TWISTING OF PROPERADS

SERGEI A. MERKULOV

Abstract. We study T. Willwacher's twisting endofunctor tw in the category of dg prop(erad)s $\mathcal{P}$ under the operad of (strongly homotopy) Lie algebras, $i : \mathcal{L}ie \to \mathcal{P}$. It is proven that if $\mathcal{P}$ is a properad under properad of Lie bialgebras $\mathcal{L}ieb$, then the associated twisted properad $\mathsf{tw}\mathcal{P}$ becomes in general a properad under quasi-Lie bialgebras (rather than under $\mathcal{L}ieb$). This result implies that the cyclic cohomology of any cyclic homotopy associative algebra has in general an induced structure of a quasi-Lie bialgebra. We show that the cohomology of the twisted properad $\mathsf{tw}\mathcal{L}ieb$ is highly non-trivial — it contains the cohomology of the so called hairy graph complex introduced and studied recently in the context of the theory of long knots and the theory of moduli spaces $\mathcal{M}_{g,n}$ of algebraic curves of arbitrary genus g with n punctures.

Using a polydifferential functor from the category of props to the category of operads, we introduce and study two new twisting endofunctors, one in the category dg prop(erad)s $\mathcal{P}$ under the minimal resolution of $\mathcal{L}ieb$, and one for the involutive version of $\mathcal{L}ieb$. We compute the cohomology of the associated deformation complexes, and discuss their applications in string topology.

Mathematics Subject Classifications (2010). 18D50, 17B56, 55P50.
Key words. Operads, properads, Lie algebras, string topology

Contents

1. Introduction

In this paper we study some new aspects of a well-known twisting endofunctor tw [W1] in a certain subcategory of the category of properads, and introduce a new one. Both of them have applications in several areas of modern research — string topology, the theory of moduli spaces of algebraic curves, the theory of cyclic strongly homotopy associative algebras etc — which we discuss below.

Let $\mathcal{P}$ be a properad under the operad $\mathcal{L}ie_d$ of (degree $d \in \mathbb{Z}$ shifted) Lie algebras, that is, one equipped with a morphism

$$i : \mathcal{L}ie_d \longrightarrow \mathcal{P}.$$

Thomas Willwacher introduced in [W1] a *twisting endofunctor*

$$\mathsf{tw} : \mathcal{P} \longrightarrow \mathsf{tw}\mathcal{P}$$

in the category of such properads; the twisted properad $\mathsf{tw}\mathcal{P}$ is obtained from $\mathcal{P}$ by adding to it a new generator degree d generator with no inputs and precisely one output encoding the defining property of a Maurer-Cartan element of a generic $\mathcal{L}ie_d$-algebra. This twisting construction originated in the formality theory of the operad of chains of little disks operad [Ko2, LV, W1] and found many other important applications including the Deligne conjecture [DW], the homotopy theory of configuration spaces [CW] and the theory of moduli spaces of algebraic curves [MW1, Me2]. The twisted properad comes equipped with a canonical morphism

$$\mathcal{L}ie_d \longrightarrow \mathsf{tw}\mathcal{P}$$

so the twisting construction is indeed an endofunctor in the category $\mathsf{PROP}_{\mathcal{L}ie_d}$ of operads under $\mathcal{L}ie_d$. It can be naturally extended [W1] to the category $\mathsf{PROP}_{\mathcal{H}olie_d}$ of properads under $\mathcal{H}olie_d$, the minimal resolution of $\mathcal{L}ie_d$. It is proven in [DW] that $\mathsf{tw}\mathcal{H}olie_d$ is quasi-isomorphic to $\mathcal{L}ie_d$; the cohomologies of twisted versions of some other classical operads under $\mathcal{L}ie_d$ have been computed in [DW, DSV].

The main purpose of this paper is to study the restriction of T. Willwacher's twisting endofunctor tw,

$$\mathsf{tw} : \mathsf{PROP}_{\mathcal{L}ieb_{c,d}} \longrightarrow \mathsf{PROP}_{\mathcal{L}ie_d}$$

to the subcategory $\mathsf{PROP}_{\mathcal{L}ieb_{c,d}} \subset \mathsf{PROP}_{\mathcal{L}ie_d}$ of properads under the properad $\mathcal{L}ieb_{c,d}$ of (degree shifted) Lie bialgebras, i.e. the ones which come equipped with a non-trivial morphism of properads,

$$i : \mathcal{L}ieb_{c,d} \longrightarrow \mathcal{P}. \tag{1}$$

and then to appropriately modify $\mathsf{tw} \rightsquigarrow \mathsf{Tw}$ such that the new functor Tw which an *endofuntor* of $\mathsf{PROP}_{\mathcal{L}ieb_{c,d}}$. Everything will work, of course, in the category $\mathsf{PROP}_{\mathcal{H}olieb_{c,d}}$ of properads under $\mathcal{H}olieb_{c,d}$, the minimal resolution of $\mathcal{L}ieb_{c,d}$.

The notion of Lie bialgebra was introduced by Vladimir Drinfeld in [D1] in the context of the theory of Yang-Baxter equations and the deformation theory of universal enveloping algebras. This notion and its involutive version have since found many applications in algebra, string topology, contact homology, theory of associators and the theory of Riemann surfaces with punctures. The properad $\mathcal{L}ieb_{c,d}$ controls Lie bialgebras with Lie bracket of degree $1-d$ and Lie-cobracket of degree $1-c$;

the properads $\mathcal{L}ieb_{c,d}$ with the same parity of $c+d\in\mathbb{Z}$ are isomorphic to each other up to degree shift so that there are essentially two different types of such properads, even and odd ones.

In general, given $\mathcal{P}\in\mathsf{PROP}_{\mathcal{L}ieb_{c,d}}$, the associated twisted properad $\mathsf{tw}\mathcal{P}$ is no more a properad under $\mathcal{L}ieb_{c,d}$ — a generic Maurer-Cartan element of the Lie bracket need not to respect the Lie cobracket. Surprisingly enough, the cohomology of any twisted properad $\mathsf{tw}\mathcal{P}$ always comes equipped with an induced from i morphism of properads,

$$i^q : q\mathcal{L}ieb_{c-1,d} \longrightarrow H^\bullet(\mathsf{tw}\mathcal{P}), \tag{2}$$

where $q\mathcal{L}ieb_{c-1,d}$ is the properad of (degree shifted) *quasi*-Lie bialgebras which have been also introduced by Vladimir Drinfeld in [D1] in the context of the theory of quantum groups. The map i^q is described explicitly in Theorem **3.4** below. In the special case when $\mathcal{P}$ is the properad of ribbon graphs $\mathcal{RG}ra_d$ introduced in [MW1], the map i^q has been found (in a slightly different but equivalent form) in [Me2]. As $\mathsf{tw}\mathcal{RG}ra_d$ acts canonically (almost by its very construction), on the reduced cyclic cohomology $H^\bullet(Cyc(A))$ of an arbitrary cyclic strongly homotopy associative algebra A (equipped with the degree $-d$ scalar product), we deduce a new observation that $H^\bullet(Cyc(A))$ *is always a quasi-Lie bialgebra*, see §**3.7** for full details. It is worth noting that the twisted properad of ribbon graphs $\mathsf{tw}\mathcal{RG}ra_d$ controls [Me2] the totality of compactly supported cohomology groups $\prod_{n\geq 1, 2g+n\geq 3} H_c^{\bullet-d(2g-2+n)}(\mathcal{M}_{g,n})$ of moduli spaces $\mathcal{M}_{g,n}$ of genus g algebraic curves with m boundaries and n punctures, and the associated map

$$i^q : q\mathcal{L}ieb_{d-1,d} \longrightarrow H(\mathsf{tw}\mathcal{RG}ra_d) \simeq \prod_{n\geq 1, 2g+n\geq 3} H_c(\mathcal{M}_{g,n})$$

is non-trivial on infinitely many elements of $q\mathcal{L}ieb_{-1,0}$ (see §3.9 in [Me2]).

The deformation complex

$$\mathsf{Def}\left(q\mathcal{L}ieb_{c-1,d} \stackrel{i^q}{\longrightarrow} H^\bullet(\mathsf{tw}\mathcal{P})\right)$$

of the morphism i^q has, in general, a much richer cohomology than the complex $\mathsf{Def}(\mathcal{L}ie_d \stackrel{i}{\longrightarrow} \mathsf{tw}\mathcal{P})$; moreover, that cohomology comes always equipped with a morphism of cohomology groups,

$$H^\bullet(\mathsf{GC}^{\geq 2}_{c+d-1}) \longrightarrow H^\bullet\left(\mathsf{Def}\left(q\mathcal{L}ieb_{c-1,d} \stackrel{i^q}{\longrightarrow} H^\bullet(\mathsf{tw}\mathcal{P})\right)\right)$$

where $\mathsf{GC}^{\geq 2}_n$ is the famous Maxim Kontsevich graph complex [Ko1] (more precisely, its extension allowing graphs with bivalent vertices). The case $c+d=3$ is of special interest as the dg Lie algebra $H^\bullet(\mathsf{GC}^{\geq 2}_2)$ contains the Grothendieck–Teichmüller Lie algebra [W1]. The case $c+d=2$ is also of interest as it corresponds to the odd Kontsevich graph complex $H^\bullet(\mathsf{GC}^{\geq 2}_1)$ which contains a rich subspace generated by trivalent graphs.

Another important example of a dg properad in the subcategory $\mathsf{PROP}_{\mathcal{L}ieb_{c,d}}$ is the properad $\mathcal{L}ieb_{c,d}$ itself; the cohomology $H^\bullet(\mathsf{tw}\mathcal{L}ieb_{c,d})$ of the associated twisted properad is highly non-trivial: we show in §**3.5** that, for any natural number $N\geq 1$, there is an injection of cohomology groups,

$$H^\bullet(\mathsf{HGC}^N_{c+d}) \longrightarrow H^{\bullet+dN}(\mathsf{tw}\mathcal{L}ieb_{c,d})$$

where HGC^N_d is a version of the Kontsevich graph complex GC_d with N labelled hairs which has been introduced and studied recently in the context of the theory of moduli spaces $\mathcal{M}_{g,n}$ of algebraic curves of arbitrary genus g with n punctures [CGP] and the theory of long knots [FTW].

The functor tw is *not* an endofunctor of the category $\mathsf{PROP}_{\mathcal{H}olieb_{c,d}}$ and, contrary to the canonical projection $\mathsf{tw}\mathcal{H}olie_d \to \mathcal{H}olie_d$, the analogous "forgetful" map

$$\mathsf{tw}\mathcal{H}olieb_{c,d} \longrightarrow \mathcal{H}olieb_{c,d}$$

is *not* a quasi-isomorphism (as the above result with the hairy graph complex demonstrates). In §4 we introduce a new twisting endofunctor

$$\mathsf{Tw} : \mathsf{PROP}_{\mathcal{H}olieb_{c,d}} \longrightarrow \mathsf{PROP}_{\mathcal{H}olieb_{c,d}}$$

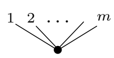
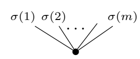
which is an enlargement of tw in the sense of the number of new generators (hence the notation) and which fixes both these "not"s . The key point is to introduce the correct notion of a Maurer-Cartan element of a generic $\mathcal{H}olieb_{c,d}$-algebra. The idea is to use a polydifferential functor [MW1]

$$\begin{array}{rccc} \mathcal{O}: & \text{Category of dg props} & \longrightarrow & \text{Category of dg operads} \\ & \mathcal{P} & \longrightarrow & \mathcal{O}\mathcal{P} \end{array}$$

whose main property is that, given any dg prop $\mathcal{P}$ and a representation of $\mathcal{P}$ in a dg vector space V, the dg operad $\mathcal{O}\mathcal{P}$ comes equipped canonically with an induced representation in the graded commutative tensor algebra $\odot^\bullet V$ given in terms of polydifferential — with respect to the standard multiplication in $\odot^\bullet V$ — operators. The point is that the dg operad $\mathcal{O}\mathcal{H}olieb_{c,d}$ comes equipped with a highly non-trivial morphism of dg properads which was discovered in [MW1],

$$\mathcal{H}olie^+_{c+d} \longrightarrow \mathcal{O}\mathcal{H}olieb_{c,d},$$

and whose image brings into play *all* generators of the properad $\mathcal{H}olieb_{c,d}$, not just the ones spanning the sub-properad $\mathcal{H}olie_d$ of $\mathcal{H}olieb_{c,d}$. Here the symbol $+$ means a slight extension of $\mathcal{H}olie_d$ which take cares about deformations of the differential in representation spaces [Me1]. It makes sense to talk about Maurer–Cartan elements of representations of $\mathcal{H}olie^+_{c+d}$ as usual, and hence it makes sense to talk about *Maurer–Cartan elements* $\gamma \in \odot^{\bullet \geq 1}(V[c])$ *of an arbitrary* $\mathcal{H}olieb_{c,d}$*-algebra* V via the above morphism. Rather surprisingly, these MC elements γ can be used to twist not only the dg operad $\mathcal{O}\mathcal{H}olieb_{c,d}$ but the dg properad $\mathcal{H}olieb_{c,d}$ itself giving thereby rise to a new twisting endofunctor Tw on the category $\mathsf{PROP}_{\mathcal{H}olieb_{c,d}}$! As explained in §4, the twisting endofunctor Tw adds to a generic properad $\mathcal{P}$ under $\mathcal{H}olieb_{c,d}$ infinitely many new (skew)symmetric generators,

 $= (-1)^{c|\sigma|}$  $\forall \sigma \in \mathbb{S}_m,\ m \geq 1.$

with the $m = 1$ corolla corresponding to the original functor tw. We show in §**4.7** that the quotient of $\mathsf{Tw}\mathcal{H}olieb_{c,d}$ by the ideal generated by  gives us a dg free properad closely related to the properad of strongly homotopy *triangular* Lie bialgebras.

It is proven in §4 that the natural projection

$$\mathsf{Tw}\mathcal{H}olieb_{c,d} \longrightarrow \mathcal{H}olieb_{c,d}$$

is a quasi-isomorphism. We also prove for any $\mathcal{P} \in \mathsf{PROP}_{\mathcal{H}olieb_{c,d}}$ the dg Lie algebra $\mathsf{Def}(\mathcal{H}olieb_{c,d} \stackrel{i}{\to} \mathcal{P})$ controlling deformations of the given morphism $i : \mathcal{H}olieb_{c,d} \to \mathcal{P}$ acts on $\mathsf{Tw}\mathcal{P}$ by derivations. In the case $\mathcal{P} = \mathcal{H}olieb_{c,d}$ the associated complex

$$\mathsf{Def}(\mathcal{H}olieb_{c,d} \stackrel{\mathrm{Id}}{\to} \mathcal{H}olieb_{c,d})$$

has cohomology equal (up to one rescaling class) to $H^\bullet(\mathsf{GC}^{\geq 2}_{c+d+1})$ [MW2] so that, for any dg properad $\mathcal{P}$ under $\mathcal{H}olieb_{c,d}$ there is always a morphism of cohomology groups

$$H^\bullet(\mathsf{GC}^{\geq 2}_{c+d+1}) \longrightarrow H^\bullet(\mathsf{Def}(\mathcal{H}olieb_{c,d} \stackrel{i}{\to} \mathcal{P}))$$

and hence an action of $H^\bullet(\mathsf{GC}^{\geq 2}_{c+d+1})$ on the cohomology of the twisted dg properad $\mathsf{Tw}\mathcal{P}$ by derivations (which can be in concrete cases homotopy trivial).

A similar trick via the polydifferential functor $\mathcal{O}$ works fine in the case of strongly homotopy *involutive* Lie bialgebras; we denote the associate properad by $\mathcal{L}ieb^\diamond_{c,d}$ and its minimal resolution by $\mathcal{H}olieb^\diamond_{c,d}$. There is again a highly non-trivial morphism of dg properads [MW1],

$$\mathcal{H}olie^{\diamond +}_{c+d} \to \mathcal{O}\mathcal{H}olieb^\diamond_{c,d},$$

where $\mathcal{H}olie^{\diamond +}$ is a *diamond* extension of $\mathcal{H}olie_d$ which was introduced and studied in [CMW]. Maurer–Cartan elements of $\mathcal{H}olie^{\diamond +}$-algebras are defined in the standard way so that the above morphism of properads can be immediately translated into the notion of *Maurer–Cartan element of a* $\mathcal{H}olieb^\diamond_{c,d}$*-algebra*. However this time we obtain essentially nothing new: this approach just re-discovers the notion which has been introduced earlier in general in [CFL], and in the special case of $\mathcal{L}ieb^{\diamond}_{c,d}$-algebras of cyclic words in [B1, B2]. Therefore the diamond extension $\mathsf{Tw}^{\diamond}$ of the twisting endofunctor Tw from §3 gives us essentially nothing new as well: we obtain just a properadic incarnation of the twisting constructions in [B1, B2, CFL]. This incarnation fits nicely the beautiful approach to string topology developed in [NW] via the so called partition functions of closed manifolds. Full details are given in §5.

Notation. We work over a field $\mathbb{K}$ of characteristic zero. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; its group of automorphisms is denoted by $\mathbb{S}_n$; the trivial (resp., sign) one-dimensional representation of $\mathbb{S}_n$ is denoted by $\mathbb{1}_n$ (resp., by sgn_n). We often abbreviate $sgn_n^d := sgn_n^{\otimes|d|}$, $d \in \mathbb{Z}$. The cardinality of a finite set A is denoted by $\#A$.

We work throughout in the category of $\mathbb{Z}$-graded vector spaces over a field $\mathbb{K}$ of characteristic zero. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[d]$ stands for the graded vector space with $V[d]^i := V^{i+d}$; the canonical isomorphism $V \to V[d]$ is denoted by $\mathfrak{s}^d$. for $v \in V^i$ we set $|v| := i$.

For a prop(erad) $\mathcal{P}$ we denote by $\mathcal{P}\{d\}$ a prop(erad) which is uniquely defined by the following property: for any graded vector space V a representation of $\mathcal{P}\{d\}$ in V is identical to a representation of $\mathcal{P}$ in $V[d]$; in particular, one has for an endomorphism properad $\mathcal{E}nd_V\{-d\} = \mathcal{E}nd_{V[d]}$. Thus a map $\mathcal{P}\{d\} \to \mathcal{E}nd_V$ is the same as $\mathcal{P} \to \mathcal{E}nd_{V[d]} \equiv \mathcal{E}nd_V\{-d\}$. The operad controlling Lie algebras with Lie bracket of degree $-d$ is denoted by $\mathcal{L}ie_{d+1}$ while its minimal resolution by $\mathcal{H}olie_{d+1}$; thus $\mathcal{L}ie_{d+1}$ is equal to $\mathcal{L}ie\{d\}$ if ones uses the standard notation $\mathcal{L}ie := \mathcal{L}ie_1$ for the ordinary operad of Lie algebras.

We often used the following elements

$$\oint_{123} := \sum_{k=1}^{3} (123)^k \in \mathbb{K}[\mathbb{S}_3], \qquad \mathrm{Alt}^d_{\mathbb{S}_n} := \sum_{\sigma\in\mathbb{S}_n} (-1)^{d|\sigma|}\sigma \in \mathbb{K}[\mathbb{S}_n]$$

as linear operators on $\mathbb{S}_3$- and, respectively, $\mathbb{S}_n$-modules.

Acknowledgement. I am grateful to the referee for useful comments and suggestions.

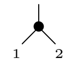
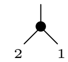
2. Twisting of operads under $\mathcal{L}ie_d$ — an overview in pictures

2.1. Introduction. This section is a more or less self-contained exposition of Thomas Willwacher's construction [W1] of the twisting endofunctor in the category of operads under the operad of Lie algebras. For purely pedagogical purposes, we consider here a new intermediate step based on the "plus" endofunctor from [Me1]. We give most of the necessary details (including elementary ones) with emphasis on the action of the deformation complexes on twisted operads. Many calculations in the later sections, where we discuss some new material, are more tedious but analogous to the ones reviewed here.

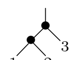
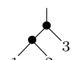
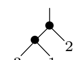
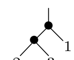
2.2. Reminder about $\mathcal{H}olie_d$. Recall that the operad of degree shifted Lie algebras is defined, for any integer $d \in \mathbb{Z}$, as a quotient,

$$\mathcal{L}ie_d := \mathcal{F}ree\langle e\rangle/\langle\mathcal{R}\rangle,$$

of the free prop generated by an $\mathbb{S}$-module $e = \{e(n)\}_{n\geq 2}$ with all $e(n) = 0$ except[1]

$e(2) := sgn_2^d \otimes \mathbb{1}_1[d-1] = \mathrm{span}\Big\langle$  $= (-1)^d$  $\Big\rangle$

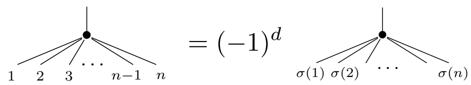
modulo the ideal generated by the following relation

(3) $\oint_{123}$  $\equiv$  $+$  $+$  $= 0.$

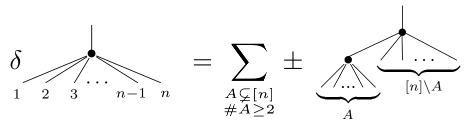
---

[1]When representing elements of all operads and props below as (decorated) graphs we tacitly assume that all edges and legs are *directed* along the flow going from the bottom of the graph to the top.

