Mathematics in the Vichten Roman Mosaic

Abstract

We explore the geometry of the renowned Vichten Roman Mosaic. The central section of the mosaic features a tessellation composed of octagons, half-squares, and symmetrical pentagons. We present a Sangaku-style exercise to investigate these polygons. Additionally, we examine the mosaic's "whirl patterns" and the optical illusion they create. Our activities offer a mathematical exploration that requires only basic school-level mathematics, highlighting the close connection between mathematics and art.

Figure 1: The central section of the Vichten Mosaic.

Luxembourg was once part of the Roman Empire. In 1995, a significant archaeological discovery was made in the town of Vichten: a 60-square-meter floor mosaic that once decorated the reception area of a large Gallo-Roman villa. Today, the Vichten Mosaic is displayed at the MNAHA Museum in Luxembourg City.

Our focus is on the mosaic's geometric structure, analyzing it through a mathematical lens. The central section of the Vichten Mosaic features a tessellation of a large square, which can be viewed as a *Sangaku*. Sangaku (a Japanese term meaning "mathematical tablet") are special types of geometry exercises typically presented as images without accompanying text. Inside the central square, there is also an intricate braid pattern. Outside the central square, two bands display a captivating "whirl pattern" that also creates an optical illusion. Our investigation will delve into the study of octagons, eight-pointed stars, and symmetrical pentagons.

A Roman Sangaku

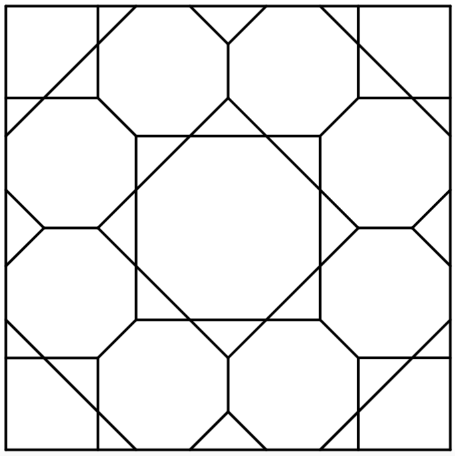

Figure 2 : The tessellation within the central square of the Vichten Mosaic.

Let us examine the central square of the Vichten Mosaic (see Figure 2). Our mathematical exploration aims to describe what we observe. What geometric figures are present? How do they relate to one another? Can we compare their sizes based on the tessellation they form, without using any measurements? In a way, the visual patterns naturally prompt geometric questions, leading to a so-called Sangaku. We follow an implicit convention in Sangaku: "What appears regular is regular." In other words, a shape that looks like a regular octagon will be a regular octagon, and an angle that appears to be 90° will indeed be a right angle.

We first describe the geometrical shapes within the central square of the Vichten Mosaic. Consider the distinct pieces of the tessellation within the square frame. Several convex polygons can be identified, including regular octagons—one in the very center and eight smaller ones surrounding it. Additionally, there are half-square triangles in two different sizes bordering the smaller octagons, and at the square frame's corners, symmetric pentagons that are "squares with a broken corner." We also observe an eight-pointed star at the center of the square, an octagon inscribed within the square frame, and four squares positioned at the corners of the square frame.

Now we can assign sizes to the geometric shapes in the tessellation, using the following designations: S for Small, M for Medium, L for Large, and XL for Extra Large. The square frame is as large as possible, so it is classified as size XL. Consequently, its inscribed octagon and also the four triangles at its corners are of size XL. The second largest octagon, located at the center, is classified as size L. Thus, the two central squares that form the eight-pointed star are classified as size L. The same holds for the eight triangles in the eight-pointed star, and also for the four triangles situated in the middle of the sides of the square frame. The eight smaller octagons can be designated as size M, and the same holds for the eight smaller triangles next to them. Finally, the four squares positioned at the corners of the square frame are classified as size S because they are smaller compared to the octagon of size M.

Without taking any measurements, we can relate the sizes of similar figures. Let $\overline{L}$ denote the side length of the square of size L, and $\overline{S}$ denote the side length of the square of size S. We will see that $\overline{L} = 2\overline{S}$. (To relate the areas of these squares, we only need to square the ratio of their side lengths, which gives us an area ratio of 4.) We denote the side length of the square frame as $\overline{XL}$. Finally, we define $\overline{M}$ as the side length of the square that circumscribes the regular octagon of size M (even though this square does not actually exist in the tessellation). To determine all the ratios between $\overline{S}$, $\overline{M}$, $\overline{L}$, and $\overline{XL}$, it suffices to find the ratios between $\overline{L}$

and the other quantities. First, we have $\overline{L} = \overline{M}\sqrt{2}$ because we can compare the octagons of sizes L and M by examining the triangle (half a square) between them. Next, note that $\overline{XL} = 2\overline{M} + \overline{L}$ because the side of the square frame can be decomposed into two sides of squares of size M (consider the height of the octagons of size M at the top and bottom) along with the side of the vertical central square. We can deduce that $\overline{XL} = (\sqrt{2} + 1)\overline{L}$. Finally, we can prove that $\overline{L} = 2\overline{S}$ by combining the equality $\overline{XL} = 2\overline{M} + \overline{L}$ with the new equality $\overline{XL} = 2\overline{M} + 2\overline{S}$, which can be derived by decomposing the side of the square frame into two sides of squares of size M (from the octagons of size M on the sides) plus two sides of squares of size S (at the corners of the square frame).

