Let's Play UNO! Training Mathematical Thinking With An Inclusive Card Game

The card game UNO celebrated 50 years in 2021 and it is very popular. The game rules can be found online, however we also recall them below. For simplicity, we suppose that there are only two players. Moreover, we remove the so-called *Wild Cards*. Now all cards have a color, that is either blue, green, red, or yellow. For each color, there is one card with the symbol 0, there are two cards with the symbols 1 to 9, and there are two cards whose symbols we call A (respectively, B and C). The card deck then has precisely 100 cards:

Blue: 0 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8 9 9 A A B B C C
Green: 0 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8 9 9 A A B B C C
Red: 0 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8 9 9 A A B B C C
Yellow: 0 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8 9 9 A A B B C C

The two players begin with seven cards in their hand. There is a card deck, called *draw pile*, from which players can draw cards. There is also a card deck, called the *discard pile,* the game beginning with one card in this pile. The card on top of the discard pile has a *color* and a *symbol* (say, there is a blue 8). The next player can then discard any card with the same color or the same symbol (in our case, any blue card or any card with value 8). For simplicity, we forbid to discard more than one card (for example, one cannot discard the two red cards with value 8 in the same turn).
If a player has no cards that can be discarded, they must draw one card, which can possibly be immediately discarded. The aim of the game is being the first to empty their hand.

The UNO cards are suitable learning material to train number recognition or elementary mathematical concepts because they display digits. We focus on more advanced mathematical concepts that build on this game.

Probability exercises.

Our card deck has precisely 100 cards, thus it is easy to count favorable cards with respect to the full deck and express probabilities in percentage. For example, the probability of drawing a yellow card from the full deck is 25%.
As an example of probability exercise, we address a fact that is counterintuitive because there are four, rather than eight, cards with value 0:

The 0 cards. *Is it twice as hard to get rid of a 0 card rather than a 1 card?*
(1) Consider some 0 card (respectively, 1 card). Count the remaining cards in the full deck after which the given card can be played.
(2) If a card is chosen at random from the full deck, what is the chance that it is possible to play a Yellow 0 after it? Or a Yellow 1?

Solution. (1) There are 24 cards of the correct color, and there are 3 (respectively, 6) cards with the correct symbol from a different color. This gives 27 (respectively, 30) favorable cards.
(2) If the chosen card has the same color of your card (25% chance), then you can play. Else (75% chance), all depends on whether the color card is a 0 (1/25 chance) or a 1 (2/25

chance). Summing up the two contributions ("probability of a scenario × probability inside the given scenario") we get
25% + (75% × 1/25) = 28% and 25% + (75% × 2/25) = 31% respectively.

Game theory exercises.

Exercises may present game scenarios and ask to compare strategies. We propose a question concerning the UNO rule that it is not compulsory to discard a card (one may draw a card instead). Why should one restrain from discarding, as the purpose of the game is to discard all cards? There are situations in which not discarding is a true necessity:

Not Discarding. *Can you think of a game scenario in which discarding a card has as a consequence losing the game, so that the best strategy is drawing a card instead?*

Solution. Suppose that your hand consists of two yellow cards, and that the other player's hand consists of just one card. Moreover, you know that this card is yellow because the other player could not end the game discarding that card when the current game color was blue, green, or red. Thus, if you discard any of your two cards, the other player wins hence it is necessary to draw a card instead.

An example of binary relation.

If binary relations and their properties have already been introduced in class, UNO naturally provides an example of such relation.