Its minimal resolution $\mathcal{H}olie_d$ is a dg free operad whose (skew)symmetric generators,

(4)  , $\forall \sigma \in \mathbb{S}_n,\ n \geq 2,$

have degrees $1 + d - nd$. The differential in $\mathcal{H}olie_d$ is given by

(5) 

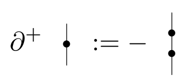
If d is even, all the signs above are equal to -1.

2.3. "Plus" extension. We also consider a dg operad $\mathcal{H}olieb_d^+$ which is an extension of $\mathcal{H}olieb_d$ by an extra degree 1 generator and the differential given by an above formula with the summation running over all possible non-empty subsets $A \subset [n]$.

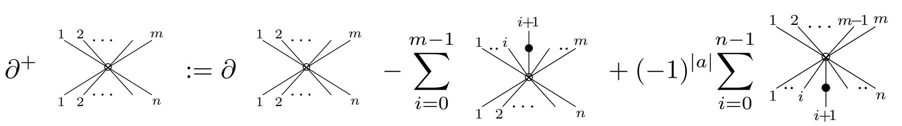
More generally, there is an endofunctor on the category of dg props (or dg operads) introduced in [Me1]

$$
\begin{array}{rccc}
{}^+ : & \text{category of dg props} & \longrightarrow & \text{category of dg props} \\
 & (\mathcal{P}, \partial) & \longrightarrow & (\mathcal{P}^+, \partial^+)
\end{array}
$$

defined as follows. For any dg prop $\mathcal{P}$, let $\mathcal{P}^+$ be the free prop generated by $\mathcal{P}$ and one other operation of arity $(1,1)$ and of cohomological degree $+1$. On $\mathcal{P}^+$ one defines a differential ∂^+ by setting its value on the new generator by



and on any other element $a \in \mathcal{P}(m,n)$ (which we identify pictorially with the (m,n)-corolla whose vertex is decorated with a) by the formula

(6) 

where ∂ is the original differential in $\mathcal{P}$. The dg prop $(\mathcal{P}^+, \partial^+)$ is uniquely characterized by the property: there is a one-to-one correspondence between representations

$$\rho : \mathcal{P}^+ \longrightarrow \mathcal{E}nd_V$$

of $(\mathcal{P}^+, \partial^+)$ in a dg vector space (V, d), and representations of $\mathcal{P}$ in the same space V but equipped with a deformed differential $d + d'$, where $d' := \rho(\,)$.

2.4. From morphisms $\mathcal{H}olie_d^+$ to twisted morphisms from $\mathcal{H}olie_d$. Given any dg operad $(\mathcal{A} = \{\mathcal{A}(n)\}_{n\geq 0}, \partial)$, it is well-known that any element $h \in \mathcal{A}(1)$ defines a derivation D_h of the (non-differential) operad $\mathcal{A}$ by the formula analogous to (6),

$$D_h a = h \circ_1 a - (-1)^{|h||a|} \sum_{i=1}^{n} a \circ_i h, \quad \forall\, a \in \mathcal{A}(n).$$

Moreover, if $|h| = 1$ and $\partial h = -h \circ_1 h$, then the operator

$$\partial_\bullet := \partial + D_h$$

is also a differential in $\mathcal{A}$ (which acts on h by $\partial_\bullet h = h \circ_1 h$). Assume we have a morphism of dg operads

(7) $$g^+ : (\mathcal{H}olie_d^+, \delta^+) \longrightarrow (\mathcal{A}, \partial)$$

Then the element $h := g^+(\overset{|}{\underset{|}{\bullet}})$ satisfies all the conditions specified above so that the sum

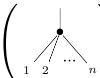
$$\partial_{\bullet} := \partial + D_{g^+(\overset{|}{\underset{|}{\bullet}})}$$

is a differential in $\mathcal{A}$. Hence we have the following

2.4.1. Proposition. *For any morphism of dg operads (7) there is an associated morphism of dg operads,*

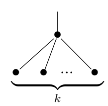
$$g : (\mathcal{H}olie_d, \delta) \longrightarrow (\mathcal{A}, \partial_{\bullet})$$

given by the restriction of g^+ to the generators of $\mathcal{H}olie_d$.

Proof. Abbreviating $C_n :=$ , we have for any $n \geq 2$,

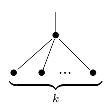
$$\partial_{\bullet} g(C_n) \equiv \partial_{\bullet} g^+(C_n) = \partial g^+(C_n) + D_{g^+(\overset{|}{\underset{|}{\bullet}})}(C_n) = g^+(\delta^+ C_n) + D_{g^+(\overset{|}{\underset{|}{\bullet}})}(C_n) = g(\delta C_n). \qquad \square$$

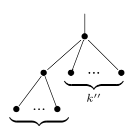
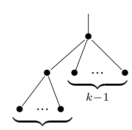
2.5. Twisting of $\mathcal{H}olie_d$ by a Maurer-Cartan element. Let $\widetilde{\mathsf{tw}}\mathcal{H}olie_d$ be a dg free operad generated by degree $1 + d - nd$ corollas (4) of type $(1, n)$ with $n \geq 2$, and also by an additional corolla $\overset{|}{\bullet}$ of type $(1, 0)$ and of degree d. The differential is defined on the $(1, n \geq 2)$ generators by the standard formula (5), while on the new generator it is defined as follows

(8) $$\delta \overset{|}{\bullet} = -\sum_{k\geq 2} \frac{1}{k!}$$ 

2.5.1. Lemma. $\delta^2 = 0$, *i.e. it is indeed a differential in $\widetilde{\mathsf{tw}}\mathcal{H}olie_d$.*

Proof. We have (assuming that d is even to simplify signs)

$$\delta^2 \overset{|}{\bullet} = -\sum_{k\geq 2} \frac{1}{k!} \delta \Big($$  $$\Big)$$

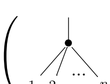
$$= +\sum_{k\geq 2} \frac{1}{k!} \Big(\sum_{\substack{k=k'+k'' \\ k'\geq 2, k''\geq 1}} \frac{k!}{k'!k''!}$$  $$\Big) - \sum_{k\geq 2} \frac{1}{k!} \Big(\sum_{l\geq 2} \frac{k}{l!}$$  $$\Big) = 0,$$

where the first summand comes from (5) and the second one from (8). $\square$

A representation

$$\rho : \widetilde{\mathsf{tw}}\mathcal{H}olie_d \longrightarrow \mathcal{E}nd_V$$

of $\widetilde{\mathsf{tw}}\mathcal{H}olie_d$ in a dg (appropriately filtered) vector space (V, d) is given by a $\mathcal{H}olie_d$-algebra structure $\{\mu_n\}_{n\geq 1}$ on V,

$$mu_1 := d, \quad \mu_n := \rho \Big($$  $$\Big) : \odot^n (V[d]) \to V[d+1], \quad n \geq$$

together with a special element $m := \rho(\overset{|}{\bullet})$ satisfying the equation (a filtration on V is assumed to be such that this infinite in general sum makes sense)

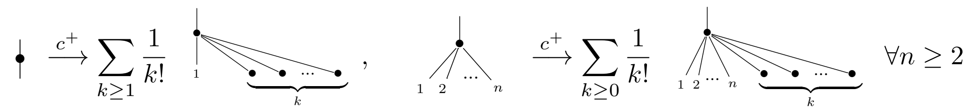
$$dm + \sum_{k\geq 2} \frac{1}{k!} \mu_k(m, \ldots, m) = 0.$$

Such an element is called the Maurer-Cartan element of the given $\mathcal{H}olie_d$-algebra structure on V.

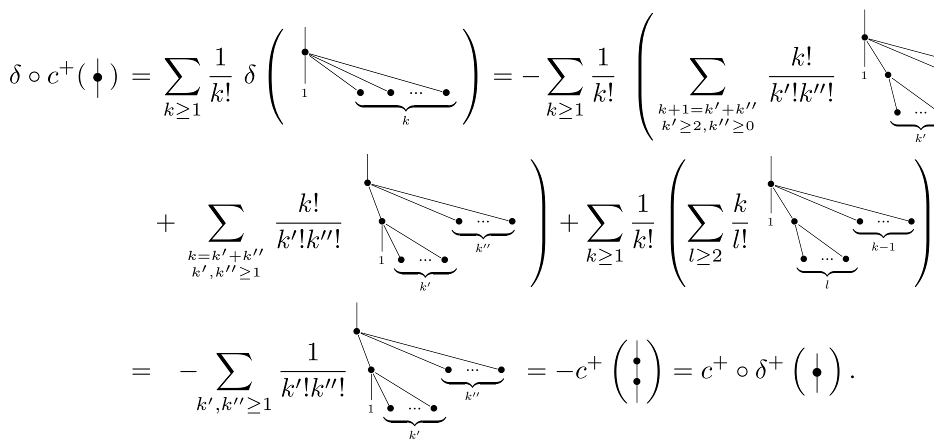
2.5.2. Proposition. *There is a morphism of dg operads*

$$c^+ : (\mathcal{H}olie_d^+, \delta^+) \longrightarrow (\widetilde{\mathsf{tw}}\mathcal{H}olie_d, \delta)$$

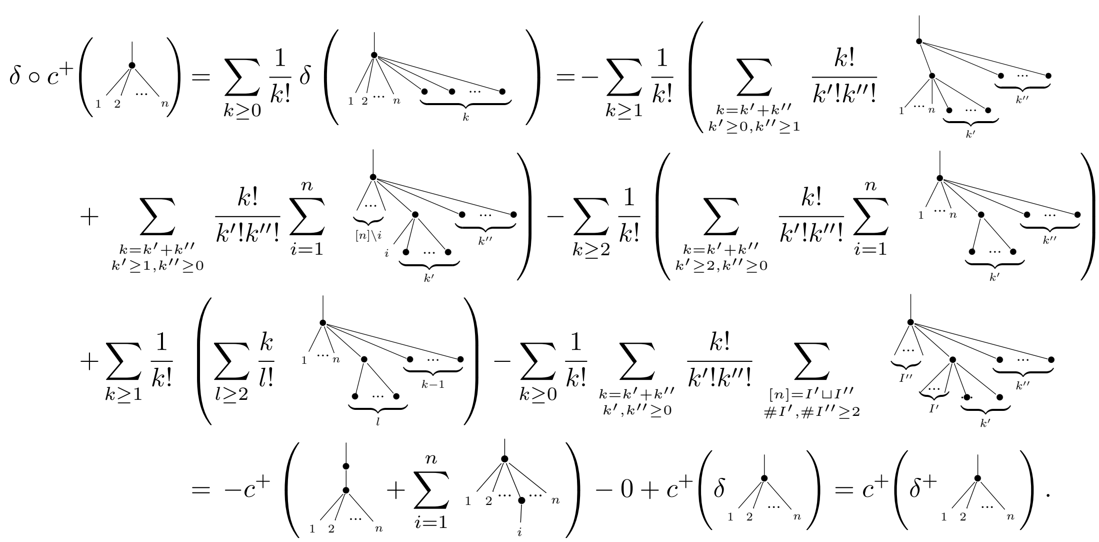
given on the generators as follows:

(9) 

Proof. One has to check that $\delta \circ c^+ = c^+ \circ \delta^+$. One has (assuming for simplicity of signs again that d is even)



Similarly one checks the required equality for any $n \geq 2$



□

Hence by Proposition **2.4.1**, the differential in the operad $\widetilde{\mathsf{tw}}\mathcal{H}olie_d$ can be twisted,

$$\delta \to \delta_\bullet = \delta + D_{c^+(\bullet)}.$$

The operad $\widetilde{\mathsf{tw}}\mathcal{H}olie_d$ equipped with the twisted differential $\delta_\bullet$ is denoted from now on by $\mathsf{tw}\mathcal{H}olie_d$.

2.5.3. Definition-proposition. The data $\mathsf{tw}\mathcal{H}olie_d := \{\mathsf{tw}\mathcal{H}olie_d(n)\}_{n\geq 0}, \delta_\bullet)$ is called the *twisted operad of strongly homotopy Lie algebras.* It comes equipped with a monomorphism

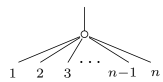
$$c : (\mathcal{H}olie_d, \delta) \longrightarrow (\mathsf{tw}\mathcal{H}olie_d, \delta_\bullet)$$

given on the generators of $\mathcal{H}olie_d$ by the second expression in formula (9).

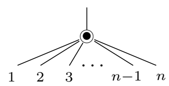
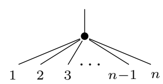
2.6. Twisting of operads under $\mathcal{H}olie_d$ [W1]**.** Let $(\mathcal{A}, \partial)$ be a dg operad equipped with a non-trivial morphism of operads

$$i : \mathcal{H}olieb_d \longrightarrow \mathcal{A}$$

Such an operad is called an *operad under* $\mathcal{H}olie_d$*. Generic* elements of $\mathcal{A} = \{\mathcal{A}(n)\}_{n\geq 0}$ are denoted in this paper as decorated corollas with, say, white vertices (to distinguish them from generators of $\mathcal{H}olie_d$),

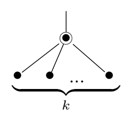
 $\in \mathcal{A}(n), \quad n \geq 1.$

The images of the generators (4) of $\mathcal{H}olie_d$ under the map i are denoted by decorated corollas with vertices shown as $\circledcirc$ (to emphasize the special status of these elements of $\mathcal{A}$),

 $:= i\Big($  $\Big) \in \mathcal{A}(n), \quad n \geq 2.$

It is worth noting that some of these elements can stand for the zero vector in $\mathcal{A}(n)$ as we do not assume in general that the map i is an injection on every generator.

We define a dg operad $\widetilde{\mathsf{tw}}\mathcal{A} = \{\widetilde{\mathsf{tw}}\mathcal{A}(n)\}_{n\geq 0}$ as an operad generated freely by $\mathcal{A}$ and one new generator  of type $(1, 0)$ and of cohomological degree d. The differential ∂ in $\widetilde{\mathsf{tw}}\mathcal{A}$ is equal to ∂ when acting on elements of $\mathcal{A}$, and its action on the new generator is defined by

(10) ∂ $= -\sum_{k\geq 2} \frac{1}{k!}$ 

There is a chain of operadic morphisms,

$$i^+ : (\mathcal{H}olie_d^+, \delta^+) \xrightarrow{c^+} (\widetilde{\mathsf{tw}}\mathcal{H}olie_d, \delta) \xrightarrow{i} (\widetilde{\mathsf{tw}}\mathcal{A}, \partial)$$

where the map i is extended to the extra generator as the identity map.

Using Proposition **2.4.1**, one concludes that the differential ∂ in $\widetilde{\mathsf{tw}}\mathcal{A}$ can be twisted,

$$\partial \to \partial_\bullet := \partial + D_{i^+(\downarrow)}$$

This makes the $\mathbb{S}$-module $\widetilde{\mathsf{tw}}\mathcal{A}$ into a new dg operad denoted from now on by $\mathsf{tw}\mathcal{A}$.

2.6.1. Definition-proposition. For any dg operad $(\mathcal{A}, \delta)$ under $\mathcal{H}olie_d$, the associated dg operad

$$\mathsf{tw}\mathcal{A} := \{\mathsf{tw}\mathcal{A}(n), \partial_\bullet)\}_{n\geq 0}$$

is called the *twisted extension of* $\mathcal{A}$ or the *twisted operad of* $\mathcal{A}$. There is

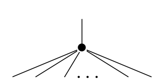
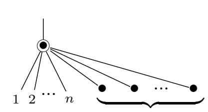
(i) a morphism of dg operads

$$\iota : (\mathcal{H}olie_d, \delta) \longrightarrow (\mathsf{tw}\mathcal{A}, \partial_\bullet)$$

which factors through the composition

$$(\mathcal{H}olie_d, \delta) \xrightarrow{c} (\mathsf{tw}\mathcal{H}olie_d), \delta_\bullet) \xrightarrow{\mathsf{tw}(i)} (\mathsf{tw}\mathcal{A}, \partial_\bullet)$$

and hence is given explicitly by

(11)  $\stackrel{\iota}{\longrightarrow} \sum_{k\geq 0} \frac{1}{k!}$  $\forall\ n \geq 2.$

(ii) a natural epimorphism of dg operads

$$p : (\mathsf{tw}\mathcal{A}, \partial_\bullet) \longrightarrow (\mathcal{A}, \partial)$$

which sends the extra generator  to zero.

2.6.2. Proposition [DW]**.** *The endofunctor* tw *in the category of operads under* $\mathcal{H}olie_d$ *is exact, i.e. any diagram*

$$\mathcal{H}olie_d \longrightarrow \mathcal{A} \stackrel{g}{\longrightarrow} \mathcal{A}'$$

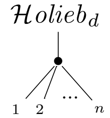
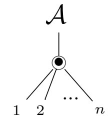
with g *being a quasi-isomorphism, the map* $\mathsf{tw}g$ *in the associated diagram*

$$\mathcal{H}olie_d \longrightarrow \mathsf{tw}\mathcal{A} \stackrel{\mathsf{tw}(g)}{\longrightarrow} \mathsf{tw}\mathcal{A}'$$

is also a quasi-isomorphism.

2.7. An action of the deformation complex of $\mathcal{H}olie_d \stackrel{i}{\to} \mathcal{A}$ **on** $\mathsf{tw}\mathcal{A}$**.** Given a dg operad $(\mathcal{A}, \partial)$ under $\mathcal{H}olie_d$,

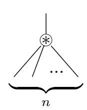
$$i: \quad \mathcal{H}olieb_d \longrightarrow \mathcal{A}$$

 $\longrightarrow$ 

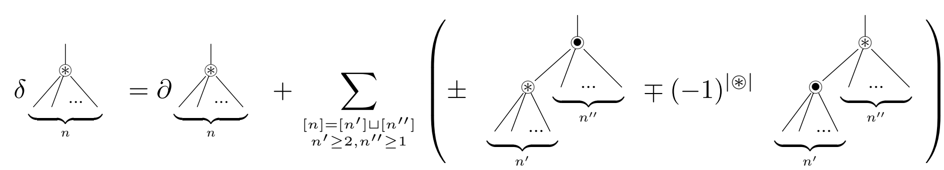
Consider a dg Lie algebra controlling deformations of the morphism i (see [MeVa] for several equivalent constructions of such a dg Lie algebra),

$$\mathsf{Def}\left(\mathcal{H}olie_d \stackrel{i}{\to} \mathcal{A}\right) = \prod_{n\geq 2} \mathcal{A}(n) \otimes_{\mathbb{S}_n} sgn_n^{|d|}[d(1-n)]$$

Its Maurer-Cartan elements are in 1-1 correspondence with morphisms $\mathcal{H}olie_d \to \mathcal{A}$ which are deformations of i; in particular the zero MC element corresponds to i itself. An element γ of the above complex can be represented pictorially as a collection of $(1,n)$-corollas,

$$\gamma = \{$$  $$\}_{n\geq 2},$$

of corollas whose vertices are decorated with elements of $\mathcal{A}(n) \otimes_{\mathbb{S}_n} sgn_n^{|d|}$ and whose input legs are (skew)symmetrized (so that we can omit their labels); the degrees of decorations of vertices are shifted by $d(1-n)$. To distinguish these elements from the generic elements of $\mathcal{A}$ as well as from the images of $\mathcal{H}olie_d$-generators under i, we denote the vertices of such corollas from now on by $\circledast$. A formal sum of such corollas is homogeneous of degree p if and only if the degree of each contributing $(1,n)$-corolla is equal to $p + d - dn$. The differential δ in the deformation complex $\mathsf{Def}\left(\mathcal{H}olie_d \stackrel{i}{\to} \mathcal{A}\right)$ can then be given explicitly by

(12) 

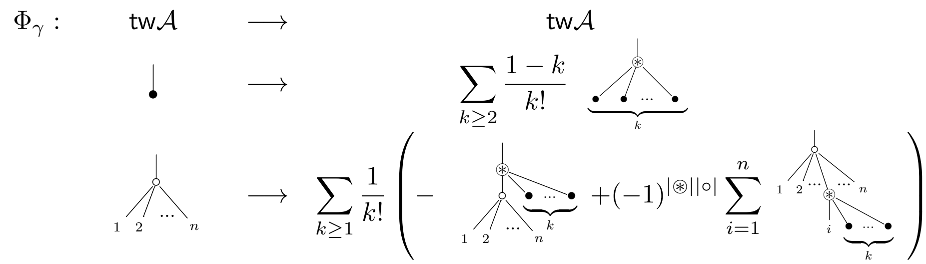
where the rule of signs depends on d and is read from (5); for d even the first $\pm$-symbol is $+1$, while the second one is -1.

Let $(\mathrm{Der}(\mathsf{tw}\mathcal{A}), [\ ,\ ])$ be the Lie algebra of derivations of the non-differential operad $\mathsf{tw}\mathcal{A}$. The differential $\partial_\bullet$ in $\mathsf{tw}\mathcal{A}$ is, of course, its MC element and hence makes $\mathrm{Der}(\mathsf{tw}\mathcal{A})$ into a dg Lie algebra with the differential given by the commutator $[\partial_\bullet,\ ]$.