Squares with broken corners

The regular octagon has a circumscribed square, which allows it to be viewed as "a square with four broken corners" (see Figure 4). Other octagons exhibit the same characteristic; for example, in Figure 3, we illustrate the "almost regular octagon" with vertices on a 3x3 square grid, which is easy to draw on graph paper. This almost regular octagon features four lines of symmetry and possesses 90° rotational symmetry. There are two distinct side lengths, with a ratio of $\sqrt{2}$; the shorter and longer sides alternate, and the height is three times the length of the shorter side.

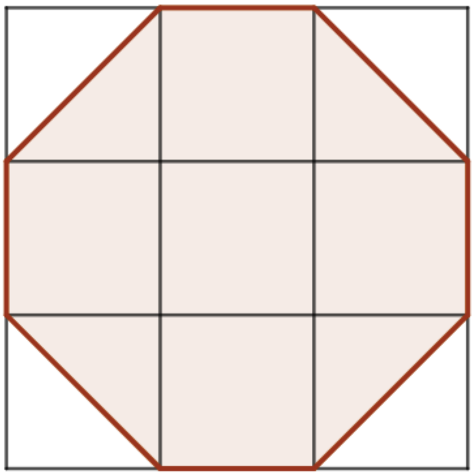

Figure 3: Octagon at the 3x3 square.

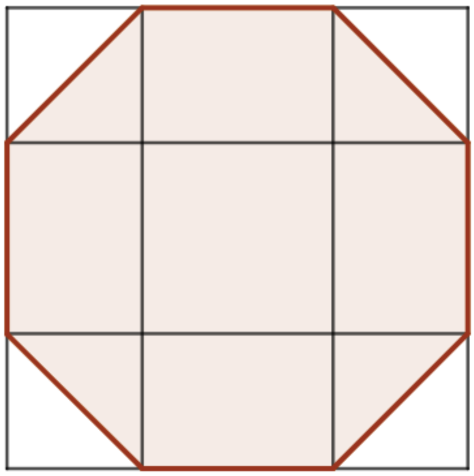

Figure 4: The regular octagon.

We now turn our attention to the tessellation piece of the Vichten Mosaic located at the corner of the square frame, specifically the pentagon that can be described as "a square with a broken corner." We will also consider the analogous pentagon derived from a 2x2 square grid. While both pentagons are irregular, they are convex and symmetric. Each has three 90° angles and two 135° angles, but they are not similar (see Figures 5 and 6). The first pentagon has side lengths that are proportional to $1, \sqrt{2}, 2$, while the second has side lengths that are proportional to $1, 2, 1 + \sqrt{2}$ (notably, this pentagon is a quarter of a regular octagon). In fact, by cutting a symmetric corner off a square, we can create an infinite family of symmetric pentagons with the same angles, where the side lengths are proportional to t, $(1 - t)\sqrt{2}$, 1,with the parameter t strictly between 0 and 1.

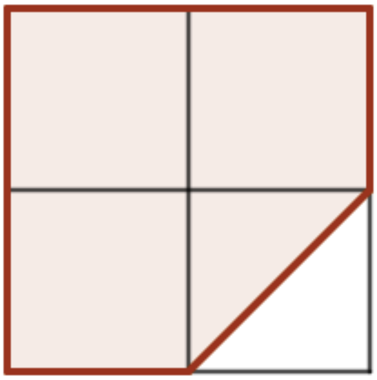

Figure 5: Pentagon within the 2x2 square grid.

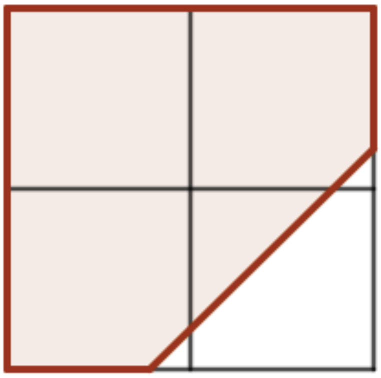

Figure 6: Pentagon from the Vichten Mosaic.

