, and it can be adapted to primary school (shape and size recognition) and to secondary school (computation of ratios). Each educator may focus on the aspects that they find most fascinating.

References

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