2.7.1. Proposition. *There is a canonical morphism of dg Lie algebras*

$$\begin{array}{rccc} \Phi: & \mathsf{Def}\left(\mathcal{H}olie_d \xrightarrow{i} \mathcal{A}\right) & \longrightarrow & \mathrm{Der}(\mathsf{tw}\mathcal{A}) \\ & \gamma & \longrightarrow & \Phi_\gamma \end{array} \tag{13}$$

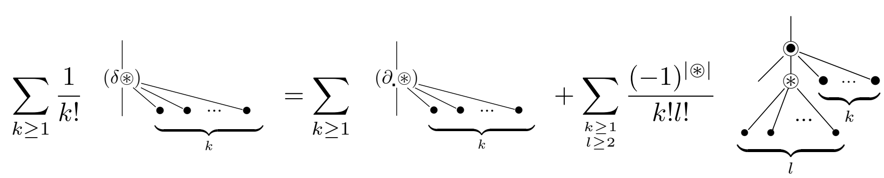
where the derivation Φ_γ *is given on the generators by*



Proof. Any derivation of $\mathsf{tw}\mathcal{A}$ is uniquely determined by its values on the generators, i.e. on and on every element of $\mathcal{A}$. The first value can be chosen arbitrary, while the second ones are subject to the condition that they are derivations of the operad structure in $\mathcal{A}$; as Φ_γ applied to is precisely of the form D_h discussed §**2.4**, we conclude that the above formulae do define a derivation of $\mathsf{tw}\mathcal{A}$ as a non-differential operad. Hence it remains to show that Φ_γ respects differentials in both dg Lie algberas, that is, satisfies the equation

$$\Phi_{\delta\gamma} = [\partial_\bullet, \Phi_\gamma]. \tag{14}$$

It is straightforward to check that the operator equality (14) holds true when applied to generators of $\mathsf{tw}\mathcal{A}$ if and only if one has the equality,



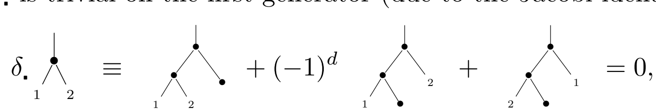
which is indeed the case due to (12). The compatibility of the map Φ with Lie brackets is almost obvious. □

Thus the Lie algebra $H^\bullet(\mathsf{Def}(\mathcal{H}olie_d \to \mathcal{A}))$ acts on the cohomology of the twisted operad $\mathsf{tw}\mathcal{A}$ by derivations. For some operads (see Example in §**2.10**) below) this cohomology Lie algebra can be extremely rich and interesting.

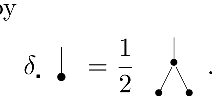
2.8. Twisting of $\mathcal{L}ie_d$. Assume, in the above notation, that $\mathcal{A}$ is $\mathcal{L}ie_d$ and the morphism

$$i : \mathcal{H}olie_d \longrightarrow \mathcal{L}ie_d$$

is the canonical quasi-isomorphism. Then $\mathsf{tw}\mathcal{L}ie_d$ is a dg operad generated by the degree $1-d$ corolla $= (-1)^d$ (modulo the Jacobi relations) and the degree d $(1,0)$-corolla . The twisted differential $\delta_\bullet$ is trivial on the first generator (due to the Jacobi identities)



while it acts on the second generator by



The twisted morphism $\mathsf{tw}(i) : \mathcal{L}ie_d \longrightarrow \mathsf{tw}\mathcal{L}ie_d$ becomes in this case the obvious inclusion. This example is important for understanding those twisted operads $\mathsf{tw}\mathcal{A}$ for which the map $\mathcal{H}olie_d \to \mathcal{A}$ factors through the above projection,

$$\mathcal{H}olie_d \longrightarrow \mathcal{L}ie_d \longrightarrow \mathcal{A}.$$

We call such dg operads *operads under* $\mathcal{L}ie_d$. The deformation complex of the epimorphism i has almost trivial cohomology, $H^\bullet(\mathsf{Def}(\mathcal{H}olieb_d \to \mathcal{L}ie_d)) = \mathbb{R}[-1]$; the only non-trivial cohomology class acts on $\mathsf{tw}\mathcal{L}ieb_d$ by rescaling its generators.

2.9. Twisting of (prop)operads under $\mathcal{L}ie_d$. Let $(\mathcal{A}, \partial)$ be a dg operad equipped with an operadic morphism

$$\imath : \quad (\mathcal{L}ie_d, 0) \quad \longrightarrow \quad (\mathcal{A}, \partial)$$

$$\longrightarrow$$

where $\mathcal{L}ie_d$ is understood as a differential operad with the trivial differential. Then $(\mathsf{tw}\mathcal{A}, \partial_\bullet)$ is an operad freely generated by $\mathcal{A}$ and one new generator of degree d. The differential $\partial_\bullet$ acts, by definition, on an element a of $\mathcal{A}(n)$ (identified with the a-decorated $(1,n)$-corolla) by a formula similar to (6)

(15) $$\partial_\bullet \quad := \partial \quad + \quad - (-1)^{|a|} \sum_{i=1}^{n}$$

and on the extra generator as follows

(16) $$\partial_\bullet \quad = \frac{1}{2} \quad .$$

By Proposition **2.6.2**, the canonical epimorphism

$$\mathsf{tw}\mathcal{H}olie_d \longrightarrow \mathsf{tw}\mathcal{L}ie_d$$

is a quasi-isomorphism. Moreover, it is proven in [DW] that the natural projections,

(17) $$\mathsf{tw}\mathcal{H}olie_d \longrightarrow \mathcal{H}olie_d, \quad \mathsf{tw}\mathcal{L}ie_d \longrightarrow \mathcal{L}ie_d$$

are quasi-isomorphisms as well.

2.9.1. Example: Twisting of $\mathcal{A}ss$. The operad of associative algebras $\mathcal{A}ss$ is obviously an operad under $\mathcal{L}ie_1$ and hence can be twisted. It is proven in [CL] that the natural projection

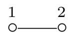
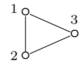
$$\mathsf{tw}\mathcal{A}ss \longrightarrow \mathcal{A}ss$$

is a quasi-isomorphism.

2.10. Example: M. Kontsevich's operad of graphs. Here is an example of the twisting procedure used in [W1] to reproduce an important dg operad of graphs $\mathcal{G}raphs_d$ which has been invented by M. Kontsevich in [Ko2] in the context of a new proof of the formality of the little disks operad, and which was further studied in [LV, W1]. By a *graph* Γ we understand a 1-dimensional CW complex whose 0-cells are called vertices and 1-cells are called edges; the set of vertices of Γ is denoted by $V(\Gamma)$ and the set of edges by $E(\Gamma)$. Let $\mathcal{G}ra_d(n)$, $d \in \mathbb{Z}$, stand for the graded vector space generated by graphs Γ such that

(i) Γ has precisely n vertices which are labelled, that is an isomorphism $V(\Gamma) \to [n]$ is fixed;
(ii) Γ is equipped with an orientation which for d even is defined as an ordering of edges (up the sign action of $\mathbb{S}_{\#E(\Gamma)}$), while for d odd it is defined as a choice of the direction on each edge (up to the sign action of $\mathbb{S}_2$ whose generator flips the direction).
(iii) Γ is assigned the cohomological degree $(1-d)\#E(\Gamma)$.

For example,

 $\in \mathcal{G}ra_d(2)$,  $\in \mathcal{G}ra_d(3)$

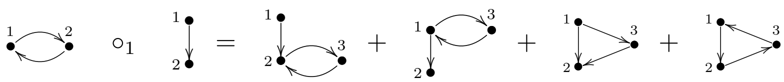
where for d odd one should assume a choice of directions on edges (defined up to a flip and multiplication by -1). The $\mathbb{Z}$-graded vector space $\mathcal{G}ra_d(n)$ is an $\mathbb{S}_n$-module with the permutation group acting on graphs by relabeling their vertices. The $\mathbb{S}$-module

$$\mathcal{G}ra_d := \{\mathcal{G}ra_d(n)\}$$

is an operad [W1] with the operadic compositions

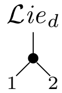
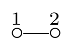
$$\begin{array}{rccc} \circ_i: & \mathcal{G}ra(n) \otimes \mathcal{G}ra(m) & \longrightarrow & \mathcal{G}ra(m+n-1) \\ & \Gamma_1 \otimes \Gamma_2 & \longrightarrow & \Gamma_1 \circ_i \Gamma_2 \end{array} \tag{18}$$

defined as follows: $\Gamma_1 \circ_i \Gamma_2$ is the linear combination of graphs obtained by substituting the graph Γ_2 into the i-labeled vertex of Γ_1 and taking a sum over all possible re-attachments of dangling edges (attached earlier to that vertex) to the vertices of Γ_2. Here is an example (for d odd),

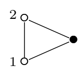


There is a morphism of operads [W1]

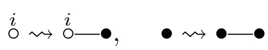
$$\begin{array}{ccc} \mathcal{L}ie_d & \longrightarrow & \mathcal{G}ra_d \end{array}$$

 $\longrightarrow$ 

so that one can apply the twisting endofunctor to $\mathcal{G}ra_d$. The resulting dg operad $\mathsf{tw}\mathcal{G}ra_d$ is generated by graphs with two types of vertices, white ones which are labelled and black ones which are unlabelled and assigned the cohomological degree d, e.g.

 $\in \mathcal{G}raphs_d(2)$

The differential acts on white vertices and black vertices by splitting them,

(19) 

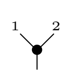
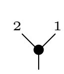
and re-attaching edges. The dg sub-operad of $\mathsf{tw}\mathcal{G}ra_d$ generated by graphs with at least one white vertex is denoted by $\mathcal{G}raphs_d$. It is proven in [Ko2, LV] that its cohomology $H^\bullet(\mathcal{G}raphs_d)$ is the operad of d-algebras. The case $d = 2$ is of special interest as 2-algebras are precisely the Gerstenhaber algebras which have many applications in algebra, geometry and mathematical physics.

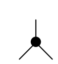
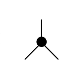
3. Partial twisting of properads under $\mathcal{L}ieb_d$ and quasi-Lie bialgebras

3.1. Reminder on the properads of (degree shifted) Lie bialgebras and quasi-Lie bialgebras. The properad of degree shifted Lie bialgebras is defined, for any pair of integer $c, d \in \mathbb{Z}$, as the quotient

$$\mathcal{L}ieb_{c,d} := \mathcal{F}ree\langle E_0\rangle / \langle \mathcal{R}\rangle,$$

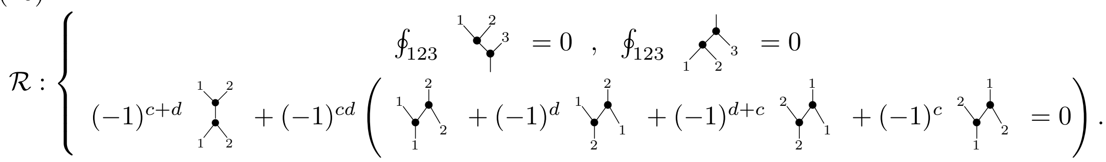
of the free prop generated by an $\mathbb{S}$-bimodule $E_0 = \{E_0(m,n)\}_{m,n\geq 0}$ with all $E_0(m,n) = 0$ except

$E_0(2,1) := \mathbb{1}_1 \otimes sgn_2^c[c-1] = \text{span}\Big\langle$  $= (-1)^c$  $\Big\rangle$

$E_0(1,2) := sgn_2^d \otimes \mathbb{1}_1[d-1] = \text{span}\Big\langle$  $= (-1)^d$  $\Big\rangle$

by the ideal generated by the following relations

(20)

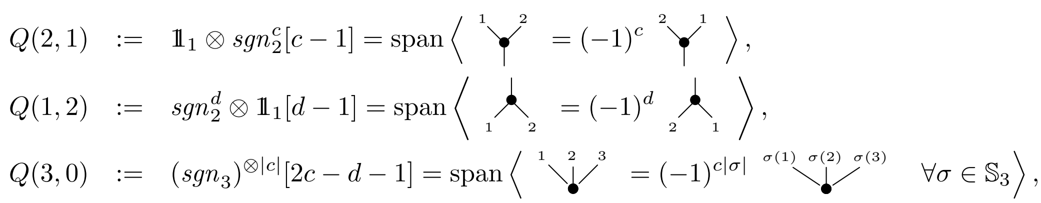


where the vertices are ordered implicitly in such a way that the ones on the top come first.

V. Drinfeld introduced [D2] the notion of *quasi-Lie bialgebra* or *Lie quasi-bialgebra*. The prop(erad) $q\mathcal{L}ieb_{c,d}$ controlling degree shifted quasi-Lie bialgebras can be defined, for any pair of integer $c, d \in \mathbb{Z}$, as the quotient

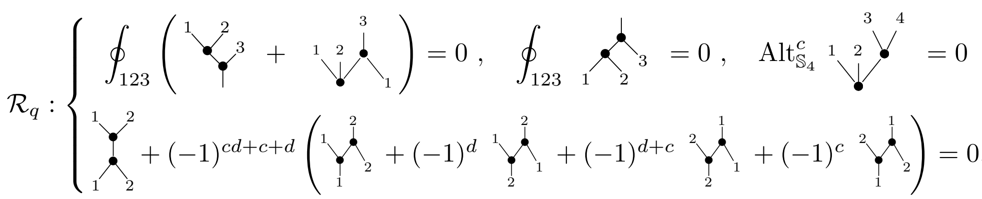
$$q\mathcal{L}ieb_{c,d} := \mathcal{F}ree\langle E_q\rangle / \langle \mathcal{R}_q\rangle,$$

of the free prop(erad) generated by an $\mathbb{S}$-bimodule $Q = \{Q(m,n)\}_{m,n\geq 0}$ with all $Q(m,n) = 0$ except

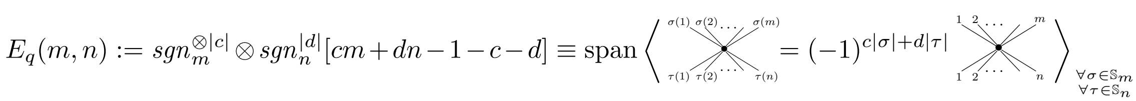


modulo the ideal generated by the following relations

(21)

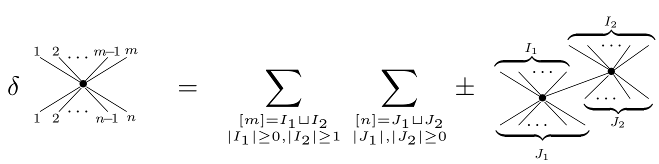


Its minimal resolution $\mathcal{H}oqlieb_{c,d}$ is a free operad $\mathcal{H}oqlieb_{c,d} := \mathcal{F}ree\langle E\rangle$ generated by an $\mathbb{S}$-bimodule $E_q = \{E_q(m,n)\}_{m\geq 1, n\geq 0, m+n\geq 3}$ with



The differential in $\mathcal{H}oqlieb_{c,d}$ is given on the generators by

(22)



where the signs on the r.h.s are uniquely fixed for $c + d \in 2\mathbb{Z}$ by the fact that they all equal to -1 if c and d are even integers. Taking the quotient of $\mathcal{H}oqlieb_{c,d}$ by the ideal generated by all $(m,0)$-corollas, $m \geq 3$, gives us the minimal model $\mathcal{H}olieb_{c,d}$ of the properad $\mathcal{L}ieb_{c,d}$.

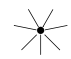
The properads $\mathcal{H}olieb_{c,d}$ and $q\mathcal{H}olieb_{c,d}$ with the same parity of $c+d$ are isomorphic to each other up to degree shift,

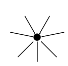
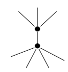
$$\mathcal{H}olieb_{c,d} = \mathcal{H}olieb_{c+d,0}\{d\}, \quad q\mathcal{H}olieb_{c,d} = q\mathcal{H}olieb_{c+d,0}\{d\},$$

i.e. there are essentially two different types of the (quasi-)Lie bialgebra properads, even and odd ones.

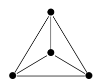
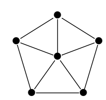
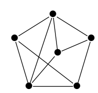
3.2. A short reminder on graph complexes. The M. Kontsevich graph complexes come in a family GC_d parameterized by an integer $d \in \mathbb{Z}$. The complex GC_d for fixed d is generated by arbitrary graphs Γ with valencies $|v|$ of vertices v of Γ satisfying $|v| \geq 3$, and with the orientation *or* defined on each graph $\Gamma \in \mathsf{GC}_d$ as an ordering of edges (up to an even permutation) for d even, and an ordering of vertices and half edges (again up to even permutation); each graph Γ has precisely two different orientations, *or* and $-or$, and one identifies $(\Gamma, or) = -(\Gamma, -or)$ and abbreviates the pair (Γ, or) to Γ. The cohomological degree of $\Gamma \in \mathsf{GC}_d$ is defined by

$$|\Gamma| = d(\#V(\Gamma) - 1) + (1-d)\#E(\Gamma)$$

The differential δ on GC_d is given by an action, $\delta\Gamma = \sum_v \delta_v \Gamma$, on each vertex $v =$  of a graph $\Gamma \in \mathsf{GC}_d$ by splitting v into two new vertices connected by an edge, and then re-attaching the edges attached earlier to v to the new vertices in all possible ways,

$\delta_v :$  $\longrightarrow \sum$  .

It is very hard to compute the cohomology classes GC_d explicitly. Here are two examples of degree zero cycles in GC_2

$\mathfrak{w}_3 =$  , $\mathfrak{w}_5 =$  $+ \frac{5}{2}$  ,

which represent non-trivial cohomology classes. It has been proven in [W1] that $H^0(\mathsf{GC}_2) = \mathfrak{grt}_1$, the Lie algebra of the Grothendieck-Teichmüller group GRT_1. Interestingly in the present context, the graph complexes GC_{c+d+1} control [MW2] the homotopy theory of properads $\mathcal{H}olieb_{c,d}$ and $\mathcal{L}ieb_{c,d}$, i.e. there is a quasi-isomorphism, up to one rescaling class, of dg Lie algebras

$$\mathsf{GC}_{c+d+1} \longrightarrow \mathrm{Der}(\mathcal{H}olieb_{c,d}) \simeq \mathsf{Def}(\mathcal{L}ieb_{c,d} \stackrel{\mathrm{Id}}{\to} \mathcal{L}ieb_{c,d})[1]$$

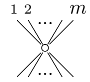
where $\mathrm{Der}(\mathcal{H}olieb_{c,d})$ is the derivation complex of the completion of the properad $\mathcal{H}olieb_{c,d}$ with respect to the filtration by the genus (or the loop number) of the generating graphs.

The graph complex HGC_d^N with N labelled hairs is defined similarly — the only novelty is that each graph Γ in HGC_d^N has precisely N hairs (or legs) attached to its vertex or vertices. Again each vertex must be at least trivalent (with hairs counted), and the differential δ acts on vertices as before. One can understand hairs as kind of special univalent vertices on which δ does not act; they are assigned the same cohomological degree $1-d$ as edges. The hairy graph complexes has been introduced and studied recently in the context of the theory of moduli spaces $\mathcal{M}_{g,N}$ of algebraic curves of arbitrary genus g with N punctures [CGP] and the theory of long knots [FTW]. It has been proven in [CGP] that there is an isomorphism of cohomology groups

$$H^\bullet(\mathsf{HGC}_0^N) = \prod_{2g+N\geq 4} W_0 H_c^{\bullet - N} \mathcal{M}_{g,N}$$

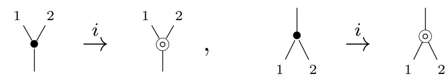
where $W_0 H_c^\bullet \mathcal{M}_{g,N}$ stands for the weight zero summand of the compactly supported cohomology of the moduli space $\mathcal{M}_{g,N}$.

3.3. Partial twisting of properads under $\mathcal{L}ieb_{c,d}$. Let $\mathcal{P} = \{\mathcal{P}(m,n), \partial\}_{m,n\geq 0}$ be a dg properad. We represent its generic elements pictorially as (m,n)-corollas

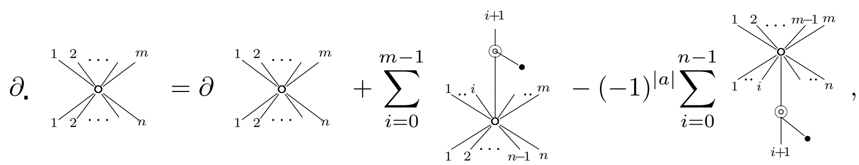
(23) 

whose white vertex is decorated by an element of $\mathcal{P}(m,n)$. Properadic compositions in $\mathcal{P}$ are represented pictorially by gluing out-legs of such decorated corollas to in-legs of another decorated corollas.