Eight-Pointed Stars

Mathematically, there are various types of regular stars (see [Stars]), which are represented by the Schläfli symbol—a notation that consists of two numbers. For example, the stars with the symbols $8/2$ and $8/3$ are both eight-pointed stars, but they differ in structure (see Figures 7 and 8, respectively). The $8/2$ star, known as *Achtort* in German, is created by superimposing two squares. In contrast, the $8/3$ star, also called an octagram, appears more intertwined. Both stars are formed from selected diagonals of the regular octagon. The $8/2$ star consists of the eight smallest diagonals, connecting vertices that have two sides in between. Meanwhile, the $8/3$ star is constructed using the eight intermediate diagonals, connecting vertices that have three sides in between. This description also provides a method for drawing these stars: start by sketching the regular octagon in which they are inscribed.

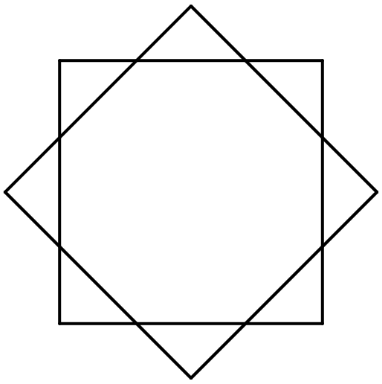

Figure 7: The $8/2$ star.

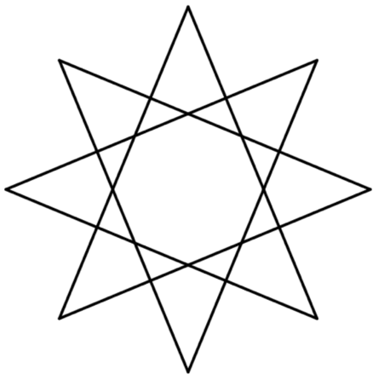

Figure 8: The $8/3$ star.

In fact, we can derive an $8/2$ star from an $8/3$ star and vice versa. The shape of the $8/3$ star consists of eight lines, and the segments in the interior of the star form an $8/2$ star. Conversely, we can extend the eight lines that make up an $8/2$ star. This extension creates eight additional intersection points, which serve as the vertices of an $8/3$ star (see Figure 9).

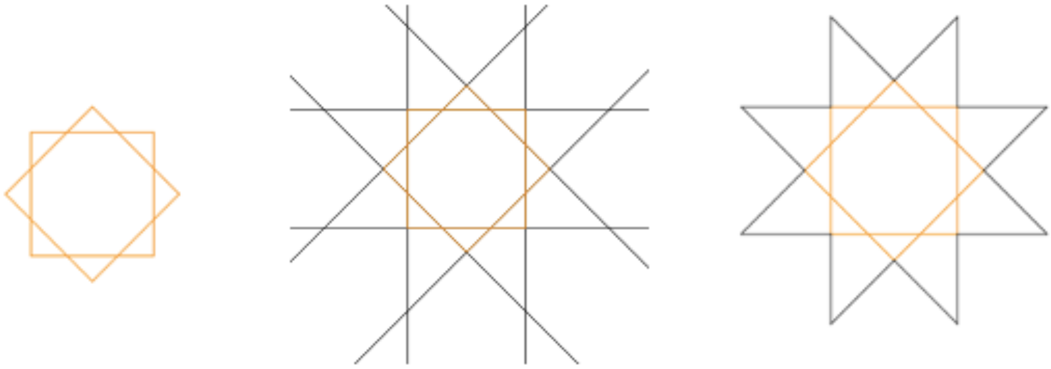

Figure 9: Constructing the 8/3 star out of the 8/2 star.

Braid Patterns

In the central part of the Vichten Mosaic, various elements of the geometrical tessellation are separated by braids. These braids consist of endless ropes (closed loops). Notably, the central braid encircling the octagon is composed of just one rope: although it appears as three separate ropes, it is actually a single rope that wraps around the octagon three times (see Figure 10).

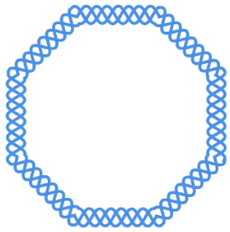

Figure 10: Braid pattern at the center of the mosaic.

The additional braids are separate from the central braid and (locally) consist of two ropes. Interestingly, these braids converge at various corners, where two or even three braids come together. Excluding the central braid, there appear to be five ropes arranged in a non-symmetrical manner (see Figure 11).