The UNO binary relation. *We say that a UNO card C is connected to a UNO card C′ if C can be played right after the card C′. Which cards are connected to which cards? Is this binary relation on the set of UNO cards reflexive, or symmetric, or transitive?*

Solution. The UNO cards (from our deck without the Wild Cards) are connected to the cards of the same color or with the same symbol. The binary relation is then symmetric. As any card is connected to itself, the binary relation is reflexive (the zero cards are unique but in principle one could play a Red 0 after a Red 0). The game is spiced up by the fact that the binary relation is not transitive: for example, Red 2 can be played after Red 1, and Blue 2 can be played after Red 2, however Blue 2 cannot be played directly after Red 1:

$$\text{Red } 1 \longrightarrow \text{Red } 2 \longrightarrow \text{Blue } 2$$

We did not consider the *Wild Cards,* however they (at least, the non-customizable ones) are connected to all cards: considering them as well, the relation on the cards is not symmetric because a Wild Card can be played after Red 1, however the converse does not hold (if e.g. the Wild Card sets the game color to be green).

Some UNO algebra

We now analyse a UNO game with two players (again, the deck is without the Wild Cards and in each round the players have discarded at most one card). A round then consisted in either of the following actions: play one card, draw one card, draw&play one card. At the beginning of the game, each player was given 7 cards, and there was one card in the discard pile.

Algebraic relations. *Consider the following quantities:*
R *= number of rounds in the game*
P *= number of play rounds*
D *= number of draw rounds*
Dp *= number of draw&play rounds*
Rem *= number of cards remaining in the hand of the losing player*
Disc *= number of cards in the discard pile.*
Can you explain the following algebraic relations?

$$R = P + D + Dp$$
$$Disc - 1 = P + Dp$$
$$P = 14 + D - Rem.$$

Solution. The first relation is because in each round we can either play, draw, or draw&play. The second relation is because there is initially one card in the discard pile, and a player adds one card to this pile if either they play or draw&play. For the third relation, consider that the number of cards that were played (namely, P+Dp) are the initial 14 cards plus the cards that have been drawn during the game (namely, D+Dp) minus the cards remaining in the hand of the losing player. Thus we get the algebraic relation $P + Dp = 14 + D + Dp - Rem$, which is equivalent to the third relation.

Further investigations

If the students are familiar with the UNO game, it is also possible to discuss game strategies with the standard card

deck, for example: *When is it best to play a Wild Draw Four Card?* (Such a card forces the other player to draw four cards.) The answer is not to be motivated by exact mathematical reasoning. Nevertheless, students could be challenged to *outline a precise strategy*, for example: when the other player has only one or two cards left; when I have other Wild Draw Four Cards in my hand and the other player has fewer cards than me.
Students should also understand the difference between common sense (i.e., reasonable choices made during the game) and mathematical results. Moreover, considerations may only be valid up to a certain extent (contrarily to what some internet websites say, claiming to reveal the very best strategy for any game).
Students should understand that it is possible to study game statistics. With them, one could (statistically) assess for example how much advantage the first player has. In case of a clear advantage or a clear disadvantage, one could introduce the following rule: if the losing player could empty their hand right after the winning player, then there is a draw.

UNO as a model of inclusive games

We argument that UNO is an extremely inclusive game and may serve as a model:

- *UNO is logistically easy to play.* The game is meant for 2 to 10 players, that play until the end of the game (and they may choose to quit the game without jeopardizing it). Players may speak different languages because they only need to call UNO! or select a color. One does not necessarily need a card table. The game can also be played with traditional packs of cards (as the card game Mau-Mau) or with the App UNO!.

- *UNO is a simple game that allows for different players' levels*. The basic game rules are extremely simple. Since luck plays a big role, unexperienced players have a chance to win. Moreover, alliances among players do not play a big role. Finally, the game outcome is not known in advance (for example, having just one card left in the hand does not ensure winning the game).
- *UNO can be adapted to meet special needs.* Speaking/ hearing impairments are not an issue for this game, and there can be special adjustments for drawing/holding/ playing cards. A fraction of color-blind people can play as the cards use very distinct colors. Moreover, there are special UNO card decks with Color ADD symbols to distinguish colors, or with the tactile writing system Braille.

Conclusions

This article is primarily intended for all teachers and educators who enjoy the card game UNO. If you find any of the suggested activities inspiring and suitable for your class, feel free to use them in whole or in part.

Beyond being playful exercises, our activities demonstrate that mathematical thinking can be applied in a variety of contexts.