Assume $\mathcal{P}$ comes equipped with a non-trivial morphism

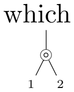
(24) $i : \mathcal{L}ieb_{c,d} \longrightarrow \mathcal{P} :$ 

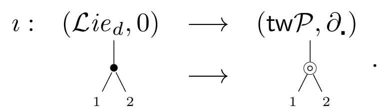
The images under i of the generators of $\mathcal{L}ieb_{c,d}$ are special elements of $\mathcal{P}$ and hence we reserve a special notation $\circledcirc$ for the decoration of the associated corollas. In particular, $\mathcal{P}$ is a properad under $\mathcal{L}ie_d$ and hence can be twisted in the full analogy to the case of operads discussed in the previous section: applying T. Willwacher twisting endofunctor to $(\mathcal{P}, \partial)$ we obtain a dg properad $(\mathsf{tw}\mathcal{P}, \partial_\bullet)$ called the *partial twisting of a properad $\mathcal{P}$ under $\mathcal{H}olieb_{c,d}$*. The latter is freely generated by $\mathcal{P}$ and an extra generator of degree d. The twisted differential $\partial_\bullet$ acts on the latter generator by the standard formula (16), while its action on elements of $\mathcal{P}$ is given by the following obvious analogue of (15),

(25) 

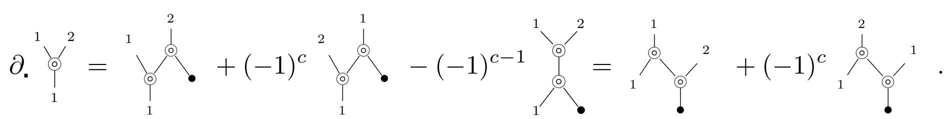
The twisted properad comes equipped with a natural epimorphism of dg properads

$$(\mathsf{tw}\mathcal{P}, \partial_\bullet) \longrightarrow (\mathcal{P}, \partial)$$

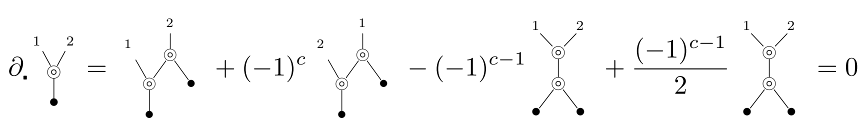
which sends the MC generator to zero. According to the general twisting machinery, the element  remains a cocycle in $\mathcal{P}$ even after the twisting of the original differential so that the original morphism i extends to the twisted version by the same formula,

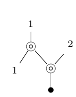
(26) 

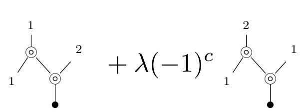
However the image of the co-Lie generator in $\mathcal{P}$ is *not*, in general, respected by the twisted differential,



where we used the image under i of the third relation in (20) (and ordered vertices from bottom to the top). The first equality in the formula just above, the formula (16) and the Drinfeld compatibility condition (that is, the bottom relation in (21)) imply



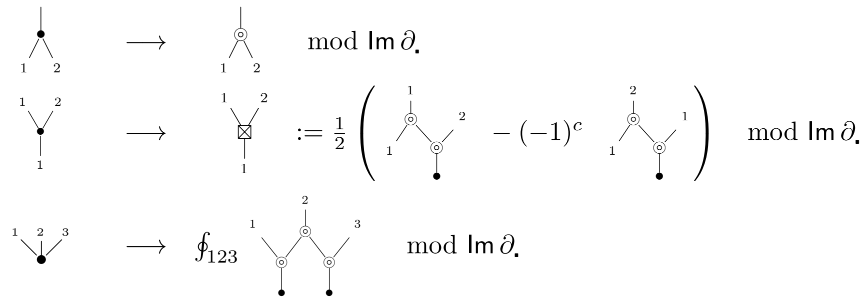
which in turn implies that the element  $\in \mathsf{tw}\mathcal{P}$ is a cycle with respect to the twisted differential $\partial_\bullet$. The linear combination



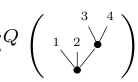
is a $\partial_\bullet$-coboundary for $\lambda = 1$, but for other values of the parameter λ, say for $\lambda = -1$, it represents, in general, a non-trivial cohomology class in $H^\bullet(\mathsf{tw}\mathcal{P}, \partial_\bullet)$ of cohomological degree $2 - c$.

3.4. Theorem on partial twisting and quasi-Lie bialgebras. *Let $\mathcal{P}$ be a dg properad under $\mathcal{L}ieb_{c,d}$ and $\mathsf{tw}\mathcal{P}$ the associated twisting of $\mathcal{P}$ as a properad under $\mathcal{L}ie_d$. Then there is an explicit morphism of properads*

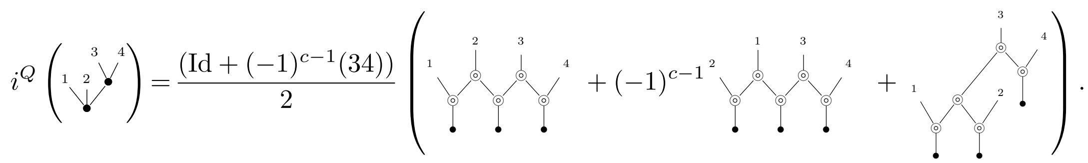
$$i^Q: \quad q\mathcal{L}ieb_{c-1,d} \longrightarrow H^\bullet(\mathsf{tw}\mathcal{P}, \partial_\bullet)$$



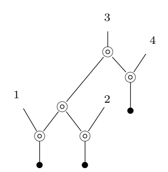
Proof. Proof is a straightforward but rather tedious calculation. Remarkably, the first and and the fourth relations in the list $\mathcal{R}_Q$ above hold true exactly. However, the remaining third relation holds true only up to $\partial_\bullet$-exact terms. Let us check it in full details that the map i^Q satisfies

(27) $\mathrm{Alt}^{c-1}_{\mathbb{S}_4}\ i^Q$  $= 0 \bmod \mathsf{Im}\,\partial_\bullet$.

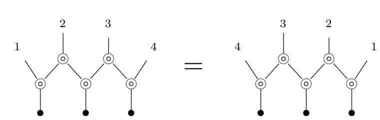
We have



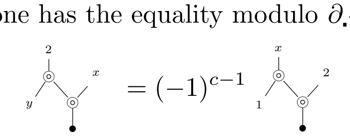
The Jacobi identity for the Lie generator implies the following vanishing

$\mathrm{Alt}^{c-1}_{\mathbb{S}_4}$  $= 0$

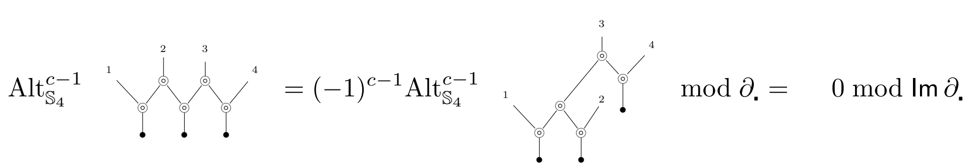
The symmetry properties of the generators imply, for any $c, d \in \mathbb{Z}$, the equality



Therefore the first two summands in the above formula do not cancel out upon (skew)symmetrization. However one has the equality modulo $\partial_\bullet$-exact terms,



which can be used to transform first two terms in the above formula into the third one (up to a permutation) which has been just considered. Hence



and the claim follows. $\square$

3.5. Hairy graph complexes and $\mathsf{tw}\mathcal{L}ieb_{c,d}$. The epimorphism

$$\mathsf{tw}\mathcal{H}olieb_{c,d} = \{\mathsf{tw}\mathcal{H}olieb_{c,d}(N,M)\} \longrightarrow \mathsf{tw}\mathcal{L}ieb_{c,d} = \{\mathsf{tw}\mathcal{L}ieb_{c,d}(N,M)\}$$

is a quasi-isomorphism for any $M, N \geq 1$ as tw is an exact functor. It is a straightforward inspection to see the complex $\mathsf{tw}\mathcal{H}olieb_{c,d}(N,0)$ is identical to the oriented hairy graph complex $\mathsf{HHOGC}^N_{c+d+1}[-dN]$ introduced in §3.4.2 of [AWZ]. One of the main results in that paper says that there is an isomorphism of cohomology groups

$$H^\bullet(\mathsf{HGC}^N_d) \simeq H^\bullet(\mathsf{HHOGC}^N_{d+1})$$

Hence we can conclude that *for any natural number $N \geq 1$ one has an isomorphism of cohomology groups*

$$H^\bullet\left(\mathsf{tw}\mathcal{H}olieb_{c,d}(N,0)\right) \simeq H^\bullet(\mathsf{tw}\mathcal{L}ieb_{c,d}(N,0)) \simeq H^{\bullet - dN}(\mathsf{HGC}^N_{c+d}).$$

In particular, we have an induced morphism

$$H^\bullet(\mathsf{HGC}^N_{c+d}) \longrightarrow H^{\bullet + dN}(\mathsf{tw}\mathcal{P}(N,0))$$

for any dg properad $\mathcal{P} \in \mathsf{PROP}_{\mathcal{H}olieb_{c,d}}$.

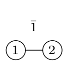
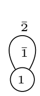
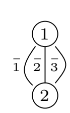
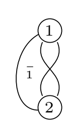
3.6. An example: (chain) gravity properad. A ribbon graph Γ is a graph with an extra structure: the set of half-edges attached to each vertex comes equipped with a cyclic ordering (a detailed definition can be found, e.g., in §4.1 of the paper [MW1] to which we refer often in this subsection). Thickening each vertex $v \in V(\Gamma)$ of a ribbon graph Γ into a closed disk, and every edge $e \in \Gamma$ attached to v into a 2-dimensional strip glued to that disk, one associates to Γ a unique topological 2-dimensional-surface with boundaries; the set of such boundaries is denoted by $B(\Gamma)$. Shrinking 2-strips back into 1-dimensional edges, one represents each boundary b as a closed path comprising some vertices and edges of Γ. We work with *connected* ribbon graphs only, their genus is defined by

$$g = 1 + \frac{1}{2}\left(\#E(\Gamma) - \#V(\Gamma) - \#B(\Gamma)\right). \tag{28}$$

Let $\mathcal{RG}ra_d(m,n)$, $d \in \mathbb{Z}$, stand for the graded vector space generated by ribbon graphs Γ such that

(i) Γ has precisely n vertices and m boundaries which are labelled, i.e. some isomorphisms $V(\Gamma) \to [n]$ and $B(\Gamma) \to [\bar{m}] := \{\bar{1}, \ldots, \bar{m}\}$ are fixed;
(ii) Γ is equipped with an orientation which is for d even is defined as an ordering of edges (up the sign action of $\mathbb{S}_{\#E(\Gamma)}$), while for d odd it is defined as a choice of the direction on each edge (up to the sign action of $\mathbb{S}_2$).
(iii) Γ is assigned the cohomological degree $(1-d)\#E(\Gamma)$.

For example,

 $\in \mathcal{RG}ra_d(1,2)$,  $\in \mathcal{RG}ra_d(2,1)$,  $\in \mathcal{RG}ra_d(3,2)$,  $\in \mathcal{RG}ra_d(1,2)$.

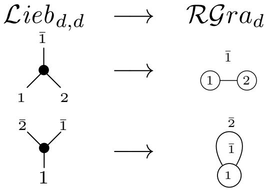
The subspace of $\mathcal{RG}ra_d(m,n)$ spanned by ribbon graphs of genus g is denoted by $\mathcal{RG}ra_d(g;m,n)$. The permutation group $\mathbb{S}^{op}_\cdot \times \mathbb{S}_n$ acts on $\mathcal{RG}ra_d(m,n)$ by relabelling vertices and boundaries. The $\mathbb{S}$-bimodule

$$\mathcal{RG}ra_d = \{\mathcal{RG}ra_d(m,n)\}$$

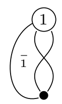
has the structure of a properad [MW1] given by substituting a boundary b of one ribbon graph into a vertex v of another one, and reattaching half-edges (attached earlier to v) among the vertices belonging to b in all possible ways while respecting the cyclic orders of both sets. One of the main motivation behind this definition of $\mathcal{RG}ra_d$ is that it comes with a morphism of operads,

$$i: \quad \mathcal{L}ieb_{d,d} \longrightarrow \mathcal{RG}ra_d$$

(29)



In particular, $\mathcal{RG}ra_d$ is a properad under $\mathcal{L}ie_d$ and hence can be twisted: $\mathsf{tw}\mathcal{RG}ra_d$ is generated by ribbon graphs with two types of vertices, white ones which are labelled and black ones which are unlabelled and assigned the cohomological degree d (cf. §**2.10**), e.g.

 $\in \mathsf{tw}\mathcal{RG}ra_d(1,1).$

The differential $\delta_\bullet$ in $\mathsf{tw}\mathcal{RG}ra_d$ is determined by its action on vertices as in (19). One of the main results in [Me2] is the proof of the following

3.6.1. Theorem. (i) *For any $g \geq 0$, $m \geq 1$ and $n \geq 0$ with $2g+m+n \geq 3$ one has an isomorphism of $\mathbb{S}_m^{op} \times \mathbb{S}_n$-modules,*

$$H^\bullet(\mathsf{tw}\mathcal{RG}ra_d(g;m,n)) = H_c^{\bullet-m+d(2g-2+m+n)}(\mathcal{M}_{g,m+n} \times \mathbb{R}^m)$$

where $\mathcal{M}_{g,m+n}$ is the moduli space of genus algebraic curves with $m+n$ marked points, and $H_c^\bullet$ stands for the compactly supported cohomology functor.

(ii) *For any $g \geq 0$, $m \geq 1$ and $n \geq 0$ with $2g+m+n < 3$ one has*

$$H^k(\mathsf{tw}\mathcal{RG}ra_d(g;m,n)) = \begin{cases} \mathbb{K} & \text{if } g=n=0, m=2, k=(1-d)p \text{ with } p \geq 1 \text{ and } p \equiv 2d+1 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

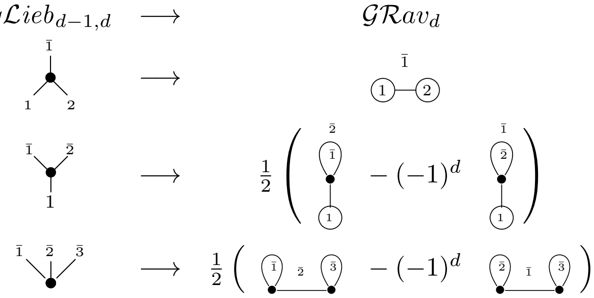
where $\mathbb{K}$ is generated by the unique polytope-like ribbon graph with p edges and p bivalent vertices which are all black.

This result says that the most important part of $\mathsf{tw}\mathcal{RG}ra_d$ is the dg sub-properad $\mathcal{C}h\mathcal{G}rav_d$ spanned by ribbon graphs with black vertices at least trivalent; it is called the *chain gravity properad.* Its cohomology

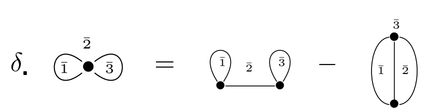
$$\mathcal{G}rav_d := \left\{ \prod_{\substack{g\geq 0 \\ 2g \geq 3-m-n}} H_c^{\bullet-m+d(2g-2+m+n)}(\mathcal{M}_{g,m+n} \times \mathbb{R}^m) \right\}_{m\geq 1, n\geq 0}$$

is called the *gravity* properad. The general morphism i^Q from Theorem **3.4** reads in this concrete situation as follows

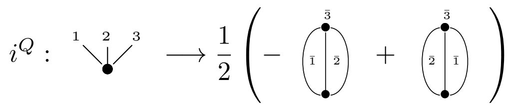
$$i^Q: \quad q\mathcal{L}ieb_{d-1,d} \longrightarrow \mathcal{GR}av_d$$



We have in $\mathsf{tw}\mathcal{RG}ra_d$



so that the above map can be re-written exactly in form first found in [Me2] via a completely independent calculation using ribbon graphs only,



This version of the map i^Q was used in [Me2] to show that this map is injective on infinitely elements of $q\mathcal{L}ieb_{d-1,d}$ constructing thereby infinitely many higher genus cohomology classes in $H^\bullet(\mathcal{M}_{g,m+n})$ from the unique cohomology class in $H^\bullet(\mathcal{M}_{0,3})$ via properadic compositions in $\mathcal{G}rav_d \subset H^\bullet(\mathsf{tw}\mathcal{RG}ra_d)$.

3.7. Quasi-Lie bialgebra structures on Hochschild cohomologies of cyclic A_∞-algebras. Let A be a graded vector space equipped with a degree $-n$ non-degenerate scalar product

$$\begin{array}{rccc} \langle\ ,\ \rangle: & A\odot A & \longrightarrow & \mathbb{K}[-n] \\ & a\odot b & \longrightarrow & \langle a,b\rangle = (-1)^{|a||b|}\langle a,b\rangle. \end{array} \tag{30}$$

One has an associated isomorphism of graded vector spaces,

$$A \simeq A^*[-n] := \mathrm{Hom}(A,\mathbb{K})[-n],$$

and an induced non-degenerate pairing

$$\begin{array}{rccc} \Theta: & \otimes^2(A[n-1]) & \longrightarrow & \mathbb{K}[n-2]\equiv\mathbb{K}[1-(3-n)] \\ & (a'=\mathfrak{s}^{n-1}a, b'=\mathfrak{s}^{n-1}b) & \longrightarrow & \mathfrak{s}^{2n-2}\langle a,b\rangle \end{array}$$

which satisfies the following equation (cf. §2.3 in [MW1] with $d=3-n$ in the notation of that paper)

$$\begin{array}{rcl} \Theta(b',a') &=& \mathfrak{s}^{2n-2}\langle b,a\rangle \\ &=& (-1)^{|a||b|}\mathfrak{s}^{2n-2}\langle a,b\rangle \\ &=& (-1)^{(|a'|+n-1)(|b'|+n-1)}\Theta(a',b') \\ &=& (-1)^{|a'|+|b'|+(3-n)}\Theta(a',b') \end{array}$$

where we used the fact that $\Theta(a,b)=0$ unless $|a|+|b|=n$. By Theorem 4.2.2 in [MW1] this symmetry equation implies that the (reduced) space of cyclic word

$$Cyc(A) := \bigoplus_{p\geq 2}(\otimes^p(A[n-1]))^{\mathbb{Z}_p} \simeq \bigoplus_{p\geq 2}(\otimes^p(A[n-1]))^{\mathbb{Z}_p}$$

carries canonically a representation of the properad $\mathcal{RG}ra_{3-n}$ discussed in the previous subsection. In particular this space is a $\mathcal{L}ieb_{3-n,3-n}$-algebra (see (29)) with the Lie bracket given by a simple formula,

$$\{(a_1'\otimes ...\otimes b_k')^{\mathbb{Z}_k}, (b_1'\otimes...\otimes b_l')^{\mathbb{Z}_n}\} :=$$
$$\sum_{i=1}^k\sum_{j=1}^l \pm\Theta(a_i',b_j')(a_1'\otimes ...\otimes a_{i-1}'\otimes b_{j+1}'\otimes ...\otimes b_l'\otimes b_1'\otimes ...\otimes b_{j-1}'\otimes a_{i+1}'\otimes\ldots\otimes a_k')^{\mathbb{Z}_{k+l-2}}$$

Maurer Cartan elements elements of this $\mathcal{L}ie_d$-algebra are degree $d=3-n$ elements $\gamma\in Cyc(A)$ such that $\{\gamma,\gamma\}=0$. There is a one-to-one correspondence between such Maurer-Cartan elements and[2] *degree n cyclic strongly homotopy algebra structures in A*. The dg Lie algebra

$$CH(A) := (Cyc(A), d_\gamma := \{\gamma,\ \})$$

is precisely the (reduced) cyclic Hochschild complex of the cyclic A_∞-algebra (A,γ).

By the very definition of the twisting endofunctor tw, the chain gravity properad $\mathcal{C}h\mathcal{G}rav_{3-n}$ admits a canonical representation in $CH(A)$ for any degree n cyclic A_∞-algebra A. In particular,

[2]One can use this statement as a definition of a degree n cyclic A_∞-algebra structure on A.

the gravity properad $\mathcal{G}rav_{3-n}$ acts on its cohomology $H^\bullet(CH(A))$ implying, by Theorem **3.4**, the following corollary.

3.7.1. Corollary. *The Hochschild cohomology $H^\bullet(CH(A))$ of any degree n cyclic A_∞ algebra is a quasi-Lie bialgebra; more precisely it carries a representation of the properad $q\mathcal{L}ieb_{2-n,3-n}$.*

If A is a dg Poincaré model of some compact n-dimensional manifold, then there is a linear map

$$\bar{H}^{S^1}_\bullet(LM) \longrightarrow H^\bullet(CH(A))$$

from the reduced equivariant homology $\bar{H}^{S^1}_\bullet(LM)$ of the free loop space LM of M. If M is simply connected, this map is an isomorphism so that the gravity properad $\mathcal{G}rav_d$ acts on $\bar{H}^{S^1}_\bullet(LM)$. However the Maurer–Cartan element associated to any Poincaré model is a relatively simple linear combination of cyclic words in the letters, and the action we just mentioned is much trivialized. That "trivialization" is studied in detail in [Me4] where it is shown that the action of $\mathcal{C}h\mathcal{G}rav_{3-n}$ on $CH(A)$ factors through a quotient properad $\mathcal{ST}_{3-n}$ which contains $\mathcal{L}ieb_{d,d}$ [CS], the gravity operad [G, We] and the four $\mathcal{H}olieb^\diamond_{d-1}$-operations found in [Me3].

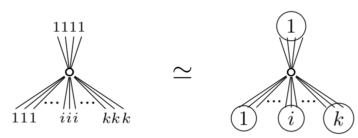
One has to consider a less trivial class (comparing to the class of Poincare models) of cyclic A_∞-algebras A in order to get a chance to see a less trivial action of the gravity properad on the associated cyclic Hochschild cohomologies.