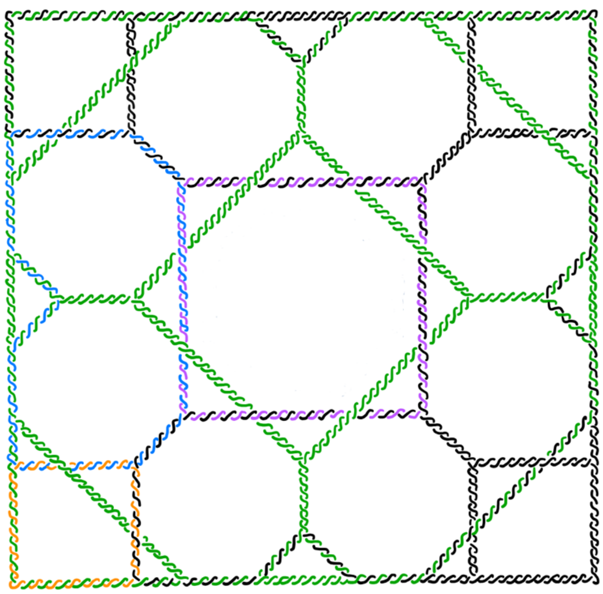

Figure 11: Braid patterns from the central square of the Vichten Mosaic.

The Whirl Pattern

Beyond the central part of the Vichten Mosaic, there are two rectangular stripes that feature several repetitions of a "whirl" shape (see Figure 12). The whirl shape is formed by the union of four congruent figures, each differing by a rotation of 90°. Each figure consists of two components. The first component is a large half-circle with two small half-circles removed, where the ratio of the diameters is 2. The diameters of the two small half-circles align with the diameter of the large half-circle. The second component is a half-annulus, with its larger arc corresponding to the arc of one of the small half-circles. There is also a rectangle. Please note that some of the segments along the borders of these figures are not drawn in the image (see Figure 12).

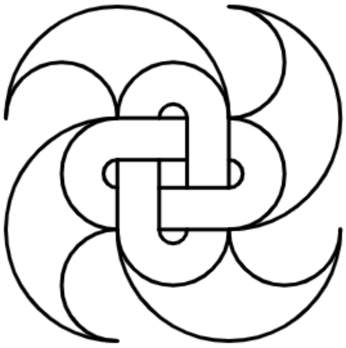

Figure 12: Whirl shape.

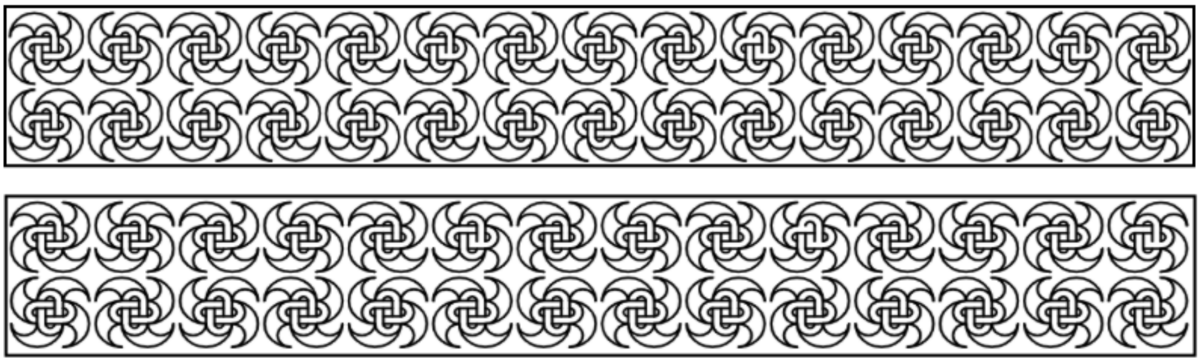

Figure 13: The two stripes featuring whirl shapes.

The two rectangular stripes in the Vichten Mosaic (see Figure 13) have the same area. However, the differing number of whirls within the two rectangles is evident when comparing the orientation of the first and last whirls in a row. Specifically, the orientation of the whirls alternates, resulting in an odd number of whirls in the upper stripe and an even number in the lower stripe. In each row, there are 15 whirls in the upper stripe and 14 in the lower stripe. The optical illusion is created by the fact that the whirls in the upper stripe are slightly smaller.

Conclusions

Roman mosaics exhibit a rich geometric diversity, providing visually engaging examples of geometric patterns and tessellations, captivating us with their symmetry and the complexity of their structures. Exploring Roman mosaics in the classroom serves as an interdisciplinary subject that bridges history, art, and mathematics. The mathematical exploration we present is based entirely on school-level mathematics and can be adapted for primary education, focusing on shape and size recognition, as well as for secondary education, which includes ratio calculations. Each educator can highlight the aspects they find most engaging. Our exploration is freely available on a dedicated website [Vichten]. Additionally, online programs like [Polypad] and [GeoGebra] enable users to interact with the geometric shapes found in the mosaic and to construct geometric figures using small mosaic tiles. Pupils can, of course, recreate the geometric shapes from the Vichten Mosaic using materials like paper, cardboard, or wood, and they also have the option of 3D-printing them.

Acknowledgements

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