4. A full twisting of properads under $\mathcal{H}olieb_{c,d}$

4.1. Reminder on the polydifferential functor $\mathcal{O}$. There is an exact polydifferential functor [MW1]

$$\begin{array}{cccc} \mathcal{O}: & \textsf{Category of dg props} & \longrightarrow & \textsf{Category of dg operads} \\ & \mathcal{P} & \longrightarrow & \mathcal{O}\mathcal{P} \end{array}$$

which has the following property: given any dg prop $\mathcal{P}$ and an arbitrary representation $\rho: \mathcal{P} \to \mathcal{E}nd_V$ in a dg vector space V, the associated dg operad $\mathcal{O}\mathcal{P}$ has an associated representation, $\mathcal{O}\rho: \mathcal{O}(\mathcal{P}) \to \mathcal{E}nd_{\odot^\bullet V}$, in the graded commutative tensor algebra $\odot^\bullet V$ given in terms of polydifferential (with respect to the standard multiplication in $\odot^\bullet V$) operators. Roughly speaking the functor $\mathcal{O}$ symmetrizes all outputs of elements of $\mathcal{P}$, and splits all inputs into symmetrized blocks; pictorially, if we identify elements of $\mathcal{P}$ with decorated corollas as in (23), then every element of $\mathcal{O}(\mathcal{P})$ can be identified with a decorated corolla which is allowed to have the *same* numerical labels assigned to its different in-legs, and also with the same label 1 assigned to its all outgoing legs[3]



Since we want to apply the above construction to dg props $\mathcal{P}$ under $\mathcal{H}olieb_{c,d}$, we are more interested in this paper in its degree shifted version, $\mathcal{O}_{c,d}$, which was also introduced in [MW1],

$$\mathcal{O}_{c,d}\mathcal{P} := \mathcal{O}(\mathcal{P}\{c\})$$

The notation may be slightly misleading as $\mathcal{O}_{c,d}$ does not depend on d but it suits us well in the context of this paper. We refer to [MW2] for more details about the functor $\mathcal{O}_{c,d}$ (and to [MW3] for its extension $\mathcal{D}$) and discuss next the particular example, the dg operad

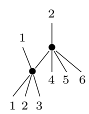
$$\mathcal{O}_{c,d}\mathcal{H}olieb_{c,d} \simeq \mathcal{O}\mathcal{H}olieb_{0,c+d} \equiv \{\mathcal{O}\mathcal{H}olieb_{0,c+d}(k)\}_{k\geq 1}.$$

The $\mathbb{S}_k$-module $\mathcal{O}\mathcal{H}olieb_{0,c+d}(k)$, $k \geq 1$, is generated by graphs γ constructed from arbitrary decorated graphs Γ from $\mathcal{H}olieb_{0,c+d}(m,n)$, $\forall m,n \geq 1$, as follows:

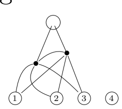
[3] Treating out- and inputs legs in this procedure on equal footing, one gets a polydifferential functor $\mathcal{D}$ in the category of dg props such that $\mathcal{O}(\mathcal{P})$ a sub-operad of $\mathcal{D}(\mathcal{P})$. It was introduced and studied in [MW3].

(i) draw new k big white vertices labelled from 1 to k (these will be inputs of γ) and one extra output big white vertex,
(ii) symmetrize all m outputs legs of Γ and attach them to the unique output white vertex;
(iii) partition the set $[n]$ of input legs of Γ into k ordered disjoint (not necessary non-empty) subsets
$$[n] = I_1 \sqcup \ldots \sqcup I_k, \qquad \# I_i \geq 0, i \in [k],$$
then symmetrize the legs in each subset I_i and attach them (if any) to the i-labelled input white vertex.

For example, the element

$$\Gamma =$$  $$\in \mathcal{H}olieb_{0,c+d}(2,6)$$

can produce the following generator

$$\gamma =$$  $$\in \mathcal{OH}olieb_{0,c+d}(4) \simeq \mathcal{O}_{c,d}\mathcal{H}olieb_{c,d}(4)$$

in the associated polydifferential operad (note that one and the same element $\Gamma \in \mathcal{H}olieb_{0,c+d}$ can give rise to several different generators of $\mathcal{OH}olieb_{0,c+d}$). The labelled white vertices of elements of $\mathcal{O}(\mathcal{H}olieb_{0,c+d}$ are called *external*, while unlabelled black vertices (more, precisely, the vertices of the underlying elements of $\mathcal{H}olieb_{0,c+d}$) are called *internal*. The same terminology can applied to $\mathcal{OP}$ for any dg prop $\mathcal{P}$.

For any $k, l \geq 1$ and $i \in [k]$ the operadic composition

$$\begin{array}{rccc} \circ_i : & \mathcal{OH}olieb_{0,c+d}(k) \otimes \mathcal{OH}olieb_{0,c+d}(k) & \longrightarrow & \mathcal{OH}olieb_{0,c+d}(k+l-1) \\ & \Gamma_1 \otimes \Gamma_2 & \longrightarrow & \Gamma_1 \circ_i \Gamma_2 \end{array}$$

is defined by

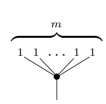
(i) substituting the graph Γ_2 (with the output external vertex erased so that all edges connected to that external vertex are hanging at this step loosely) inside the big circle of the i-labelled external vertex of Γ_1,
(ii) erasing that big i-th labelled external circle (so that all edges of Γ_1 connected to that i-th external vertex, if any, are also hanging loosely), and
(iii) finally taking the sum over all possible ways to do the following three operations in any order,
  (a) glue some (or all or none) hanging edges of Γ_2 to the same number of hanging edges of Γ_1,
  (b) attach some (or all or none) hanging edges of Γ_2 to the output external vertex of Γ_1,
  (c) attach some (or all or none) hanging edges of Γ_1 to the external input vertices of Γ_2,
in such a way that no hanging edges are left.

We refer to [MW1, MW3] for concrete examples of such compositions.

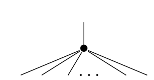
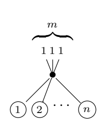
4.1.1. Proposition [MW1]**.** *There is a morphism of dg operads*

$$\mathcal{H}olie^+_{c+d} \to \mathcal{O}_{c,d}\mathcal{H}olieb_{c,d} \tag{31}$$

given explicitly on the $(1,1)$*-generator by*

 $$\longrightarrow \sum_{m\geq 2}$$ 

and on the remaining $(1,n)$*-generators with* $n \geq 2$ *by*

 $$\longrightarrow \sum_{m\geq 1}$$ 

Proof is a straightforward calculation (cf. §5.5 and §5.7 in [MW1]).

Using T. Willwacher's twisting endofuctor discussed in §2 one obtains via the morphism (31) a dg operad $\mathsf{tw}\,\mathcal{O}_{c,d}\mathcal{H}olieb_{c,d}$.

4.1.2. Properads under $\mathcal{H}olieb_{c,d}$. Assume $\mathcal{P}$ is a dg properad *under* $\mathcal{H}olieb_{c,d}$, i.e. the one which comes equipped with a non-trivial morphism

$$
i: \quad \begin{array}{ccc} \mathcal{H}olieb_{c,d} & \longrightarrow & \mathcal{P} \\ \begin{array}{c} {\scriptstyle 1\,2\,\ldots\,m} \\ \times\!\!\!\bullet \\ {\scriptstyle 1\,2\,\ldots\,n} \end{array} & \longrightarrow & \begin{array}{c} {\scriptstyle 1\,2\,\ldots\,m} \\ \times\!\!\!\circledcirc \\ {\scriptstyle 1\,2\,\ldots\,n} \end{array} \end{array} \tag{32}
$$

Note that corollas on the right hand side (the ones with $\circledcirc$ as the vertex) stand from now on for images of the generators of $\mathcal{H}olieb_{c,d}$ under the map i so that *some (or all) of them can in fact be equal to zero.*

Applying the functor $\mathcal{O}_{c,d}$ and using the above proposition we obtain an associated chain of morphism of dg operads,

$$
\iota : \mathcal{H}olie^{+}_{c+d} \longrightarrow \mathcal{O}_{c,d}\mathcal{H}olieb_{c,d} \longrightarrow \mathcal{O}_{c,d}\mathcal{P}
$$

and hence a morphism of the associated twisted dg operads,

$$
\mathsf{tw}\mathcal{O}(i) : \mathsf{tw}\mathcal{O}_{c,d}\mathcal{H}olieb_{c,d} \simeq \mathsf{tw}\mathcal{O}(\mathcal{H}olieb_{0,c+d}) \longrightarrow \mathsf{tw}\mathcal{O}_{c,d}\mathcal{P}
$$

The degree of the generating (m,n)-corolla $\begin{array}{c} {\scriptstyle\ldots} \\ \times\!\!\!\bullet \\ {\scriptstyle\ldots} \end{array}$ of $\mathcal{H}olieb_{0,c+d}$ is equal to $1+c+d-(c+d)n$, so its m out-legs are carry trivial representation of $\mathbb{S}_m$ (and are assigned degree 0), while its in-legs are (skew)symmetrized according to the parity of $c+d \in \mathbb{Z}$ (and assigned degree $(c+d)$; the vertex is assigned the degree $1+c+d$. *Hence it is only the sum $c+d$ of our integer parameters which plays a role in this story.* Therefore we can assume without loss of generality that

$$
c = 0, \quad d \text{ is an arbitrary integer,}
$$

from now on, i.e. work solely with dg props under $\mathcal{H}olieb_{0,d}$.

4.2. Maurer-Cartan elements of strongly homotopy Lie bialgebras. Given a $\mathcal{H}olieb_{0,d}$-algebra structure in a dg vector space (V,δ), i.e. a morphism of properads

$$
\rho : \mathcal{H}olieb_{0,d} \longrightarrow \mathcal{E}nd_V.
$$

Its *Maurer-Cartan element* is, by definition, a Maurer-Cartan element $\gamma \in \odot^{\geq 1}V$ of the associated $\mathcal{H}olie^{+}_{d}$ structure induced on $\odot^{\geq 1}V$ via the canonical monomorphism

$$
\mathcal{H}olie^{+}_{d} \to \mathcal{O}\mathcal{H}olieb_{0,d}
$$

described explicitly in Proposition **4.1.1**. Let us describe it in more detail. The $\mathcal{H}olieb_{0,d}$-structure on V is given by a collection of linear maps of cohomological degree $1+d-dn$,

$$
\rho\left(\begin{array}{c} {\scriptstyle 1\,2\,\ldots\,m} \\ \times\!\!\!\bullet \\ {\scriptstyle 1\,2\,\ldots\,n} \end{array} \right) =: \mu_{m,n} : \otimes^n V \longrightarrow \odot^m V
$$

satisfying compatibility conditions. Each such linear map gives rise to a map

$$
\hat{\mu}_{m,n} : \otimes^n(\odot^{\geq 1}V) \longrightarrow \odot^{\geq 1}V
$$

given, in arbitrary basis $\{p_\alpha\}$ of V as follows

$$
\hat{\mu}_{m,n}(f_1,\ldots,f_n) := \sum \pm \mu_{m,n}(p_{\alpha_1} \otimes p_{\alpha_2} \otimes \ldots \otimes p_{\alpha_n}) \cdot \frac{\partial f_1}{\partial p_{\alpha_1}} \frac{\partial f_2}{\partial p_{\alpha_2}} \cdots \frac{\partial f_n}{\partial p_{\alpha_n}}, \quad \forall f_1,\ldots,f_n \in \odot^{\geq 1}V.
$$

Then a degree d element $\gamma \in \odot^{\geq 1}V$ is a Maurer-Cartan element of the (appropriately filtered or nilpotent) $\mathcal{H}olieb_{0,d}$-algebra on structure V if and only if the following equation holds,

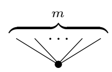
$$
\delta\gamma + \sum_{m,n\geq 1} \frac{1}{n!} \hat{\mu}_{m,n}(\underbrace{\gamma,\ldots,\gamma}_{n}) = 0. \tag{33}
$$

The operator

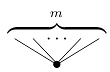
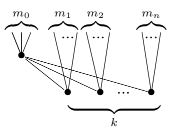
$$\delta_\gamma v := \delta v + \sum_{n\geq 1} \frac{1}{n!}\mu_{1,n+1}(\underbrace{\gamma_1,\ldots,\gamma_1}_{n}, v), \qquad \forall\, v\in V, \tag{34}$$

with γ_1 being the image of γ under the projection $\odot^{\geq 1}V \to V$, is a twisted differential on V.

4.2.1. Combinatorial incarnation. Maurer-Cartan elements of $\mathcal{H}olieb_{0,d}$-algebras admit a simple combinatorial description as (a representation in V of) an infinite linear combination

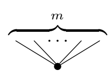
$$\gamma \simeq \sum_{m\geq 1}\frac{1}{m!}$$ 

of degree d $(m,0)$-corollas with symmetrized outgoing legs. One extends the standard differential δ in $\mathcal{H}olieb_{0,d}$ to such new generating corollas as follows

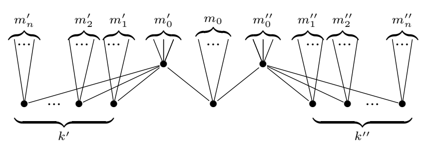
(35) δ  $= - \sum\limits_{\substack{k\geq 1, [m]=\sqcup[m_\bullet]\\ m_0\geq 1, k+m_0\geq 3\\ m_1,\ldots,m_k\geq 0}} \frac{1}{k!}$  $\forall\, m\geq 1.$

where we take the sum over all partitions of the ordered set $[m]$ into $k+1$ ordered subsets. For $k=1$ we recover the standard formula (cf. (8)).

Let us check first that the above definition makes sense.

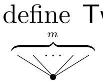
4.2.2. Lemma. δ^2  $\equiv 0$ *for any* $m\geq 1$.

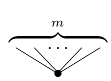
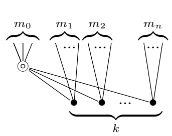
Proof. The proof is based on a straightforward calculation which is similar to the one made in the proof of Lemma **2.5.1** above. The only really new phenomenon is the appearance in δ^2 of summands of the form

(36) $\sum\limits_{[m]=[m'_\bullet]\sqcup[m''_\bullet]\sqcup[m_0]} \frac{1}{k'!}\frac{1}{k''!}$ 

which cancel each other for symmetry reasons. □

4.3. Full twisting of properads under $\mathcal{H}olieb_{c,d}$. Let $\mathcal{P}$ be a properad under $\mathcal{H}olieb_{0,d}$ (as in (32)). We construct the associated *fully twisted* properad $(\mathsf{Tw}\mathcal{P}, \partial_\bullet)$ in several steps.

First we define $\widetilde{\mathsf{Tw}}\mathcal{P}$ to be be the properad generated freely by $\mathcal{P}$ and a family of new $(m,0)$-corollas,  , $m\geq 1$, of cohomological degree d which are called *MC generators*. We make this properad differential by using the original differential ∂ on elements of $\mathcal{P}$ and extending its action on the new generators by (cf. (35))

(37) ∂  $= - \sum\limits_{\substack{k\geq 1, [m]=\sqcup[m_\bullet]\\ m_0\geq 1, k+m_0\geq 3\\ m_1,\ldots,m_k\geq 0}} \frac{1}{k!}$  $\forall\, m\geq 1.$

Note that corollas with $\circledcirc$-vertices are images of the geneartors of $\mathcal{H}olieb_{0,d}$ in $\mathcal{P}$ under the morphism i (and hence some of them can, in principle, be zero). The map (32) extends to a morphism

$$\widetilde{\mathsf{Tw}}(i) : \widetilde{\mathsf{Tw}}\mathcal{H}olieb_{0,d} \longrightarrow \widetilde{\mathsf{Tw}}\mathcal{P}$$

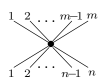
which restricts on the MC generators as the identoty map.

Next we notice the following surprising result which tells us essentially the MC elements originating in $\mathcal{O}(\mathcal{H}olieb_{c,d})$ can be used to twist not only the operad $\mathcal{O}(\mathcal{H}olieb_{c,d})$ (which is obvious) but also the properad $\mathcal{H}olieb_{c,d}$ itself!

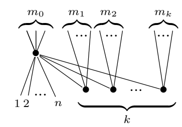
4.3.1. Theorem. *There is a canonical monomorphism of dg props*

$$\mathcal{H}olieb^{+}_{0,d} \longrightarrow \widetilde{\mathsf{Tw}}\mathcal{H}olieb_{0,d}$$

given on the generating (m,n)*-corollas with* $m,n \geq 1$ *as follows,*



$$\longrightarrow \sum_{\substack{k\geq 0,[m]=\sqcup[m_\bullet],\\ m_0\geq 1,k+m_0+n\geq 3\\ m_1,\dots,m_k\geq 0}} \frac{1}{k!}$$



Proof. One has to check that the above explicit map of properads respects their differentials, δ^+ on the l.h.s. and ∂ on the r.h.s. This is a straightforward calculation which is analogous to (but much more tedious than) the one used in the proof of Theorem **2.5.2**. The only really new aspect is again the appearance of terms (36) which cancel out for symmetry reasons. We omit full details. □

Hence for any properad $\mathcal{P}$ under $\mathcal{H}olieb_{0,d}$ there is an associated morphism of dg properads

$$i^+ : \mathcal{H}olieb^{+}_{0,d} \longrightarrow \widetilde{\mathsf{Tw}}\mathcal{P}$$

which factors through the morphism described in the Theorem just above.

4.3.2. Twisting of the differential. The argument in §**2.4** about twisting of differentials in operads extends straightforwardly to properads. Indeed, given any dg prop $(\mathcal{P} = \{\mathcal{P}(m,n)\}, \partial)$, and any $h \in \mathcal{A}(1,1)$, there is an associated derivation D_h of the (non-differential) prop $\mathcal{P}$ by the formula analogous to (6),

$$D_h(a) = \sum_{i=1}^{m} h_1 \circ_i a - (-1)^{|h||a|} \sum_{j=1}^{n} a_j \circ_1 h, \quad \forall\, a \in \mathcal{P}(m,n). \tag{38}$$

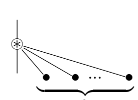
Moreover, if $|h| = 1$ and $h \circ_1 h = -\partial h$, then the sum $\partial_h := \partial + D_h$ is a differential in $\mathcal{P}$.

Assume we have a morphism of dg props

$$g^+ : (\mathcal{H}olieb^{+}_{0,d}, \delta^+) \longrightarrow (\mathcal{P}, \partial) \tag{39}$$

Then the element

$$\blacklozenge := g^+(\bullet) = \sum_{k=1}^{\infty} \frac{1}{k!} \tag{40}$$



$$\in \widetilde{\mathsf{Tw}}\mathcal{P}$$

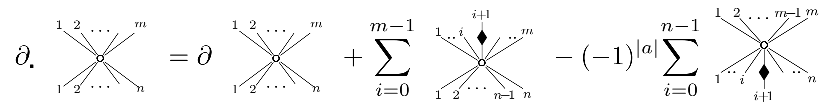
satisfies all the conditions specified above for h so that the sum

$$\partial_\bullet := \partial + D_{\blacklozenge}$$

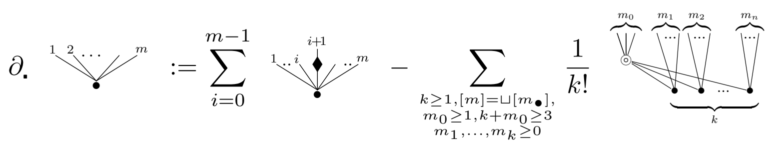
is a differential in $\widetilde{\mathsf{Tw}}\mathcal{P}$.

4.3.3. Main definition. *Let $\mathcal{P}$ be a dg properad under $\mathcal{H}olieb_{0,d}$. The full twisting, $\mathsf{Tw}\mathcal{P}$, of $\mathcal{P}$ is a dg properad defined as the properad $\widetilde{\mathsf{Tw}}\mathcal{P}$ equipped with the twisted differential $\partial_\bullet$.*

Thus $\mathsf{Tw}\mathcal{P}$ is identical to $\widetilde{\mathsf{Tw}}\mathcal{P}$ as a non-differential properad, i.e. it is generated freely by $\mathcal{P}$ and the family of extra generators, $m \geq 1$, of degree d. If we represent elements of $\mathcal{P}$ as decorated corollas (23), then the twisted differential $\partial_\bullet$ acts on elements of $\mathcal{P}$ as follows,

(41) 

where the black diamond is given by (40). On the other hand, the action of $\partial_\bullet$ on the MC generators is given by,

(42)  $\forall\, m \geq 1.$

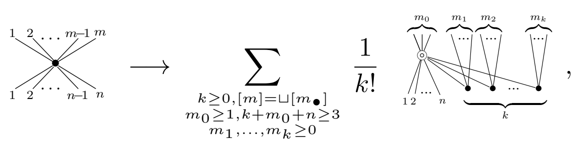
Note that for $m \geq 2$ the first sum on the r.h.s. of (42) cancels out with all the summands corresponding to $k \geq 2, m_0 = 1, m_i = m-1, i \in [k]$, in the second sum.

By its very construction, this twisted prop $\mathsf{Tw}\mathcal{P}$ has the following properties:

(a) There is a canonical chain of morphisms of dg prop(erad)s

$$(\mathcal{H}olieb_{0,d}, \delta) \longrightarrow (\mathsf{Tw}\mathcal{H}olieb_{0,d}, \delta_\bullet) \longrightarrow (\mathsf{Tw}\mathcal{P}, \partial_\bullet).$$

given explicitly by

(43)  $m, n \geq 1, m+n \geq 3$

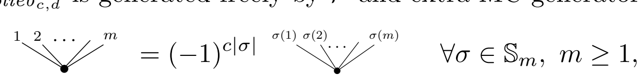
(b) There is a canonical epimorphism of of dg prop(erad)s,

$$(\mathsf{Tw}\mathcal{P}, \partial_\bullet) \longrightarrow (\mathcal{P}, \partial)$$

which sends all the MC generators, $m \geq 1$, to zero. Note that the natural inclusion of $\mathbb{S}$-bimodules $\mathcal{P} \to \mathsf{Tw}\mathcal{P}$ is not, in general, a morphism of *dg* properads.

4.3.4. Remark. Assume $\mathcal{P}$ is a properad under $\mathcal{L}ieb_{0,d}$, i.e. all corollas on the r.h.s. of the map (32) vanish except the following two, and. Then the associated twisted properad $\mathsf{Tw}\mathcal{P}$ is, in general, a properad under the minimal resolution $\mathcal{H}olieb_{0,d}$ of $\mathcal{L}ieb_{0,d}$, not just under $\mathcal{L}ieb_{0,d}$! Put another way, *the full twisting of $\mathcal{P}$ produces, in general, higher homotopy Lie bialgebras operations*, a new phenomenon comparing to what we get in $\mathsf{tw}\mathcal{P}$ under the partial twisting of $\mathcal{P}$.

4.3.5. Full twisting for general values of the integer parameters c and d. *The full twisting $\mathsf{Tw}\mathcal{P}$ of a dg properad $\mathcal{P}$ under $\mathcal{H}olieb_{c,d}$* is defined as $(\mathsf{Tw}\mathcal{P}\{c\})\{-c\}$; note that $\mathcal{P}\{c\}$ is a dg properad under $\mathcal{H}olieb_{0,c+d}$ so that the above twisting functor Tw applies. Thus the full twisting $\mathsf{Tw}\mathcal{P}$ of $\mathcal{P} \in \mathsf{PROP}_{\mathcal{H}olieb_{c,d}}$ is generated freely by $\mathcal{P}$ and extra MC generators

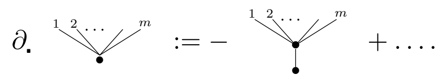


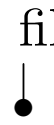
of cohomological degree $(1-m)c + d$. It comes equipped with a canonical morphism $\mathcal{H}olieb_{c,d} \to \mathsf{Tw}\mathcal{P}$ given by (43).

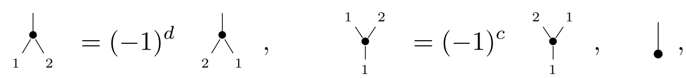
Next we show that quasi-isomorphisms (17) extend to their full twisting analogues.

4.3.6. Theorem. *The canonical projection $\pi : \mathsf{Tw}\mathcal{H}olieb_{c,d} \to \mathcal{H}olieb_{c,d}$ is a quasi-isomorphism, i.e. $H^\bullet(\mathsf{Tw}\mathcal{H}olieb_{c,d}) = \mathcal{L}ieb_{c,d}$.*

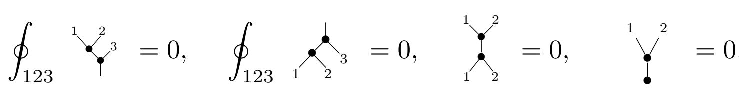
Proof. For any $m \geq 2$ the r.h.s. of formula (42) applied to $\mathcal{P} = \mathcal{H}olieb_{c,d}$ contains a unique summand of the form



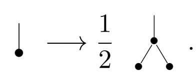
Let us call the unique edge of such a summand *special*, and consider a filtration of $\mathsf{Tw}\mathcal{H}olieb_{c,d}$ by the number of non-special edges plus the total number of MC generators. On the initial page E_0 of the associated spectral sequence the induced differential acts only on MC generators with $m \geq 2$ by the formula given just above (with no additional terms). Hence the next page E_1 of the spectral sequence is equal to the quotient subcomplex $\mathsf{tw}\mathcal{H}olieb'_{c,d}$ of the partially twisted properad $\mathsf{tw}\mathcal{H}olieb_{c,d}$ by the differential ideal generated by graphs with at least one special edge; the induced differential acts only on the generators of $\mathcal{H}olieb_{c,d}$ by the standard formula (22). We consider next a filtration of E_1 by the total number of paths connecting in-legs and univalent MC generators  to the out-legs of elements of E_1. The induced differential d on the associated graded complex grE_1 is precisely the $\frac{1}{2}$-prop differential[4] in $\mathcal{H}olieb_{c,d}$ given explicitly by those summands in (22) whose lower (or upper) corolla has type $(1, p \geq 2)$ (or, resp., $(p \geq 2, 1)$) only. The point is that such summands never create new special edges which have to be set to zero by hand, i.e. the fact that we have to take the quotient by graphs with at least one special edge does not complicate the action of the induced differential any more. Hence the next page $E_2 \sim H^\bullet(grE_1)$ of our spectral sequence is spanned by graphs generated by the following three corollas



subject to the relations



The induced differential acts only on the MC generator by the standard formula



As $H^\bullet(\mathsf{tw}\mathcal{L}ie_d) = \mathcal{L}ie_d$, we conclude that the cohomology is spanned by the standard two generators of $\mathcal{L}ieb_{c,d}$ modulo the above three relations. Hence $H^\bullet(\mathsf{Tw}\mathcal{H}olieb_{c,d}) = \mathcal{L}ieb_{c,d}$ and the Theorem is proven. $\square$

4.4. An action of the deformation complex $\mathsf{Def}(\mathcal{H}olieb_{c,d} \to \mathcal{P})$ on $\mathsf{Tw}\mathcal{P}$. Given a dg properad $\mathcal{P}$ under $\mathcal{H}olieb_{c,d}$ (see (32)), one can consider $\mathcal{P}$ as a dg properad under $\mathcal{H}olieb^+_{c,d}$ using the composition

$$i^+ : \mathcal{H}olieb^+_{c,d} \longrightarrow \mathcal{H}olieb_{c,d} \xrightarrow{i} \mathcal{P}$$

where the first arrow is the unique morphism which sends the $(1,1)$-generator to zero and is the identity on all other generators. Following [MW1] we define the deformation complex of the morphism i in (32) as the deformation complex of the morphism i^+,

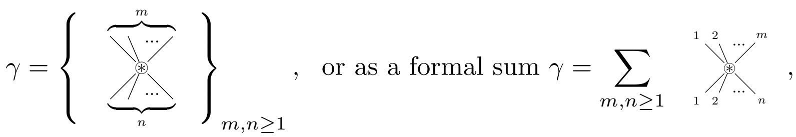
$$\mathsf{Def}\left(\mathcal{H}olieb^+_{c,d} \xrightarrow{i^+} \mathcal{P}\right) = \prod_{m,n\geq 1} \mathcal{P}(m,n) \otimes_{\mathbb{S}^{op}_m \times \mathbb{S}_n} \left(sgn_m^{|c|} \otimes sgn_n^{|d|}\right)[c(1-m) + d(1-n)].$$

Note that even in the case when the properad $\mathcal{P}$ is generated by (m,n)-operations with $m, n \geq 1$ and $m + n \geq 3$ (as, e.g., in the case $\mathcal{P} = \mathcal{H}olieb_{c,d}$) the term $\mathcal{P}(1,1)$ is often non-zero and should

[4]The notion of $\frac{1}{2}$-prop (as well the closely related notion of path filtration) was introduced by Maxim Kontsevich [Ko3]. A nice exposition of this theory can be found in [MaVo].

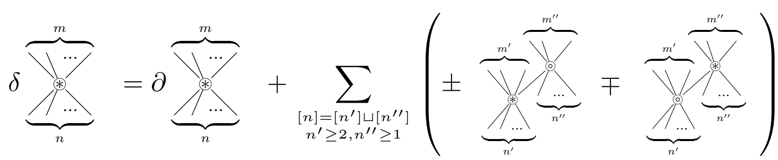
not be ignored in the deformation theory (cf. [MW3]); hence the +-extension. As in [MW2], we abuse notation and re-denote $\mathsf{Def}(\mathcal{H}olieb_{c,d}^{+} \xrightarrow{i^+} \mathcal{P})$ by $\mathsf{Def}(\mathcal{H}olieb_{c,d} \xrightarrow{i} \mathcal{P})$.

Following [MeVa] one can describe the differential (and Lie brackets) in $\mathsf{Def}\left(\mathcal{H}olieb_{c,d} \xrightarrow{i} \mathcal{P}\right)$ very explicitly. Let us represent a generic element of this complex pictorially as a collection of (m,n)-corollas,



of corollas whose vertices are decorated with elements[5] of $\mathcal{P}(m,n) \otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_n} (sgn_m^{|c|} \otimes sgn_n^{|d|})$ and denoted by $\circledast$ in order to distinguish them from the generic elements of $\mathcal{P}$ which are represented as corollas (23) and also from the images of $\mathcal{H}olie_d$-generators under i which are represented pictorially as $\circledcirc$-vertex corollas. Since labels of in- and out legs are (skew)symmetrized, one cam omit them in pictures. The differential in the deformation complex is given by

(44)



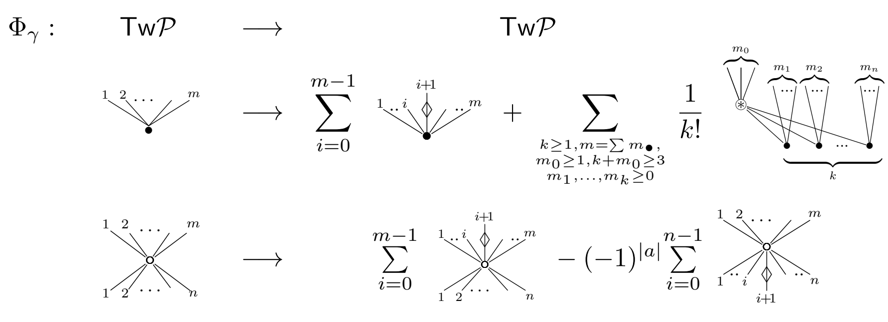
where the rule of signs depends on d and is read from (5); for d even the first ambiguous sign symbol on the r.h.s. is $+1$, while the second one is $-(-1)^{|\circledast|}$.

Let $(\mathrm{Der}(\mathsf{Tw}\mathcal{P}), [\ ,\ ])$ be the Lie algebra of derivations of the properad $\mathsf{Tw}\mathcal{P}$. The differential $\partial_{\bullet}$ is its MC element making $\mathrm{Der}(\mathsf{Tw}\mathcal{P})$ into a dg Lie algebra with the differential $[\partial_{\bullet},\ ]$.

4.4.1. Theorem. *There is a morphism of dg Lie algebras*

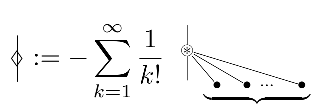
$$
\begin{array}{rccc}
\Phi: & \mathsf{Def}\left(\mathcal{H}olieb_{c,d} \xrightarrow{i} \mathcal{P}\right) & \longrightarrow & \mathrm{Der}(\mathsf{Tw}\mathcal{P}) \\
 & \gamma & \longrightarrow & \Phi_\gamma
\end{array}
\tag{45}
$$

where the derivation Φ_γ is given on the generators by

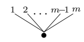


where (cf. (40))

(46)



[5]A formal sum γ of such (m,n)-corollas is homogeneous of degree k as an element of the deformation complex if and only if their degrees $|\circledast|$ as elements of $\mathcal{P}(m,n)\otimes_{\mathbb{S}_m^{op}\times\mathbb{S}_n} (sgn_m^{|c|}\otimes sgn_n^{|d|})$ are equal to $k + c(1-m) + d(1-n)$. This explains the grading consistency of the explicit formulae shown in Theorem **4.4.1** below.

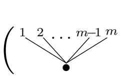
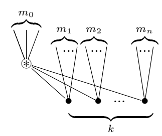
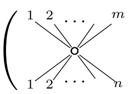
Proof. (A sketch). Any derivation of $\mathsf{Tw}\mathcal{P}$ (viewed as a non-differential properad) is uniquely determined by its values on the MC generators  and on arbitrary elements of $\mathcal{P}$. The first values can be chosen arbitrary, while the second ones must be compatible with the properad compositions; as the second values are, by the definition, of the form (38), we conclude that the above formulae do define a derivation of $\mathsf{Tw}\mathcal{P}$ as a non-differential prop(erad). Hence the main point is to show that Φ_γ respects differentials in both dg Lie algebras, i.e. satisfies the equation

$$\Phi_{\delta\gamma} = [\partial_\bullet, \Phi_\gamma]. \tag{47}$$

Consider first a simpler morphism of non-differential graded Lie algebras,

$$\begin{array}{rccc} \widetilde{\Phi}: & \mathsf{Def}\left(\mathcal{H}olieb_{c,d} \xrightarrow{i} \mathcal{P}\right) & \longrightarrow & \mathrm{Der}(\mathsf{Tw}\mathcal{P}) \\ & \gamma & \longrightarrow & \widetilde{\Phi}_\gamma \end{array}$$

where the derivation $\widetilde{\Phi}_\gamma \in \mathrm{Der}(\mathsf{Tw}\mathcal{P})$ is given on the generators by

$\widetilde{\Phi}_\gamma\left(\right.$  $\left.\right) = \sum_{\substack{k\geq 1, m=\sum m_\bullet,\\ m_0\geq 1, k+m_0\geq 3\\ m_1,\ldots,m_k\geq 0}} \frac{1}{k!}$  , $\quad\widetilde{\Phi}_\gamma\left(\right.$  $\left.\right) = 0.$

The map $\widetilde{\Phi}$ respects the Lie bracket while the obstruction for this map to respect the differentials is given by the derivation of type (38),

$[\partial_\bullet, \widetilde{\Phi}_\gamma] - \widetilde{\Phi}_\gamma = D_{\gamma_1}, \quad \gamma_1 = \widetilde{\Phi}_\gamma\left(\right.$  $\left.\right) \in \mathsf{Tw}\mathcal{P}(1,1),$

with  given by (40). It is a straightforward calculation to check that the adjustment of the derivation $\widetilde{\Phi}_\gamma$ with an extra term of the type (38),

$\widetilde{\Phi}_\gamma \longrightarrow \Phi_\gamma = \widetilde{\Phi}_\gamma + D$ 

solves the problem of the compatibility with the differentials. □

4.5. Grothendieck-Teichmüller group and twisted properads. Let $\widehat{\mathcal{H}olieb}_{c,d}$ be the genus completion of the properad $\mathcal{H}olieb_{c,d}$. It was proven in [MW2] that for any $c, d \in \mathbb{Z}$ there is a morphism of dg Lie algebras

$$F\colon \mathsf{GC}^{or}_{c+d+1} \to \mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d})$$

which is a quasi-isomorphism up to one rescaling class (which controls the automorphism of $\mathcal{H}olieb_{c,d}$ given by rescaling each (m,n) generator by λ^{m+n-2} for any $\lambda \in K^*$). Here GC^{or}_{c+d+1} stands for the oriented version of the Kontsevich graph complex from §**3.2** which was studied in [W2] and where it was proven that

$$H^\bullet(\mathsf{GC}^{or}_3) = H^\bullet(\mathsf{GC}_2) = \mathfrak{grt}_1,$$

This result implies that for any $c, d \in \mathbb{Z}$ with $c + d = 2$, one has an isomorphism of Lie algebras,

$$H^0(\mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d})) = \mathfrak{grt}$$

where $\mathfrak{grt}$ is the Lie algebra of the "full" Grothendieck-Teichmüller group GRT_1 [D2].

Let $\widehat{\mathcal{P}}$ be a dg properad under $\widehat{\mathcal{H}olieb}_{c,d}$ and let $\mathsf{Tw}\widehat{\mathcal{P}}$ be the associated twisted properad. One has morphisms

$$\mathsf{Tw}(i) : (\widehat{\mathcal{H}olieb}_{c,d}, \delta) \longrightarrow (\mathsf{Tw}\widehat{\mathcal{P}}, \partial_\bullet), \qquad \Phi : \mathsf{Def}\left(\widehat{\mathcal{H}olieb}_{c,d} \xrightarrow{i} \widehat{\mathcal{P}}\right) \longrightarrow \mathrm{Der}(\mathsf{Tw}\widehat{\mathcal{P}})$$

given explicitly by the same formulae as in (43) and in Theorem **4.4.1**.

4.5.1. Proposition. *For any dg properad $\widehat{\mathcal{P}}$ under $\widehat{\mathcal{H}olieb}_{c,d}$ there is an associated morphism of complexes,*

$$\mathcal{F}: \mathsf{GC}^{or}_{c+d+1} \longrightarrow \mathrm{Der}(\mathsf{Tw}\widehat{\mathcal{P}})[1],$$

where GC^{or}_{c+d+1} is the oriented version of the Kontsevich graph complex. If $c+d=2$, there is an associated linear map $\mathfrak{grt} \longrightarrow H^1\left(\mathrm{Der}(\mathsf{Tw}\widehat{\mathcal{P}})\right)$.

Proof. The morphism $\mathsf{Tw}(i)$ induces a morphism of dg Lie algebras,

$$\mathsf{Def}(\widehat{\mathcal{H}olieb}_{c,d} \stackrel{\mathrm{Id}}{\to} \widehat{\mathcal{H}olieb}_{c,d}) \longrightarrow \mathsf{Def}(\widehat{\mathcal{H}olieb}_{c,d} \stackrel{\mathsf{Tw}(i)}{\longrightarrow} \mathsf{Tw}\widehat{\mathcal{P}})$$

The l.h.s. can be identified as a complex (but not as a Lie algebra) with the degree shifted derivation complex $\mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d})[-1]$ while the r.h.s. can be mapped, according to Theorem **4.4.1**, into the complex $\mathrm{Der}(\mathsf{Tw}(\widehat{\mathcal{P}})$. Thus we obtain a chain of morphisms of complexes

$$\mathcal{F}: \mathsf{GC}^{or}_{c+d+1} \stackrel{F}{\longrightarrow} \mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d}) \longrightarrow \mathrm{Der}(\mathsf{Tw}\widehat{\mathcal{P}})[1]$$

which proves the claim. □

Thus fully twisted completed properads under $\mathcal{H}olieb_{c,d}$ can have potentially a highly non-trivial homotopy theory depending on the properties of the above map $\mathcal{F}$ at the cohomology level.

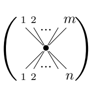
4.6. From representations of $\mathcal{P}$ to representations of $\mathsf{Tw}\mathcal{P}$. Assume a dg properad $\mathcal{P}$ under $\mathcal{H}olieb_{c,d}$ admits a representation in a dg space (V,d). Then the graded vector space $V[-c]$ is a $\mathcal{H}olieb_{0,c+d}$-algebra. For any Maurer-Cartan element $\gamma \in \odot^{\geq 1}(V[-c])$, that is, for any a solution of the equation (33), we obtain a presentation of $\mathsf{Tw}\mathcal{P}$ in V equipped with the twisted differential (34). Let us consider such twisted representations in the case $\mathcal{P} = \mathcal{H}olieb_{c,d}$ in detail.

4.6.1. Twisted $\mathcal{H}olieb_{c,d}$-algebra structures: an explicit description. Let (V,d) be a graded vector space equipped with a basis $\{e_\alpha\}$ and V^* its dual equipped with the dual basis $\{e^\alpha\}$. Consider a graded commutative tensor algebra

$$\odot^{\geq 1}(V[-c]) \otimes \odot^{\geq 1}(V^*[-d]) \subset \odot^\bullet\left(V[-c] \oplus V^*[-d]\right) \simeq \mathbb{K}[x^\alpha, p_\alpha]$$

where $x^\alpha := \mathfrak{s}^{-d} e^\alpha$, $p_\alpha := \mathfrak{s}^{-c} e_\alpha$. The paring $V[-c] \otimes V^*[-d] \to \mathbb{K}[-c-d]$ makes this space into a Lie algebra with respect to the Poisson type brackets $\{\ ,\ \}$ (of degree $-c-d$).

There is a 1-1 correspondence representations of $\mathcal{H}olieb_{c,d}$ in V,

$\rho: \mathcal{H}olieb_{c,d} \longrightarrow \mathcal{E}nd_V \Rightarrow \Big\{\rho\Big($  $\Big) =: \pi^m_n(x,p) \in \odot^n(V^*[-d]) \otimes \odot^m(V[-c])\Big\}_{m,n\geq 1, m+n\geq 3}$,

and Maurer-Cartan elements of the Lie algebra $(\odot^{\geq 1}(V[c]) \bigotimes \odot^{\geq 1}(V^*[d]), \{\ ,\ \})$, that is, with degree $1+c+d$ elements

$$\pi(x,p) = \sum_{n,m\geq 1} \pi^m_n(x,p) = \sum_{m\geq 1}\sum_{\alpha_\bullet,\beta_\bullet} \frac{1}{m!n!} \pi^{\alpha_1\ldots\alpha_m}_{\beta_1\ldots\beta_n} p_{\alpha_1}\cdots p_{\alpha_m} x^{\beta_1}\cdots x^{\beta_n}$$

such that

$$\{\pi,\pi\} = 2\sum_\alpha \pm \frac{\partial \pi}{\partial x^\alpha}\frac{\partial \pi}{\partial p_\alpha} = 0$$

and the degree 1 summand $\pi^1_1 \in V \otimes V^*$ is precisely the given differential d in V. The representation ρ induces a representation of the associated polydifferential operad

$$\mathcal{O}_{c,d}(\rho): \quad \mathcal{O}_{c,d}(\mathcal{H}olieb_{c,d}) \longrightarrow \mathcal{E}nd_{\odot^\bullet(V[-c])}$$

and hence a $\mathcal{H}olie_{c+d}$-algebra structure on $\mathcal{E}nd_{\odot^\bullet(V\{-c\})}$ via the composition of $\mathcal{O}_{c,d}(\rho)$ with the map (31). Let

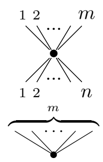
$$\gamma(p) = \sum_{m\geq 1} \gamma_m(p), \quad \gamma_m \in \odot^m(V[-c]) \subset \mathbb{K}[p_\alpha]$$

be a Maurer-Cartan element of that $\mathcal{H}olie_{c+d}$-algebra structure, that is, a degree $c+d$ solution of the following explicit coordinate incarnation of the Maurer-Cartan equation (33),

$$\sum_{n\geq 1} \pm \frac{1}{n!}\frac{\partial^n \pi}{\partial x^{\alpha_1}\ldots \partial x^{\alpha_n}}|_{x=0} \frac{\partial \gamma(p)}{\partial p_{\alpha_1}}\ldots \frac{\partial \gamma(p)}{\partial p_{\alpha_n}} = 0.$$

Then the data $\pi(x,p)$ and $\gamma(p)$ give rise to a representation,

$$\rho^{Tw}: \quad \mathsf{Tw}\mathcal{H}olieb_{c,d} \longrightarrow \mathcal{E}nd_V$$



$$\longrightarrow \pi_n^m \qquad\qquad \longrightarrow \gamma_m$$

of the twisted prop $\mathsf{Tw}\mathcal{H}olieb_{c,d}$ in $V = \mathrm{span}\langle e_\alpha\rangle$ equipped with the deformed differential

$$d_\bullet = d + \sum_{k\geq 1} \pm e_\beta \frac{1}{k!}\frac{\partial^{k+2}\pi}{\partial p_\beta \partial x^{\alpha_0} x^{\alpha_1}\ldots \partial x^{\alpha_k}}_{x=p=0} \frac{\partial \gamma(p)}{\partial p_{\alpha_1}}|_{p=0}\cdots \frac{\partial \gamma(p)}{\partial p_{\alpha_k}}|_{p=0} \frac{\partial}{\partial e_{\alpha_0}} \tag{48}$$

The associated twisted $\mathcal{H}olieb_{c,d}$ structure on V is given explicitly by (cf. (43))

$$\pi^{\mathsf{Tw}} := \sum_{\substack{m,n\geq 1\\ \alpha_\bullet,\beta_\bullet}} \frac{1}{m!n!}\pi^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n} p_{\beta_1}\ldots p_{\beta_m}(x^{\alpha_1} + \frac{\partial \gamma}{\partial p_{\alpha_1}})\ldots(x^{\alpha_n} + \frac{\partial\gamma}{\partial p_{\alpha_n}})$$

The MC equation for γ ensures that $\pi^{\mathsf{Tw}}|_{x=0} = 0$. As π^{Tw} is produced from π by the change of variables $x^\alpha \to x^\alpha + \frac{\partial\gamma(p)}{\partial p_\alpha}$, it is easy to check — using the vanishing of the sum

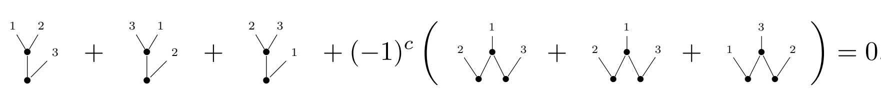
$$\sum \pm \frac{\partial^2\gamma(p)}{\partial p_\alpha \partial p_\beta})\frac{\partial \pi(x,p)}{\partial x^\alpha}\frac{\partial\pi(x,p)}{\partial x^\beta} \equiv 0$$

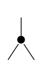
solely for degree+symmetry reasons — that that the equation $\{\pi^{\mathsf{Tw}}, \pi^{\mathsf{Tw}}\} = 0$ holds true indeed. Finally, one notices that the $(1,1)$ summand in π^γ (which is responsible for the differential on V) is precisely the twisted differential (48) or, equivalently, $d + \sum_{k\geq 2}\hat{\mu}_{k,1}$ in the notation of §**4.2)**. This gives a short and independent "local coordinate" check of many "properadic" claims made above.

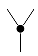
4.7. Homotopy triangular Lie bialgebras and Lie trialgebras. Assume V is a vector space concentrated in degree zero, say, $V = \mathbb{K}^N$ for some $N \in \mathbb{N}$, and let $\mathcal{H}olieb_{1,1}$ be the prop of ordinary Lie bialgebras. Any representation ρ of $\mathsf{Tw}\mathcal{H}olieb_{1,1}$ in V is uniquely determined by its values on generators of cohomological degree zero only, i.e. only on the following three generators of $\mathsf{Tw}\mathcal{H}olieb_{1,1}$,

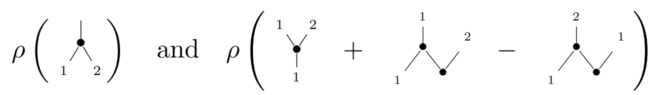
$\rho($  $): V \to \wedge^2 V, \quad \rho($  $): \wedge^2 V \to V, \quad \rho($  $) \in \wedge^2 V$

which satisfy the standard relations (20) as well as the following one,

(49) 

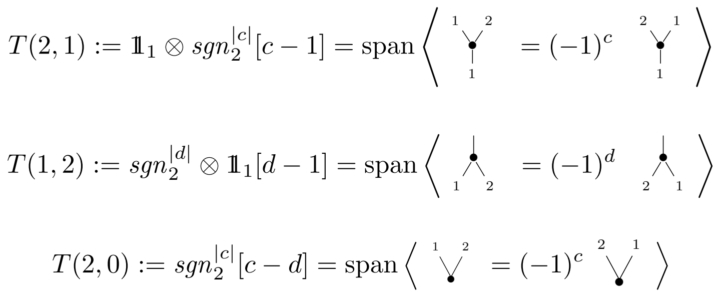
If $\rho($  $)$ happens to be zero, then the new relation (49) reduced to the classical Yang-Baxter equation so that associated $\mathsf{Tw}\mathcal{H}olieb_{1,1}$-algebra structure in V becomes precisely a so called *triangular Lie bialgebra structure* on V [D1]. Thus a generic $\mathsf{Tw}\mathcal{H}olieb_{1,1}$-algebra structure on

$\mathbb{K}^N$ is a version of that notion in which V has two Lie bialgebra structures, one is given by the pair $\rho($  $)$ and $\rho($  $)$ and one is given by a pair

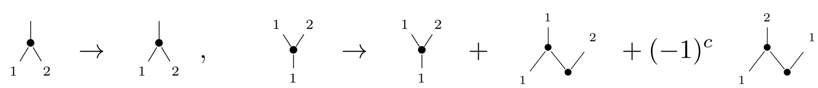


in which the Lie cobracket is twisted by the coboundary term.

Motivated by the above observation, we introduce a properad of *Lie trialgebras* $\mathcal{L}ieb^{\vee}_{c,d}$ which is generated by the $\mathbb{S}$-bimodule $T = \{T(m,n)\}_{m,n\geq 0}$ with all $T(m,n) = 0$ except



modulo relations (20) and (49). This properad comes equipped with two morphisms from $\mathcal{L}ieb_{c,d}$, the one which is identity on the generators of $\mathcal{L}ieb_{c,d}$ and the twisted one given by

(50) 

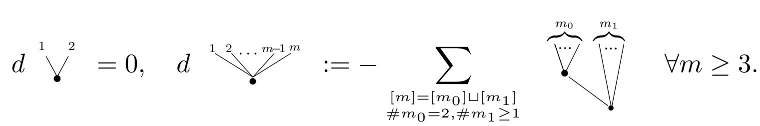
The full twisting construction gives us a minimal resolution of $\mathcal{L}ieb^{\vee}_{c,d}$ as follows. Consider a quotient dg properad

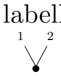
$$\mathcal{H}olieb^{\vee}_{c,d} := \mathsf{Tw}\mathcal{H}olieb_{c,d}/\langle \quad \rangle$$ 

by the ideal generated by the univalent MC generator.

4.7.1. Theorem. *The canonical projection $\mathcal{H}olieb^{\vee}_{c,d} \to \mathcal{L}ieb^{\vee}_{c,d}$ is a quasi-isomorphism.*

Proof. Consider a filtration of $\mathcal{H}olieb^{\vee}_{c,d}$ by the number of MC generators. The differential d in the associated graded complex $gr\mathcal{H}olieb^{\vee}_{c,d}$ acts on the generators coming from $\mathcal{H}olieb_{c,d}$ by the standard formula (22) while on the MC generators by



Since the number of the MC generators is preserved, we can assume that they are distinguished, say, labelled by integers. Then the direct summand of $gr\mathcal{H}olieb^{\vee}_{c,d}$ with, say, k MC generators (labelled by integers from $[k]$) can by identified with a direct summand in $\mathcal{H}olieb_{c,d}$ whose first in-legs (labelled by integers from $[k]$) are attached to "operadic type" $(m_i, 1)$-corollas with $m_i \geq 2$, $i \in [k]$. The cohomology of this summand is spanned by trivalent corollas only; trivalent $(2,1)$ corollas whose unique in-legs are labelled by integers from $[k]$ correspond in this approach precisely to k copies of the MC generator  . This result proves the claim. $\square$

4.7.2. Properad of triangular Lie bialgebras and its minimal resolution. Triangular Lie bialgebras appear naturally in the representation theory of the twisted properad $\mathsf{Tw}\mathcal{H}olieb_{c,d}$. Consider a quotient properad

$$\mathcal{L}ieb^{\triangle}_{c,d} := \mathcal{L}ieb^{\vee}_{c,d}/I$$

of the defined above properad $\mathcal{L}ieb^{\vee}_{c,d}$ by the ideal I generated by the coLie corolla. Thus $\mathcal{L}ieb^{\triangle}_{c,d}$ governs two operations of degrees $1-d$ and $d-c$ respectively,

$$= (-1)^d \quad , \qquad = (-1)^c$$

which are subject to the following relations

(51) $$\mathcal{R}^{\triangle}: \left\{ \quad + \quad + \quad = 0, \qquad + \quad + \quad = 0. \right.$$

Its representations in a dg vector space V are precisely degree shifted triangular Lie bialgebra structures in V, the case $c = d = 1$ corresponding to he ordinary triangular Lie bialgebras [D1]. There is a morphism of properads

$$f : \mathcal{L}ieb_{c,d} \longrightarrow \mathcal{L}ieb^{\triangle}_{c,d}$$

given on the generators by

$$f(\ ) = \quad , \qquad f(\ ) = \quad + (-1)^c$$

Consider an ideal $I^{\triangle}$ in $\mathsf{Tw}\mathcal{H}olieb_{c,d}$ generated by all (m,n)-corollas with $m \geq 2$, $n \geq 1$. This ideal is differential, and, moreover, the quotient properad $\mathsf{Tw}\mathcal{H}olieb_{c,d}/I^{\triangle}$ is a dg *free* properad with generators

$$= (-1)^d \quad \forall \sigma \in \mathbb{S}_{n\geq 2} \quad , \qquad = (-1)^{c|\tau|} \quad \forall \tau \in \mathbb{S}_{m\geq 1},$$

of degrees $1+d-nd$ and $d+c-mc$ respectively. This is an extension of $\mathsf{Tw}\mathcal{H}olie_d$ by the MC $(m,0)$-generators with $m \geq 2$. The induced differential acts on the unique $(1,2)$- and $(2,0)$-generators trivially, i.e. they are cohomology classes. In fact every cohomology class in $\mathsf{Tw}\mathcal{H}olieb_{c,d}/I^{\triangle}$ is generated by this pair via properadic compositions. Indeed, consider a further quotient of $\mathsf{Tw}\mathcal{H}olieb_{c,d}/I^{\triangle}$ by the ideal generated by and denote that quotient by $\mathcal{H}olieb^{\triangle}_{c,d}$; notice that the induced differential in $\mathcal{H}olieb^{\triangle}_{c,d}$ is much simplified: this properad is freely generated by the operad $\mathcal{H}olie_d$ equipped with standard differential (5) and the MC elements , $m \geq 2$ with the differential given by

$$\partial_{\bullet} \quad = 0, \quad \partial_{\bullet} \quad = - \sum_{\substack{k\geq 2, [m]=\sqcup [m_{\bullet}],\\ m_0=1, m_1,\ldots,m_k\geq 1}} \frac{1}{k!} \qquad \forall\, m \geq 3.$$

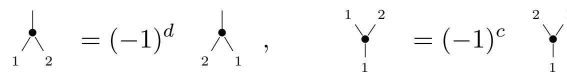
Let show that the projection

$$\mathsf{Tw}\mathcal{H}olieb_{c,d}/I^{\triangle} \longrightarrow \mathcal{H}olieb^{\triangle}_{c,d}$$

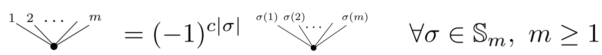
is a quasi-isomorphism. Consider a filtration of both sides by the number of the MC $(m,0)$-generators with $m \geq 2$ (this number can not decreese). The induced differential on the associated graded of the right hand side acts only on $\mathcal{H}olie_d$ generators so that its cohomology is a properad, say P, generated freely by $\mathcal{L}ie_d$ and the MC generators with $m \geq 2$. On the other hand, the associated graded of the left hand side is isomorphic to tensor products (modulo the action of finite permutation groups) of complexes $\mathsf{tw}\mathcal{H}olie_d$ with the trivial complex spanned by $(m,0)$ generators with $m \geq 2$. According to [DW], the $H^{\bullet}(\mathsf{tw}\mathcal{H}olie_d) = \mathcal{L}ie_d$ so that on the l.h.s. we get the same properad P. Thus at the second page of the spectral sequence the above map becomes the identity map implying, by the Comparison Theorem, the resquired quasi-isomorphism.

It has been proven [Kh] using Gröbner basis techniques (cf. [DK]) that $\mathcal{H}olieb^{\triangle}_{c,d}$ is a minimal resolution of $\mathcal{L}ieb^{\triangle}_{c,d}$ at the dioperadic level. Perhaps this result holds true at the properadic level as well. Homotopy triangular Lie bialgebras (in a different, non-properadic, context) have been studied in [LST] where their relation to homotopy Rota–Baxter Lie algebras has been established.

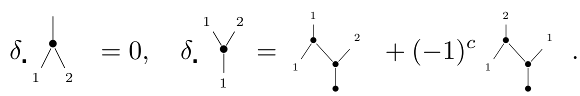
4.8. Full twisting of properads under $\mathcal{L}ieb_{c,d}$. Assume $\mathcal{P} = \mathcal{L}ieb_{c,d}$ and the map $i : \mathcal{H}olieb_{c,d} \to \mathcal{L}ieb_{c,d}$ is the canonical projection. The associated twisted dg prop(erad) $(\mathsf{Tw}\mathcal{L}ieb_{c,d}, \delta_\bullet)$ is generated by the standard corollas



of degrees $1-c$ and $1-d$ respectively modulo relations (20), as well by the family of extra generators,

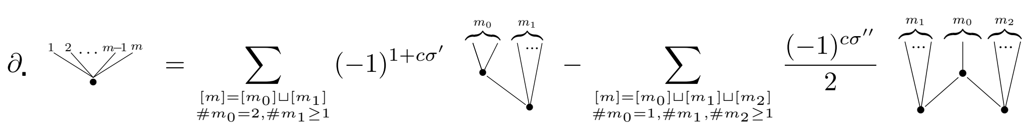


of cohomological degree $(1-m)c+d$. The twisted differential $\delta_\bullet$ is given explicitly on the first pair of generators by



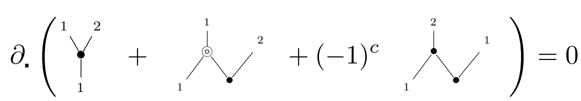
where we used relations (20) (and an ordering of vertices in the second line just above goes from the bottom to the top). The action of $\partial_\bullet$ on the remaining MC generators is given by

and, for $m \geq 3$, by

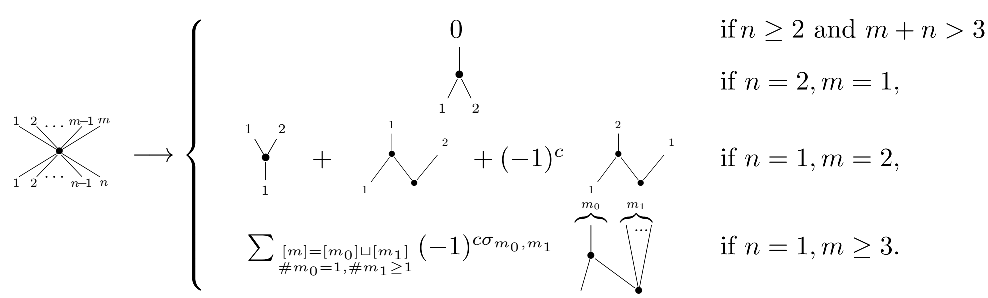


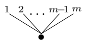
where σ' (resp. σ'') is the parity of the permutation $[m] \to [m_0] \sqcup [m_1]$ (resp. $[m] \to [m_0] \sqcup [m_1] \sqcup [m_2]$) associated with the partition of the ordered set $[m]$ into two (resp. three) disjoint ordered subsets.

These relations imply



which is in agreement with the general result saying that $\mathsf{Tw}\mathcal{L}ieb_{c,d}$ is a properad under $\mathcal{H}olieb_{c,d}$; the morphism (43) takes in this case the following form

(52) 

Note that the quotient of the dg operad $\mathsf{Tw}\mathcal{L}ieb_{c,d}$ by the (differential) ideal generated by corollas  with $m \neq 2$ gives us precisely the properad of Lie trialgebras $\mathcal{L}ieb^{\vee}_{c,d}$. Theorem **4.3.6** implies that the canonical projection $\mathsf{Tw}\mathcal{L}ieb_{c,d} \to \mathcal{L}ieb_{c,d}$ is a quasi-isomorphism.

Similarly one can describe explicitly the twisted properad $\mathsf{Tw}\mathcal{P}$ associated to any properad $\mathcal{P} \in \mathsf{PROP}_{\mathcal{L}ieb_{c,d}}$. Note that $\mathsf{Tw}\mathcal{P}$ is *not* in general a properad under $\mathcal{L}ieb_{c,d}$ as higher homotopy operations of type $(m \geq 3, 1)$ can be non-trivial.

5. Full twisting endofunctor in the case of involutive Lie bialgebras

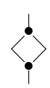
5.1. Introduction. This section adopts the full twisting endofunctor Tw in the category of properads under $\mathcal{H}olieb_{c,d}$ to the case when (strongly homotopy) Lie bialgebras satisfy the *involutivity or diamond* condition (which is often satisfied in applications). The corresponding twisting endofunctor

$$\mathsf{Tw}^{\diamond} : \mathsf{PROP}_{\mathcal{H}olieb^{\diamond}_{c,d}} \longrightarrow \mathsf{PROP}_{\mathcal{H}olieb^{\diamond}_{c,d}}$$

admits a much shorter and nicer formulation than Tw due to the equivalence of $\mathcal{H}olieb^{\diamond}_{c,d}$-algebra structures and the so called homotopy $\mathcal{BV}^{com}$-structures which are used heavily in the Batalin-Vilkovisky formalism of the mathematical physics and QFT.

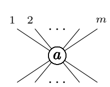
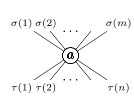
We give many formulae explicitly but omit all the calculations proving them because proofs are analogous to the ones given in the previous sections. This *diamond* version $\mathsf{Tw}^{\diamond}$ of Tw gives us a properadic incarnation the well-known and very nice constructions from [CFL, NW]; more precisely that constructions are recovered in our approach via representations of the twisted properad $\mathsf{Tw}^{\diamond}\mathcal{H}olieb^{\diamond}_{c,d}$.

5.2. Reminder on involutive Lie bialgebras. Given any pair of integers c, d of the same parity, $c = d \bmod 2\mathbb{Z}$, the properad $\mathcal{L}ieb^{\diamond}_{c,d}$ of *involutive* Lie bialgebras is defined as the quotient of $\mathcal{L}ieb_{c,d}$ by the ideal generated by the involutivity, or "diamond", relation

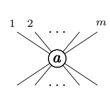
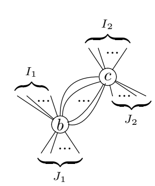
 $= 0.$

Note that this relation is void in $\mathcal{L}ieb_{c,d}$ for c and d of opposite parities.

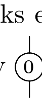
It was proven in [CMW] that the minimal resolution $\mathcal{H}olieb^{\diamond}_{c,d}$ of the properad $\mathcal{L}ieb^{\diamond}_{c,d}$ is a free properad generated by the following (skew)symmetric corollas of degree $1+c(1-m-a)+d(1-n-a)$

(53)  $= (-1)^{(d+1)(\sigma+\tau)}$  $\quad \forall \sigma \in \mathbb{S}_m, \forall \tau \in \mathbb{S}_n,$

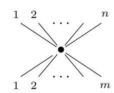
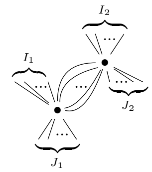
where $m+n+a \geq 3$, $m \geq 1$, $n \geq 1$, $a \geq 0$. The differential in $\mathcal{H}olieb^{\diamond}_{d}$ is given on the generators by

(54) δ  $= \sum\limits_{l \geq 1} \ \sum\limits_{a=b+c+l-1} \ \sum\limits_{\substack{[m]=I_1 \sqcup I_2 \\ [n]=J_1 \sqcup J_2}} \pm$ 

where the summation parameter l counts the number of internal edges connecting the two vertices on the right hand side, and the signs are fixed by the fact that they all equal to -1 for c and d odd integers.

The "plus" extension (see §**2.3**), $\mathcal{H}olieb^{\diamond+}_{c,d}$, of this properad looks especially natural — one adds just one extra $(1,1)$-generator (which we denote from now on by ) to the list while keeping the differential (54) formally the same.

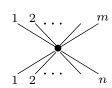
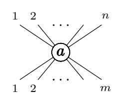
Let $\hbar$ be a formal variable of degree $c+d$ and, for a vector space V. Let $V[[\hbar]]$ stand for the topological vector space of formal power series with coefficients in V; it is a module over the topological ring $\mathbb{K}[[\hbar]]$ of formal power series in $\hbar$. Consider a dg properad $\mathcal{H}olieb^{\hbar+}_{c,d}$ which is identical to $\mathcal{H}olieb^{+}_{c,d}[[\hbar]]$ as a topological $\mathbb{K}[[\hbar]]$-module but is equipped with a different $\hbar$-dependent differential

(55) δ  $= \sum_{l\geq 1} \sum_{[m]=I_1\sqcup I_2 \atop [n]=J_1\sqcup J_2} \pm \hbar^{l-1}$ 

where l counts the number of internal edges connecting the two vertices on the r.h.s. The symbol $\pm$ stands for -1 in the case $c,d\in 2\mathbb{Z}$. There is a morphism of dg properads (cf. [CMW])

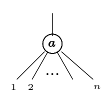
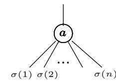
$$\mathcal{F}^{+}: \mathcal{H}olieb^{\hbar+}_{c,d} \longrightarrow \mathcal{H}olieb^{\diamond+}_{c,d}[[\hbar]]$$

given on the generators as follows (cf. [CMW])

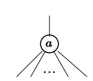
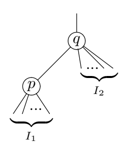
(56) $\mathcal{F}^{+}:$  $\longrightarrow \sum_{a=0}^{\infty} \hbar^a$  $\forall\ m,n\geq 1.$

There is obviously a 1-1 correspondence between morphisms of dg properads $\mathcal{H}olieb^{\diamond}_{c,d} \longrightarrow \mathcal{P}$ in the category of graded vector spaces over $\mathbb{K}$, and continuous morphisms of dg properads $\mathcal{H}olieb^{\hbar+}_{c,d} \longrightarrow \mathcal{P}[[\hbar]]$ in the category of topological $\mathbb{K}[[\hbar]]$-modules.

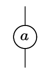
Let $\mathcal{H}olie^{\diamond}_{d}$ be the quotient of $\mathcal{H}olieb^{\diamond}_{c,d}$ by the (differential) ideal generated by all corollas with the number of outgoing legs ≥ 2. It is generated by the following family of (skew)symmetric corollas with $a\geq 0$, $n\geq 1$ and $a+n\geq 2$,

 $= (-1)^{d|\sigma|}$  $\forall\ \sigma\in\mathbb{S}_n,$

which are assigned degree $1-d(n-1+a)$; the induced differential acts as follows

δ  $= \sum_{a=p+q \atop [n]=I_1\sqcup I_2} \pm$ 

Representations, $\rho: \mathcal{H}olie^{\diamond}_{d} \to \mathcal{E}nd_V$, of this operad in a dg vector space (V,∂) can be identified with continuous representations of the topological operad $\mathcal{H}olie_d[[\hbar]]$ in the topological vector space $V[[\hbar]]$ equipped with the differential

$\partial + \sum_{p\geq 1} \hbar^p \Delta_p, \quad \Delta_p := \rho\Big($  $\Big).$

Here the formal parameter $\hbar$ is assumed to have homological degree d.

It is easy to see that the quotient of the dg properad $\mathcal{H}olieb^{\hbar+}_{c,d}$ by the (differential) ideal generated by all corollas with the number of outgoing legs ≥ 2 is identical to $\mathcal{H}olie^{+}_{d}[[\hbar]]$ as a dg properad. Hence we obtain from (56) a canonical morphism of dg properads

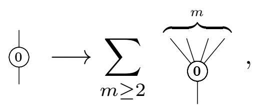
$$f^{+}: \mathcal{H}olie^{+}_{d}[[\hbar]] \longrightarrow \mathcal{H}olie^{\diamond+}_{d}[[\hbar]] \tag{57}$$

It gives us a compact presentation of any morphism $\mathcal{H}olie^{\diamond+}_{d} \to \mathcal{P}$ as an associated continuous morphism of properads $\mathcal{H}olie_d[[\hbar]] \to \mathcal{P}[[\hbar]]$ in the category of topological $\mathbb{K}[[\hbar]]$-modules.

5.2.1. Proposition. *There is a morphism of dg operads*

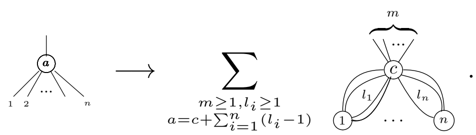
$$F^+ : \mathcal{H}olie^{\diamond +}_{c+d} \to \mathcal{O}_{c,d}\mathcal{H}olieb^{\diamond}_{c,d}$$

given explicitly on the $(1,1)$*-generators by*



and on the remaining $(1,n)$*-generators with* $a+n\geq 2$ *by*

(58)

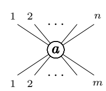


Proof is a straightforward direct calculation (cf. §**4.1.1**). The existence of such a map follows also from Proposition 5.4.1 and Lemma B.4.1 proven in [CMW].

Given a $\mathcal{H}olieb^{\diamond}_{c,d}$-algebra structure,

$$\rho : \mathcal{H}olieb^{\diamond}_{c,d} \longrightarrow \mathcal{E}nd_V,$$

$$\mu^a_{m,n} := \rho\left(\quad\right) : \odot^n(V[-c]) \longrightarrow (\odot^m(V[-c]))[1+(c+d)(1-n-a)],$$



in a graded vector space V, there is an associated $\mathcal{O}_{c,d}\mathcal{H}olieb_{c,d}$-algebra structure in $\odot^\bullet(V[-c])$ given in terms of polydifferential operators, and hence a continuous $\mathcal{H}olie^+_d[[\hbar]]$-algebra structure on $\odot^\bullet(V[-c])[[\hbar]]$ given by the composition

$$\mathcal{H}olie^+_d[[\hbar]] \xrightarrow{f^+} \mathcal{H}olie^{\diamond +}_d[[\hbar]] \xrightarrow{F^+} \mathcal{O}_{c,d}\mathcal{H}olieb_{c,d}[[\hbar]] \to \mathcal{E}nd_{\odot^\bullet(V[-c])}[[\hbar]]$$

Assuming that the latter is nilpotent (or appropriately filtered which is often the case in applications), one defines a *Maurer-Cartan element* γ *of the given* $\mathcal{H}olieb_{c,d}$*-algebra structure in* V as a Maurer–Cartan of the induced continuous $\mathcal{H}olie^{\hbar +}_{c+d}$-algebra structure in $\odot^\bullet(V[-c])[[\hbar]]$. Using (58) one can describe such an MC element as a homogeneous (of degree $c+d$) formal power series

$$\gamma = \sum_{a\geq 0, m\geq 0} \hbar^a \gamma_{a,m} \in \odot^{\bullet\geq 1}(V[-c])[[\hbar]], \quad \gamma_{a,m} \in \odot^m(V[-c]), \tag{59}$$

satisfying the equation

$$\Delta_\rho\left(e^{\frac{\gamma}{\hbar}}\right) = 0 \tag{60}$$

where Δ_ρ is a degree $+1$ polydifferential operator on $\odot^\bullet(V[-c])[[\hbar]]$ given, in an arbitrary basis $\{p_\alpha\}$ of $V[-c]$ as a sum (cf. §**4.2**)

$$\Delta_\rho := \sum_{\substack{a\geq 0\\ m,n\geq 1}} \pm \hbar^{a+n-1} \mu^a_{m,n}(p_{\alpha_1}\otimes \ldots \otimes p_{\alpha_n}) \frac{\partial^n}{\partial p_{\alpha_1}\ldots \partial p_{\alpha_n}} \tag{61}$$

Here the differential in V is encoded as $\mu^0_{1,1}$. The operator Δ_ρ encodes fully the given $\mathcal{H}olieb^{\diamond}_{c,d}$-algebra structure ρ in V: there is a *one-to-one correspondence* [CMW, Me3] between $\mathcal{H}olieb^{\diamond}_{c,d}$-algebra structures in V and degree 1 operators on $\odot^\bullet(V[-c])[[\hbar]]$ of the form

$$\Delta = \sum_{a\geq 0} \hbar^a \Delta_a$$

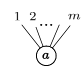
such that Δ_a is a derivation of the graded commutative algebra $\odot^\bullet(V[-c])$ of order $\leq a+1$ (such structures are often called $\mathcal{BV}^{com}_\infty$-algebra structures in the literature [Kr]).

The sum

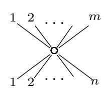
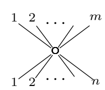
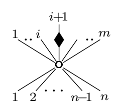
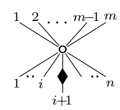
$$\delta_\gamma v := \sum_{n\geq 0} \frac{1}{n!}\mu^0_{1,n+1}(\underbrace{\gamma_1,\ldots,\gamma_1}_{n},v), \tag{62}$$

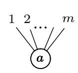
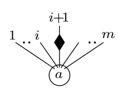
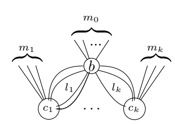
is a twisted differential on V which is used in the following definition-proposition.

5.3. Diamond twisting endofunctor. The *diamond twisting*, $(\mathsf{Tw}^\diamond\mathcal{P},\partial_\bullet)$, of a dg properad $(\mathcal{P},\partial)$ under $\mathcal{H}olieb^\diamond_{c,d}$ is, by definition, a properad freely generated by $\mathcal{P}$ and the $\mathbb{S}$-bimodule $M=\{M(m,n)\}$ such that $M(m,n)=0$ for $n\geq 1$ and $M(m,0)=\oplus_{a\geq 0}M_a(0,m)$ with $M_a(0,m)$ being the following 1-dimensional representations of $\mathbb{S}_m$,

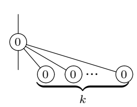
$$M_a(0,m) := sgn_m^{|c|}[(c+d)(1-a)-cm] = \mathrm{span}\left\langle\right.$$  $$\left.\right\rangle$$

The differential in $\mathsf{Tw}^\diamond\mathcal{P}$ is defined on the generators as follows

(63) $\partial_\bullet$  $= \partial$  $+\sum_{i=0}^{m-1}$  $-(-1)^{|a|}\sum_{i=0}^{n-1}$ 

(64) $\partial_\bullet$  $:= \sum_{i=0}^{m-1}$  $- \sum_{\substack{k\geq 1, [m]=\sqcup[m_\bullet]\\ a=b+\sum_{i=1}^k(c_i+l_i-1)}} \frac{1}{k!}$  $\forall\ a,m\geq 1.$

where  is given by

(65)  $:= \sum_{k=1}^{\infty}\frac{1}{k!}$ 

Note that for $m+a\geq 1$ the first sum on the r.h.s. of (42) cancels out with all the summands corresponding to $k\geq 2, m_0=1, m_i=m-1, c_i=a, i\in[k]$, in the second sum.

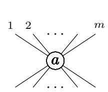
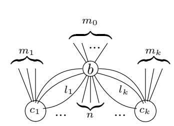
5.3.1. Theorem. *For any dg properad $\mathcal{P}$ equipped with a map*

$$f:\mathcal{H}olieb^\diamond_{c,d}\longrightarrow\mathcal{P}$$

there is an associated map of dg properads

$$\mathsf{Tw}f:\mathcal{H}olieb^\diamond_{c,d}\longrightarrow\mathsf{Tw}^\diamond\mathcal{P}$$

given explicitly by

 $\longrightarrow \sum_{\substack{k\geq 1, [m]=\sqcup[m_\bullet]\\ a=b+\sum_{i=1}^k(c_i+l_i-1)}} \pm\frac{1}{k!}$ 

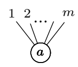
(In the case $c,d\in 2\mathbb{Z}$ the symbol $\pm$ above can be replaced by $+1$.)
To prove this statement one has to check the compatibility of $\mathsf{Tw}f$ with the differentials on both sides. This can be done either by a direct (but tedious) computation or by studying generic representations of both properads involved in the above statement as is done briefly in §**5.4** below in the most important and illustrative case $\mathcal{P}=\mathcal{H}olieb^\diamond_{c,d}$.

In a full analogy to §**4.4.1** the deformation complex of any morphism f as above acts on $\mathsf{Tw}^\diamond\mathcal{P}$ by derivations, that is, there is a morphism of dg Lie algebras

$$\mathsf{Def}\left(\mathcal{H}olieb^\diamond_{c,d}\xrightarrow{f}\mathcal{P}\right)\longrightarrow \mathrm{Der}(\mathsf{Tw}\mathcal{P})$$

5.4. Representations of $\mathsf{Tw}^\diamond\mathcal{H}olieb^\diamond_{c,d}$. Let $\rho : \mathcal{H}olieb^\diamond_{c,d} \to \mathcal{E}nd_V$ be a homotopy involutive Lie bialgebra structure in a graded vector space V and let Δ_ρ be its equivalent incarnation as a differential operator (61) on $\odot^\bullet(V[-c][[\hbar]]$. Assume a Maurer-Cartan element γ of this $\mathcal{H}olieb^\diamond_{c,d}$-structure is fixed, that is, a formal power series (59) satisfying the equation (60). This datum (ρ, γ) gives us

(i) a representation of $\mathsf{Tw}\mathcal{H}olieb^\diamond_{c,d}$ in V which sends the MC generators,



$$\longrightarrow \gamma_{a,m} \in \odot^m(V[-c])$$

to the corresponding summands of the MC series (59).

(ii) a *twisted* $\mathcal{H}olieb^\diamond_{c,d}$-algebra structure on V which can be encoded as the following γ-twisted differential operator

$$\Delta_\gamma := e^{-\frac{\gamma}{\hbar}} \circ \Delta_\rho \circ e^{\frac{\gamma}{\hbar}} =: \sum_{a\geq 0} \hbar^a \Delta_{(a)\gamma}$$

The MC equation (60) guarantees that summands $\Delta_{(a)\gamma}$ are differential operators of order $\leq a+1$ so that Δ_γ induces some $\mathcal{H}olieb^\diamond_{c,d}$-algebra structure on V indeed. A straightforward combinatorial inspection of Δ_γ recovers the universal properadic formula shown in Theorem §**5.3.1**.

Thus, contrary to the twisting endofunctor Tw introduced in the previous section, its diamond version $\mathsf{Tw}^\diamond$ gives us essentially nothing new — it reproduces in a different language the well-known twisting construction introduced in §9 of [CFL] in terms of generic representations of the properads $\mathcal{H}olieb^\diamond_{c,d}$ and $\mathsf{Tw}^\diamond\mathcal{H}olieb^\diamond_{c,d}$. In the special class of representations of $\mathcal{L}ieb^\diamond_{c,d}$ on the spaces of cyclic words, the MC equation (60) has been introduced and studied by S. Barannikov [B1, B2] in the context of the deformation theory of modular operads and its applications in the theory of Kontsevich moduli spaces.

A beautiful concrete solution Γ of the MC equation (60) has been constructed by F. Näef and T. Willwacher in [NW] when studying string topology of not necessarily simply connected manifolds M; that MC element γ has been obtained in [NW] from the so called *partition function* Z_M on M which has been constructed earlier by R. Campos and T. Willwacher in [CW] when studying new graph models of configuration spaces of manifolds. Thus the $\mathcal{H}olieb^\diamond_{3-n}$-algebra structure constructed in [NW] on the space of cyclic words $Cyc(H^\bullet(M)[1])$ of the de Rham cohomology $H^\bullet(M)$ of an n-dimensional closed manifold M gives us an example of the action of the twisting endofunctor $\mathsf{Tw}^\diamond$ on standard $\mathcal{L}ieb^\diamond_{3-n}$-algebra structure on $Cyc(H^\bullet(M)[1])$.

References

[AWZ] A. Andersson, T. Willwacher and M. Živković, *Oriented hairy graphs and moduli spaces of curves*, arXiv:2005.00439 (2020).

[B1] S. Barannikov, *Modular operads and Batalin-Vilkovisky geometry.* Int. Math. Res. Not. IMRN (2007), no. 19, Art. ID rnm075, 31 pp.

[B2] S. Barannikov, *Noncommutative Batalin-Vilkovisky geometry and matrix integrals.* C. R. Math. Acad. Sci. Paris **348** (2010), 3591-7362

[CMW] R. Campos, S. Merkulov and T. Willwacher *The Frobenius properad is Koszul*, Duke Math. J. **165**, No.1 (2016), 2921-2989.

[CW] R. Campos and T. Willwacher, *A model for configuration spaces of points*, to appear in Algebraic & Geometric Topology,

[CS] M. Chas and D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, in: The legacy of Niels Henrik Abel, pp. 771–784, Springer, Berlin, 2004.

[CFL] K. Cieliebak, K. Fukaya and J. Latschev, *Homological algebra related to surfaces with boundary*, Quantum Topology, **11**, No.4 (2020) 691-837.

[CGP] M. Chan, S. Galatius and S. Payne, *Topology of moduli spaces of tropical curves with marked points*, preprint arXiv:1903.07187 (2019)

[CL] J. Chuang and A. Lazarev, Combinatorics and formal geometry of the master equation, Lett. Math. Phys. **103**, 1 (2013) 79-112.

[C1] K. Costello, *The A_∞ operad and the moduli space of curves*, arXiv: math.AG/0402015 (2004)

[C2] K. Costello, *A dual version of the ribbon graph decomposition of moduli space*, Geometry & Topology **11** (2007) 1637-1652.

[DK] V. Dotsenko and A. Khoroshkin, *Gröbner bases for operads*, Duke Math. J. **153**, no. 2 (2010), 363-396

[D1] V. Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations*, Soviet Math. Dokl. **27** (1983) 68–71.

[D2] V. Drinfeld, *On quasitriangular quasi-Hopf algebras and a group closely connected with* $Gal(\bar{Q}/Q)$, Leningrad Math. J. **2**, No. 4 (1991), 829–860.

[DW] V. Dolgushev and T. Willwacher, *Operadic twisting — with an application to Deligne's conjecture*, Journal of Pure and Applied Algebra **219** (2015) 1349-1428

[DSV] V. Dotsenko S. Shadrin B. Vallette, *The twisting procedure*, arXiv:1810.02941 (2018)

[FTW] B. Fresse, V. Turchin and T. Willwacher, *The rational homotopy of mapping spaces of* E_n *operads* Preprint, arXiv:1703.06123, 2017.

[G] E. Getzler, *Two-dimensional topological gravity and equivariant cohomology*, Comm. Math. Phys. **163** (1994), no. 3, 473-489.

[Kh] A. Khoroshkin, private communication.

[Ko1] M. Kontsevich, *Formality Conjecture*, In: D. Sternheimer et al. (eds.), Deformation Theory and Symplectic Geometry, Kluwer 1997, 139-156.

[Ko2] M. Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys. **48**(1) (1999), 351-772.

[Ko3] M. Kontsevich, unpublished.

[Kr] O. Kravchenko, *Deformations of Batalin-Vilkovysky algebras*, in "Poisson Geometry (Warsaw, 1998)", Banach Center Publ., vol. 51, Polish Acad. Sci., Warsaw, 2000, 131-139.

[LV] P. Lambrechts and I. Volic, *Formality of the little N-disks operad*, Memoirs of the AMS **230** (2013), 116pp.

[LST] A. Lazarev, Y. Sheng and R. Tang, *Homotopy relative Rota-Baxter Lie algebras, triangular* L_∞ *-bialgebras and higher derived brackets*, arXiv:2008.00059 (2020).

[MaVo] M. Markl and A.A. Voronov, *PROPped up graph cohomology*. In: "Algebra, Arithmetic and Geometry - Manin Festschrift" (eds. Yu. Tschinkel and Yu. Zarhin), Vol. II, Progr. Math. vol. 270, Birkhauser (2010) pp. 249-281.

[Me1] S.A. Merkulov, *Formality theorem for quantizations of Lie bialgebras*, Lett. Math. Phys. **106** (2016) 169-195

[Me2] S.A. Merkulov, *Gravity prop and moduli spaces* $\mathcal{M}_{g,n}$, arXiv:2108.10644 (2021)

[Me3] S.A. Merkulov, *Prop of ribbon hypergraphs and strongly homotopy involutive Lie bialgebras*, Internat. Math. Res. Notices (2022) rnac023.

[Me4] S.A. Merkulov, *From gravity to string topology*, arXiv:2201.01122 (2012)

[MeVa] S. Merkulov and B. Vallette, *Deformation theory of representations of prop(erad)s I & II*, J. für die reine und angewandte Mathematik (Qrelle) **634**, 51-106, & **636**, 123-174 (2009)

[MW1] S.A. Merkulov and T. Willwacher, *Props of ribbon graphs, involutive Lie bialgebras and moduli spaces of curves*, preprint arXiv:1511.07808 (2015) 51pp.

[MW2] S. Merkulov and T. Willwacher, *Deformation theory of Lie bialgebra properads*, In: Geometry and Physics: A Festschrift in honour of Nigel Hitchin, Oxford University Press 2018, pp. 219-248.

[MW3] S.A. Merkulov and T. Willwacher, *Classification of universal formality maps for quantizations of Lie bialgebras*, Compositio Mathematica, **156** (2020) 2111-2148

[NW] F. Naef and T. Willwacher, *String topology and configuration spaces of two points*, preprint arXiv:1911.06202 (2019).

[P] R.C. Penner, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. **113** (1987), 299-339.

[We] C. Westerland, *Equivariant operads, string topology, and Tate cohomology*, Math. Ann. **340** (2008), no. 1, 97-142.

[W1] T. Willwacher, *M. Kontsevich's graph complex and the Grothendieck-Teichmueller Lie algebra*, Invent. Math. **200** (2015), 671-760.

[W2] T. Willwacher, *The oriented graph complexes*, Comm. Math. Phys. 334 (2015), no. 3, 1649–1666.

Sergei Merkulov: Department of Mathematics, University of Luxembourg, Grand Duchy of Luxembourg
Email address: sergei.merkulov@uni.lu